

RESEARCH ARTICLE

Computation of Robust Control Invariant Sets with Predefined Complexity for Uncertain Systems[†]

Ankit Gupta*¹ | Hakan Koroğlu² | Paolo Falcone¹

¹Department of Electrical Engineering,
Chalmers University of Technology,
Gothenburg, Sweden

²Faculty of Engineering Technology,
Department of Mechanics of Solids,
Surfaces, and Systems, University of
Twente, Enschede, Netherlands

Correspondence

*Ankit Gupta, Email:
ankit.gupta@chalmers.se

Summary

This paper presents an algorithm that computes polytopic robust control-invariant (RCI) sets for rationally parameter-dependent systems with additive disturbances. By means of novel LMI feasibility conditions for invariance along with a newly developed method for volume maximization, an iterative algorithm is proposed for the computation of RCI sets with maximized volumes. The obtained RCI sets are symmetric around the origin by construction and have a user-defined level of complexity. Unlike many similar approaches, fixed state feedback structure is not imposed. In fact, a specific control input is obtained from the LMI problem for each extreme point of the RCI set. The outcomes of the proposed algorithm can be used to construct a piecewise-affine controller based on offline computations.

KEYWORDS:

Linear matrix inequalities (LMI), Invariant set, Semi-definite program (SDP), Linear fractional transformation (LFT), Linear systems

1 | INTRODUCTION

RCI sets have been an important area of research within the controls community for the past six decades^{1,2}. For a controlled dynamical system, these sets are bounded regions of the state-space, wherein the state trajectories can always be restricted by the application of some admissible control inputs¹. Invariant sets can be also used for controller synthesis (e.g. model predictive control (MPC)³) and stability analysis of closed-loop systems¹. Consequently, a lot of research has been done specifically focusing on the computation of these sets.

The algorithms to compute an RCI set for a linear system with polytopic constraints are well established in the literature^{1,2,4,5,6}. A widely used algorithm to compute an RCI set for a linear system belongs to the class of the so-called geometric approach, in which a recursive method based on one-step backward reach operator is employed until some termination condition is matched. In each backward reach operation, numerical procedures like Minkowski sum, projection and finding minimal set representation are performed on polytopes, which can be computationally very demanding^{1,7}. Depending upon the choice of the initial set, the result of such a recursive method is the arbitrarily close outer/inner approximation of the maximal RCI set^{8,9}. Although effective, the geometric approach does not guarantee finite time termination of the procedure, and also the obtained set may have a very high representational complexity^{2,8,9,10,11}. For polytopic systems, the computational complexity grows exponentially with each additional vertex and system dimension¹.

Recently, many algorithms have been proposed for the efficient computation of RCI sets with restricted representational complexity. In these algorithms, the candidate RCI sets are assumed to be symmetric ellipsoids or polytopes, and the computation thereof generally involves solving an optimization problem to find a set with maximum or minimum volume. For example,

[†]This work is supported by the Vinnova FFI Complex Control Program, under the grant No. 2015-02309. The second author is supported by University of Twente.

algorithms are proposed by^{12,13} to compute ellipsoidal RCI sets of maximum volumes for uncertain systems. For linear systems with polytopic constraints, it is well known from the literature that the maximal RCI set is also polytopic¹⁴. Therefore, using a polytopic candidate RCI set would qualify to be an obvious choice. As a result,^{15,16,17,18,11} proposed different approaches for the computation of restricted complexity or low complexity polytopic RCI sets. Algorithms developed by^{15,17,16} are applicable to polytopic linear systems, whereas those in^{18,11} can be applied to affinely parameter-dependent systems. It is important to mention here that all these algorithms assume the invariance inducing controller to be based on linear state feedback, which is computed by the algorithm while optimizing the volume of the set. In¹⁹, an approach is proposed to compute RCI sets using a more general feedback structure, which is though only applicable to LTI systems. Imposing a predefined feedback structure on the controller can be a major source of conservatism in the above algorithms, as also emphasized by^{20,11,19}.

In this paper, we present a novel algorithm to compute an RCI set with a predefined complexity for an uncertain system with additive disturbances. Similar to^{20,10}, we consider rationally parameter-dependent systems, which are more general than other linear system descriptions discussed above in that they can be used to approximately describe many nonlinear engineering systems²¹. We start by proposing a new necessary and sufficient condition for invariance. By then using this condition along with the full block S-procedure²², a sufficient LMI feasibility condition for invariance is derived. Also, an approach to directly maximize the volume of the RCI sets is presented, which is less conservative than using an indirect approach used in^{10,11}. In contrast to^{15,17,18,20,10,11}, we do not impose any structure on the control inputs while computing the RCI sets, and thus potentially reduce conservatism via the proposed formulation. The algorithm generates invariant sets of monotonically increasing volume at each iteration until convergence. Hence, it can be terminated at any iteration, unlike the geometric approach which needs a termination condition to be satisfied to eventually obtain a maximal RCI set. Moreover, we have also shown that solutions obtained from the proposed algorithm can be aptly combined with the approach in^{23,24,25} to construct a piece-wise affine controller for the simple constrained control of the system. We demonstrate the applicability of the proposed algorithm by using numerical examples, where considerably larger RCI sets were obtained in comparison with other similar algorithms.

2 | PRELIMINARIES

For completeness, we recall some basic definitions and well-established results from^{1,6}, which will be useful in the later parts.

Definition 1. A set is a C-set if it is convex, compact and contains the origin in its interior.

Definition 2. A C-set \mathcal{S} is 0-symmetric if

$$x \in \mathcal{S} \Rightarrow -x \in \mathcal{S}. \quad (1)$$

Definition 3. The convex hull of a set $\mathcal{S} \subset \mathbb{R}^n$ is the smallest convex set containing \mathcal{S} . For any finite set $\mathcal{S} = \{s^1, s^2, \dots, s^r\}$, where $s^i \in \mathbb{R}^n, i = 1, 2, \dots, r$, the convex hull of \mathcal{S} is given by

$$\mathcal{S} = \text{conv}(\mathcal{S}) \triangleq \left\{ \sum_{i=1}^r \alpha_i s^i : \sum_{i=1}^r \alpha_i = 1, \alpha_i \in [0, 1] \right\}. \quad (2)$$

Definition 4. An n -simplex $\mathcal{S} \subset \mathbb{R}^n$ is an n -dimensional polytope, which is the convex hull of $n + 1$ vertices.

Consider a dynamical system

$$x(k+1) = f(x(k), u(k), w(k)), \quad (3)$$

where $x(k) \in \mathcal{X}, u(k) \in \mathcal{U}, w(k) \in \mathcal{W}$ are the state, input and disturbance (or uncertainty) vectors.

Definition 5. A set $\Omega \subseteq \mathcal{X}$ is an RCI set for the system (3) if

$$x(k) \in \Omega \Rightarrow \exists u(k) \in \mathcal{U} : x(k+1) \in \Omega, \forall w(k) \in \mathcal{W}. \quad (4)$$

Definition 6. A set $\Omega_\infty \subseteq \mathcal{X}$ is said to be the maximal RCI set for the system (3) if it is RCI and contains all other RCI sets.

Definition 7. Given $\lambda, 0 < \lambda \leq 1$, a C-set $\mathcal{S} \subseteq \mathcal{X}$ is a λ -contractive set for the system (3) if

$$x(k) \in \mathcal{S} \Rightarrow \exists u(k) \in \mathcal{U} : x(k+1) \in \lambda \mathcal{S}, \forall w(k) \in \mathcal{W} \quad (5)$$

For $\lambda = 1$, we say \mathcal{S} is an RCI set.

Lemma 1. A set \mathcal{S} is λ -contractive for the system (3) if and only if it is controlled-invariant for the modified system

$$x(k+1) = \frac{f(x(k), u(k), w(k))}{\lambda}. \quad (6)$$

3 | PROBLEM STATEMENT

We consider a discrete-time uncertain system described by

$$x^+ = \mathcal{A}(\Delta)x + \mathcal{B}(\Delta)u + \mathcal{E}(\Delta)w, \quad (7)$$

where $x \in \mathbb{R}^{n_x}$ is the current state vector, x^+ is the successor state vector, and $u \in \mathbb{R}^{n_u}$ and $w \in \mathbb{R}^{n_w}$ are the control input and the (additive) disturbance vectors respectively. Furthermore, $\mathcal{A}(\Delta)$, $\mathcal{B}(\Delta)$ and $\mathcal{E}(\Delta)$ are the rationally parameter dependent system matrices expressed as an LFT of the form

$$\left[\mathcal{A}(\Delta) \mid \mathcal{B}(\Delta) \mid \mathcal{E}(\Delta) \right] = \left[A \mid B \mid E \right] + B_p \Delta (I - D_p \Delta)^{-1} \left[A_d \mid B_d \mid E_d \right], \quad (8)$$

where A , B , E , A_d , B_d , E_d , B_p and D_p are known matrices of appropriate dimensions. Δ is an uncertain (and possibly) time-varying parameter matrix that satisfies

$$\Delta(k) \in \mathbf{\Delta}, \quad \forall k \geq 0, \quad (9)$$

where $\mathbf{\Delta}$ is a known compact polytopic set expressed as the convex-hull of finitely many given matrices $\Delta_v = \{\Delta^1, \dots, \Delta^n\}$:

$$\mathbf{\Delta} = \text{conv}(\Delta_v). \quad (10)$$

For now, we assume that the term $(I - D_p \Delta)^{-1}$ is invertible $\forall \Delta \in \mathbf{\Delta}$, which will be later guaranteed by **Lemma 4**.

The system is subject to the following polytopic state and input constraints, respectively:

$$\begin{aligned} \mathcal{X} &= \{x : Hx \leq \mathbf{1}\}, \\ \mathcal{U} &= \{u : Gu \leq \mathbf{1}\}. \end{aligned} \quad (11)$$

Here $H \in \mathbb{R}^{n_h \times n_x}$ and $G \in \mathbb{R}^{n_g \times n_u}$ are given matrices and $\mathbf{1}$ represents the vector of ones of compatible dimension. We assume $w \in \mathcal{W}$, with \mathcal{W} as a C-set and represented by the convex-hull of a set of finitely many known vertices $\mathcal{W}_v = \{w^1, w^2, \dots, w^r\}$:

$$\mathcal{W} = \text{conv}(\mathcal{W}_v). \quad (12)$$

We allow the sets \mathcal{X} , \mathcal{U} and \mathcal{W} to be non-symmetric, unlike^{17,18,10,11} where some or all the sets are assumed to be symmetric. The affine parameter dependence of the system, as considered in e.g.^{18,11}, is just a special case that corresponds to $D_p = 0$ in (8). Moreover, the algorithm proposed in this paper does not impose any structural restriction on the matrix Δ .

The goal in this paper is to compute a 0-symmetric RCI set with a predefined complexity n_p described as

$$\mathcal{C} = \{x \in \mathbb{R}^{n_x} : -\mathbf{1} \leq PW^{-1}x \leq \mathbf{1}\}, \quad (13)$$

where $P \in \mathbb{R}^{n_p \times n_x}$ and $W \in \mathbb{R}^{n_x \times n_x}$. We tacitly assume that W is invertible, which would be later guaranteed by the LMI conditions for invariance.

From **Definition 5**, a set \mathcal{C} is RCI if for each x

$$x \in \mathcal{C} \Rightarrow x \in \mathcal{X}, \quad (14)$$

$$x \in \mathcal{C} \Rightarrow \exists u \in \mathcal{U} : x^+ \in \mathcal{C}, \quad \forall (w, \Delta) \in (\mathcal{W}, \mathbf{\Delta}). \quad (15)$$

Notice that we do not impose any structure on the control input. The problem of finding an RCI set can now be formulated as follows:

Problem 1. Given a matrix P and the discrete-time system (7) subject to constraints (10), (11) and (12) with given Δ_v , H and G , find a matrix W such that

1. the controlled system in (7) and (8) satisfies (15);
2. all elements of the set \mathcal{C} in (13) satisfy (14).

The representational complexity of the set \mathcal{C} (i.e. n_p) is decided by the choice of the matrix P . For $P = I$, the candidate RCI set \mathcal{C} has a complexity similar to the one considered in^{10,18,17}. Systematic choices of P with increasing degrees of complexity

will be exemplified briefly in the sequel. Thus, in the next section, we derive tractable feasibility conditions for the solvability of **Problem 1**. The obtained feasibility conditions will be then utilized to develop an iterative algorithm to compute RCI sets with potentially increased volume at each step.

4 | SUFFICIENT CONDITIONS FOR INVARIANCE

Let us first introduce a state transformation

$$\theta \triangleq W^{-1}x \Leftrightarrow x = W\theta. \quad (16)$$

Using (16), the controlled system (7) can be expressed in the transformed space as

$$\theta^+ = \underbrace{W^{-1}\mathcal{A}(\Delta)W}_{\bar{\mathcal{A}}(W,\Delta)}\theta + \underbrace{W^{-1}\mathcal{B}(\Delta)}_{\bar{\mathcal{B}}(W,\Delta)}u + \underbrace{W^{-1}\mathcal{E}(\Delta)}_{\bar{\mathcal{E}}(W,\Delta)}w. \quad (17)$$

For notational simplicity, we will suppress the arguments of the matrices $\mathcal{A}(\Delta)$, $\mathcal{B}(\Delta)$, $\mathcal{E}(\Delta)$, $\bar{\mathcal{A}}(W, \Delta)$, $\bar{\mathcal{B}}(W, \Delta)$ and $\bar{\mathcal{E}}(W, \Delta)$ in the later parts. Using (16), we can also express the set \mathcal{C} as

$$\mathcal{C} = \{W\theta \in \mathbb{R}^{n_x} : \theta \in \Theta\}, \quad (18)$$

where Θ is a symmetric set defined as follows:

$$\Theta \triangleq \{\theta \in \mathbb{R}^{n_x} : -\mathbf{1} \leq P\theta \leq \mathbf{1}\}. \quad (19)$$

We have thus introduced a θ -state-space in which the candidate RCI set is a known 0-symmetric set identified as in (19). The RCI set \mathcal{C} in the x -state-space will be determined as in (18) based on the choice of W .

According to the **Definition 5**, for invariance of Θ , we have to show that for each $\theta \in \Theta$, there exists a $u \in \mathcal{U}$ for which $\theta^+ \in \Theta$, $\forall (w, \Delta) \in (\mathcal{W}, \Delta)$. Verifying the existence of u for each $\theta \in \Theta$ for invariance is an intractable problem. To obtain a tractable formulation, we first note that the set Θ can always be expressed as the convex hull of finitely many (known) vertices. The vertices (and thus the number of them) are determined by the choice of the matrix P . With the set of vertices represented as $\Theta_v = \{\theta^1, \dots, \theta^{2\sigma}\}$, we can write

$$\Theta = \text{conv}(\Theta_v). \quad (20)$$

For now, we ignore the state constraints on θ obtained using (11) and (16), and propose a necessary and sufficient condition for the invariance of the set Θ .

Lemma 2. Consider following statements:

- i. Θ in (19) is an RCI set for the system (17), i.e. for each $\theta \in \Theta$,

$$\exists u \in \mathcal{U} : \bar{\mathcal{A}}\theta + \bar{\mathcal{B}}u + \bar{\mathcal{E}}w \in \Theta, \forall (w, \Delta) \in (\mathcal{W}, \Delta). \quad (21)$$

- ii. for each $\theta^j \in \Theta_v$,

$$\exists u^j \in \mathcal{U} : \bar{\mathcal{A}}\theta^j + \bar{\mathcal{B}}u^j + \bar{\mathcal{E}}w \in \Theta, \forall (w, \Delta) \in (\mathcal{W}, \Delta). \quad (22)$$

The above two statements are equivalent.

Proof. It can be clearly seen that (i) \Rightarrow (ii) since $\theta^j \in \Theta$. To prove the converse statement (i.e. ii) \Rightarrow (i), we first assume the existence of u^j as in (22) and aim to construct u for any given $\theta \in \Theta$ such that the invariance condition expressed in (21) is satisfied. As implied by the notation, the idea would be to compute the control input as a convex combination of u^j s. This convex combination is identified by first expressing θ in terms of the extreme points θ^j as

$$\theta = \sum_{j=1}^{2\sigma} \alpha_j \theta^j, \text{ where } \sum_{j=1}^{2\sigma} \alpha_j = 1, \alpha_j \in [0, 1]. \quad (23)$$

By using the α^j s identified from this decomposition, the control input is then constructed as

$$u = \sum_{j=1}^{2\sigma} \alpha_j u^j \in \mathcal{U}. \quad (24)$$

With this input, the transformed state vector in the next step would be obtained based on (17) as

$$\theta^+ = \underbrace{\bar{A} \left(\sum_{j=1}^{2\sigma} \alpha_j \theta^j \right)}_{\theta} + \underbrace{\bar{B} \left(\sum_{j=1}^{2\sigma} \alpha_j u^j \right)}_u + \underbrace{\bar{E} \left(\sum_{j=1}^{2\sigma} \alpha_j \right)}_1 w = \sum_{j=1}^{2\sigma} \alpha_j \underbrace{(\bar{A}\theta^j + \bar{B}u^j + \bar{E}w)}_{y(w,\Delta)}. \quad (25)$$

We know from (22) that $y(w, \Delta) \in \Theta, \forall (w, \Delta) \in (\mathcal{W}, \Delta)$. Since θ^+ is obtained as a convex combination of $y(w, \Delta) \in \Theta$ and as Θ is a convex set, it then necessarily follows that $\theta^+ \in \Theta, \forall (w, \Delta) \in (\mathcal{W}, \Delta)$. \square

Lemma 2 shows that for invariance of the set Θ , existence of $u \in \mathcal{U}$ needs to be verified only for the finite set of points $\theta^j \in \Theta_v$. This is a crucial observation that paves the way towards a tractable solution for **Problem 1**.

We next derive matrix inequality conditions for the invariance of the set Θ based on (22). For each $\theta^j \in \Theta_v$, we can see that condition (22) can be rewritten as

$$\exists u^j \in \mathcal{U} : -1 \leq P(\bar{A}\theta^j + \bar{B}u^j + \bar{E}w) \leq 1, \forall (w, \Delta) \in (\mathcal{W}, \Delta). \quad (26)$$

Condition (26) must be satisfied element-wise. Using (17), we hence express (26) equivalently as

$$\exists u^j \in \mathcal{U} \text{ and } \phi_{j,i} \in \mathbb{R}_+ : \phi_{j,i}(1 - (e_i^T P W^{-1} (A W \theta^j + B u^j + E w))^2) \geq 0, \forall (w, \Delta) \in (\mathcal{W}, \Delta); i = 1, \dots, n_p, \quad (27)$$

where e_i represents the i -th column of the identity matrix of size $n_p \times n_p$ and $\phi_{j,i}$ s are introduced for the convenience of our derivations in the sequel. Condition (27) is a nonlinear parameter-dependent necessary and sufficient condition for the invariance of the set Θ , which has to be satisfied for each $\theta^j \in \Theta_v$.

Having obtained the condition for invariance of the set Θ , we now proceed to present the main result of this paper, which gives feasibility conditions for **Problem 1**. In order to deal with uncertain parameter dependence, we employ a specific version of the full-block S-procedure from¹⁰, which is cited in **Appendix A** for ease of reference. As ingredients that emerge through this procedure, we introduce a set of multiplier matrices (associated with Δ) and an inner approximation thereof respectively as

$$\mathcal{M} \triangleq \{M : (A3)\} \text{ and } \mathcal{M}_{pol} \triangleq \{M : (A5)\}. \quad (28)$$

Note that $\mathcal{M}_{pol} \subseteq \mathcal{M}$ is characterized by LMI conditions (as inherited from²⁶) and is hence used to express the theorem statement.

Theorem 1. Problem 1 is feasible if there exist $u^j \in \mathbb{R}^{n_u}, V_{j,i} = V_{j,i}^T \in \mathbb{R}^{n_x \times n_x}, \phi_{j,i} \in \mathbb{R}_+, M_{j,i} \in \mathcal{M}_{pol}; i = 1, \dots, n_p; j = 1, \dots, 2\sigma$ and $W \in \mathbb{R}^{n_x \times n_x}$ with which (29), (30) and (31) are satisfied for $i = 1, \dots, n_p, j = 1, \dots, 2\sigma$ and $k = 1, \dots, \gamma$. A robust control invariant set can then be obtained as in (13).

$$H W \theta^j \leq 1, G u^j \leq 1, \quad (29)$$

$$\begin{bmatrix} W^T V_{j,i}^{-1} W & * \\ \phi_{j,i} e_i^T P & \phi_{j,i} \end{bmatrix} > 0, \quad (30)$$

$$\begin{bmatrix} \phi_{j,i} & * & * \\ A W \theta^j + B u^j + E w^k & V_{j,i} + B_p R_{j,i} B_p^T & * \\ A_d W \theta^j + B_d u^j + E_d w^k & S_{j,i} B_p^T + D_p R_{j,i}^T B_p^T & Q_{j,i} + S_{j,i} D_p^T + D_p S_{j,i}^T + D_p R_{j,i} D_p^T \end{bmatrix} \geq 0. \quad (31)$$

Proof. Recalling **Lemma 2**, we aim for finding $u^j \in \mathcal{U}; j = 1, \dots, 2\sigma$ that satisfy (22). In this fashion we will have established that Θ in (19) and thus \mathcal{C} in (18) are RCI sets. We hence assume that the control input is formed as in (24) and observe in reference to (16) that the state and control input constraints in (11) read as (29). With the intention to resolve the nonlinearity (27), we now introduce a positive-definite matrix variable $V_{j,i}$ that satisfies

$$V_{j,i}^{-1} - \phi_{j,i} W^{-T} P^T e_i e_i^T P W^{-1} > 0. \quad (32)$$

Note that this condition can be expressed equivalently as in (30) by applying first a congruence transformation with W and then the Schur complement lemma. With $V_{j,i}$ satisfying (32), a sufficient condition for (27) can be formulated as

$$\phi_{j,i} - (A W \theta^j + B u^j + E w)^T V_{j,i}^{-1} (A W \theta^j + B u^j + E w) \geq 0. \quad (33)$$

This reads after an application of the Schur complement lemma as the parameter-dependent LMI

$$\begin{bmatrix} \phi_{j,i} & * \\ \mathcal{A}(\Delta)W\theta^j + \mathcal{B}(\Delta)u^j + \mathcal{E}(\Delta)w & V_{j,i} \end{bmatrix} > 0. \quad (34)$$

By now applying **Lemma 4** with

$$\mathcal{Y} = \left[\begin{array}{c|cc} D_p & A_d W \theta^j + B_d u^j + E_d w & 0 \\ 0 & \frac{1}{2} \phi_{j,i} & 0 \\ \hline B_p & A W \theta^j + B u^j + E w & \frac{1}{2} V_{j,i} \end{array} \right], \quad (35)$$

we arrive at an LMI condition in terms of $M_{j,i} \in \mathcal{M}$, which needs to be satisfied for all $w \in \mathcal{W}$. Thanks to the fact that dependence on w is affine and \mathcal{W} in (12) is polytopic, we arrive at the condition in (31) expressed in terms of the vertices v^j . \square

It should be noted at this point that (1, 1) block of (30) has nonlinear dependence on $(W, V_{j,i})$. This issue was also faced in¹⁰ and resolved by adapting a successive linearization approach from¹⁸. We follow similar lines to develop an iterative scheme for volume optimization in the next section.

5 | ITERATIVE RCI SET COMPUTATION

In this section, we first briefly recall the linearization approach from^{18,10}. We then formulate a cost based on which successive volume optimization can be performed.

5.1 | Linearization

Consider the following inequality

$$W^T V_{j,i}^{-1} W = \underbrace{(W - V_{j,i} Y_{j,i})^T}_{\mathcal{V}_{j,i}^T} V_{j,i}^{-1} \underbrace{(W - V_{j,i} Y_{j,i})}_{\mathcal{V}_{j,i}} + Y_{j,i}^T W + W^T Y_{j,i} - \underbrace{Y_{j,i}^T V_{j,i} Y_{j,i}}_{\mathcal{N}_{j,i}} \geq Y_{j,i}^T W + W^T Y_{j,i} - Y_{j,i}^T V_{j,i} Y_{j,i}, \quad (36)$$

where $Y_{j,i}$ is an arbitrary matrix of compatible dimension. A sufficient condition for (30) can be obtained by replacing its (1, 1) block with $\mathcal{N}_{j,i}$ in (36). Nevertheless, the resulting condition is an LMI only if $Y_{j,i}$ fixed. Assuming that an iterative scheme can be started with an initial choice, the approach developed in¹⁰ is based on fixing the choice at the $n + 1$ -st step as

$$Y_{j,i} = Y_{j,i}^n \triangleq (V_{j,i}^n)^{-1} W^n, \quad (37)$$

where $(W^n, V_{j,i}^n)$ are the values of $(W, V_{j,i})$ obtained at the n -th step. Hence, the sufficient LMI condition for (30) to be used at the $n + 1$ -st step of our iterative scheme is

$$\begin{bmatrix} (Y_{j,i}^n)^T W + W^T Y_{j,i}^n - (Y_{j,i}^n)^T V_{j,i}^n Y_{j,i}^n & e_i P^T \phi_{j,i} \\ \phi_{j,i} e_i^T P & \phi_{j,i} \end{bmatrix} > 0. \quad (38)$$

In¹⁰, it has been shown that such an iterative scheme reduces the conservatism introduced by (36). Note that (38) still holds if $W = W^n$ and $V_{j,i} = V_{j,i}^n$, which confirms that the solutions obtained at the n -th step are also feasible at the $n + 1$ -st step. We also recall that the invertibility of matrix W is implied from the (1, 1) block of (38).

5.2 | Iterative Volume Maximization

We now develop an iterative scheme based on **Theorem 1** for the computation of RCI sets of potentially large volumes at each step. Some recent works like^{11,20} adopt an indirect approach to maximize the volume of the RCI set by iteratively maximizing the volume of the ellipsoidal set enclosed therein. In contrast, we present in this paper a direct approach in which the volume of the actual RCI set is aimed to be maximized.

5.3 | Volume Computation

As the first major step towards developing our algorithm, we first show that the volume of the set \mathcal{C} is proportional to $|\det(W)|$ once P is fixed. For this, we use the fact that any 0-symmetric polytopic set \mathcal{C} can be decomposed into n_x -simplices \mathcal{C}_m , $m =$

$1, \dots, 2\mu$, which are identified in terms of their vertex points x_m^i as

$$\mathcal{C}_m = \left\{ \sum_{i=0}^{n_x} \alpha_i x_m^i : \sum_{i=0}^{n_x} \alpha_i = 1, \alpha_i \in [0, 1] \right\}. \quad (39)$$

These simplices would have the following properties:

- i. \mathcal{C}_m is non-empty;
- ii. $\text{int}(\mathcal{C}_{m_1} \cap \mathcal{C}_{m_2}) = \emptyset$, if $m_1 \neq m_2$;
- iii. $\bigcup_{m=1}^{2\mu} \mathcal{C}_m = \mathcal{C}$.

Each \mathcal{C}_m would have origin as a common vertex point (i.e. $x_m^0 = 0, \forall m = 1, \dots, 2\mu$). The remaining n_x vertices of \mathcal{C}_m can be represented as the columns of a matrix

$$X_m = [x_m^1 \ x_m^2 \ \dots \ x_m^{n_x}]. \quad (40)$$

From²⁷, it's known that the volume of the simplex \mathcal{C}_m is given by the Cayley-Menger determinant:

$$\text{Volume}(\mathcal{C}_m) = \frac{1}{n_x!} |\det(X_m)|. \quad (41)$$

Let the matrix Θ_m be defined as

$$\Theta_m = W^{-1} X_m \Leftrightarrow X_m = W \Theta_m. \quad (42)$$

From (16), we know that the columns of the matrix Θ_m represent n_x independent vertices from the set Θ_v .

Lemma 3. The volume of the set \mathcal{C} identified as in (13) with a fixed P is proportional to $|\det(W)|$.

Proof. Since the simplices \mathcal{C}_m have disjoint interiors, the volume of the set \mathcal{C} can be written as

$$\text{Volume}(\mathcal{C}) = \sum_{m=1}^{2\mu} \text{Volume}(\mathcal{C}_m). \quad (43)$$

It then follows from (41) and (42) that

$$\text{Volume}(\mathcal{C}) = \frac{1}{n_x!} \sum_{m=1}^{2\mu} |\det(X_m)| = \frac{1}{n_x!} \sum_{m=1}^{2\mu} |\det(W \Theta_m)|. \quad (44)$$

Since the set \mathcal{C} is symmetric, we can always order Θ_m s as

$$\Theta_{m+\mu} = -\Theta_m, \quad m = 1, \dots, \mu. \quad (45)$$

With $|\det(\Theta_m)| = |\det(-\Theta_m)|$, the summation in (44) can be rewritten as

$$\text{Volume}(\mathcal{C}) = \frac{2}{n_x!} |\det(W)| \sum_{m=1}^{\mu} |\det(\Theta_m)|. \quad (46)$$

For a fixed P matrix, Θ_m s will be fixed too, which implies that

$$\text{Volume}(\mathcal{C}) \propto |\det(W)|. \quad (47)$$

□

Note that the simplicial decomposition of the set \mathcal{C} is not unique, which is though not a problem for our purposes. Once P is fixed, **Lemma 3** basically justifies to maximize $|\det(W)|$ to obtain a set \mathcal{C} of desirably large volume. Simplicial decomposition will also be useful when we consider controller design in the next section.

5.3.1 | Iterative Volume Maximization

In order to obtain an RCI set with a desirably large volume, we hence need to solve a determinant maximization problem under LMI conditions presented in **Theorem 1**. Such a problem reads as a generalized semi-definite program only when W is required to be symmetric²⁸, which would necessarily introduce potential conservatism. An iterative volume maximization approach was developed in¹⁰ without enforcing symmetry on W , which we will also use in this paper.

The basic idea is to maximize the determinant of a different matrix \bar{Z} , which is required to be symmetric and positive-definite. We relate \bar{Z} to W by a condition as

$$W^T W \succeq \bar{Z} > 0, \quad (48)$$

which would ensure that $\det(\bar{Z}) \leq |\det(W)|^2$. Since (48) is not an LMI, it needs to be replaced with a sufficient condition. This is done within an iterative scheme in which the solution of W at the n -th step is represented as W^n . A sufficient condition for (48) is formulated by¹⁰ in terms of W^n as

$$W^T W^n + (W^n)^T W - (W^n)^T W^n \succeq \bar{Z} > 0. \quad (49)$$

Note that this condition is necessarily satisfied with $W = W^n$. As a result, maximization of $\det(\bar{Z})$ under (49) would lead to a solution W^{n+1} that satisfies

$$|\det(W^{n+1})| \geq |\det(W^n)|. \quad (50)$$

This allows us to develop the following iterative algorithm to compute RCI sets of increased volume at each step for a priori chosen matrix P :

Algorithm 1: [$n + 1$ -st step]

$$\begin{aligned} & \max && \log \det(\bar{Z}) \\ & W, \bar{Z}, V_{j,i}, M_{j,i}, \phi_{j,i}, u^j \\ & \text{subject to:} && (29), (31), (38) \text{ and } (49) \end{aligned}$$

Initial Optimization to Compute W^0 : Condition (49) is removed and $\log \det(\bar{Z})$ is changed to $\log \det(W + W^T)$; (38) is imposed with $Y_{j,i}^n \rightarrow \psi I$, where ψ is selected by a line search to find an initial feasible solution.

It has been already shown before that the solution from step n is feasible at step $n + 1$ in (38) and (49). This ensures that **Algorithm 1** is recursively feasible. Hence the volume of set \mathcal{C} increases iteratively until it converges to some stationary point. The algorithm should be terminated in practice when the change in $|\det(W)|$ is below a certain value or when infeasibility occurs due to numerical reasons.

One might obtain different RCI sets from **Algorithm 1** for different P choices (as well as different initial solutions) and these can in fact be combined to construct larger RCI sets. As a matter of fact, the invariance conditions of (31) and (38) are only sufficient and hence might be conservative. By updating the matrix W at each iteration, we are actually scaling and rotating the initial polyhedron to a desirably large volume. As is known from²⁹, the convex hull and union of different RCI sets is a larger RCI set with higher complexity. RCI sets with larger volumes can hence be obtained simply as the convex hulls or union of the sets obtained from **Algorithm 1**.

Remark 1. **Algorithm 1** needs a user-defined matrix P , which determines the representational complexity of the set \mathcal{C} . We can always choose matrix P in (13), while assuming $W = I$, such that it defines an initial polytope of desired shape and complexity. A simple approach to select the initial polytope could be to distribute the hyperplanes on the unit sphere uniformly. For example, a possible choice in the two-dimensional case could be selecting the i -th row as

$$P_i = \left[\cos \left(\frac{\pi(i-1)}{n_p} \right) \sin \left(\frac{\pi(i-1)}{n_p} \right) \right], \quad i = 1, \dots, n_p. \quad (51)$$

Furthermore, having an approximate knowledge of the shape of the RCI set can be very helpful while selecting the initial polytope. Thus alternatively one could select the initial polytope by performing a few iterations of the geometric approach⁶ (convergence not needed) to get an initial estimate of the RCI set. Note that this could be still beneficial in the case of a higher-dimensional system where the geometric approach is known to be computationally inefficient and may not even converge.

Remark 2. In **Algorithm 1**, the SDP consists of $\left(2\sigma \times n_p \times \left(\gamma + 1 + \frac{\eta(\eta+1)}{2} \right) + 1 \right)$ LMIs due to (31), (38), (49); and $(2\sigma \times (n_h + n_g))$ scalar inequalities representing the system constraints in (29). Furthermore, these inequalities are in terms of $(2 + 2\sigma \times (1 + 4n_p))$ matrix variables and $(2\sigma \times n_p)$ scalar variables. The complexity due to system and invariance constraints can in fact be reduced significantly in the case when the sets \mathcal{X} , \mathcal{U} and \mathcal{W} are 0-symmetric. Because of the symmetry, the invariance condition needs to be verified only on non-symmetric elements of Θ_v . In this way the number of LMIs and scalar inequalities reduce to $\left(\sigma \times n_p \times \left(\gamma + 1 + \frac{\eta(\eta+1)}{2} \right) + 1 \right)$ and $\sigma \times (n_h + n_g)$ respectively. We then also have less number of matrix and scalar variables as $(2 + \sigma(1 + 4n_p))$ and $(\sigma \times n_p)$ respectively. The computational complexity can be reduced by also considering alternative relaxation schemes or structured multipliers at the cost of potential conservatism. For instance, Pólya relaxation might be replaced with the so-called convex hull relaxation (see e.g.²⁶) to reduce the number of LMIs. The number of variables can be reduced by using D-scales, DG scales or block-diagonal sub-blocks in the multiplier matrix M^{21} .

Remark 3. An algorithm to obtain desirably small RCI sets can be formulated by minimizing $|\det(W)|$ iteratively in a similar fashion as done in **Algorithm 1**. This can be achieved by minimizing $\text{trace}(\underline{Z})$ at each iteration and replacing (49) by $\begin{bmatrix} \underline{Z} & W^T \\ W & I \end{bmatrix} \succeq 0$ in **Algorithm 1**. For further details, we refer the reader to¹⁸.

Algorithm 1 would provide RCI sets for which control inputs that ensure invariance are known to exist. But to realize constrained control, a method needs to be devised to compute a feasible input at each step. The use of RCI sets in MPC control (in which a cost is optimized online with a terminal set) is well known and is not within the scope of this paper. Instead, we present in the next section an offline approach to constrained controller design.

6 | CONTROLLER DESIGN

The purpose of this section is to design a piecewise affine controller based on the approach presented in^{23,24} for the constrained control of the system in (7). The approach requires that at each vertex of the set \mathcal{C} , there should exist an admissible control action that brings the state to the interior of the set \mathcal{C} within finite time. To be able to design such a controller, we consider using **Lemma 1** together with the RCI set computation algorithm developed in the previous section. For a specific $\lambda \in (0, 1)$ choice, we would then obtain an RCI set \mathcal{C} for the modified system in (6), which would serve as a λ -contractive set for the original system of (3). A controller designed for the modified system of (6) would hence serve as a stabilizing piecewise-affine controller for the original system (3), thus ensuring asymptotic convergence to origin in the absence of any disturbance input. This controller will also keep the trajectory of (3) in \mathcal{C} for any disturbance input within the considered disturbance set.

To this end, we first recall the decomposition of the RCI set \mathcal{C} into n_x -simplicies \mathcal{C}_m as in (39). The control law is then formulated as a domain-dependent state feedback of the form

$$x(k) \in \mathcal{C}_m \Rightarrow u(k) = K_m x(k). \quad (52)$$

If $x(k)$ happens to be at a point that is common to \mathcal{C}_i and \mathcal{C}_j (e.g. a point on a common edge), then the control input can be computed with either K_i or K_j . To identify the state-feedback gain matrices K_m , we first use the control inputs associated with the vertices x_m^j to form a matrix as

$$U_m = [u_m^1 \ u_m^2 \ \dots \ u_m^{n_x}]. \quad (53)$$

The associated state feedback gain matrix is then computed as

$$K_m = U_m X_m^{-1}, \quad m = 1, \dots, 2\mu. \quad (54)$$

In^{23,24}, it is shown that the piece-wise affine control law obtained in this fashion is stabilizing and is recursively feasible, which indeed imply that it is an invariance-inducing controller. Indeed, if $x(k)$ is expressed as a convex combination of the vertices of \mathcal{C}_m , $u(k)$ of (52) can then also be expressed as a convex combination of the associated vertices u_m^j , which basically corresponds to the control law proposed in (24):

$$x(k) = X_m \alpha \Rightarrow u(k) = K_m X_m \alpha = U_m \alpha. \quad (55)$$

Assuming that the system starts from an initial state within the computed RCI set, it is also possible to develop a control algorithm based on online optimization by using the associated W matrix and the vertices θ^j , $j = 1, \dots, 2\sigma$ ²⁵. At each time step, a linear program would then be solved to obtain a set of weights, with which the control input would be constructed. With k 'th time step indicated with a superscript k , the weights are obtained and used to construct the control input as follows:

$$\beta^k = \arg \min_{\beta \in \mathbb{R}^{2\sigma}} \left\{ \sum_{j=1}^{2\sigma} \beta_j : \sum_{j=1}^{2\sigma} \beta_j W \theta^j = x(k), 0 \leq \beta_j \leq 1, j = 1, \dots, 2\sigma \right\} \Rightarrow u(k) = \sum_{j=1}^{2\sigma} \beta_j^k u^j. \quad (56)$$

A controller implemented in this fashion would ensure asymptotic stability, as established in detail by²⁵. An online approach might be preferable especially when working with high dimensional systems, for which obtaining a simplicial decomposition of the set \mathcal{C} would be difficult.

The closed-loop transient performance of the system can (to some extent) be shaped by the choice of the contraction factor λ , which also affects the size of the set \mathcal{C} . However, a known weakness of such a control structure is that a full range of control inputs are utilized only at the boundary of the set \mathcal{C} . In the interior of the set \mathcal{C} , progressively smaller control inputs are applied as the system trajectories approach origin. Thus it may take a long time for a controller to stabilize the system. It follows from²⁵

that this weakness could easily be overcome by modifying the proposed controller into a so-called interpolation-based controller, which suboptimally minimizes a predefined cost function.

7 | ILLUSTRATIVE EXAMPLES

In this section, we provide some examples to illustrate the potential of the approach proposed in this paper. The algorithm was implemented in Matlab on a 3.1 GHz Intel Core i7-555U macOS computer with 8 GB RAM with YALMIP³⁰ and the solver SeDuMi³¹. The computation of the volume, vertices and projections of the polytope was done using MPT³².

7.1 | Double Integrator

In this section, the proposed algorithm is demonstrated with a discrete-time double integrator that has rational parameter dependence. The system dynamics are described as in (7) with

$$\mathcal{A} = \begin{bmatrix} 1 + \delta_1 & 1 + \delta_1 \\ 0 & 1 + \frac{\delta_2}{1 + \delta_1} \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (57)$$

where $|\delta_{1,2}| \leq 0.25$ represent the uncertain parameters. The state and control input constraints and the input disturbance bound are expressed as

$$|x| \leq \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \quad |u| \leq 3, \quad |w| \leq 0.6. \quad (58)$$

In order to make comparisons with the existing algorithms^{18,11} and the geometric approach, we first introduce a new parameter as $\delta_3 = \delta_2/(1 + \delta_1)$ and thus view (57) as a system that has affine dependence on (δ_1, δ_3) . We illustrate the proposed algorithm by computing the sets \mathcal{C}_4 and \mathcal{C}_5 associated with $n_p = 4$ and $n_p = 5$ respectively. These sets are compared with the maximal RCI set Ω_∞ obtained by using an existing geometric approach, in which the control inputs are also assumed to be free. **FIGURE 1(a)** shows the generated RCI sets. We were unable to compute any RCI set with the methods from^{17,18,20,10,11}, in which the control input is assumed to have a linear structure and/or a conservative system description is used to account for parametric uncertainty. It can be observed that the maximal RCI set from the geometric approach (Volume=40.2445) is described by 10 hyperplanes and has a slightly larger volume than the one obtained using the proposed approach with the same complexity (Volume=39.9720). The non-symmetric vertices of the set \mathcal{C}_5 and the corresponding inputs are given by

$$\begin{aligned} X &= \begin{pmatrix} x^1 & x^2 & x^3 & x^4 & x^5 \\ 3.3198 & 5.0000 & 5.0000 & 2.2185 & -1.6636 \\ -3.6425 & -3.4323 & -1.4801 & 1.2336 & 3.2409 \end{pmatrix} \\ U &= \begin{pmatrix} u^1 & u^2 & u^3 & u^4 & u^5 \\ 2.9922 & 2.6635 & -1.1764 & -2.9825 & -3.0000 \end{pmatrix} \end{aligned} \quad (59)$$

Notice that the vertices $\{x^5, -x^5\} \notin \Omega_\infty$, this is due to the approximation of (57) with an affine parameter dependent system while computing Ω_∞ . Lastly, in **FIGURE 1(b)**, we demonstrate the controller proposed in the Section 6 in closed-loop with the system (57). Also, the proposed decomposition of the set \mathcal{C} into simplices is shown. The closed-loop trajectory, when starting from one of the vertices, can be seen in black (dot-dashed) is produced by randomly varying disturbances and uncertainties within their bounds.

7.2 | 4-D Vehicle Lateral Dynamics

We now compare the proposed algorithm with a recent work¹⁰ using a 4-dimensional bicycle model for vehicle lateral dynamics¹⁰. We assume the longitudinal velocity of the vehicle to be an uncertain parameter as $V_x \in [50, 70]$ km/h, which enters the system dynamics rationally. For a fair comparison, we keep the complexity of the RCI set same as¹⁰ by selecting $P = I$. Using **Algorithm 1**, the nonsymmetric vertices of the set \mathcal{C} and corresponding control inputs were found to be

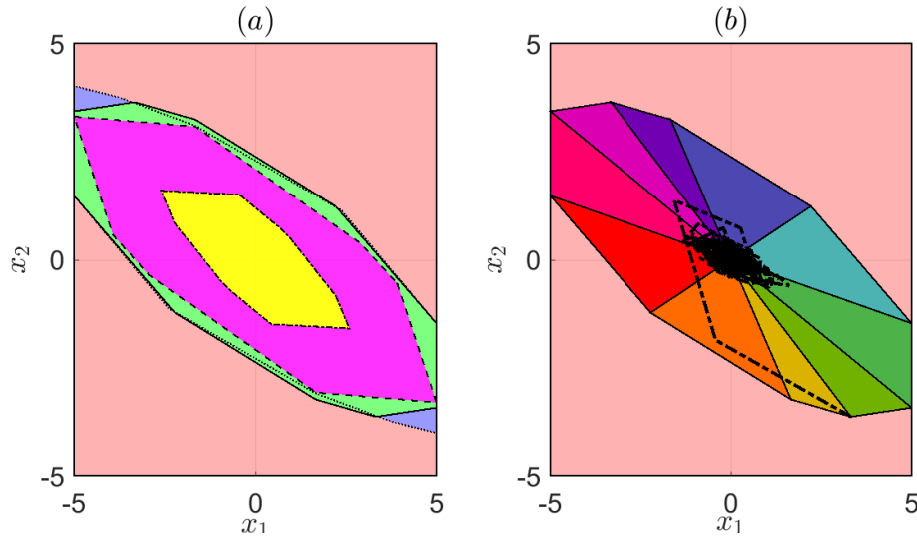


FIGURE 1 (a) Admissible states \mathcal{X} (red) and maximal RCI set Ω_∞ geometric approach (blue; dotted), and maximum volume RCI set \mathcal{C}_5 (green; solid) and \mathcal{C}_4 (magenta; dashed) from **Algorithm 1** with complexity $n_p=5$ and $n_p=4$, respectively, and minimum volume RCI set from **Remark 3** with $n_p=5$ (yellow; dot-dashed) and (b) Admissible states \mathcal{X} (red) and Simplex decomposition of the maximum volume RCI set (with $\lambda = 0.99$) from **Algorithm 1** with $n_p=5$ (colored).

$$\begin{aligned}
 X &= \begin{pmatrix} x^1 & x^2 & x^3 & x^4 & x^5 & x^6 & x^7 & x^8 \\ 0.3714 & 0.4000 & -0.2800 & -0.2514 & 0.2514 & 0.2800 & -0.4000 & -0.3714 \\ -3.0000 & -2.9754 & -2.3478 & -2.3233 & -1.3182 & -1.2937 & -0.6661 & -0.6415 \\ 0.0485 & 0.0035 & 0.1136 & 0.0687 & 0.0681 & 0.0232 & 0.1332 & 0.0883 \\ -0.2040 & -0.1502 & -0.0704 & -0.0165 & -1.3353 & -1.2815 & -1.2017 & -1.1479 \end{pmatrix} \\
 U &= \begin{pmatrix} u^1 & u^2 & u^3 & u^4 & u^5 & u^6 & u^7 & u^8 \\ -0.0791 & 0.0776 & -0.0873 & 0.0873 & 0.0517 & 0.0873 & -0.0426 & 0.0873 \\ -0.1235 & 0.1407 & -1.0000 & -0.9999 & 1.0000 & 0.9991 & 0.3675 & -0.9999 \end{pmatrix} \quad (60)
 \end{aligned}$$

The plots in **FIGURE 2**-(a), (b), (c) and (d) show the projections of the admissible states \mathcal{X} , and the maximum volume RCI set \mathcal{C} obtained from **Algorithm 1** after 3000 iterations, and the algorithm in¹⁰, respectively. The RCI set obtained from the proposed algorithm is much larger than the compared approach. Furthermore, even though the computational complexity is high (see **Remark 2**), each iteration of the proposed algorithm took 24 seconds on average compared to 14 seconds taken by¹⁰. We also tried to compute the RCI set by using the approximate polytopic model, but were unable to compute any RCI set with the methods from^{18,11}. On the other hand, the geometric approach did not converge even after more than 24 hours.

8 | CONCLUSION

We have developed an iterative algorithm to compute a desirably large 0-symmetric RCI set for a rationally parameter-dependent system with additive disturbances. Sufficient LMI conditions guaranteeing invariance have been derived by favor of a state transformation, the full block S-procedure and a Newton update technique. These LMI conditions and the reformulated system constraints are then integrated into an algorithm where an SDP is solved to obtain RCI sets of successively larger volumes at every iteration. A relatively less conservative formulation has been achieved by rendering the control inputs independent of any predefined feedback structure along with the employment of a direct approach for volume maximization. These claims are supported by including relevant numerical examples where the proposed algorithm was able to compute considerably larger RCI sets in comparison with other similar approaches.

It has been observed in our numerical exercises and is also emphasized in Remark 1 that the choice of initial polytope affects the final outcome, thereby encouraging investigations to identify an appropriate approach to identify this initial set. Moreover, a future research direction could be to extend the proposed formulation to compute asymmetric polytopic RCI sets. It requires

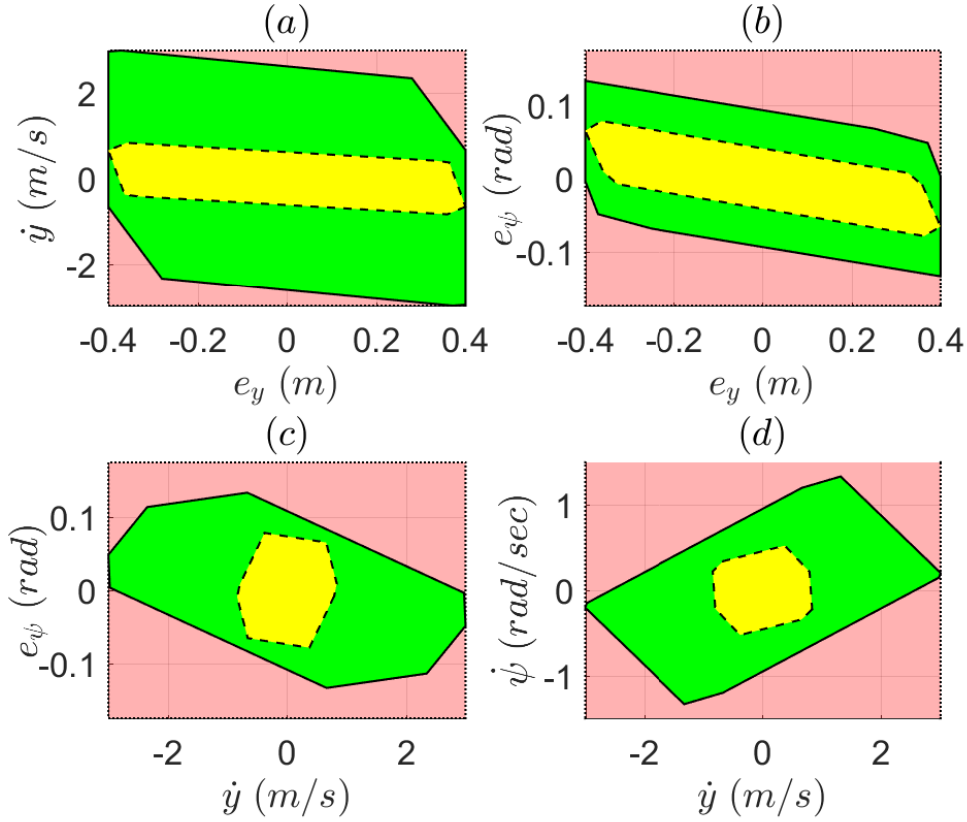


FIGURE 2 (a) Projection of admissible set \mathcal{X} (red; dotted), maximum volume RCI set \mathcal{C} from **Algorithm 1** (green; solid) and using¹⁰ (yellow; dashed).

some further investigations to develop parameter-dependent set computations along with control algorithms that achieve certain performance objectives.

9 | ACKNOWLEDGMENT

The authors thank Dr. Aniket Deshpande for his valuable inputs.



APPENDIX

A A FULL-BLOCK S-PROCEDURE

The following Lemma is a special case of the full block S-procedure²² as presented in¹⁰. We first introduce the LFT

$$\Delta \star \underbrace{\begin{bmatrix} \mathcal{Y}_{11} & \mathcal{Y}_{12} \\ \mathcal{Y}_{21} & \mathcal{Y}_{22} \end{bmatrix}}_y \triangleq \mathcal{Y}_{22} + \mathcal{Y}_{21} \Delta (I - \mathcal{Y}_{11} \Delta)^{-1} \mathcal{Y}_{12}, \quad (\text{A1})$$

which is said to be well-posed when $I - \mathcal{Y}_{11} \Delta$ is nonsingular $\forall \Delta \in \Delta$.

Lemma 4. The LFT $\Delta \star \mathcal{Y}$ is well-posed and

$$\text{He}\{\Delta \star \mathcal{Y}\} \triangleq \Delta \star \mathcal{Y} + (\Delta \star \mathcal{Y})^T > 0, \forall \Delta \in \Delta \quad (\text{A2})$$

holds if there exists a scaling matrix M that satisfies

$$\begin{bmatrix} \Delta^T \\ I \end{bmatrix}^T \underbrace{\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}}_M \begin{bmatrix} \Delta^T \\ I \end{bmatrix} \preceq 0, \quad \forall \Delta \in \Delta, \quad (\text{A3})$$

$$\begin{bmatrix} \mathcal{Y}_{21} R \mathcal{Y}_{21}^T + \text{He}\{\mathcal{Y}_{22}\} & \mathcal{Y}_{21} R \mathcal{Y}_{11}^T + \mathcal{Y}_{21} S^T + \mathcal{Y}_{12}^T \\ * & Q + \mathcal{Y}_{11} R \mathcal{Y}_{11}^T + \text{He}\{\mathcal{Y}_{11} S^T\} \end{bmatrix} > 0. \quad (\text{A4})$$

The condition (A3) is a semi-infinite LMI condition. Since we consider a polytopic uncertainty region, one can use Pólya's method to relax (A3). Hence using zeroth order Pólya relaxation in^{26,10}, we replace (A3) by a sufficient condition given by

$$\left. \begin{array}{l} \Omega_{jj}(M) \preceq 0, \quad j = 1, \dots, \eta \\ \text{He}\{\Omega_{ji}(M)\} \preceq 0, \quad j = 1, \dots, \eta; \quad i = j + 1, \dots, \eta \end{array} \right\} \quad (\text{A5})$$

where

$$\Omega_{ij}(M) \triangleq \begin{bmatrix} (\Delta^i)^T \\ I \end{bmatrix}^T M \begin{bmatrix} (\Delta^j)^T \\ I \end{bmatrix}, \quad i, j = 1, \dots, \eta. \quad (\text{A6})$$

Note that, it is possible to obtain potentially less conservative conditions at the cost of a combinatorial increase in the number of LMIs by employing higher-order Pólya relaxations; see¹⁰. Lastly, for the uncertainty regions described by polynomial inequalities, the sum-of-squares (SOS) approach can be used; see^{33,26,34}.

References

1. Blanchini F, Miani S. *Set-Theoretic Methods in Control*. Boston, MA: Birkhäuser . 2015.
2. Blanchini F. Set invariance in control. *Automatica* 1999; 35(11): 1747–1767.
3. Mayne D, Rawlings J, Rao C, Scokaert P. Constrained model predictive control: Stability and optimality. *Automatica* 2000; 36(6): 789–814.
4. Gilbert E, Tan K. Linear Systems with State and Control Constraints: The Theory and Application of Maximal Output Admissible Sets. *IEEE Transactions on Automatic Control* 1991; 36(9): 1008–1020.
5. Dórea CET, Hennes JC. (A, B)-Invariant Polyhedral Sets of Linear Discrete-Time Systems. *Journal of Optimization Theory and Application* 1999; 103(3): 521–542.
6. Borrelli F, Bemporad A, Morari M. *Predictive Control for Linear and Hybrid Systems*. Cambridge University Press . 2017
7. Klintberg E, Nilsson M, Mårdh LJ, Gupta A. A Primal Active-Set Minimal-Representation Algorithm for Polytopes with Application to Invariant-Set Calculations. *Conference on Decision and Control* 2018: 6862–6867.
8. Rungger M, Tabuada P. Computing Robust Controlled Invariant Sets of Linear Systems. *IEEE Transactions on Automatic Control* 2017; 62(7): 3665–3670.
9. Fiacchini M, Alamir M. Computing control invariant sets is easy. *CoRR* 2017; abs/1708.04797. <http://arxiv.org/abs/1708.04797>.
10. Gupta A, Koroğlu H, Falcone P. Computation of low-complexity control-invariant sets for systems with uncertain parameter dependence. *Automatica* 2019; 101: 330 - 337.
11. Liu C, Tahir F, Jaimoukha IM. Full-complexity polytopic robust control invariant sets for uncertain linear discrete-time systems. *International Journal of Robust and Nonlinear Control* 2019; 29(11): 3587–3605.

12. Wen-Hua C, Ballance D, O'Reilly J. Optimisation of attraction domains of nonlinear MPC via LMI methods. *American Control Conference* 2001: 3067–3072.
13. Yu S, Zhou Y, Qu T, Xu F, Ma Y. Control Invariant Sets of Linear Systems with Bounded Disturbances. *International Journal of Control, Automation and Systems* 2018; 16(2): 622–629.
14. Pluymers B, Rossiter JA, Suykens JAK, Moor BD. The efficient computation of polyhedral invariant sets for linear systems with polytopic uncertainty. *American Control Conference* 2005: 804–809.
15. Cannon M, Deshmukh V, Kouvaritakis B. Nonlinear model predictive control with polytopic invariant sets. *Automatica* 2003; 39(8): 1487–1494.
16. Athanasopoulos N, Bitsoris G, Lazar M. Construction of invariant polytopic sets with specified complexity. *International Journal of Control* 2014; 87(8): 1681–1693.
17. Blanco T, Cannon M, De Moor B. On efficient computation of low-complexity controlled invariant sets for uncertain linear systems. *International Journal of Control* 2010; 83(7): 1339–1346.
18. Tahir F, Jaimoukha I. Low-Complexity Polytopic Invariant Sets for Linear Systems Subject to Norm-Bounded Uncertainty. *IEEE Transactions on Automatic Control* 2015; 60(5): 1416–1421.
19. Raković S, Baric M. Parameterized Robust Control Invariant Sets for Linear Systems: Theoretical Advances and Computational Remarks. *IEEE Transactions on Automatic Control* 2010; 55(7): 1599–1614.
20. Gupta A, Falcone P. Full-Complexity Characterization of Control-Invariant Domains for Systems with Uncertain Parameter Dependence. *IEEE Control System Letter* 2019; 3(1): 19–24.
21. Boyd S, El Ghaoui L, Feron E, Balakrishnan V. *Linear Matrix Inequalities in System and Control Theory*. Society for Industrial and Applied Mathematics . 1994.
22. Scherer C. A Full Block S-Procedure with applications. *Conference on Decision and Control* 1997: 2602–2607.
23. Gutman P, Cwikel M. Admissible sets and feedback control for discrete-time linear dynamical systems with bounded controls and states. *IEEE Transactions on Automatic Control* 1986; 31(4): 373-376.
24. Blanchini F. Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions. *IEEE Transactions on Automatic Control* 1994; 39(2): 428-433.
25. Nguyen HN, Gutman PO, Olaru S, Hovd M. Explicit improved vertex control for uncertain, time-varying linear discrete-time systems with state and control constraints. *International Journal of Robust and Nonlinear Control* 2016; 26(12): 2652-2667.
26. Koroğlu H, Scherer C. Robust Stability Analysis against Perturbations of Smoothly Time-Varying Parameters. *Conference on Decision and Control* 2006: 2895–2900.
27. Büeler B, Enge A, Fukuda K. *Exact Volume Computation for Polytopes: A Practical Study*: 131–154; Basel: Birkhäuser Basel . 2000.
28. Vandenberghe L, Boyd S, Wu S. Determinant Maximization with Linear Matrix Inequality Constraints. *SIAM Journal on Matrix Analysis and Application* 1998; 19(2): 499–533.
29. Raković S, Kerrigan E, Mayne D, Kouramas K. Optimized robust control invariance for linear discrete-time systems: Theoretical foundations. *Automatica* 2007; 43(5): 831-841.
30. Löfberg J. YALMIP : A Toolbox for Modeling and Optimization in MATLAB. In: ; 2004; Taipei, Taiwan.
31. Sturm J. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software* 1999; 11–12: 625–653. Version 1.05 available from <http://fewcal.kub.nl/sturm>.
32. Herceg M, Kvasnica M, Jones C, Morari M. Multi-Parametric Toolbox 3.0. In: ; 2013; Zürich, Switzerland: 502–510. <http://control.ee.ethz.ch/~mpt>.

33. Scherer C. LMI Relaxations in Robust Control. *European Journal of Control* 2006; 12(1): 3–29.
34. Scherer C, Weiland S. Linear matrix inequalities in control. *Lecture Notes, Dutch Institute for Systems and Control* 2015.