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Kim, I., Wan, X., Zhang, G. (2020). Plurisuperharmonicity of reciprocal energy function on Teichmüller space and Weil-Petersson
metric. Journal des Mathematiques Pures et Appliquees, 141: 316-341.
http://dx.doi.org/10.1016/j.matpur.2020.01.009

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J. Math. Pures Appl. 141 (2020) 316–341

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

Plurisuperharmonicity of reciprocal energy function on Teichmüller space and Weil-Petersson metric ☆

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ARTICLE INFO

Article history:
Received 16 September 2019
Available online 29 January 2020

MSC:
53C43
53C21
53C25

Keywords:
Harmonic map
Teichmüller space
Energy function
Weil-Petersson metric

ABSTRACT

We consider harmonic maps $u(z) : \mathcal{X}_z \to N$ in a fixed homotopy class from Riemann surfaces $\mathcal{X}_z$ of genus $g \geq 2$ varying in the Teichmüller space $\mathcal{T}$ to a Riemannian manifold N with non-positive Hermitian sectional curvature. The energy function $E(z) = E(u(z))$ can be viewed as a function on $\mathcal{T}$ and we study its first and the second variations. We prove that the reciprocal energy function $E(z)^{-1}$ is plurisuperharmonic on Teichmüller space. We also obtain the (strict) plurisubharmonicity of $\log E(z)$ and $E(z)$. As an application, we get the following relationship between the second variation of logarithmic energy function and the Weil-Petersson metric if the harmonic map $u(z)$ is holomorphic or anti-holomorphic and totally geodesic, i.e.,

$$\sqrt{-1}\partial\bar{\partial} \log E(z) = \frac{\omega_{WP}}{2\pi(g-1)}. \tag{0.1}$$

We consider also the energy function $E(z)$ associated to the harmonic maps from a fixed compact Kähler manifold M to Riemann surfaces $\{\mathcal{X}_z\}_{z\in\mathcal{T}}$ in a fixed homotopy class. If $u(z)$ is holomorphic or anti-holomorphic, then (0.1) is also proved.
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RÉSUMÉ

Nous étudions les fonctions harmoniques $u(z) : \mathcal{X}_z \to N$ dans la classe homotopie fixée des surfaces Riemann $\mathcal{X}_z$ de genre $g \geq 2$ variant dans l'espace Teichmüller $\mathcal{T}$ à une variété riemannienne N de courbure Hermitienne sectionnelle non-positive. La fonctionnelle $E(z) = E(u(z))$ peut être considerée comme une application sur $\mathcal{T}$ et nous étudions sa premiére et le seconde variation. Nous montrons que la fonctionnelle d'énergie reciproque $E(z)^{-1}$ est plurisubharmonique. De plus, nous obtenons la (strictement) plurisubharmonicité des $\log E(z)$ et $E(z)$. Comme application, nous obtenons la relation entre la seconde variation de la fontionnelle d'énergie logarithmique et la métrique Weil-Petersson si la function harmonique $u(z)$ est holomorphe ou anti-holomorphe et totalement géodésique, i.e.

☆ Research by Inkang Kim is partially supported by Grant NRF-2019R1A2C1083865 and KIAS Individual Grant (MG031408), and Xueyuan Wan is supported by KIAS Individual Grant (MG075801) at Korea Institute for Advanced Study. Genkai Zhang is partially supported by Swedish Research Council (VR), 2018-03402.
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https://doi.org/10.1016/j.matpur.2020.01.009
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$$\sqrt{-1}\partial\bar{\partial}\log E(z) = \frac{\omega_{WP}}{2\pi(g-1)}. \tag{0.1}$$

Nous considérons la fontionnelle d'énergie $E(z)$ associée aux applications harmoniques de la variété Kählérienne compacte fixé aux surfaces Riemann $\{\mathcal{X}_z\}_{z\in\mathcal{T}}$ dans la classe homotopie fixée. Si $u(z)$ est holomorphe ou anti-holomorphe, (0.1) est obtenu encore.

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Contents

Introduction

Recently, the Weil-Petersson metric and other Kähler metrics on Teichmüller space of a surface have been studied extensively [14,15,24,32]. The Weil-Petersson metric has several interesting properties; it is Kähler [1], incomplete [4,30], geodesically convex [29] and negatively curved [27,31], and the energy function of harmonic maps between Riemann surfaces is a Kähler potential of it [8,28].

In this paper we consider the log-plurisubharmonicity and the plurisuperharmonicity of reciprocal energy function of harmonic maps between Riemann surfaces and a general Riemannian manifold, and we compare the second variation with the Weil-Petersson metric. Plurisubharmonicity of $\log E(z)$ and $E(z)$ gives mapping class group invariant Kähler metrics on Teichmüller space. We also show that for some cases, these metrics agree with the Weil-Petersson metric up to scaling. See Theorems 0.6, 0.8. Hence we introduce new way of constructing Kähler metrics on Teichmüller space. By varying Riemannian manifolds, it would be interesting to see which kinds of Kähler metrics would arise.

Let Σ be a Riemann surface of genus $g \geq 2$ equipped with hyperbolic metric, and M and N Riemannian manifolds. There are two kinds of harmonic maps whose variations are of interests, the maps $u : M \to \Sigma$ and maps $u : \Sigma \to N$. The primary examples of the first kind are closed geodesics in Σ viewed as harmonic maps from the circle to Σ. Now as the hyperbolic metric varies in the Teichmüller space $\mathcal{T}$ we get Riemann surfaces $\mathcal{X}_z$ and the geodesic length can be viewed as a function of $z \in \mathcal{T}$, the variation formulas of the geodesic length function, i.e., the energy function, have been obtained in Axelsson and Schumacher's formulas [2,3]. In a recent paper [11] we find general variational formulas for harmonic maps $u : M \to \mathcal{X}_z$ and we prove the logarithmic plurisubharmonicity of the energy function, thus generalizing the results in [2,3]. The another kind of harmonic maps $u : \Sigma \to N$ appear also naturally in the study of rigidity [25] and in Hitchin components [13]. If N is also a negatively curved Riemann surface Tromba [26] showed that this energy function is strictly plurisubharmonic. For variational formulas and (strict) convexity of energy functions between surfaces, see [12]. When N has non-positive Hermitian sectional curvature, Toledo [25] proved that the energy function is also plurisubharmonic. A natural question is whether the logarithm of energy function is also plurisubharmonic. In this paper, we give an affirmative answer to this question.

Let (N, g) be a Riemannian manifold with non-positive Hermitian sectional curvature (see Definition 2.4). In particular N has non-positive sectional curvature. Let $\mathcal{T}$ be Teichmüller space of a surface of genus $g \geq 2$, and $\pi : \mathcal{X} \to \mathcal{T}$ Teichmüller curve over $\mathcal{T}$, namely it is the holomorphic family of Riemann surfaces over $\mathcal{T}$, the fiber $\mathcal{X}_z := \pi^{-1}(z)$ being exactly the Riemann surface given by the complex structure $z \in \mathcal{T}$, see e.g. [1, Section 5]. Let $u_0 : (\mathcal{X}_z, \Phi_z) \to (N, g)$ be a continuous map, where Φ_z is the hyperbolic metric on the Riemann surface $\mathcal{X}_z$. We assume that for each $z \in \mathcal{T}$, there is a unique harmonic map $u(z) : (\mathcal{X}_z, \Phi_z) \to (N, g)$ homotopic to u_0. Then we get a smooth map $u(z, v) : \mathcal{X} \to N$ and the energy

$$E(z) = E(u(z)) = \frac{1}{2} \int_{\mathcal{X}_z} |du(z)|^2 d\mu_{\Phi_z} \tag{0.2}$$

is a smooth function on Teichmüller space, see [6,19,26] for proofs of smooth dependence in several contexts.

Our first main theorem is

Theorem 0.1. *Let (N, g) be a Riemannian manifold with non-positive Hermitian sectional curvature and fix a smooth map $u_0 : \Sigma \to N$. If there is a unique harmonic map $u(z) : \mathcal{X}_z \to N$ in the homotopy class $[u_0]$ for each $z \in \mathcal{T}$, then the reciprocal energy function $E(z)^{-1}$ is plurisuperharmonic.*

Note that the uniqueness assumption is typically satisfied. For instance, if (N, g) has strictly negative sectional curvature, then the harmonic map is unique unless its image is either a point or a closed geodesic [9]. If N is a locally symmetric space of non-compact type, then u is also unique unless $u_*(\pi_1(\Sigma))$ is centralized by a semi-simple element in the group of isometries of the universal cover of N [21].

We also obtain the strictly plurisubharmonicity of $\log E(z)$. More precisely,

Theorem 0.2. *Under the conditions of Theorem 0.1, the logarithm of energy function $\log E(z)$ of $u(z) : \mathcal{X}_z \to N$ is plurisubharmonic. Moreover, if (N, g) has strictly negative Hermitian sectional curvature and $d(u(z_0))$ is never zero on $\mathcal{X}_{z_0}$ for some $z_0 \in \mathcal{T}$, then $\log E(z)$ is strictly plurisubharmonic at z_0.*

As a corollary, we obtain the following result of Toledo.

Corollary 0.3 *([25, Theorem 1, 3]). Under the conditions of Theorem 0.1, the energy function $E(z)$ is plurisubharmonic. Moreover, if (N, g) has strictly negative Hermitian sectional curvature and $d(u(z_0))$ is never zero on $\mathcal{X}_{z_0}$ for some $z_0 \in \mathcal{T}$, then $E(z)$ is strictly plurisubharmonic at z_0.*

The (strict) plurisubharmonicity of energy function is proved in [25] by using a formula of Micallef-Moore [16]. More precisely, let D be a small disk in $\mathbb{C}$ centered at 0, and let $J = J(s, t)$ be a family of complex structures on Σ compatible with the orientation and depending holomorphically on the complex parameter $z = s + \sqrt{-1}t \in D$. Then $E(z) = E(s, t) = E(J(s, t))$, and the complex variation can be obtained from the real variation, i.e.,

$$\Delta E(0) = \frac{\partial^2 E}{\partial s^2}|_{z=0} + \frac{\partial^2 E}{\partial t^2}|_{z=0},$$

where $\Delta = 4\partial_z \partial_{\bar{z}}$. The family $J = J(s, t)$ satisfies certain Cauchy-Riemann equations [25] and the variation can be computed in terms of J. Our method is completely different from Toledo's. We shall treat the energy function as the push-forward of a differential form on Teichmüller curve $\mathcal{X}$, by using the canonical decomposition of the holomorphic cotangent bundle $T^*\mathcal{X}$, and we obtain a precise and somewhat more concrete formula on the second variation of the function.

We proceed to explain further details of our results and methods. Let $u(z) := (\mathcal{X}_z, \Phi_z) \to (N, g)$ be a family of harmonic maps considered as a smooth map $u : \mathcal{X} \to N$. Let $(z; v) = (z^1, \cdots, z^m; v)$ be local holomorphic coordinates of $\mathcal{X}$ with $\pi(z,v)=z$, where (z) denotes the local coordinates of $\mathcal{T}$ and (v) denotes the local coordinate of Riemann surface $\mathcal{X}_z$, $m=3g-3=\dim_{\mathbb{C}}\mathcal{T}$. Note that $du\in A^1(\mathcal{X},u^*TN)$ can be decomposed as

$$du=\partial u+\bar\partial u\in A^1(\mathcal{X},u^*TN),$$

where ∂u denotes the $(1,0)$-component of du and $\bar\partial u=\overline{\partial u}$ denotes the $(0,1)$-component of du. Let $\langle\partial u\wedge\bar\partial u\rangle$ denote the (1,1)-form on $\mathcal{X}$ obtained by combining the wedge product in $\mathcal{X}$ with the Riemannian metric $\langle\cdot,\cdot\rangle$ on u^*TN. Then the energy function $E(z)$ can be expressed as

$$E(z)=\sqrt{-1}\int\limits_{\mathcal{X}/\mathcal{T}}\langle\partial u\wedge\bar\partial u\rangle;$$

see (2.2) below. Here $\int_{\mathcal{X}/\mathcal{T}}$ denotes the integral along fibers. Then the first and the second variations of the energy function are given by

$$\partial E(z)=\sqrt{-1}\int\limits_{\mathcal{X}/\mathcal{T}}\partial\langle\partial u\wedge\bar\partial u\rangle,\quad \partial\bar\partial E(z)=\sqrt{-1}\int\limits_{\mathcal{X}/\mathcal{T}}\partial\bar\partial\langle\partial u\wedge\bar\partial u\rangle.$$

The holomorphic cotangent bundle $T^*\mathcal{X}$ has the following decomposition:

$$T^*\mathcal{X}=\mathcal{H}^*\oplus\mathcal{V}^*,$$

where $\mathcal{H}^*$ and $\mathcal{V}^*$ are defined in (1.5). By using the above decomposition, the first and second variational formulas are obtained as follows; see Subsection 1.2 for the definition of the connection ∇ and the notations.

Theorem 0.4. *The first variation of the energy is*

$$\begin{aligned}\frac{\partial E(z)}{\partial z^\alpha}&=\int\limits_{\mathcal{X}/\mathcal{T}}\sqrt{-1}\langle\partial^V u\wedge\nabla_{\frac{\delta}{\delta z^\alpha}}\bar\partial^V u\rangle\\&=-\langle A_\alpha,du\rangle,\end{aligned}$$

where $A_\alpha=A^v_{\alpha\bar v}u^j_v d\bar v\otimes\frac{\partial}{\partial x^j}\in A^1(\mathcal{X}_z,u^*TN)$, $A^v_{\alpha\bar v}=\partial_{\bar v}(-\phi_{\alpha\bar v}\phi^{v\bar v})$.

Theorem 0.5. *The second variation of energy is*

$$\frac{\partial^2E(z)}{\partial z^\alpha\partial\bar z^\beta}=-2\int\limits_{\mathcal{X}/\mathcal{T}}R\left(\frac{\partial u}{\partial v},\frac{\delta u}{\delta z^\alpha},\frac{\partial u}{\partial\bar v},\frac{\delta u}{\delta\bar z^\beta}\right)\sqrt{-1}\delta v\wedge\delta\bar v+2\int\limits_{\mathcal{X}/\mathcal{T}}\langle\nabla_{\frac{\delta}{\delta\bar z^\beta}}\partial^V u\wedge\nabla_{\frac{\delta}{\delta z^\alpha}}\bar\partial^V u\rangle.$$

By using Cauchy-Schwarz inequality, for any $\xi=\xi^\alpha\frac{\partial}{\partial z^\alpha}\in T_z\mathcal{T}$, one has

$$\left|\xi^\alpha\frac{\partial E(z)}{\partial z^\alpha}\right|^2\le E(z)\cdot\int\limits_{\mathcal{X}/\mathcal{T}}\langle\nabla_{\bar\xi^\beta\frac{\delta}{\delta\bar z^\beta}}\partial^V u\wedge\nabla_{\xi^\alpha\frac{\delta}{\delta z^\alpha}}\bar\partial^V u\rangle,$$

see Lemma 2.7. If (N,g) has non-positive Hermitian sectional curvature, then

$$\frac{\partial^2E(z)^{-1}}{\partial z^\alpha\partial\bar z^\beta}\xi^\alpha\bar\xi^\beta=-\frac{1}{E^2}\left(\frac{\partial^2E(z)}{\partial z^\alpha\partial\bar z^\beta}-\frac{2}{E}\frac{\partial E(z)}{\partial z^\alpha}\frac{\partial E(z)}{\partial\bar z^\beta}\right)\xi^\alpha\bar\xi^\beta\le0,$$

which completes the proof of Theorem 0.1. By using the following two identities

$$\sqrt{-1}\partial\bar{\partial}\log E = -E\sqrt{-1}\partial\bar{\partial}E^{-1} + E^{-2}\sqrt{-1}\partial E \wedge \bar{\partial}E$$

and

$$\sqrt{-1}\partial\bar{\partial}E = E\sqrt{-1}\partial\bar{\partial}\log E + E^{-1}\sqrt{-1}\partial E \wedge \bar{\partial}E,$$

we obtain the plurisubharmonicity of $\log E(z)$ and $E(z)$.

As applications we find that the second variation of the logarithmic energy function is related to the Weil-Petersson metric. More precisely

Theorem 0.6. *Let (N, h) be a Hermitian manifold and fix a smooth map $u_0 : \Sigma_g \to N$. If there is a unique harmonic map $u(z) : (\mathcal{X}_z, \Phi_z) \to (N, g = Re\ h)$ in the homotopy class $[u_0]$ for each $z \in \mathcal{T}$, moreover $u(z_0)$ is holomorphic (resp. anti-holomorphic) and totally geodesic on $\mathcal{X}_{z_0}$, then*

$$\sqrt{-1}\partial\bar{\partial}\log E(z)|_{z=z_0} = \frac{\omega_{WP}}{2\pi(g-1)}.$$

Corollary 0.7 *([8, Theorem 2.6]). If $u(z_0) = Id : (\mathcal{X}_{z_0}, \Phi_{z_0}) \to (\mathcal{X}_{z_0}, \Phi_{z_0})$ is identity, then*

$$\sqrt{-1}\partial\bar{\partial}E(z)|_{z=z_0} = 2\omega_{WP}.$$

In [11], we considered a general Riemannian manifold M, harmonic maps $u : M \to \mathcal{X}_z$, and showed that both $\log E(z)$ and $E(z)$ are strictly plurisubharmonic on Teichmüller space. Consequently we constructed many possible mapping class group invariant Kähler metrics on $\mathcal{T}$. For some special cases, these Kähler metrics agree with the Weil-Petersson metric up to scaling.

We consider also the harmonic maps $u : M \to \mathcal{X}_z$ from a Riemannian manifold M to Riemann surfaces $(\mathcal{X}_z, \Phi_z)$ as in [11], but with further assumption that M is a Kähler.

The energy function $E(z)$ is again defined on Teichmüller space [11,34]. We show that the variation is again related to the Weil-Petersson metric.

Theorem 0.8. *Let (M, ω_g) be a compact Kähler manifold and fix a smooth map $u_0 : M \to \Sigma$, let $E(z)$ denote the energy function of harmonic maps from (M, g) to $(\mathcal{X}_z, \Phi_z)$ in the class $[u_0]$, where g is the Riemannian metric associated to ω_g. If $u(z_0)$ is holomorphic or anti-holomorphic for some $z_0 \in \mathcal{T}$, then*

$$\sqrt{-1}\partial\bar{\partial}\log E(z)|_{z=z_0} = \frac{\omega_{WP}}{2\pi(g(\Sigma)-1)},$$

where $g(\Sigma)$ is the genus of Σ.

As a corollary, we obtain

Corollary 0.9. *If M is a Riemann surface, and $u(z_0)$ is holomorphic or anti-holomorphic, then*

$$\sqrt{-1}\partial\bar{\partial}E(z)|_{z=z_0} = |\deg u(z_0)| \cdot 2\omega_{WP}.$$

Here $\deg u(z_0)$ is the degree of $u(z_0)$.

In particular, if $u(z_0)$ is the identity map, then

$$\sqrt{-1}\partial\bar{\partial}E(z)|_{z=z_0} = 2\omega_{WP},$$

which was proved by M. Wolf [28, Theorem 5.7].

Remark 0.10. In many situations, the harmonic maps are ± holomorphic (i.e. holomorphic or anti-holomorphic) automatically. For example,

(i) (Eells and Wood [5]) Let X and Y be compact Riemann surfaces and f a harmonic map from X to Y with respect to some Kähler metrics. If f satisfies the following condition then f is ± holomorphic:

$$e(X) + |\deg f \cdot e(Y)| > 0$$

where $e(X)$ and $e(Y)$ are the Euler numbers of X and Y respectively and $\deg(f)$ is the degree of the map $f : X \to Y$.

(ii) (Ono [17]) If (M^n, ω) is a compact Kähler manifold with negative first Chern class and satisfies

$$n|f^*c_1(N) \cdot c_1(M)^{n-1}[M]| > |c_1(M)^n[M]|,$$

and f is a harmonic from M to a compact hyperbolic Riemann N, then f is ± holomorphic.

(iii) (Siu [22]) Let M and N be compact Kähler manifolds and assume that N has strongly negative curvature in the sense of Siu. Let f be a harmonic map from M to N with respect to the Kähler metrics. If there is a point in M where the rank of df is greater than or equal to four, then f is ± holomorphic.

(iv) (Siu and Yau [23]) Let (M, h) be a compact Kähler manifold of dimension $n \geq 2$ with positive holomorphic bisectional curvature. Then any energy minimizing map $f : \mathbb{P}^1 \to M$ must be ± holomorphic.

This article is organized as follows. In Section 1, we fix notations and recall some basic facts on Teichmüller curve and harmonic maps. In Section 2, we compute the first and the second variations of the energy function (0.2) and prove Theorem 0.4, 0.5. In Subsection 2.3 we show the plurisuperharmonicity of reciprocal energy and prove Theorem 0.1, 0.2 and Corollary 0.3. In the last two sections, we study the relationship between the energy function and the Weil-Petersson metric, and prove Theorem 0.6, 0.8 and Corollary 0.7, 0.9.

Acknowledgment

Part of this work was done while X. Wan and G. Zhang were visiting KIAS during June 2018 with G. Zhang as a KIAS Scholar. We thank KIAS for its support and for providing excellent working conditions.

1. Preliminaries

In this section, we shall fix the notations and recall some basic facts on Teichmüller curve and harmonic maps. The results in this section are well-known.

1.1. Teichmüller curve

Let $\mathcal{T}$ be Teichmüller space of a fixed surface of genus $g \geq 2$. Let $\pi : \mathcal{X} \to \mathcal{T}$ be Teichmüller curve over $\mathcal{T}$, namely the holomorphic family of Riemann surfaces over $\mathcal{T}$, the fiber $\mathcal{X}_z := \pi^{-1}(z)$ being exactly the Riemann surface given by the complex structure $z \in \mathcal{T}$; see e.g. [1, Section 5]. Denote by

$$(z; v) = (z^1, \cdots, z^m; v)$$

local holomorphic coordinates of $\mathcal{X}$ with $\pi(z, v) = z$, where $z = (z^1, \cdots z^m)$ denotes local coordinates of $\mathcal{T}$ and v local coordinate of $\mathcal{X}_z$, $m = 3g - 3 = \dim_{\mathbb{C}} \mathcal{T}$. Let $K_{\mathcal{X}/\mathcal{T}}$ denote the relative canonical line bundle over $\mathcal{X}$, so $K_{\mathcal{X}/\mathcal{T}}|_{\mathcal{X}_z} = K_{\mathcal{X}_z}$. The fibers $\mathcal{X}_z$ are equipped with hyperbolic metric

$$\sqrt{-1}\phi_{v\bar{v}}dv \wedge d\bar{v}$$

depending smoothly on the parameter z and having negative constant curvature -1, namely,

$$\partial_v \partial_{\bar{v}} \log \phi_{v\bar{v}} = \phi_{v\bar{v}}, \tag{1.1}$$

where $\phi_{v\bar{v}} := \partial_v \partial_{\bar{v}} \phi$. From (1.1), up to a scaling function on $\mathcal{T}$ a metric (weight) ϕ on $K_{\mathcal{X}/\mathcal{T}}$ can be chosen such that

$$e^{\phi} = \phi_{v\bar{v}}. \tag{1.2}$$

For convenience, we denote

$$\phi_\alpha := \frac{\partial \phi}{\partial z^\alpha}, \quad \phi_{\bar{\beta}} := \frac{\partial \phi}{\partial \bar{z}^\beta}, \quad \phi_v := \frac{\partial \phi}{\partial v}, \quad \phi_{\bar{v}} := \frac{\partial \phi}{\partial \bar{v}},$$

where $1 \leq \alpha, \beta \leq m$, as well as $\phi_{\alpha\bar{v}}$, $\phi_{\alpha\bar{\beta}}$ in a similar way. With respect to the $(1,1)$-form $\sqrt{-1}\partial\bar{\partial}\phi$, we have a canonical horizontal-vertical decomposition of $T\mathcal{X}$, $T\mathcal{X} = \mathcal{H} \oplus \mathcal{V}$, where

$$\mathcal{H} = \text{Span}\left\{ \frac{\delta}{\delta z^\alpha} = \frac{\partial}{\partial z^\alpha} + a^v_\alpha \frac{\partial}{\partial v}, 1 \leq \alpha \leq m \right\}, \quad \mathcal{V} = \text{Span}\left\{ \frac{\partial}{\partial v} \right\}, \tag{1.3}$$

where

$$a^v_\alpha = -\phi_{\alpha\bar{v}}\phi^{v\bar{v}}, \tag{1.4}$$

and $\phi^{v\bar{v}} = (\phi_{v\bar{v}})^{-1}$. By duality, $T^*\mathcal{X} = \mathcal{H}^* \oplus \mathcal{V}^*$, where

$$\mathcal{H}^* = \text{Span}\left\{ dz^\alpha, 1 \leq \alpha \leq m \right\}, \quad \mathcal{V}^* = \text{Span}\left\{ \delta v = dv - a^v_\alpha dz^\alpha \right\}. \tag{1.5}$$

Moreover, the differential operators

$$\partial^V = \frac{\partial}{\partial v} \otimes \delta v, \quad \partial^H = \frac{\delta}{\delta z^\alpha} \otimes dz^\alpha, \quad \bar{\partial}^V = \frac{\partial}{\partial \bar{v}} \otimes \delta\bar{v}, \quad \bar{\partial}^H = \frac{\delta}{\delta \bar{z}^\alpha} \otimes d\bar{z}^\alpha$$

are well-defined and satisfy

$$d = \partial + \bar{\partial}, \quad \partial = \partial^H + \partial^V, \quad \bar{\partial} = \bar{\partial}^V + \bar{\partial}^H$$

when acting on smooth functions of $\mathcal{X}$. The following two lemmas can be proved by direct computations.

Lemma 1.1 *([7, Lemma 1.1]). The $(1,1)$-form $\sqrt{-1}\partial\bar{\partial}\phi$ on $\mathcal{X}$ has the following horizontal-vertical decomposition:*

$$\sqrt{-1}\partial\bar{\partial}\phi = c(\phi) + \sqrt{-1}\phi_{v\bar{v}}\delta v \wedge \delta\bar{v},$$

where $c(\phi) = \sqrt{-1}c(\phi)_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^\beta$, $c(\phi)_{\alpha\bar{\beta}} = \phi_{\alpha\bar{\beta}} - \phi^{v\bar{v}}\phi_{\alpha\bar{v}}\phi_{v\bar{\beta}}$.

Lemma 1.2. *For any smooth function f on $\mathcal{X}$ we have*

$$\partial\bar{\partial}f = (f_{\alpha\bar{\beta}} + f_{\alpha\bar{v}}\overline{a^v_\beta} + f_{v\bar{\beta}}a^v_\alpha + f_{v\bar{v}}a^v_\alpha\overline{a^v_\beta})dz^\alpha \wedge d\bar{z}^\beta + f_{v\bar{v}}\delta v \wedge \delta\bar{v}$$
$$+ \frac{\delta}{\delta z^\alpha}\left(\frac{\partial f}{\partial \bar{v}}\right) dz^\alpha \wedge \delta\bar{v} + \frac{\delta}{\delta \bar{z}^\beta}\left(\frac{\partial f}{\partial v}\right) \delta v \wedge d\bar{z}^\beta.$$

Consider the following tensor

$$\bar\partial^V \frac{\delta}{\delta z^\alpha} = (\partial_{\bar v} a^v_\alpha)\frac{\partial}{\partial v} \otimes \delta \bar v \in A^0(\mathcal{X}, \mathrm{End}(\mathcal{V})). \tag{1.6}$$

We denote its component and its dual with respect to the metric $\sqrt{-1}\phi_{v\bar v}\delta v \wedge \delta \bar v$ as

$$A^v_{\alpha\bar v} = \partial_{\bar v} a^v_\alpha = \partial_{\bar v}(-\phi^{v\bar v}\phi_{\alpha\bar v}), \quad A_{\alpha\bar v\bar v} = A^v_{\alpha\bar v}\phi_{v\bar v}. \tag{1.7}$$

Lemma 1.3 (i) below shows that its restriction to each fiber is a harmonic element representing the Kodaira-Spencer class $\rho(\frac{\partial}{\partial z^\alpha})$,

$$\rho : T_z\mathcal{T} \to H^1(\mathcal{X}_z, T_{\mathcal{X}_z})$$

being the Kodaira-Spencer map.

Lemma 1.3. *[18, Proposition 2, 3] The following identities hold:*

(i) $\partial_v A_{\alpha\bar v\bar v} = 0$;
(ii) $(\Box + 1)c(\phi)_{\alpha\bar\beta} = A^v_{\alpha\bar v}A^{\bar v}_{\bar\beta v}$ *where* $\Box = -\phi^{v\bar v}\partial_v\partial_{\bar v}$.

Definition 1.4. The Weil-Petersson metric ω_{WP} on Teichmüller space $\mathcal{T}$ is defined by

$$\omega_{WP} = \sqrt{-1}G_{\alpha\bar\beta}dz^\alpha \wedge d\bar z^\beta, \quad G_{\alpha\bar\beta}(z) = \int_{\mathcal{X}_z} A^v_{\alpha\bar v}\overline{A^v_{\beta\bar v}}\sqrt{-1}\phi_{v\bar v}dv \wedge d\bar v. \tag{1.8}$$

By Lemma 1.3 (ii) and Stokes' theorem, the Weil-Petersson metric can also be expressed as

$$G_{\alpha\bar\beta}(z) = \int_{\mathcal{X}_z} c(\phi)_{\alpha\bar\beta}\sqrt{-1}\phi_{v\bar v}dv \wedge d\bar v. \tag{1.9}$$

1.2. Harmonic maps from Riemann surfaces to a Riemannian manifold

Let

$$\Phi_z = \phi_{v\bar v}(dv \otimes d\bar v + d\bar v \otimes dv)$$

denote the Riemannian metric on $\mathcal{X}_z$ associated to the fundamental $(1,1)$-form

$$\sqrt{-1}\phi_{v\bar v}dv \wedge d\bar v = \sqrt{-1}\phi_{v\bar v}(dv \otimes d\bar v - d\bar v \otimes dv).$$

Let $T\mathcal{X}_z$ be the holomorphic tangent bundle of $\mathcal{X}_z$ and $T_{\mathbb{C}}\mathcal{X}_z = T\mathcal{X}_z \oplus \overline{T\mathcal{X}_z}$ the complex tangent bundle. For any smooth map $u : (\mathcal{X}_z, \Phi_z) \to (N, g)$ the differential du is a section of the bundle $T^*_{\mathbb{C}}\mathcal{X}_z \otimes u^*TN$. Let $\{x^i\}_{1\le i\le \dim N}$ denote a local coordinate system of N and v a local complex coordinate on $\mathcal{X}_z$. Then $du \in A^0(\mathcal{X}_z, T^*_{\mathbb{C}}\mathcal{X}_z \otimes u^*TN)$ is locally expressed as

$$du = \frac{\partial u^i}{\partial v}dv \otimes \frac{\partial}{\partial x^i} + \frac{\partial u^i}{\partial \bar v}d\bar v \otimes \frac{\partial}{\partial x^i}.$$

The energy density and the energy are defined by

$$
\begin{aligned}
|du|^2 &:= (du, du) = 2g_{ij}u_v^i u_{\bar{v}}^j \phi^{v\bar{v}}, \\
E(u) &:= \frac{1}{2}\|du\|^2 := \frac{1}{2}\int_{\mathcal{X}_z} |du|^2 d\mu_{\Phi_z} \\
&= \int_{\mathcal{X}_z} (g_{ij}u_v^i u_{\bar{v}}^j \phi^{v\bar{v}})\sqrt{-1}\phi_{v\bar{v}} dv \wedge d\bar{v} \\
&= \int_{\mathcal{X}_z} g_{ij}u_v^i u_{\bar{v}}^j \sqrt{-1} dv \wedge d\bar{v},
\end{aligned} \tag{1.10}
$$

where $u_v^i := \frac{\partial u^i}{\partial v}$ and $d\mu_{\Phi_z} = \sqrt{-1}\phi_{v\bar{v}} dv \wedge d\bar{v}$ is the Riemannian volume form of Φ_z. The harmonic equation is

$$
\partial_{\bar{v}} u_v^i + \Gamma_{jk}^i u_v^j u_{\bar{v}}^k = 0; \tag{1.11}
$$

see e.g. [33, (1.2.10)]. Here

$$
\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij}\right)
$$

denotes the Christoffel symbols on (N, g).

Let $\{u(z)\}_{z\in\mathcal{T}}$ be a smooth family of harmonic maps $u(z) : (\mathcal{X}_z, \Phi_z) \to (N, g)$, $z \in \mathcal{T}$. We shall treat it as a smooth map u,

$$
u : \mathcal{X} \to N, \quad (z, v) \mapsto u(z, v) := (u(z))(v).
$$

Note that $\mathcal{V}^*$ defined in (1.5) is a holomorphic line bundle over $\mathcal{X}$ with holomorphic frame $\{\delta v\}$, which is equipped with a Hermitian metric ($\phi_{v\bar{v}}^{-1} = e^{-\phi}$), thus there is a natural induced connection ∇ on $\mathcal{V}^* \otimes u^*TN$ from the Chern connection of $\mathcal{V}^*$ and the Levi-Civita connection of TN, i.e. for $X \in T_{\mathbb{C}}X$,

$$
\nabla_X : A^0(\mathcal{X}, \mathcal{V}^* \otimes u^*TN) \to A^1(\mathcal{X}, \mathcal{V}^* \otimes u^*TN).
$$

By conjugation, we obtain a connection ∇ on $\overline{\mathcal{V}}^* \otimes u^*TN$. More precisely for any $f = f_v^i \delta v \otimes \frac{\partial}{\partial x^i} + f_{\bar{v}}^i \delta\bar{v} \otimes \frac{\partial}{\partial x^i} \in A^0(\mathcal{X}, (\mathcal{V}^* \oplus \overline{\mathcal{V}}^*) \otimes u^*TN)$ and any vector $X \in T_{\mathbb{C}}\mathcal{X}$,

$$
\nabla_X f = (\nabla_X f_v^i)\delta v \otimes \frac{\partial}{\partial x^i} + (\nabla_X f_{\bar{v}}^i)\delta\bar{v} \otimes \frac{\partial}{\partial x^i},
$$

where

$$
\nabla_X f_v^i := X(f_v^i) + \Gamma_{kl}^i f_v^k X(u^l) - (\partial\phi)(X) f_v^i
$$

and

$$
\nabla_X f_{\bar{v}}^i := X(f_{\bar{v}}^i) + \Gamma_{kl}^i f_{\bar{v}}^k X(u^l) - (\bar{\partial}\phi)(X) f_{\bar{v}}^i.
$$

Denote $\nabla_v := \nabla_{\frac{\partial}{\partial v}}$, $\nabla_{\bar{v}} := \nabla_{\frac{\partial}{\partial \bar{v}}}$ for notational convenience. In particular for $u_v^i \delta v \otimes \frac{\partial}{\partial x^i} \in A^0(\mathcal{X}, \mathcal{V}^* \otimes u^*TN)$ we have

$$
\nabla_{\bar{v}}(u_v^i \delta v \otimes \frac{\partial}{\partial x^i}) = (\nabla_{\bar{v}} u_v^i)\delta v \otimes \frac{\partial}{\partial x^i}, \quad \nabla_{\bar{v}} u_v^i := \partial_{\bar{v}} u_v^i + \Gamma_{jk}^i u_v^j u_{\bar{v}}^k.
$$

By (1.11), u is a harmonic map if and only if

$$\nabla_{\bar{v}} u_v^i = 0. \tag{1.12}$$

2. Variations of energy on Teichmüller space

In this section we will calculate the first and the second variations of the energy $E(u(z))$ for harmonic maps $u(z) : \mathcal{X}_z \to N$. Fix a smooth map $u_0 : \Sigma \to N$ from a surface Σ of genus g to N. We assume that $u : \mathcal{X} \to N$ is a smooth map such that $u(z) : \mathcal{X}_z \to N$ is a harmonic map in the homotopy class $[u_0]$. Then the following function

$$E(z) := E(u(z))$$

is smooth on Teichmüller space $\mathcal{T}$.

2.1. The first variation

The differential du of a smooth map

$$u : \mathcal{X} \to N, \quad (z, v) \mapsto u(z, v),$$

is

$$du = \partial u + \bar{\partial} u \in A^1(\mathcal{X}, u^*TN),$$

with

$$\partial u := \partial u^i \otimes \frac{\partial}{\partial x^i} = (u_z^i dz + u_v^i dv) \otimes \frac{\partial}{\partial x^i} \in A^{1,0}(\mathcal{X}, u^*TN)$$

the $(1, 0)$-component of du, and $\bar{\partial} u = \overline{\partial u}$ the $(0, 1)$-component of du. Let

$$\langle \partial u \wedge \bar{\partial} u \rangle = g_{ij}(u(z, v)) \partial u^i \wedge \bar{\partial} u^j \in A^{1,1}(\mathcal{X}) \tag{2.1}$$

denote the two-form on $\mathcal{X}$ obtained by combining the wedge product in $\mathcal{X}$ with the Riemannian metric $\langle , \rangle$ on u^*TN. The corresponding the energy $E(z)$ function (1.10) can be written then as

$$E(z) = \sqrt{-1} \int\limits_{\mathcal{X}/\mathcal{T}} \langle \partial u \wedge \bar{\partial} u \rangle. \tag{2.2}$$

Here we view $\int_{\mathcal{X}/\mathcal{T}}$ as the integral along fibers (see e.g. [18, Section 2.1]),

$$\int\limits_{\mathcal{X}/\mathcal{T}} : A^{2+k}(\mathcal{X}) \to A^k(\mathcal{T});$$

moreover ∂, $\bar{\partial}$-operators commute with $\int_{\mathcal{X}/\mathcal{T}}$.

The variations of $E(z)$ are

$$\partial E(z) = \sqrt{-1} \int\limits_{\mathcal{X}/\mathcal{T}} \partial \langle \partial u \wedge \bar{\partial} u \rangle, \quad \partial\bar{\partial} E(z) = \sqrt{-1} \int\limits_{\mathcal{X}/\mathcal{T}} \partial\bar{\partial} \langle \partial u \wedge \bar{\partial} u \rangle. \tag{2.3}$$

Note that $\partial\langle\partial u\wedge\bar\partial u\rangle \in A^3(\mathcal{X})$, which can be decomposed in terms of the frame $\wedge^3\{dz^\alpha, d\bar z^\beta, \delta v, \delta\bar v\}$ obtained from the basis vectors. Denote by $\left[\partial\langle\partial u\wedge\bar\partial u\rangle\right]^{(\delta v\wedge\delta\bar v)}$ the component of $\partial\langle\partial u\wedge\bar\partial u\rangle$ containing $\delta v\wedge\delta\bar v$. We shall need the following

Lemma 2.1. *The $\delta v\wedge\delta\bar v$-component is*

$$\left[\partial\langle\partial u\wedge\bar\partial u\rangle\right]^{(\delta v\wedge\delta\bar v)} = \langle\partial^V u\wedge\nabla_{\frac{\delta}{\delta z^\alpha}}\bar\partial^V u\rangle\wedge dz^\alpha,$$

where

$$\partial^V u := u^i_v\delta v\otimes\frac{\partial}{\partial x^i}\in A^0(\mathcal{X},\mathcal{V}^*\otimes u^*TN)\subset A^{1,0}(\mathcal{X},u^*TN)$$

and

$$\bar\partial^V u := u^i_{\bar v}\delta\bar v\otimes\frac{\partial}{\partial x^i}\in A^0(\mathcal{X},\overline{\mathcal{V}}^*\otimes u^*TN)\subset A^{0,1}(\mathcal{X},u^*TN).$$

Proof. For any fixed point $(z_0,v_0)\in\mathcal{X}$, we choose a normal coordinate system $\{x^i\}$ around $u(z_0,v_0)$ such that

$$g_{ij}(u(z_0,v_0)) = \delta_{ij},\quad (dg_{ij})(u(z_0,v_0)) = 0. \tag{2.4}$$

From (1.11), one has $\partial_{\bar v}u^i_v(z_0,v_0)=0$. Lemma 1.2 implies that

$$\partial\bar\partial u^i = (u_{\alpha\bar\beta}+u_{\alpha\bar v}\overline{a^v_\beta}+u_{v\bar\beta}a^v_\alpha)dz^\alpha\wedge d\bar z^\beta + \frac{\delta}{\delta z^\alpha}u^i_{\bar v}dz^\alpha\wedge\delta\bar v + \frac{\delta}{\delta\bar z^\beta}u^i_v\delta v\wedge d\bar z^\beta. \tag{2.5}$$

Thus at $(z_0,v_0)\in\mathcal{X}$, one has

$$\begin{aligned}\left[\partial\langle\partial u\wedge\bar\partial u\rangle\right]^{(\delta v\wedge\delta\bar v)} &= \left[\partial g_{ij}\wedge\partial u^i\wedge\bar\partial u^j - g_{ij}\partial u^i\wedge\partial\bar\partial u^j\right]^{(\delta v\wedge\delta\bar v)}\\ &= \left[-\partial u^i\wedge\partial\bar\partial u^i\right]^{(\delta v\wedge\delta\bar v)}\\ &= \left[-\partial u^i\wedge(\frac{\delta}{\delta z^\alpha}u^i_{\bar v})dz^\alpha\wedge\delta\bar v\right]^{(\delta v\wedge\delta\bar v)}\\ &= g_{ij}u^i_v\overline{\nabla_{\frac{\delta}{\delta\bar z^\alpha}}u^j_v}\delta v\wedge\delta\bar v\wedge dz^\alpha\\ &= \langle\partial^V u\wedge\nabla_{\frac{\delta}{\delta z^\alpha}}\bar\partial^V u\rangle\wedge dz^\alpha.\end{aligned}$$

Since both $\left[\partial\langle\partial u\wedge\bar\partial u\rangle\right]^{(\delta v\wedge\delta\bar v)}$ and $\langle\partial^V u\wedge\nabla_{\frac{\delta}{\delta z^\alpha}}\bar\partial^V u\rangle\wedge dz^\alpha$ are globally defined, independent of the normal coordinate system $\{x^i\}$, and the point (z_0,v_0) is arbitrary,

$$\left[\partial\langle\partial u\wedge\bar\partial u\rangle\right]^{(\delta v\wedge\delta\bar v)} = \langle\partial^V u\wedge\nabla_{\frac{\delta}{\delta z^\alpha}}\bar\partial^V u\rangle\wedge dz^\alpha$$

on $\mathcal{X}$. □

Theorem 2.2. *The first variation of the energy is*

$$\frac{\partial E(z)}{\partial z^\alpha} = \int_{\mathcal{X}/\mathcal{T}}\sqrt{-1}\langle\partial^V u\wedge\nabla_{\frac{\delta}{\delta z^\alpha}}\bar\partial^V u\rangle.$$

Proof. By (2.3) and Lemma 2.1,

$$\begin{aligned}\partial E(z) &= \int\limits_{\mathcal{X}/\mathcal{T}} \sqrt{-1}\partial\langle \partial u \wedge \bar{\partial} u\rangle \\ &= \int\limits_{\mathcal{X}/\mathcal{T}} \sqrt{-1}\left[\partial\langle \partial u \wedge \bar{\partial} u\rangle\right]^{(\delta v\wedge\delta\bar{v})} \\ &= \left(\int\limits_{\mathcal{X}/\mathcal{T}} \sqrt{-1}\langle \partial^V u \wedge \nabla_{\frac{\delta}{\delta z^\alpha}} \bar{\partial}^V u\rangle\right) dz^\alpha,\end{aligned}$$

which completes the proof. □

Now we will give another formula on the first variation of the energy function. Denote

$$A_\alpha = A^v_{\alpha\bar{v}} u^j_v d\bar{v} \otimes \frac{\partial}{\partial x^j} \in A^1(\mathcal{X}_z, u^*TN), \quad A^v_{\alpha\bar{v}} = \partial_{\bar{v}}(-\phi_{\alpha\bar{v}}\phi^{v\bar{v}}). \tag{2.6}$$

Then

$$\begin{aligned}\langle A_\alpha, du\rangle &= \langle A^v_{\alpha\bar{v}} u^j_v d\bar{v} \otimes \frac{\partial}{\partial x^j}, u^i_v dv \otimes \frac{\partial}{\partial x^i} + u^i_{\bar{v}} d\bar{v} \otimes \frac{\partial}{\partial x^i}\rangle \\ &= \int\limits_{\mathcal{X}_z} g_{ij} u^i_v u^j_v A^v_{\alpha\bar{v}} \sqrt{-1} dv \wedge d\bar{v}.\end{aligned}$$

Theorem 2.3. *The first variation of the energy function is*

$$\frac{\partial E(z)}{\partial z^\alpha} = -\langle A_\alpha, du\rangle.$$

Proof. From Theorem 2.2 we find

$$\begin{aligned}\frac{\partial E(z)}{\partial z^\alpha} &= \int\limits_{\mathcal{X}/\mathcal{T}} \sqrt{-1}\langle \partial^V u \wedge \nabla_{\frac{\delta}{\delta z^\alpha}} \bar{\partial}^V u\rangle \\ &= \int\limits_{\mathcal{X}_z} g_{ij} u^i_v \nabla_{\frac{\delta}{\delta z^\alpha}} u^j_{\bar{v}} \sqrt{-1} dv \wedge d\bar{v} \\ &= \int\limits_{\mathcal{X}_z} g_{ij} u^i_v \left(\nabla_{\bar{v}} \frac{\delta u^j}{\delta z^\alpha} - A^v_{\alpha\bar{v}} u^j_v\right) \sqrt{-1} dv \wedge d\bar{v} \\ &= -\int\limits_{\mathcal{X}_z} g_{ij} u^i_v u^j_v A^v_{\alpha\bar{v}} \sqrt{-1} dv \wedge d\bar{v} \\ &= -\langle A_\alpha, du\rangle,\end{aligned}$$

where the fourth equality follows from Stokes' theorem and the harmonic equation (1.12), and the third equality holds by

$$
\begin{aligned}
\nabla_{\frac{\delta}{\delta z^\alpha}} u^j_{\bar v} &= \frac{\delta}{\delta z^\alpha} u^j_{\bar v} + \Gamma^j_{kl} u^l_{\bar v} \frac{\delta u^k}{\delta z^\alpha} \\
&= \frac{\partial}{\partial \bar v}(\phi_{\alpha\bar v}\phi^{\bar v v}) u^j_v + \frac{\partial}{\partial \bar v}\left(\frac{\delta u^j}{\delta z^\alpha}\right) + \Gamma^j_{kl} u^l_{\bar v}\frac{\delta u^k}{\delta z^\alpha} \qquad (2.7)\\
&= -A^v_{\alpha\bar v} u^j_v + \nabla_{\bar v}\frac{\delta u^j}{\delta z^\alpha}. \quad \square
\end{aligned}
$$

2.2. The second variation

We first recall the definition of Hermitian sectional curvature on a Riemannian manifold (N, g). Let ∇^N be the Levi-Civita connection of Riemannian manifold (N, g). Recall that the Riemann curvature endomorphism $R \in A^2(N, \mathrm{End}(TN))$ is

$$R(X, Y)Z = \nabla^N_X \nabla^N_Y Z - \nabla^N_Y \nabla^N_X Z - \nabla^N_{[X,Y]} Z.$$

Recall also the notation

$$R(X, Y, Z, W) = -\langle R(X, Y)Z, W\rangle,$$

and

$$R_{ikjl} := R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^l}\right).$$

By direct computations one has the known formula

$$R_{ikjl} = -\frac{1}{2}\left(\frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^j} - \frac{\partial^2 g_{kj}}{\partial x^i \partial x^l}\right) - g_{mn}\left(\Gamma^m_{ij}\Gamma^n_{kl} - \Gamma^m_{il}\Gamma^n_{kj}\right).$$

The sectional curvature is defined by

$$K(X \wedge Y) = \frac{R(X, Y, X, Y)}{\|X\|^2\|Y\|^2 - \langle X, Y\rangle^2}.$$

The Riemann curvature tensor R can be extended on the complexified tangent bundle $TN \otimes \mathbb{C}$. We recall the following curvature condition of Siu [22] and Sampson [20].

Definition 2.4 *([22,20,25]).* For any $X, Y \in TN \otimes \mathbb{C}$, the *Hermitian sectional curvature* on the plane $X \wedge Y$ is defined by

$$K_{\mathbb{C}}(X \wedge Y) := \frac{R(X, Y, \overline{X}, \overline{Y})}{\|X\|^2\|Y\|^2 - |\langle X, \overline{Y}\rangle|^2}.$$

The Riemannian manifold (N, g) is said to have *non-positive (resp. strictly negative) Hermitian sectional curvature* if

$$K_{\mathbb{C}}(X \wedge Y) \le 0 \quad (\textit{resp.} < 0)$$

for any $X, Y \in TN \otimes \mathbb{C}$ with $X \wedge Y \neq 0$.

Recall the notation $\left[\partial\bar\partial\langle\partial u \wedge \bar\partial u\rangle\right]^{(\delta v \wedge \delta \bar v)}$, the part of $\partial\bar\partial\langle\partial u \wedge \bar\partial u\rangle$ containing $\delta v \wedge \delta\bar v$. Then

Lemma 2.5. *It holds*

$$\left[\partial\bar{\partial}\langle\partial u\wedge\bar{\partial}u\rangle\right]^{(\delta v\wedge\delta\bar{v})} = -2R\left(\frac{\partial u}{\partial v},\frac{\delta u}{\delta z^\alpha},\frac{\partial u}{\partial\bar{v}},\frac{\delta u}{\delta\bar{z}^\beta}\right)\delta v\wedge\delta\bar{v}\wedge dz^\alpha\wedge d\bar{z}^\beta + 2\langle\nabla_{\frac{\delta}{\delta\bar{z}^\beta}}\partial^V u\wedge\nabla_{\frac{\delta}{\delta z^\alpha}}\bar{\partial}^V u\rangle\wedge dz^\alpha\wedge d\bar{z}^\beta,$$

where

$$\frac{\partial u}{\partial v}=du(\frac{\partial}{\partial v})=u^i_v\frac{\partial}{\partial x^i},\quad \frac{\delta u}{\delta z^\alpha}=\frac{\delta u^i}{\delta z^\alpha}\frac{\partial}{\partial x^i},\quad \frac{\partial u}{\partial\bar{v}}=u^i_{\bar{v}}\frac{\partial}{\partial x^i},\quad \frac{\delta u}{\delta\bar{z}^\beta}=\frac{\delta u^i}{\delta\bar{z}^\beta}\frac{\partial}{\partial x^i}.$$

Proof. From (2.1), one has

$$\partial\bar{\partial}\langle\partial u\wedge\bar{\partial}u\rangle = (\partial\bar{\partial}g_{ij}\wedge\partial u^i-\partial g_{ij}\wedge\partial\bar{\partial}u^i)\wedge\bar{\partial}u^j+(\bar{\partial}g_{ij}\wedge\partial u^i-g_{ij}\partial\bar{\partial}u^i)\wedge\partial\bar{\partial}u^j.$$

By taking a normal coordinates system $\{x^i\}$ around $u(z_0,v_0)$ for any fixed point $(z_0,v_0)\in\mathcal{X}$ as in (2.4), we get that, at the point (z_0,v_0),

$$\partial\bar{\partial}\langle\partial u\wedge\bar{\partial}u\rangle = \partial\bar{\partial}g_{ij}\wedge\partial u^i\wedge\bar{\partial}u^j-(\partial\bar{\partial}u^i)^2.$$

By (2.5) we have further

$$\begin{aligned}
&\left[\partial\bar{\partial}\langle\partial u\wedge\bar{\partial}u\rangle\right]^{(\delta v\wedge\delta\bar{v})}\\
&=\left[\partial\bar{\partial}g_{ij}\wedge\partial u^i\wedge\bar{\partial}u^j-(\partial\bar{\partial}u^i)^2\right]^{(\delta v\wedge\delta\bar{v})}\\
&=\left[(\partial_k\partial_l g_{ij})\partial u^k\wedge\bar{\partial}u^l\wedge\partial u^i\wedge\bar{\partial}u^j-(\frac{\delta}{\delta z^\alpha}u^i_{\bar{v}}dz^\alpha\wedge\delta\bar{v}+\frac{\delta}{\delta\bar{z}^\beta}u^i_v\delta v\wedge d\bar{z}^\beta)^2\right]^{(\delta v\wedge\delta\bar{v})}\\
&=\Bigg((\partial_k\partial_l g_{ij})(\frac{\delta u^k}{\delta z^\alpha}\frac{\delta u^l}{\delta\bar{z}^\beta}u^i_v u^j_{\bar{v}}-\frac{\delta u^k}{\delta z^\alpha}\frac{\delta u^j}{\delta\bar{z}^\beta}u^i_v u^l_{\bar{v}}\\
&\quad-\frac{\delta u^i}{\delta z^\alpha}\frac{\delta u^l}{\delta\bar{z}^\beta}u^k_v u^j_{\bar{v}}+\frac{\delta u^i}{\delta z^\alpha}\frac{\delta u^j}{\delta\bar{z}^\beta}u^k_v u^l_{\bar{v}})+2(\frac{\delta}{\delta z^\alpha}u^i_{\bar{v}})(\frac{\delta}{\delta\bar{z}^\beta}u^i_v)\Bigg)dz^\alpha\wedge d\bar{z}^\beta\wedge\delta v\wedge\delta\bar{v}\\
&=2\left(-R_{ikjl}\frac{\delta u^k}{\delta z^\alpha}\frac{\delta u^l}{\delta\bar{z}^\beta}u^i_v u^j_{\bar{v}}+g_{ij}\nabla_{\frac{\delta}{\delta\bar{z}^\beta}}u^j_v\overline{\nabla_{\frac{\delta}{\delta\bar{z}^\alpha}}u^i_v}\right)\delta v\wedge\delta\bar{v}\wedge dz^\alpha\wedge d\bar{z}^\beta\\
&=-2R\left(\frac{\partial u}{\partial v},\frac{\delta u}{\delta z^\alpha},\frac{\partial u}{\partial\bar{v}},\frac{\delta u}{\delta\bar{z}^\beta}\right)\delta v\wedge\delta\bar{v}\wedge dz^\alpha\wedge d\bar{z}^\beta\\
&\quad+2\langle\nabla_{\frac{\delta}{\delta\bar{z}^\beta}}\partial^V u\wedge\nabla_{\frac{\delta}{\delta z^\alpha}}\bar{\partial}^V u\rangle\wedge dz^\alpha\wedge d\bar{z}^\beta,
\end{aligned}$$

where the second equality follows from (2.5) and note that $[\partial\bar{\partial}u^i]^{(\delta v\wedge\delta\bar{v})}=0$ at the point (z_0,v_0), the fourth equality holds since

$$\nabla_{\frac{\delta}{\delta\bar{z}^\beta}}u^j_v=\frac{\delta}{\delta\bar{z}^\beta}u^j_v+\Gamma^j_{kl}u^k_v\frac{\delta u^l}{\delta\bar{z}^\beta}=\frac{\delta}{\delta\bar{z}^\beta}u^j_v$$

at the point (z_0,v_0) and

$$R_{ikjl}=-\frac{1}{2}\left(\frac{\partial^2 g_{ij}}{\partial x^k\partial x^l}+\frac{\partial^2 g_{kl}}{\partial x^i\partial x^j}-\frac{\partial^2 g_{il}}{\partial x^k\partial x^j}-\frac{\partial^2 g_{kj}}{\partial x^i\partial x^l}\right)$$

at the point $u(z_0,v_0)$. Since the point (z_0,v_0) is arbitrary, we complete the proof. □

Theorem 2.6. *The second variation of the energy is*

$$\frac{\partial^2 E(z)}{\partial z^\alpha \partial \bar{z}^\beta} = 2 \int\limits_{\mathcal{X}/\mathcal{T}} -R\left(\frac{\partial u}{\partial v}, \frac{\delta u}{\delta z^\alpha}, \frac{\partial u}{\partial \bar{v}}, \frac{\delta u}{\delta \bar{z}^\beta}\right)\sqrt{-1}\delta v \wedge \delta \bar{v} + 2 \int\limits_{\mathcal{X}/\mathcal{T}} \langle \nabla_{\frac{\delta}{\delta \bar{z}^\beta}} \partial^V u \wedge \nabla_{\frac{\delta}{\delta z^\alpha}} \bar{\partial}^V u \rangle.$$

Proof. By (2.3) and Lemma 2.5 we have

$$\begin{aligned}
\partial\bar{\partial}E(z) &= \sqrt{-1} \int\limits_{\mathcal{X}/\mathcal{T}} \partial\bar{\partial}\langle \partial u \wedge \bar{\partial} u\rangle \\
&= \sqrt{-1} \int\limits_{\mathcal{X}/\mathcal{T}} \left[\partial\bar{\partial}\langle \partial u \wedge \bar{\partial} u\rangle\right]^{(\delta v \wedge \delta \bar{v})} \\
&= 2 \int\limits_{\mathcal{X}/\mathcal{T}} \left(-R\left(\frac{\partial u}{\partial v}, \frac{\delta u}{\delta z^\alpha}, \frac{\partial u}{\partial \bar{v}}, \frac{\delta u}{\delta \bar{z}^\beta}\right) \delta v \wedge \delta \bar{v} \wedge dz^\alpha \wedge d\bar{z}^\beta \right. \\
&\quad \left. + \langle \nabla_{\frac{\delta}{\delta \bar{z}^\beta}} \partial^V u \wedge \nabla_{\frac{\delta}{\delta z^\alpha}} \bar{\partial}^V u \rangle \wedge dz^\alpha \wedge d\bar{z}^\beta \right) \\
&= 2 \int\limits_{\mathcal{X}/\mathcal{T}} \left(-R\left(\frac{\partial u}{\partial v}, \frac{\delta u}{\delta z^\alpha}, \frac{\partial u}{\partial \bar{v}}, \frac{\delta u}{\delta \bar{z}^\beta}\right)\sqrt{-1} \delta v \wedge \delta \bar{v} \right. \\
&\quad \left. + \langle \nabla_{\frac{\delta}{\delta \bar{z}^\beta}} \partial^V u \wedge \nabla_{\frac{\delta}{\delta z^\alpha}} \bar{\partial}^V u \rangle \right) \cdot dz^\alpha \wedge d\bar{z}^\beta,
\end{aligned}$$

as claimed. □

2.3. Plurisuperharmonicity

In this subsection, we will prove the strict plurisubharmonicity of logarithmic energy function $\log E(z)$ and plurisuperharmoncity for the reciprocal energy function $E(z)^{-1}$.

Firstly, we will show the reciprocal energy function $E(z)^{-1}$ is plurisuperharmonic.

Lemma 2.7. *For any* $\xi = \xi^\alpha \frac{\partial}{\partial z^\alpha} \in T_z\mathcal{T}$ *it holds*

$$\left|\xi^\alpha \frac{\partial E(z)}{\partial z^\alpha}\right|^2 \leq E(z) \cdot \int\limits_{\mathcal{X}/\mathcal{T}} \langle \nabla_{\bar{\xi}^\beta \frac{\delta}{\delta \bar{z}^\beta}} \partial^V u \wedge \nabla_{\xi^\alpha \frac{\delta}{\delta z^\alpha}} \bar{\partial}^V u \rangle.$$

Proof. This follows directly from Theorem 2.2 and Cauchy-Schwarz inequality:

$$\begin{aligned}
\left|\xi^\alpha \frac{\partial E(z)}{\partial z^\alpha}\right|^2 &= \left|\xi^\alpha \int\limits_{\mathcal{X}/\mathcal{T}} \sqrt{-1}\langle \partial^V u \wedge \nabla_{\frac{\delta}{\delta z^\alpha}} \bar{\partial}^V u \rangle \right|^2 \\
&= \left| \int\limits_{\mathcal{X}_z} g_{ij} u_v^i \overline{\nabla_{\bar{\xi}^\alpha \frac{\delta}{\delta \bar{z}^\alpha}} u_v^j} \sqrt{-1} dv \wedge d\bar{v} \right|^2 \\
&\leq \int\limits_{\mathcal{X}_z} g_{ij} u_v^i u_{\bar{v}}^j \sqrt{-1} dv \wedge d\bar{v} \cdot \int\limits_{\mathcal{X}_z} g_{ij} \nabla_{\bar{\xi}^\alpha \frac{\delta}{\delta \bar{z}^\alpha}} u_v^i \overline{\nabla_{\bar{\xi}^\alpha \frac{\delta}{\delta \bar{z}^\alpha}} u_v^j} \sqrt{-1} dv \wedge d\bar{v}
\end{aligned}$$

$$= E(z) \cdot \int\limits_{\mathcal{X}/\mathcal{T}} \langle \nabla_{\bar{\xi}^\beta \frac{\delta}{\delta \bar{z}^\beta}} \partial^V u \wedge \nabla_{\xi^\alpha \frac{\delta}{\delta z^\alpha}} \bar{\partial}^V u \rangle. \quad \square$$

Theorem 2.8. *If (N, g) has non-positive Hermitian sectional curvature, then the function $E(z)^{-1}$ is plurisuperharmonic, i.e.*

$$\sqrt{-1}\partial\bar{\partial}E(z)^{-1} \leq 0.$$

Proof. For any vector $\xi = \xi^\alpha \frac{\partial}{\partial z^\alpha} \in T_z\mathcal{T}$,

$$\frac{\partial^2 E(z)^{-1}}{\partial z^\alpha \partial \bar{z}^\beta} \xi^\alpha \bar{\xi}^\beta = -\frac{1}{E^2} \left(\frac{\partial^2 E(z)}{\partial z^\alpha \partial \bar{z}^\beta} - \frac{2}{E} \frac{\partial E(z)}{\partial z^\alpha} \frac{\partial E(z)}{\partial \bar{z}^\beta} \right) \xi^\alpha \bar{\xi}^\beta. \tag{2.8}$$

The first term above can be treated using Theorem 2.6,

$$\begin{aligned} \frac{\partial^2 E(z)}{\partial z^\alpha \partial \bar{z}^\beta} \xi^\alpha \bar{\xi}^\beta &= 2 \int\limits_{\mathcal{X}/\mathcal{T}} -R\left(\frac{\partial u}{\partial v}, \xi^\alpha \frac{\delta u}{\delta z^\alpha}, \frac{\partial u}{\partial \bar{v}}, \bar{\xi}^\beta \frac{\delta u}{\delta \bar{z}^\beta} \right) \sqrt{-1}\delta v \wedge \delta \bar{v} \\ &\quad + 2 \int\limits_{\mathcal{X}/\mathcal{T}} \langle \nabla_{\bar{\xi}^\beta \frac{\delta}{\delta \bar{z}^\beta}} \partial^V u \wedge \nabla_{\xi^\alpha \frac{\delta}{\delta z^\alpha}} \bar{\partial}^V u \rangle \\ &\geq 2 \int\limits_{\mathcal{X}/\mathcal{T}} \langle \nabla_{\bar{\xi}^\beta \frac{\delta}{\delta \bar{z}^\beta}} \partial^V u \wedge \nabla_{\xi^\alpha \frac{\delta}{\delta z^\alpha}} \bar{\partial}^V u \rangle \end{aligned} \tag{2.9}$$

by the non-positivity of Hermitian sectional curvature. Furthermore Lemma 2.7 implies that

$$\frac{\partial^2 E(z)}{\partial z^\alpha \partial \bar{z}^\beta} \xi^\alpha \bar{\xi}^\beta \geq \frac{2}{E} \left| \xi^\alpha \frac{\partial E(z)}{\partial z^\alpha} \right|^2 = \frac{2}{E} \frac{\partial E(z)}{\partial z^\alpha} \frac{\partial E(z)}{\partial \bar{z}^\beta} \xi^\alpha \bar{\xi}^\beta. \tag{2.10}$$

Substituting (2.10) into (2.8), one has

$$\frac{\partial^2 E(z)^{-1}}{\partial z^\alpha \partial \bar{z}^\beta} \xi^\alpha \bar{\xi}^\beta \leq 0.$$

Thus

$$\sqrt{-1}\partial\bar{\partial}E(z)^{-1} = \frac{\partial^2 E(z)^{-1}}{\partial z^\alpha \partial \bar{z}^\beta} \sqrt{-1} dz^\alpha \wedge d\bar{z}^\beta \leq 0. \quad \square$$

Next, using Theorem 2.8 we get the following (strict) plurisubharmonicity of logarithmic energy $\log E(z)$.

Theorem 2.9. *If (N, g) has non-positive Hermitian sectional curvature, then the logarithmic energy function $\log E(z)$ is plurisubharmonic on Teichmüller space $\mathcal{T}$, i.e.*

$$\sqrt{-1}\partial\bar{\partial} \log E(z) \geq 0.$$

Moreover, if (N, g) has strictly negative Hermitian sectional curvature and $d(u(z))$ is never zero on $\mathcal{X}_z$, then $\log E(z)$ is strictly plurisubharmonic, i.e.

$$\sqrt{-1}\partial\bar{\partial} \log E(z) > 0.$$

Proof. From Theorem 2.8, we get

$$\sqrt{-1}\partial\bar{\partial}\log E(z) = -E(z)\sqrt{-1}\partial\bar{\partial}E(z)^{-1} + E(z)^{-2}\sqrt{-1}\partial E(z)\wedge\bar{\partial}E(z) \geq 0, \tag{2.11}$$

which yields the plurisubharmonicity of $\log E(z)$. To prove the strict plurisubharmonicity we let $\xi = \xi^\alpha \frac{\partial}{\partial z^\alpha} \in T_z\mathcal{T}$ such that

$$\frac{\partial^2 \log E(z)}{\partial z^\alpha \partial \bar{z}^\beta}\xi^\alpha\bar{\xi}^\beta = 0.$$

Then, in view of (2.9-2.11),

$$R\left(\frac{\partial u}{\partial v}, \xi^\alpha \frac{\delta u}{\delta z^\alpha}, \frac{\partial u}{\partial \bar{v}}, \bar{\xi}^\beta \frac{\delta u}{\delta \bar{z}^\beta}\right) = 0, \quad \nabla_{\xi^\alpha \frac{\delta}{\delta z^\alpha}}\bar{\partial}^V u = 0.$$

If (N, g) has strictly negative Hermitian sectional curvature, then

$$\frac{\partial u}{\partial v}\wedge \xi^\alpha \frac{\delta u}{\delta z^\alpha} = 0, \quad \nabla_{\xi^\alpha \frac{\delta}{\delta z^\alpha}}\bar{\partial}^V u = 0. \tag{2.12}$$

Since

$$d(u(z)) = u^i_v dv \otimes \frac{\partial}{\partial x^i} + \overline{u^i_v} d\bar{v} \otimes \frac{\partial}{\partial x^i}$$

is never zero on $\mathcal{X}_z$, so u^i_v is also never zero. From the first equation of (2.12), there exists a vector filed $W = W^v \frac{\partial}{\partial v} \in A^0(\mathcal{X}_z, T\mathcal{X}_z)$ such that

$$\xi^\alpha \frac{\delta u^i}{\delta z^\alpha} = W^v u^i_v.$$

The second equation of (2.12) is

$$\begin{aligned} 0 &= \nabla_{\xi^\alpha \frac{\delta}{\delta z^\alpha}}\bar{\partial}^V u \\ &= \xi^\alpha \left(\nabla_{\frac{\delta}{\delta z^\alpha}} u^i_{\bar{v}}\right)\delta\bar{v}\otimes\frac{\partial}{\partial x^i} \\ &= \xi^\alpha \left(\nabla_{\bar{v}}\frac{\delta u^i}{\delta z^\alpha} - A^v_{\alpha\bar{v}}u^i_v\right)\delta\bar{v}\otimes\frac{\partial}{\partial x^i} \\ &= \left(\nabla_{\bar{v}}W^v u^i_v - \xi^\alpha A^v_{\alpha\bar{v}}u^i_v\right)\delta\bar{v}\otimes\frac{\partial}{\partial x^i} \\ &= \left(\partial_{\bar{v}}W^v - \xi^\alpha A^v_{\alpha\bar{v}}\right)u^i_v\delta\bar{v}\otimes\frac{\partial}{\partial x^i}, \end{aligned}$$

where the third equality follows from (2.7) and the last equality follows from harmonic equation $\nabla_{\bar{v}}u^i_v = 0$. Thus

$$\xi^\alpha A^v_{\alpha\bar{v}}d\bar{v}\otimes\frac{\partial}{\partial v} = \bar{\partial}W \in A^{0,1}(\mathcal{X}_z, T\mathcal{X}_z).$$

This implies that

$$\rho\left(\xi^\alpha\frac{\partial}{\partial z^\alpha}\right) = \left[\xi^\alpha A^v_{\alpha\bar{v}}d\bar{v}\otimes\frac{\partial}{\partial v}\right] = [\bar{\partial}W] = 0 \in H^1(\mathcal{X}_z, T\mathcal{X}_z).$$

Since $\rho : T_z\mathcal{T} \to H^1(\mathcal{X}_z, T\mathcal{X}_z)$ is injective, so $\xi = 0$. This proves the strict plurisubharmonicity. □

The following result was obtained by D. Toledo [25, Theorem 1, 3].

Corollary 2.10 *([25, Theorem 1, 3]). If (N, g) has non-positive Hermitian sectional curvature, then the energy function $E(z)$ is plurisubharmonic on Teichmüller space $\mathcal{T}$. Moreover, if (N, g) has strictly negative Hermitian sectional curvature and $d(u(z))$ is never zero on $\mathcal{X}_z$, then $E(z)$ is strictly plurisubharmonic.*

Proof. Note that

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial}E(z) &= E(z)\sqrt{-1}\partial\bar{\partial}\log E(z) + E(z)^{-1}\sqrt{-1}\partial E(z)\wedge\bar{\partial}E(z)\\ &\geq E(z)\sqrt{-1}\partial\bar{\partial}\log E(z).\end{aligned}$$

The claim follows immediately from Theorem 2.9. □

We give another application of our results on the variation of the energy function in the context of Hitchin representations. Let $\Gamma = \pi_1(\Sigma)$ be the fundamental group of a closed surface Σ of genus $g \geq 2$. Let G be a real semisimple Lie group and consider the space of all reductive representations $\rho : \Gamma \to G$ of Γ in G modulo the conjugations by elements in G. It can be identified with a subset in G^{2g-2} modulo the diagonal action of G. When G is a split real form of a complex semisimple Lie group there is a distinguished component [10] called Hitchin component. Given any reductive representation ρ of Γ and given a hyperbolic structure on Σ, i.e., given a point z in the Teichmüller space $\mathcal{T}$, there is a $\rho(\Gamma)$-equivariant harmonic map $u : \mathbb{H}^2 \to G/K$ from the hyperbolic plane $\mathbb{H}^2$ to the Riemannian symmetric space G/K, the map is unique up to the action of G. In particular the energy function $E_\rho(z) = E(u) = \int_{\mathcal{X}_z} |du|^2$ is well-defined. When G is $SL(n, \mathbb{R})$ it is conjectured by Labourie that for each element ρ in the Hitchin component there is a unique minimizing point of $E_\rho(z)$ in the Teichmüller space $\mathcal{T}$. Recall [20] that the Riemannian symmetric space has non-positive Hermitian curvature. We have thus

Corollary 2.11. *Let ρ be a reductive representation of Γ in G. The energy function $E_\rho(z)$ is plurisubharmonic on $\mathcal{T}$.*

It might be interesting to pursue the study of Labourie's conjecture using our variational formulas.

3. Energy functions and potentials of Weil-Petersson metric

We assume in this section that N is a complex manifold with a Hermitian metric h. It turns out that in this case there is a close relation between the second variation of the energy of $u(z) : \mathcal{X}_z \to N$ and Weil-Petersson metric.

Let $\{s^i\}_{1\leq i\leq \dim_{\mathbb{C}} N}$ be a local holomorphic coordinates system of N. The Riemannian metric $g = \mathrm{Re}\, h$ is

$$g = g_{i\bar{j}}(ds^i \otimes d\bar{s}^j + d\bar{s}^j \otimes ds^i)$$

where

$$g_{i\bar{j}} = g\left(\frac{\partial}{\partial s^i}, \frac{\partial}{\partial \bar{s}^j}\right).$$

Then $\omega = -\mathrm{Im}\, h$ is a two form so that $h = g - \sqrt{-1}\omega$, and

$$g_{jk} = g_{\bar{j}\bar{k}} = 0, \quad g_{j\bar{k}} = g_{\bar{k}j}, \quad g_{\bar{j}k} = \bar{g}_{j\bar{k}}.$$

For any smooth map $u : (\mathcal{X}_z, \Phi_z) \to (N, g)$,

$$du = u^i_v dv \otimes \frac{\partial}{\partial s^i} + u^i_{\bar v} d\bar v \otimes \frac{\partial}{\partial s^i} + \overline{u^j_{\bar v}} dv \otimes \frac{\partial}{\partial \bar s^j} + \overline{u^j_v} d\bar v \otimes \frac{\partial}{\partial \bar s^j} \in A^1(\mathcal{X}_z, u^*TN).$$

Hence

$$|du|^2 = \phi^{v\bar v} g_{i\bar j}(u^i_v \overline{u^j_v} + u^i_{\bar v}\overline{u^j_{\bar v}} + \overline{u^j_{\bar v}} u^i_{\bar v} + \overline{u^j_v} u^i_v) = 2\phi^{v\bar v} g_{i\bar j}(u^i_v \overline{u^j_v} + u^i_{\bar v}\overline{u^j_{\bar v}}). \tag{3.1}$$

So the energy is given by

$$E(u) = \int_{\mathcal{X}_z} \frac{1}{2}|du|^2 \sqrt{-1}\phi_{v\bar v} dv \wedge d\bar v = \int_{\mathcal{X}_z} g_{i\bar j}(u^i_v \overline{u^j_v} + u^i_{\bar v}\overline{u^j_{\bar v}})\sqrt{-1} dv \wedge d\bar v. \tag{3.2}$$

Now we assume that $u : \mathcal{X} \to N$ is a smooth map such that each $u(z) : \mathcal{X}_z \to N$ is a harmonic map, and $u(z_0)$ is a holomorphic map (and is then harmonic by the definition (1.11)). For notational convenience we write $z_0 = o$. Then $E(z) = E(u(z))$ is a smooth map on Teichmüller map $\mathcal{T}$ and from Theorem 2.3 we have

$$\begin{aligned}\frac{\partial E(z)}{\partial z^\alpha} &= -\langle A_\alpha, du\rangle \\ &= -\langle A^v_{\alpha\bar v} u^i_v d\bar v \otimes \frac{\partial}{\partial s^i} + A^v_{\alpha\bar v}\overline{u^j_{\bar v}} d\bar v \otimes \frac{\partial}{\partial \bar s^j}, du\rangle \\ &= -2\int_{\mathcal{X}_z} g_{i\bar j} u^i_v \overline{u^j_{\bar v}} A^v_{\alpha\bar v}\sqrt{-1} dv \wedge d\bar v.\end{aligned}$$

Evaluating at $o \in \mathcal{T}$ and using $u(o)$ is holomorphic we get

$$\frac{\partial E(z)}{\partial z^\alpha}|_{z=o} = 0. \tag{3.3}$$

The second variation of energy at the point $o \in \mathcal{T}$ is

$$\begin{aligned}\frac{\partial^2 E(z)}{\partial z^\alpha \partial \bar z^\beta}|_o &= -2\int_{\mathcal{X}_o} g_{i\bar j} u^i_v \frac{\partial}{\partial \bar z^\beta}\overline{u^j_{\bar v}} A^v_{\alpha\bar v}\sqrt{-1} dv \wedge d\bar v \\ &= -2\int_{\mathcal{X}_o} g_{i\bar j} u^i_v \overline{\nabla_{\frac{\delta}{\delta z^\beta}} u^j_{\bar v}} A^v_{\alpha\bar v}\sqrt{-1} dv \wedge d\bar v \\ &= -2\int_{\mathcal{X}_o} g_{i\bar j} u^i_v \overline{\nabla_{\bar v}\frac{\delta u^j}{\delta z^\beta} - A^v_{\beta\bar v} u^j_v} A^v_{\alpha\bar v}\sqrt{-1} dv \wedge d\bar v \\ &= 2\int_{\mathcal{X}_o} g_{i\bar j} u^i_v \overline{u^j_v} A^v_{\alpha\bar v}\overline{A^v_{\beta\bar v}}\sqrt{-1} dv \wedge d\bar v \\ &\quad + 2\int_{\mathcal{X}_o} g_{i\bar j}\nabla_v(u^i_v) A^v_{\alpha\bar v}\overline{\frac{\delta u^j}{\delta z^\beta}}\sqrt{-1} dv \wedge d\bar v,\end{aligned} \tag{3.4}$$

where the first equality follows from the holomorphicity of $u(o)$, the second equality follows from harmonic equation (1.12) and the definition of horizontal subbundle (1.3), the third equality holds by (2.7), and the last equality holds by Stokes' theorem and Lemma 1.3 (i),

$$\nabla_v A^v_{\alpha\bar v} = \nabla_v(A_{\alpha\bar v\bar v}\phi^{\bar v v}) = \partial_v A_{\alpha\bar v\bar v}\phi^{\bar v v} = 0.$$

Here

$$\nabla_v u_v^i = \partial_v u_v^i + \Gamma_{kl}^i u_v^k u_v^l - \phi_v u_v^i.$$

Since $u(o) : \mathcal{X}_z \to N$ is holomorphic,

$$d(u(o)) = u_v^i(o) dv \otimes \frac{\partial}{\partial x^i} \in A^0(\mathcal{X}_z, T^*\mathcal{X}_z \otimes u(o)^* TN).$$

Let ∇ denote the metric connection on the bundle $T^*\mathcal{X}_z \otimes u(o)^*TN$ induced from the Chern connection of $(T^*\mathcal{X}_z, e^{-\phi})$ and the pullback of Levi-Civita connection (N, g). By conjugation we get a connection on $\overline{T^*\mathcal{X}_z} \otimes u(o)^*TN$, denoted also by ∇. Then

$$\begin{aligned}\nabla d(u(o)) &= \left(\partial_v u_v^i(o) + \Gamma_{kl}^i u_v^k(o) u_v^l(o) - \phi_v u_v^i(o)\right) dv \otimes dv \otimes \frac{\partial}{\partial x^i} \\ &= (\nabla_v u_v^i)(o) dv \otimes dv \otimes \frac{\partial}{\partial x^i}.\end{aligned} \tag{3.5}$$

Now we assume further that $u(z) : \mathcal{X}_o \to N$ is totally geodesic (see e.g. [33, Definition 1.2.1] for the definition), i.e.

$$\nabla d(u(o)) \equiv 0. \tag{3.6}$$

The equation (3.4) becomes

$$\frac{\partial^2 E(z)}{\partial z^\alpha \partial \bar{z}^\beta}|_{z=o} = 2 \int_{\mathcal{X}_o} g_{i\bar{j}} u_v^i \overline{u_v^j} A_{\alpha\bar{v}}^v \overline{A_{\beta\bar{v}}^v} \sqrt{-1} dv \wedge d\bar{v}. \tag{3.7}$$

By the harmonic equation $\nabla_{\bar{v}} u_v^i = 0$ and the assumption $\nabla_v u_v^i = 0$ on $\mathcal{X}_o$,

$$\nabla_v (g_{i\bar{j}} u_v^i \overline{u_v^j} \phi^{v\bar{v}}) = \frac{\partial}{\partial v}(g_{i\bar{j}} u_v^i \overline{u_v^j} \phi^{v\bar{v}}) = g_{i\bar{j}}(\nabla_v u_v^i \overline{u_v^j} + u_v^i \overline{\nabla_{\bar{v}} u_v^j}) \phi^{v\bar{v}} = 0.$$

This implies that $(g_{i\bar{j}} u_v^i \overline{u_v^j} \phi^{v\bar{v}})$ is a constant on $\mathcal{X}_o$ and it equals

$$g_{i\bar{j}} u_v^i \overline{u_v^j} \phi^{v\bar{v}}(o) = \frac{\int_{\mathcal{X}_o} (g_{i\bar{j}} u_v^i \overline{u_v^j} \phi^{v\bar{v}}) \sqrt{-1} \phi_{v\bar{v}} dv \wedge d\bar{v}}{\int_{\mathcal{X}_o} \sqrt{-1} \phi_{v\bar{v}} dv \wedge d\bar{v}} = \frac{E(o)}{2\pi(2g-2)}. \tag{3.8}$$

Substituting (3.8) into (3.7), one has

$$\begin{aligned}\frac{\partial^2 E(z)}{\partial z^\alpha \partial \bar{z}^\beta}|_{z=o} &= \frac{E(o)}{2\pi(g-1)} \int_{\mathcal{X}_o} A_{\alpha\bar{v}}^v \overline{A_{\beta\bar{v}}^v} \sqrt{-1} \phi_{v\bar{v}} dv \wedge d\bar{v} \\ &= \frac{E(o)}{2\pi(g-1)} G_{\alpha\bar{\beta}}(o),\end{aligned} \tag{3.9}$$

where $G_{\alpha\bar{\beta}}$ is defined in (1.8). By (3.3), the first variation of the energy at o vanishes, so the second variation of $\log E$ satisfies

$$\sqrt{-1}\partial\bar{\partial} \log E(z)|_{z=o} = \frac{1}{E(o)} \sqrt{-1}\partial\bar{\partial} E(z)|_{z=o} = \frac{1}{2\pi(g-1)} \omega_{WP}. \tag{3.10}$$

Namely we have

Theorem 3.1. *If $u(o)$ is holomorphic (resp. anti-holomorphic) and totally geodesic on $\mathcal{X}_o$, then*

$$\sqrt{-1}\partial\bar{\partial}\log E(z)|_{z=o} = \frac{\omega_{WP}}{2\pi(g-1)}.$$

Specifying to the case when N is also a Riemann surface we obtain Fischer and Tromba's theorem; see [8] and [28, Corollary 5.8].

Corollary 3.2 *([8, Theorem 2.6]). If $u(o) = Id : (\mathcal{X}_o, \Phi_o) \to (\mathcal{X}_o, \Phi_o)$ is identity, then*

$$\sqrt{-1}\partial\bar{\partial}E(z)|_{z=o} = 2\omega_{WP}.$$

Proof. In this case, $u(o)$ is holomorphic, $u^i_v(o) = \delta^i_v$ and

$$\Gamma^i_{jk} = \partial_v \log \phi_{v\bar{v}} = \phi_v.$$

So

$$(\nabla_v u^i_v)(o) = (\partial_v u^i_v + \Gamma^i_{kl} u^k_v u^l_v - \phi_v u^i_v)(o) = 0$$

and

$$E(o) = \int\limits_{\mathcal{X}_o} \sqrt{-1}\phi_{v\bar{v}} dv \wedge d\bar{v} = 2\pi(2g-2).$$

So the identity (3.10) becomes

$$\sqrt{-1}\partial\bar{\partial}E(z)|_{z=o} = E_0\sqrt{-1}\partial\bar{\partial}\log E(z)|_{z=o} = 2\omega_{WP}$$

completing the proof. □

We may also apply our result above, as in Section 3, to the energy function related to a reductive representation $\rho : \Gamma = \pi_1(\Sigma) \to G$. Recall the definition of the energy function $E_\rho(z)$ for a general element z in the Teichmüller space in Section 3. Now let G be a Hermitian semisimple Lie group with G/K a non-compact Hermitian symmetric space. Let $\rho : PSL(2,\mathbb{R}) \to G$ be a fixed representation with the induced totally geodesic map $\mathbb{H}^2 = PSL(2,\mathbb{R})/SO(2) \to G/K$ being holomorphic. Let $\mathcal{X}_o = \mathbb{H}^2/\Gamma_o$ be fixed Riemann surface with Γ_o a representation of Γ in $PSL(2,\mathbb{R})$. The representation ρ then defines also a representation of Γ, also denoted by ρ, i.e. $\rho : \Gamma \to \Gamma_0 \subset PSL(2,\mathbb{R}) \xrightarrow{\rho} G$. The (lifted) $\rho(\Gamma)$-equivariant map $u(z)$ for $z = o$ is then holomorphic and totally geodesic $u(o) : \mathbb{H}^2 = PSL(2,\mathbb{R})/SO(2) \to G/K$. We can compute the second variation of $E_\rho(z)$ at $z = o$.

Corollary 3.3. *Let ρ be the reductive representation of $\pi_1(\mathcal{X}_0)$ in G obtained from a representation of $PSL(2,\mathbb{R})$ in G with the totally geodesic map $\mathbb{H}^2 \to G/K$ being holomorphic. Then the second variation of $E_\rho(z)$ at $z = o$ is*

$$\sqrt{-1}\partial\bar{\partial}\log E_\rho(z)|_{z=o} = \frac{\omega_{WP}}{2\pi(g-1)}.$$

4. The second variation of the energy of $u(z) : M \to \mathcal{X}_z$ and Weil-Petersson metric

As we explained in the introduction we may also consider harmonic maps $u(z) : (M, \omega_g) \to \mathcal{X}_z$; see [11] and references therein. We assume further that (M, ω_g) is a compact Kähler manifold, i.e. ω_g is a closed and positive $(1,1)$-form. Let $\{s^i\}_{1\le i\le n}$ denote local coordinates of M, $n = \dim_{\mathbb{C}} M$. Locally, ω_g can be expressed as

$$\omega_g = \sqrt{-1} g_{i\bar{j}} ds^i \wedge d\bar{s}^j$$

for some positive definite hermitian matrix $(g_{i\bar{j}})$. The associated Riemannian metric g is given by

$$g = g_{i\bar{j}}(ds^i \otimes d\bar{s}^j + d\bar{s}^j \otimes ds^i).$$

For any smooth map $u(z) : (M, g) \to (\mathcal{X}_z, \Phi_z)$, du is the section of bundle $T^*M \otimes u^* T_{\mathbb{C}} \mathcal{X}_z$, for which there is an induced metric $g^* \otimes \Phi_z$ from (M^n, g) and $(\mathcal{X}_z, \Phi_z)$. Let $\{v\}$ denote a holomorphic coordinate of Riemann surface $\mathcal{X}_z$. In the same way as in (3.1), (3.2), one has

$$|du|^2 = 2g^{\bar{j}i}(u_i^v \overline{u_j^v} + u_{\bar{j}}^v \overline{u_{\bar{i}}^v})\phi_{v\bar{v}} \tag{4.1}$$

and the energy is given by

$$E(u) = \frac{1}{2}\int_M |du|^2 d\mu_g = \int_M g^{\bar{j}i}(u_i^v \overline{u_j^v} + u_{\bar{j}}^v \overline{u_{\bar{i}}^v})\phi_{v\bar{v}} d\mu_g.$$

Here

$$d\mu_g = \frac{\omega_g^n}{n!}$$

denotes Riemannian volume form determined by g. The harmonic equation is

$$g^{\bar{j}i}\nabla_i u_{\bar{j}}^v = g^{\bar{j}i}(\partial_i u_{\bar{j}}^v + \phi_v u_i^v u_{\bar{j}}^v) = 0, \tag{4.2}$$

see e.g. [11, (1.20)]. We assume that $u : M \to \mathcal{X}$ is a smooth map such that $u(z) : M \to \mathcal{X}_z$ is a harmonic map and we put $E(z) := E(u(z))$ the energy function on Teichmüller space $\mathcal{T}$. Similar to (2.6) we define (with some abuse of notation)

$$A_\alpha = A_{\alpha\bar{v}\bar{v}}\overline{u_j^v}\phi^{v\bar{v}} d\bar{s}^j \otimes \frac{\partial}{\partial v} + A_{\alpha\bar{v}\bar{v}}\overline{u_{\bar{i}}^v}\phi^{v\bar{v}} ds^i \otimes \frac{\partial}{\partial v} \in A^1(M, u^*T\mathcal{X}_z);$$

Let $\Delta = \nabla\nabla^* + \nabla^*\nabla$ be the Hodge-Laplace operator on $A^\ell(M, u^*T\mathcal{X}_z)$ (see e.g. [11, Subsection 1.2]), and set

$$\mathcal{L} = \Delta + \frac{1}{2}|du|^2, \quad \mathcal{G} = 2g^{\bar{j}i}\phi_{v\bar{v}} u_i^v u_{\bar{j}}^v \frac{\partial}{\partial v} \otimes d\bar{v} \in \mathrm{Hom}(u^*\overline{T\mathcal{X}_z}, u^*T\mathcal{X}_z).$$

Theorem 4.1 *([11, Theorem 0.5, 0.6]). The first and the second variation of the energy are given by*

$$\frac{\partial E(z)}{\partial z^\alpha} = \langle A_\alpha, du\rangle = 2\int_M A_{\alpha\bar{v}\bar{v}}\overline{u_i^v u_{\bar{j}}^v} g^{\bar{j}i} d\mu_g \tag{4.3}$$

and

$$\frac{\partial^2 E(z)}{\partial z^\alpha \partial \bar{z}^\beta} = \frac{1}{2}\int_M c(\phi)_{\alpha\bar{\beta}} |du|^2 d\mu_g + \langle (Id - \nabla \left(\mathcal{L} - \mathcal{G}\mathcal{L}^{-1}\overline{\mathcal{G}}\right)^{-1} \nabla^*) A_\alpha, A_\beta \rangle. \tag{4.4}$$

Now we assume that at $o \in \mathcal{T}$ the map $u(o)$ is holomorphic. It satisfies the harmonic equation (4.2) automatically. By (4.3), one has

$$\frac{\partial E(z)}{\partial z^\alpha}|_{z=o} = 0. \tag{4.5}$$

We recall [11, (1.22)] that

$$\begin{aligned}
\nabla^* A_\alpha &= \left(-g^{\bar{j}i} \nabla_i (A_{\alpha\bar{v}\bar{v}} \overline{u^v_j} \phi^{v\bar{v}}) - g^{\bar{j}i} \nabla_{\bar{j}} (A_{\alpha\bar{v}\bar{v}} \overline{u^v_{\bar{i}}} \phi^{v\bar{v}})\right) \frac{\partial}{\partial v} \\
&= \left(-g^{\bar{j}i} A_{\alpha\bar{v}\bar{v}} \overline{\nabla_{\bar{i}} u^v_j} - g^{\bar{j}i} u^v_i \partial_v (A_{\alpha\bar{v}\bar{v}}) \overline{u^v_j} - g^{\bar{j}i} A_{\alpha\bar{v}\bar{v}} \overline{\nabla_j u^v_{\bar{i}}}\right) \phi^{v\bar{v}} \frac{\partial}{\partial v} \\
&= 0,
\end{aligned} \tag{4.6}$$

where the second equality holds since $u^v_{\bar{i}} = 0$, the third equality follows from the harmonic equation (4.2) and Lemma 1.3 (i). Substituting (4.6) into (4.4) we find

$$\frac{\partial^2 E(z)}{\partial z^\alpha \partial \bar{z}^\beta}|_{z=o} = \frac{1}{2}\int_M c(\phi)_{\alpha\bar{\beta}} |du|^2 d\mu_g + \langle A_\alpha, A_\beta \rangle. \tag{4.7}$$

Lemma 4.2. *The following identity holds for any smooth real two form α on $\mathcal{X}_o$*

$$\int_M u^*\alpha \wedge \omega_g^{n-1} = \frac{\deg_{\omega_g}(u^* K_{\mathcal{X}_o})}{2g-2} \int_{\mathcal{X}_o} \alpha,$$

where

$$\deg_{\omega_g}(u^* K_{\mathcal{X}_o}) = \int_M u^* c_1(K_{\mathcal{X}_o}) \wedge \omega_g^{n-1}.$$

Proof. Let ω_o be the area form on $\mathcal{X}_o$ such that $\int_{\mathcal{X}_o} \omega_o = c_1(K_{\mathcal{X}_0})[\mathcal{X}_0] = 2g-2$. Then $H^2(\mathcal{X}_o, \mathbb{R}) = \mathbb{R}\omega_0$ and we need only to check the identity for ω_0. We have

$$\begin{aligned}
\int_M u^*\omega_o \wedge \omega^{n-1} &= (u^*[\omega_o][\omega]^{n-1})[M] \\
&= (u^* c_1(K_{\mathcal{X}_o})[\omega]^{n-1})[M] \\
&= \frac{\deg_{\omega_g}(u^* K_{\mathcal{X}_o})}{2g-2} \int_{\mathcal{X}_o} \omega_0. \quad \square
\end{aligned}$$

By (4.1) and holomorphicity of $u(o)$, the first term in the RHS of (4.7) is

$$\frac{1}{2}\int_M c(\phi)_{\alpha\bar{\beta}} |du|^2 d\mu_g = \int_M c(\phi)_{\alpha\bar{\beta}} (g^{i\bar{j}} \phi_{v\bar{v}} u^v_i \overline{u^v_j}) \frac{\omega_g^n}{n!}$$

$$
\begin{aligned}
&= \int_M u^*(c(\phi)_{\alpha\bar{\beta}}\sqrt{-1}\phi_{v\bar{v}}dv \wedge d\bar{v}) \wedge \frac{\omega_g^{n-1}}{(n-1)!} \\
&= \frac{1}{(n-1)!}\frac{\deg_{\omega_g}(u^*K_{\mathcal{X}_o})}{2g-2}G_{\alpha\bar{\beta}},
\end{aligned} \tag{4.8}
$$

where the last equality follows from Lemma 4.2 and (1.9), the second equality follows from holomorphicity of $u(o)$ and the following elementary fact that

$$
n\alpha \wedge \omega_g^{n-1} = (tr_{\omega_g}\alpha)\omega_g^n, \tag{4.9}
$$

for any $(1,1)$-form $\alpha = \sqrt{-1}\alpha_{i\bar{j}}ds^i \wedge d\bar{s}^j$ with $tr_{\omega_g}\alpha := g^{i\bar{j}}\alpha_{i\bar{j}}$, $\omega_g = \sqrt{-1}g_{i\bar{j}}ds^i \wedge d\bar{s}^j$.

Similarly, by (4.9) the second term in the RHS of (4.7) is

$$
\begin{aligned}
\langle A_\alpha, A_\beta\rangle &= \int_M (A^v_{\alpha\bar{v}}\overline{A^v_{\beta\bar{v}}u^v_j}u^v_i g^{i\bar{j}}\phi_{v\bar{v}})\frac{\omega_g^n}{n!} \\
&= \int_M u^*(A^v_{\alpha\bar{v}}\overline{A^v_{\beta\bar{v}}}\sqrt{-1}\phi_{v\bar{v}}dv \wedge d\bar{v}) \wedge \frac{\omega_g^{n-1}}{(n-1)!} \\
&= \frac{1}{(n-1)!}\frac{\deg_{\omega_g}(u^*K_{\mathcal{X}_o})}{2g-2}G_{\alpha\bar{\beta}}.
\end{aligned} \tag{4.10}
$$

Substituting (4.8) and (4.10) into (4.7) we have

$$
\frac{\partial^2 E(z)}{\partial z^\alpha \partial \bar{z}^\beta}|_{z=o} = \frac{1}{(n-1)!}\frac{\deg_{\omega_g}(u^*K_{\mathcal{X}_o})}{g-1}G_{\alpha\bar{\beta}}. \tag{4.11}
$$

The energy for $u(o)$ is now

$$
\begin{aligned}
E(o) &= \int_M g^{\bar{j}i}u^v_i\overline{u^v_j}\phi_{v\bar{v}}\frac{\omega_g^n}{n!} \\
&= \int_M u^*(\sqrt{-1}\phi_{v\bar{v}}dv \wedge d\bar{v}) \wedge \frac{\omega_g^{n-1}}{(n-1)!} \\
&= 2\pi\int_M u^*c_1(K_{\mathcal{X}_o}) \wedge \frac{\omega_g^{n-1}}{(n-1)!} \\
&= \frac{2\pi}{(n-1)!}\deg_{\omega_g}(u^*K_{\mathcal{X}_o}).
\end{aligned} \tag{4.12}
$$

Therefore the second variation of $\log E(z)$ at $z = o$, in view of (4.5)-(4.11)-(4.12) above, is

$$
\begin{aligned}
\frac{\partial^2 \log E(z)}{\partial z^\alpha \partial \bar{z}^\beta}|_{z=o} &= \left(\frac{1}{E(z)}\frac{\partial^2 E(z)}{\partial z^\alpha \partial \bar{z}^\beta} - \frac{1}{E(z)^2}\frac{\partial E(z)}{\partial z^\alpha}\frac{\partial E(z)}{\partial \bar{z}^\beta}\right)|_{z=o} \\
&= \frac{1}{2\pi(g-1)}G_{\alpha\bar{\beta}}.
\end{aligned} \tag{4.13}
$$

Similarly, for anti-holomorphic map $u(o)$, we also can get (4.13). Thus

Theorem 4.3. *If $u(o)$ is a holomorphic or anti-holomorphic map, then*

$$\sqrt{-1}\partial\bar{\partial}\log E(z)|_{z=o} = \frac{\omega_{WP}}{2\pi(g-1)}.$$

As a corollary, we obtain

Corollary 4.4. *If M is a Riemann surface, and $u(o)$ is holomorphic or anti-holomorphic, then*

$$\sqrt{-1}\partial\bar{\partial}E(z)|_{z=o} = |\deg u(o)| \cdot 2\omega_{WP},$$

where $\deg u(o)$ *is the degree of* $u(o)$.

Proof. If M is a Riemann surface, from (4.11)

$$\begin{aligned}\frac{\partial^2 E(z)}{\partial z^\alpha \partial \bar{z}^\beta}|_{z=o} &= \frac{1}{(n-1)!}\frac{|\deg_{\omega_g}(u^* K_{\mathcal{X}_o})|}{g-1}G_{\alpha\bar{\beta}} \\ &= \frac{|\int_M u^* c_1(K_{\mathcal{X}_o})|}{g-1}G_{\alpha\bar{\beta}} \\ &= |\deg u(o)| \cdot 2G_{\alpha\bar{\beta}}.\end{aligned}$$

Thus

$$\sqrt{-1}\partial\bar{\partial}E(z)|_{z=o} = \frac{\partial^2 E(z)}{\partial z^\alpha \partial \bar{z}^\beta}|_{z=o}\sqrt{-1}dz^\alpha \wedge d\bar{z}^\beta = |\deg u(o)| \cdot 2\omega_{WP}. \quad \square$$

Remark 4.5. In particular, if $u(o)$ is the identity map, then

$$\sqrt{-1}\partial\bar{\partial}E(z)|_{z=o} = 2\omega_{WP},$$

which was proved by M. Wolf [28, Theorem 5.7].

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