Tensor hierarchy algebras and extended geometry. Part I. Construction of the algebra

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ABSTRACT: Tensor hierarchy algebras constitute a class of non-contragredient Lie superalgebras, whose finite-dimensional members are the "Cartan-type" Lie superalgebras in Kac's classification. They have applications in mathematical physics, especially in extended geometry and gauged supergravity. We further develop the recently proposed definition of tensor hierarchy algebras in terms of generators and relations encoded in a Dynkin diagram (which coincides with the diagram for a related Borcherds superalgebra). We apply it to cases where a grey node is added to the Dynkin diagram of a rank $r+1$ Kac-Moody algebra $\mathfrak{g}^+$, which in turn is an extension of a rank r finite-dimensional semisimple simply laced Lie algebra $\mathfrak{g}$. The algebras are specified by $\mathfrak{g}$ together with a dominant integral weight λ. As a by-product, a remarkable identity involving representation matrices for arbitrary integral highest weight representations of $\mathfrak{g}$ is proven. An accompanying paper [1] describes the application of tensor hierarchy algebras to the gauge structure and dynamics in models of extended geometry.

Contents

1   Introduction

Simple finite-dimensional Lie superalgebras were classified by Kac in ref. [2]. Among them are the peculiar Cartan-type superalgebras $W(n)$ and $S(n)$, where $W(n)$ is the derivation algebra of the associative superalgebra of (point-wise) forms in n dimensions under the wedge product (the Grassmann algebra on n generators), and $S(n) \subset W(n)$ is a scale-preserving subalgebra. These superalgebras are non-contragredient, meaning that they do not have a presentation in terms of generators and relations which is symmetric, up to signs, under the interchange of generators at positive and negative levels.

In ref. [3], we introduced a set of generators and relations for $W(n)$ and $S(n)$, with an antisymmetry between positive and negative levels, by modifying the presentation of the contragredient Lie superalgebra $A(0, n-1) = \mathfrak{sl}(1|n)$. This construction starts with the Dynkin diagram of $A(0, n-1) = \mathfrak{sl}(1|n)$ but can be applied to other similar Dynkin diagram as well. In general it leads to a so called tensor hierarchy algebra (THA) [4], a Lie superalgebra that is an infinite-dimensional super-extension of a Kac-Moody algebra $\mathfrak{g}$. The Cartan-type superalgebras of Kac are obtained as the special cases $W(A_{n-1}) = W(n)$ and $S(A_{n-1}) = S(n)$. The corresponding contragredient Lie superalgebra is a Borcherds superalgebra $\mathscr{B}(\mathfrak{g})$ such that $\mathscr{B}(A_{n-1}) = A(0, n-1)$. In ref. [3] we studied in detail the case of finite-dimensional $\mathfrak{g}$. The main purpose of the present paper is to extend this study to the case where $\mathfrak{g}$ is extended by an additional node in the Dynkin diagram to a possibly infinite-dimensional Kac-Moody algebra $\mathfrak{g}^+$ (the precise definition depends in addition to $\mathfrak{g}$ on the choice of a dominant integral weight λ, in a way that will be clarified later).

The invention of the THA's was motivated by the need to accommodate the embedding tensor of gauged supergravities in the algebra [4, 5]. It has subsequently become clear [6–9] that they are also needed as an algebraic basis for models of extended geometry [10]. In certain simple cases, where so called ancillary transformations do not appear, only the corresponding Borcherds superalgebra is needed. In ref. [8] we derived an L_∞-algebra from it, encoding the gauge structure in the absence of ancillary transformations. The more general situation demands that a THA is used. We refer to the accompanying paper [1] for details on extended geometry, and for details about gauge transformations (generalised diffeomorphisms) and dynamics in such models.

The paper is organised as follows. In section 2, we review the Chevalley-Serre construction of the corresponding Borcherds superalgebras $\mathscr{B}(\mathfrak{g}^+)$. This presentation is then generalised, using the same Dynkin diagram, to the THA's in section 3. Section 4 deals with the tensor product between the adjoint of $\mathfrak{g}$ and any highest weight representation, using the multiplicity formula of Parthasarathy, Ranga Rao and Varadarajan [11]. This tensor product is needed to determine the content of a THA in a double grading, where each grade forms a $\mathfrak{g}$-module. A $\mathfrak{g}$-covariant description is then given in section 5, and a sequence of subalgebra embeddings of THA's is described in section 6. The $\mathfrak{g}$-covariant description leads to a remarkable algebraic identity involving projectors on irreducible submodules of the tensor product $R(\lambda) \otimes \mathbf{adj}$, which is verified explicitly in a series of examples in section 7. We end with conclusions in section 8.

The accompanying paper [1] deals with the application of the tensor hierarchy algebras $S(\mathfrak{g}^+)$ constructed here to extended geometry, both the gauge structure (in the form of an L_∞ algebra) and the dynamics. In order for both papers to be reasonably self-contained, their contents have a certain overlap.

2   The Borcherds superalgebra $\mathscr{B}$

We start with a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ or rank r, which we assume to be simply laced, and a dominant integral weight λ, which we assume satisfies $(\lambda, \lambda) \neq 1$ in a normalisation where the simple roots α_i of $\mathfrak{g}$ have length squared $(\alpha_i, \alpha_i) = 2$. The
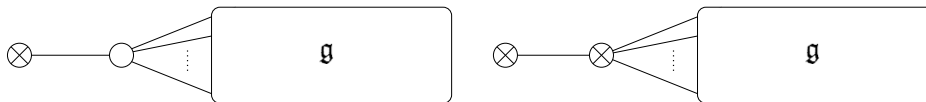
Figure 1. Two equivalent Dynkin diagrams for $\mathscr{B}(\mathfrak{g}^+)$, $W(\mathfrak{g}^+)$ and $S(\mathfrak{g}^+)$. Removing the "grey" node in the first diagram yields the Dynkin diagram of $\mathfrak{g}^+$.

assumption that λ is dominant integral means that the Dynkin labels $\lambda_i = (\lambda, \alpha_i)$ are non-negative integers (not all zero). The Dynkin labels are the coefficients of λ in the basis of fundamental weights Λ_i, defined by $(\Lambda_i, \alpha_j) = \delta_{ij}$. A dominant integral weight λ defines a highest weight representation, which is denoted $R(\lambda)$, with λ as highest weight, The dual (conjugate) representation with lowest weight $-\lambda$ is denoted $R(-\lambda) = \overline{R(\lambda)}$. We use the same notation for the representations and the corresponding modules. In concrete examples they may also be denoted by their dimension, written in boldface.

The Borcherds superalgebra $\mathscr{B} = \mathscr{B}(\mathfrak{g}^+)$ can be constructed by adding two nodes to the Dynkin diagram of $\mathfrak{g}$. This can be done in two different but equivalent ways, related by an "odd Weyl reflection" [12] as shown in figure 1. In ref. [8] we considered $\mathscr{B}$ as constructed from a Dynkin diagram of the second type, with two grey nodes. Here we will instead construct $\mathscr{B}$ from a Dynkin diagram of the first type, with only one grey node. A difference in notation compared to ref. [8] is that we label the r nodes in the Dynkin diagram of $\mathfrak{g}$ (or the corresponding simple roots) by an index i that takes the values $i = 2, \ldots, r+1$ rather than $i = 1, 2, \ldots, r$.

Thus, to the Dynkin diagram of $\mathfrak{g}$ we first add a white node 1 connected to node i by λ_i lines, extending $\mathfrak{g}$ to $\mathfrak{g}^+$. This first extending node corresponds to a simple root α_1 of the same length as the simple roots of $\mathfrak{g}$ (even when it is connected with multiple lines, which means that there are no arrows). Then, the Dynkin diagram of $\mathfrak{g}^+$ is extended by a grey node 0 connected to node 1 with a single line, and disconnected from all nodes i of $\mathfrak{g}$. The corresponding simple root α_0 is a null root. This Dynkin diagram corresponds to a Cartan matrix B_{ab} ($a, b = 0, 1, \ldots, r+1$) where B_{ij} ($i, j = 2, 3, \ldots, r+1$) is the Cartan matrix of $\mathfrak{g}$ and

$$B_{1i} = B_{i1} = -\lambda_i, \qquad B_{01} = B_{10} = -1, \qquad B_{11} = 2, \qquad B_{0i} = B_{i0} = B_{00} = 0. \qquad (2.1)$$

To each node a we associate three generators e_a, f_a, h_a. Among these $3(r+2)$ generators, e_0 and f_0 are odd, the others even. Now $\mathscr{B}$ is defined as the Lie superalgebra generated by the set $\{e_a, f_a, h_a\}$ modulo the Chevalley-Serre relations

$$[h_a, e_b] = B_{ab} e_b, \qquad [h_a, f_b] = -B_{ab} f_b, \qquad [e_a, f_b] = \delta_{ab} h_b, \qquad (2.2)$$
$$(\text{ad } e_a)^{1-B_{ab}} (e_b) = (\text{ad } f_a)^{1-B_{ab}} (f_b) = 0. \qquad (2.3)$$

Note that we use the notation $[\cdot, \cdot]$ for the brackets, also between two odd (fermionic) elements, when it is symmetric.

When we extend $\mathfrak{g}$ to $\mathscr{B}$ we also extend the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ to a Cartan subalgebra $\mathscr{H}$ of $\mathscr{B}$. The set of simple roots α_a constitute a basis of the dual space $\mathscr{H}^*$ with an inner product given by the Cartan matrix, $(\alpha_a, \alpha_b) = B_{ab}$. Since we assume that

$\mathfrak{g}$ is simply laced, the Cartan matrix B_{ij} is symmetric and all the simple roots have the same length squared, which we normalise to 2. It should however be straightforward to generalise our results to situations where $\mathfrak{g}$ is not simply laced, as long as λ has vanishing Dynkin labels for the short roots (i.e., if node 1 is disconnected from nodes representing short roots). We write $\psi(\alpha) = h_\alpha$ for the isomorphism $\psi : \mathfrak{h}^* \to \mathfrak{h}$ given by $\psi(\alpha_i) = h_i$.

The Borcherds superalgebra $\mathscr{B}$ can be decomposed into a direct sum of subspaces, labelled by pairs of integers (p, q) where p and q are the eigenvalues of $-h_0$ and

$$\big((\lambda, \lambda) - 1\big)h_0 - h_1 - h_\lambda \,, \tag{2.4}$$

respectively. We will refer to them as *level* and *height*, respectively.[1] This is a consistent $(\mathbb{Z} \times \mathbb{Z})$-grading in the sense that the subspaces at even and odd $p + q$ belong to the even and odd part of the Lie superalgebra, respectively. Since $\mathfrak{g}$ is a subalgebra at $(p, q) = (0, 0)$, the subspace at any definite pair of integers (p, q) forms an $\mathfrak{g}$-module. Our notation for these modules is given in table 1. As can be seen there, all modules come in pairs, except for those at level $p = 0$. For all other pairs of integers, any irreducible module that appears at (p, q) also appears at either $(p, q + 1)$ or $(p, q - 1)$. This "doublet structure" follows from the fact that e_0, f_0, h_0 form a Heisenberg superalgebra,

$$[e_0, f_0] = [f_0, e_0] = h_0 \,, \qquad\qquad [h_0, e_0] = [h_0, f_0] = 0 \,, \tag{2.5}$$

that commutes with $\mathfrak{g}$. In ref. [8] we defined corresponding raising and lowering operators. In the notation that we use here, the definitions take the form[2]

$$\sharp \; : \quad A \mapsto A^\sharp = -\frac{1}{p}[A, f_0] \,,$$
$$\flat \; : \quad A \mapsto A^\flat = -[A, e_0] \,, \tag{2.6}$$

for any element A at level $p \neq 0$. It follows from the Chevalley-Serre relations that they satisfy

$$\sharp^2 = \flat^2 = 0 \,, \qquad\qquad \sharp\flat + \flat\sharp = \mathbb{1} \,, \tag{2.7}$$

and commute with the adjoint action of any element in $\mathfrak{g}$. We introduce basis elements E_M and F^M for the odd subspaces at $(p, q) = (1, 0)$ and $(p, q) = (-1, 0)$, respectively, which form the $\mathfrak{g}$-modules $R(-\lambda)$ and $R(\lambda)$. Accordingly, $E_M^\sharp$ and $F^{\flat M}$ (denoted $\widetilde{E}_M$ and $\widetilde{F}^M$ in refs. [8, 13]) are basis elements for the even subspaces at $(p, q) = (1, 1)$ and $(p, q) = (-1, -1)$. For the subalgebra $\mathfrak{g}$ at $(p, q) = (0, 0)$ we introduce basis elements T_α, where the adjoint index can be raised by the inverse $\eta^{\alpha\beta}$ of the Killing form $\eta_{\alpha\beta} = (T_\alpha, T_\beta)$. At $(p, q) = (0, 0)$ we also have a two-dimensional abelian subalgebra that commutes with $\mathfrak{g}$. As basis elements, it is convenient to choose $k = h_0 + h_1 + h_\lambda$ and $\widetilde{k} = h_1 + h_\lambda$. Also the

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[1]We will occasionally talk about "levels" with respect to other $\mathbb{Z}$-gradings too, and also about the "height" of a root or a weight in the usual meaning.

[2]Note that, unlike in ref. [8], the raising operator is here associated with an "f generator", and the lowering operator with an "e generator". This is a consequence of the "odd Weyl reflection" that relates the two diagrams in figure 1 to each other.

	$p=-1$	$p=0$	$p=1$	$p=2$	$p=3$	$\cdots$
$\cdots$						$\boxed{n=0}$
$q=3$					$\widetilde{\widetilde{R}}_3$	$\boxed{n=1}$
$q=2$				$\widetilde{R}_2$	$\widetilde{R}_3 \oplus \widetilde{\widetilde{R}}_3$	$\boxed{n=2}$
$q=1$		$\mathbf{1}$	R_1	$R_2 \oplus \widetilde{R}_2$	$R_3 \oplus \widetilde{R}_3$	$\boxed{n=3}$
$q=0$	$\overline{R}_1$	$\mathbf{1} \oplus \mathbf{adj} \oplus \mathbf{1}$	R_1	R_2	R_3	$\cdots$
$\cdots$	$\overline{R}_1$	$\mathbf{1}$				

Table 1. The general structure of the superalgebra $\mathscr{B}(\mathfrak{g}^+)$. Red lines are the usual levels $n = p - q$ in the level decomposition of $\mathscr{B}(\mathfrak{g}^+)$, and form $\mathfrak{g}^+$-modules.

	$p=-1$	$p=0$	$p=1$
$q=1$		f_0	$E_M^\sharp$
$q=0$	F^M	$k \quad T_\alpha \quad \widetilde{k}$	E_M
$q=-1$	$F^{\flat M}$	e_0	

Table 2. Basis elements for $\mathscr{B}(\mathfrak{g}^+)$ at $p = -1, 0, 1$.

generators e_0 and f_0 at $(p, q) = (0, -1)$ and $(p, q) = (0, 1)$, respectively, are clearly singlets under $\mathfrak{g}$ since node 0 is disconnected from the nodes $2, 3, \ldots, r+1$. At levels $p = 0, \pm 1$ (the *local part* of the Lie superalgebra with respect to this $\mathbb{Z}$-grading) we thus have the basis elements shown in table 2.

Of particular interest are modules R_2 and $\widetilde{R}_2$. R_2 contains the symmetric tensor product of two R_1's, except the lowest one, which is removed by the relation $[e_0, e_0] = 0$, so that

$$R_2 = \vee^2 R(-\lambda) \ominus R(-2\lambda). \tag{2.8}$$

$\widetilde{R}_2$ contains the antisymmetric tensor product of two $R(-\lambda)$'s, with the modules corresponding to Serre relations in $\mathfrak{g}^+$ containing two e_1's removed, i.e.,

$$\widetilde{R}_2 = \wedge^2 R(-\lambda) \ominus \bigoplus_{i:\lambda_i=1} R(-(2\lambda - \alpha_i)) \tag{2.9}$$

(we use $\vee$ and $\wedge$ for symmetric and antisymmetric tensor products).

The (anti-)commutation relations are

$$[T_\alpha, E_M] = -(t_\alpha)_M{}^N E_N \,, \qquad\qquad [T_\alpha, E_M^\sharp] = -(t_\alpha)_M{}^N E_N^\sharp \,,$$

$$[k, E_M] = -(\lambda, \lambda) E_M \,, \qquad\qquad [\widetilde{k}, E_M^\sharp] = (2 - (\lambda, \lambda)) E_M^\sharp \,,$$

$$[\widetilde{k}, E_N] = (1 - (\lambda, \lambda)) E_N \,, \qquad\qquad [k, E_N^\sharp] = (1 - (\lambda, \lambda)) E_N^\sharp \,,$$

$$[f_0, E_N] = -E_N^\sharp \,, \qquad\qquad\qquad [f_0, E_N^\sharp] = 0 \,,$$

$$[e_0, E_N] = 0 \,, \qquad\qquad\qquad\quad [e_0, E_N^\sharp] = E_N \,, \qquad\qquad (2.10)$$

$$[T_\alpha, F^N] = (t_\alpha)_M{}^N F^M \,, \qquad\qquad [T_\alpha, F^{\flat N}] = (t_\alpha)_M{}^N F^{\flat M} \,,$$

$$[k, F^N] = (\lambda, \lambda) F^N \,, \qquad\qquad\quad [\widetilde{k}, F^{\flat N}] = ((\lambda, \lambda) - 2) F^{\flat N} \,,$$

$$[\widetilde{k}, F^N] = ((\lambda, \lambda) - 1) F^N \,, \qquad\quad [k, F^{\flat N}] = ((\lambda, \lambda) - 1) F^{\flat N} \,,$$

$$[f_0, F^N] = 0 \,, \qquad\qquad\qquad\quad [f_0, F^{\flat N}] = -F^N \,,$$

$$[e_0, F^N] = -F^{\flat N} \,, \qquad\qquad\qquad [e_0, F^{\flat N}] = 0 \,, \qquad\qquad (2.11)$$

$$[E_M, F^N] = -(t^\alpha)_M{}^N T_\alpha + \delta_M{}^N k \,, \qquad [E_M^\sharp, F^{\flat N}] = -(t^\alpha)_M{}^N T_\alpha + \delta_M{}^N \widetilde{k} \,,$$

$$[E_M, F^{\flat N}] = \delta_M{}^N e_0 \,, \qquad\qquad\quad [E_M^\sharp, F^N] = \delta_M{}^N f_0 \,. \qquad\qquad (2.12)$$

3 Modifying $\mathscr{B}$ to a tensor hierarchy algebra

In the Borcherds superalgebra, there is never a nontrivial module $\widetilde{R}_1$. A direct motivation from extended geometry to introduce a tensor hierarchy algebra comes from the need for such a module in order to describe ancillary transformations [1].

In ref. [3], two different Lie superalgebras $W(\mathfrak{g}^+)$ and $S(\mathfrak{g}^+)$, both called tensor hierarchy algebras, were defined in the case of finite-dimensional $\mathfrak{g}^+$. We will here give a slightly different definition, valid also for infinite-dimensional $\mathfrak{g}^+$ (but still finite-dimensional $\mathfrak{g}$). The algebra needed in extended geometry [1] is $S(\mathfrak{g}^+)$, but in accordance with ref. [3] we first give the definition of $W(\mathfrak{g}^+)$, and then explain how the definition of $S(\mathfrak{g}^+)$ is obtained from it.

Our investigation will exclude the case $(\lambda, \lambda) = 1$, which happens when $\mathfrak{g} = D_r$ and $\lambda = \Lambda_1$, so that $R(\lambda)$ is the vector representation. This case is somewhat degenerate (see eqs. (5.13) and (6.4)), for example in the sense that the embeddings of section 6 are not valid. The corresponding tensor hierarchy algebras are still well-defined, and should be relevant for double geometry. However, some aspects, especially the identification of ideals, require a special treatment, which we will not deal with here.

3.1 The tensor hierarchy algebra W

The tensor hierarchy algebras $W = W(\mathfrak{g}^+)$ and $S = S(\mathfrak{g}^+)$ are defined from the same Dynkin diagram and Cartan matrix B_{ab} as $\mathscr{B}$, corresponding to an $(r + 2)$-dimensional vector space with a basis consisting of simple roots α_a and inner product $(\alpha_a, \alpha_b) = B_{ab}$. However, the assignments of generators to the nodes in the Dynkin diagram is different.

The generators of W are obtained from those of $\mathscr{B}$ in the following way. The even generators e_i, f_i, h_a and the odd generator e_0 are kept, but the other odd generator f_0

is replaced by $r + 1$ odd generators f_{0a}, where $a = 0, 2, \ldots, r + 1$. Henceforth, whenever f_{0a} appears we assume $a \neq 1$, and whenever f_a appears we assume $a \neq 0$. Otherwise, if nothing else explicitly stated, the indices $a, b, \ldots$ will take the values $0, 1, 2, \ldots, r + 1$. The default values of the indices $i, j, \ldots$ will be $2, 3, \ldots, r + 1$. We introduce a consistent $(\mathbb{Z} \times \mathbb{Z})$-grading with level p and height q as for $\mathscr{B}$.

In the definition of W we now first define an auxiliary algebra $\widetilde{W}$ as the Lie superalgebra generated by the set $\{e_a, f_a, f_{0a}, h_a\}$ modulo the relations

$$[h_a, e_b] = B_{ab} e_b, \qquad [h_a, f_b] = -B_{ab} f_b, \qquad [e_a, f_b] = \delta_{ab} h_b, \qquad (3.1)$$

$$(\text{ad } e_a)^{1 - B_{ab}}(e_b) = (\text{ad } f_a)^{1 - B_{ab}}(f_b) = 0. \qquad (3.2)$$

$$[e_0, f_{0a}] = h_a, \qquad [h_a, f_{0b}] = -B_{a0} f_{0b}, \qquad [e_i, [f_j, f_{0a}]] = \delta_{ij} B_{aj} f_{0j}, \qquad (3.3)$$

$$[e_1, f_{0a}] = [f_1, [f_1, f_{0a}]] = [f_{0a}, f_{0b}] = 0. \qquad (3.4)$$

In the first two lines we recognise the relations (2.2) (but now with the assumption that the single index on f does not take the value 0).

Let $\widetilde{W}_{(i,j)}$ be the subspace of $\widetilde{W}$ spanned by all elements of the form

$$[x_1, [x_2, \ldots, [x_{N-1}, x_N] \cdots]] \qquad (3.5)$$

for some integer N, where each $x_j \in \{e_a, f_a, f_{0a}, h_a\}$ $(j = 1, 2, \ldots, N)$ and among the N elements x_j, the generators e_1 and f_1 appear i and j, times, respectively. (Henceforth, we will occasionally write a multi-bracket of the form (3.5) simply as $[x_1, \ldots, x_{N-1}, x_N]$.) The algebra $\widetilde{W}$ has a $\mathbb{Z}$-grading $\widetilde{W} = \bigoplus_{p \in \mathbb{Z}} \widetilde{W}_p$ where $\widetilde{W}_p$ is the sum of all subspaces $\widetilde{W}_{(i,j)}$ such that $i - j = p$. Let J be the maximal ideal of $\widetilde{W}$ intersecting $\widetilde{W}_0$ trivially (obtained by taking the sum of all ideals with this property). We define W as the quotient obtained from $\widetilde{W}$ by factoring out this ideal, $W = \widetilde{W}/J$.

We will see that $\widetilde{W}_{(1,0)} = \widetilde{W}_1$ and $\widetilde{W}_{(0,1)} = \widetilde{W}_{-1}$. This is not obvious. Since there are no relations $[e_i, f_{0a}] = 0$ for $i = 2, 3, \ldots, r + 1$, the Lie superalgebra $\widetilde{W}$ does not admit a triangular decomposition. When we consider basis elements of the form (3.5) for $N \geqslant 2$, we cannot assume that either all $x_j \in \{e_a\}$ or all $x_j \in \{f_a, f_{0b}\}$. Moreover, if one of the elements x_j is equal to e_a and another one is equal to f_a (if $a \neq 0$) or some f_{0b} (if $a = 0$), then it is in general not possible to rewrite any such expression using $[e_a, f_b] = \delta_{ab} f_b$ or $[e_0, f_{0b}] = h_b$ so that both disappear. It is however possible in special cases: for any a when $\mathfrak{g}$ is finite-dimensional [3] and, as we will see, for $a = 1$ when $\mathfrak{g}^+$ is finite-dimensional. (What we will show explicitly is the corresponding statement for the subalgebra S, but it can be shown in the same way for W.)

In ref. [3], where $\mathfrak{g}$ was assumed to be finite-dimensional and λ a fundamental weight $\lambda = \Lambda_k$, the tensor hierarchy algebra W was defined similarly from an auxiliary algebra $\widetilde{W}$, but with the $\mathbb{Z}$-grading associated to node 0 rather than to node 1. It was then shown that, in the case of $\mathfrak{g} = A_r$ and $\lambda = \Lambda_2$, where W is the finite-dimensional Lie superalgebra of Cartan type $W(r + 2)$, the ideal J intersecting the local part trivially was generated by the relations

$$[f_{0a}, f_{0b}] = [f_{0i}, [f_{0j}, f_1]] = [(f_{02} - f_{00}), [f_{0j}, f_1]] = 0 \qquad (3.6)$$

for $i, j = 3, \ldots, r + 1$. Here we have instead included the relations $[f_{0a}, f_{0b}] = 0$ already in the definition of $\widetilde{W}$ and the ideal that we factor out is the maximal one intersecting $\widetilde{W}_0$ trivially, where the $\mathbb{Z}$-grading is associated to node 1 rather than node 0 (the relations involving f_1 are contained in this ideal). The reason is that we have $\widetilde{W}_{(1,0)} = \widetilde{W}_1$ and $\widetilde{W}_{(0,1)} = \widetilde{W}_{-1}$ in this $\mathbb{Z}$-grading, as discussed above.

Another difference in comparison with the relations in ref. [3] is that, among the relations

$$[e_a, [e_a, f_{0b}]] = [f_a, [f_a, f_{0a}]] = 0 \,, \tag{3.7}$$

there, we have only included $[f_1, [f_1, f_{0a}]] = 0$ here. The other ones follow in fact from the relations above. For $[e_a, [e_a, f_{0b}]] = 0$ with $a = 0$ or $a = 1$, this was noted already in ref. [3], and also that $[e_i, f_{0a}] = [f_i, f_{0a}] = 0$ if $B_{ia} = 0$. Suppose now that $B_{ij} = -1$. Then

$$\begin{aligned}
2[e_i, [e_i, f_{0j}]] &= [e_i, [e_i, [e_j, [f_j, f_{0j}]]]] \\
&= 2[e_i, [e_j, [e_i, [f_j, f_{0j}]]]] - [e_j, [e_i, [e_i, [f_j, f_{0j}]]]] = 0
\end{aligned} \tag{3.8}$$

and finally

$$\begin{aligned}
-[e_i, [e_i, f_{0i}]] &= [e_i, [e_i, [e_i, [f_i, f_{0j}]]]] \\
&= [e_i, [e_i, [f_i, [e_i, f_{0j}]]]] \\
&= [e_i, [h_i, [e_i, f_{0j}]]] + [e_i, [f_i, [e_i, [e_i, f_{0j}]]]] = 0 \,.
\end{aligned} \tag{3.9}$$

In the same way, one can show that $[f_i, [f_i, f_{0a}]] = 0$.

3.2 The tensor hierarchy algebra S

It is easy to see that if we remove the generators f_{0i} and the relations that involve them, but keep f_{00}, then we recover $\mathscr{B}$ from W (identifying f_{00} with f_0). Conversely, we can remove the generators f_{00} and h_0 and the relations that involve them, but keep f_{0i}. Then we obtain the tensor hierarchy algebra S. Thus S is defined, via an auxiliary algebra $\widetilde{S}$, in the same way as W above but without the generators f_{00} and h_0 and the relations that involve them. We assign values of p and q to the generators as in W.

In W we can define operators $\sharp$ and $\flat$ satisfying (2.7), by replacing f_0 by f_{00} in (2.6). In S this is not possible since there is no generator f_{00} in S that could be identified with f_0 in $\mathscr{B}$. One might think that this would mean an absence in S of the "doublet structure" present in $\mathscr{B}$ at nonzero levels. However, it is in fact still present in S (we do not have a proof to all levels in the general case, but the opposite seems extremely unlikely), and it even extends to level $p = 0$.

We will show that it is indeed possible to define an operator $\sharp$ on the subalgebra of S generated by $\{e_i, f_i, e_0, f_{0i}, h_a\}$ such that $\sharp$ satisfies (2.7), with $\flat$ still defined by (2.6). First we set

$$h_i{}^\sharp = -f_{0i} \,, \qquad h_1{}^\sharp = f_{0\lambda} \,, \qquad e_0{}^\sharp = \widetilde{k} = h_1 + h_\lambda \,, \qquad f_{0i}{}^\sharp = 0 \,. \tag{3.10}$$

It then follows that $\sharp^2 = 0$ and $\sharp\flat + \flat\sharp = \mathbb{1}$ on these generators, and that $[x, e_0]^\sharp = [x, e_0{}^\sharp]$, where x is any element in $\mathfrak{g}$.

Let us write $h_\alpha{}^\sharp = -f_{0\alpha}$. In order to extend the operator $\sharp$ to the root vectors e_α of $\mathfrak{g}$ (corresponding to positive or negative roots), we note that

$$(\alpha, \beta)[e_\alpha, f_{0\gamma}] = (\alpha, \gamma)[e_\alpha, f_{0\beta}] \tag{3.11}$$

for any root α of $\mathfrak{g}$ and $\beta, \gamma \in \mathfrak{h}^*$. This was shown in ref. [3] in the case when α, β, γ are simple roots, and it is straightforward to show it also in this general case. We can then unambiguously set

$$e_\alpha{}^\sharp = \frac{1}{(\alpha, \beta)}[e_\alpha, f_{0\beta}] \tag{3.12}$$

for any root α of $\mathfrak{g}$ and any $\beta \in \mathfrak{g}^*$ such that $(\alpha, \beta) \neq 0$. As shown in ref. [3] (with $\beta = \varrho$), this implies that

$$[x, y^\sharp] = [x, y]^\sharp \tag{3.13}$$

for any $x, y \in \mathfrak{g}$. We also set $e_1{}^\sharp = 0$ and

$$f_1{}^\sharp = \frac{1}{(\lambda, \lambda)}[f_1, f_{0\lambda}]. \tag{3.14}$$

Another result from ref. [3] that we will use is

$$[e_\alpha, [e_{-\alpha}, f_{0\beta}]] = (\alpha, \beta)f_{0\alpha} \tag{3.15}$$

for any root α of $\mathfrak{g}$ and any $\beta \in \mathfrak{h}^*$.

3.3 Local part of S

We will now study the subspaces of S at levels $p = 0, \pm 1$ and decompose each of them further into subspaces at different heights q. It will be useful to consider also a $\mathbb{Z}$-grading of $\mathfrak{g}$ with respect to λ. We let $\mathfrak{g}_{(\ell)}$ be the subspace of $\mathfrak{g}$ spanned by all root vectors e_α corresponding to roots α such that $(\alpha, \lambda) = \ell$, and, if $\ell = 0$, the Cartan generators h_i of $\mathfrak{g}$. We thus have $\mathfrak{g} = \bigoplus_{\ell \in \mathbb{Z}} \mathfrak{g}_{(\ell)}$. We also write, for example, $\mathfrak{g}_{(\leqslant 1)} = \bigoplus_{\ell \leqslant 1} \mathfrak{g}_{(\ell)}$. For homogeneous elements x in $\mathfrak{g}$ with respect to this $\mathbb{Z}$-grading we call this degree λ-level and denote it by $\ell(x)$, so that $x \in \mathfrak{g}_{(\ell(x))}$. The Dynkin diagram of $\mathfrak{g}_{(0)} \subseteq \mathfrak{g}$ is obtained by removing the nodes i in the Dynkin diagram of $\mathfrak{g}$ that are connected to node 1 in the extension to $\mathfrak{g}^+$, i.e., the nodes with $\lambda_i \neq 0$.

In the notation introduced above for $\widetilde{W}$, the algebra $\widetilde{S}$ contains subspaces $\widetilde{S}_{(0,0)}$, $\widetilde{S}_{(1,0)}$ and $\widetilde{S}_{(0,1)}$ at levels 0, 1 and -1, respectively. The subspace $S_{(0,0)}$ is the subalgebra generated by all generators but e_1 and f_1. We will also denote it by S' below. The subspace $S_{(1,0)}$ is spanned by multi-brackets that contain precisely one e_1 and no f_1, whereas, conversely, $S_{(0,1)}$ is spanned by multi-brackets that contain precisely one f_1 and no e_1. In the multi-brackets that span $S_{(1,0)}$, the only e_1 generator can always be put in the innermost position by the Jacobi identity. When considered as spanned by such multi-brackets, we say that

the $S_{(1,0)}$ is the S'-module generated by e_1, and denote it by $S'(e_1)$. We will use the corresponding notation, $\mathfrak{g}(a)$, for the $\mathfrak{g}$-module generated by some element a in S (or in the algebra currently under investigation).

At this point it is not clear that the algebras $\widetilde{W}$ and $\widetilde{S}$ are non-trivial, i.e. that the relations (3.1)–(3.4) generate a proper ideal of the free Lie superalgebra generated by $\{e_a, f_a, f_{0a}, h_a\}$ and not the whole free Lie superalgebra itself. This will be shown in section 5. We will anticipate this result and proceed under the assumption that $\widetilde{W}$ and $\widetilde{S}$ indeed are non-trivial.

3.3.1 The subalgebra S'

We start by examining the contents of the subalgebra S' of S. At height $q = 0$, it contains the subalgebra generated by $\{e_i, f_i, h_1\}$. This is $\mathfrak{g} \oplus \langle \widetilde{k} \rangle$ of $\mathfrak{g}$, the direct sum of $\mathfrak{g}$ and a one-dimensional Lie algebra spanned by $\widetilde{k}$. At height $q = 1$ and $q = -1$ it contains the $\mathfrak{g}$-modules $\mathfrak{g}(e_0)$ and $\mathfrak{g}(\mathfrak{h}^\sharp)$ generated by e_0 and all f_{0i}, respectively. The first one is a singlet since e_0 commutes with all e_i, f_i. The second one is $\mathfrak{g}^\sharp$, which is isomorphic to $\mathfrak{g}$ itself, the adjoint module, according to (3.13). Since $[e_0, \mathfrak{g}^\sharp] = \mathfrak{g}$, there is no other $\mathfrak{g}$-module in S' at height $q = 0$ or $q = \pm 1$. Furthermore, since $[e_0, e_0] = 0$, the algebra S' does not contain any non-trivial element at height $q \leqslant -2$. To see that S' does not contain any non-trivial element at height $q \geqslant 2$ either, we use (3.11). We then get $[[e_\alpha, f_{0\beta}], f_{0\gamma}] = [e_\alpha, [f_{0\beta}, f_{0\gamma}]] = 0$ if $(\alpha, \gamma) = 0$, and otherwise

$$[[e_\alpha, f_{0\beta}], f_{0\gamma}] = \frac{(\alpha, \beta)}{(\alpha, \gamma)}[[e_\alpha, f_{0\gamma}], f_{0\gamma}] = \frac{(\alpha, \beta)}{2(\alpha, \gamma)}[e_\alpha, [f_{0\gamma}, f_{0\gamma}]] = 0. \tag{3.16}$$

From this it easily follows that $[\mathfrak{g}^\sharp, \mathfrak{g}^\sharp] = 0$. We summarise:

$$S' = \langle e_0 \rangle \oplus \langle \widetilde{k} \rangle \oplus \mathfrak{g} \oplus \mathfrak{g}^\sharp, \tag{3.17}$$

where the $\mathfrak{g}$-modules on the right hand side appear at heights $q = -1$, 0, 0 and 1, respectively. At this point, it is not yet clear that (3.17) is the full content of S at level $p = 0$ since *a priori* there might be elements in $\widetilde{S}_{(1,1)}, \widetilde{S}_{(2,2)}, \ldots$ that are not contained in $S' = \widetilde{S}_{(0,0)}$. We will however see that this is not the case. It suffices to show that $[f_1, \widetilde{S}_{(1,0)}] \subset \widetilde{S}_{(0,0)}$.

3.3.2 The subspace S_1

Before studying the full subspace S_1 we first study $\widetilde{S}_{(1,0)} \subseteq S_1$ in order to show that $[f_1, \widetilde{S}_{(1,0)}] \subseteq \widetilde{S}_{(0,0)}$, which implies that $S_1 = \widetilde{S}_{(1,0)}$.

The subspace $\widetilde{S}_{(1,0)}$ of $\widetilde{S}$ is spanned by all elements of the form

$$[s_1, [s_2, \ldots, [s_{N-1}, [s_N, e_1]] \cdots]] \tag{3.18}$$

where $s_1, \ldots, s_{N-1} \in S'$ for some $N \geqslant 0$. It follows from the relations in S' that any such expression can be written as a sum of other ones, which are "normal-ordered" in the following sense:

$$s_1, \ldots, s_P \in \mathfrak{g},$$
$$s_{P+1}, \ldots, s_{P+Q} \in \mathfrak{g}^\sharp,$$
$$s_{P+Q+1}, \ldots, s_{P+Q+R} = e_0, \tag{3.19}$$

where $P, Q, R \geqslant 0$ and $P + Q + R = N$. We note that any such nonzero expression is antisymmetric in $s_{P+1}, \ldots, s_{P+Q}$ since $[\mathfrak{g}^\sharp, \mathfrak{g}^\sharp] = 0$ and we may also without loss of generality assume that it is symmetric in $s_1, \ldots, s_P$. Furthermore, because of the relation $[e_0, [e_0, e_1]] = 0$ we can assume R to be either 0 or 1. We will see that this holds also for Q, and we will also restrict the λ-levels of the elements in $\mathfrak{g}$ and $\mathfrak{g}^\sharp$. First we will show that $[\mathfrak{g}_{(\leqslant 1)}^\sharp, e_1] = 0$.

Consider $[x^\sharp, e_1]$, where $x \in \mathfrak{g}$. From the relations $[h_i^\sharp, e_1] = -[f_{0i}, e_1] = 0$ we know that this is zero if x belongs to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. If x is a root vector e_α of a root α such that $(\alpha, \lambda) \leqslant 0$, then $[e_\alpha, e_1] = 0$ and

$$[x^\sharp, e_1] = \frac{1}{(\alpha, \beta)}[[e_\alpha, f_{0\beta}], e_1] = \frac{1}{(\alpha, \beta)}\big([e_\alpha, [f_{0\beta}, e_1]] - [f_{0\beta}, [e_\alpha, e_1]]\big) = 0\,, \qquad (3.20)$$

for some $\beta \in \mathfrak{h}^*$ such that $(\alpha, \beta) \neq 0$. Thus $[\mathfrak{g}_{(\leqslant 0)}^\sharp, e_1] = 0$. If x is a root vector e_α of a root α such that $(\alpha, \lambda) = 1$, then $[e_\alpha, e_1] \neq 0$, but still $[e_\alpha, [e_\alpha, e_1]] = 0$. This implies

$$(\operatorname{ad} e_\alpha)^2(\operatorname{ad} e_1) - 2(\operatorname{ad} e_\alpha)(\operatorname{ad} e_1)(\operatorname{ad} e_\alpha) + (\operatorname{ad} e_1)(\operatorname{ad} e_\alpha)^2 = 0\,, \qquad (3.21)$$

and then, using (3.15),

$$\begin{aligned}
[e_1, e_\alpha{}^\sharp] &= \frac{1}{2}[e_1, [e_\alpha, f_{0\alpha}]] \\
&= \frac{1}{4}[e_1, [e_\alpha, [e_\alpha, [e_{-\alpha}, f_{0\alpha}]]]] \\
&= \frac{1}{2}[e_\alpha, [e_1, [e_\alpha, [e_{-\alpha}, f_{0\alpha}]]]] - \frac{1}{4}[e_\alpha, [e_\alpha, [e_1, [e_{-\alpha}, f_{0\alpha}]]]] \\
&= [e_\alpha, [e_1, f_{0\alpha}]] - \frac{1}{4}[e_\alpha, [e_\alpha, [e_{-\alpha}, [e_1, f_{0\alpha}]]]] = 0\,.
\end{aligned} \qquad (3.22)$$

Thus $[\mathfrak{g}_{(\leqslant 1)}^\sharp, e_1] = 0$. Since $[\mathfrak{g}_{(\leqslant 0)}, e_1] = 0$ and $[\mathfrak{g}_{(\ell)}, \mathfrak{g}_{(\ell')}^\sharp] = \mathfrak{g}_{(\ell+\ell')}^\sharp$ we can now refine (3.19) to

$$\begin{aligned}
s_1, \ldots, s_P &\in \mathfrak{g}_{(\geqslant 1)}\,, \\
s_{P+1}, \ldots, s_{P+Q} &\in \mathfrak{g}_{(\geqslant 2)}{}^\sharp\,, \\
s_{P+Q+1}, \ldots, s_{P+Q+R} &= e_0\,.
\end{aligned} \qquad (3.23)$$

Next we will show that $[f_1, \mathfrak{g}_{(\geqslant 1)}^\sharp] = 0$. Acting on (3.22), where $(\alpha, \lambda) = 1$, twice with f_1 we get

$$\begin{aligned}
0 = [f_1, f_1, e_1, e_\alpha{}^\sharp] &= -[f_1, h_1, e_\alpha{}^\sharp] - [h_1, f_1, e_\alpha{}^\sharp] + [e_1, f_1, f_1, e_\alpha{}^\sharp] \\
&= (2((\lambda, \alpha) - 1) + 2)[f_1, e_\alpha{}^\sharp] \\
&= 2(\lambda, \alpha)[f_1, e_\alpha{}^\sharp]\,,
\end{aligned} \qquad (3.24)$$

where we have used that

$$[f_1, f_1, e_\alpha{}^\sharp] = \frac{1}{(\alpha, \beta)}[f_1, f_1, e_\alpha, f_{0\beta}] = \frac{1}{(\alpha, \beta)}[e_\alpha, f_1, f_1, f_{0\beta}] = 0 \qquad (3.25)$$

for some β such that $(\alpha, \beta) \neq 0$. Thus $[f_1, \mathfrak{g}_{(1)}{}^\sharp] = 0$. If $x \in \mathfrak{g}_{(\ell)}$ for $\ell \geqslant 2$, then $x^\sharp$ is a sum of terms $[x_1, \ldots, x_\ell, f_{0\gamma}]$ where $x_1, \ldots, x_\ell \in \mathfrak{g}_{(1)}$ and

$$[f_1, x_1, \ldots, x_\ell, f_{0\gamma}] = [x_1, \ldots, x_{\ell-1}, f_1, x_\ell, f_{0\gamma}] = 0. \tag{3.26}$$

Thus $[f_1, \mathfrak{g}_{(\geqslant 1)}{}^\sharp] = 0$. Since in particular $[f_1, \mathfrak{g}_{(\geqslant 2)}{}^\sharp] = 0$, and also $[f_1, \mathfrak{g}_{(\geqslant 1)}] = 0$, we get

$$[f_1, s_1, \ldots, s_N, e_1] = [s_1, \ldots, s_N, f_1, e_1] = -[s_1, \ldots, s_N, h_1] \in S' \tag{3.27}$$

when we act with f_1 on (3.18), assuming (3.23). We conclude that $[f_1, \widetilde{S}_{(1,0)}] \subseteq \widetilde{S}_{(0,0)}$ and it follows that

$$\widetilde{S}_{-1} = \widetilde{S}_{(0,1)}, \qquad\qquad \widetilde{S}_0 = \widetilde{S}_{(0,0)}, \qquad\qquad \widetilde{S}_1 = \widetilde{S}_{(1,0)}. \tag{3.28}$$

In the same way as in (3.27), for any $x, y \in \mathfrak{g}_{(\geqslant 2)}$, we get

$$[f_1, s_1, \ldots, s_N, x^\sharp, y^\sharp, e_1] = -[s_1, \ldots, s_N, x^\sharp, y^\sharp, h_1]. \tag{3.29}$$

This is proportional to $[s_1, \ldots, s_N, x^\sharp, y^\sharp]$, which is zero, since $[x^\sharp, y^\sharp]$ is. It follows that $[x^\sharp, [y^\sharp, e_1]]$ generates an ideal of $\widetilde{S}$ that is contained in $\bigoplus_{p \geqslant 1} \widetilde{S}_p$, and then it must be zero in S since S is obtained from $\widetilde{S}$ by factoring out the maximal ideal that intersects $\widetilde{S}_0$ trivially. Thus $[\mathfrak{g}^\sharp, [\mathfrak{g}^\sharp, e_1]] = 0$ in S. Furthermore, since

$$\begin{aligned}
[x^\sharp, y^\sharp, e_0, e_1] &= -[x^\sharp, y, e_1] - [x^\sharp, e_0, y^\sharp, e_1] \\
&= -[x^\sharp, y, e_1] + [x, y^\sharp, e_1] + [e_0, x^\sharp, y^\sharp, e_1] \\
&= -[[x, y]^\sharp, e_1] - [y, x^\sharp, e_1] + [x, y^\sharp, e_1] = 0,
\end{aligned} \tag{3.30}$$

we have $[\mathfrak{g}^\sharp, \mathfrak{g}^\sharp, e_0, e_1] \subseteq \mathfrak{g}([\mathfrak{g}^\sharp, e_1])$ and $[\mathfrak{g}^\sharp, \mathfrak{g}^\sharp, \mathfrak{g}^\sharp, e_0, e_1] = 0$. We get

$$\begin{aligned}
S_1 &= \mathfrak{g}([e_0, e_1]) + \mathfrak{g}(e_1) + \mathfrak{g}([\mathfrak{g}^\sharp, e_0, e_1]) + \mathfrak{g}([\mathfrak{g}^\sharp, e_1]) \\
&= \mathfrak{g}(e_1{}^\flat) + \mathfrak{g}(e_1) + \mathfrak{g}([\mathfrak{g}^\sharp, e_1{}^\flat]) + \mathfrak{g}([\mathfrak{g}^\sharp, e_1]).
\end{aligned} \tag{3.31}$$

Here we can replace $\mathfrak{g}([\mathfrak{g}^\sharp, e_1{}^\flat])$ by $\mathfrak{g}([\mathfrak{g}^\sharp, e_1]^\flat)$. We then get

$$S_1 = \mathfrak{g}(e_1{}^\flat) + \mathfrak{g}(e_1) + \mathfrak{g}\big([\mathfrak{g}^\sharp, e_1]^\flat\big) + \mathfrak{g}\big([\mathfrak{g}^\sharp, e_1]\big), \tag{3.32}$$

where $\mathfrak{g}$ can be replaced by $\mathfrak{g}_{(\geqslant 1)}$ and $\mathfrak{g}^\sharp$ can be replaced by $\mathfrak{g}_{(\geqslant 2)}{}^\sharp$. We will see later that this sum of $\mathfrak{g}$-modules is direct. The $\mathfrak{g}$-modules on the right hand side appear at heights $q = 0, 1, 1$ and 2, respectively.

3.3.3 The subspace S_{-1}

We now turn to level $p = -1$ and the subspace $S_{-1} = S'(f_1)$. It is spanned by all elements of the form

$$[s_1, \ldots, s_N, f_1] \tag{3.33}$$

where $s_1, \ldots, s_{N-1} \in S'$ for some $N \geqslant 0$, and, according to what we have already shown, we may assume

$$
\begin{aligned}
s_1, \ldots, s_P &\in \mathfrak{g}_{(\leqslant -1)}, \\
s_{P+1}, \ldots, s_{P+Q} &\in \mathfrak{g}_{(\leqslant 0)}{}^\sharp,
\end{aligned}
\tag{3.34}
$$

where $P, Q \geqslant 0$ and $P + Q = N$. We will show that $[\mathfrak{g}^\sharp, \mathfrak{g}^\sharp, \mathfrak{g}^\sharp, f_1] = 0$, which means that $Q \leqslant 2$ in any nonzero expression of the form (3.33). According to the results above, it is sufficient to show that $[e_1, S'([\mathfrak{g}^\sharp, \mathfrak{g}^\sharp, \mathfrak{g}^\sharp, f_1])] = 0$ since that implies that $S'([\mathfrak{g}^\sharp, \mathfrak{g}^\sharp, \mathfrak{g}^\sharp, f_1])$ generates an ideal of S contained in $\bigoplus_{p \leqslant -1} S_p$, which must be trivial. The vector space $S'([\mathfrak{g}^\sharp, \mathfrak{g}^\sharp, \mathfrak{g}^\sharp, f_1])$ is spanned by elements of the form

$$
[e_1, s_1, \ldots, s_N, x^\sharp, y^\sharp, z^\sharp, f_1]
\tag{3.35}
$$

where $x, y, z \in \mathfrak{g}$ and $s_1, \ldots, s_N$ are elements in S' that we can assume satisfy (3.23) with $R = 0, 1$. If $R = 0$ (that is, if $s_N \neq e_0$), then we may as well assume that $s_1, \ldots, s_N$ satisfy (3.34). Since $[e_1, \mathfrak{g}_{(\leqslant 1)}{}^\sharp] = 0$ (in particular $[e_1, \mathfrak{g}_{(\leqslant 0)}{}^\sharp] = 0$, see (3.20)) and $[e_1, \mathfrak{g}_{(\leqslant -1)}] = 0$ we then get

$$
\begin{aligned}
[e_1, s_1, \ldots, s_N, x^\sharp, y^\sharp, z^\sharp, f_1] &= [s_1, \ldots, s_N, x^\sharp, y^\sharp, z^\sharp, e_1, f_1] \\
&= [s_1, \ldots, s_N, x^\sharp, y^\sharp, z^\sharp, h_1] = 0.
\end{aligned}
\tag{3.36}
$$

If $R = 1$ (that is, if $s_N = e_0$), then the expression (3.35) is equal to

$$
\begin{aligned}
[e_1, s_1, \ldots, s_{N-1}, e_0, x^\sharp, y^\sharp, z^\sharp, f_1] = &-[e_1, s_1, \ldots, s_{N-1}, x, y^\sharp, z^\sharp, f_1] \\
&+ [e_1, s_1, \ldots, s_{N-1}, x^\sharp, y, z^\sharp, f_1] \\
&- [e_1, s_1, \ldots, s_{N-1}, x^\sharp, y^\sharp, z, f_1],
\end{aligned}
\tag{3.37}
$$

which in turn can be written as a sum of terms of the form

$$
[e_1, s_1, \ldots, s_N, x^\sharp, y^\sharp, f_1]
\tag{3.38}
$$

where $x, y \in \mathfrak{g}$ and $s_1, \ldots, s_N$ satisfy (3.34). This can be shown to be zero in the same way as $[e_1, s_1, \ldots, s_N, x^\sharp, y^\sharp, z^\sharp, f_1]$ in (3.36).

Thus we have $[\mathfrak{g}^\sharp, \mathfrak{g}^\sharp, \mathfrak{g}^\sharp, f_1] = 0$, and it follows that

$$
S_{-1} = \mathfrak{g}(f_1) + \mathfrak{g}([\mathfrak{g}^\sharp, f_1]) + \mathfrak{g}([\mathfrak{g}^\sharp, \mathfrak{g}^\sharp, f_1]).
\tag{3.39}
$$

As we will see, it is convenient to rewrite this sum of $\mathfrak{g}$-modules. First, since

$$
\begin{aligned}
[x^\sharp, y^\sharp, f_1] &= -[x^\sharp, y^\sharp, e_0, f_1{}^\sharp] \\
&= [x^\sharp, y, f_1{}^\sharp] - [x, y^\sharp, f_1{}^\sharp] - [e_0, x^\sharp, y^\sharp, f_1{}^\sharp] \\
&= [[x, y]^\sharp, f_1{}^\sharp] + [y, x^\sharp, f_1{}^\sharp] - [x, y^\sharp, f_1{}^\sharp],
\end{aligned}
\tag{3.40}
$$

we have $\mathfrak{g}([\mathfrak{g}^\sharp, \mathfrak{g}^\sharp, f_1]) = \mathfrak{g}([\mathfrak{g}^\sharp, f_1{}^\sharp])$. Second, it will turn out to be convenient to write $\mathfrak{g}([\mathfrak{g}^\sharp, f_1])$ as a sum of the two submodules $\mathfrak{g}(f_1{}^\sharp)$ and $\mathfrak{g}([\mathfrak{g}^\sharp, f_1{}^\sharp]^\flat)$. We thus arrive at

$$
S_{-1} = \mathfrak{g}(f_1) + \mathfrak{g}(f_1{}^\sharp) + \mathfrak{g}\big([\mathfrak{g}^\sharp, f_1{}^\sharp]^\flat\big) + \mathfrak{g}\big([\mathfrak{g}^\sharp, f_1{}^\sharp]\big),
\tag{3.41}
$$

where $\mathfrak{g}$ can be replaced by $\mathfrak{g}_{(\leqslant -1)}$ and $\mathfrak{g}^\sharp$ can be replaced by $\mathfrak{g}_{(\leqslant 0)}{}^\sharp$. The $\mathfrak{g}$-modules on the right hand side appear at heights $q = -1$, 0, 0 and 1, respectively.

In the above derivation of the content of $S_{\pm 1}$ in terms of $\mathfrak{g}$-modules, we have relied on the definition of S as the superalgebra obtained by factoring out the maximal ideal in $\widetilde{S}$ intersecting $\widetilde{S}_0$ trivially. We know that in many cases [3], the ideal contains a part generated by $[f_{0i}, [f_{0j}, f_1]]$ for all i, j such that $\lambda_i = \lambda_j = 0$. We also know that in some cases there is an additional part generated by elements at positive levels $p \geqslant 2$, see section 5.4. Although we have not been able to derive the content of $S_{\pm 1}$ in terms of $\mathfrak{g}$-modules using only the defining relations (and $[f_{0i}, [f_{0j}, f_1]]$ for $\lambda_i = \lambda_j = 0$) we have no proof that it is impossible. This possibility of course does not affect the results (3.32) and (3.41).

4 The tensor product $R(\lambda) \otimes \mathbf{adj}$

We will now determine the $\mathfrak{g}$ modules that appear in the local part of S, that is, on the right hand sides of (3.17), (3.32) and (3.41). At $p = 0$ we already know that $\mathfrak{g}^\sharp$ is an adjoint $\mathfrak{g}$ module, and that e_0 and $\widetilde{k}$ span two singlets. At $p = 1$ it is easy to see that $\mathfrak{g}(e_1{}^\flat)$ and $\mathfrak{g}(e_1)$ are lowest-weight modules with lowest weights $-\lambda$,

$$\mathfrak{g}(e_1{}^\flat) \simeq \mathfrak{g}(e_1) \simeq R(-\lambda). \tag{4.1}$$

Likewise, at $p = -1$ it is easy to see that $\mathfrak{g}(f_1)$ and $\mathfrak{g}(f_1{}^\sharp)$ are highest-weight modules with highest weights λ,

$$\mathfrak{g}(f_1) \simeq \mathfrak{g}(f_1{}^\sharp) \simeq R(\lambda). \tag{4.2}$$

It remains to determine the modules

$$\mathfrak{g}\big([\mathfrak{g}^\sharp, e_1]^\flat\big) \simeq \mathfrak{g}\big([\mathfrak{g}^\sharp, e_1]\big) \tag{4.3}$$

at $p = 1$ and

$$\mathfrak{g}\big([\mathfrak{g}^\sharp, f_1{}^\sharp]^\flat\big) \simeq \mathfrak{g}\big([\mathfrak{g}^\sharp, f_1{}^\sharp]\big) \tag{4.4}$$

at $p = -1$. These modules must be contained in the tensor products $R(-\lambda) \otimes \mathbf{adj}$ and $R(\lambda) \otimes \mathbf{adj}$, respectively. We will therefore in this section study the tensor product $R(\lambda) \otimes \mathbf{adj}$ and its decomposition into a direct sum of irreducible submodules. (It is of course sufficient to study one of the two tensor products in detail.)

We thus consider the tensor product $R(\lambda) \otimes \mathbf{adj}$, where λ is an arbitrary dominant integral weight. Clearly, all irreducible representations occurring in the tensor products with non-zero multiplicity are $R(\lambda + \gamma)$, where γ lies in the root lattice.

Denote the multiplicity of $R(\nu)$ in $R(\mu) \otimes R(\lambda)$ by $\mathrm{mult}(R(\mu) \otimes R(\lambda), R(\nu))$. The multiplicity formula of Parthasarathy, Ranga Rao and Varadarajan (PRV) [11] reads

$$\mathrm{mult}(R(\lambda) \otimes R(\mu), R(\nu)) = \dim \left\{ v \in R(\mu)_{\nu - \lambda} : e_i^{\lambda_i + 1} v = 0 \text{ for all } i \right\}, \tag{4.5}$$

where $R(\mu)_\nu$ denotes the subspace of $R(\mu)$ at weight ν. The rôles of λ and μ can of course be interchanged in the formula. A state $v \in R(\mu)_{\nu - \lambda}$ such that $e_i^{\lambda_i + 1} v = 0$ for all i will

be called PRV state below. A PRV state is in general not a highest weight state for the corresponding irreducible representation in the tensor product, but always a part of it. Applied to the tensor product under consideration, we get

$$\text{mult}(R(\lambda) \otimes \mathbf{adj}, R(\lambda + \gamma)) = \dim \left\{ v \in \mathbf{adj}_\gamma : (\text{ad}\, e_i)^{\lambda_i + 1} v = 0 \text{ for all } i \right\}. \qquad (4.6)$$

This shows that the multiplicity can only be non-zero when $\gamma \in \Gamma \cup \{0\}$, where Γ is the root space of $\mathfrak{g}$. It also immediately follows that non-zero multiplicities of $R(\lambda + \gamma)$, $\gamma \neq 0$, equal 1, due to the non-degeneracy of the root decomposition of the (finite-dimensional) Lie algebra $\mathfrak{g}$.

First consider PRV states v for $\gamma \neq 0$. For i such that $\lambda_i = 0$, we need $e_i v = 0$, which means that γ is a highest root at some λ-level. For i such that $\lambda_i \neq 0$, we have $[e_i, [e_i, v]] = 0$ for such roots, with the only exception $\ell = -1$, $v = f_i$ for an i with $\lambda_i = 1$. When $\gamma = 0$, we need elements in the Cartan algebra, which are annihilated by all e_i for which $\lambda_i = 0$. These are linear combinations of h_j for $\lambda_j \neq 0$, namely the fundamental weights Λ_j, and they trivially satisfy the remaining conditions.

Thus, we have shown that

$$R(\lambda) \otimes \mathbf{adj} = NR(\lambda) \oplus \bigoplus_{\substack{\ell = -(\lambda,\theta) \\ \ell \neq -1}}^{(\lambda,\theta)} \bigoplus_{\gamma^{(\ell)} \in H_\ell} R(\lambda + \gamma^{(\ell)}), \qquad (4.7)$$

where H_ℓ is the set of highest roots at λ-level ℓ, and N is the number of non-zero λ_i ($N = 1$ for λ a multiple of a fundamental weight).

At a given ℓ, there may be several roots in H_ℓ. All λ-levels from $-(\lambda,\theta)$ to (λ,θ) occur, except $\ell = -1$, assuming λ is not a multiple of a smaller integral dominant weight. If $\lambda = n\lambda'$, the relevant λ-levels are $-n(\lambda',\theta), -n((\lambda',\theta)+1), \ldots, n(\lambda',\theta)$.

We introduce the notation

$$R_{(\ell)} = \bigoplus_{\gamma^{(\ell)} \in H_\ell} R(\lambda + \gamma^{(\ell)}) \qquad (4.8)$$

for $\ell \neq 0, -1$ and

$$R_{(0)} = NR(\lambda) \oplus \bigoplus_{\gamma^{(0)} \in H_0} R(\lambda + \gamma^{(0)}), \qquad R_{(-1)} = \{0\} \qquad (4.9)$$

so that $R(\lambda) \otimes \mathbf{adj} = \bigoplus_{\ell=-(\lambda,\theta)}^{(\lambda,\theta)} R_{(\ell)}$. We will show that

$$\mathfrak{g}(v_\lambda \otimes \mathfrak{g}_{(\geqslant 1)}) = \bigoplus_{\ell=1}^{(\lambda,\theta)} R_{(\ell)}, \qquad (4.10)$$

where v_λ is a lowest weight state in $R(\lambda)$. Any element in $v_\lambda \otimes \mathfrak{g}_{(\geqslant 1)}$ must belong to the module on the right hand side of (4.10), since the complementary submodule $\bigoplus_{\ell=-(\lambda,\theta)}^{0} R_{(\ell)}$ of $R(\lambda) \otimes \mathfrak{g}$ is spanned by states of lower weights. Thus the left hand side of (4.10) is

contained in the module on the right hand side. Conversely, the highest weight state in any submodule $R_{(\ell)}$ must be a linear combination of $v_\lambda \otimes e_{\gamma^{(\ell)}}$ and elements in

$$\mathfrak{g}_-(v_\lambda) \otimes \mathfrak{g}_+\big(e_{\gamma^{(\ell)}}\big) \subseteq \mathfrak{g}_-\big(v_\lambda \otimes \mathfrak{g}_+(e_{\gamma^{(\ell)}})\big) + v_\lambda \otimes \mathfrak{g}_-\big(\mathfrak{g}_+(e_{\gamma^{(\ell)}})\big) \qquad (4.11)$$

of weight $\lambda + \gamma^{(\ell)}$, where $\mathfrak{g}_\pm$ denote the Borel subalgebras of $\mathfrak{g}$ spanned by $\{e_i\}$ and $\{f_i\}$, respectively. But the only elements in $v_\lambda \otimes \mathfrak{g}_-\big(\mathfrak{g}_+(e_{\gamma^{(\ell)}})\big)$ of weight $\lambda + \gamma^{(\ell)}$ are multiples of $v_\lambda \otimes e_{\gamma^{(\ell)}}$. Thus the highest weight state in any module $R_{(\ell)}$ belongs to $\mathfrak{g}(v_\lambda \otimes \mathfrak{g}_{(\geqslant \ell)})$ and it follows that the module on the right hand side of (4.10) is contained in the module on the left hand side. We conclude that (4.10) holds.

Let us now return to the modules in (4.3) and (4.4). We have seen that $[x^\sharp, f_1{}^\sharp]$ is zero for $x \in \mathfrak{g}_{(\geqslant 1)}$. Also, if $x \in \mathfrak{h}_\lambda$, where $\mathfrak{h}_\lambda$ is the subspace of $\mathfrak{h}$ spanned by h_λ, then we have $[x^\sharp, f_1{}^\sharp] = 0$ since $2[f_{0\lambda}, [f_{0\lambda}, f_1]] = [[f_{0\lambda}, f_{0\lambda}], f_1] = 0$. On the other hand, if x is a root vector $x \in \mathfrak{g}_{(0)}$ or $x \in \mathfrak{g}_{(\leqslant -2)}$, then $[x^\sharp, f_1{}^\sharp]$ is nonzero. This can be seen by acting with first e_0 and then e_1. We then get

$$[e_1, e_0, x^\sharp, f_1{}^\sharp] = \left(\left(1 - \frac{1}{(\lambda,\lambda)}\right)\ell(x) - 1\right)x^\sharp, \qquad (4.12)$$

where $\ell(x)$ is the λ-level of x. If $\ell(x) \leqslant -2$ for some nonzero $x \in \mathfrak{g}$ then $(\lambda,\lambda) \geqslant 1$ since the only case where $(\lambda,\lambda) < 1$ is $\mathfrak{g} = A_r$, $\lambda = \Lambda_1$ (or $\lambda = \Lambda_r$), which leads to a 3-grading $\mathfrak{g} = \mathfrak{g}_{(-1)} \oplus \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)}$ (this can be checked by inspecting the inverse Cartan matrices for simply laced Lie algebras $\mathfrak{g}$) and then

$$\left(1 - \frac{1}{(\lambda,\lambda)}\right)\ell(x) - 1 \leqslant -1. \qquad (4.13)$$

Also for $x \in \mathfrak{h}'$, where $\mathfrak{h}'$ is a subspace of $\mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{h}_\lambda \oplus \mathfrak{h}'$ (if $N > 1$) it is easy to check that $[x^\sharp, f_1{}^\sharp] \neq 0$. It follows that

$$\mathfrak{g}\big([\mathfrak{g}^\sharp, f_1{}^\sharp]^\flat\big) \simeq \mathfrak{g}\big([\mathfrak{g}_{(\leqslant 0)}{}^\sharp, f_1{}^\sharp]\big) \simeq \frac{R(\lambda) \otimes \mathfrak{g}}{\mathfrak{g}(v_\lambda \otimes \mathfrak{g}_{(\geqslant 1)}) \oplus \mathfrak{g}(v_\lambda \otimes \mathfrak{h}_\lambda)}$$

$$\simeq \bigoplus_{\ell=-(\lambda,\theta)}^{0} R_{(\ell)} \ominus R(\lambda). \qquad (4.14)$$

Similarly, at level $p = 1$ we find that

$$\mathfrak{g}\big([\mathfrak{g}^\sharp, e_1]^\flat\big) \simeq \mathfrak{g}\big([\mathfrak{g}^\sharp, e_1]\big) \simeq \frac{R(-\lambda) \otimes \mathfrak{g}}{\mathfrak{g}(u_{-\lambda} \otimes \mathfrak{g}_{(\leqslant 1)})} \simeq \bigoplus_{\ell=-(\lambda,\theta)}^{-2} \overline{R_{(\ell)}}, \qquad (4.15)$$

where $u_{-\lambda}$ is a lowest weight state in $R(-\lambda)$. This is the representation $\widetilde{R}_1$.

5 Construction from $\mathfrak{g}$-representations

5.1 Local superalgebra in terms of $\mathfrak{g}$-modules

We have shown that if S is non-trivial, then its local part decomposes into a sum of $\mathfrak{g}$-modules according to (3.17), (3.32) and (3.41). In order to show that S indeed is non-trivial

we will now construct a non-trivial Lie superalgebra that satisfies the relations (3.1)–(3.4) if the generators are identified with certain elements in it. In this construction we use the fact that there is a $\mathbb{Z}$-graded Lie superalgebra $\mathscr{U} = \bigoplus_{p \in \mathbb{Z}} \mathscr{U}_p$ associated to any $\mathbb{Z}_2$-graded vector space $\mathscr{U}_1$, generalising the universal $\mathbb{Z}$-graded Lie algebra associated to a vector space [14, 15]. The subspaces $\mathscr{U}_{-p}$ for $p \geqslant 0$ are defined recursively as consisting of all linear maps $\mathscr{U}_1 \to \mathscr{U}_{-p+1}$, and the brackets are such that $[A, a] = A(a)$ for $A \in \mathscr{U}_{-p}$ ($p \geqslant 0$) and $a \in \mathscr{U}_1$. In particular, $\mathscr{U}_0 = \mathfrak{gl}(\mathscr{U}_1)$. The subalgebra $\bigoplus_{p \geqslant 0} \mathscr{U}_p$ is freely generated by $\mathscr{U}_1$.

In this case, we let $\mathscr{U}_1$ be the direct sum of four $\mathfrak{g}$-modules, pairwise isomorphic with an isomorphism $\sharp$. Two of the four $\mathfrak{g}$-modules transform in the representation $R_1 = R(-\lambda)$ and are denoted by U and $U^\sharp$, respectively. The other two transform in $\widetilde{R}_1$ and are denoted by $\widetilde{U}$ and $\widetilde{U}^\sharp$, respectively. Thus

$$\mathscr{U}_1 = U \oplus U^\sharp \oplus \widetilde{U} \oplus \widetilde{U}^\sharp \,. \tag{5.1}$$

According to the discussion in the preceding section, we consider the module $\widetilde{U}$ as the quotient

$$\widetilde{U} = \frac{U \otimes \mathfrak{g}}{\mathfrak{g}(e_1 \otimes \mathfrak{g}_{(\leqslant 1)})} \,, \tag{5.2}$$

where e_1 is a lowest weight state of U. We let L be the natural map $U \otimes \mathfrak{g} \to \widetilde{U}$, so that $L(u \otimes x) = 0$ if and only if $x \in \mathfrak{g}_{(\leqslant 1)}$.

Since $\mathscr{U}_1$ is a $\mathfrak{g}$-module, we can consider $\mathfrak{g}$ as a subalgebra of $\mathscr{U}_0 = \mathfrak{gl}(\mathscr{U}_1)$. We then define an odd subspace $\mathfrak{g}^\sharp$ of $\mathscr{U}_0$ isomorphic to $\mathfrak{g}$, an odd element $e_0 \in \mathscr{U}_0$ and an even element $\widetilde{k} \in \mathscr{U}_0$ by

$$\begin{aligned}
[x^\sharp, L(u \otimes y)^\sharp] &= 0 \,, \\
[x^\sharp, L(u \otimes y)] &= [x, L(u \otimes y)]^\sharp = L([x, u] \otimes y)^\sharp + L(u \otimes [x, y])^\sharp \,, \\
[x^\sharp, u^\sharp] &= -L(u \otimes x)^\sharp \,, \\
[x^\sharp, u] &= -[x, u]^\sharp - L(u \otimes x) \,,
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
[e_0, L(u \otimes x)^\sharp] &= -L(u \otimes x) \,, \\
[e_0, L(u \otimes x)] &= 0 \,, \\
[e_0, u^\sharp] &= u \,, \\
[e_0, u] &= 0 \,,
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
[\widetilde{k}, L(u \otimes x)^\sharp] &= \big(3 - (\lambda, \lambda)\big) L(u \otimes x)^\sharp \,, \\
[\widetilde{k}, L(u \otimes x)] &= \big(2 - (\lambda, \lambda)\big) L(u \otimes x) \,, \\
[\widetilde{k}, u^\sharp] &= \big(2 - (\lambda, \lambda)\big) u^\sharp \,, \\
[\widetilde{k}, u] &= \big(1 - (\lambda, \lambda)\big) u \,.
\end{aligned} \tag{5.5}$$

It is then easy to check that the subspace $\langle e_0 \rangle \oplus \langle \widetilde{k} \rangle \oplus \mathfrak{g} \oplus \mathfrak{g}^\sharp$ of $\mathscr{U}_0$ closes under the super-commutator and thus form a subalgebra. The brackets are given by

$$[x, y^\sharp] = [x, y]^\sharp \,, \qquad [e_0, x^\sharp] = -x \,, \qquad [\widetilde{k}, x^\sharp] = x^\sharp \,, \qquad [\widetilde{k}, e_0] = -e_0 \tag{5.6}$$

and $[e_0, e_0] = [\mathfrak{g}^\sharp, \mathfrak{g}^\sharp] = [e_0, \mathfrak{g}] = [\widetilde{k}, \mathfrak{g}] = 0$.

We define $e_1^\flat$ by $e_1 = (e_1{}^\flat)^\sharp$ and define an element $f_1 \in \mathscr{U}_{-1}$ recursively by

$$[f_1, e_1{}^\flat] = -e_0$$

$$[f_1, e_1] = h_\lambda - \widetilde{k}$$

$$[f_1, L(e_1 \otimes x)] = \begin{cases} (\ell(x) - 1)x & \text{if} \quad \ell(x) \geqslant 2\,, \\ 0 & \text{if} \quad \ell(x) \leqslant 1\,, \end{cases}$$

$$[f_1, L(e_1 \otimes x)^\sharp] = \begin{cases} (\ell(x) - 1)x^\sharp & \text{if} \quad \ell(x) \geqslant 2\,, \\ 0 & \text{if} \quad \ell(x) \leqslant 1\,, \end{cases} \tag{5.7}$$

where $\ell(x)$ is the λ-level of x, and

$$[f_1, e_i(u)] = 0\,,$$
$$[f_1, e_i(u^\sharp)] = [e_i, [f_1, u^\sharp]]\,,$$
$$[f_1, L([e_i, u] \otimes x)] = [e_i, [f_1, L(u \otimes x)]] - [f_1, L(u \otimes [e_i, x])]\,,$$
$$[f_1, L([e_i, u] \otimes x)^\sharp] = [e_i, [f_1, L(u \otimes x)^\sharp]] - [f_1, L(u \otimes [e_i, x])^\sharp]\,. \tag{5.8}$$

It is straightforward to show that f_1 is well defined and then that all the relations (3.1)–(3.4) are satisfied with $f_{0i} = -h_i{}^\sharp$ and $h_1 = h_\lambda - \widetilde{k}$. Thus there is a surjective isomorphism from $\widetilde{S}$ to the subalgebra of $\mathscr{U}$ generated by $f_1 \in \mathscr{U}_{-1}$ and $\mathscr{U}_1$. It follows that the Lie superalgebra S indeed is non-trivial, the sums in (3.17), (3.32) and (3.41) are direct, and the $\mathfrak{g}$-modules that appear can be decomposed into highest and lowest weight modules according to the discussion in the preceding section.

5.2 Covariant description

Let E_M be a basis of U. We set $L_{\alpha M} = -L(E_M \otimes T_\alpha)$, so that $L_{\alpha M}$ is a basis of $\widetilde{U}$ (as before, T_α is a basis of $\mathfrak{g}$). Similarly to $S'(e_1) = \mathscr{U}_1$ at $p = 1$, we decompose the subspace $S'(f_1)$ of $\mathscr{U}_{-1}$ at $p = -1$ into $\mathfrak{g}$-modules as

$$S'(f_1) = V \oplus V^\sharp \oplus \widetilde{V} \oplus \widetilde{V}^\sharp\,, \tag{5.9}$$

where V transforms in $R(\lambda)$ with lowest weight state f_1 and basis F^M. Then we can identify $\widetilde{V}$ with the quotient

$$\frac{V \otimes \mathfrak{g}}{\mathfrak{g}(f_1 \otimes \mathfrak{g}_{(\geqslant 1)}) \oplus \mathfrak{g}(f_1 \otimes \mathfrak{h}_\lambda)} \tag{5.10}$$

and let Φ be the natural map $V \otimes \mathfrak{g} \to \widetilde{V}$. If we now set $\Phi(F^M \otimes T_\alpha) = -\Phi_\alpha{}^M$, then the brackets in S involving $\mathfrak{g}^\sharp$ and the modules at level $p = \pm 1$ can be written on tensorial

form as

$$[T_\alpha^\sharp, E_M] = t_{\alpha M}{}^N E_N^\sharp + L_{\alpha M} \,,$$
$$[T_\alpha^\sharp, E_M^\sharp] = L_{\alpha M}^\sharp \,,$$
$$[T_\alpha^\sharp, L_{\beta M}] = [T_\alpha, L_{\beta M}]^\sharp = f_{\alpha\beta}{}^\gamma L_{\gamma M}^\sharp - t_{\alpha M}{}^N L_{\beta N}^\sharp \,,$$
$$[T_\alpha^\sharp, L_{\beta M}^\sharp] = 0 \,, \tag{5.11}$$

$$[T_\alpha^\sharp, H^{\flat M}] = t_{\alpha N}{}^M H^N + \Phi_\alpha{}^M \,,$$
$$[T_\alpha^\sharp, H^M] = -\Phi_\alpha^{\sharp\,M} \,,$$
$$[T_\alpha^\sharp, \Phi_\beta{}^M] = -[T_\alpha, \Phi_\beta{}^M]^\sharp = -f_{\alpha\beta}{}^\gamma \Phi_\gamma^{\sharp\,M} - t_{\alpha N}{}^M \Phi_\alpha^{\sharp\,N} \,,$$
$$[T_\alpha^\sharp, \Phi_\beta^{\sharp\,M}] = 0 \,, \tag{5.12}$$

$$[H^{\flat M}, E_N] = -\delta_N^M e_0 \,,$$
$$[H^{\flat M}, E_N^\sharp] = t_{\alpha N}{}^M T^\alpha - \delta_N^M \widetilde{k} \,,$$
$$[H^M, E_N] = -\left(1 - \frac{1}{(\lambda,\lambda)}\right) t^\alpha{}_N{}^M T_\alpha + \delta_N^M \widetilde{k} \,,$$
$$[H^M, E_N^\sharp] = \frac{1}{(\lambda,\lambda)} t^\alpha{}_N{}^M T_\alpha^\sharp \,, \tag{5.13}$$

$$[H^{\flat M}, L_{\alpha N}] = -\ell_{\alpha N}{}^{\beta M} T_\beta \,,$$
$$[H^{\flat M}, L_{\alpha N}^\sharp] = -\ell_{\alpha N}{}^{\beta M} T_\beta^\sharp \,,$$
$$[H^M, L_{\alpha N}] = -\ell_{\alpha N}{}^{\beta M} T_\beta^\sharp \,,$$
$$[H^M, L_{\alpha N}^\sharp] = 0 \,, \tag{5.14}$$

$$[\Phi_\alpha{}^M, E_N] = \varphi^\beta{}_{N,\alpha}{}^M T_\beta \,,$$
$$[\Phi_\alpha{}^M, E_N^\sharp] = \varphi^\beta{}_{N,\alpha}{}^M T_\beta^\sharp \,,$$
$$[\Phi_\alpha^{\sharp\,M}, E_N] = -\varphi^\beta{}_{N,\alpha}{}^M T_\beta^\sharp \,,$$
$$[\Phi_\alpha^{\sharp\,M}, E_N^\sharp] = 0 \,, \tag{5.15}$$

$$[L_{\alpha M}, \Phi_\beta{}^N] = [L_{\alpha M}, \Phi_\beta^{\sharp\,N}] = [L_{\alpha M}^\sharp, \Phi_\beta{}^N] = [L_{\alpha M}^\sharp, \Phi_\beta^{\sharp\,N}] = 0 \,, \tag{5.16}$$

for some invariant tensors $\ell_{\alpha N}{}^{\beta M}$ and $\varphi^\alpha{}_{M,\beta}{}^N$. These tensors will be some linear combinations of projectors on the modules appearing in L and Φ. The coefficients in these linear combinations are completely determined. One may think of L and Φ as defined by their appearances in the first equations in (5.11) and (5.12). The normalisation is then fixed, and the tensors ℓ and φ are determined. As we will see in section 5.3, they are even seemingly over-determined, and exist thanks to a peculiar identity.

An alternative way of deriving the content of S_{-1} is to note that the basis elements E_M for $R_1 = R(-\lambda)$ have a covariant Serre relation in $\mathfrak{g}([e_0, e_0]) = R(-2\lambda)$, so that the bracket $[E_M, E_N]$ lies in $R_2 = \vee^2 R_1 \ominus R(-2\lambda)$. Any element at $(p,q) = (-1,0)$ must respect the ideal in $R(-2\lambda)$. This allows for the introduction of generators $\Phi_\alpha{}^M$ with brackets $[E_N, \Phi_\alpha{}^M] = \varphi^\beta{}_{N,\alpha}{}^M T_\beta$, where φ is a linear combination of projection operators

	$p = -1$	$p = 0$	$p = 1$
$q = 2$			$L^\sharp_{\alpha M}$
$q = 1$	$\Phi^{\sharp M}_\alpha \quad G^{\sharp M}$	$f_{00} \quad T^\sharp_\alpha$	$E^\sharp_M \quad L_{\alpha M}$
$q = 0$	$F^M \quad \Phi_\alpha{}^M \quad G^M$	$k \quad T_\alpha \quad \widetilde{k}$	E_M
$q = -1$	$F^{\flat M}$	e_0	

Table 3. Basis elements for $W(\mathfrak{g}^+)$ at $p = -1, 0, 1$.

	$p = -1$	$p = 0$	$p = 1$
$q = 2$			$L^\sharp_{\alpha M}$
$q = 1$	$\Phi^{\sharp\, M}_\alpha$	$T^\sharp_\alpha$	$E^\sharp_M \qquad L_{\alpha M}$
$q = 0$	$\Phi_\alpha{}^M \qquad H^M$	$\widetilde{k} \qquad T_\alpha$	E_M
$q = -1$	$H^{\flat M}$	e_0	

Table 4. Basis elements for $S(\mathfrak{g}^+)$ at $p = -1, 0, 1$.

on the irreducible modules in Φ. They respect the ideal in $R(-2\lambda)$ if

$$t_{\beta\langle M}{}^P \varphi^\beta{}_{N\rangle,\alpha}{}^Q = (t_\beta \otimes \varphi^\beta{}_\alpha)_{\langle MN\rangle}{}^{PQ} = 0\,, \tag{5.17}$$

or equivalently, $(\varphi^\alpha{}_\beta \otimes t^\beta)_{MN}{}^{\langle PQ\rangle} = 0$, where $\langle MN\rangle$ denotes projection on $R(\pm 2\lambda)$. Eq. (5.17) is the condition for the representation of the embedding tensor, or the "big torsion representation" in extended geometry.[3]

Let us check which of the representations in $R(\lambda) \otimes \mathbf{adj}$ that respect the Serre relations. Consider an irreducible submodule $R(\lambda + \gamma^{(\ell)})$, where $\gamma^{(\ell)}$ is a highest root at λ-level ℓ. The Serre relations will automatically have vanishing bracket with an element in this module if $R(\lambda + \gamma^{(\ell)}) \otimes R(-2\lambda) \not\supset R(-\lambda)$, i.e., if

$$R(\lambda + \gamma^{(\ell)}) \otimes R(\lambda) \not\supset R(2\lambda)\,. \tag{5.18}$$

Applying the PRV formula (4.5) for the multiplicity of $R(2\lambda)$ in the tensor product on the

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[3]Although we have not performed a complete analysis, we have noted that in cases when λ is attached to a short root, there is typically no solution to this algebraic condition.

left hand side, we obtain

$$
\begin{aligned}
\text{mult}&(R(\lambda + \gamma^{(\ell)}) \otimes R(\lambda), R(2\lambda)) \\
&= \dim \left\{ v \in R(\lambda)_{\lambda - \gamma^{(\ell)}} : e_i^{(\lambda + \gamma^{(\ell)})_i + 1} v = 0 \text{ for all } i \right\}.
\end{aligned}
\tag{5.19}
$$

This multiplicity is obviously 0 for $\ell \geqslant 0$, since $R(\lambda)$ does not contain any states with the same or higher λ-level than the highest weight state. For the module $R(\lambda)$ we have $\text{mult}(R(\lambda) \otimes R(\lambda), R(2\lambda)) = 1$. We arrive at the statement that Φ respects the Serre relations at level 2 if it contains the irreducible modules

$$
(N-1)R(\lambda) \oplus \bigoplus_{\gamma \in H_0} R(\lambda + \gamma) \oplus \bigoplus_{\ell = 2}^{(\lambda, \theta)} \bigoplus_{\beta \in L_\ell} R(\lambda - \beta),
\tag{5.20}
$$

where H_0 is the set of highest roots at λ-level 0 and L_ℓ the set of lowest roots at λ-level ℓ. This is the same sum of irreducible modules as was already shown to constitute $\mathfrak{g}\big([\mathfrak{g}^\sharp, f_1{}^\sharp]^\flat\big)$ in eq. (4.14).

Using the covariant brackets, one can also check explicitly that Φ respects the Serre relations in $\bigoplus_{i:\lambda_i=1} R(-(2\lambda - \alpha_i))$ in $[E_M^\sharp, E_N^\sharp]$. The condition becomes

$$
L^\sharp_{\beta\{N} \varphi^\beta{}_{P\},\alpha}{}^M = 0,
\tag{5.21}
$$

where $\{NP\}$ denotes projection on $\bigoplus_{i:\lambda_i=1} R(\pm(2\lambda - \alpha_i))$. This is automatically satisfied, since the highest modules in Φ and L are $R(\lambda + \gamma_0)$ and $R(\lambda - \beta_2)$, where γ_0 is a highest root at level 0. The tensor product can not contain $R(2\lambda - \alpha_i)$, where $(\lambda, \alpha_i) = 1$, since $2\lambda - \alpha_i \succ 2\lambda + \gamma_0 - \beta_2$.

5.3 A remarkable identity

Consider the Jacobi identity between $T_\alpha^\sharp$, E_M and $H^{\flat N}$. This turns out to be the only non-trivial Jacobi identity within the local superalgebra at $p = -1, 0, 1$, in the sense that all others can be obtained from it by raising and lowering operations. A short calculation leads to the necessary and sufficient condition for this Jacobi identity to be fulfilled:

$$
\varphi^\beta{}_{M,\alpha}{}^N - \ell_{\alpha M}{}^{\beta N} = \delta_\alpha^\beta \delta_M^N - f_\alpha{}^{\beta\gamma} t_{\gamma M}{}^N - \frac{1}{(\lambda, \lambda)} (t^\beta t_\alpha)_M{}^N \equiv Q_{\alpha M}{}^{\beta N},
\tag{5.22}
$$

i.e.,

$$
\varphi^\beta{}_\alpha - \ell_\alpha{}^\beta = \delta_\alpha^\beta - f_\alpha{}^{\beta\gamma} t_\gamma - \frac{1}{(\lambda, \lambda)} t^\beta t_\alpha \equiv Q_\alpha{}^\beta.
\tag{5.23}
$$

If we now make use of the algebraic condition (5.17) on φ, the part of this relation only involving ℓ becomes

$$
\begin{aligned}
\ell_{\beta M}{}^{\alpha\langle P} t^\beta{}_N{}^{Q\rangle} &= f^\alpha{}_{\beta\gamma} t^\beta{}_M{}^{\langle P} t^\gamma{}_N{}^{Q\rangle} + t^\alpha{}_M{}^{\langle P} \delta_N^{Q\rangle} - \delta_M^{\langle P} t^\alpha{}_N{}^{Q\rangle} \\
&= (f^\alpha{}_{\beta\gamma} t^\beta \otimes t^\gamma + t^\alpha \otimes \mathbb{1} - \mathbb{1} \otimes t^\alpha)_{MN}{}^{\langle PQ\rangle}.
\end{aligned}
\tag{5.24}
$$

The right hand side is recognised as the "S tensor" of ref. [10]. (There, a non-vanishing S tensor was shown to be equivalent to the presence of ancillary transformations in the commutator of two generalised diffeomorphisms. Here, it is related to the existence of a module $\widetilde{R}_1$. See also ref. [1].) We thus have

$$(\ell_\beta{}^\alpha \otimes t^\beta)_{MN}{}^{\langle PQ \rangle} = S^\alpha{}_{MN}{}^{PQ} \,. \tag{5.25}$$

The tensor S is antisymmetric in its lower indices. In addition, it satisfies $S_{\{MN\}}{}^{PQ} = 0$, thanks to the identity

$$S^\alpha{}_{MN}{}^{PQ} = \left(\frac{1-\sigma}{2} Y(\mathbb{1} \otimes t^\alpha) \right)_{MN}{}^{\langle PQ \rangle} \,, \tag{5.26}$$

where σ is the permutation operator and Y is the tensor that appears in the expression for generalised diffeomorphisms in extended geometry,

$$\sigma Y = -\eta^{\alpha\beta} t_\alpha \otimes t_\beta + (\lambda, \lambda) - 1 + \sigma \,. \tag{5.27}$$

The existence of the THA shows that there is always a solution to eq. (5.23). The difficulty with directly analysing this equation lies in the translation between the projections on irreducible modules in $\mathbf{adj} \otimes R(\lambda)$ of the types $P_{\alpha M}{}^{\beta N}$ and $P^\beta{}_{M,\alpha}{}^N$, used to characterise ℓ and φ, respectively. We are not aware of any explicit translation table in the general case, although an analysis of the eigenvalues in eq. (4.12) and the corresponding ones for $p = 1$ may provide an answer.

Let us do a counting, which shows that the matrix Q must be degenerate. Assume that λ is a fundamental weight (the statements may hold in a wider setting). All irreducible modules in $\mathbf{adj} \otimes R(\lambda)$ appear with multiplicity 1. There is a single module at each level $-(\lambda, \theta) \leqslant \ell \leqslant (\lambda, \theta)$ in the grading with respect to λ, except at $\ell = -1$ where there are none, and at level 0, where there is $R(\lambda)$ and in addition a number of modules $R(\lambda + \gamma_0)$. The number of highest roots at level 0 equals the number of disjoint components of the Dynkin diagram of $\mathfrak{g}$ when the root dual to λ is deleted. The modules not in Φ are $R(\lambda)$ and $R(\lambda + \gamma_\ell)$ for $\ell \geqslant 1$. Their total number is $(\lambda, \theta) + 1$. The irreducible modules in ℓ are $R(\lambda + \gamma_\ell)$ for $\ell \leqslant -2$, giving a total number $(\lambda, \theta) - 1$. An equation like (5.23) would, for a generic Q, be over-determined by 2 equations. In order for a solution to exist, Q must show some degeneracy, which in general will involve projections of the two types. Namely, a linear combination of $P^{R(\lambda + \gamma_\ell)}{}_{\alpha M}{}^{\beta N}$ for $\ell \leqslant -2$ must have a decomposition in terms of $P^{R(\lambda + \gamma'_{\ell'})\beta}{}_{M,\alpha}{}^N$, where the coefficients for the terms with $\gamma' = 0$ and $\ell' \leqslant 1$ agree with those of Q.

The existence of the tensor hierarchy algebra thus relies on, and implies, a quite non-trivial algebraic identity involving representation matrices for arbitrary highest weight representations of finite-dimensional simply laced Lie algebras, which we have not been able to prove in an alternative way. In section 7, this identity is verified for a number of examples, and classes of examples. To this end, we need the eigenvalues of Q when it acts on irreducible modules in the tensor product $\mathbf{adj} \otimes R(\lambda)$. They can be calculated in either picture. We choose the φ picture (simply because Φ contains a larger number of irreducible modules than L).

The first term, $\delta_\alpha^\beta \delta_M^N$, has eigenvalue 1 on all modules. The second term in Q, $-f_\alpha{}^{\beta\gamma}t_{\gamma M}{}^N$, has eigenvalues that can be calculated using the quadratic Casimir operator. We have, for any representation $R(\Lambda) \ni v$, the eigenvalue

$$C_2(\Lambda)v = \frac{1}{2}T^\alpha \cdot T_\alpha \cdot v = \frac{1}{2}(\Lambda, \Lambda + 2\varrho)v\,. \tag{5.28}$$

For $v^\alpha{}_M$ in $R(\lambda + \gamma)$, this gives the eigenvalue of the second term as

$$-C_2(\lambda + \gamma) + C_2(\lambda) + g^\vee = -(\lambda + \varrho, \gamma) + g^\vee + 1 + \delta_{\gamma,0}\,. \tag{5.29}$$

The last term, $-\frac{1}{(\lambda,\lambda)}(t^\beta t_\alpha)_M{}^N$, has eigenvalue $-\frac{2C_2(\lambda)}{(\lambda,\lambda)}$ on the $R(\lambda)$ which is not in Φ and 0 on the rest (including other $R(\lambda)$'s, if λ is not a multiple of a fundamental weight). The total eigenvalue of Q on the module $R(\lambda + \gamma)$ becomes

$$Q|_{R(\lambda+\gamma)} = g^\vee - (\lambda + \varrho, \gamma) + \delta_{\gamma,0} - \frac{2C_2(\lambda)}{(\lambda,\lambda)}\varepsilon\,, \tag{5.30}$$

where $\varepsilon = 1$ on the $R(\lambda)$ not in Φ and 0 otherwise.

5.4 Comparison between $\mathscr{B}$, W and S at positive levels

Consider the level decompositions of $\mathscr{B}$, W and S in the $\mathbb{Z}$-grading where the levels $n = p-q$ form $\mathfrak{g}^+$-modules (the red lines in table 1). The modification, described in section 3.1, taking us from $\mathscr{B}$ to W, only involves the addition of the odd generators f_{0i} at level -1. The generator at e_0 at level 1, remains. The generator f_{00} in W is identified with f_0 in $\mathscr{B}$ and S is obtained from W by removing the generator f_{00}.

Since the modification only involves generators at level -1, it would seem that the subalgebras containing the positive levels, which we denote $\mathscr{B}_+$, W_+ and S_+, are unaffected, and all isomorphic. There are however two subtleties.

First, *a priori*, there might be elements in W or S formed as multibrackets with M generators e_0 and N generators f_{0a} for $M \geqslant N$ where it is not possible to cancel the N generators f_{0a} against N of the generators e_0. *A posteriori*, this turns out to not happen in the present case, where $\mathfrak{g}$ is finite-dimensional. It follows that $\mathscr{B}_+ \simeq W_+$.

Second, the removal of f_{00} in the construction of S may lead to the appearance of new ideals at positive levels. Suppose there is a $\mathfrak{g}^+$-module $\mu \subset W$ at some definite positive level n which does not vanish using only $[e_0, e_0] = 0$, and which furthermore obeys $[f_{0i}, \mu] = 0$ but $[f_{00}, \mu] \neq 0$. Then μ, seen as a subspace of $\widetilde{S}$, will generate an ideal, that according to our definitions has to be factored out to obtain the (simple) superalgebra S. The positive subalgebras are isomorphic, $S_+ \simeq \mathscr{B}_+$, only if there is no such ideal, and in general $S_+ = \mathscr{B}_+/K$, where K is the maximal ideal of this kind.

We have no general recipe for determining whether or not the ideal K of $\widetilde{S}(\mathfrak{g}^+)$ is non-trivial, but it is straightforward to find examples where this is the case. Take for example $\mathfrak{g}^+ = E_8$, and the fermionic node attached to the fundamental (adjoint) node. The level expansion of $\mathscr{B}_+(E_8)$ (see refs. [16–18]) is

$$\mathscr{B}_+(E_8) = \mathbf{248}_1 \oplus (\mathbf{1} \oplus \mathbf{3875})_2 \oplus (\mathbf{248} \oplus \mathbf{3875} \oplus \mathbf{147250})_3 \oplus \cdots \tag{5.31}$$

where the subscripts denote the level n. The elements at level -1 in $\mathscr{B}(E_8)$ consist of the module $\mathbf{248}$, while $W(E_8)$ contains $\mathbf{248} \oplus \mathbf{3875}$ and $S(E_8)$ only $\mathbf{3875}$ at level -1. It is then obvious, just by considering tensor products of E_8 representations, that the singlet $\mathbf{1}_2 \in \mathscr{B}(E_8)$ generates an ideal in $S(E_8)$, to be factored out. A similar example occurs for $S(E_6)$. There [16–18],

$$\mathscr{B}_+(E_6) = \mathbf{27}_1 \oplus \overline{\mathbf{27}}_2 \oplus \mathbf{78}_3 \oplus \overline{\mathbf{351}}_4 \oplus (\overline{\mathbf{27}} \oplus \overline{\mathbf{1728}})_5$$
$$\oplus \, (\mathbf{1} \oplus \mathbf{78} \oplus \mathbf{650} \oplus \mathbf{2430} \oplus \overline{\mathbf{5824}})_6 \oplus \cdots \qquad (5.32)$$

At level -1, $S(E_6)$ contains $\mathbf{351}$, but not the $\overline{\mathbf{27}}$ present in $\mathscr{B}(E_6)$ and $W(E_6)$. The singlet at level 6 generates an ideal.

6 The embeddings $\widetilde{W}(\mathfrak{g}) \subset \widetilde{S}(\mathfrak{g}^+) \subset \widetilde{W}(\mathfrak{g}^+)$

Suppose that λ is a fundamental weight, which we take to be Λ_2 for simplicity. Thus node 1 is connected to node 2 with a single line but disconnected from nodes $3, 4, \ldots, r+1$. We will here show that in this case $\widetilde{S}(\mathfrak{g})$ and $\widetilde{W}(\mathfrak{g})$ can be embedded in $\widetilde{S}(\mathfrak{g}^+)$ as subalgebras at height $q = 0$ with node 2 in $\widetilde{S}(\mathfrak{g}^+)$ as "node 1" in $\widetilde{S}(\mathfrak{g})$ and $\widetilde{W}(\mathfrak{g})$. First we set

$$e_0{}' = [e_0, e_1]\,, \qquad f_{0i}{}' = -[f_{0i}, f_1]\,, \qquad e_j{}' = e_j\,, \qquad f_j{}' = f_j \qquad (6.1)$$

for $i = 3, \ldots, r+1$ for $j = 2, 3, \ldots, r+1$. This already gives an embedding of $\widetilde{S}(\mathfrak{g})$ in $\widetilde{S}(\mathfrak{g}^+)$. In order to extend it $\widetilde{S}(\mathfrak{g})$ to $\widetilde{W}(\mathfrak{g})$, we have to find elements $f_{00}{}'$ and $h_0{}'$ in $\widetilde{S}(\mathfrak{g}^+)$. They will have the form $f_{00}{}' = f_{0\alpha}$ and $h_0{}' = h_\mu$ for some $\alpha, \mu \in \mathfrak{h}^*$, where μ must satisfy

$$(\mu, \alpha_0 + \alpha_1) = 0\,, \qquad (\mu, \alpha_2) = -1\,, \qquad (\mu, \alpha_3) = \cdots = (\mu, \alpha_{r+1}) = 0\,. \qquad (6.2)$$

From the relation $[e_0{}', f_{00}{}'] = h_0{}'$ we then get

$$\alpha - (\alpha, \alpha_1)\alpha_1 = \mu\,. \qquad (6.3)$$

If we now set (recall that we assume $(\lambda, \lambda) \neq 1$)

$$\mu = \frac{\lambda + (\lambda, \lambda)\alpha_1}{(\lambda, \lambda) - 1}\,, \qquad\qquad \alpha = \frac{\lambda}{(\lambda, \lambda) - 1}\,, \qquad (6.4)$$

then it is easy to show that these element satisfy the conditions (6.2) and (6.3), and then the defining relations for $\widetilde{W}(\mathfrak{g})$ follow. Thus $\widetilde{W}(\mathfrak{g}) \subset \widetilde{S}(\mathfrak{g}^+)$. Since clearly also $\widetilde{S}(\mathfrak{g}) \subset \widetilde{W}(\mathfrak{g})$ and $\widetilde{S}(\mathfrak{g}^+) \subset \widetilde{W}(\mathfrak{g}^+)$ we have a chain of embeddings

$$\widetilde{W}(\mathfrak{g}^+) \supset \widetilde{S}(\mathfrak{g}^+) \supset \widetilde{W}(\mathfrak{g}) \supset \widetilde{S}(\mathfrak{g}) \qquad (6.5)$$

that can be continued to lower rank at least as long as the grey node is connected to only one white node, so that chain of embedding corresponds to a chain of white nodes, but presumably our definition of the tensor hierarchy algebras can be generalised in order to allow for more than one "node 1" so that the chain could be continued in general (and of course also to higher rank with the definitions that we have already). The procedure is

similar to the one giving rise to a chain of embeddings for the corresponding Borcherds superalgebras, described in ref. [19].

To what extent do the embeddings (6.5) hold if we "remove the tildes", i.e., if we factor out the maximal ideal intersecting the subalgebra at $p = 0$ trivially? Ideals at negative levels will not affect the subalgebra embeddings, since level 1 is identical in S and W. We need to investigate what happens when there is ideal $K \subset \widetilde{S}$ (see section 5.4) at positive levels which is not an ideal in $\widetilde{W}$. Then there is not a subalgebra embedding $S(\mathfrak{g}^+) \subset W(\mathfrak{g}^+)$. If the ideal K is non-trivial, one instead has

$$S(\mathfrak{g}^+) \ltimes K \subset W(\mathfrak{g}^+). \tag{6.6}$$

We already know that $W_+(\mathfrak{g}) \simeq \mathscr{B}_+(\mathfrak{g})$ (see section 5.4). The only ideal factored out at positive levels to arrive at the simple superalgebra $\mathscr{B}(\mathfrak{g})$ is the one generated by $[e_0, e_0]$ [17]. This implies that $W_+(\mathfrak{g}) = \widetilde{W}_+(\mathfrak{g})$, so the ideal K in $\widetilde{S}(\mathfrak{g}^+)$ intersects $\widetilde{W}(\mathfrak{g})$ trivially. We thus have a subalgebra embedding

$$W(\mathfrak{g}) \subset S(\mathfrak{g}^+) \tag{6.7}$$

This can be observed in the examples of section 5.4. In both examples, the singlet generating the ideal appears at $q = 2$, and the ideal does not intersect $q = 0$ (the locus of the $W(\mathfrak{g})$ subalgebra).

7 Examples

In this section, we give a number of examples of tensor hierarchy algebras. Focus is put on the identity (5.22), which is the crucial test for the existence of the algebras. Even if it follows from the construction that the Jacobi identities are satisfied, the proof is quite implicit. Therefore, we want to verify it explicitly in some concrete cases. We give them by increasing value of (λ, θ) (and subsequently, increasing degree of complication), from 1 to 3.

7.1 $(\lambda, \theta) = 1$

Consider the situation when $(\lambda, \theta) = 1$, i.e., when $\widetilde{R}_1 = \{0\}$. Then, $\ell_{\alpha M}{}^{\beta N} = 0$ and $\varphi = Q$. The invariant tensor φ will have vanishing projections on $R(\lambda + \theta)$ and $R(\lambda)$. We calculate the eigenvalues on these modules using eq. (5.30), and get

$$\varphi|_{R(\lambda+\theta)} = g^\vee - 1 - (\varrho, \theta) = 0,$$
$$\varphi|_{R(\lambda)} = g^\vee + 1 - \frac{2C_2(\lambda)}{(\lambda, \lambda)}. \tag{7.1}$$

The vanishing of the latter expression can be shown as follows. The condition $(\lambda, \theta) = 1$ means that λ must be a fundamental weight Λ_i corresponding to a simple root α_i (and furthermore that the associated Coxeter label is 1). Let $\mathfrak{g}^-$ be the simple subalgebra of $\mathfrak{g}$ with Dynkin diagram obtained by removing node i from the diagram of $\mathfrak{g}$. The grading of $\mathfrak{g}$ with respect to λ is a 3-grading:

$$\mathfrak{g} = \mathfrak{g}_{(-1)} \oplus \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)} = \mathfrak{g}_{(-1)} \oplus (\mathfrak{g}^- \oplus \mathbb{R}) \oplus \mathfrak{g}_{(1)}, \tag{7.2}$$

where $\mathfrak{g}_{(1)}$ is a module for a $\mathfrak{g}^-$ representation $R(-\nu) = R(\overline{\nu})$ with e_i as a lowest weight state and e_θ as a highest weight state. Thus $\theta - \alpha_i = \overline{\nu} - (-\nu)$. However, θ and α are roots of $\mathfrak{g}$, whereas $\overline{\nu}$ and ν are weights of $\mathfrak{g}^-$ and thus linear combinations of only the simple roots α_j such that $j \neq i$. For ν we can determine this linear combination by the conditions $(\nu, \lambda) = 0$, which means that ν has zero coefficient for α_i in the basis of simple roots, and $(\nu, \alpha_j) = -(\alpha_i, \alpha_j)$ if $i \neq j$. We then get

$$\nu = \frac{\lambda}{(\lambda, \lambda)} - \alpha_i \tag{7.3}$$

which gives

$$(\varrho, \nu) = \frac{(\varrho, \lambda)}{(\lambda, \lambda)} - 1 . \tag{7.4}$$

Now $\overline{\nu}$ is the image of ν under an isometry of the weight lattice that permutes the simple roots of $\mathfrak{g}^-$ (which is just the identity map, $\overline{\nu} = \nu$, unless the symmetry group of the Dynkin diagram of $\mathfrak{g}^-$ is $\mathbb{Z}_2$), and since the Weyl vector ϱ of $\mathfrak{g}$ has the property $(\varrho, \alpha_j) = 1$ for all simple roots α_j of $\mathfrak{g}$ (in particular those of $\mathfrak{g}^-$), we have $(\varrho, \overline{\nu}) = (\varrho, \nu)$. Taking the inner product of ϱ with $\theta = \alpha_i + \nu + \overline{\nu}$ we then get

$$(\varrho, \theta) = (\varrho, \alpha_i) + 2(\varrho, \nu) = 1 + 2\frac{(\varrho, \lambda)}{(\lambda, \lambda)} - 2(\varrho, \alpha_i) = 2\frac{(\varrho, \lambda)}{(\lambda, \lambda)} - 1 . \tag{7.5}$$

Using the expression for the second Casimir of a representation $R(\Lambda)$ with highest weight Λ, $C_2(\Lambda) = \frac{1}{2}(\Lambda, \Lambda + 2\varrho)$, this relation may be expressed as

$$\frac{2C_2(\lambda)}{(\lambda, \lambda)} = g^\vee + 1 , \tag{7.6}$$

or equivalently, using the Freudenthal-de Vries "strange formula",

$$\frac{6(\lambda, \lambda)(\varrho, \varrho)}{(\lambda, \varrho)} = \dim \mathfrak{g} . \tag{7.7}$$

7.2 $(\lambda, \theta) = 2$: the THA over an affine algebra

We consider the case when $R(\lambda)$ is the adjoint of $\mathfrak{g}$ so that $\mathfrak{g}^+$ is the affine extension of $\mathfrak{g}$ (for example $\mathfrak{g} = E_8$ and $\mathfrak{g}^+ = E_9$). We thus take $\lambda = \theta$, i.e., $(\lambda, \theta) = 2$.

The representations in **adj** $\otimes$ **adj** which are not in Φ are $R(2\theta)$, $\bigoplus_i R(2\theta - \alpha_i)$ and $R(\theta)$, at levels 2, 1 and 0 respectively, where $\{\alpha_i\}$ is the set of simple roots with $(\theta, \alpha_i) = 1$. The eigenvalues of Q on these representations are -1, 1 and 1, respectively.

The modules in Φ are a number of $R(\theta + \gamma_0)$, where γ_0 are the highest roots at level 0, and $R(0) = \mathbf{1}$. Each γ_0 defines a subalgebra $\mathfrak{g}_{\gamma_0}$, the Dynkin diagram of which is a component of the Dynkin diagram of $\mathfrak{g}$ with the node(s) corresponding to θ removed, and $\lambda + \gamma_0$ is the highest root of $\mathfrak{g}_{\gamma_0}$.

For the example $\mathfrak{g} = E_8$, $\mathfrak{g}^+ = E_9$, we have **adj** $= \mathbf{248}$. There is a single root γ_0, and $\mathbf{248} \otimes \mathbf{248} = \mathbf{27000} \oplus \mathbf{3875} \oplus \mathbf{1} \oplus \mathbf{30380} \oplus \mathbf{248}$. Of these, $\mathbf{3875} \oplus \mathbf{1}$ are contained in Φ.

	$p = -1$	$p = 0$	$p = 1$
$q = 2$			$L^\sharp$
$q = 1$	$\varphi^\sharp \qquad \Phi^\sharp_{\alpha\beta} \qquad G^\sharp_\alpha$	$f_{00} \qquad T^\sharp_\alpha$	$E^\sharp_\alpha \qquad L$
$q = 0$	$F_\alpha \quad \varphi \quad \Phi_{\alpha\beta} \quad G_\alpha$	$k \quad \tilde{k} \quad T_\alpha$	E_α
$q = -1$	$F^\flat_\alpha$	e_0	

Table 5. Basis elements of W when $\mathfrak{g}^+$ is the affine extension of $\mathfrak{g}$.

	$p = -1$	$p = 0$	$p = 1$
$q = 2$			$L^\sharp$
$q = 1$	$\varphi^\sharp \qquad \Phi^\sharp_{\alpha\beta}$	$T^\sharp_\alpha$	$E^\sharp_\alpha \qquad L$
$q = 0$	$H_\alpha \quad \varphi \quad \Phi_{\alpha\beta}$	$\tilde{k} \quad T_\alpha$	E_α
$q = -1$	$H^\flat_\alpha$	e_0	

Table 6. Basis elements of S when $\mathfrak{g}^+$ is the affine extension of $\mathfrak{g}$.

Tables 5 and 6 show the local ($p = -1, 0, 1$) parts of $W(\mathfrak{g}^+)$ and $S(\mathfrak{g}^+)$. Tables 7 and 8 give the corresponding decompositions of $W(E_9)$ and $S(E_9)$ into E_8 modules.

The eigenvalue of Q on $R(\theta + \gamma_0)$ is $g^\vee - g^\vee_{\gamma_0} + 1$. The eigenvalue on $\mathbf{1}$ is $2g^\vee + 1$. The projector on $\mathbf{1}$ in the ℓ picture is $\frac{1}{\dim \mathfrak{g}} \eta_{\alpha M} \eta^{\beta N}$, and its eigenvalues on the modules in the φ picture are $\pm \frac{1}{\dim \mathfrak{g}}$, depending on whether it is in the symmetric or antisymmetric part of the tensor product. We saw that Q has eigenvalue -1 on the symmetric module not in Φ, and 1 on the antisymmetric ones.

Equation (5.23) is solved with

$$\varphi = \sum_{\gamma_0 \in H_0} (g^\vee - g^\vee_{\gamma_0} + 2) P_{R(\theta + \gamma_0)} + 2(g^\vee + 1) P_{\mathbf{1}} \,,$$
$$\ell = \dim \mathfrak{g} \, P_{\mathbf{1}} \,, \tag{7.8}$$

where the projectors in φ and ℓ are expressed in their respective bases. In the example with $S(E_9)$, $g^\vee = 30$ and $\mathfrak{g}_{\gamma_0} = E_7$ with $g^\vee_{\gamma_0} = 18$, and we get $\varphi = 14 P_{\mathbf{3875}} + 62 P_{\mathbf{1}}$, i.e. (see eq. (7.10)),

$$\varphi_{\alpha\beta}{}^{\gamma\delta} = 2 \delta^\gamma_{(\alpha} \delta^\delta_{\beta)} - f_{(\alpha}{}^{\gamma\epsilon} f_{\beta)}{}^\delta{}_\epsilon \,. \tag{7.9}$$

– 27 –

	$p=-1$	$p=0$	$p=1$	$p=2$
$q=2$			**1**	**248**
$q=1$	**1** $\oplus$ **3875** $\oplus$ **248**	**1** $\oplus$ **248**	**248** $\oplus$ **1**	**1** $\oplus$ **3875** $\oplus$ **248**
$q=0$	**248** $\oplus$ **1** $\oplus$ **3875** $\oplus$ **248**	**1** $\oplus$ **248** $\oplus$ **1**	**248**	**1** $\oplus$ **3875**
$q=-1$	**248**	**1**		

Table 7. Basis elements of $W(E_9)$. The modules not present in $\mathscr{B}(E_9)$ are marked blue. Note the presence of $\widetilde{R}_1 = \mathbf{1}$.

	$p=-1$	$p=0$	$p=1$	$p=2$
$q=2$			**1**	**248**
$q=1$	**1** $\oplus$ **3875**	**248**	**248** $\oplus$ **1**	**1** $\oplus$ **3875** $\oplus$ **248**
$q=0$	**248** $\oplus$ **1** $\oplus$ **3875**	**1** $\oplus$ **248**	**248**	**1** $\oplus$ **3875**
$q=-1$	**248**	**1**		

Table 8. Basis elements of $S(E_9)$. Note the symmetry under $(p,q) \leftrightarrow (1-p, 1-q)$ associated with existence of a bilinear form.

This latter expression is generic in the present class of examples. This can be shown by inserting this expression for φ, together with $\ell_{\alpha\beta}{}^{\gamma\delta} = \eta_{\alpha\beta}\eta^{\gamma\delta}$, into eq. (5.22) with $(t_\alpha)_\beta{}^\gamma = -f_{\alpha\beta}{}^\gamma$ and using the Jacobi identity.

In this series of examples, $\mathfrak{g}^+$ is the (untwisted) affine algebra over $\mathfrak{g}$. At level 1, there is an anti-fundamental module, whose lowest weight state is e_0. At level 0, there is, in addition to the adjoint, a single generator L, which can be identified with the Virasoro generator L_1. At level -1, we find a shifted fundamental module, with highest weight state $L^\sharp$.

As can be seen in table 7, there is a symmetry in the representation content of $S(E_9)$ under $(p,q) \leftrightarrow (1-p, 1-q)$, associated with the existence of an invariant non-degenerate bilinear form [4, 6]. This symmetry occurs for $S(\mathfrak{g}^+)$ whenever $\mathfrak{g}$ is an affine algebra. In general, if there is an affine Kac-Moody algebra $\mathfrak{g}^{(k)}$ obtained by adding a chain of k white nodes to the Dynkin diagram of $\mathfrak{g}$ (for example if $\mathfrak{g} = E_{9-k}$), then there is such a symmetry under $(p,q) \leftrightarrow (k-p, 1-q)$ in $S(\mathfrak{g})$ [4], and this seems to hold even for negative k (if "adding a chain of k white nodes" is interpreted as "removing a chain of $-k$ white nodes") [6].

7.3 $(\lambda, \theta) = 2$: another series

Another series of examples, also with $(\lambda, \theta) = 2$, is D_r with $R(\lambda)$ a 3-form, $(0010\ldots0)$. Then, $\lambda + \theta = (0110\ldots0)$, $\lambda + \gamma_1 = (10010\ldots0)$ and $\lambda - \beta_2 = (10\ldots0)$. The modules that are not part of Φ are $R(\lambda + \theta)$, $R(\lambda + \gamma_1)$ and $R(\lambda)$. The eigenvalues of Q on them are -1, 2 and 2, respectively. The projector on $R(\lambda - \beta_2)$ is proportional to $\delta_{mn,[pq} \delta^{st,[uv} \delta^{w]}_{r]}$. Letting this tensor act on states $\Psi_{st}{}^{pqr}$ in the three modules that do not appear in Φ, one finds the eigenvalues $\frac{1}{3}$, $-\frac{2}{3}$ and $-\frac{2}{3}$, which with a factor 3 cancels the contribution from Q, and eq. (5.23) holds. The extended algebra $\mathfrak{g}^+$ is hyperbolic for $r \leqslant 9$.

7.4 An example with $(\lambda, \theta) = 3$

Finally, we would like to give an example where $(\lambda, \theta) = 3$. With $\mathfrak{g} = E_8$ and $\mathbf{adj} = \mathbf{248} = \binom{0}{1000000}$, we take $\lambda = \binom{0}{0100000}$, $R(\lambda) = \mathbf{30\,380}$.

The construction makes use of the projections on the irreducible representations in $\mathbf{adj} \otimes \mathbf{adj}$, which are

$$P^{\mathbf{27\,000}}{}_{\alpha\beta}{}^{\gamma\delta} = \frac{6}{7}\delta^\gamma_{(\alpha}\delta^\delta_{\beta)} + \frac{1}{14}f_{(\alpha}{}^{\gamma\epsilon}f_{\beta)}{}^\delta{}_\epsilon + \frac{3}{217}\eta_{\alpha\beta}\eta^{\gamma\delta},$$

$$P^{\mathbf{30\,380}}{}_{\alpha\beta}{}^{\gamma\delta} = \delta^{\gamma\delta}_{\alpha\beta} + \frac{1}{60}f_{\alpha\beta}{}^\epsilon f^{\gamma\delta}{}_\epsilon,$$

$$P^{\mathbf{3\,875}}{}_{\alpha\beta}{}^{\gamma\delta} = \frac{1}{7}\delta^\gamma_{(\alpha}\delta^\delta_{\beta)} - \frac{1}{14}f_{(\alpha}{}^{\gamma\epsilon}f_{\beta)}{}^\delta{}_\epsilon - \frac{1}{56}\eta_{\alpha\beta}\eta^{\gamma\delta},$$

$$P^{\mathbf{248}}{}_{\alpha\beta}{}^{\gamma\delta} = -\frac{1}{60}f_{\alpha\beta}{}^\epsilon f^{\gamma\delta}{}_\epsilon,$$

$$P^{\mathbf{1}}{}_{\alpha\beta}{}^{\gamma\delta} = \frac{1}{248}\eta_{\alpha\beta}\eta^{\gamma\delta}. \tag{7.10}$$

The only identity, not following from the Jacobi identities, that is needed for verification of the projector properties is

$$q_{\alpha\beta}{}^{\kappa\lambda}q_{\kappa\lambda}{}^{\gamma\delta} = 24\delta^\gamma_{(\alpha}\delta^\delta_{\beta)} - 10q_{\alpha\beta}{}^{\gamma\delta} + 12\eta_{\alpha\beta}\eta^{\gamma\delta}, \tag{7.11}$$

where $q_{\alpha\beta}{}^{\gamma\delta} = f_{(\alpha}{}^{\gamma\epsilon}f_{\beta)}{}^\delta{}_\epsilon$. Define $\star P^R{}_{\alpha\beta}{}^{\gamma\delta} = \eta_{\alpha\epsilon}\eta^{\gamma\varphi}P^R{}_{\varphi\beta}{}^{\epsilon\delta}$. Then, $\star P^{R_i} = \sum_{j=1}^5 M_{ij}P^{R_j}$, where $\{R_i, i = 1, \ldots, 5\}$ are the representations in the order listed above, and

$$M = \begin{pmatrix} \frac{23}{62} & \frac{90}{217} & \frac{27}{31} & \frac{225}{62} & \frac{3375}{31} \\ \frac{7}{15} & \frac{1}{2} & \frac{7}{10} & 0 & -\frac{245}{2} \\ \frac{1}{8} & \frac{5}{56} & -\frac{3}{8} & -\frac{25}{8} & \frac{125}{8} \\ \frac{1}{30} & 0 & -\frac{1}{5} & \frac{1}{2} & -1 \\ \frac{1}{248} & -\frac{1}{248} & \frac{1}{248} & -\frac{1}{248} & \frac{1}{248} \end{pmatrix} \tag{7.12}$$

This translation matrix is used in some of the following calculations.

The representations in $\mathbf{adj} \otimes R(\lambda)$ obtained from the roots in eq. (4.7) are:

$$
\begin{aligned}
\theta = \gamma_3 &= \left[{}_{2345642}^{\quad 3}\right] & \lambda + \gamma_3 &= \left({}_{1100000}^{\quad 0}\right) & R(\lambda + \gamma_3) &= \mathbf{4\,096\,000} \\
\gamma_2 &= \left[{}_{1245642}^{\quad 3}\right] & \lambda + \gamma_2 &= \left({}_{0010000}^{\quad 0}\right) & R(\lambda + \gamma_2) &= \mathbf{2\,450\,240} \\
\gamma_1 &= \left[{}_{1123432}^{\quad 2}\right] & \lambda + \gamma_2 &= \left({}_{1000001}^{\quad 0}\right) & R(\lambda + \gamma_1) &= \mathbf{779\,247} \\
\gamma_0 &= \left[{}_{0012321}^{\quad 2}\right] & \lambda + \gamma_0 &= \left({}_{0000000}^{\quad 1}\right) & R(\lambda + \gamma_0) &= \mathbf{147\,250} \\
\gamma_0' &= \left[{}_{1000000}^{\quad 0}\right] & \lambda + \gamma_0' &= \left({}_{2000000}^{\quad 0}\right) & R(\lambda + \gamma_0') &= \mathbf{27\,000} \\
& & \lambda &= \left({}_{0100000}^{\quad 0}\right) & R(\lambda) &= \mathbf{30\,380} \\
\beta_2 &= \left[{}_{1222210}^{\quad 1}\right] & \lambda - \beta_2 &= \left({}_{0000001}^{\quad 0}\right) & R(\lambda - \beta_2) &= \mathbf{3\,875} \\
\beta_3 &= \left[{}_{1345642}^{\quad 3}\right] & \lambda - \beta_3 &= \left({}_{1000000}^{\quad 0}\right) & R(\lambda - \beta_3) &= \mathbf{248}
\end{aligned}
$$

(we use the notation $\left[{}_{i_1 i_2 i_3 i_4 i_5 i_6 i_7}^{\qquad i_8}\right]$ for coefficients in root basis and $\left({}_{j_1 j_2 j_3 j_4 j_5 j_6 j_7}^{\qquad j_8}\right)$ in weight basis).

To distinguish the projectors from the ones for $\mathbf{adj} \otimes \mathbf{adj}$, we denote them $\mathbb{P}^R{}_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi}$. They satisfy $P^{\mathbf{248}}{}_{\beta\gamma}{}^{\kappa\lambda}\mathbb{P}^R{}_{\alpha,\kappa\lambda}{}^{\delta,\epsilon\varphi} = 0$, which can be implemented by letting

$$
\mathbb{P}^R{}_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi} = P^{\mathbf{30\,380}}{}_{\beta\gamma}{}^{\kappa\lambda}\Pi^R{}_{\alpha,\kappa\lambda}{}^{\delta,\rho\sigma} P^{\mathbf{30\,380}}{}_{\rho\sigma}{}^{\epsilon\varphi}. \tag{7.13}
$$

The Π's are equivalent modulo combinations of an antisymmetric pair into $\mathbf{248}$, which we will treat as equality. The relevant product and trace on the Π's are

$$
\begin{aligned}
(\Pi \circ \Pi')_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi} &= \Pi_{\alpha,\beta\gamma}{}^{\kappa,\lambda\mu} P^{\mathbf{30\,380}}{}_{\lambda\mu}{}^{\rho\sigma}\Pi'_{\kappa,\rho\sigma}{}^{\delta,\epsilon\varphi}, \\
\operatorname{tr}\Pi &= \Pi_{\alpha,\beta\gamma}{}^{\alpha,\epsilon\varphi} P^{\mathbf{30\,380}}{}_{\epsilon\varphi}{}^{\beta\gamma}.
\end{aligned} \tag{7.14}
$$

The explicit forms of the Π^R's are

$$
\begin{aligned}
\Pi^{\mathbf{4\,096\,000}}{}_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi} &= \frac{4}{3}\delta^\delta_{(\alpha}\delta^{\epsilon\varphi}_{\beta)\gamma} - \Big(\Pi^{\mathbf{779\,247}} + \Pi^{\mathbf{147\,250}} + \frac{14}{45}\Pi^{\mathbf{27\,000}} \\
&\qquad\qquad + \frac{1}{3}\Pi^{\mathbf{30\,380}} + \frac{7}{15}\Pi^{\mathbf{3\,875}} + \Pi^{\mathbf{248}} \Big)_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi}, \\
\Pi^{\mathbf{2\,045\,240}}{}_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi} &= \delta^{\delta\epsilon\varphi}_{\alpha\beta\gamma} - \Big(\frac{31}{45}\Pi^{\mathbf{27\,000}} + \frac{2}{3}\Pi^{\mathbf{30\,380}} + \frac{8}{15}\Pi^{\mathbf{3\,875}} \Big)_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi}, \\
\Pi^{\mathbf{779\,247}}{}_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi} &= \frac{49}{26}(U + 2V)_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi}, \\
\Pi^{\mathbf{147\,250}}{}_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi} &= \frac{1}{6}(U - 14V)_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi}, \\
\Pi^{\mathbf{27\,000}}{}_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi} &= -\frac{15}{434}f_{\alpha\beta}{}^\rho f^{\delta\epsilon}{}_\sigma P^{\mathbf{27\,000}}{}_{\gamma\rho}{}^{\varphi\sigma}, \\
\Pi^{\mathbf{30\,380}}{}_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi} &= -\frac{1}{30}f_{\alpha\beta}{}^\rho f^{\delta\epsilon}{}_\sigma \delta^{\varphi\sigma}_{\gamma\rho}, \\
\Pi^{\mathbf{3\,875}}{}_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi} &= -\frac{5}{168}f_{\alpha\beta}{}^\rho f^{\delta\epsilon}{}_\sigma P^{\mathbf{3\,875}}{}_{\gamma\rho}{}^{\varphi\sigma}, \\
\Pi^{\mathbf{248}}{}_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi} &= \frac{2}{245}\eta_{\alpha\beta}\eta^{\delta\epsilon}\delta^\varphi_\gamma,
\end{aligned} \tag{7.15}
$$

where

$$U_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi} = P^{\mathbf{3\,875}}{}_{\alpha\beta}{}^{\delta\epsilon}\delta^{\varphi}_{\gamma} - \left(\frac{1}{28}\Pi^{\mathbf{30\,380}} + \frac{7}{20}\Pi^{\mathbf{3\,875}} + \frac{1}{16}\Pi^{\mathbf{248}}\right)_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi},$$

$$\begin{aligned}V_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi} &= P^{\mathbf{3\,875}}{}_{\alpha\beta}{}^{\varphi\rho}P^{\mathbf{3\,875}}{}_{\gamma\rho}{}^{\delta\epsilon} \\ &\quad - \left(\frac{11}{392}\Pi^{\mathbf{30\,380}} - \frac{7}{40}\Pi^{\mathbf{3\,875}} + \frac{3}{128}\Pi^{\mathbf{248}}\right)_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi}.\end{aligned} \tag{7.16}$$

It is relatively straightforward to show that $U \circ U = \frac{1}{2}(U - V)$. The remaining identity needed is $U \circ V = -\frac{1}{56}(U - 40V)$, from which it then follows that $V \circ V = \frac{1}{392}(10U - 127V)$. We have not checked it explicitly, but it is needed for the projection operators to work and to give the correct dimensions of the representations.

We now want to translate between the two "pictures", i.e., express $\star\mathbb{P}^{R}{}_{\alpha,\beta\gamma}{}^{\delta,\epsilon\varphi} \equiv \eta_{\alpha\rho}\eta^{\delta\sigma}\mathbb{P}^{R}{}_{\sigma,\beta\gamma}{}^{\rho,\epsilon\varphi}$ in terms of the $\mathbb{P}^{R}$'s. This needs to be done for $R(\lambda - \beta_3) = \mathbf{248}$ and $R(\lambda - \beta_2) = \mathbf{3\,875}$. A lengthy calculation yields

$$\begin{aligned}\star\mathbb{P}^{\mathbf{248}} &= \frac{1}{245}\bigg(\mathbb{P}^{\mathbf{4\,096\,000}} - 2\mathbb{P}^{\mathbf{2\,450\,240}} + \mathbb{P}^{\mathbf{779\,247}} + \mathbb{P}^{\mathbf{147\,250}} \\ &\qquad\qquad - \frac{16}{15}\mathbb{P}^{\mathbf{27\,000}} - \mathbb{P}^{\mathbf{30\,380}} - \frac{3}{5}\mathbb{P}^{\mathbf{3\,875}} + \mathbb{P}^{\mathbf{248}}\bigg), \\ \star\mathbb{P}^{\mathbf{3\,875}} &= \frac{5}{784}\bigg(\mathbb{P}^{\mathbf{4\,096\,000}} + 0\mathbb{P}^{\mathbf{2\,450\,240}} - \frac{23}{3}\mathbb{P}^{\mathbf{779\,247}} + 15\mathbb{P}^{\mathbf{147\,250}} \\ &\qquad\qquad + \frac{98}{5}\mathbb{P}^{\mathbf{27\,000}} - 7\mathbb{P}^{\mathbf{30\,380}} - \frac{49}{5}\mathbb{P}^{\mathbf{3\,875}} - 6\mathbb{P}^{\mathbf{248}}\bigg).\end{aligned} \tag{7.17}$$

A good check on the result is that the dimensions add up correctly.

In order for eq. (5.22) to have a solution, i.e., for the tensor hierarchy algebra to exist, it must be possible to cancel the contribution from Q to the representations $\mathbf{4\,096\,000}$, $\mathbf{2\,045\,240}$, $\mathbf{779\,247}$ and $\mathbf{30\,380}$ by a linear combination of the right hand sides of eq. (7.17). The decomposition of Q is given by eq. (5.30), and we have

$$\begin{aligned}Q &= -2\mathbb{P}^{\mathbf{4\,096\,000}} + \mathbb{P}^{\mathbf{2\,045\,240}} + 11\mathbb{P}^{\mathbf{779\,247}} + 19\mathbb{P}^{\mathbf{147\,250}} \\ &\quad + 29\mathbb{P}^{\mathbf{27\,000}} + 11\mathbb{P}^{\mathbf{30\,380}} + 43\mathbb{P}^{\mathbf{3\,875}} + 61\mathbb{P}^{\mathbf{248}}.\end{aligned} \tag{7.18}$$

The coefficients of the projectors on the representation not present in Φ cancel by adding

$$\ell = \frac{1176}{5}\star\mathbb{P}^{\mathbf{3\,875}} + \frac{245}{2}\star\mathbb{P}^{\mathbf{248}} \tag{7.19}$$

as $(-2, 1, 11, 11) + (\frac{3}{2}, 0, -\frac{23}{2}, -\frac{21}{2}) + (\frac{1}{2}, -1, \frac{1}{2}, -\frac{1}{2}) = 0$. The remainder is

$$\varphi = Q + \ell = 42\mathbb{P}^{\mathbf{147\,250}} + \frac{868}{15}\mathbb{P}^{\mathbf{27\,000}} + 28\mathbb{P}^{\mathbf{3\,875}} + \frac{105}{2}\mathbb{P}^{\mathbf{248}}. \tag{7.20}$$

The extended algebra $\mathfrak{g}^+$ in this example is the hyperbolic Lie algebra D_7^{++}. In the tensor hierarchy algebra S, level 1 contains $R(-\Lambda)$, where $\Lambda = \left(\begin{smallmatrix} 0 & 0 \\ 1000000 \end{smallmatrix}\right)$. At level 0, there is of course the adjoint, but also (at least) two lowest weight representations $R(-\mu)$, $R(-\nu)$, with $\mu = \left(\begin{smallmatrix} 1 & 0 \\ 0000000 \end{smallmatrix}\right)$, $\nu = \left(\begin{smallmatrix} 0 & 0 \\ 0000001 \end{smallmatrix}\right)$, whose lowest representations in a grading with respect to the extending node are the $\mathbf{248}$ and $\mathbf{3\,875}$ in L.

8 Conclusions

We have given definitions of the tensor hierarchy algebras $W(\mathfrak{g}^+)$ and $S(\mathfrak{g}^+)$ in terms of generators and relations, when $\mathfrak{g}^+$ is a Lie algebra obtained by extending the finite-dimensional (simply laced) Lie algebra $\mathfrak{g}$ by a single node. A number of examples are given, of which some are relevant to physical applications.

One main difficulty with deriving the content of the superalgebras is associated with the appearance of "mixed" elements; the root space contains roots where the coefficients for the simple roots are not all positive or all negative. This phenomenon is also associated with the appearance of "extra" elements together with $\mathfrak{g}^+ \oplus \mathbb{R}$ at level $n = 0$ (beginning with the generators $L_{\alpha M}$). This is seen e.g. in tables 3 and 4. Such elements are significant in the application to extended geometry, as explained in ref. [1].

The definition should be good also for infinite-dimensional $\mathfrak{g}$. The derivations in the present paper will then not be valid. For example, there will typically also appear some elements in $\widetilde{R}_0$, i.e., a pair of isomorphic modules at $(p,q) = (0,1)$ and $(0,2)$ in the double grading. We have verified this for affine $\mathfrak{g}$, where $\widetilde{R}_0$ is a singlet, and $\widetilde{R}_p = \{0\}$, $p < 0$. For "more infinite-dimensional" algebras, e.g. hyperbolic $\mathfrak{g}$, also $\widetilde{R}_{-1}$ etc. can appear. Even if the definitions remain formally identical, the implications seem to differ drastically, also in the local subalgebra. It would be desirable to design a method that determines the "extra" elements in a more direct way. For infinite-dimensional $\mathfrak{g}$, there may also be "extra" elements at positive levels n, so that it would no longer be true that $W_+ \simeq \mathscr{B}_+$, as stated for finite-dimensional $\mathfrak{g}$ in section 5.4.

A topic we have not touched is representation theory for THA's. In particular, the construction of non-trivial representations would be a more efficient and general method to prove that the tensor hierarchy algebra is non-trivial. A denominator formula for positive levels for $W(\mathfrak{g}^+)$ coincides with the one for Borcherds superalgebras [17]; we do not yet have such a formula for $S(\mathfrak{g}^+)$ in situations where the ideal K is non-trivial. Neither do we have a denominator formula for negative levels. In situations described in the end of section 7.2, where an invariant bilinear form exists, the negative level generators can be deduced from the positive ones. This invariant bilinear form is interesting for many other reasons too, and needs to be better understood.

Acknowledgments

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