Infinite sums and the calculation of $\ldots$, as presented by the Swedish mathematician Anders Gabriel Duhre in the early 18th century

Downloaded from: https://research.chalmers.se, 2020-01-31 06:40 UTC

Citation for the original published paper (version of record):
Pejlare, J. (2019)
Infinite sums and the calculation of $\ldots$, as presented by the Swedish mathematician Anders Gabriel Duhre in the early 18th century
Proceedings of the Eighth European Summer University on History and Epistemology in Mathematics

N.B. When citing this work, cite the original published paper.
INFINITE SUMS AND THE CALCULATION OF $\pi$, AS PRESENTED BY THE SWEDISH MATHEMATICIAN ANDERS GABRIEL DUHRE IN THE EARLY 18TH CENTURY

Johanna PEJLARE
Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, SE-412 96 Gothenburg, Sweden
pejlare@chalmers.se

ABSTRACT
Anders Gabriel Duhre, an important mathematician and mathematics educator in Sweden during the 18th century, contributed with two textbooks in mathematics, one in algebra and one in geometry. Among others, he treats infinitesimals based on Nieuwentijt’s theories from Analysis infinitorum and infinite sums based on Wallis’ method of induction from Arithmetica infinitorum. Based on these results, Duhre develops an ingenious method to determine the area enclosed by curves by constructing a corresponding curve. He applies his method to the circle in order to find an expression of $\pi$ as an infinite series. The series he finds is a modified version of the Gregory-Leibniz’ series. In the present paper we consider in detail Duhre’s presentation in order to further investigate the influence upon him as well as his influence on the Swedish mathematical society of his time.

1 Introduction
The Swedish mathematician and mathematics educator Anders Gabriel Duhre (c. 1680–1739) was an important and influential person in the Swedish mathematical society in the early 18th century (Rodhe, 2002). He studied mathematics at Uppsala University, Sweden, and for some time he was a student of the Swedish scientist, inventor and industrialist Christopher Polhem (1661–1751) at his school Laboratorium Mechanicum in Stjärnsund. For some years Duhre taught mathematics to engineering students at Bergskollegium (a central agency in the mining industry) and to prospective officers at the Royal Fortification Office in Stockholm. In 1723 he opened his own school, Laboratorium Mathematico-Oeconomicum, outside Uppsala, where theoretical and practical subjects were taught to young boys (Hebbe, 1933). Of particular interest is that mathematics was taught in this school; Duhre had knowledge of mathematics that was not yet taught at the university, and students at the university turned to him to learn more on modern mathematics. Among his students were several of the Swedish mathematicians to be established during the 1720s and 1730s (Rodhe, 2002). Duhre taught in Swedish and early on planned to write mathematical textbooks in Swedish in order to introduce the Swedish youth to new and modern mathematics.

Duhre contributed with two textbooks in mathematics – one in algebra and one in geometry. Both were based on his lecture notes from his teaching at Bergskollegium and the Royal Fortification Office. The first book, En Grundelig Inledning til Mathesin Universalem och Algebraen (“A thorough introduction to universal mathematics and algebra”), was edited by Georg Brandt and published in 1718. In this book, modern algebra based on Descartes’ notation is presented, as well as examples from Newton’s, Wallis’ and Nieuwentijt’s theories from the end of the 17th century. For example, he treats infinitesimals based on Nieuwentijt’s theory as presented in Analysis infinitorum (1695) and utilizes Wallis’ method of induction, as presented in Arithmetica infinitorum (1656), to determine the quotient of infinite series. In his second

In this paper, we will consider Duhre’s utilization of infinitesimals and infinite sums to determine the quotient between the circumference and the diameter of a circle, in order to find $\pi$ expressed as an infinite series. We will first give a short introduction to Nieuwentijt’s *Analysis infinitorum* and his utilization of infinitesimals, before we consider Duhre’s interpretation of Nieuwentijt’s work. Thereafter we will consider Wallis’ *Arithmetica infinitorum* and how Duhre utilizes his method of induction to determine the quotient of infinite series. Following that, we will consider Duhre’s method to find the area enclosed by curves. Finally, we will consider how Duhre utilizes this method on a circle and how he determines an expression for $\pi$.

2 **Infinitesimals in Nieuwentijt’s *Analysis infinitorum***

The Dutch philosopher and mathematician Bernard Nieuwentijt (1654–1718) is, in particular, known for his critique on the foundations of Leibniz’ infinitesimal calculus. In 1695 he published *Analysis infinitorum*, a book “written by a beginner for beginners”\(^1\) on elementary infinitesimal calculus. This book is primarily of a didactic character; he attempted at presenting mathematics in a systematic way as a coherent unit (Vermij, 1989). In the prologue he presents three definitions and two axioms which enable him to deduce rules for calculating with the infinite and infinitesimal quantities through more than 50 lemmas. In the chapters following the introduction, these lemmas lead to the propositions on infinitesimal calculus.

For Nieuwentijt, a quantity is infinitesimal if it is smaller than any arbitrary given quantity and it is infinite if it is greater than any arbitrary given quantity. The word infinitesimal is however not used in the definitions, axioms or lemmas. Instead, Nieuwentijt uses the expression “datâ minor” which can be translated into “the given smallest”. Of central importance is his first axiom:

Anything that when multiplied, however many times, does not equal another given quantity, however small, cannot be considered a quantity, geometrically it is absolutely nothing.\(^2\)

The main peculiarity of Nieuwentijt’s approach to infinitesimals is represented in Lemma 10, where it is stated that if an infinitesimal quantity is multiplied by an infinitesimal quantity, then the product is zero or nothing. The product of two infinitesimal quantities, or “the infinite small of the infinite small”, can be interpreted as Leibniz’ second differential. However, whereas Nieuwentijt considered squares of infinitesimals to be equal to zero, this is generally not the case with Leibniz’ differentials (Mancosu, 1996).

---

\(^1\)“Tyroni scriptum tyronibus” (Nieuwentijt, 1695, præfatio).

\(^2\)“Qui cquid toties sumi, hoc est per tantum numerum multiplicari non potest, ut datum ullam quantitatem, ut ut exiguum, magnitudine suă æquare valeat, quantitas non est, sed in re geometricâ merum nihil” (Nieuwentijt, 1695, p. 2).
3 Infinitely small quantities in Duhre’s textbook on algebra

In Chapter XXVI of his book on algebra, Duhre presents an interpretation of the prologue of Nieuwentijt’s Analysis infinitorum (1695). An infinitely small quantity is defined by Duhre as:

If a quantity is divided by an infinitely big number, one should consider the received quotient to be infinitely small; it is something that is smaller than the smallest quantity that can ever be given.\(^5\)

Thus, according to Duhre, if \(\mathfrak{D}\) is an infinitely big number then the quotient \(\frac{a}{\mathfrak{D}}\) is infinitely smaller than the quantity \(a\). Duhre considers the nature of an infinitely big number to be that it is bigger than every given number and that it thus can be seen as “ceaselessly growing with no return”.\(^4\) From this it follows that \(\frac{a}{\mathfrak{D}}\) is smaller than the smallest quantity that can ever be given. Duhre gives a proof by contradiction that \(\frac{a}{\mathfrak{D}}\) really is “smaller than the smallest”: if \(c\) is a quantity that is smaller than \(\frac{a}{\mathfrak{D}}\), then the given quantity \(a\) is bigger than \(\mathfrak{D}c\) and the quotient \(\frac{a}{c}\) is bigger than the infinitely big quantity \(\mathfrak{D}\), but this “contradicts all truth”.\(^5\) Therefore, \(\frac{a}{\mathfrak{D}}\) must be smaller than the smallest quantity, i.e., an infinitely small quantity.

The arguments above show that handling the infinite is problematic. Duhre treats the infinite as a fixed number, but this is in conflict with his earlier statement that an infinite number grows ceaselessly. Also, it seems easier to accept the infinitely big than the infinitely small, since the existence of the infinitely small is proven with the help of a given existence of the infinitely big.

After introducing infinitely small quantities, Duhre continues with 14 lemmas with rules for calculating with them; 10 of these are also found in Nieuwentijt’s Analysis infinitorum. Among Duhre’s lemmas we find, among others, that the sum of two infinitely small quantities is an infinitely small quantity (Lemma 1) and that the product of any number and an infinitely small quantity is an infinitely small quantity (Lemma 3). Of great importance for his later presentation on infinite sums is Lemma 4, which corresponds to Nieuwentijt’s Lemma 10:

If an infinitely small part \(\frac{a}{\mathfrak{D}}\) is either multiplied by itself or by another infinitely small part \(\frac{d}{\mathfrak{D}}\); then the received product \(\frac{aa}{\mathfrak{D}\mathfrak{D}}\) or \(\frac{ad}{\mathfrak{D}\mathfrak{D}}\) is nothing or no quantity.\(^6\)

Thus, Duhre, just as Nieuwentijt, considers the square of infinitely small quantities to be equal to zero. In the proof of this lemma Duhre uses Nieuwentijt’s first axiom: If the product of two infinitely small quantities is multiplied by an infinite number, this will be equal to an infinitely small quantity, i.e., \(\mathfrak{D} \times \frac{aa}{\mathfrak{D}\mathfrak{D}} = \frac{aa}{\mathfrak{D}}\) and \(\mathfrak{D} \times \frac{ad}{\mathfrak{D}\mathfrak{D}} = \frac{ad}{\mathfrak{D}}\), and since something multiplied by an infinite

\(^3\) “Om en föreståld quantitet hälles före vara fördehalad utaf ett oändligen stort tahl; bör man anse then ther af komma quotienten för oändeligen lijen thet är för en ting som är mindre än then allerminsta quantitet som någonsin kan gifwas” (Brandt, 1718, p. 212).

\(^4\) “[…] ophörölingen växande utan någon återvända” (Brandt, 1718, p. 213).

\(^5\) “[…] stridande emot all sanning” (Brandt, 1718, p. 213).

\(^6\) “Om en oändligen lijen deh\(\frac{d}{\mathfrak{D}}\) antingen warder multiplocrad med sig sielf eller med någon annan oändligen lijen deh\(\frac{a}{\mathfrak{D}}\) at then ther af komma producten \(\frac{aa}{\mathfrak{D}}\) eller \(\frac{ad}{\mathfrak{D}}\) måtte wara alsintet eller ingen quantitet” (Brandt, 1718, p. 214).
number is equal to an infinitely small number then this something is not a quantity and geometrically is nothing.

In this proof Duhre does not seem to have a problem handling the infinite; it is no problem for him to shorten the expression with the infinitely big number $\mathfrak{O}$. He uses Lemma 4 in Lemma 14 where he deals with how infinitely small quantities can be handled in equations. He concludes that in an equation involving infinitely small quantities, the infinitely small quantities can be omitted, since, if the equation is divided by an infinitely big number $\mathfrak{O}$, then it follows from Lemma 4 that these can be considered as nothing. Algebraically this lemma can be interpreted as $x + \frac{a}{\mathfrak{O}} = x$ since $\frac{x}{\mathfrak{O}} + \frac{a}{\mathfrak{O}^2} = \frac{x}{\mathfrak{O}}$.

4 Wallis’ *Arithmetica infinitorum*

After considering the introduction of Nieuwentijt’s *Analysis infinitorum*, Duhre, in Chapter XXVII of his book on algebra, proceeds with studying John Wallis’ (1616–1703) *Arithmetica infinitorum* from 1656. *Arithmetica infinitorum* was an important text in the 17th century, in particular regarding the transition from geometry to algebra and regarding infinite series (Stedall, 2005). For example, Isaac Newton (1642–1727) was influenced by Wallis in his work towards integral calculus. Introducing new methods and concepts, Wallis’ purpose was to find a general method of quadrature, i.e., finding the area enclosed by curves, or rather the ratios of those areas to inscribed or circumscribed rectangles. He achieved this by drawing together ideas from René Descartes’ (1596–1650) algebraic geometry and Bonaventura Cavalieri’s (1598–1647) theory of indivisibles. Wallis’ results were based on the summation of indivisibles or infinitesimal quantities, where an indivisible can be considered to have at least one dimension equal to zero, as for example a line or a plane, while an infinitesimal is considered to have an arbitrarily non-zero width or thickness. Wallis was however not concerned with the distinction between indivisibles and infinitesimals and generally spoke of infinitely small quantities.

In order to find the area enclosed by curves, Wallis reduced the geometric problem to the summation of arithmetic sequences (Stedall, 2004). Two important mathematical methods he developed were *induction* and *interpolation*. Wallis’ method of induction relied on intuition; he believed that if a pattern was established for a few cases then it could be assumed to continue indefinitely. Also, in his method of interpolation he relied on intuition; for example, he assumed continuity regarding sequences of numbers in order to interpolate intermediate values. One example of this is when he used his method of interpolation between the triangular numbers 1, 3, 6, 10 … Another example of interpolation is when he, in Proposition 191, found the ratio of a square to an inscribed circle: $\frac{4}{\pi} = \frac{3\times3\times5\times5\times7\times7\text{etc.}}{2\times4\times4\times6\times6\times8\text{etc.}}$.

5 Infinite sums in Duhre’s textbook on algebra

We now turn our attention to Duhre’s textbook on algebra again. We will here only consider those parts when Duhre uses Wallis’ method of induction in order to deal with infinite sums. Duhre begins Chapter XXVII by determining that the proportion of the sum of infinitely many squares with the roots 1, 2, 3, 4, 5 et cetera to the *summan totidem terminorum maximo aequalium* equals the proportion of 1 to 3. The *summan totidem terminorum maximo aequalium*
is explained to be “the sum of the greatest term as many times as there are terms in the progression”? Thus, in modern notation the proportion to be determined can be interpreted as:

$$\lim_{n \to \infty} \frac{\sum_{k=0}^{n} k^2}{(n+1)n^2} = \frac{1}{3}$$

Duhre proves this proportion using Wallis’ method of induction, as presented in *Arithmetica infinitiorum*. To do this, he first examines the proportion when \( n \) equals 1, 2, 3, 4, and 5 in the expression above:

\[
\begin{align*}
0 + 1 &= \frac{1}{3} + \frac{1}{6} \\
0 + 1 + 4 &= \frac{1}{3} + \frac{1}{12} \\
0 + 1 + 4 + 9 &= \frac{1}{3} + \frac{1}{18} \\
0 + 1 + 4 + 9 + 16 &= \frac{1}{3} + \frac{1}{24} \\
0 + 1 + 4 + 9 + 16 + 25 &= \frac{1}{3} + \frac{1}{30}
\end{align*}
\]

Duhre examines the pattern of the partial proportions and concludes that the denominators 6, 12, 18, 24, 30 et cetera form an arithmetical sequence. As long as the number of squares is finite the proportion is bigger than \( \frac{1}{3} \). However, if we have infinitely many (\( \mathfrak{O} \)) squares, the proportion will be \( \frac{1}{3} + \frac{1}{\mathfrak{O}} \). But since \( \frac{1}{3} + \frac{1}{\mathfrak{O}} = \frac{1}{3} \) according to Lemma 14 in Chapter XXVI (see Section 3), the proportion will be \( \frac{1}{3} \). Therefore, he concludes, the proportion of the sum of infinitely many squares with the roots 1, 2, 3, 4, 5 et cetera to the *summan todidem terminorum maximo aequalium* equals the proportion of 1 to 3.

In this presentation, Duhre closely follows Wallis, but unlike Wallis who in his following propositions offers geometrical interpretations of this result, Duhre does not do so. According to Wallis, the above proportion 1 to 3 geometrically corresponds to the proportion of the complement of half a parabola to the parallelogram completed by the same half parabola and its complement (Wallis, 1656, Prop. XXIII). Furthermore, Wallis’ method of induction would not be an accepted method of induction today, since only a limited number of cases for \( n = 1, 2, 3, \ldots \) were tested and the induction step (i.e., if the property is assumed to be true for \( n = k \) it should be proven to be true for \( n = k + 1 \)) was not included.

Duhre proceeds by proving the corresponding proportion for cubes with the help of Wallis’ method of induction. In modern notation, he proves the following:

$$\lim_{n \to \infty} \frac{\sum_{k=0}^{n} k^3}{(n+1)n^3} = \frac{1}{4}$$

---

7 “[…] en summa innehållande then största ledamoten så ofta som progressionens ledamöter äre” (Brandt, 1718, p. 77).
After these two proofs, using Wallis method of induction, Duhre states that, again interpreted in modern notation, the following proportions are true:

\[
\lim_{n \to \infty} \frac{\sum_{k=0}^{n} k^4}{(n + 1)n^4} = \frac{1}{5}
\]

\[
\lim_{n \to \infty} \frac{\sum_{k=0}^{n} k^5}{(n + 1)n^5} = \frac{1}{6}
\]

\[
\lim_{n \to \infty} \frac{\sum_{k=0}^{n} k^6}{(n + 1)n^6} = \frac{1}{7}
\]

6  Duhre’s method of finding the area enclosed by curves

Let us now turn to Duhre’s textbook on geometry. We will consider Duhre’s method of finding the area enclosed by curves in order to see how he uses the proportions including infinite sums that he considered in his *Algebra*. In Chapter XXX Duhre formulates a proposition where he considers the curve $ABCD$ and from it constructs the curve $AIOM$ such that the area of the segment $ADCBA$ is equal to half of the area $AEMOIA$ (see Figure 1). The curve $AIOM$ is constructed in the following way: Let $AS$ be a tangent at the point $A$, parallel to the ordinate $DE$ and for every point $C$ on $ABCD$ with a tangent $CG$ where $G$ is a point on $AS$, the ordinate $OK$ is equal to the line $AG$.

![Figure 1: The area of the segment $ADCBA$ is equal to half of the area $AEMOIA$ (Duhre, 1721, p. 572).](image)

Duhre proves this proposition without using algebra, only considering geometrical properties. First, he draws a few helplines. He draws the line $AQ$ parallel to $DG$ such that $ADGQ$ is a parallelogram. If the point $C$ is considered to be infinitely close to the point $D$, he concludes that the line $CD$ can be considered to be a straight line and thus it can be considered to be a part of the tangent $DG$. Then he draws the line $CL$ parallel to $DM$ and the lines $CR$, $LM$ and $NP$ parallel to $AE$. Finally, he draws the line $AC$. The proof of the proposition follows:

Since the two parallel lines $DM$ and $CL$ are infinitely close to each other, the points $L$ and $O$ are infinitely close to each other, and thus the mixed lines figure $EMOK$ must be the same as the parallelogram $EMLK$. Furthermore, the lines $EM$, $AG$ and $CN$ are equal to each other and hence the parallelogram $EMLK$ equals the parallelogram $PNCR$, which in turn equals the parallelogram $QNCD$. Now, if $CD$ is considered as a base, the parallelogram $QNCD$ is twice as big as the triangle $ACD$, since the lines $CD$ and $AQ$ are parallel. This implies that also the mixed lines figure $EMOK$ and the parallelogram $PNCR$ are twice as big as the triangle $ACD$. Finally, if other lines parallel to the line $DM$ are drawn, each of the resulting mixed lines figures are
twice as big as the corresponding triangles for the same reason that the mixed lines figure EMOK is twice as big as the triangle ACD. Therefore, the figure AEMOI A, which is the composite of the mixed lines figures, equals twice the sum of the corresponding triangles that forms the segment ADCB, which is what Duhre wanted to prove.

7 Duhre’s method applied to the circle

In order to calculate the decimals of π, or more specifically, in order to show that the proportion between the diameter and the circumference of a circle is approximately the same as 100 to 314, Duhre now wants to apply the proposition from Chapter XXX to a circle, i.e., instead of considering the circumference he considers the area of a circle. He begins Chapter XXXI with considering a half circle; the area under the corresponding curve to a half circle should be equal to the area of a full circle (see Figure 2). However, the corresponding curve ASM to the half circle ACB in fact is an asymptote to the line BV, and thus the “indescribable width” of the area contained by the “indescribable” line ASM is equal to the area of the circle. However, the “undescribable width” is too difficult for Duhre to consider further. Therefore, he instead considers a quarter of a circle ACD and its corresponding curve ASR. Doing this, the area ADRH equals twice of the area of the segment ACE according to the proposition in Chapter XXX. By adding half of this area to the area of the triangle ADC and multiplying the expression by four, an expression of the area of the circle will be given.

![Figure 2: The area ADRH equals twice of the area of the segment ACE (Duhre, 1721, p. 574).](image)

Instead of calculating the area of the figure ADRH, Duhre’s idea is to calculate the area of the figure ARQ. He states that the line AQ, which is equal to the line AD, can be divided into infinitely many equal parts, and the lines NT, OH, PS et cetera proceeding from these points of intersection will fill up the figure ARQ.

Now Duhre introduces the variables a, x and y. He lets AB = 2a, i.e., the radius of the circle equals a, the ordinate GH = AF = DI = x and AG = y. He wants to find an expression for y, which can be considered as a length that varies. He does this using proportional reasoning: He first concludes that BG = 2a − y and, because of properties of the circle the square of GE equals AG · BG which is the same as 2ay − y². Considering the two uniform triangles BDI and BGE, Duhre concludes that since BD, DI, BG and GE are geometrical proportional, i.e.,

8 “[…] obeskrifweliga widden” (Duhre, 1721, p. 110).
BD, DI :: BG, GE, the squares BDq, DIq, BGq and GEq will also be geometrical proportional, i.e., BDq, DIq :: BGq, GEq.\(^9\) From this it follows that \(aa, xx :: 4aa - 4ay + yy, 2ay - yy\), which can be simplified into \(aa, xx :: 2a - y, y\). He now uses the fact that the product of the two utmost in a geometrical progression equals the product of the two inners, i.e., \(aay = 2axx - xy\). By adding \(xy\) and dividing by \(aa + xx\) on both sides, Duhre now finally finds the expression \(y = \frac{2axx}{aa + xx} = AG\). This quotient can be expressed as an infinite series:

\[
AG = y = \frac{2axx}{aa + xx} = \frac{2xx}{a} - \frac{2x^4}{a^3} + \frac{2x^6}{a^5} - \frac{2x^8}{a^7} - \cdots
\]

Furthermore, he concludes that if \(GH = 2x\) then

\[
AG = \frac{8xx}{a} - \frac{32x^4}{a^3} + \frac{128x^6}{a^5} - \frac{512x^8}{a^7} - \cdots
\]

if \(GH = 3x\) then

\[
AG = \frac{18xx}{a} - \frac{162x^4}{a^3} + \frac{1458x^6}{a^5} - \frac{13122x^8}{a^7} - \cdots
\]

and so on. Since \(AQ = a\) is divided into infinitely many equal parts, where the first one is \(AN = x, AO = 2x, AP = 3x\), and so on, the expressions above give the corresponding lengths of \(AG = y\). These lengths could also be denoted \(NT, OH, PS\) according to Figure 2. The last of these lengths is \(QR = a\). The infinitely many lengths together fill up the figure \(AQR\), and therefore Duhre now has to compute the infinite sum of these infinitely many series. In order to compute the sum, i.e., the area of the figure \(AQR\), Duhre now collects all terms of the same power of \(x\). Thus, the area \(AQR\) will be:

\[
\frac{2}{a} (xx + 4xx + 9xx &c.) - \frac{2}{a^3} (x^4 + 16x^4 + 81x^4 &c.) + \frac{2}{a^5} (x^6 + 64x^6 + 729x^6 &c.) &c.
\]

In modern notation this expression can be interpreted as

\[
\frac{2}{a} \lim_{n \to \infty} \sum_{k=1}^{n} (kx)^2 - \frac{2}{a^3} \lim_{n \to \infty} \sum_{k=1}^{n} (kx)^4 + \frac{2}{a^5} \lim_{n \to \infty} \sum_{k=1}^{n} (kx)^6 - \cdots
\]

To compute these sums, Duhre uses the results on infinite sums from his text book on algebra (see Section 5). First, he has to determine the summa totidem terminorum maximo æqualium. The summa totidem terminorum maximo æqualium to the infinite sum \(xx + 4xx + 9xx&c\). must be \(a \cdot aa\), since he considers \(a\) to be the number of terms in the infinite sum and \(aa\) to be the biggest term in the sum. It follows that, in modern notation, \(\lim_{n \to \infty} \sum_{k=1}^{n} (kx)^2 = \frac{1}{3}a^3\). In the same way \(\lim_{n \to \infty} \sum_{k=1}^{n} (kx)^4 = \frac{1}{5}a^5\), \(\lim_{n \to \infty} \sum_{k=1}^{n} (kx)^6 = \frac{1}{7}a^7\) and so on. Therefore, the infinite sum of the infinite series above, i.e., the area of the figure \(AQR\), will be equal to

\[\text{\textit{In modern notation:}} \frac{BD}{DI} = \frac{BG}{GE} \text{ i.e.,} \frac{BD^2}{DI^2} = \frac{BG^2}{GE^2}\]
\[
\frac{2}{a} \left( \frac{1}{3}a^3 \right) - \frac{2}{a^5} \left( \frac{1}{5}a^5 \right) + \frac{2}{a^7} \left( \frac{1}{7}a^7 \right) 
&= \\
\frac{2}{3}aa - \frac{2}{5}aa + \frac{2}{7}aa - \frac{2}{9}aa &\text{c.}
\]

Duhre can now easily find an expression for the area of the figure \(ARD\); he just has to take the area of the square of \(AQ\), i.e., \(a^2\), and subtract the area of the figure \(AQR\). Thus, the area of the figure \(ARD\) will be

\[
aa - \frac{2}{3}aa + \frac{2}{5}aa - \frac{2}{7}aa + \frac{2}{9}aa &\text{c.}
\]

According to the method presented in Chapter XXX (see Section 6), the area of the figure \(ARD\) is twice the area of the segment \(ACE\), and therefore it follows that the area of the segment \(ACE\) will be

\[
\frac{1}{2}aa - \frac{1}{3}aa + \frac{1}{5}aa - \frac{1}{7}aa + \frac{1}{9}aa &\text{c.}
\]

Now, adding the area of the triangle \(ADC\) to this expression and multiply with four will finally give an expression for the area of the circle with radius \(a\):

\[
4aa - \frac{4}{3}aa + \frac{4}{5}aa - \frac{4}{7}aa + \frac{4}{9}aa &\text{c.}
\]

Duhre modifies this expression even further, in order to find an expression for the circumference of the circle. Since the area of a circle equals the area of a triangle where the base equals the circumference of the circle and the height equals the radius of the circle, he concludes that he will find an expression of the circumference of the circle if he divides the area of the circle with half of its radius, i.e., \(\frac{1}{2}a\). Thus, he gets the following series expressing the circumference of the circle:

\[
8a - \frac{8}{3}a + \frac{8}{5}a - \frac{8}{7}a + \frac{8}{9}a &\text{c.}
\]

Duhre now lets the diameter of the circle, i.e., \(2a\), equal 1 and finds that the proportion between the diameter of a circle and its circumference is as one to the following series:

\[
4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} &\text{c.}
\]

He finally modifies this series by merging the terms pairwise:

\[
\frac{8}{3} + \frac{8}{35} + \frac{8}{99} + \frac{8}{195} + \frac{8}{323} + \frac{8}{483} + \frac{8}{675} &\text{c.}
\]

In modern notation we can interpret this result as
\[ \pi = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{8}{(4k-2)^2 - 1}. \]

8 Duhre’s calculation of \( \pi \)

After finding the proportion of the diameter of the circle to its circumference, Duhre proceeds with computing this proportion. He starts with constructing a table (see Figure 3) with the first 315 denominators of the series \( \sum_{k=1}^{n} \frac{8}{(4k-2)^2 - 1} \). This table is actually not completely correct, possibly due to typesetting errors. For example, for \( k=100 \) it says 258.403 instead of 158.403 and for \( k=50 \) it says 39.204 instead of 39.203.

Duhre proceeds with constructing a second table, containing the first 315 terms and partial sums of the series (see Figure 4). However, he does not want to consider decimals and therefore he considers a circle with diameter 100.000.000 instead of 1, i.e., the general numerator in the series will be 800.000.000 instead of 8. In modern notation this new series can be written as

\[ 100.000.000\pi = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{800.000.000}{(4k-2)^2 - 1}. \]
In this way the partial sums, after approximations, will be natural numbers. In the table in Figure 4 we can see that the proportion of the diameter of a circle to its circumference will be approximately as 100,000,000 to 314,000,528, or as 100 to 314.

Figure 4: Duhre’s table showing the first 315 approximated terms and partial sums in the series
\[ \sum_{n=1}^{\infty} \frac{(4k+2)^2}{(4k+1)^2} - 1 \] (Duhre, 1721, pp. 119–121).

Duhre concludes Chapter XXX by noting that in practice, when minor computations have to be made, the proportion 100 to 314 or the Archimedean proportion 7 to 22 can be used, the requested proportion being smaller than the former and bigger than the latter. If larger computations have to be performed, however, he suggests that the proportion 100,000 to 314,159 should be used. Nevertheless, he does not perform the computations needed to find this proportion.

9 Concluding remarks

The series \(4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} + \ldots\) which Duhre received before he merged the terms pairwise, we recognize as a Maclaurin series for \(4 \tan^{-1} x\) for \(x = 1\). Since \(4 \tan^{-1} 1 = \pi\), we can conclude that Duhre’s series is correct. However, it converges very slowly. This series is known as the Gregory–Leibniz’ series after James Gregory (1638–1675) and Gottfried Wilhelm Leibniz (1646–1716). Leibniz was concerned with the quadrature and when he applied his method to the circle he received the series \(\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\). Leibniz found this result in 1673, but already in 1671 Gregory, who was concerned with infinite series representations of transcendental functions, had found the corresponding Taylor series. Also, an Indian mathematician, whose identity is not definitely known, found the series for \(\tan^{-1} x\) during the 15th century (Roy, 1990). This series, written in Sanskrit verse, is usually ascribed to Kerala Gargya Nilakantha (c.1450–c.1550) and can be found in the book Tantrasangraha composed around 1500.
Since Duhre follows Wallis’ method of induction when he considers the infinite series, it may be surprising that he in his book on geometry does not proceed with studying Wallis’ interpolation method to find the area of a circle in order to find an expression for \( \pi \). However, Duhre’s method, where he from the circle constructs a corresponding curve where he can use the previously found infinite sums to find the enclosed area, is indeed ingenious. In his search for \( \pi \) Duhre also uses modern algebra that cannot be found in Wallis’ *Arithmetica infinitorum*. Duhre considers algebra to be helpful, since it enables complicated expressions to be transformed into simpler ones, and thus convenience in calculations is obtained.

While Duhre primarily was an educator, his main pioneering achievement was that he brought knowledge of modern mathematics into the Swedish mathematical community. Of particular value is his choice to write in Swedish in order to find a greater audience. Twice he applied for a position as professor at Uppsala University, without success, but he still succeeded in inspiring several among the next generation of Swedish mathematicians. Certainly, also his students at Bergskollegium and the Royal Fortification Office had the opportunity to be introduced into modern mathematics thanks to Duhre.

**Acknowledgement**

This work was supported by the Swedish Research Council [Grant no. 2015-02043].

**REFERENCES**


