Thesis for the Degree of Doctor of Philosophy

On the Einstein-Vlasov system: Stationary Solutions and Small Data Solutions with Charged and Massless Particles

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Abstract

The Vlasov matter model describes an ensemble of collisionless particles moving through space-time. These particles interact via the gravitational field which they create collectively. In the framework of General Relativity this gravitational field is described by space-time curvature. Mathematically the situation is captured by the Einstein-Vlasov system. If the particles are charged an electro-magnetic field is created as well and the Maxwell equations are coupled to the system in addition. In astrophysics Vlasov matter is widely used to describe galaxies, globular clusters or galaxy clusters. Also in cosmology or plasma physics the Vlasov matter model plays an important role.

In this thesis a collection of results on the Einstein-Vlasov system is presented. The Papers I to IV are concerned with stationary solutions and the Papers V and VI contain stability results for Minkowski space-time (the trivial solution of Einstein's field equations describing an empty, flat space-time), i.e. global existence results for the time evolution problem with small initial data.

In Paper I, spherically symmetric, static solutions of the Einstein-Vlasov system with massless particles are constructed. These solutions constitute very thin and highly dense shells of matter with a vacuum region at the center. One can think of these shells as highly energetic, bent light which keeps itself together through the strong gravitational field created by itself. In Paper II, charged particles are considered and the existence of spherically symmetric, static solutions of the Einstein-Vlasov-Maxwell system is proven. It is possible to obtain the large variety of different spherically symmetric, static solutions that are known in the uncharged case, as for example balls, shells and multi-shells. Paper III is concerned with isotropic solutions, i.e. solutions where the momenta are equally distributed among the particles. In this case Vlasov matter resembles a perfect fluid in many respects. It is shown that a uniqueness result for perfect fluids can be applied to Vlasov matter. This implies that every isotropic, static solution is uniquely determined by the surface potential and in particular spherically symmetric, if its overall pressure is not too high. In Paper IV solutions are constructed where the momenta are not equally distributed among the particles. These solutions have preferred axes of rotation or even an overall angular momentum. This way

axially symmetric (but not spherically symmetric), stationary solutions of the Einstein-Vlasov-Maxwell system are obtained.

In Paper V, exploiting the convenient conformal invariance properties of massless Vlasov matter, this matter model is integrated into the framework of the conformal Einstein field equations. In this framework, via a conformal rescaling, the physical space-time, which might be a perturbation of Minkowski space-time or de Sitter space-time, is identified with a compact portion of the Einstein-cylinder (or perturbations thereof). This way global Cauchy problems are turned into local Cauchy problems for which methods to obtain local existence are available. A semi-global stability result for Minkowski space-time and a global stability result for de Sitter space-time is obtained this way. In Paper VI the stability of Minkowski space-time for perturbations with massless Vlasov matter is proved with a completely different method, the vector field method for relativistic transport equations. Thereby an asymptotic stability result with very weak assumptions on the initial data is obtained, in particular no compact support assumptions of any kind are necessary for the initial data.

Keywords: General Relativity, Vlasov matter, Static Solutions, massless Einstein-Vlasov system, Einstein-Vlasov-Maxwell system, Conformal Einstein Field Equations, Vector Field Method, non-linear wave equation, symmetric hyperbolic system, Minkowski stability

List of appended papers

Paper I H. Andréasson, D. Fajman, M. Thaller, *Models for Self-Gravitating Photon Shells and Geons*, Ann. Henri Poincaré **18**, 681-705 (2017)

Paper II M. Thaller, *Existence of Static Solutions of the Einstein–Vlasov–Maxwell System and the Thin Shell Limit*, SIAM J. Math. Anal. **51** (3), 2231–2260 (2019)

Paper III T. Harada, M. Thaller, *Uniqueness of static, isotropic low-pressure solutions of the Einstein-Vlasov system*, preprint, ArXiv:1806.10539

Paper IV M. Thaller, *Rotating Clouds of Charged Vlasov Matter in General Relativity*, preprint, ArXiv:1907.10517 (accepted for publication in *Class. Quantum Gravity*)

Paper V J. Joudioux, M. Thaller, J. Valiente Kroon, *The Conformal Einstein Field Equations with Massless Vlasov Matter*, preprint, ArXiv:1903.12251

Paper VI L. Bigorgne, D. Fajman, J. Joudioux, J. Smulevici, M. Thaller, *Asymptotic Stability of Minkowski Space-Time with non-compactly supported massless Vlasov-matter*, preprint, unpublished

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Maximilian Thaller
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Contents

Introduction

1 A model for collisionless particles

In this PhD thesis the Vlasov matter model plays the central role. This matter model describes an ensemble of particles which are moving through space time. The particles' motion is thereby governed by fields that the particles create themselves collectively. For applications in astrophysics, the most important one of these fields is the gravitational field which is due to the particles' mass and energy. If the particles are charged an electro-magnetic field is also generated. Collisions however are not part of the model. Even though this model is based on relatively simple assumptions it constitutes a realistic matter model describing complex structures. These modelled structures mirror observable behaviour and many physical effects. For this reason Vlasov matter has become a well established model to describe structures like galaxies, galaxy clusters, or galactic nebulae. Phenomena as for example matter accretion by black holes [41], black hole formation [2] or star dynamics in a single galaxy [13] can be described. Also on a cosmological scale, as a model representing all the matter in the universe, the Vlasov matter model has been used [39]. In the model of *Conformal Cyclic Cosmology*, advanced by Penrose [33], radiative matter, as for example Vlasov matter with massless particles, plays the role of the dominant form of matter in the universe in certain time-periods.

The assumption of an absence of collisions between the particles seems to be fairly justified. Even though a galaxy typically contains several hundreds of billions of stars the stars make up only roughly a fraction of the millionth part of the volume of a galaxy. Even when our galaxy, the Milky Way, will collide with the Andromeda galaxy, which is predicted to happen in about 4.5 billion years, collisions will very rarely happen since they are so unlikely. Galaxies are very dilute structures. They rather fly through one another and deform one another via the gravitational fields that their stars create collectively. The larger the matter structures become, that one considers, the diluter they usually are. So, for example, on the level of galaxy clusters, where individual galaxies are modelled as single particles, collisions are even more unlikely [44].

There are many more matter models that have been studied in general relativity [38]. Examples are scalar fields, general relativistic dust or perfect fluids. The study of these matter models is interesting for many reasons. However, for the purpose of the understanding of the phenomena mentioned above the Vlasov matter model is particularly suitable. Interesting structures and physically relevant phenomena are covered by this model. In contrast to dust or scalar fields, non-trivial static solutions can be constructed. In contrast to dust the formation of *naked singularities* is less obvious [32] and maybe does even not occur. Perfect fluids are very suitable to describe individual stars. In some situations their dynamics is however very complex and hardly accessible with the available mathematical tools.

Vlasov matter with charged particles in the setting where gravity is modelled by General Relativity is described by the quadruple $(\mathscr{M}, g, f, F)$. Here $\mathscr{M}$ is a four dimensional manifold which is endowed with the Lorentzian metric g, f is the *particle distribution function* which is defined on the tangent bundle $T\mathscr{M}$ of $\mathscr{M}$, and F is the electro-magnetic field tensor.

The tuple $(\mathscr{M}, g)$ we will refer to as a *space-time*. The metric g encodes the curvature of the space-time $(\mathscr{M}, g)$. This space-time typically has a *causal structure*, such as a *time-orientation* and a notion of causal future and causal past of space-time events $x \in \mathscr{M}$. See for example [40, Section 10.2.4, ff.] for an introduction of these concepts. Basic causal notions are also the following. Let $p \in T_x\mathscr{M}$, $x \in \mathscr{M}$ be a vector. The vector p is *time-like* if $g(p,p) < 0$, it is *space-like* if $g(p,p) > 0$ and it is *light-like* if $g(p,p) = 0$. A *causal* vector is a vector which is either time-like or light-like. If a manifold is time orientable there exists a smooth time-like vector field T which encodes the time orientation. A causal vector $p \in T_x\mathscr{M}$, $p \neq 0$ is *future pointing* if $g(p,T) < 0$.

A freely falling particle is described by a future pointing, causal geodesic $\Gamma = \{(X^0(s), \ldots, X^3(s)) \in \mathscr{M} \ : \ s \in \mathbb{R}\}$. If at each point along the geodesic Γ the tangent vector $P = \dot{X}^\mu \partial_{x^\mu}$ is time-like it is the trajectory of a freely falling massive particle. Since $g(P,P)$ is constant along physically possible trajectories of particles, cf. [43], the particle rest mass m_p can be defined by

$$g(P,P) = -c^2 m_p^2, \tag{1.1}$$

where c is the speed of light. If there holds $g(P,P) = 0$ along the curve then the curve is said to be light like and describes the trajectory of a massless particle.

The particle rest mass m_p gives rise to the definition of the mass shell $\mathscr{P}_{m_p}$ via

$$\mathscr{P}_{m_p} = \{(x,p) \in T\mathscr{M} \ : \ g(p,p) = -c^2 m_p^2, \ p \text{ is future pointing}\}. \tag{1.2}$$

It can be shown [43] that it is a seven dimensional submanifold of $T\mathscr{M}$. The mass shell relation (1.1) can often be seen as a way to express the coordinate p^0 in terms of the coordinates x^μ, p^i, $\mu = 0, \ldots, 3$, $i = 1, 2, 3$.

In the Vlasov matter model the particles are not described individually but one considers an ensemble of particles which is described by the regular distribution function f. If particles of a certain rest mass m_p are considered (usually one only distinguishes between the massive case where $m_p = 1$ and the massless case where $m_p = 0$) the corresponding distribution function is defined on the corresponding mass shell $\mathscr{P}_{m_p}$. (A distribution function defined on the whole tangent bundle $T\mathscr{M}$ would describe a particle ensemble with a continuum of different rest masses.) If one integrates f over a volume on the mass shell one obtains the number of particles in the corresponding volume of the space-time manifold $\mathscr{M}$, the momenta p^i of which are in the corresponding volume of momentum space.

The particle distribution gives rise to an energy momentum tensor $T[f]$ via

$$T_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} \frac{c}{m_p} \int_{\mathscr{P}_x} f(x,p) p^\alpha p^\beta \, \mathrm{dvol}_{\mathscr{P}_x}, \tag{1.3}$$

where $\mathscr{P}_x$ denotes the fibre of the mass shell $\mathscr{P}_{m_p}$ over the space-time event $x \in \mathscr{M}$. This fibre is a three dimensional manifold equipped with the volume form $\mathrm{dvol}_{\mathscr{P}_x}$. A derivation of this form can be found in [39] or [46].

The electro-magnetic field tensor F encodes the electric field E and the magnetic field $\mathscr{B}$. The electric field E is defined by the splitting $F = E \wedge \mathrm{d}t + B$, where the two form B includes no term with $\mathrm{d}t$. The magnetic field is defined by the splitting $\star F = \mathscr{E} - \mathscr{B} \wedge \mathrm{d}t$, where $\star : \Lambda^2(\mathscr{M}) \to \Lambda^2(\mathscr{M})$ is the Hodge star operator and $\mathscr{E}$ is a two-form with no $\mathrm{d}t$-term. Cf. [17] for details. Like the particle distribution function f the electro-magnetic field tensor F gives rise to an energy momentum tensor. This energy momentum tensor is given by

$$\tau_{\mu\nu} = \frac{1}{4\pi} \left(-\frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} + F_{\nu\alpha} F_\mu{}^\alpha \right). \tag{1.4}$$

The Einstein-Vlasov-Maxwell system (EVM-system) consists in the following equations:

$$G = \frac{8\pi}{c^4} \left(T + \tau \right), \tag{1.5}$$

$$\mathfrak{T} f = 0, \tag{1.6}$$

$$\mathrm{d}F = 0, \tag{1.7}$$

$$\nabla_\alpha F^{\alpha\beta} = -4\pi q J^\beta, \quad J^\beta = \frac{1}{c} \int_{\mathscr{P}_x} f(x,p) p^\beta \mathrm{dvol}_{\mathscr{P}_x}. \tag{1.8}$$

The constant $q \in \mathbb{R}$ is the particle charge parameter. The equation (1.5) is Einstein's field equations. On the left hand side we have the Einstein tensor G, a curvature tensor. On the right hand side we have the energy momentum tensors (1.3) and (1.4).

The equation (1.6) is the Vlasov equation. It is equivalent to the statement that the particle distribution function stays constant along the characteristic curves $\{(X^\mu(s), P^i(s)) : s \in \mathbb{R}\} \subset T\mathscr{M}$ of the transport operator

$$\mathfrak{T} = p^\mu \partial_{x^\mu} + \left(q F^\mu{}_\alpha p^\alpha - \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta \right) \partial_{p^\mu}. \tag{1.9}$$

These characteristic curves are solutions to the system

$$\dot{X}^\mu(s) = P^\mu(s), \tag{1.10}$$

$$\dot{P}^\mu(s) = q F^\mu{}_\alpha(X(s)) P^\alpha(s) - \Gamma^\mu_{\alpha\beta}(X(s)) \, P^\alpha(s) \, P^\beta(s). \tag{1.11}$$

It is interesting to remark that the characteristic curves (1.10)–(1.11) never leave the mass shell. This is equivalent to the statements that the transport operator $\mathfrak{T}$, seen as vector field on the tangent bundle $T\mathscr{M}$, is tangent to the mass shell and that the rest mass parameter m_p, given via (1.1), is conserved along the characteristic curves of $\mathfrak{T}$. See [43] for details.

Furthermore it is interesting to remark the following. In the absence of an electro-magnetic field, i.e. if $F \equiv 0$, the projections of the characteristic curves of $\mathfrak{T}$ onto the space-time manifold $\mathscr{M}$ are geodesics. If there is however a non-vanishing electro-magnetic field present the projections of the characteristic curves of $\mathfrak{T}$ onto $\mathscr{M}$ are not geodesics.

The equations (1.7)–(1.8) are the Maxwell equations. The first Maxwell equation (1.7) is trivially satisfied if one assumes that the electro-magnetic field tensor F is induced by an electro-magnetic four potential $A \in \Lambda^1(T\mathscr{M})$ via $F = \mathrm{d}A$. The second Maxwell equation (1.8) describes how the particle current $J[f]$ determines the sources of the electro-magnetic field.

The Vlasov matter model has an equivalent formulation on the co-tangent bundle $T^*\mathscr{M}$. The characteristic curves for Vlasov particles on the co-tangent bundle can be obtained from (1.10)–(1.11) via the relation $P_\mu = g_{\mu\nu}P^\nu$. The characteristic curves of particles of a rest mass m_p are then lying in the co-mass shell

$$\mathscr{P}^*_{m_p} = \{(x,p) \in T^*\mathscr{M} \ : \ g^{\alpha\beta}p_\alpha p_\beta = -c^2 m_p^2, \ p \text{ is future pointing}\}. \tag{1.12}$$

For the problems described in the Sections 6 and 7 the formulation on the co-tangent bundle is particularly advantageous.

An overview over known results on the Einstein-Vlasov system can be found in [6]. The articles [43, 42] give a rigorous introduction to the system and its equations. The book [13] is a standard reference for galactic dynamics.

This thesis consists in several articles containing results on solutions of the Einstein-Vlasov or the EVM-system. In the following sections the problems which are considered in these articles are introduced and the main contributions are highlighted. The notation in the individual section corresponds to the notation in the corresponding article.

2 Shells with massless particles

The Einstein-Vlasov system admits static solutions with compactly supported matter quantities [6]. Such solutions solutions describe finitely extended clouds of Vlasov matter with a time-independent particle distributions. These particle clouds are sufficiently concentrated such that the gravitational field keeps the particles in a confined region in space. In the literature such solutions are referred to as *steady states*.

Steady states have been mainly investigated in spherical symmetry. Steady states which are not spherically symmetric have so far only been investigated numerically or analytically as perturbations of spherically symmetric steady states, cf. Section 5. In a spherically symmetric, static space-time there exist coordinates t, x^1, x^2, x^3 such that the metric can be written in diagonal form and is entirely determined by two spherically symmetric functions μ and λ. Denoting $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ the metric can be written as

$$g = -e^{2\mu(r)}\mathrm{d}t^2 + e^{2\lambda(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{2.1}$$

where $\mathrm{d}\Omega^2$ is the standard flat metric on the unit two-sphere and units are chosen such that $G = c = 1$. The Vlasov field f is said to be spherically symmetric and static if f is independent of t and $f(\vec{x}, \vec{p}) = f(A \cdot \vec{x}, A \cdot \vec{p})$ for all $A \in SO(3)$, where we denote $\vec{x} = (x^1, x^2, x^3)$ and $\vec{p} = (p^1, p^2, p^3)$.

Symmetries of solutions of the Einstein-Vlasov system (and also the EVM-system) entail the existence of conserved quantities. So corresponds the symmetry in time to a conserved particle energy E, and rotational symmetry to conserved components of the angular momentum L. These quantities are conserved along the characteristic curves (1.10)–(1.11) of the transport operator $\mathfrak{T}$, given in (1.9). As a consequence these conserved quantities can be used to construct solutions of the Vlasov equations. This technique is standard [37], also for time dependent problems, and has been used in all papers on stationary solutions in this thesis.

The Vlasov field gives rise to matter quantities. of which the energy density ϱ and the radial pressure p are two particularly important ones. Taking, for the distribution function f, an ansatz in terms of the conserved quantities, one can express the matter quantities ϱ and p in terms of explicitly known functions g and h via $\varrho(r) = g(r, \mu(r))$ and $p(r) = h(r, \mu(r))$. The analysis of the Einstein-Vlasov system in a spherically symmetric, static setting reduces to the analysis of the integro-differential equation

$$\mu'(r) = \frac{4\pi}{1 - \frac{8\pi}{r}\int_0^r s^2 g(s, \mu(s))\mathrm{d}s}\left(rh(r, \mu(r)) + \frac{1}{r^2}\int_0^r s^2 g(s, \mu(s))\mathrm{d}s\right), \tag{2.2}$$

$$\mu(0) = \mathring{\mu}, \tag{2.3}$$

where $\mathring{\mu} < 0$ is a prescribed value of μ at $r = 0$. See for example [46] for details.

In the paper "Models for Self-Gravitating Photon Shells and Geons" solutions of (2.2)–(2.3) with massless particles are constructed. The particle distribution function of these solutions is of compact support. This means that the solutions describe spatially confined ensembles of massless particles (photons) that are kept together by their own gravity. This result was somehow surprising since compared to massive particles, massless particles tend much less to form spatially confined objects. This can be illustrated in the following way.

In the construction of static solutions of the Einstein-Vlasov system the property that the matter quantities are of compact support is related to the presence of a cut-off energy E_0, i.e. it is assumed that no particle has more energy than this value E_0. The particle energy is given by

$$E = e^{\mu}\sqrt{m_p^2 + w^2 + \frac{L}{r^2}}, \tag{2.4}$$

where μ is a metric function, cf. (2.1), m_p is the rest mass of the particles, w denotes the radial momentum of the particles and L denotes the angular momentum of the particles. The function μ is increasing in r. If one now shows for $m_p > 0$ that at a certain radius there holds $e^{\mu} m_p \geq E_0$ one has found an upper bound for the support of the matter quantities since particles further out would have more energy than the maximally allowed value E_0. The proofs for compact support in the massive case rely on this mechanism [34]. If $m_p = 0$ this argument cannot work and indeed the solutions tend to have infinitely extended "atmospheres" of massless particles.

It was however realised in a different context, in work which was concerned with *Buchdahl-type inequalities*, that certain spherically symmetric, static solutions of the massive Einstein-Vlasov system contain shells of matter which are separated by vacuum [5]. A very similar analysis could then be carried out in the massless case. This analysis yields static solutions with matter regions and vacuum regions. A priori the support of the matter quantities of these solutions is not confined to a bounded domain. However, one can use that if a solution has a vacuum region at a sphere of a certain radius, there exists a solution which coincides with this solution on the inside of this sphere and which is purely vacuum on the outside of this sphere. This construction is done in the paper and thereby self-gravitating, thin shells of massless Vlasov particles are obtained.

For the construction of solutions which contain matter regions separated by vacuum regions it is very important that these solutions are highly relativistic, i.e. that these solutions contain regions with a very high energy density. So all compactly supported steady states constructed in the article are highly relativistic. The paper then discusses further the conjecture that all spherically symmetric, static solutions with massless particles are necessarily highly relativistic and consequently probably also unstable.

Finally, a parallel to in many respects similar solutions of the Einstein-Maxwell system is observed. These solutions describe a standing electromagnetic wave confined to a thin shell. Wheeler studied these solutions as models for individual particles and he called them *geons*. This observation opens another interesting way of interpreting the constructed steady states.

3 Shells with charged particles

The static, spherically symmetric Einstein-Vlasov system for massive, uncharged particles is understood already quite well [6]. Various different solutions with matter quantities of compact support have been observed numerically. Among these solutions there are ball shaped solutions, shells and multi-shells. Their existence has been proven analytically. Furthermore the Buchdahl-type inequality

$$\frac{2m(r)}{r} \leq \frac{8}{9}, \quad m(r) = 4\pi \int_0^r s^2 \varrho(s)\mathrm{d}s \tag{3.1}$$

is satisfied by these solutions, where $m(r)$ is called the *Hawking mass* and ϱ is the energy density of the particle distribution [4]. This Buchdahl-type inequality is sharp in the sense that one can construct a sequence of thinner and thinner shell solutions in whose limit (3.1) is saturated [5]. This analytical result, that the Buchdahl-type inequality holds for all static, spherically symmetric solutions, has been generalised to the Einstein-Vlasov system with charged particles, i.e. the EVM-system [3]. The corresponding inequality reads

$$\sqrt{\frac{m_g(r)}{r}} \leq \frac{1}{3} + \sqrt{\frac{1}{9} + \frac{q(r)^2}{r^2}}, \tag{3.2}$$

where the mass function $m_g(r)$ contains the Hawking mass and some additional charge terms, and the function $q(r)$ describes the charge contained in a sphere of radius r. However, the existence of steady states with charged particles has priorly not been proven.

The paper "Existence of Spherically Symmetric Solutions of the Einstein-Vlasov-Maxwell System and the Thin Shell Limit" provides some analytical understanding of this question. First the existence of spherically symmetric steady states is proved for small particle charges. Then, generalising the uncharged case, a sequence of thin shell solutions is constructed in order to show that the Buchdahl type inequality for charged Vlasov matter (3.2) can be saturated, too. In the thin shell limit it turns out that the effects of the particle charge become negligible compared to the gravitational effects. For this reason this sequence can be constructed for arbitrary values of the particle charge.

As for the uncharged case, the symmetry assumptions simplify the EVM-system. The metric is determined by the two functions μ and λ, cf. (2.1), and the electro magnetic field tensor F is entirely determined by the function $q(r)$ [31, 46]. In contrast to the uncharged case there is not only one integro-differential equation but a system of three coupled integro-differential equations for the functions μ, λ, and q has to be analysed.

The particle charge has been considered as parameter for a family of solutions in numerical studies [7]. "Turning on" the particle charge by choosing small positive values of the charge parameter does not qualitatively change the appearance of the solutions. The repelling effect of the particle charge can however be seen by a slightly reduced density of the particles. Increasing the particle charge parameter further one reaches a critical value beyond which no static solution with matter quantities of compact support have been observed. Motivated by this observation the proof of existence of solutions works by a perturbation technique and is formulated for small particle charges.

4 Fluid-like solutions

In certain situations the Einstein-Vlasov system and the Einstein-Euler system for a perfect fluid show considerable resemblances. An example of such a situation is the analysis of static solutions of the systems. A static space-time containing a perfect fluid has to be spherically symmetric and the matter is supported on a ball centred around the centre of symmetry [28]. It was widely assumed that the same holds true for Vlasov matter if the particles are evenly distributed in momentum space.

In the paper "Uniqueness of static, low-pressure solutions of the Einstein-Vlasov system" the link between Vlasov matter and the matter model of a perfect fluid is made explicit and is exploited by applying a uniqueness result for fluids developed by Beig and Simon [12]. The similarities of fluid matter and kinetic matter have already been investigated in other work, e.g. in the non-relativistic case [35]. However, there are some differences between the relativistic and the non-relativistic setting. In contrast to the non-relativistic case (Vlasov-Poisson system) in the general relativistic setting (Einstein-Vlasov system) it is not known if steady states with Vlasov matter can be constructed with variational methods [1].

The Einstein-Vlasov system is considered under the assumptions that the space-time is static and that the particle distribution is isotropic in the momentum variables. This means that topologically the space-time is $\mathbb{R} \times \Sigma$ and there are coordinates t, x^1, x^2, x^3 such that the metric can be written in the form

$$g = -V^2(x^1, x^2, x^3)\mathrm{d}t^2 + \gamma_{ab}(x^1, x^2, x^3)\mathrm{d}x^a\mathrm{d}x^b, \tag{4.1}$$

where γ is a Riemannian metric on Σ. Furthermore this means that $f(t, \vec{x}, \vec{p}) = f(\vec{x}, |\vec{p}|)$, where $\vec{x} = (x^1, x^2, x^3)$ and $\vec{p} = (p^1, p^2, p^3)$. The Einstein equations in this setting can be written as

$$R_{ab} = \frac{1}{V} D_a D_b V + 4\pi(\varrho - p)g_{ab}, \tag{4.2}$$

$$\Delta V = 4\pi V(\varrho + 3p), \tag{4.3}$$

where R_{ab} is the Ricci tensor of γ, D is the Levi-Civita connection of γ and $\Delta = \gamma^{ab} D_a D_b$. The matter quantities $\varrho = \varrho(x^1, x^2, x^3)$ (energy density) and $p = p(x^1, x^2, x^3)$ (radial pressure) are defined as certain components of the energy-momentum tensor.

In the article it is shown that under these assumptions Vlasov matter can actually be described as a perfect fluid, i.e. there exists a unit timelike vector field such that the energy-momentum tensor can be written as

$$T^{\mu\nu} = \varrho u^\mu u^\nu + p(u^\mu u^\nu + g^{\mu\nu}). \tag{4.4}$$

As the analysis of [12] shows the level sets of the function V can be considered as radial coordinate and ϱ and p can be expressed as sufficiently regular bijective functions in V only. Consequently one can express one in terms of the other, i.e. there exists a function $p(\varrho)$. This function $p = p(\varrho)$, i.e. the *equation of state*, cannot be explicitly written down but it is implicitly given by the Vlasov equation and additional assumptions on the particle distribution function f (an ansatz for f). For Vlasov matter, an ansatz for the particle distribution function is sometimes referred to as "microscopic equation of state".

If the result of Beig and Simon [12] can be applied to a fluid model (the model is characterised by the equation of state) then for a given surface potential the corresponding static solution is unique and thereby in particular also spherically symmetric. However, this result does not apply to any fluid model but only if the equation of state satisfies a technical condition. Thus the implicitly given "equation of state" for Vlasov matter, i.e. the relation between p and ϱ, is analysed and it is shown that it matches the assumptions of the result by Beig and Simon if the matter distribution is not too concentrated, i.e. not highly relativistic. A numerical study then gives evidence for that if the matter distribution is very concentrated, i.e. highly relativistic, then several different solutions can be constructed with the same surface potential. Consequently uniqueness does not hold.

This last observation has its parallel to the fluid case since the result by Beig and Simon cannot be applied to fluid models that describe very relativistic objects as for example neutron stars. Finally, the regime where the result by Beig and Simon applies to isotropic Vlasov matter is discussed and it is remarked that objects like globular clusters are typically sufficiently little concentrated.

5 Rotating clouds of charged particles

So far the known analytical results on stationary solutions are for rather symmetric ones, like solutions in spherical symmetry, cylindrical symmetry, or axial symmetry [6]. On the one hand, an important factor here is of course the question which situations are accessible form a technical point of view. Symmetry assumptions simplify the analysis. Since the Einstein-Vlasov system and the EVM-system constitute a fairly complicated system of equations these simplifying assumptions have in many situations been very helpful for its study.

On the other hand, if there are no other elements in the model, gravitation "prefers" spherically symmetric configurations. Non-rotating, static fluid solutions, for example, are always spherically symmetric [28]. This is also true for Vlasov matter if the momenta are distributed isotropically among the particles (and their concentration is not too high), cf. Section 4. It is an interesting question which symmetry properties are necessary for stationary solutions of the Einstein-Vlasov system under different circumstances. The solutions deviate from spherical symmetry if a corresponding physical mechanism is present. An example for such a mechanism is rotation, either in the form of an overall angular momentum, or just preferred axes around which the particles move. If the particles are charged similar statements apply to the electro-magnetic field. The symmetries of the electric field follow the symmetries of the particle distribution. Magnetic fields are induced by currents of charged particles which then in turn determine what sorts of magnetic fields can be observed.

The paper "Rotating Clouds of Charged Vlasov Matter in General Relativity" generalises earlier work by Andréasson, Kunze, and Rein [9, 8]. This earlier work is the first result where a stationary solution of the Einstein-Vlasov system is constructed which is not spherically symmetric. The starting point of this construction is a spherically symmetric, static solution of the Vlasov-Poisson system, which in the paper is called ζ_0. Such solutions are analytically well understood and can be constructed [11]. The idea, which has priorly been used for fluids [25, 24, 21, 20] and the Vlasov-Poisson system [36], is to consider a suitable function space $\mathcal{X}$ which contains this solution. In this function space the solution ζ_0 is embedded into continuous families of functions which, if they are solutions of the Einstein-Vlasov system, are axially symmetric but not spherically symmetric and relativistic.

In axial symmetry there are considerably more degrees of freedom than in spherical symmetry. In an axially symmetric, stationary space-time one can find coordinates t, ϱ, ϑ, φ such that the metric can be written in the form

$$\begin{aligned} g = &-e^{2\nu(\varrho,z)}\mathrm{d}t^2 + e^{2\mu(\varrho,z)}\mathrm{d}\varrho^2 + e^{2\mu(\varrho,z)}\mathrm{d}z^2 \\ &+ \varrho^2 H(\varrho,z)^2 e^{-2\nu(\varrho,z)}\left(\mathrm{d}\varphi - \omega(\varphi,z)\mathrm{d}t\right)^2, \end{aligned} \tag{5.1}$$

where ν, μ, H and ω are axially symmetric functions and units have been chosen such that $G = c = 1$ [10]. The EVM-system can in this situation be cast as a system of coupled elliptic partial differential equations. In the work in axial symmetry presented in this thesis the EVM-system is written as a system of non-linear Poisson equations in different dimensions and a first order partial differential equation. The electro-magnetic field tensor has more independent components than in the previously discussed spherically symmetric setting. However, as mentioned above, the the electro-magnetic field must always match the particle distribution and its symmetries.

Now the technique how axially but not spherically symmetric solutions can be constructed is briefly explained before we highlight some key ideas in the article which is part of this thesis. If one does not choose units such that the speed of light c is normalised to 1, the speed of light can be considered as a variable parameter in the system. Given certain boundary conditions the system can be solved for different values of this parameter. Now, if one considers a sequence of such solutions where the parameter c goes to infinity this sequence converges to the solution of the Vlasov-Poisson system with the same boundary conditions. A solution to the Einstein-Vlasov system with $c > 0$ is a relativistic solution (albeit not “highly relativistic” in the sense of the previous sections). Via a scaling argument it can be transformed into a solution of the system with $c = 1$.

A deviation of spherical symmetry is caused by according physical effects like rotation. The presence of rotation in the considered model, i.e. the considered solution of the Einstein-Vlasov system, is anchored in the a priori assumptions on the form of the particle distribution function (the ansatz). As for Newtonian and relativistic solutions a continuous transition from spherically symmetric to axially symmetric, but not spherically symmetric solutions, can be achieved. To this end a family of ansatz functions is considered which is parameterised by a parameter λ which “turns on” the dependency of the particle distribution on the angular momentum.

Then a solution operator $\mathfrak{F} : \mathcal{X} \times [0, \delta) \times (-\delta, \delta) \to \mathcal{X}$ to the Einstein-Vlasov system is constructed. It is defined on the aforementioned function space $\mathcal{X}$ and takes values of the parameters $\gamma := c^{-2}$ and λ in the interval $[0, \delta) \times (-\delta, \delta)$. It is constructed such that $\mathfrak{F}(\zeta; \gamma, \lambda) = 0$ if ζ is a solution of the Einstein-Vlasov system with the parameters γ and λ. It is shown that this solution operator satisfies the assumptions of the implicit function theorem for Banach spaces. This then implies that there is an open neighbourhood $\mathcal{U} \subset \mathcal{X}$ of the spherically symmetric Newtonian solution ζ_0 where all contained elements of $\mathcal{X}$ are solutions of the Einstein-Vlasov system.

The addition of the particle charge to the picture entails several difficulties but also interesting new physical effects that can be modelled. First it has to be understood in what sense the Vlasov-Poisson system is the non-relativistic limit of the EVM-system. In fact, in the Newtonian picture, the repulsive forces due to the particles’ charge and the attractive forces due to their mass are of the same nature. The forces can be described by a potential which satisfies a Poisson equation for an effective mass parameter $m_p^2 - q^2$. It turns out that indeed in the limit where $c \to \infty$ the equations for most of the unknowns become trivial and the equations for the lapse function ν and the t-component A_t of the electro-magnetic four-potential become Poisson equations. The quantity $\nu+qA_t$ corresponds to the Newtonian potential U_N of the Vlasov Poisson system with effective mass $m_p^2 - q^2$.

The right definition of the function space $\mathcal{X}$ in which the constructed axially symmetric but not spherically symmetric solution will lie is key. This definition has to be changed compared to the uncharged case in order to deal with the additional electro-magnetic field terms. The reason is the following.

As already mentioned the EVM-system can be written as a system of coupled, non-linear Poisson equations. We focus on the equations for the φ-component A_φ of the electro-magnetic four-potential. After the change of variables given by $a = \varrho^{-2} A_\varphi$ the equation reads

$$\Delta_5 a = \text{source} = \frac{2}{1+h} \frac{a}{\varrho} \frac{\partial h}{\partial \varrho} + \frac{2}{4\pi^2 c^2} \frac{a}{\varrho} \frac{\partial \nu}{\partial \varrho} + \dots . \tag{5.2}$$

where $1 + h = H$ is an unknown metric field, and Δ_5 denotes the Laplace operator in five dimensions. (Viewing the differential operator $\partial_\varrho^2 + \partial_z^2 + \frac{3}{\varrho}$ as the Laplace operator acting on an axially symmetric function in $\mathbb{R}^5$ is a trick that has already been used by Andréasson, Kunze, and Rein [9]. This trick permits to include rotation intoq the picture.) The solution operator shall be constructed with the fundamental solution of the Poisson equation. Schematically the solution operator reads

$$\mathfrak{F}(\zeta; \gamma, \lambda) = \zeta - \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{R}^n} \frac{\text{source}}{|x-y|^{n-2}} \mathrm{d}y, \tag{5.3}$$

where n is the dimension of the Laplace operator Δ_n. The problem with the source term of equation (5.2) is that it appears singular at $\varrho = 0$. In order to shown that (5.3) is of a certain regularity the fraction $(a\partial_\varrho h)/\varrho$ needs to be controlled.

If a and h are not seen as axially symmetric function on $\mathbb{R}^5$ or $\mathbb{R}^3$ but if they are extended to negative values for the radial coordinate ϱ and thereby seen as functions on $\mathbb{R}^2$ a useful regularity property becomes accessible. The needed regularity can be deduced if h is of higher regularity than a. A corresponding hierarchy in regularity needs to be built into the function space $\mathcal{X}$.

If the particles are charged rotating solutions of the EVM-system show interesting additional properties compared to the uncharged case. An axially symmetric electric field is automatically generated. If there is an overall rotation then a poloidal magnetic field is induced. Under the symmetry assumptions the equations describing the toroidal components of the magnetic field only have the trivial solution. The absence of a toroidal magnetic field is intuitively expected since for this not only particle currents that are rotating around the axis of symmetry are necessary but also particle currents that move along the axis of symmetry.

6 Vlasov matter and the conformal method

An important "test" for the explicitly known vacuum space-times is the study of the future development of small perturbations and the question whether these vacuum space-times are stable under these perturbations. In a first instance, perturbations of the gravitational field in vacuum have been studied. Perturbations containing fields of physically relevant matter models are interesting, too. In reality there exists no perfect vacuum. One has to assume that there are always small fluctuations in the gravitational field or that some small amount of particles is present. It would be fatal if the equations of the Vlasov matter model would predict a time evolution in which small perturbations cause a fundamental change of the space-time's character or even the creation of singularities. In other words, since small perturbations are always present, the model would predict that the vacuum space-time would collapse in finite time. This clearly is in contradiction with observations. This motivates the study of small data problems.

In [33] Penrose describes the theory of *Conformal Cyclic Cosmology*, which also contributes to the motivation of studying the conformal properties of massless Vlasov matter since radiative, conformally invariant matter models, as massless Vlasov matter, play a crucial role int this model: In a late stage of the "life-cycle" of the universe, an *aeon*, most of the matter is transformed into radiative matter due to Hawking radiation. Then, due to the conformal invariance, the infinitely expanded "end-state" of the universe can be identified with the big bang state of the next aeon cycle.

The paper "The Conformal Einstein Field Equations with Massless Vlasov Matter" contains a small data result for the massless Einstein-Vlasov system. The result builds on the well developed machinery for the Einstein equations with a conformally invariant matter model whose energy momentum tensor is trace-free. The development of this machinery has been initiated by Friedrich. The book [47] by Valiente Kroon gives an introduction to the field. The result of the paper yields a global stability proof for de Sitter space-time and a semi-global stability proof for Minkowski space-time. The initial data of the Vlasov field which can be treated by the method is compactly supported in the momentum variables.

For Minkowski space-time global stability in the case of massless Vlasov matter has been shown earlier by Taylor [45]. In [45] the initial data for the distribution function f is compactly supported in space and momentum, however the metric perturbations does not need to be compactly supported. For this reason, in contrast to the alternative proof presented here, [45] provides a global stability result. For de Sitter space-time there exists already the stability proof in the case of massive particles by Ringström [39]. Most likely the methods of [39] yield the corresponding result also in the massless case.

The fundamental idea of the conformal method is not to consider the global existence problem of the Einstein-matter system directly but to study a related local existence problem. This idea is motivated by the fact that many interesting vacuum solutions of the Einstein equations, such as Minkowski space-time and de Sitter space-time, can be identified with a compact submanifold of another space-time, the *Einstein cylinder* $(\mathscr{E}, \boldsymbol{g}_{\mathscr{E}})$. The topology of the manifold $\mathscr{E}$ is $\mathbb{R} \times \mathbb{S}^3$. Let τ be a time function on $\mathscr{E}$ and $\boldsymbol{\partial}_\tau$ the vector field constituting the time-orientation. Since there is no global chart of coordinates on $\mathscr{E}$ tensorial quantities are easiest expressed in terms of an orthonormal frame $(\boldsymbol{\partial}_\tau, \boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3)$ which can be defined globally on $(\mathscr{E}, \boldsymbol{g}_{\mathscr{E}})$. The metric $\boldsymbol{g}_{\mathscr{E}}$ can be written as

$$\boldsymbol{g}_{\mathscr{E}} = \mathbf{d}\tau \otimes \mathbf{d}\tau - \boldsymbol{\sigma}, \tag{6.1}$$

where $\boldsymbol{\sigma}$ is the flat metric on the unit three-sphere $\mathbb{S}^3$. (A word on notation: tensors and frame components are written in bold face in this paper. Furthermore, in contrast to the other papers of this thesis, the convention $(+,-,-,-)$ is used for the Lorentzian metrics.)

Different space-times, as the Minkowski space-time $(\mathcal{M}_{\mathrm{M}}, \boldsymbol{g}_{\mathrm{M}})$ and the de Sitter space-time $(\mathcal{M}_{\mathrm{d}S}, \mathbf{g}_{\mathrm{dS}})$ can be identified with portions of the Einstein cylinder via the conformal factor Ξ (we denote Ξ_{M} for Minkowski and Ξ_{dS} for de Sitter). The conformal factor is a scalar function defined on the Einstein cylinder $\mathscr{E}$. In the interior of the portions of $\mathscr{E}$ which are identified with $\mathcal{M}_{\mathrm{M}}$ or $\mathcal{M}_{\mathrm{dS}}$, respectively, there holds

$$\boldsymbol{g}_{\mathscr{E}} = \Xi_{\mathrm{M}}^2 \boldsymbol{g}_{\mathrm{M}}, \quad \text{or} \quad \boldsymbol{g}_{\mathscr{E}} = \Xi_{\mathrm{dS}}^2 \boldsymbol{g}_{\mathrm{dS}}, \tag{6.2}$$

respectively. The boundary of these portions of $\mathscr{E}$ is characterised by $\{\Xi = 0\}$.

If a metric $\tilde{\boldsymbol{g}}$, henceforth referred to as the *physical metric*, which satisfies the Einstein equations is conformally related to another metric $\boldsymbol{g}$, the *unphysical metric*, via

$$\boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}} \tag{6.3}$$

then the unphysical metric $\boldsymbol{g}$ satisfies the *conformal Einstein field equations* (CEF). The converse is also true.

The CEF are, as the Einstein equations themselves, a system of partial differential equations of second order. Taking several curvature quantities, as for example components of the Weyl tensor, which contain first order derivatives of the metric as additional unknowns it is possible to cast the CEF as an equivalent Cauchy problem for a quasi-linear symmetric hyperbolic system of first order.

The unknowns of this Cauchy problem are a collection of curvature quantities which we will refer to by $\boldsymbol{u}_g$. The components $\{\boldsymbol{e}_{\boldsymbol{a}}^{\mu}\}_{\boldsymbol{a}=0}^{3}$ of an orthonormal frame and the conformal factor Ξ are among the unknowns of this Cauchy problem. This Cauchy problem can now be studied with initial data given on a three-sphere. "Trivial" initial data for the curvature quantities on the hypersurface $\{\tau = 0\}$ and the values $\Xi_{\mathrm{M}}|_{\tau=0}$ or $\Xi_{\mathrm{dS}}|_{\tau=0}$ for the conformal factor gives the Einstein cylinder and the conformal embeddings of Minkowski space-time or de Sitter space-time, respectively. These conformal embeddings (of Minkowski or de Sitter space-time) only require a solution of the Cauchy problem until the finite evolution time where the conformal factor vanishes. It is now interesting to study what happens if these Minkowski- or de Sitter initial data on the three-sphere $\{\tau = 0\}$ are slightly perturbed.

The local existence and stability theory developed by Kato [22] yields the desired answers. The time evolution yields a curved space-time which is close to the Einstein cylinder. As in the unperturbed case the conformal factor vanishes in finite time which permits to identify a perturbed version of the investigated embedded vacuum space-time. In the vacuum setting the corresponding stability results have been obtained in [19]. Matter models like a "trace free scalar field" or a Maxwell field have been considered, too [18, 30]. In the present paper massless Vlasov matter is included. Massless Vlasov matter resembles the upper fields in the respect that the energy momentum tensor is trace free and behaves nicely under conformal transformations, and that the Vlasov equation is conformally invariant. However, there is an important difference. The Vlasov field is not defined on the space-time manifold but on its co-tangent bundle and this is the level where the problem needs to be understood.

To understand this problem it is much more convenient to work on the co-tangent bundle $T^*\mathcal{M}$ instead of the tangent bundle $T\mathcal{M}$. Furthermore, we work with the coordinates $v_{\boldsymbol{a}}$, $\boldsymbol{a} = 0, \ldots, 3$ induces by an orthonormal co-frame $\{\boldsymbol{\alpha}^{\boldsymbol{a}}\}_{\boldsymbol{a}=0}^{3}$ of the orthonormal frame $\{\boldsymbol{e}_{\boldsymbol{a}}\}_{\boldsymbol{a}=0}^{3}$ via $p = v_{\boldsymbol{a}}\boldsymbol{\alpha}^{\boldsymbol{a}}$ for any $p \in T^*\mathcal{M}$, $x \in \mathcal{M}$. On the co-tangent bundle the Vlasov particles follow curves $\Gamma = \{(X^\mu(s), V_{\boldsymbol{a}}(s)) : s \in \mathbb{R}\} \subset T^*\mathcal{M}$ which solve the differential equations

$$\dot{X}^\mu = \eta^{\boldsymbol{ab}} V_{\boldsymbol{a}} e_{\boldsymbol{b}}{}^{\mu}(X(s)), \tag{6.4}$$

$$\dot{V}_{\boldsymbol{d}} = \eta^{\boldsymbol{ab}} V_{\boldsymbol{a}} V_{\boldsymbol{b}} \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}(X(s)), \qquad \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{b}} = \boldsymbol{\alpha}^{\boldsymbol{c}} \left(\nabla_{\boldsymbol{e_a}} \boldsymbol{e_b}\right). \tag{6.5}$$

These curves are the characteristic curves of the Liouville vector field

$$\mathcal{L} = \eta^{\boldsymbol{ab}} v_{\boldsymbol{a}} e_{\boldsymbol{b}}{}^{\mu} \boldsymbol{\partial}_{x^\mu} + \eta^{\boldsymbol{ab}} v_{\boldsymbol{a}} v_{\boldsymbol{c}} \Gamma_{\boldsymbol{b}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}} \boldsymbol{\partial}_{v_{\boldsymbol{d}}} \tag{6.6}$$

(in the formulation on the tangent bundle it was referred to as transport operator $\mathfrak{T}$ in the sections above). The Vlasov equations reads $\mathcal{L}f = 0$.

The projection of the curves (6.4)–(6.5) onto the space-time manifold $\mathcal{M}$ are null geodesics. As sets null geodesics are conformally invariant. However the "speed" that the particles follow these curves with is not conformally invariant. This can be illustrated as follows. Null geodesics in Minkowski space-time are infinitely extended and the particles move with constant speed (if the parameterisation is chosen accordingly). In the conformally compactified picture the whole Minkowski space-time is identified with a finitely extended portion of the Einstein cylinder (measured with the unphysical metric $\boldsymbol{g}$). Consequently the particles need to "slow down" as they approach the conformal boundary.

Let $\tilde{\mathcal{L}}$ be the Liouville vector field corresponding to the physical metric $\tilde{\boldsymbol{g}}$ and $\mathcal{L}$ the Liouville vector field corresponding to the conformally rescaled metric $\boldsymbol{g}$, cf. (6.3). In the formulation on the co-tangent bundle there holds

$$\tilde{\mathcal{L}} = \Xi^2 \mathcal{L} \tag{6.7}$$

on the mass shell for massless particles. If the problem is formulated on the tangent bundle and $\mathcal{L}$ is defined as a vector field on $T\mathcal{M}$ then (6.7) only holds if one takes into account the change of parameterisation of the characteristic curves by an appropriate rescaling of the momentum variables $v_{\boldsymbol{a}}$.

The source terms of the symmetric hyperbolic system which is equivalent to the Einstein-Vlasov system contain the expressions $\nabla_{\boldsymbol{a}} T_{\boldsymbol{bc}}[f]$. As a consequence the matter field cannot be described only by the unknown f since if that was the case there would be derivatives of the unknowns in the source terms and the local existence and stability theory could not be applied. Some derivatives of f need to be added to the set of unknowns. The way out is to add further quantities to the set of unknowns. Since

$$\nabla_{\boldsymbol{a}} T_{\boldsymbol{bc}}[f] = \int_{\mathcal{P}_x} \boldsymbol{e}_{\boldsymbol{a}}(f) v_{\boldsymbol{b}} v_{\boldsymbol{c}} \mathrm{dvol}_{\mathcal{P}_x} - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{b}} T_{\boldsymbol{dc}}[f] - \Gamma_{\boldsymbol{a}}{}^{\boldsymbol{d}}{}_{\boldsymbol{c}} T_{\boldsymbol{bd}}[f] \tag{6.8}$$

the fields $\boldsymbol{e}_{\boldsymbol{a}}(f)$, $\boldsymbol{a} = 0, \ldots, 3$ would be naive candidates for these additional unknowns. (The energy-momentum tensor was defined in (1.3) and the co-mass shell in (1.12). $\mathcal{P}_x$ denotes the fibre of $\mathcal{P}_0^*$ over the space-time event $x \in \mathcal{M}$.)

For additional unknowns we also need additional equations. These additional equations can be obtained by evaluating the expressions $\mathcal{L}(\boldsymbol{e}_{\boldsymbol{a}}(f))$, $\boldsymbol{a} = 0, \ldots, 3$. It turns however out that these additional equations cannot be expressed in terms of the unknowns that are available due to derivatives acting on the components $\boldsymbol{e}_{\boldsymbol{a}}{}^{\mu}$, $\mu = 0, \ldots, 3$, of the frame fields $\boldsymbol{e}_{\boldsymbol{a}}$, $\boldsymbol{a} = 0, \ldots, 3$. So a closed system of equations cannot be obtained this way because $\boldsymbol{e}_{\boldsymbol{a}}$ are not the right notion of vectors on the co-tangent bundle (seen as manifold with a tangent bundle).

Let $b \in T^*\mathcal{M}$. For the tangent bundle of $T\mathcal{M}$, and consequently also on the tangent space $\mathcal{I}_b$ at $b \in T^*\mathcal{M}$, the Levi-Civita connection ∇ gives rise to a split

$$\mathfrak{I}_b = \mathfrak{P}_b \oplus \mathfrak{Q}_b \tag{6.9}$$

of $\mathfrak{I}_b$ into the 4-dimensional horizontal subspace $\mathfrak{Q}_b$ (not tangent to the fibre $\mathfrak{I}_b$) and the vertical subspace $\mathfrak{P}_b$. A coordinate-invariant way of lifting $\boldsymbol{e_a}$, $\boldsymbol{a} = 0, \ldots, 3$ to the tangent bundle of $T^*\mathcal{M}$ is then given by the *horizontal lift*

$$\hat{\boldsymbol{e}}_{\boldsymbol{a}} = \boldsymbol{e_a} + v_{\boldsymbol{c}}\Gamma_{\boldsymbol{a}}{}^{\boldsymbol{c}}{}_{\boldsymbol{d}}\partial_{v_{\boldsymbol{d}}}, \quad \boldsymbol{a} = 0, \ldots, 3. \tag{6.10}$$

With (6.10) there holds $\hat{\boldsymbol{e}}_{\boldsymbol{a}} \in \mathfrak{Q}_b$ and clearly $\pi_*(\hat{\boldsymbol{e}}_{\boldsymbol{a}}) = \boldsymbol{e_a}$ where $\pi_* : \mathfrak{I}_b \to T_x\mathcal{M}$ is the push-forward of the projection map $\pi : T\mathcal{M} \to \mathcal{M}$, and $\pi(b) = x$. The tangent vectors of the characteristic curves (6.4)–(6.5) are everywhere horizontal. The horizontal lifts $\hat{\boldsymbol{e}}_{\boldsymbol{a}}$ are parallelly transported along such curves. More details can be found in [29, 42]. The fields $\hat{\boldsymbol{e}}_{\boldsymbol{a}}(f)$, $\boldsymbol{a} = 0, \ldots, 3$ are the right choice for the additional unknowns that avoid the problems described in the preceding paragraphs.

The last consequence of the fact that the Vlasov field "lives" on $T^*\mathcal{M}$ and not on $\mathcal{M}$ which should be discussed here, is concerned with the application of Kato's existence and stability theorems. The unknowns describing the geometric fields (living on $\mathcal{M}$) and the unknowns describing the Vlasov field (living on $T^*\mathcal{M}$) can be grouped into the two collections $\mathbf{u}_g$ and $\mathbf{u}_f$, respectively. The conformal Einstein-massless-Vlasov system can be written as two coupled symmetric hyperbolic systems that schematically take the form

$$\mathbf{A}^0_g[\mathbf{u}_g] \cdot \partial_\tau \mathbf{u}_g + A^i_g[\mathbf{u}_g] \cdot \partial_{x^i} \mathbf{u}_g = \mathbf{F}_g[\mathbf{u}_g, \mathbf{u}_f], \tag{6.11}$$

$$\mathbf{A}^0_f[\mathbf{u}_g] \cdot \partial_\tau \mathbf{u}_f + \mathbf{A}^i_f[\mathbf{u}_g] \cdot \partial_{x^i} \mathbf{u}_f + \mathfrak{A}_{\mathbf{c}}[\mathbf{u}_g] \cdot \partial_{v_{\mathbf{c}}} \mathbf{u}_f = \mathbf{F}_f[\mathbf{u}_g, \mathbf{u}_f], \tag{6.12}$$

where $i = 1, 2, 3$, and $\mathbf{A}^\mu_g[\mathbf{u}_g(\tau, \cdot)](\underline{x}, v_{\boldsymbol{a}})$, $\mathbf{A}^\mu_f[\mathbf{u}_g(\tau, \cdot)](\underline{x}, v_{\boldsymbol{a}})$, $\mathfrak{A}_{\mathbf{c}}[\mathbf{u}_g(\tau, \cdot)](\underline{x}, v_{\boldsymbol{a}})$, $\mu, \boldsymbol{c}, \boldsymbol{a} = 0, \ldots, 3$, are coefficient matrices and $\mathbf{F}_g[\mathbf{u}_g(\tau, \cdot), \mathbf{u}_f(\tau, \cdot)](\underline{x}, v_{\boldsymbol{a}})$, $\mathbf{F}_f[\mathbf{u}_g(\tau, \cdot), \mathbf{u}_f(\tau, \cdot)](\underline{x}, v_{\boldsymbol{a}})$ are vectors. Here we use the notation $\underline{x} = (x^1, x^2, x^3)$. One assumptions of Kato's theorem is that the matrix $\mathbf{A}^0_f[\mathbf{u}_g(\tau, \cdot)]$, when the entries are seen as functions of $\underline{x}, v_{\boldsymbol{a}}$, stays uniformly bounded from below and from above. Now, one finds

$$\mathbf{A}^0_f[\mathbf{u}_g(\tau, \cdot)](\underline{x}, v_{\boldsymbol{a}}) = \eta^{\boldsymbol{ab}} v_{\boldsymbol{a}} e_{\boldsymbol{b}}{}^0 \mathbb{1}_7, \tag{6.13}$$

where $\mathbb{1}_7$ denotes the unit matrix in seven dimensions. Inspecting (6.13) one notices immediately that $\mathbf{A}^0_f[\mathbf{u}_g(\tau, \cdot)](\underline{x}, v_{\boldsymbol{a}})$ is uniformly bounded only if $|v|$ is bounded from below and from above. To ensure this it is necessary to restrict the initial data to compact support in the v-variables, bounded away from 0. An analysis of the characteristic system (6.4)–(6.5) of the Liouville vector field then ensures for that the v-variables stay bounded (away from 0 and ∞) sufficiently long in the perturbed space-time.

7 The vector field method for relativistic transport equations

There are several approaches to the problem of proving stability of Minkowski space-time under perturbations with Vlasov matter. The conformal method has already been discussed in the preceding section. Thanks to very powerful tools, with the conformal method it was possible to obtain stability results for massless Vlasov matter with a relatively concise proof. The current disadvantages of the conformal method are that the treatment of massive Vlasov matter is a much more difficult and open question since the massive Vlasov equation is not conformally invariant. Moreover, as far as Minkowski space-time is concerned, only a special type of initial data can be treated due to the "problem at spatial infinity", cf. [47].

Taylor has given a proof for global non-linear stability of Minkowski space-time with massless Vlasov matter using a different, non-conformal method [45]. Later, independently, Lindblad and Taylor [27] and Fajman, Joudioux, and Smulevici [15] gave a proof of stability of Minkowski space-time as solution of the Einstein-Vlasov system with massive particles. These results for Vlasov fields make use of stability results for the vacuum case, i.e. for the Einstein equations without coupling to any matter model. The first proof of global stability of Minkowski space-time for pure metric-perturbations is due to the breakthrough result by Christodoulou and Klainerman [14]. Later, Lindblad and Rodnianski gave a shorter proof using so called *wave coordinates* [26]. The paper "Stability of Minkowski space-time with non-compactly supported massless Vlasov matter" continues the investigation of these stability problems. The *vector field method for relativistic transport equations* developed by Fajman, Joudioux, and Smulevici [16] is used to obtain a proof of stability of Minkowski space-time with massless Vlasov matter. This proof is shorter than the previously known proof by Taylor and it can treat more general initial data. In Taylor's work the initial data for the Vlasov field is assumed to be compactly supported both in the space and the momentum variables. His work relies on the fact that in the time evolution all particles remain in the *wave zone*, i.e. close to the light cone. With the vector field method for relativistic transport equations the compact support assumptions can be removed and replaced by decay conditions that are optimal.

In the paper presented in this thesis the Einstein-Vlasov system is considered in wave coordinates. Splitting the metric g into the Minkowski part η and a (small) perturbation h, i.e. writing $g = \eta + h$, one can write the Einstein-Vlasov system as a non-linear wave equation coupled to a transport equation,

$$g^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h) - 2T[f]_{\mu\nu}, \tag{7.1}$$

$$\mathbf{T}_g f = 0. \tag{7.2}$$

Here $F_{\mu\nu}(h)(\partial h, \partial h)$ is the non-linearity coming from the Einstein equations and $\mathbf{T}_g$ is the transport operator, given by

$$\mathbf{T}_g = g^{\mu\nu} v_\mu \partial_{x^\nu} - \frac{1}{2}\partial_{x^i} g^{\mu\nu} v_\mu v_\nu \partial_{v_i}, \tag{7.3}$$

where x^μ, v_i, $\mu = 0, \dots, 3$, $i = 1, 2, 3$ denote the coordinates on the co-mass shell, defined in (1.12). The energy momentum tensor for Vlasov matter is given by

$$T[f]_{\mu\nu} = \int_{\mathcal{P}_x} f v_\mu v_\nu \mathrm{dvol}_{\mathcal{P}_x}, \tag{7.4}$$

where $\mathcal{P}_x$ is the fibre of the co-mass shell over $x \in \mathcal{M}$ and $\mathrm{dvol}_{\mathcal{P}_x}$ is the volume element on this fibre.

The system (7.1)–(7.2) has to be supplemented with initial data on a Cauchy hypersurface. This initial data has to satisfy the *Einstein constraint equations*. Non-trivial initial data for the Vlasov field entails, via the constraint equations, perturbed initial data for the metric. The metric fields can be analysed with the methods developed by Lindblad and Rodnianski [26]. However the source term which consists in the energy-momentum tensor of the Vlasov field needs of course to be analysed separately.

The Vlasov field is analysed with the vector field method for relativistic transport equations. In order to describe the new ideas of the paper we first recall some basic notions of this method. As for the vector field method for non-linear wave equations developed by Klainerman and others in the '80s the Minkowski vector fields play an important role. These vector fields are the generators of the symmetries of Minkowski space-time. The set of these vector fields is denoted by $\mathbb{K}$. We assume that they are numbered (from 1 to 11) such that a multi index $I = (I_1, I_2, \dots)$, where $I_i = 1, \dots, 11$ for all i, describes a unique sequence of Minkowski fields. To denote a general element of $\mathbb{K}$ the variable Z is used and by Z^I we denote the aforementioned sequence of Minkowski fields.

For the vector field method for relativistic transport equations, extensions of the Minkowski vector fields to the co-tangent bundle are needed. These extensions are given by the *complete lifts* of the Minkowski fields, and they are denoted by $\hat{Z}$, see [16] for details. Energy norms for the Vlasov field of the form

$$\mathbb{E}_N[f](t) = \sum_{|I|\le N} \int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v} \left|\hat{Z}^I f(t,x,v)\right| |v| \,\mathrm{d}v\mathrm{d}x, \quad |v| := \sqrt{v_1^2 + v_2^2 + v_3^2}, \tag{7.5}$$

can be controlled. $|I|$ denotes the length of the multi-index I, i.e. the number of involved, lifted Minkowski fields $\hat{Z}$. This in turn gives pointwise decay estimate for velocity averages of the Vlasov field via the Klainerman-Sobolev type inequality

$$\int_{\mathbb{R}^3_v} |f(t,x,v)||v|\mathrm{d}v \lesssim \frac{\mathbb{E}_3[f](t)}{(1+t+r)^2(1+|t-r|)}, \tag{7.6}$$

where $r = |x|$. We point out that these functionals $\mathbb{E}_N$ are not the energy of the space-time or its mass. The name results from the technical analogy to the wave equation. The key ingredients for the control of these energy norms are energy estimates, decay estimates and a commutator formula. The energy estimate for Vlasov fields reads

$$\mathbb{E}[f](t) \lesssim \mathbb{E}[f](0) + \int_0^t \int_{\mathbb{R}^3_x} \int_{\mathbb{R}^3_v} |\mathbf{T}_g f| \,\mathrm{d}v\mathrm{d}x\mathrm{d}\tau. \tag{7.7}$$

So in order to control the energy $\mathbb{E}_N[f]$ one needs to control the term

$$\sum_{|I|\leq N} \int_0^t \int_{\mathbb{R}^3_x} \int_{\mathbb{R}^3_v} |\mathbf{T}_g \hat{Z}^I f| \mathrm{d}v\mathrm{d}x\mathrm{d}\tau.$$

Since $\mathbf{T}_g f = 0$ there holds $\mathbf{T}_g \hat{Z}^I f = [\mathbf{T}_g, \hat{Z}^I] f$. This is where the commutator formula comes into play. The commutator formula gives (very schematically) terms like

$$\left[\mathbf{T}_g, \hat{Z}^I\right] f = \sum_{\substack{|J|+|K|\leq|I|+1 \\ |K|\geq 1}} \kappa_{J,K}(t,x,v) \left(\partial Z^J h\right) \left(\hat{Z}^K f\right),$$

where $\kappa_{J,K}(t,x,v)$ is a coefficient function. The important mechanism is that, if the total number N of derivatives is high enough, in each commutator term there are either "many" derivatives (i.e. Z-fields) acting on h or there are "many" derivatives (i.e. $\hat{Z}$-fields) acting on f. Never both cases occur simultaneously. Consequently, there is always one factor with "not many" derivatives which can be estimated pointwise using either the ordinary Klainerman-Sobolev inequality for h or the Klainerman-Sobolev type inequality (7.6) for f. These Klainerman-Sobolev estimates are the decay estimates. The other factor can then be estimated in energy.

The decay coming form the Klainerman-Sobolev inequality is however not enough to control these energies, at least not for four dimensional space-times. One needs to exploit that the energy of the gravitational perturbations stays concentrated close to the light cone. A consequence of this is that the derivatives of the gravitational perturbation tangent to the light cone have improved decay properties. For massless Vlasov matter a similar effect can be exploited. Most of the particles move outwards along the light cone very fast and the momenta in the other directions of these particles decay. Now, in the space-time integral which has to be controlled in the energy estimate (7.7), there will be in each term either a "good" component of the gravitational perturbation h or of the momenta v of the particles. This structure is referred to as *null structure*.

In order to exploit this structure it is convenient to work with an adapted null frame $\{L, \underline{L}, A, B\}$, where L is pointing along the light cone, $\underline{L}$ is pointing transversal to the light cone and A and B are orthogonal unit vectors on the two-sphere. At the analysis of the space-time integrals of the energy estimate (7.7) for the highest order energy $\mathbb{E}_N[f]$ one encounters the following "problematic" term:

$$\sum_{|I|\leq N} \int_0^t \int_{\mathbb{R}^3_x} \int_{\mathbb{R}^3_v} \tau \left|\not\nabla Z^I h\right|_{\mathcal{L}\mathcal{L}} \left|\not\nabla f\right| \,\mathrm{d}v\mathrm{d}x\mathrm{d}\tau, \tag{7.8}$$

where $\not\nabla$ denotes the gradient tangential to the two-sphere. This gradient contains the "good" derivatives A, B. We are clearly in the situation where there are "many" derivatives acting on the metric perturbation h and only one derivative on f, i.e. h must be estimated in energy and f can be estimated pointwise which yields some decay, cf. (7.6).

We call this term (7.8) "problematic" because it cannot be controlled with tools that have been known in prior work. On the one hand there is a growth in t which needs to be compensated for. On the other hand there is a lot of favourable structure in the h-factor. First the "good" derivative $\not\nabla$ is present. Second one deals with the "good" LL-component. The question is how to make use of this structure.

To this end we consider separate energies for different components of the metric perturbation h. These energies are of the form

$$\mathcal{E}^{a,b}_{k,\mathcal{V}\mathcal{W}}[h](t) = \sum_{|I|\leq k} \int_{\mathbb{R}^3_x} \left|\nabla Z^I h\right|^2_{\mathcal{V}\mathcal{W}} \omega^b_a \mathrm{d}x + \sum_{|I|\leq k} \int_0^t \int_{\mathbb{R}^3_x} \frac{\left|\nabla Z^I h\right|^2_{\mathcal{V}\mathcal{W}}}{1+|t-r|} \omega^b_a \mathrm{d}x\mathrm{d}\tau, \tag{7.9}$$

where $\omega^b_a = \omega^b_a(u)$ is a weight function in the variable $u = t-r$ with parameters $a, b \geq 0$. We work with the energy $\mathcal{E}^{a,b}_{k,\mathcal{U}\mathcal{U}}$, where $\mathcal{U}$ stands for an arbitrary component of the frame $\{L, \underline{L}, A, B\}$, with the energy $\mathcal{E}^{a,b}_{k,\mathcal{T}\mathcal{U}}$ where $\mathcal{T}$ stands for the components $\{L, A, B\}$ which are tangential to the light cone, and with the energy $\mathcal{E}^{a,b}_{k,\mathcal{L}\mathcal{L}}$. We can make use of improved behaviour of the "good" components of the source terms of (7.7) to show that the energy $\mathcal{E}^{a,b}_{k,\mathcal{L}\mathcal{L}}$ grows much slower on top order than the energy $\mathcal{E}^{a,b}_{k,\mathcal{U}\mathcal{U}}$.

Of course, the different decay properties of the different components of the metric perturbation h have been exploited in energy in prior work as well. The concept of energies for different frame components is however new as far as we know. In this prior work the way to use favourable components of h in energy is more indirect and works with weighted energies, like the energy

$$\mathcal{E}_{\text{Morawetz}} = \int_{\mathbb{R}^3_x} \Big((1 + |t+r|^2) \left(|Lh|^2 + |Ah|^2 + |Bh|^2 \right) + (1 + |t-r|^2) |\underline{L}h|^2 \Big) \mathrm{d}x. \quad (7.10)$$

One obtains this energy by taking the Morawetz vector field as multiplier. In [23] this has been used for the analysis of a charged scalar field on a fixed, curved background, for example.

The strategy relying on weighted energies is however not appropriate in the present situation due to the Vlasov related source term (basically the energy momentum tensor $T[f]$ of the Vlasov field) in (7.7). It is very difficult to avoid a certain growth of the problematic term (7.8). Using an energy for h with no weights in $|t+r|$ to estimate the problematic term (7.8) leads to a growth of a bit more than $\sqrt{1+t}$. This in turn leads to a growth of the Vlasov related source terms of (7.7). Energies with stronger weights in $|t+r|$ yield better bounds of the problematic term (7.8), but they are more difficult to control. Consequently the source terms need to have better properties. The critical mechanism is that any growth of the problematic term (7.8) leads to growth of the energy $\mathcal{E}^{a,b}_{N,\mathcal{UU}}[h]$ which then in turn would yield even worse bounds of (7.8). So the energy estimates could not be closed. Resorting to energies for different components of h eliminates this back coupling. Even though the problematic term (7.8) grows, the "good" component of the Vlasov related source term of (7.7) does not grow. Consequently the growth of the problematic term does not affect the behaviour of the energy $\mathcal{E}^{a,b}_{k,\mathcal{LL}}[h]$ which is precisely the energy needed to estimate it.

Controlling energies for individual components of the metric perturbation h requires special technical tools. Commuting the wave operator $g^{\alpha\beta}\partial_\alpha\partial_\beta$ with the "good" frame fields L, A, B produces additional source terms with factors $1/r$. One needs to extract decay from these factors. Here the weight functions ω^b_a (cf. the definition (7.9)) play an important role. When controlling the energy $\mathcal{E}^{a,b}_{N,\mathcal{LL}}[h]$ one can extract decay from the factor $1/r$ via $1/r \lesssim (1+|t-r|)/(1+t+r)$. The factor $1+|t-r| = 1+|u|$ needs then to be absorbed by the weight function of a different energy, for example $\mathcal{E}^{a,b}_{N,\mathcal{UU}}[h]$, that is used when controlling $\mathcal{E}^{a,b}_{N,\mathcal{LL}}[h]$.

Among the problems that have been solved with the vector field method, it is a peculiarity of the problem at hand, that the null structure has to be crucially exploited on the highest order of the energy estimates to deal with some strong growth on this order.

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