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Citation for the original published paper (version of record):
Pelaez, J., Perälä, A., Raettyae, J. (2020). Hankel operators induced by radial Bekolle-Bonami weights on Bergman spaces. Mathematische Zeitschrift, 296(1-2): 211-238.
http://dx.doi.org/10.1007/s00209-019-02412-8

Mathematische Zeitschrift (2020) 296:211–238
https://doi.org/10.1007/s00209-019-02412-8

Mathematische Zeitschrift

Hankel operators induced by radial Bekollé–Bonami weights on Bergman spaces

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Received: 26 June 2018 / Accepted: 4 October 2019 / Published online: 7 November 2019

Abstract

We study big Hankel operators $H_f^\nu : A_\omega^p \to L_\nu^q$ generated by radial Bekollé–Bonami weights ν, when $1 < p \le q < \infty$. Here the radial weight ω is assumed to satisfy a two-sided doubling condition, and A_ω^p denotes the corresponding weighted Bergman space. A characterization for simultaneous boundedness of H_f^ν and $H_{\overline{f}}^\nu$ is provided in terms of a general weighted mean oscillation. Compared to the case of standard weights that was recently obtained by Pau et al. (Indiana Univ Math J 65(5):1639–1673, 2016), the respective spaces depend on the weights ω and ν in an essentially stronger sense. This makes our analysis deviate from the blueprint of this more classical setting. As a consequence of our main result, we also study the case of anti-analytic symbols.

Keywords Hankel operator · Bekollé–Bonami weight · Bergman space · Bergman projection · doubling weight

Mathematics Subject Classification Primary 47B35; Secondary 32A36

This research was supported in part by Ministerio de Economía y Competitivivad, Spain, projects MTM2014-52865-P and MTM2015-69323-REDT; La Junta de Andalucía, project FQM210; Academy of Finland Project No. 268009. AP acknowledges financial support from the Spanish Ministry of Economy and Competitiveness, through the María de Maeztu Programme for Units of Excellence in R&D (MDM-2014-0445), Academy of Finland Project No. 268009, and the Grant MTM2017-83499-P (Ministerio de Educación y Ciencia).

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1 Introduction and main results

Let $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. A function $\omega : \mathbb{D} \to [0, \infty)$, integrable over the unit disc $\mathbb{D}$, is called a weight. It is radial if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. For $0 < p < \infty$ and a weight ω, the Lebesgue space L^p_ω consists of (equivalence classes of) complex-valued measurable functions f in $\mathbb{D}$ such that

$$\|f\|_{L^p_\omega} = \left(\int_{\mathbb{D}} |f(z)|^p \omega(z)\, dA(z)\right)^{\frac{1}{p}} < \infty,$$

where $dA(z) = dx\, dy/\pi$ denotes the normalized Lebesgue area measure on $\mathbb{D}$. The weighted Bergman space A^p_ω is the space of analytic functions in L^p_ω. As usual, A^p_α denotes the weighted Bergman space induced by the standard radial weight $(\alpha+1)(1-|z|^2)^\alpha$. If ν is a radial weight then A^2_ν is a closed subspace of L^2_ν, and the orthogonal projection from L^2_ν to A^2_ν is given by

$$P_\nu(f)(z) = \int_{\mathbb{D}} f(\zeta)\overline{B^\nu_z(\zeta)}\nu(\zeta)\, dA(\zeta), \quad z \in \mathbb{D},$$

where B^ν_z are the reproducing kernels of A^2_ν; $f(z) = \langle f, B^\nu_z\rangle_{A^2_\nu}$ for all $z \in \mathbb{D}$ and $f \in A^2_\nu$.

The study of the boundedness of weighted Bergman projections on L^p-spaces is a compelling topic that has attracted a considerable amount of attention during the last decades. A well known result due to Bekollé and Bonami [4,5] describes the weights ω such that the Bergman projection P_η, induced by the standard weight $(\eta+1)(1-|z|^2)^\eta$, is bounded on L^q_ω for $1 < q < \infty$. We denote this class of weights by $B_q(\eta)$, and write $B_q = \cup_{\eta>-1} B_q(\eta)$ for short. In the case of a standard weight, the Bergman reproducing kernels are given by the neat formula $(1 - \overline{z}\zeta)^{-(2+\eta)}$. However, for a general radial weight ν the Bergman reproducing kernels B^ν_z may have zeros [18] and such explicit formulas for the kernels do not necessarily exist. This is one of the main obstacles in dealing with P_ν [9,16]. Nonetheless, we shall prove in Proposition 6 below that if $\nu \in B_q$ is radial, then $P_\nu : L^q_\nu \to L^q_\nu$ is bounded for each $1 < q < \infty$. The proof of this relies on accurate estimates for the integral means of B^z_ν recently obtained in [16, Theorem 1], and the result itself plays an important role in the proof of the main discovery of this paper.

All the above makes the class of radial weights in B_q an appropriate framework for the study of the big Hankel operator

$$H^\nu_f(g)(z) = (I - P_\nu)(fg)(z), \quad f \in L^1_\nu, \quad z \in \mathbb{D},$$

on weighted Bergman spaces. For an analytic function f, the Hankel operator $H^\beta_{\overline{f}}$, induced by a standard projection, has been widely studied on Bergman spaces since the pioneering work of Axler [3], which was later extended in [1]. In the case of a rapidly decreasing weight ν and $f \in \mathcal{H}(\mathbb{D})$, Galanopoulos and Pau [10] did an extensive research on $H^\nu_{\overline{f}}$ on A^2_ν; see [2] for further results. For general symbols, Zhu [21] was the first to build up a bridge between Hankel operators and the mean oscillation of the symbols in the Bergman metric, and this idea has been further developed in several contexts [6–8,22]; see [23] and the references therein for further information on the theory of Hankel operators. More recently, Pau et al. [12] described the complex valued symbols f such that the Hankel operators H^β_f and $H^\beta_{\overline{f}}$ are simultaneously bounded from A^p_α to L^q_β, provided $1 < p \le q < \infty$. Our primary aim is to extend this last-mentioned result to the context of radial B_q-weights. To do this, some definitions are needed. For a radial weight ω, we assume throughout the paper that $\widehat{\omega}(z) = \int_{|z|}^1 \omega(s)\, ds > 0$ for all $z \in \mathbb{D}$, for otherwise the Bergman space A^p_ω would contain all analytic functions in $\mathbb{D}$. A radial weight ω belongs to the class $\widehat{\mathcal{D}}$ if there exists a constant $C = C(\omega) > 1$ such that $\widehat{\omega}(r) \leq C\widehat{\omega}(\frac{1+r}{2})$ for all $0 \leq r < 1$. Moreover, if there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that

$$\widehat{\omega}(r) \geq C\widehat{\omega}\left(1 - \frac{1-r}{K}\right), \quad 0 \leq r < 1, \tag{1.1}$$

then $\omega \in \check{\mathcal{D}}$. We write $\mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}}$ for short. For basic properties of these classes of weights and more, see [13,14] and the references therein. Let $\beta(z, \zeta)$ denote the hyperbolic distance between $z, \zeta \in \mathbb{D}$, $\Delta(z, r)$ the hyperbolic disc of center z and radius $r > 0$, and $S(z)$ the Carleson square associated to z. For $0 < p, q < \infty$ and radial weights ω, ν, define

$$\gamma(z) = \gamma_{\omega,\nu,p,q}(z) = \frac{\widehat{\nu}(z)^{\frac{1}{q}}(1-|z|)^{\frac{1}{q}}}{\widehat{\omega}(z)^{\frac{1}{p}}(1-|z|)^{\frac{1}{p}}}, \quad z \in \mathbb{D}. \tag{1.2}$$

Further, for $f \in L^1_{\nu,\mathrm{loc}}$, write $\widehat{f}_{r,\nu}(z) = \frac{\int_{\Delta(z,r)} f(\zeta)\nu(\zeta)\,dA(\zeta)}{\nu(\Delta(z,r))}$ and

$$\mathrm{MO}_{\nu,q,r}(f)(z) = \left(\frac{1}{\nu(\Delta(z,r))}\int_{\Delta(z,r)} |f(\zeta) - \widehat{f}_{r,\nu}(z)|^q \nu(\zeta)\,dA(\zeta)\right)^{\frac{1}{q}}$$

for all $z \in \mathbb{D}$. It is worth noticing that for prefixed $r > 0$, the quantity $\nu(\Delta(z, r))$ may equal to zero for some z arbitrarily close to the boundary if $\nu \in \widehat{\mathcal{D}}$. However, if $\nu \in \mathcal{D}$, then there exists $r_0 = r_0(\nu) > 0$ such that $\nu(\Delta(z, r)) \asymp \nu(S(z)) > 0$ for all $z \in \mathbb{D}$ if $r \geq r_0$. The space $\mathrm{BMO}(\Delta)_{\omega,\nu,p,q,r}$ consists of $f \in L^q_{\nu,\mathrm{loc}}$ such that

$$\|f\|_{\mathrm{BMO}(\Delta)_{\omega,\nu,p,q,r}} = \sup_{z\in\mathbb{D}} \left(\mathrm{MO}_{\nu,q,r}(f)(z)\gamma(z)\right) < \infty.$$

We will show that if $\nu \in \mathcal{D}$, then $\mathrm{BMO}(\Delta)_{\omega,\nu,p,q,r}$ does not depend on r for $r \geq r_0$. In this case, we simply write $\mathrm{BMO}(\Delta)_{\omega,\nu,p,q}$. The main result of this study reads as follows and it will be proved in Sect. 5.

Theorem 1 *Let $1 < p \leq q < \infty$, $\omega \in \mathcal{D}$, $\nu \in B_q$ a radial weight and $f \in L^q_\nu$. Then $H^\nu_f, H^\nu_{\overline{f}} : A^p_\omega \to L^q_\nu$ are bounded if and only if $f \in \mathrm{BMO}(\Delta)_{\omega,\nu,p,q}$.*

The approach employed in the proof of this result follows the guideline of [12, Thorem 4.1], however a good number of steps cannot be adapted straightforwardly and need substantial modifications. In Sect. 2 we prove some results concerning the classes of weights involved in this work and the boundedness of the Bergman projection P_ν, while in Sect. 3 we introduce and study two spaces of functions on $\mathbb{D}$. One of them is denoted as $\mathrm{BA}(\Delta)_{\omega,\nu,p,q}$, and although its initial definition depends on r, it can be described in terms of an appropriate Berezin transform or simply observing that $f \in \mathrm{BA}(\Delta)_{\omega,\nu,p,q}$ if and only the multiplication operator $M_f(g) = fg$ is bounded from A^p_ω to L^q_ν [15]. The second one, denoted by $\mathrm{BO}(\Delta)_{\omega,\nu,p,q}$, consists of continuous functions on $\mathbb{D}$ such that the oscillation in the Bergman metric is bounded in terms of the auxiliary function γ given in (1.2). We show that $f \in \mathrm{BO}(\Delta)_{\omega,\nu,p,q}$ if and only if

$$|f(z) - f(\zeta)| \lesssim \|f\|_{\mathrm{BO}(\Delta)_{\omega,\nu,p,q}}(1 + \beta(z, \zeta))\Gamma_\tau(z, \zeta) \quad z, \zeta \in \mathbb{D},$$

where

$$\Gamma_\tau(z,\zeta)=\frac{\left(\frac{|1-\overline{z}\zeta|^2}{\max\{1-|z|^2,1-|\zeta|^2\}}\right)^{\frac{1}{p}-\frac{\tau+1}{q}}\widehat{\omega}\left(1-\frac{2|1-\overline{z}\zeta|^2}{\max\{1-|z|^2,1-|\zeta|^2\}}\right)^{\frac{1}{p}}}{\min\left\{\frac{\widehat{\nu}(z)}{(1-|z|)^\tau},\frac{\widehat{\nu}(\zeta)}{(1-|\zeta|)^\tau}\right\}^{\frac{1}{q}}},\quad z,\zeta\in\mathbb{D},$$

for an appropriate (small) constant $\tau=\tau(\omega,\nu)>0$. If ω and ν are standard weights, then Γ_τ does not coincide with the function playing the corresponding role in [12, Lemma 3.2]; in the latter case the function is simpler in many aspects and does not depend on the additional parameter τ. Then, we show that

$$\mathrm{BMO}(\Delta)_{\omega,\nu,p,q}=\mathrm{BA}(\Delta)_{\omega,\nu,p,q}+\mathrm{BO}(\Delta)_{\omega,\nu,p,q}. \tag{1.3}$$

In order to prove this decomposition, due to the complex nature of $\Gamma_\tau(z,\zeta)$, we are forced to split $\mathbb{D}$ into several regions depending on z, establish sharp estimates for $\Gamma_\tau(z,\zeta)$ in each region and then apply properties of weights in $\mathcal{D}$. The identity (1.3) together with a description of the boundedness of the integral operator

$$T_{b,c}f(z)=\int_{\mathbb{D}}f(\zeta)\frac{(1-|\zeta|^2)^b}{(1-z\overline{\zeta})^c}\,dA(\zeta)$$

and its maximal counterpart from A^p_ω to L^q_ν, see Sect. 4 below, are key tools to prove that each $f\in\mathrm{BMO}(\Delta)_{\omega,\nu,p,q}$ induces a bounded Hankel operator from A^p_ω to L^q_ν. Theorem 1 will be proved in Sect. 5.

Finally, in Sect. 6, as a byproduct of Theorem 1, we describe the analytic symbols such that $H_{\overline{f}}:A^p_\omega\to L^q_\nu$ is bounded. The space $\mathcal{B}_{d\gamma}$ consists of $f\in\mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}_{d\gamma}}=\sup_{z\in\mathbb{D}}|f'(z)|(1-|z|)\gamma(z)+|f(0)|<\infty,$$

where γ is given by (1.2).

Theorem 2 *Let $1<p\le q<\infty$, $\omega\in\mathcal{D}$, $\nu\in B_q$ a radial weight and $f\in A^1_\nu$. Then $H^\nu_{\overline{f}}:A^p_\omega\to L^q_\nu$ is bounded if and only if $f\in\mathcal{B}_{d\gamma}$.*

Throughout the paper $\frac{1}{p}+\frac{1}{p'}=1$ for $1<p<\infty$. Further, the letter $C=C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a\lesssim b$ if there exists a constant $C=C(\cdot)>0$ such that $a\le Cb$, and $a\gtrsim b$ is understood in an analogous manner. In particular, if $a\lesssim b$ and $a\gtrsim b$, then we will write $a\asymp b$.

2 Auxiliary results

For a radial weight ω, $K>1$ and $0\le r<1$, let $\rho^r_n=\rho^r_n(\omega,K)$ be defined by $\widehat{\omega}(\rho^r_n)=\widehat{\omega}(r)K^{-n}$ for all $n\in\mathbb{N}\cup\{0\}$. Write $\rho_n=\rho^0_n$ for short. For $x\ge 1$, write $\omega_x=\int_0^1 r^x\omega(r)\,dr$. Denote

$$\omega^\star(z)=\int_{|z|}^1\log\frac{s}{|z|}\omega(s)s\,ds,\quad z\in\mathbb{D}\setminus\{0\}.$$

Throughout the proofs we will repeatedly use several basic properties of weights in the classes $\widehat{\mathcal{D}}$ and $\check{\mathcal{D}}$. For a proof of the first lemma, see [13, Lemma 2.1]; the second one can be proved by similar arguments.

Lemma A *Let ω be a radial weight. Then the following statements are equivalent:*

(i) $\omega \in \widehat{\mathcal{D}}$;
(ii) *There exist $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$ such that*

$$\widehat{\omega}(r) \le C \left(\frac{1-r}{1-t}\right)^{\beta} \widehat{\omega}(t), \quad 0 \le r \le t < 1;$$

(iii) *There exist $C = C(\omega) > 0$ and $\gamma = \gamma(\omega) > 0$ such that*

$$\int_0^t \left(\frac{1-t}{1-s}\right)^{\gamma} \omega(s)\, ds \le C\widehat{\omega}(t), \quad 0 \le t < 1;$$

(iv) *There exists $\lambda = \lambda(\omega) \ge 0$ such that*

$$\int_{\mathbb{D}} \frac{dA(z)}{|1-\overline{\zeta}z|^{\lambda+1}} \asymp \frac{\widehat{\omega}(\zeta)}{(1-|\zeta|)^{\lambda}}, \quad \zeta \in \mathbb{D};$$

(v) *There exist $K = K(\omega) > 1$ and $C = C(\omega, K) > 1$ such that $1 - \rho_n^r(\omega, K) \ge C(1-\rho_{n+1}^r(\omega, K))$ for some (equivalently for all) $0 \le r < 1$ and for all $n \in \mathbb{N} \cup \{0\}$.*

Lemma B *Let ω be a radial weight. Then $\omega \in \check{\mathcal{D}}$ if and only if there exist $C = C(\omega) > 0$ and $\alpha = \alpha(\omega) > 0$ such that*

$$\widehat{\omega}(t) \le C \left(\frac{1-t}{1-r}\right)^{\alpha} \widehat{\omega}(r), \quad 0 \le r \le t < 1.$$

Two more results on weights of more general nature than Lemmas A and B are also needed.

Lemma 3 *Let ω be a radial weight. Then the following statements are equivalent:*

(i) $\omega \in \widehat{\mathcal{D}}$;
(ii) *For some (equivalently for each) $\nu \in \mathcal{D}$ there exists a constant $C = C(\omega, \nu) > 0$ such that*

$$\int_r^1 \frac{\omega(t)\widehat{\nu}(t)}{\widehat{\omega}(t)}\, dt \le C\widehat{\nu}(r), \quad 0 \le r < 1;$$

(iii) *For some (equivalently for each) $\nu \in \mathcal{D}$ there exists a constant $C = C(\omega, \nu) > 0$ such that*

$$\int_0^r \frac{\omega(t)}{\widehat{\omega}(t)\widehat{\nu}(t)}\, dt \le \frac{C}{\widehat{\nu}(r)}, \quad 0 \le r < 1.$$

Proof Let first $\omega \in \widehat{\mathcal{D}}$ and $0 \le r < 1$, and consider $\rho_n^r = \rho_n^r(\omega, K)$ for all $n \in \mathbb{N} \cup \{0\}$. Then Lemma B, applied to $\nu \in \mathcal{D} \subset \check{\mathcal{D}}$, and Lemma A(v), applied to ω, imply

$$\begin{aligned}
\int_r^1 \frac{\omega(t)\widehat{\nu}(t)}{\widehat{\omega}(t)}\, dt &= \sum_{n=0}^{\infty} \int_{\rho_n^r}^{\rho_{n+1}^r} \frac{\omega(t)\widehat{\nu}(t)}{\widehat{\omega}(t)}\, dt \le \sum_{n=0}^{\infty} \widehat{\nu}(\rho_n^r) \int_{\rho_n^r}^{\rho_{n+1}^r} \frac{\omega(t)}{\widehat{\omega}(t)}\, dt \\
&\lesssim \log K \frac{\widehat{\nu}(\rho_0^r)}{(1-\rho_0^r)^{\beta}} \sum_{n=0}^{\infty} (1-\rho_n^r)^{\beta} \\
&\le \widehat{\nu}(r) \log K \sum_{n=0}^{\infty} \frac{1}{(C^{\beta})^n} = \widehat{\nu}(r) \log K \frac{C^{\beta}}{C^{\beta}-1}, \quad 0 \le r < 1,
\end{aligned}$$

for a suitably fixed $K = K(\omega) > 1$, and thus (ii) is satisfied. Conversely, (ii) implies

$$C\widehat{\nu}(r) \geq \int_r^1 \frac{\omega(t)\widehat{\nu}(t)}{\widehat{\omega}(t)}\,dt \geq \int_r^{\frac{1+r}{2}} \frac{\omega(t)\widehat{\nu}(t)}{\widehat{\omega}(t)}\,dt \geq \widehat{\nu}\left(\frac{1+r}{2}\right)\log\frac{\widehat{\omega}(r)}{\widehat{\omega}\left(\frac{1+r}{2}\right)}, \quad 0 \leq r < 1,$$

and since $\nu \in \mathcal{D} \subset \widehat{\mathcal{D}}$ by the hypothesis, we deduce $\widehat{\omega}(r) \lesssim \widehat{\omega}\left(\frac{1+r}{2}\right)$ for all $0 \leq r < 1$. Thus $\omega \in \widehat{\mathcal{D}}$.

Let $\omega \in \widehat{\mathcal{D}}$ and $0 \leq r < 1$, and consider $\rho_n = \rho_n(\omega, K)$ for all $n \in \mathbb{N} \cup \{0\}$. Fix $k = k(\omega, K) \in \mathbb{N} \cup \{0\}$ such that $\rho_k \leq r < \rho_{k+1}$. Then

$$\int_0^r \frac{\omega(t)}{\widehat{\omega}(t)\widehat{\nu}(t)}\,dt = \sum_{n=0}^{k-1}\int_{\rho_n}^{\rho_{n+1}} \frac{\omega(t)}{\widehat{\omega}(t)\widehat{\nu}(t)}\,dt + \int_{\rho_k}^r \frac{\omega(t)}{\widehat{\omega}(t)\widehat{\nu}(t)}\,dt, \quad 0 \leq r < 1,$$

where, by Lemma B, applied to $\nu \in \mathcal{D} \subset \check{\mathcal{D}}$, and Lemma A(v), applied to ω,

$$\begin{aligned}
\sum_{n=0}^{k-1}\int_{\rho_n}^{\rho_{n+1}} \frac{\omega(t)}{\widehat{\omega}(t)\widehat{\nu}(t)}\,dt &\leq \sum_{n=0}^{k-1}\frac{1}{\widehat{\nu}(\rho_{n+1})}\int_{\rho_n}^{\rho_{n+1}}\frac{\omega(t)}{\widehat{\omega}(t)}\,dt \\
&\lesssim \sum_{n=0}^{k-1}\frac{(1-\rho_k)^\alpha}{\widehat{\nu}(\rho_k)}\frac{1}{(1-\rho_{n+1})^\alpha}\log\left(\frac{\widehat{\omega}(\rho_n)}{\widehat{\omega}(\rho_{n+1})}\right) \\
&\leq \log K\,\frac{(1-\rho_k)^\alpha}{\widehat{\nu}(r)}\sum_{n=0}^{k-1}\frac{1}{(C^\alpha)^{k-1-n}(1-\rho_k)^\alpha} \\
&\leq \frac{\log K}{\widehat{\nu}(r)}\sum_{n=0}^{\infty}\frac{1}{(C^\alpha)^n} = \frac{\log K}{\widehat{\nu}(r)}\frac{C^\alpha}{C^\alpha - 1}, \quad k \in \mathbb{N},
\end{aligned}$$

for some $\alpha = \alpha(\nu) > 0$ and for a suitably fixed $K = K(\omega) > 1$, and similarly,

$$\int_{\rho_k}^r \frac{\omega(t)}{\widehat{\omega}(t)\widehat{\nu}(t)}\,dt \leq \frac{1}{\widehat{\nu}(r)}\log\left(\frac{\widehat{\omega}(\rho_k)}{\widehat{\omega}(r)}\right) \leq \frac{\log K}{\widehat{\nu}(r)}, \quad k \in \mathbb{N} \cup \{0\}.$$

The statement (iii) follows from these estimates.

Conversely, by replacing r by $\frac{1+r}{2}$ in (iii) we obtain

$$\frac{C}{\widehat{\nu}\left(\frac{1+r}{2}\right)} \geq \int_0^{(1+r)/2} \frac{\omega(t)}{\widehat{\omega}(t)\widehat{\nu}(t)}\,dt \geq \int_r^{(1+r)/2} \frac{\omega(t)}{\widehat{\omega}(t)\widehat{\nu}(t)}\,dt \geq \frac{1}{\widehat{\nu}(r)}\log\frac{\widehat{\omega}(r)}{\widehat{\omega}\left(\frac{1+r}{2}\right)}, \quad 0 \leq r < 1,$$

and since $\nu \in \mathcal{D} \subset \widehat{\mathcal{D}}$ by the hypothesis, we deduce $\widehat{\omega}(r) \lesssim \widehat{\omega}\left(\frac{1+r}{2}\right)$ for all $0 \leq r < 1$. Thus $\omega \in \widehat{\mathcal{D}}$. □

Lemma 4 *Let $\omega, \nu \in \mathcal{D}$, and denote $\sigma = \sigma_{\omega,\nu} = \omega\widehat{\nu}/\widehat{\omega}$. Then $\widehat{\sigma} \asymp \widehat{\nu}$ on $[0, 1)$, and hence $\sigma \in \mathcal{D}$.*

Proof Lemma 3(ii) implies $\widehat{\sigma} \lesssim \widehat{\nu}$ on $[0, 1)$. The argument used to prove (i) ⇒ (ii) in the said lemma shows that $\widehat{\sigma} \gtrsim \widehat{\nu}$ on $[0, 1)$, provided $\omega \in \check{\mathcal{D}}$ and $\nu \in \mathcal{D}$. Thus $\widehat{\sigma} \asymp \widehat{\nu}$, and $\sigma \in \mathcal{D}$ by Lemmas A(ii) and B. □

The next lemma says that in many instances concerning A^p-norms we may replace ω by $\widetilde{\omega} = \widehat{\omega}/(1 - |\cdot|)$ if $\omega \in \mathcal{D}$. This result has the flavor of radial Carleson measures and indeed can be established by appealing to the characterization of Carleson measures for the Bergman space A^p_ω induced by $\omega \in \widehat{\mathcal{D}}$ given in [15]. That approach requires showing that the involved weights belong to $\widehat{\mathcal{D}}$, which is of course the case, and thus involves more calculations than the simple proof given below.

Lemma 5 *Let $0 < p < \infty$, $\omega \in \mathcal{D}$ and $-\alpha < \kappa < \infty$, where $\alpha = \alpha(\omega) > 0$ is that of Lemma B. Then*

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|)^\kappa \omega(z)\,dA(z) \asymp \int_{\mathbb{D}} |f(z)|^p (1-|z|)^{\kappa-1} \widehat{\omega}(z)\,dA(z), \quad f \in \mathcal{H}(\mathbb{D}). \tag{2.1}$$

Proof The function $(1-|\cdot|)^{\kappa-1}\widehat{\omega}$ is a weight for each $\kappa > -\alpha$ by Lemma B. Therefore an integration by parts shows that (2.1) is equivalent to

$$\int_0^1 \frac{\partial}{\partial r} M_p^p(r,f) \left(\int_r^1 (1-t)^\kappa \omega(t)\,dt \right) dr \asymp \int_0^1 \frac{\partial}{\partial r} M_p^p(r,f) \left(\int_r^1 (1-t)^{\kappa-1} \widehat{\omega}(t)\,dt \right) dr.$$

Another integration by parts reveals that both integrals from r to 1 above are bounded by a constant times $\widehat{\omega}(r)(1-r)^\kappa$. But Lemma A(ii) implies

$$\int_r^1 (1-t)^{\kappa-1}\widehat{\omega}(t)\,dt \gtrsim \frac{\widehat{\omega}(r)}{(1-r)^{\beta(\omega)}} \int_r^1 (1-t)^{\kappa-1+\beta(\omega)}\,dt \asymp \widehat{\omega}(r)(1-r)^\kappa, \quad 0 \le r < 1,$$

and

$$\int_r^1 (1-t)^\kappa \omega(t)\,dt \gtrsim \frac{\widehat{\omega}(r)}{(1-r)^{\beta(\omega)}} \int_r^1 \frac{\omega(t)(1-t)^{\kappa+\beta(\omega)}}{\widehat{\omega}(t)}\,dt \asymp \widehat{\omega}(r)(1-r)^\kappa, \quad 0 \le r < 1,$$

by Lemma 4. The assertion follows. □

The last auxiliary results shows that each radial weight in the Bekollé–Bonami class B_q belongs to $\mathcal{D}$, and for each $\nu \in \mathcal{D}$ the maximal Bergman projection

$$P_\nu^+(f)(z) = \int_{\mathbb{D}} f(\zeta)|B_z^\nu(\zeta)|\nu(\zeta)\,dA(\zeta), \quad z \in \mathbb{D},$$

is bounded on L^q_ν. It is worth noticing that obviously $\mathcal{D} \not\subset \cup_{1<q<\infty} B_q$ because $\nu \in \mathcal{D}$ may vanish on a set of positive measure.

Proposition 6 *Let $1 < q < \infty$ and $\nu \in B_q$ a radial weight. Then $\nu \in \mathcal{D}$. Moreover, $P_\nu^+ : L^q_\nu \to L^q_\nu$ is bounded for all $\nu \in \mathcal{D}$.*

Proof If $\nu \in B_q$, then by [5] there exists $\beta > -1$ such that

$$\left(\int_{S(a)} \nu(z)\,dA(z) \right)^{\frac{1}{q}} \left(\int_{S(a)} \left(\frac{(1-|z|)^\beta}{\nu(z)} \right)^{\frac{q'}{q}} (1-|z|)^\beta\,dA(z) \right)^{\frac{1}{q'}} \lesssim (1-|a|)^{(2+\beta)}, \quad a \in \mathbb{D}.$$

Since ν is radial, this condition easily implies $\nu \in \mathcal{D}$.

Let now $1 < q < \infty$ and $\nu \in \mathcal{D}$, and define $h = \widehat{\nu}^{-\frac{1}{qq'}}$. Then $\int_t^1 h(s)^{q'}\nu(s)\,ds \asymp \widehat{\nu}(t)^{\frac{1}{q'}}$ for all $0 \le t < 1$. Therefore Lemma B yields

$$\int_0^r \frac{\int_t^1 h(s)^{q'}\nu(s)\,ds}{\widehat{\nu}(t)(1-t)}\,dt \asymp \int_0^r \frac{dt}{\widehat{\nu}(t)^{\frac{1}{q}}(1-t)} \lesssim \frac{1}{\widehat{\nu}(r)^{\frac{1}{q}}} = h^{q'}(r), \quad 0 \le r < 1. \tag{2.2}$$

Moreover, by symmetry, (2.2) with q' in place of q is satisfied. Since $\nu \in \widehat{\mathcal{D}}$, we may apply [16, Theorem 1] and (2.2) to deduce

$$\int_{\mathbb{D}} |B_z^\nu(\zeta)| h^{p'}(\zeta)\nu(\zeta)\,dA(\zeta) \lesssim h^{p'}(z), \quad z \in \mathbb{D},$$

and

$$\int_{\mathbb{D}} |B_z^\nu(\zeta)| h^{p}(z)\nu(z)\,dA(z) \lesssim h^{p}(\zeta), \quad \zeta \in \mathbb{D}.$$

It follows from Schur's test [23, Theorem 3.6] that the maximal Bergman projection $P_\nu^+ : L_\nu^p \to L_\nu^p$ is bounded. □

3 Some spaces of functions

Recall that

$$\gamma(z) = \gamma_{\omega,\nu,p,q}(z) = \frac{\widehat{\nu}(z)^{\frac{1}{q}}(1-|z|)^{\frac{1}{q}}}{\widehat{\omega}(z)^{\frac{1}{p}}(1-|z|)^{\frac{1}{p}}}, \quad z \in \mathbb{D}, \tag{3.1}$$

and $\widehat{f}_{r,\nu}(z) = \frac{\int_{\Delta(z,r)} f(\zeta)\nu(\zeta)\,dA(\zeta)}{\nu(\Delta(z,r))}$ for $f \in L^1_{\nu,\mathrm{loc}}$, and

$$\mathrm{MO}_{\nu,q,r}(f)(z) = \left(\frac{1}{\nu(\Delta(z,r))}\int_{\Delta(z,r)} |f(\zeta) - \widehat{f}_{r,\nu}(z)|^q \nu(\zeta)\,dA(\zeta)\right)^{\frac{1}{q}}$$

for all $z \in \mathbb{D}$. If $\nu \in \check{\mathcal{D}}$, then by the definition there exist $K = K(\nu) > 1$ and $C = C(\nu) > 1$ such that

$$\int_r^{1-\frac{1-r}{K}} \nu(s)\,ds \geq (C-1)\widehat{\nu}\left(1 - \frac{1-r}{K}\right) > 0, \quad 0 \leq r < 1.$$

It follows that there exists $r_\nu \in (0,\infty)$ such that $\nu(\Delta(z,r)) > 0$ for all $z \in \mathbb{D}$ if $r \geq r_\nu$.

The space $\mathrm{BMO}(\Delta) = \mathrm{BMO}(\Delta)_{\omega,\nu,p,q,r}$ consists of $f \in L^q_{\nu,\mathrm{loc}}$ such that

$$\|f\|_{\mathrm{BMO}(\Delta)} = \sup_{z\in\mathbb{D}} \big(\mathrm{MO}_{\nu,q,r}(f)(z)\gamma(z)\big) < \infty.$$

The following lemma is easy to establish; see [12, Lemma 3.1] for a similar result.

Lemma 7 *Let $1 \leq p, q < \infty$, ω a radial weight, $\nu \in \check{\mathcal{D}}$ and $r_\nu \leq r < \infty$. Then*

$$\mathrm{MO}_{\nu,q,r}(f)(z) \leq 2\left(\frac{1}{\nu(\Delta(z,r))}\int_{\Delta(z,r)} |f(\zeta) - \lambda|^q \nu(\zeta)\,dA(\zeta)\right)^{\frac{1}{q}}, \quad z \in \mathbb{D},\ \lambda \in \mathbb{C},\ f \in L^q_\nu,$$

and therefore $f \in L^q_\nu$ belongs to BMO(Δ) *if and only if for each $z \in \mathbb{D}$ there exists $\lambda_z \in \mathbb{C}$ such that*

$$\sup_{z\in\mathbb{D}} \left(\frac{\gamma(z)^q}{\nu(\Delta(z,r))}\int_{\Delta(z,r)} |f(\zeta) - \lambda_z|^q \nu(\zeta)\,dA(\zeta)\right) < \infty.$$

For $0 < p, q < \infty$, $0 \le \tau < \infty$ and radial weights ω, ν, let

$$\Gamma_\tau(z,\zeta) = \frac{\left(\frac{|1-\overline{z}\zeta|^2}{\max\{1-|z|^2,1-|\zeta|^2\}}\right)^{\frac{1}{p}-\frac{\tau+1}{q}} \widehat{\omega}\left(1-\frac{2|1-\overline{z}\zeta|^2}{\max\{1-|z|^2,1-|\zeta|^2\}}\right)^{\frac{1}{p}}}{\min\left\{\frac{\widehat{\nu}(z)}{(1-|z|)^\tau}, \frac{\widehat{\nu}(\zeta)}{(1-|\zeta|)^\tau}\right\}^{\frac{1}{q}}}, \quad z,\zeta \in \mathbb{D}, \quad (3.2)$$

with the understanding that $\widehat{\omega}(t) = \widehat{\omega}(0)$ when $t < 0$. The following lemma explains the behavior of Γ_τ near the diagonal.

Lemma 8 *Let $0 < p, q, r < \infty$, $0 \le \tau < \infty$ and $\omega, \nu \in \widehat{\mathcal{D}}$. Then*

$$\Gamma_\tau(z,\zeta) \asymp \gamma(z)^{-1} \asymp \gamma(\zeta)^{-1}, \quad \beta(z,\zeta) \le r.$$

Proof Clearly

$$|1-\overline{z}\zeta| \asymp 1-|z| \asymp 1-|\zeta|, \quad \beta(z,\zeta) \le r,$$

and hence there exist $0 < m_r < 1 < M_r < \infty$ such that

$$m_r(1-|z|) \le \frac{2|1-\overline{z}\zeta|^2}{\max\{1-|z|^2, 1-|\zeta|^2\}} \le M_r(1-|z|), \quad \beta(z,\zeta) \le r.$$

Since $\omega \in \widehat{\mathcal{D}}$ by the hypothesis, and $\widehat{\omega}(t) = \widehat{\omega}(0)$ for $t < 0$, Lemma A(ii) implies

$$\widehat{\omega}(z) \le \frac{C}{m_r^\beta}\widehat{\omega}(1-m_r(1-|z|)) \le \frac{C}{m_r^\beta}\widehat{\omega}\left(1-\frac{2|1-\overline{z}\zeta|^2}{\max\{1-|z|^2, 1-|\zeta|^2\}}\right), \quad \beta(z,\zeta) \le r,$$

and

$$\widehat{\omega}\left(1-\frac{2|1-\overline{z}\zeta|^2}{\max\{1-|z|^2, 1-|\zeta|^2\}}\right) \le CM_r^\beta\widehat{\omega}(1-M_r(1-|z|)) \le CM_r^\beta\widehat{\omega}(z), \quad \beta(z,\zeta) \le r,$$

for some $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$. Further, $\widehat{\nu}(z) \asymp \widehat{\nu}(\zeta)$ and $\widehat{\omega}(z) \asymp \widehat{\omega}(\zeta)$ if $\beta(z,\zeta) \le r$ by Lemma A(ii). The assertion follows from these estimates. □

For continuous $f : \mathbb{D} \to \mathbb{C}$ and $0 < r < \infty$, define

$$\Omega_r f(z) = \sup\{|f(z) - f(\zeta)| : \beta(z,\zeta) < r\}, \quad z \in \mathbb{D},$$

and let $\mathrm{BO}(\Delta) = \mathrm{BO}(\Delta)_{\omega,\nu,p,q,r}$ denote the space of those f such that

$$\|f\|_{\mathrm{BO}(\Delta)} = \sup_{z\in\mathbb{D}}(\Omega_r f(z)\gamma(z)) < \infty.$$

Lemma 9 shows that the space $\mathrm{BO}(\Delta) = \mathrm{BO}(\Delta)_{\omega,\nu,p,q,r}$ is independent of r.

Lemma 9 *Let $0 < p \le q < \infty$, $0 < r < \infty$, $\omega, \nu \in \check{\mathcal{D}}$ and $\gamma(z) = \gamma_{\omega,\nu,p,q}(z) = \frac{\widehat{\nu}(z)^{\frac{1}{q}}(1-|z|)^{\frac{1}{q}}}{\widehat{\omega}(z)^{\frac{1}{p}}(1-|z|)^{\frac{1}{p}}}$. Let $f : \mathbb{D} \to \mathbb{C}$ be continuous, and $0 < \tau < \min\{q\alpha(\omega)/p, \alpha(\nu)\}$, where $\alpha(\nu)$ and $\alpha(\omega)$ are those from Lemma B. Then the following statements are equivalent:*

(i) *$f \in \mathrm{BO}(\Delta)$;*
(ii) *$|f(z) - f(\zeta)| \lesssim \|f\|_{\mathrm{BO}(\Delta)}(1+\beta(z,\zeta))\Gamma_\tau(z,\zeta)$ for all $z, \zeta \in \mathbb{D}$.*

Proof Lemma 8 shows that (ii) implies (i). For the converse, assume (i), that is,

$$|f(z)-f(\zeta)|\gamma(z)\le\|f\|_{\mathrm{BO}(\Delta)},\quad \beta(z,\zeta)<r. \tag{3.3}$$

The estimate (ii) for $\beta(z,\zeta)\le r$ then follows from Lemma 8. If $\beta(z,\zeta)>r$, let $N=\max\{n\in\mathbb{N}:n\le\beta(z,\zeta)/r+1\}$, and pick up $N+1$ points from the geodesic joining z and ζ such that $\beta(z_j,z_{j+1})=\beta(z,\zeta)/N<r$ for all $j=0,\dots,N-1$. Then, as the hyperbolic distance is additive along geodesics, (3.3) yields

$$|f(z)-f(\zeta)|\le\sum_{j=0}^{N-1}|f(z_j)-f(z_{j+1})|\le\|f\|_{\mathrm{BO}(\Delta)}\sum_{j=0}^{N-1}\frac{\widehat{\omega}(z_j)}{\widehat{\nu}(z_j)}(1-|z_j|)^{\frac{1}{p}-\frac{1}{q}}.$$

Next, observe that

$$1-|z_j|\le\frac{2|1-\overline{z}\zeta|^2}{\max\{1-|z|^2,1-|\zeta|^2\}},\quad j=0,\dots,N; \tag{3.4}$$

see the proof of [12, Lemma 3.2] for details. This together with the inequality $\frac{1}{p}-\frac{1}{q}\ge0$ gives

$$\begin{aligned}|f(z)-f(\zeta)|&\le\|f\|_{\mathrm{BO}(\Delta)}\left(\frac{2|1-\overline{z}\zeta|^2}{\max\{1-|z|^2,1-|\zeta|^2\}}\right)^{\frac{1}{p}-\frac{1}{q}}\sum_{j=0}^{N-1}\frac{\widehat{\omega}(z_j)^{\frac{1}{p}}}{\widehat{\nu}(z_j)^{\frac{1}{q}}}\\&=\|f\|_{\mathrm{BO}(\Delta)}\left(\frac{2|1-\overline{z}\zeta|^2}{\max\{1-|z|^2,1-|\zeta|^2\}}\right)^{\frac{1}{p}-\frac{1}{q}}\sum_{j=0}^{N-1}\frac{\widehat{\omega}(z_j)^{\frac{1}{p}}}{(1-|z_j|)^{\frac{\tau}{q}}}\frac{(1-|z_j|)^{\frac{\tau}{q}}}{\widehat{\nu}(z_j)^{\frac{1}{q}}}.\end{aligned}$$

The election of τ together with Lemma B shows that the functions $\widehat{\omega}(r)/(1-r)^{\frac{p\tau}{q}}$ and $\widehat{\nu}(r)/(1-r)^{\tau}$ are essentially decreasing on $[0,1)$. Therefore the inequalities (3.4) and $|z_j|\le\max\{|z|,|\zeta|\}$ yield

$$\begin{aligned}|f(z)-f(\zeta)|&\lesssim\|f\|_{\mathrm{BO}(\Delta)}\left(\frac{2|1-\overline{z}\zeta|^2}{\max\{1-|z|^2,1-|\zeta|^2\}}\right)^{\frac{1}{p}-\frac{\tau+1}{q}}\\&\quad\cdot\widehat{\omega}\left(1-\frac{2|1-\overline{z}\zeta|^2}{\max\{1-|z|^2,1-|\zeta|^2\}}\right)^{\frac{1}{p}}\sum_{j=0}^{N-1}\frac{(1-|z_j|)^{\frac{\tau}{q}}}{\widehat{\nu}(z_j)^{\frac{1}{q}}}\\&\lesssim\|f\|_{\mathrm{BO}(\Delta)}\Gamma_\tau(z,\zeta)N\lesssim\|f\|_{\mathrm{BO}(\Delta)}(1+\beta(z,\zeta))\Gamma_\tau(z,\zeta),\quad\beta(z,\zeta)>r.\end{aligned}$$

Therefore (ii) is satisfied. □

For $0<p,q<\infty$, $0<r<\infty$ and radial weights ω,ν, the space $\mathrm{BA}(\Delta)=\mathrm{BA}(\Delta)_{\omega,\nu,p,q,r}$ consists of $f\in L^q_{\nu,\mathrm{loc}}$ such that

$$\|f\|_{\mathrm{BA}(\Delta)}=\sup_{z\in\mathbb{D}}\left(\left(\frac{1}{\nu(\Delta(z,r))}\int_{\Delta(z,r)}|f(\zeta)|^q\nu(\zeta)\,dA(\zeta)\right)^{\frac{1}{q}}\gamma(z)\right)<\infty.$$

For $c,\sigma\in\mathbb{R}$ and a radial weight ν, the general Berezin transform of $\varphi\in L^1_{\nu(1-|\cdot|)^\sigma}$ is defined by

$$B(\varphi)(z)=B_{\nu,c,\sigma}(\varphi)(z)=\frac{(1-|z|^2)^{c+1}}{\widehat{\nu}(z)}\int_{\mathbb{D}}\varphi(\zeta)\frac{(1-|\zeta|^2)^\sigma}{|1-z\overline{\zeta}|^{2+c+\sigma}}\nu(\zeta)\,dA(\zeta),\quad z\in\mathbb{D}.$$

The next lemma shows, in particular, that the space $\mathrm{BA}(\Delta) = \mathrm{BA}(\Delta)_{\omega,\nu,p,q,r}$ is independent of r as long as r is sufficiently large depending on $\nu \in \mathcal{D}$.

Lemma 10 *Let $0 < p \le q < \infty$, $0 < r < \infty$ and $\omega, \nu \in \mathcal{D}$, $\gamma(z) = \gamma_{\omega,\nu,p,q}(z) = \frac{\widehat{\nu}(z)^{\frac{1}{q}}(1-|z|)^{\frac{1}{q}}}{\widehat{\omega}(z)^{\frac{1}{p}}(1-|z|)^{\frac{1}{p}}}$. If $f \in L^q_\nu$, then the following statements are equivalent:*

(i) *There exists $r_0 = r_0(\nu) > 0$ such that $f \in \mathrm{BA}(\Delta) = \mathrm{BA}(\Delta)_{\omega,\nu,p,q,r}$ for all $r \ge r_0$;*
(ii) *$|f|^q \nu dA$ is a q-Carleson measure for A^p_ω;*
(iii) *The identity operator $Id : A^p_\omega \to L^q_{|f|^q\nu}$ is bounded;*
(iv) *The multiplication operator $M_f(g) = fg$ is bounded from A^p_ω to L^q_ν;*
(v) $\sup_{z\in\mathbb{D}} \gamma(z)^q B(|f|^q)(z) < \infty$ *for all $\sigma > 1 - \frac{q}{p}(1+\alpha)$ and $c > \max\{-1-\sigma, \frac{q}{p}(1+\beta) - 2\}$, where $\alpha = \alpha(\omega) > 0$ and $\beta = \beta(\omega) > 0$ are those of Lemmas A(ii) and B.*

Proof It is obvious that (ii), (iii) and (iv) are equivalent by the definitions. Assume (ii) is satisfied, that is,

$$\left(\int_{\mathbb{D}} |g(\zeta)|^q |f(\zeta)|^q \nu(\zeta)\, dA(\zeta)\right)^{\frac{1}{q}} \lesssim \|g\|_{A^p_\omega}, \quad g \in A^p_\omega. \tag{3.5}$$

For $z \in \mathbb{D}$, let $g_z(\zeta) = \left(\frac{1-|z|}{1-\bar{z}\zeta}\right)^{\frac{\lambda+1}{p}}$, where $\lambda = \lambda(\omega) > 0$ is that of Lemma A(iv). Further, since $\nu \in \check{\mathcal{D}}$ by the hypothesis, there exists $r_\nu \in (0,\infty)$ such that $\nu(\Delta(z,r)) > 0$ for all $r \ge r_\nu$. For $g = g_z$ and $r \ge r_\nu$, (3.5) yields

$$\left(\frac{1}{\nu(\Delta(z,r))}\int_{\Delta(z,r)} |f(\zeta)|^q \nu(\zeta)\, dA(\zeta)\right)^{\frac{1}{q}} \lesssim \frac{\|g_z\|_{A^p_\omega}}{\nu(\Delta(z,r))^{\frac{1}{q}}} \lesssim \frac{(\widehat{\omega}(z)(1-|z|))^{\frac{1}{p}}}{\nu(\Delta(z,r))^{\frac{1}{q}}}, \quad z \in \mathbb{D}.$$

But since $\nu \in \mathcal{D}$, applications of Lemmas A(ii) and B show that

$$\nu(\Delta(z,r)) \asymp \widehat{\nu}(z)(1-|z|), \quad z \in \mathbb{D}, \tag{3.6}$$

if r is sufficiently large. It follows that $f \in \mathrm{BA}(\Delta) = \mathrm{BA}(\Delta)_{\omega,\nu,p,q,r}$ for all such r, and thus (i) is satisfied.

Conversely, if (i) is satisfied, then by using (3.6) we deduce

$$\left(\int_{\Delta(z,r)} |f(\zeta)|^q \nu(\zeta)\, dA(\zeta)\right)^{\frac{1}{q}} \lesssim \widehat{\omega}(z)^{\frac{1}{p}}(1-|z|)^{\frac{1}{p}}, \quad z \in \mathbb{D}.$$

Therefore $|f|^q \nu dA$ is a q-Carleson measure for A^p_ω by [17, Theorem 3].

By integrating only over $\Delta(z,r)$ in (v) and using (3.6) we obtain (i) from (v). To complete the proof of the lemma, it remains to show the converse implication. To do this, pick up a sequence $\{a_j\}$ and $0 < r < \infty$ in accordance with [23, Lemma 4.7], and observe that $\widehat{\omega}$ is essentially constant in each hyperbolically bounded region by Lemma A(ii). Then by using (3.6), the hypothesis (i), the election of c and σ, and finally Lemmas A(ii) and B, we deduce

$$\begin{aligned}
\frac{\widehat{\nu}(z)B(|f|^q)(z)}{(1-|z|^2)^{c+1}} &\lesssim \sum_{j=1}^{\infty}\int_{\Delta(a_j,r)}|f(\zeta)|^q\frac{(1-|\zeta|^2)^{\sigma}}{|1-z\overline{\zeta}|^{2+c+\sigma}}\nu(\zeta)\,dA(\zeta)\\
&\lesssim \sum_{j=1}^{\infty}\frac{(1-|a_j|^2)^{\sigma}}{|1-z\overline{a_j}|^{2+c+\sigma}}\int_{\Delta(a_j,r)}|f(\zeta)|^q\nu(\zeta)\,dA(\zeta)\\
&\lesssim \sum_{j=1}^{\infty}\frac{(1-|a_j|^2)^{\sigma+1}\widehat{\nu}(a_j)}{|1-z\overline{a_j}|^{2+c+\sigma}\nu(\Delta(a_j,r))}\int_{\Delta(a_j,r)}|f(\zeta)|^q\nu(\zeta)dA(\zeta)\\
&\lesssim \sum_{i=1}^{\infty}\frac{(1-|a_i|^2)^{\sigma+1}\widehat{\nu}(a_i)}{|1-z\overline{a_i}|^{2+c+\sigma}\gamma(a_i)^q}\asymp\sum_{j=1}^{\infty}\frac{(1-|a_j|^2)^{\sigma+\frac{q}{p}}\widehat{\omega}(a_j)^{\frac{q}{p}}}{|1-z\overline{a_j}|^{2+c+\sigma}}\\
&\lesssim \int_{\mathbb{D}}\frac{(1-|u|^2)^{\sigma+\frac{q}{p}-2}\widehat{\omega}(u)^{\frac{q}{p}}}{|1-z\overline{u}|^{2+c+\sigma}}\,dA(u)\\
&\lesssim \int_0^{|z|}\frac{\widehat{\omega}(t)^{\frac{q}{p}}}{(1-t)^{c+3-\frac{q}{p}}}\,dt+\frac{1}{(1-|z|)^{c+\sigma+1}}\int_{|z|}^1(1-t)^{\sigma+\frac{q}{p}-2}\widehat{\omega}(t)^{\frac{q}{p}}\,dt\\
&\lesssim \frac{\widehat{\omega}(|z|)^{\frac{q}{p}}}{(1-|z|)^{c+2-\frac{q}{p}}}\asymp\frac{\widehat{\nu}(z)}{(1-|z|^2)^{c+1}\gamma(z)^q},\quad z\in\mathbb{D},
\end{aligned}$$

and thus (v) is satisfied. □

With these preparations we are ready to show that $\mathrm{BMO}(\Delta)=\mathrm{BA}(\Delta)+\mathrm{BO}(\Delta)$. This follows from the case (ii) of the next theorem.

Theorem 11 *Let* $1\le p\le q<\infty$, $\omega,\nu\in\mathcal{D}$, $\gamma(z)=\gamma_{\omega,\nu,p,q}(z)=\frac{\widehat{\nu}(z)^{\frac{1}{q}}(1-|z|)^{\frac{1}{q}}}{\widehat{\omega}(z)^{\frac{1}{p}}(1-|z|)^{\frac{1}{p}}}$ *and* $f\in L^q_\nu$. *Further, let* $r\ge r_\nu$, $\sigma>0$ *and*

$$c>2\frac{q}{p}\left(\beta(\omega)+1\right)+\sigma+\max\left\{2\beta(\nu),\gamma(\nu)\right\},$$

where $\beta(\omega),\beta(\nu),\gamma(\nu)>0$ *are associated to* ν *and* ω *via Lemma A(ii), (iii). Then the following statements are equivalent:*

- (i) *There exists* $r_0=r_0(\nu)\ge r_\nu$ *such that* $f\in\mathrm{BMO}(\Delta)=\mathrm{BMO}(\Delta)_{\omega,\nu,p,q,r}$ *for all* $r\ge r_0$;
- (ii) $f=f_1+f_2$, *where* $f_1\in\mathrm{BA}(\Delta)$ *and* $f_2=\widehat{f}_{r,\nu}\in\mathrm{BO}(\Delta)$;
- (iii) $\sup_{z\in\mathbb{D}}\left(B(|f-\widehat{f}_{r,\nu}(z)|^q)\gamma(z)^q\right)<\infty$;
- (iv) *For each* $z\in\mathbb{D}$ *there exists* $\lambda_z\in\mathbb{C}$ *such that* $\sup_{z\in\mathbb{D}}\left(B(|f-\lambda_z|^q)\gamma(z)^q\right)<\infty$.

Proof Obviously, (iii) implies (iv). Next assume (iv). The relation (3.6) shows that there exists $r_0=r_0(\nu)>0$ such that

$$\begin{aligned}
&\frac{1}{\nu\left(\Delta(z,r)\right)}\int_{\Delta(z,r)}|f(\zeta)-\lambda_z|^q\nu(\zeta)\,dA(\zeta)\\
&\quad\lesssim\frac{(1-|z|)^{c+1}}{\widehat{\nu}(z)}\int_{\mathbb{D}}|f(\zeta)-\lambda_z|^q\frac{(1-|\zeta|^2)^{\sigma}}{|1-z\overline{\zeta}|^{2+c+\sigma}}\nu(\zeta)\,dA(\zeta),\quad z\in\mathbb{D},\quad r_0\le r<\infty,
\end{aligned}$$

which together with Lemma 7 shows that (i) is satisfied.

Assume now (i), and let $f_2=\widehat{f}_{r,\nu}$. Since $f\in L^q_\nu$, $q\ge1$ and $r\ge r_\nu$, the function f_2 is well defined and continuous. Since $\omega,\nu\in\mathcal{D}$ by the hypothesis, one may use Lemmas A(ii) and B together with the argument in [12, 1651–1652] with minor modifications to show that $f_2 = \widehat{f}_{r,\nu} \in \mathrm{BO}(\Delta)$ and $f_1 = f - \widehat{f}_{r,\nu} \in \mathrm{BA}(\Delta)$. Thus (ii) is satisfied.

To complete the proof it suffices to show that (ii) implies (iii), so assume $f = f_1 + f_2$, where $f_1 \in \mathrm{BA}(\Delta)$ and $f_2 = \widehat{f}_{r,\nu} \in \mathrm{BO}(\Delta)$. Since $\widehat{f}_{r,\nu} = \widehat{f_1}_{r,\nu} + \widehat{f_2}_{r,\nu}$, it suffices to prove the condition in (iii) for f_1 and f_2 separately. First observe that by Lemma A(iii) the constant function 1 satisfies

$$B(1)(z) \lesssim \frac{(1-|z|)^{c+1}}{\widehat{\nu}(z)} \left(\int_0^{|z|} \frac{\nu(t)}{(1-t)^{1+c}}\,dt + \frac{1}{(1-|z|)^{1+c+\sigma}} \int_{|z|}^1 (1-t)^{\sigma}\nu(t)\,dt \right) \lesssim 1, \quad z \in \mathbb{D},$$

because $c > \max\{\gamma(\nu), \sigma\} - 1$ by the hypothesis. This together with Hölder's inequality and Lemma 10 yields

$$\begin{aligned} B\left(\left|f_1 - \widehat{f_1}_{r,\nu}(z)\right|^q\right)\gamma(z)^q &\lesssim \left(B(|f_1|^q)(z) + |\widehat{f_1}_{r,\nu}(z)|^q\right)\gamma(z)^q \\ &\le \left(B(|f_1|^q)(z) + \widehat{|f_1|^q}_{r,\nu}(z)\right)\gamma(z)^q \lesssim 1, \quad z \in \mathbb{D}, \end{aligned}$$

and thus (iii) for $f_1 \in \mathrm{BA}(\Delta)$ is satisfied.

To deal with $f_2 \in \mathrm{BO}(\Delta)$, pick up τ satisfying the hypothesis of Lemma 9. Then

$$\begin{aligned} |f_2(\zeta) - \widehat{f_2}_{r,\nu}(z)| &= \left| \frac{1}{\nu(\Delta(z,r))} \int_{\Delta(z,r)} (f_2(\zeta) - f_2(u))\nu(u)\,dA(u) \right| \\ &\le \frac{1}{\nu(\Delta(z,r))} \int_{\Delta(z,r)} |f_2(\zeta) - f_2(u)|\nu(u)\,dA(u) \\ &\lesssim \frac{1}{\nu(\Delta(z,r))} \int_{\Delta(z,r)} (1+\beta(\zeta,u))\Gamma_\tau(\zeta,u)\nu(u)\,dA(u) \\ &\lesssim (1+\beta(z,\zeta))\Gamma_\tau(z,\zeta), \quad z,\zeta \in \mathbb{D}, \end{aligned}$$

because $\Gamma_\tau(\zeta,u) \asymp \Gamma_\tau(z,\zeta)$ for all $u \in \Delta(z,r)$ by Lemma A(ii); see the proof of Lemma 8 for similar estimates. Hence it suffices to show that

$$\frac{(1-|z|)^{c+1}\gamma(z)^q}{\widehat{\nu}(z)} \int_{\mathbb{D}} |(1+\beta(z,\zeta))\Gamma_\tau(z,\zeta)|^q \frac{(1-|\zeta|^2)^{\sigma}}{|1-z\overline{\zeta}|^{2+c+\sigma}} \nu(\zeta)\,dA(\zeta) \lesssim 1, \quad z \in \mathbb{D}, \tag{3.7}$$

to obtain (iii) for $f_2 \in \mathrm{BO}(\Delta)$. The proof of (3.7) is involved and will be divided into four separate cases. Before dealing with each case, we observe that since $\beta(z,\zeta)$ grows logarithmically, we may pick up $0 < \delta < \min\left\{\sigma, \frac{q}{p}\beta(\omega) + \beta(\nu) + \frac{\sigma}{2}\right\}$ and a constant $C = C(\delta) > 0$ such that

$$1 + \beta(z,\zeta) \le C\,|(1-|\varphi_z(\zeta)|)^{-\frac{\delta}{q}} = C\left(\frac{|1-\overline{z}\zeta|^2}{(1-|z|)(1-|\zeta|)}\right)^{\frac{\delta}{q}}, \quad z,\zeta \in \mathbb{D}. \tag{3.8}$$

Case 1 If

$$\zeta \in D_1(z) = \left\{ w \in \mathbb{D} : 1 - \frac{2|1-z\overline{w}|^2}{1-|z|^2} \le 0 \right\},$$

then $1-|z|\lesssim|1-z\overline{\zeta}|^2$ and

$$\Gamma_\tau(z,\zeta)^q\le\frac{\left(\frac{|1-\overline{z}\zeta|^2}{\max\{1-|z|^2,1-|\zeta|^2\}}\right)^{\frac{q}{p}-\tau-1}}{\min\left\{\frac{\widehat{\nu}(z)}{(1-|z|)^\tau},\frac{\widehat{\nu}(\zeta)}{(1-|\zeta|)^\tau}\right\}}\widehat{\omega}(0)^{\frac{q}{p}}\lesssim\left(\frac{|1-z\overline{\zeta}|^2}{1-|z|^2}\right)^{\frac{q}{p}-\tau-1}\frac{(1-|z|)^\tau}{\widehat{\nu}(z)}\chi_{D(0,|z|)}(\zeta)$$
$$+\left(\frac{|1-z\overline{\zeta}|^2}{1-|z|^2}\right)^{\frac{q}{p}-\tau-1}\frac{(1-|\zeta|)^\tau}{\widehat{\nu}(\zeta)}\chi_{\mathbb{D}\setminus D(0,|z|)}(\zeta),\quad z\in\mathbb{D},\quad \zeta\in D_1(z),$$

because of how τ is chosen in Lemma 9. Therefore (3.8) together with Lemmas A(ii) and 3 (ii) yields

$$\begin{aligned}
&\frac{(1-|z|)^{c+1}\gamma(z)^q}{\widehat{\nu}(z)}\int_{D_1(z)}|(1+\beta(z,\zeta))\Gamma_\tau(z,\zeta)|^q\frac{(1-|\zeta|^2)^\sigma}{|1-z\overline{\zeta}|^{2+c+\sigma}}\nu(\zeta)\,dA(\zeta)\\
&\lesssim\frac{(1-|z|)^{c+2+2\tau-\delta-\frac{q}{p}}\gamma(z)^q}{\widehat{\nu}(z)^2}\int_{D_1(z)\cap D(0,|z|)}\frac{(1-|\zeta|^2)^{\sigma-\delta}}{|1-z\overline{\zeta}|^{4+c+\sigma-2\left(\frac{q}{p}+\delta-\tau\right)}}\nu(\zeta)\,dA(\zeta)\\
&\quad+\frac{(1-|z|)^{c+2+\tau-\delta-\frac{q}{p}}\gamma(z)^q}{\widehat{\nu}(z)}\int_{D_1(z)\setminus D(0,|z|)}\frac{(1-|\zeta|^2)^{\sigma-\delta+\tau}}{\widehat{\nu}(\zeta)|1-z\overline{\zeta}|^{4+c+\sigma-2\left(\frac{q}{p}+\delta-\tau\right)}}\nu(\zeta)\,dA(\zeta)\\
&\lesssim\frac{(1-|z|)^{\frac{c}{2}+\tau-\frac{\sigma}{2}}\gamma(z)^q}{\widehat{\nu}(z)^2}\int_0^{|z|}(1-s)^{\sigma-\delta}\nu(s)\,ds\\
&\quad+\frac{(1-|z|)^{\frac{c}{2}-\frac{\sigma}{2}}\gamma(z)^q}{\widehat{\nu}(z)}\int_{|z|}^1(1-s)^{\sigma-\delta+\tau}\frac{\nu(s)}{\widehat{\nu}(s)}\,ds\\
&\lesssim\frac{(1-|z|)^{\frac{c}{2}+\tau-\frac{\sigma}{2}+1-\frac{q}{p}}}{\widehat{\nu}(z)\widehat{\omega}(z)^{\frac{q}{p}}}+\frac{(1-|z|)^{\frac{c}{2}+\frac{\sigma}{2}+1+\tau-\delta-\frac{q}{p}}}{\widehat{\omega}(z)^{\frac{q}{p}}}\\
&\lesssim(1-|z|)^{\frac{c}{2}+\tau-\frac{\sigma}{2}+1-\frac{q}{p}-\beta(\nu)-\frac{q}{p}\beta(\omega)}\lesssim1,\quad z\in\mathbb{D},
\end{aligned}$$

where the last estimate is an immediate consequence of the choices of c and δ.

Case 2 If

$$\zeta\in D_2(z)=\left\{w\in\mathbb{D}:1-\frac{2|1-z\overline{w}|^2}{1-|z|^2}\ge|z|\ge|w|\right\},$$

then $|1-z\overline{\zeta}|\asymp1-|z|^2\le1-|\zeta|^2$, which together the fact that $\frac{\widehat{\nu}(t)}{(1-t)^\tau}$ and $\frac{\widehat{\omega}(r)}{(1-r)^{\tau\frac{p}{q}}}$ are essentially decreasing on $[0,1)$ gives

$$\Gamma_\tau(z,\zeta)^q\lesssim\gamma(z)^{-q},\quad z\in\mathbb{D},\quad\zeta\in D_2(z).$$

Therefore (3.8) and Lemma A(iii) yield

$$\begin{aligned}
&\frac{(1-|z|)^{c+1}\gamma(z)^q}{\widehat{\nu}(z)}\int_{D_2(z)}|(1+\beta(z,\zeta))\Gamma_\tau(z,\zeta)|^q\frac{(1-|\zeta|^2)^\sigma}{|1-z\overline{\zeta}|^{2+c+\sigma}}\nu(\zeta)\,dA(\zeta)\\
&\lesssim\frac{(1-|z|)^{c+1-\delta}}{\widehat{\nu}(z)}\int_{D_2(z)}\frac{(1-|\zeta|^2)^{\sigma-\delta}}{|1-z\overline{\zeta}|^{2+c+\sigma-2\delta}}\nu(\zeta)\,dA(\zeta)\\
&\lesssim\frac{(1-|z|)^{c+1-\delta}}{\widehat{\nu}(z)}\int_0^{|z|}\frac{\nu(r)}{(1-r)^{c+1-\delta}}\,dr\lesssim1,\quad z\in\mathbb{D}.
\end{aligned}$$

Case 3 If

$$\zeta \in D_3(z)=\left\{w\in\mathbb{D}:\min\left\{1-\frac{2|1-z\overline{w}|^2}{1-|z|^2},|w|\right\}\geq|z|\right\},$$

then $|1-z\overline{\zeta}|\asymp 1-|z|^2\geq 1-|\zeta|^2$, which together the fact that $\frac{\widehat{\nu}(t)}{(1-t)^\tau}$ and $\frac{\widehat{\omega}(r)}{(1-r)^{\tau\frac{p}{q}}}$ are essentially decreasing on $[0,1)$ implies

$$\Gamma_\tau(z,\zeta)^q\lesssim\frac{\widehat{\omega}(z)^{\frac{q}{p}}(1-|z|)^{\frac{q}{p}-1}}{\widehat{\nu}(\zeta)},\quad z\in\mathbb{D},\quad \zeta\in D_3(z).$$

Therefore (3.8) and Lemma 3(ii) imply

$$\begin{aligned}
&\frac{(1-|z|)^{c+1}\gamma(z)^q}{\widehat{\nu}(z)}\int_{D_3(z)}|(1+\beta(z,\zeta))\Gamma_\tau(z,\zeta)|^q\frac{(1-|\zeta|^2)^\sigma}{|1-z\overline{\zeta}|^{2+c+\sigma}}\nu(\zeta)\,dA(\zeta)\\
&\lesssim(1-|z|)^{c+1-\delta}\int_{D_3(z)}\frac{(1-|\zeta|^2)^{\sigma-\delta}}{|1-z\overline{\zeta}|^{2+c+\sigma-2\delta}}\frac{\nu(\zeta)}{\widehat{\nu}(\zeta)}dA(\zeta)\\
&\lesssim(1-|z|)^{\delta-\sigma}\int_{|z|}^1\frac{(1-s)^{\sigma-\delta}\nu(s)}{\widehat{\nu}(s)}ds\lesssim 1,\quad z\in\mathbb{D}.
\end{aligned}$$

Case 4 If

$$\zeta\in D_4(z)=\left\{w\in\mathbb{D}:1-\frac{2|1-z\overline{w}|^2}{1-|z|^2}<|z|\right\},$$

then Lemma A(ii) gives

$$\widehat{\omega}\left(1-\frac{2|1-z\overline{\zeta}|^2}{1-|z|^2}\right)\lesssim\left(\frac{|1-z\overline{\zeta}|}{1-|z|}\right)^{2\beta(\omega)}\widehat{\omega}(z),\quad z\in\mathbb{D},\quad\zeta\in D_4(z),$$

and hence

$$\begin{aligned}
\Gamma_\tau(z,\zeta)^q\lesssim&\left(\frac{|1-z\overline{\zeta}|}{1-|z|}\right)^{2\beta(\omega)\frac{q}{p}}\widehat{\omega}(z)^{\frac{q}{p}}\left(\left(\frac{|1-z\overline{\zeta}|^2}{1-|\zeta|}\right)^{\frac{q}{p}-\tau-1}\frac{(1-|z|)^\tau}{\widehat{\nu}(z)}\chi_{D(0,|z|)}(\zeta)\right.\\
&\left.+\left(\frac{|1-z\overline{\zeta}|^2}{1-|z|}\right)^{\frac{q}{p}-\tau-1}\frac{(1-|\zeta|)^\tau}{\widehat{\nu}(\zeta)}\chi_{\mathbb{D}\setminus D(0,|z|)}(\zeta)\right),\quad z\in\mathbb{D},\quad\zeta\in D_4(z).
\end{aligned}$$

Therefore (3.8) and Lemmas A(iii) and 3 (ii) yield

$$\begin{aligned}
&\frac{(1-|z|)^{c+1}\gamma(z)^q}{\widehat{\nu}(z)}\int_{D_4(z)}|(1+\beta(z,\zeta))\Gamma_\tau(z,\zeta)|^q\frac{(1-|\zeta|^2)^\sigma}{|1-z\overline{\zeta}|^{2+c+\sigma}}\nu(\zeta)\,dA(\zeta)\\
&\lesssim\frac{(1-|z|)^{c+2-\delta-\frac{q}{p}-2\beta(\omega)\frac{q}{p}+\tau}}{\widehat{\nu}(z)}\int_{D_4(z)\cap D(0,|z|)}\frac{(1-|\zeta|^2)^{\sigma-\delta-\frac{q}{p}+\tau+1}}{|1-z\overline{\zeta}|^{4+c+\sigma-2\delta-2\beta(\omega)\frac{q}{p}-2\frac{q}{p}+2\tau}}\nu(\zeta)\,dA(\zeta)\\
&\quad+(1-|z|)^{c+2-\delta-\frac{q}{p}-2\beta(\omega)\frac{q}{p}-\frac{q}{p}+\tau+1}\int_{D_4(z)\setminus D(0,|z|)}\frac{(1-|\zeta|)^{\sigma-\delta+\tau}}{|1-z\overline{\zeta}|^{4+c+\sigma-2\delta-2\beta(\omega)\frac{q}{p}-2\frac{q}{p}+2\tau}}\frac{\nu(\zeta)}{\widehat{\nu}(\zeta)}\,dA(\zeta)\\
&\lesssim\frac{(1-|z|)^{c+2-\delta-\frac{q}{p}-2\beta(\omega)\frac{q}{p}+\tau}}{\widehat{\nu}(z)}\int_0^{|z|}\frac{\nu(r)}{(1-r)^{2+c-\delta-2\beta(\omega)\frac{q}{p}-\frac{q}{p}+\tau}}\,dr\\
&\quad+\frac{1}{(1-|z|)^{\sigma-\delta+\tau}}\int_{|z|}^1\frac{(1-r)^{\sigma-\delta+\tau}\nu(r)}{\widehat{\nu}(r)}\,dr\lesssim 1,\quad z\in\mathbb{D}.
\end{aligned}$$

Since $\mathbb{D}=\cup_{j=1}^{4}D_j(z)$ for each $z\in\mathbb{D}$, by combining the four cases we obtain (3.7). Thus (ii) implies (iii), and the proof is complete. □

4 Boundedness of integral operators

In order to deal with the boundedness of Hankel operators, we need a technical result concerning certain integral operators. For $f\in L_b^1$ and $b,c\in\mathbb{R}$, define

$$T_{b,c}(f)(z)=\int_{\mathbb{D}} f(\zeta)\frac{(1-|\zeta|^2)^b}{(1-z\overline{\zeta})^c}\,dA(\zeta),\quad z\in\mathbb{D},$$

and

$$S_{b,c}(f)(z)=\int_{\mathbb{D}} f(\zeta)\frac{(1-|\zeta|^2)^b}{|1-z\overline{\zeta}|^c}\,dA(\zeta),\quad z\in\mathbb{D}.$$

In the analytic case the operator $T_{b,c}$ can be interpreted as a fractional differentiation or integration depending on the parameters b and c [20]. The boundedness of these operator between L^p spaces induced by standard weights has been characterized in [19].

Lemma A(ii) shows that for $\eta\in\widehat{\mathcal{D}}$ there exists a constant $c_0=c_0(\sigma)>1$ such that hypotheses (i) and (ii) of the next lemma are satisfied for all $c\geq c_0$.

Lemma 12 *Let $1<p\leq q<\infty$, $b>-1$, $c>1$ and $\sigma,\eta\in\mathcal{D}$ such that*

(i) $$\int_r^1\frac{(1-t)^{c-2}}{\widehat{\eta}(t)^{\frac{1}{q}}}\,dt\lesssim\frac{(1-r)^{c-1}}{\widehat{\eta}(r)^{\frac{1}{q}}},\quad 0\leq r<1;$$

(ii) $$\int_0^r\frac{\eta(t)}{(1-t)^{\frac{cq}{p}-1}\widehat{\eta}(t)^{\frac{1}{p'}}}\,dt\lesssim\frac{\widehat{\eta}(r)^{\frac{1}{p}}}{(1-r)^{\frac{cq}{p}-1}},\quad 0\leq r<1.$$

Then the following statements are equivalent:

1. $S_{b,c}:A_\sigma^p\to L_\eta^q$ *is bounded;*
2. $T_{b,c}:A_\sigma^p\to L_\eta^q$ *is bounded;*
3. $\sup_{0<r<1}(1-r)^{2+b-c+\frac{1}{q}-\frac{1}{p}}\frac{\widehat{\eta}(r)^{\frac{1}{q}}}{\widehat{\sigma}(r)^{\frac{1}{p}}}<\infty.$

Proof Obviously (1) implies (2). Assume now (2), and for each $\zeta\in\mathbb{D}$ and $N\in\mathbb{N}$, define $f_{\zeta,N}\in H^\infty$ by $f_{\zeta,N}(z)=\frac{z^N}{\sigma(S(\zeta))^{\frac{1}{p}}}\left(\frac{1-|\zeta|^2}{1-\overline{\zeta}z}\right)^{2+b+N}$ for all $z\in\mathbb{D}$. By differentiating the reproducing formula of A_b^2 we obtain

$$g^{(N)}(z)=M_1\int_{\mathbb{D}}\frac{\overline{u}^N g(u)(1-|u|^2)^b}{(1-\overline{u}z)^{2+b+N}}\,dA(u),\quad z\in\mathbb{D},\quad N\in\mathbb{N},\quad g\in A_b^2,\tag{4.1}$$

where $M_1=M_1(N,b)>0$ is a constant. Therefore

$$\begin{aligned}T_{b,c}(f_{\zeta,N})(z)&=\frac{(1-|\zeta|^2)^{2+b+N}}{\sigma(S(\zeta))^{\frac{1}{p}}}\int_{\mathbb{D}}\frac{u^N(1-|u|^2)^b}{(1-u\overline{\zeta})^{2+b+N}(1-\overline{u}z)^c}\,dA(u)\\&=\frac{(1-|\zeta|^2)^{2+b+N}}{\sigma(S(\zeta))^{\frac{1}{p}}}\overline{\int_{\mathbb{D}}\frac{\overline{u}^N(1-|u|^2)^b}{(1-\zeta\overline{u})^{2+b+N}(1-\overline{z}u)^c}\,dA(u)}\\&=M_2\frac{(1-|\zeta|^2)^{2+b+N}}{\sigma(S(\zeta))^{\frac{1}{p}}}\frac{z^N}{(1-z\overline{\zeta})^{c+N}},\end{aligned}$$

where $M_2 = M_2(b, c, N) > 0$. Fix $N > \max\left\{\frac{\lambda(\eta)+1}{q} - c, \frac{\lambda(\sigma)+1}{p} - b - 2\right\}$. Then Lemma A(iv) gives $\|f_{\zeta,N}\|_{L^p_\sigma} \asymp 1$ and

$$\int_{\mathbb{D}} \frac{\eta(z)}{|1-\overline{\zeta}z|^{(c+N)q}}\,dA(z) \asymp \frac{\eta(S(\zeta))}{(1-|\zeta|)^{(c+N)q}}, \quad \zeta \in \mathbb{D}.$$

Therefore (2) yields

$$\begin{aligned}\infty > \|f_{\zeta,N}\|^q_{L^p_\sigma} &\gtrsim \|T_{b,c}(f_{\zeta,N})\|^q_{L^q_\eta} \asymp \left(\frac{(1-|\zeta|^2)^{2+b+N}}{\sigma(S(\zeta))^{\frac{1}{p}}}\right)^q \int_{\mathbb{D}} \frac{\eta(z)}{|1-\overline{\zeta}z|^{(c+N)q}}\,dA(z)\\ &\asymp (1-|\zeta|^2)^{q(2+b-c)} \frac{\eta(S(\zeta))}{\sigma(S(\zeta))^{\frac{q}{p}}}, \quad \zeta \in \mathbb{D},\end{aligned}$$

thus (3) holds.

Assume (3) holds and let $h(\zeta) = \widehat{\sigma}(\zeta)^{\frac{1}{pp'}}(1-|\zeta|^2)^{\frac{b}{p}+\left(\frac{1}{p}-\frac{1}{q}\right)\frac{1}{p'}}$ for all $\zeta \in \mathbb{D}$. Then Hölder's inequality yields

$$\begin{aligned}|S_{b,c}f(z)| &\le \left(\int_{\mathbb{D}} |f(\zeta)|^p h(\zeta)^p \frac{dA(\zeta)}{|1-z\overline{\zeta}|^c}\right)^{\frac{1}{p}} \left(\int_{\mathbb{D}} \left(\frac{(1-|\zeta|^2)^b}{h(\zeta)}\right)^{p'} \frac{dA(\zeta)}{|1-z\overline{\zeta}|^c}\right)^{\frac{1}{p'}}\\ &= I_1(z)^{\frac{1}{p}} \cdot I_2(z)^{\frac{1}{p'}},\end{aligned}$$

where

$$\begin{aligned}I_2(z) &= \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{b-\frac{1}{p}+\frac{1}{q}}}{|1-z\overline{\zeta}|^c \widehat{\sigma}(\zeta)^{\frac{1}{p}}}\,dA(\zeta) \asymp \int_0^1 \frac{(1-r)^{b-\frac{1}{p}+\frac{1}{q}}}{\widehat{\sigma}(r)^{\frac{1}{p}}(1-r|z|)^{c-1}}\,dr\\ &= \int_0^{|z|} \frac{(1-r)^{b-\frac{1}{p}+\frac{1}{q}}}{\widehat{\sigma}(r)^{\frac{1}{p}}(1-r|z|)^{c-1}}\,dr + \int_{|z|}^1 \frac{(1-r)^{b-\frac{1}{p}+\frac{1}{q}}}{\widehat{\sigma}(r)^{\frac{1}{p}}(1-r|z|)^{c-1}}\,dr = J^{|z|} + J_{|z|}.\end{aligned}$$

Lemma B together with the assumption (3) yields

$$J^{|z|} \le \int_0^{|z|} \frac{(1-r)^{b-\frac{1}{p}+\frac{1}{q}+1-c}}{\widehat{\sigma}(r)^{\frac{1}{p}}}\,dr \lesssim \int_0^{|z|} \frac{dr}{\widehat{\eta}(r)^{\frac{1}{q}}(1-r)} \lesssim \frac{1}{\widehat{\eta}(z)^{\frac{1}{q}}}, \quad z \in \mathbb{D},$$

since $\eta \in \mathcal{D} \subset \check{\mathcal{D}}$ by the hypothesis. In a similar fashion, (3) together with the hypothesis (i) gives

$$J_{|z|} \le \frac{1}{(1-|z|)^{c-1}} \int_{|z|}^1 \frac{(1-r)^{b-\frac{1}{p}+\frac{1}{q}}}{\widehat{\sigma}(r)^{\frac{1}{p}}}\,dr \lesssim \frac{1}{(1-|z|)^{c-1}} \int_{|z|}^1 \frac{(1-r)^{c-2}}{\widehat{\eta}(r)^{\frac{1}{q}}}\,dr \lesssim \frac{1}{\widehat{\eta}(z)^{\frac{1}{q}}}, \quad z \in \mathbb{D},$$

and hence $I_2(z) \lesssim \widehat{\eta}(z)^{-\frac{1}{q}}$ for all $z \in \mathbb{D}$. This estimate and Minkowski's integral inequality (Fubini's theorem in the case $q = p$) now yield

$$\begin{aligned}\|S_{b,c}(f)\|^p_{L^q_\eta} &\lesssim \left(\int_{\mathbb{D}} \left(\int_{\mathbb{D}} |f(\zeta)|^p h(\zeta)^p \frac{dA(\zeta)}{|1-z\overline{\zeta}|^c}\right)^{\frac{q}{p}} \frac{\eta(z)}{\widehat{\eta}(z)^{\frac{1}{p'}}}\,dA(z)\right)^{\frac{p}{q}}\\ &\le \int_{\mathbb{D}} |f(\zeta)|^p \widetilde{\sigma}(\zeta) I_3(\zeta)\,dA(\zeta),\end{aligned}$$

where

$$I_3(\zeta)=\frac{h(\zeta)^p}{\widetilde{\sigma}(\zeta)}\left(\int_{\mathbb{D}}\frac{\eta(z)dA(z)}{|1-z\overline{\zeta}|^{\frac{cq}{p}}\widehat{\eta}(z)^{\frac{1}{p'}}}\right)^{\frac{p}{q}}\asymp\frac{h(\zeta)^p}{\widetilde{\sigma}(\zeta)}\left(\int_0^1\frac{\eta(r)}{(1-r|\zeta|)^{\frac{cq}{p}-1}\widehat{\eta}(r)^{\frac{1}{p'}}}dr\right)^{\frac{p}{q}}.$$

Since

$$\int_0^{|\zeta|}\frac{\eta(r)}{(1-r|\zeta|)^{\frac{cq}{p}-1}\widehat{\eta}(r)^{1/p'}}dr\le\int_0^{|\zeta|}\frac{\eta(r)}{(1-r)^{\frac{cq}{p}-1}\widehat{\eta}(r)^{\frac{1}{p'}}}dr\lesssim\frac{\widehat{\eta}(\zeta)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{cq}{p}-1}},\quad \zeta\in\mathbb{D},$$

by the hypothesis (ii), and

$$\int_{|\zeta|}^1\frac{\eta(r)}{(1-r|\zeta|)^{\frac{cq}{p}-1}\widehat{\eta}(r)^{\frac{1}{p'}}}dr\le\frac{1}{(1-|\zeta|)^{\frac{cq}{p}-1}}\int_{|\zeta|}^1\frac{\eta(r)}{\widehat{\eta}(r)^{\frac{1}{p'}}}dr\asymp\frac{\widehat{\eta}(\zeta)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{cq}{p}-1}},\quad \zeta\in\mathbb{D},$$

we deduce

$$I_3(\zeta)\lesssim(1-|\zeta|)^{2+b-c+\frac{1}{q}-\frac{1}{p}}\frac{\widehat{\eta}(\zeta)^{\frac{1}{q}}}{\widehat{\sigma}(\zeta)^{\frac{1}{p}}}\lesssim 1,\quad \zeta\in\mathbb{D},$$

by the assumption (3). It follows that $\|S_{b,c}(f)\|_{L^q_\eta}\lesssim\|f\|_{A^p_{\widetilde{\sigma}}}$. This finishes the proof because $\|f\|_{A^p_{\widetilde{\sigma}}}\asymp\|f\|_{A^p_\sigma}$ for all $f\in\mathcal{H}(\mathbb{D})$ by Lemma 5 provided $\sigma\in\mathcal{D}$. □

5 Proof of Theorem 1

In order to prove the sufficiency part of Theorem 1 we shall use the next result which follows from the argument used in the proof of [12, Lemma 4.5].

Lemma 13 *Let $1<q<\infty$ and ν,ω weights such that $P_\omega:L^q_\nu\to L^q_\nu$ is bounded. Then*

$$\|H^\nu_f(g)\|^q_{L^q_\nu}\le(1+\|P_\omega\|_{L^q_\nu\to L^q_\nu})\|H^\omega_f(g)\|^q_{L^q_\nu},\quad f\in L^q_\nu,\quad g\in H^\infty.$$

Proposition 14 *Let $1<p\le q<\infty$, $\nu\in B_q$ a radial weight and $\omega\in\mathcal{D}$. If $f\in\mathrm{BO}(\Delta)$, then $H^\nu_f:A^p_\omega\to L^q_\nu$ is bounded.*

Proof By [5] there exists a constant $s_0=s_0(\nu)>-1$ such that $P_s:L^q_\nu\to L^q_\nu$ is bounded for each $s>s_0$. Let $0<\tau<\min\{q\alpha(\omega)/p,\alpha(\nu)\}$, where $\alpha(\nu)$ and $\alpha(\omega)$ are those from Lemma B. Then Lemmas 9 and 13 yield

$$\begin{aligned}\|H^\nu_f(g)\|^q_{L^q_\nu}&\lesssim\|H^s_f(g)\|^q_{L^q_\nu}\le\int_{\mathbb{D}}\left(\int_{\mathbb{D}}\frac{|f(z)-f(\zeta)||g(\zeta)|}{|1-\overline{z}\zeta|^{2+s}}(1-|\zeta|^2)^s\,dA(\zeta)\right)^q\nu(z)\,dA(z)\\&\lesssim\int_{\mathbb{D}}\left(\int_{\mathbb{D}}|g(\zeta)|\frac{(\beta(z,\zeta)+1)\Gamma_\tau(z,\zeta)}{|1-\overline{z}\zeta|^{2+s}}(1-|\zeta|^2)^s\,dA(\zeta)\right)^q\nu(z)\,dA(z),\quad g\in H^\infty.\end{aligned}$$

Let $s>\max\{s_0,2(\beta(\omega)+\beta(\nu)+2\alpha(\nu))\}$, $\delta<\min\{\frac{\tau}{q},\frac{\alpha(\nu)}{q}\}$ and $K>1$ to be fixed later. Then applying (3.8), we get

$$\begin{aligned}\|H^\nu_f(g)\|^q_{L^q_\nu}&\lesssim\sum_{j=1}^5\int_{\mathbb{D}}\left(\int_{\Omega_j(z)}|g(\zeta)|\frac{\Gamma_\tau(z,\zeta)\,dA(\zeta)}{|1-\overline{z}\zeta|^{2+s-2\delta}(1-|\zeta|^2)^{\delta-s}}\right)^q\frac{\nu(z)}{(1-|z|)^{q\delta}}\,dA(z)\\&=\sum_{j=1}^5 I_j(g),\end{aligned}\tag{5.1}$$

where

$$\Omega_1(z) = \left\{\zeta \in \mathbb{D} : \frac{1}{|1-\overline{z}\zeta|^2} \le \frac{2}{\max\{1-|z|^2, 1-|\zeta|^2\}}\right\} \cap D(0,|z|),$$
$$\Omega_2(z) = \left\{\zeta \in \mathbb{D} : \frac{1}{|1-\overline{z}\zeta|^2} \le \frac{2}{\max\{1-|z|^2, 1-|\zeta|^2\}}\right\} \cap (\mathbb{D}\setminus D(0,|z|)),$$
$$\Omega_3(z,K) = \left\{\zeta \in \mathbb{D} : \frac{1-|\zeta|}{K} \ge \frac{2|1-\overline{z}\zeta|^2}{\max\{1-|z|^2, 1-|\zeta|^2\}}\right\},$$
$$\Omega_4(z,K) = \left\{\zeta \in \mathbb{D} : \frac{1-|\zeta|}{K} < \frac{2|1-\overline{z}\zeta|^2}{\max\{1-|z|^2, 1-|\zeta|^2\}} < 1\right\} \cap D(0,|z|),$$
$$\Omega_5(z,K) = \left\{\zeta \in \mathbb{D} : \frac{1-|\zeta|}{K} < \frac{2|1-\overline{z}\zeta|^2}{\max\{1-|z|^2, 1-|\zeta|^2\}} < 1\right\} \cap (\mathbb{D}\setminus D(0,|z|)).$$

The quantities $I_j(g)$, $j=1,\dots,5$, will be estimated separately.

Case $I_1(g)$ By using the definition of $\Omega_1(z)$, and the fact that $\frac{\widehat{\nu}(x)}{(1-x)^\tau}$ is essentially decreasing on $[0,1)$ we deduce

$$\Gamma_\tau(z,\zeta) \lesssim \frac{\left(\frac{|1-\overline{z}\zeta|^2}{\max\{1-|z|^2,1-|\zeta|^2\}}\right)^{\frac{1}{p}-\frac{1}{q}}}{\min\left\{\frac{\widehat{\nu}(z)}{(1-|z|)^\tau}, \frac{\widehat{\nu}(\zeta)}{(1-|\zeta|)^\tau}\right\}^{\frac{1}{q}}} \lesssim \left(\frac{|1-\overline{z}\zeta|^2}{1-|\zeta|^2}\right)^{\frac{1}{p}-\frac{1}{q}} \left(\frac{(1-|z|)^\tau}{\widehat{\nu}(z)}\right)^{\frac{1}{q}}, \quad z\in\mathbb{D}, \quad \zeta\in\Omega_1(z).$$

Then the estimate

$$M_1(r,f) \le M_p(r,f) \lesssim \|f\|_{A^p_\omega} \widehat{\omega}(r)^{-\frac{1}{p}}, \quad 0\le r<1, \quad f\in\mathcal{H}(\mathbb{D}), \tag{5.2}$$

and Lemma 3(ii) yield

$$\begin{aligned} I_1(g) &\lesssim \int_{\mathbb{D}} \left(\int_{\Omega_1(z)} |g(\zeta)| \frac{(1-|\zeta|)^{s-\delta-\frac{1}{p}+\frac{1}{q}}}{|1-\overline{z}\zeta|^{2+s-2\delta-2\left(\frac{1}{p}-\frac{1}{q}\right)}}\, dA(\zeta)\right)^q \frac{\nu(z)(1-|z|)^{\tau-\delta q}}{\widehat{\nu}(z)}\, dA(z) \\ &\lesssim \left(\int_{\mathbb{D}} |g(\zeta)|(1-|\zeta|)^{\frac{s}{2}-1}\, dA(\zeta)\right)^q \int_{\mathbb{D}} \frac{\nu(z)(1-|z|)^{\tau-\delta q}}{\widehat{\nu}(z)}\, dA(z) \\ &\lesssim \|g\|^q_{A^p_\omega} \left(\int_0^1 \frac{(1-t)^{\frac{s}{2}-1}}{\widehat{\omega}(t)^{\frac{1}{p}}}\, dt\right)^q \lesssim \|g\|^q_{A^p_\omega} \left(\int_0^1 (1-t)^{\frac{s}{2}-1-\frac{\beta(\omega)}{p}}\, dt\right)^q \lesssim \|g\|^q_{A^p_\omega}, \quad g\in H^\infty. \end{aligned}$$

Case $I_2(g)$ The definition of $\Omega_2(z)$ and the fact that $\frac{\widehat{\nu}(x)}{(1-x)^\tau}$ is essentially decreasing imply

$$\Gamma_\tau(z,\zeta) \lesssim \frac{\left(\frac{|1-\overline{z}\zeta|^2}{\max\{1-|z|^2,1-|\zeta|^2\}}\right)^{\frac{1}{p}-\frac{1}{q}}}{\min\left\{\frac{\widehat{\nu}(z)}{(1-|z|)^\tau}, \frac{\widehat{\nu}(\zeta)}{(1-|\zeta|)^\tau}\right\}^{1/q}} \lesssim \left(\frac{|1-\overline{z}\zeta|^2}{1-|\zeta|^2}\right)^{\frac{1}{p}-\frac{1}{q}} \left(\frac{(1-|\zeta|)^\tau}{\widehat{\nu}(\zeta)}\right)^{\frac{1}{q}}, \quad z\in\mathbb{D}, \quad \zeta\in\Omega_2(z).$$

Therefore (5.2) and Lemmas A and B yield

$$\begin{aligned} I_2(g) &\lesssim \int_{\mathbb{D}} \left(\int_{\Omega_2(z)} |g(\zeta)| \frac{(1-|\zeta|)^{s-\delta-\frac{1}{p}+\frac{1+\tau}{q}}}{\widehat{\nu}(\zeta)^{\frac{1}{q}} |1-\overline{z}\zeta|^{2+s-2\delta-2\left(\frac{1}{p}-\frac{1}{q}\right)}} \, dA(\zeta) \right)^q (1-|z|)^{-\delta q} \nu(z) \, dA(z) \\ &\lesssim \left(\int_{\mathbb{D}} |g(\zeta)| \frac{(1-|\zeta|)^{\frac{s}{2}-1+\frac{\tau}{q}}}{\widehat{\nu}(\zeta)^{\frac{1}{q}}} \, dA(\zeta) \right)^q \int_0^1 (1-r)^{-\delta q} \nu(r) \, dr \\ &\lesssim \|g\|_{A^p_\omega}^q \left(\int_0^1 \frac{(1-r)^{\frac{s}{2}-1+\frac{\tau}{q}}}{\widehat{\omega}(r)^{\frac{1}{p}} \widehat{\nu}(\zeta)^{\frac{1}{q}}} \, dA(\zeta) \right)^q \left(\widehat{\nu}(0) + \int_0^1 \frac{\widehat{\nu}(t)}{(1-t)^{1+q\delta}} \, dt \right) \\ &\lesssim \|g\|_{A^p_\omega}^q \left(\int_0^1 (1-t)^{\frac{s}{2}-1-\frac{\beta(\omega)}{p}-\frac{\beta(\nu)-\tau}{q}} \, dt \right)^q \lesssim \|g\|_{A^p_\omega}^q, \quad g \in H^\infty. \end{aligned}$$

Case $I_3(g)$ To deal with $I_3(g)$, note first that now $2K|1-\overline{z}\zeta|^2 \le (1-|\zeta|)\max\{1-|z|^2, 1-|\zeta|^2\} \le 2(\max\{1-|z|, 1-|\zeta|\})^2$ for all $\zeta \in \Omega_3(z,K)$. Hence $\zeta \in \Delta(z,R)$ for some $R = R(K) \in (0,\infty)$ if $K \ge 1$ is sufficiently large. Fix such a K, and note that then $\widehat{\nu}(\zeta) \asymp \widehat{\nu}(z)$ for all $\zeta \in \Omega(z,K)$ by Lemma A(ii). By using this and the fact that $\frac{\widehat{\omega}(x)}{(1-x)^{\frac{p\tau}{q}}}$ is essentially decreasing on $[0,1)$ we deduce

$$\begin{aligned} \Gamma_\tau(z,\zeta) &\lesssim \left(\frac{|1-\overline{z}\zeta|^2}{\max\{1-|z|^2, 1-|\zeta|^2\}} \right)^{\frac{1}{p}-\frac{1}{q}} \frac{\widehat{\omega}(\zeta)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{\tau}{q}}} \min\left\{ \frac{\widehat{\nu}(z)}{(1-|z|)^\tau}, \frac{\widehat{\nu}(\zeta)}{(1-|\zeta|)^\tau} \right\}^{-\frac{1}{q}} \\ &\asymp \frac{(1-|\zeta|)^{\frac{1}{p}-\frac{1}{q}} \widehat{\omega}(\zeta)^{\frac{1}{p}}}{\widehat{\nu}(\zeta)^{\frac{1}{q}}}, \quad z \in \mathbb{D}, \quad \zeta \in \Omega_3(z,K), \end{aligned}$$

and it follows that

$$\begin{aligned} I_3(g) &\lesssim \int_{\mathbb{D}} \left(\int_{\Delta(z,R)} \left(|g(\zeta)| \widehat{\omega}(\zeta)^{\frac{1}{p}} \right) \frac{\left(1-|\zeta|^2\right)^{s-\delta+\frac{1}{p}-\frac{2}{q}} dA(\zeta)}{|1-\overline{z}\zeta|^{2+s-2\delta}} \right)^q \frac{\nu(z)(1-|z|)^{1-q\delta}}{\widehat{\nu}(z)} \, dA(z) \\ &\asymp \int_{\mathbb{D}} \left(\int_{\Delta(z,R)} \left(|g(\zeta)| \widetilde{\omega}(\zeta)^{\frac{1}{p}} \right) \frac{\left(1-|\zeta|^2\right)^{s+\frac{2}{p}-\frac{2}{q}} dA(\zeta)}{|1-\overline{z}\zeta|^{2+s}} \right)^q \frac{\nu(z)(1-|z|)}{\widehat{\nu}(z)} \, dA(z) \\ &\asymp \int_{\mathbb{D}} \left(\int_{\Delta(z,R)} \left(|g(\zeta)| \widetilde{\omega}(\zeta)^{\frac{1}{p}} \right) \frac{\left(1-|\zeta|^2\right)^{s} dA(\zeta)}{|1-\overline{z}\zeta|^{2+s-\frac{2}{p}+\frac{2}{q}}} \right)^q \frac{\nu(z)(1-|z|)}{\widehat{\nu}(z)} \, dA(z) \qquad (5.3) \\ &\le \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \left(|g(\zeta)| \widetilde{\omega}(\zeta)^{\frac{1}{p}} \right) \frac{\left(1-|\zeta|^2\right)^{s} dA(\zeta)}{|1-\overline{z}\zeta|^{2+s-\frac{2}{p}+\frac{2}{q}}} \right)^q \frac{\nu(z)(1-|z|)}{\widehat{\nu}(z)} \, dA(z) \\ &= \left\| S_{s,s+2\left(1-\frac{1}{p}+\frac{1}{q}\right)} \left(|g| \widetilde{\omega}^{\frac{1}{p}} \right) \right\|_{L^q_\eta}^q = \left\| S_{b,c} \left(|g| \widetilde{\omega}^{\frac{1}{p}} \right) \right\|_{L^q_\eta}^q, \quad g \in H^\infty, \end{aligned}$$

where $\eta(z) = \frac{\nu(z)(1-|z|)}{\widehat{\nu}(z)}$ for all $z \in \mathbb{D}$. To apply Lemma 12 with $\sigma \equiv 1$, we must check that its hypotheses are satisfied. To do this, first observe that $\eta \in \mathcal{D}$ and $\widehat{\eta}(r) \asymp (1-r)$ for all

$0 \le r < 1$ by Lemma 4. Hence

$$\int_r^1 \frac{(1-t)^{c-2}}{\widehat{\eta}(t)^{\frac{1}{q}}}\,dt \asymp \int_r^1 (1-t)^{s-\frac{2}{p}+\frac{1}{q}}\,dt \asymp (1-r)^{1+s-\frac{2}{p}+\frac{1}{q}} \asymp \frac{(1-r)^{c-1}}{\widehat{\eta}(r)^{\frac{1}{q}}}, \quad 0 \le r < 1,$$

and, by Lemma 3(iii),

$$\begin{aligned}\int_0^r \frac{\eta(t)}{(1-t)^{\frac{cq}{p}-1}\widehat{\eta}(t)^{\frac{1}{p'}}}\,dt &\asymp \int_0^r \frac{\nu(t)}{\widehat{\nu}(t)(1-t)^{\frac{q}{p}\left(s+2\left(1-\frac{1}{p}+\frac{1}{q}\right)\right)-1-\frac{1}{p}}}\,dt\\ &\lesssim \frac{1}{(1-r)^{\frac{q}{p}\left(s+2\left(1-\frac{1}{p}+\frac{1}{q}\right)\right)-1-\frac{1}{p}}} \asymp \frac{\widehat{\eta}(r)^{\frac{1}{p}}}{(1-r)^{\frac{cq}{p}-1}}, \quad 0 \le r < 1,\end{aligned}$$

so the hypotheses of Lemma 12 are satisfied. Moreover,

$$(1-r)^{2+b-c+\frac{1}{q}-\frac{1}{p}}\frac{\widehat{\eta}(r)^{\frac{1}{q}}}{\widehat{\sigma}(r)^{\frac{1}{p}}} \asymp 1, \quad 0 \le r < 1,$$

and consequently (5.3) and Lemmas 12 and 5 yield $I_3(g) \lesssim \|g\|^q_{A^p_{\widetilde{\omega}}} \asymp \|g\|^q_{A^p_{\omega}}$ for all $g \in H^\infty$.

Case $I_4(g)$ By using the definition of $\Omega_4(z,K)$, Lemma A(ii) and the fact that $\frac{\widehat{\nu}(x)}{(1-x)^\tau}$ is essentially decreasing on $[0,1)$, we deduce

$$\begin{aligned}\Gamma_\tau(z,\zeta) &\lesssim \frac{\left(\frac{|1-\bar{z}\zeta|^2}{\max\{1-|z|^2,1-|\zeta|^2\}}\right)^{\frac{1}{p}-\frac{1}{q}}\widehat{\omega}\left(1-\frac{2|1-\bar{z}\zeta|^2}{\max\{1-|z|^2,1-|\zeta|^2\}}\right)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{\tau}{q}}\min\left\{\frac{\widehat{\nu}(z)}{(1-|z|)^\tau},\frac{\widehat{\nu}(\zeta)}{(1-|\zeta|)^\tau}\right\}^{\frac{1}{q}}}\\ &\lesssim \frac{\left(\frac{|1-\bar{z}\zeta|^2}{1-|\zeta|}\right)^{\frac{1}{p}-\frac{1}{q}}\widehat{\omega}\left(1-\frac{K2|1-\bar{z}\zeta|^2}{\max\{1-|z|^2,1-|\zeta|^2\}}\right)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{\tau}{q}}\min\left\{\frac{\widehat{\nu}(z)}{(1-|z|)^\tau},\frac{\widehat{\nu}(\zeta)}{(1-|\zeta|)^\tau}\right\}^{\frac{1}{q}}} \lesssim \frac{\left(\frac{|1-\bar{z}\zeta|^2}{1-|\zeta|}\right)^{\frac{1}{p}-\frac{1}{q}}\left(\frac{|1-\bar{z}\zeta|}{1-|\zeta|}\right)^{\frac{2\beta(\omega)}{p}}\widehat{\omega}(\zeta)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{\tau}{q}}\min\left\{\frac{\widehat{\nu}(z)}{(1-|z|)^\tau},\frac{\widehat{\nu}(\zeta)}{(1-|\zeta|)^\tau}\right\}^{\frac{1}{q}}}\\ &\lesssim \frac{|1-\bar{z}\zeta|^{\frac{2\beta(\omega)}{p}+\frac{2}{p}-\frac{2}{q}}}{(1-|\zeta|)^{\frac{2\beta(\omega)}{p}+\frac{1}{p}-\frac{1}{q}}}\frac{\widehat{\omega}(\zeta)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{\tau}{q}}}\left(\frac{(1-|z|)^\tau}{\widehat{\nu}(z)}\right)^{\frac{1}{q}}, \quad z \in \mathbb{D}, \quad \zeta \in \Omega_4(z,K).\end{aligned}$$

Therefore

$$\begin{aligned}I_4(g) &\lesssim \int_{\mathbb{D}}\left(\int_{\mathbb{D}}\left(|g(\zeta)|\widetilde{\omega}(\zeta)^{\frac{1}{p}}\right)\frac{(1-|\zeta|)^{s-\delta-\frac{2\beta(\omega)}{p}+\frac{1}{q}-\frac{\tau}{q}}}{|1-\bar{z}\zeta|^{2+s-2\delta-\frac{2\beta(\omega)}{p}-\frac{2}{p}+\frac{2}{q}}}dA(\zeta)\right)^q\frac{\nu(z)(1-|z|)^{\tau-\delta q}}{\widehat{\nu}(z)}\,dA(z)\\ &= \left\|S_{b,c}\left(|g|\widetilde{\omega}^{\frac{1}{p}}\right)\right\|^q_{L^q_\eta}, \quad g \in H^\infty,\end{aligned} \tag{5.4}$$

where $b = s-\delta-\frac{2\beta(\omega)}{p}+\frac{1}{q}-\frac{\tau}{q}$, $c = 2+s-2\delta-\frac{2\beta(\omega)}{p}-\frac{2}{p}+\frac{2}{q}$ and $\eta(z) = \frac{\nu(z)(1-|z|)^{\tau-\delta q}}{\widehat{\nu}(z)}$ for all $z \in \mathbb{D}$. We will appeal to Lemma 12 with $\sigma \equiv 1$. First observe that $\eta \in \mathcal{D}$ and $\widehat{\eta}(r) \asymp (1-r)^{\tau-\delta q}$ for all $0 \le r < 1$ by Lemma 4. Hence

$$\begin{aligned}\int_r^1 \frac{(1-t)^{c-2}}{\widehat{\eta}(t)^{\frac{1}{q}}}\,dt &\asymp \int_r^1 (1-t)^{s-\delta-\frac{2\beta(\omega)}{p}-\frac{\tau}{q}-\frac{2}{p}+\frac{2}{q}}\,dt \asymp (1-r)^{1+s-\delta-\frac{2\beta(\omega)}{p}-\frac{\tau}{q}-\frac{2}{p}+\frac{2}{q}}\\ &\asymp \frac{(1-r)^{c-1}}{\widehat{\eta}(r)^{\frac{1}{q}}}, \quad 0 \le r < 1,\end{aligned}$$

and, by Lemma 3(iii),

$$\int_0^r \frac{\eta(t)}{(1-t)^{\frac{cq}{p}-1}\widehat{\eta}(t)^{\frac{1}{p'}}}\,dt \asymp \int_0^r \frac{\nu(t)}{\widehat{\nu}(t)(1-t)^{\frac{q}{p}\left(2+s-2\delta-\frac{2\beta(\omega)}{p}-\frac{2}{p}+\frac{2}{q}\right)-1-\frac{\tau-q\delta}{p}}}\,dt$$

$$\lesssim \frac{1}{(1-r)^{\frac{q}{p}\left(2+s-2\delta-\frac{2\beta(\omega)}{p}-\frac{2}{p}+\frac{2}{q}\right)-1-\frac{\tau-q\delta}{p}}} \asymp \frac{\widehat{\eta}(r)^{\frac{1}{p}}}{(1-r)^{\frac{cq}{p}-1}},\quad 0\le r<1,$$

so the hypotheses of Lemma 12 are satisfied. Moreover,

$$(1-r)^{2+b-c+\frac{1}{q}-\frac{1}{p}}\frac{\widehat{\eta}(r)^{\frac{1}{q}}}{\widehat{\sigma}(r)^{\frac{1}{p}}} \asymp 1,\quad 0\le r<1,$$

and hence (5.4) and Lemmas 12 and 5 imply $I_4(g)\lesssim \|g\|^q_{A^p_{\widetilde{\omega}}} \asymp \|g\|^q_{A^p_{\omega}}$ for all $g\in H^\infty$.

Case $I_5(g)$ By using the definition of $\Omega_5(z,K)$, Lemma A(ii) and the fact that $\frac{\widehat{\nu}(x)}{(1-x)^\tau}$ is essentially decreasing on $[0,1)$ we deduce

$$\Gamma_\tau(z,\zeta)\lesssim \frac{\left(\frac{|1-\overline{z}\zeta|^2}{\max\{1-|z|^2,1-|\zeta|^2\}}\right)^{\frac{1}{p}-\frac{1}{q}}\widehat{\omega}\left(1-\frac{2|1-\overline{z}\zeta|^2}{\max\{1-|z|^2,1-|\zeta|^2\}}\right)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{\tau}{q}}\min\left\{\frac{\widehat{\nu}(z)}{(1-|z|)^\tau},\frac{\widehat{\nu}(\zeta)}{(1-|\zeta|)^\tau}\right\}^{\frac{1}{q}}}$$

$$\lesssim \frac{\left(\frac{|1-\overline{z}\zeta|^2}{1-|\zeta|}\right)^{\frac{1}{p}-\frac{1}{q}}\widehat{\omega}\left(1-\frac{K2|1-\overline{z}\zeta|^2}{\max\{1-|z|^2,1-|\zeta|^2\}}\right)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{\tau}{q}}\min\left\{\frac{\widehat{\nu}(z)}{(1-|z|)^\tau},\frac{\widehat{\nu}(\zeta)}{(1-|\zeta|)^\tau}\right\}^{\frac{1}{q}}} \lesssim \frac{\left(\frac{|1-\overline{z}\zeta|^2}{1-|\zeta|}\right)^{\frac{1}{p}-\frac{1}{q}}\left(\frac{|1-\overline{z}\zeta|}{1-|\zeta|^2}\right)^{\frac{2\beta(\omega)}{p}}\widehat{\omega}(\zeta)^{\frac{1}{p}}}{(1-|\zeta|)^{\frac{\tau}{q}}\min\left\{\frac{\widehat{\nu}(z)}{(1-|z|)^\tau},\frac{\widehat{\nu}(\zeta)}{(1-|\zeta|)^\tau}\right\}^{\frac{1}{q}}}$$

$$\lesssim \left(\frac{|1-\overline{z}\zeta|^2}{1-|\zeta|}\right)^{\frac{1}{p}-\frac{1}{q}}\left(\frac{|1-\overline{z}\zeta|}{1-|\zeta|}\right)^{\frac{2\beta(\omega)}{p}}\frac{\widehat{\omega}(\zeta)^{\frac{1}{p}}}{\widehat{\nu}(\zeta)^{\frac{1}{q}}},\quad z\in\mathbb{D},\quad \zeta\in\Omega_5(z,K).$$

Therefore Lemma A(ii) yields

$$\begin{aligned} I_5(g) &\lesssim \int_{\mathbb{D}}\left(\int_{\Omega_5(z,K)}\left(|g(\zeta)|\widetilde{\omega}(\zeta)^{\frac{1}{p}}\right)\frac{(1-|\zeta|)^{s-\delta-\frac{2\beta(\omega)}{p}+\frac{1}{q}}}{\widehat{\nu}(\zeta)^{\frac{1}{q}}|1-\overline{z}\zeta|^{2+s-2\delta-\frac{2\beta(\omega)}{p}-\frac{2}{p}+\frac{2}{q}}}dA(\zeta)\right)^q\frac{\nu(z)\,dA(z)}{(1-|z|)^{q\delta}} \\ &\lesssim \int_{\mathbb{D}}\left(\int_{\mathbb{D}}\left(|g(\zeta)|\widetilde{\omega}(\zeta)^{\frac{1}{p}}\right)\frac{(1-|\zeta|)^{s-\delta-\frac{2\beta(\omega)}{p}+\frac{1}{q}-\frac{\beta(\nu)}{q}}}{|1-\overline{z}\zeta|^{2+s-2\delta-\frac{2\beta(\omega)}{p}-\frac{2}{p}+\frac{2}{q}}}dA(\zeta)\right)^q\frac{\nu(z)\,dA(z)}{(1-|z|)^{q\delta-\beta(\nu)}\widehat{\nu}(z)^{\frac{1}{q}}} \qquad (5.5) \\ &= \left\|S_{b,c}\left(|g|\widetilde{\omega}^{\frac{1}{p}}\right)\right\|^q_{L^q_\eta},\quad g\in H^\infty, \end{aligned}$$

where $b=s-\delta-\frac{2\beta(\omega)}{p}+\frac{1}{q}-\frac{\beta(\nu)}{q}$, $c=2+s-2\delta-\frac{2\beta(\omega)}{p}-\frac{2}{p}+\frac{2}{q}$ and $\eta(z)=\frac{\nu(z)(1-|z|)^{\beta(\nu)-\delta q}}{\widehat{\nu}(z)}$ for all $z\in\mathbb{D}$. Again we will appeal to Lemma 12 with $\sigma\equiv 1$. First observe that $\eta\in\mathcal{D}$ and $\widehat{\eta}(r)\asymp(1-r)^{\beta(\nu)-\delta q}$ for all $0\le r<1$ by Lemma 4. Hence

$$\int_r^1\frac{(1-t)^{c-2}}{\widehat{\eta}(t)^{\frac{1}{q}}}\,dt \asymp \int_r^1 (1-t)^{s-\delta+\frac{2\beta(\omega)}{p}-\frac{2}{p}+\frac{2}{q}-\frac{\beta(\nu)}{q}}\,dt \asymp (1-r)^{1+s-\delta+\frac{2\beta(\omega)}{p}-\frac{2}{p}+\frac{2}{q}-\frac{\beta(\nu)}{q}}$$

$$\asymp \frac{(1-r)^{c-1}}{\widehat{\eta}(r)^{\frac{1}{q}}},\quad 0\le r<1,$$

and, by Lemma 3(iii),

$$\int_0^r \frac{\eta(t)}{(1-t)^{\frac{cq}{p}-1}\widehat{\eta}(t)^{1/p'}}\,dt \asymp \int_0^r \frac{\nu(t)}{\widehat{\nu}(t)(1-t)^{\frac{q}{p}\left(2+s-2\delta-\frac{2\beta(\omega)}{p}-\frac{2}{p}+\frac{2}{q}\right)-1-\frac{\beta(\nu)-q\delta}{p}}}\,dt$$

$$\lesssim \frac{1}{(1-r)^{\frac{q}{p}\left(2+s-2\delta-\frac{2\beta(\omega)}{p}-\frac{2}{p}+\frac{2}{q}\right)-1-\frac{\beta(\nu)-q\delta}{p}}} \asymp \frac{\widehat{\eta}(r)^{\frac{1}{p}}}{(1-r)^{\frac{cq}{p}-1}}, \quad 0 \le r < 1,$$

so the hypotheses of Lemma 12 are satisfied. Moreover,

$$(1-r)^{2+b-c+\frac{1}{q}-\frac{1}{p}} \frac{\widehat{\eta}(r)^{\frac{1}{q}}}{\widehat{\sigma}(r)^{\frac{1}{p}}} \asymp 1, \quad 0 \le r < 1,$$

and hence (5.5) together with Lemmas 5 and 12 imply $I_5(g) \lesssim \|g\|^q_{A^p_{\widetilde{\omega}}} \asymp \|g\|^q_{A^p_\omega}$ for all $g \in H^\infty$. This finishes the proof of the proposition. □

In order to prove the necessity part of Theorem 1 some definitions are needed. For $\eta > -1$ and a radial weight ω, let $b^\eta_{z,\omega} = B^\eta_z / \|B^\eta_z\|_{A^p_\omega}$ for $z \in \mathbb{D}$, where $B^\eta_z(\zeta) = (1-\bar{z}\zeta)^{-(2+\eta)}$. For each $f \in L^1_\nu$, define

$$g^\eta_{z,\omega}(\zeta) = \frac{P_\nu(\overline{f} b^\eta_{z,\omega})(\zeta)}{b^\eta_{z,\omega}(\zeta)}, \quad \zeta \in \mathbb{D},$$

and note that $g^\eta_{z,\omega}$ is a well-defined analytic function in $\mathbb{D}$ because the standard Bergman kernel $b^\eta_{z,\omega}$ has no zeros. If ν, ω are weights, $\eta > -1$ and $0 < p, q < \infty$, let us consider the global mean oscillation

$$\|f b^\eta_{z,\omega} - \overline{g^\eta_{z,\omega}(z)} b^\eta_{z,\omega}\|_{L^q_\nu}, \quad z \in \mathbb{D}.$$

Proposition 15 *Let $1 < p \le q < \infty$, $f \in L^q_\nu$, $\omega \in \widehat{\mathcal{D}}$, $\nu \in B_q$ a radial weight and $\gamma(z) = \gamma_{\omega,\nu,p,q}(z) = \frac{\widehat{\nu}(z)^{\frac{1}{q}}(1-|z|)^{\frac{1}{q}}}{\widehat{\omega}(z)^{\frac{1}{p}}(1-|z|)^{\frac{1}{p}}}$. If $H^\nu_f, H^\nu_{\overline{f}} : A^p_\omega \to L^q_\nu$ are bounded, then there exists $\eta_0 = \eta_0(\nu,\omega) > -1$ such that*

$$\sup_{z\in\mathbb{D}} \|f b^\eta_{z,\omega} - \overline{g^\eta_{z,\omega}(z)} b^\eta_{z,\omega}\|_{L^q_\nu} \le \|H^\nu_f\|_{A^p_\omega\to L^q_\nu} + \|P_\eta\|_{L^q_\nu\to L^q_\nu}\left(\|H^\nu_f\|_{A^p_\omega\to L^q_\nu} + \|H^\nu_{\overline{f}}\|_{A^p_\omega\to L^q_\nu}\right).$$

for each $\eta \ge \eta_0$. Moreover, there exists $r_0 = r_0(\nu) > 0$ such that for each fixed $r \ge r_0$ and $\eta \ge \eta_0$,

$$\sup_{z\in\mathbb{D}} \|f b^\eta_{z,\omega} - \overline{g^\eta_{z,\omega}(z)} b^\eta_{z,\omega}\|_{L^q_\nu} \gtrsim \sup_{z\in\mathbb{D}} \gamma(z)\left(\frac{1}{\nu(\Delta(z,r))}\int_{\Delta(z,r)} |f(\zeta) - \overline{g^\eta_{z,\omega}(z)}|^q \nu(\zeta)\,dA(\zeta)\right)^{\frac{1}{q}}.$$

Proof The definition of the Hankel operator along with triangle inequality gives

$$\begin{aligned}\|f b^\eta_{z,\omega} - \overline{g^\eta_{z,\omega}(z)} b^\eta_{z,\omega}\|_{L^q_\nu} &\le \|H^\nu_f(b^\eta_{z,\omega})\|_{L^q_\nu} + \|P_\nu(f b^\eta_{z,\omega}) - \overline{g^\eta_{z,\omega}(z)} b^\eta_{z,\omega}\|_{L^q_\nu}\\ &\le \|H^\nu_f\|_{A^p_\omega\to L^q_\nu}\|b^\eta_{z,\omega}\|_{A^p_\omega} + \|P_\nu(f b^\eta_{z,\omega}) - \overline{g^\eta_{z,\omega}(z)} b^\eta_{z,\omega}\|_{L^q_\nu}\\ &= \|H^\nu_f\|_{A^p_\omega\to L^q_\nu} + \|P_\nu(f b^\eta_{z,\omega}) - \overline{g^\eta_{z,\omega}(z)} b^\eta_{z,\omega}\|_{L^q_\nu}.\end{aligned}$$

If $g \in A^1_\eta$, then the reproducing formula for the standard weighted Bergman projection yields $\overline{g(z)} b^\eta_{z,\omega} = P_\eta(\overline{g} b^\eta_{z,\omega})$. Since $\nu \in B_q$ is radial and $f \in L^q_\nu$, we have $\nu \in \mathcal{D}$ and $P_\nu(f b^\eta_z) \in A^q_\nu$

by Proposition 6. Therefore $g_z^\eta \in A^q_\nu$ for all $z \in \mathbb{D}$. Moreover, $A^q_\nu \subset A^q_\eta \subset A^1_\eta$ if $\eta > \frac{\beta(\nu)}{q} - 1$ by Lemma A(ii). It follows that

$$\begin{aligned}\|P_\nu(fb^\eta_{z,\omega}) - \overline{g^\eta_{z,\omega}(z)}b^\eta_{z,\omega}\|_{L^q_\nu} &= \|P_\nu(fb^\eta_{z,\omega}) - P_\eta(\overline{g^\eta_{z,\omega}}b^\eta_{z,\omega})\|_{L^q_\nu}\\ &= \|P_\eta(P_\nu(fb^\eta_{z,\omega}) - \overline{g^\eta_{z,\omega}}b^\eta_{z,\omega})\|_{L^q_\nu}, \quad z \in \mathbb{D}.\end{aligned}$$

By [5], there exists $\eta_1 = \eta_1(\nu) > \frac{\beta(\nu)}{q} - 1$ such that $P_\eta : L^q_\nu \to L^q_\nu$ is bounded if $\eta \geq \eta_1$. Therefore

$$\|P_\nu(fb^\eta_{z,\omega}) - \overline{g^\eta_{z,\omega}(z)}b^\eta_{z,\omega}\|_{L^q_\nu} \leq \|P_\eta\|_{L^q_\nu \to L^q_\nu}\|P_\nu(fb^\eta_{z,\omega}) - \overline{g^\eta_{z,\omega}}b^\eta_{z,\omega}\|_{L^q_\nu}, \quad z \in \mathbb{D}, \quad \eta \geq \eta_1.$$

The triangle inequality yields

$$\begin{aligned}\|P_\nu(fb^\eta_{z,\omega}) - \overline{g^\eta_{z,\omega}}b^\eta_{z,\omega}\|_{L^q_\nu} &\leq \|fb^\eta_{z,\omega} - P_\nu(fb^\eta_{z,\omega})\|_{L^q_\nu} + \|fb^\eta_{z,\omega} - \overline{g^\eta_{z,\omega}}b^\eta_{z,\omega}\|_{L^q_\nu}\\ &= \|H^\nu_f(b^\eta_{z,\omega})\|_{L^q_\nu} + \|\overline{f}b^\eta_{z,\omega} - g^\eta_{z,\omega}b^\eta_{z,\omega}\|_{L^q_\nu}\\ &\leq \|H^\nu_f\|_{A^p_\omega \to L^q_\nu}\|b^\eta_{z,\omega}\|_{A^p_\omega} + \|\overline{f}b^\eta_{z,\omega} - P_\nu(\overline{f}b^\eta_{z,\omega})\|_{L^q_\nu}\\ &= \|H^\nu_f\|_{A^p_\omega \to L^q_\nu} + \|H^\nu_{\overline{f}}(b^\eta_{z,\omega})\|_{L^q_\nu} \leq \|H^\nu_f\|_{A^p_\omega \to L^q_\nu} + \|H^\nu_{\overline{f}}\|_{A^p_\omega \to L^q_\nu}.\end{aligned}$$

By combining the above estimates we deduce

$$\|fb^\eta_{z,\omega} - \overline{g^\eta_{z,\omega}(z)}b^\eta_{z,\omega}\|_{L^q_\nu} \leq \|H^\nu_f\|_{A^p_\omega \to L^q_\nu} + \|P_\eta\|_{L^q_\nu \to L^q_\nu}\left(\|H^\nu_f\|_{A^p_\omega \to L^q_\nu} + \|H^\nu_{\overline{f}}\|_{A^p_\omega \to L^q_\nu}\right),$$

for any $\eta \geq \eta_1(\nu)$.

To see the second one, first observe that [16, Corollary 2] and Lemma A(ii) give

$$\begin{aligned}\|B^\eta_z\|^p_{A^p_\omega} &\asymp \int_0^{|z|} \frac{\widehat{\omega}(t)}{(1-t)^{p(2+\eta)}}\,dt \lesssim \frac{\widehat{\omega}(|z|)}{(1-|z|)^{\beta(\omega)}} \int_0^{|z|} \frac{1}{(1-t)^{p(2+\eta)-\beta(\omega)}}\,dt\\ &\asymp \frac{\widehat{\omega}(z)}{(1-|z|)^{p(2+\eta)-1}}, \quad |z| \to 1^-,\end{aligned}$$

provided $\eta > \frac{\beta(\omega)+1}{p} - 2$. Moreover, by (3.6) there exists $r_0 = r_0(\nu) > 0$ such that $(1-|z|)\widehat{\nu}(z) \asymp \nu(\Delta(z,r_0))$ for any $r \geq r_0$. Hence, for each $r \geq r_0$ we have

$$\begin{aligned}\|fb^\eta_{z,\omega} - \overline{g^\eta_{z,\omega}(z)}b^\eta_{z,\omega}\|^q_{L^q_\nu} &\geq \int_{\Delta(z,r)} |f(\zeta) - \overline{g^\eta_{z,\omega}(z)}|^q |b^\eta_z(\zeta)|^q \nu(\zeta)\,dA(\zeta)\\ &\asymp \frac{1}{\|B^\eta_z\|^q_{A^p_\omega}(1-|z|)^{q(2+\eta)}} \int_{\Delta(z,r)} |f(\zeta) - \overline{g^\eta_{z,\omega}(z)}|^q \nu(\zeta)\,dA(\zeta)\\ &\asymp \frac{1}{\widehat{\omega}(z)^{\frac{q}{p}}(1-|z|)^{\frac{q}{p}}} \int_{\Delta(z,r)} |f(\zeta) - \overline{g^\eta_{z,\omega}(z)}|^q \nu(\zeta)\,dA(\zeta),\\ &\asymp \frac{\widehat{\nu}(z)(1-|z|)}{\widehat{\omega}(z)^{\frac{q}{p}}(1-|z|)^{\frac{q}{p}}} \frac{1}{\nu(\Delta(z,r))} \int_{\Delta(z,r)} |f(\zeta) - \overline{g^\eta_{z,\omega}(z)}|^q \nu(\zeta)dA(\zeta).\end{aligned}$$

The second claim for $\eta_0 = \max\{\eta_1, \frac{\beta(\omega)+1}{p} - 2\}$ is now proved. □

Proof of Theorem 1 If $H^\nu_f, H^\nu_{\overline{f}} : A^p_\omega \to L^q_\nu$ are bounded, then $f \in \mathrm{BMO}(\Delta)$ by Proposition 15 and Theorem 11.

Conversely, let $f \in \mathrm{BMO}(\Delta)$. Then f can be decomposed as $f = f_1 + f_2$, where $f_1 \in \mathrm{BA}(\Delta)$ and $f_2 \in \mathrm{BO}(\Delta)$, by Theorem 11(ii). Proposition 14 shows that $H^\nu_{f_2}, H^\nu_{\overline{f_2}}$:

$A^p_\omega \to L^q_\nu$ are bounded. Moreover, since $\nu \in B_q$ is radial, $\nu \in \mathcal{D}$ and $P_\nu : L^q_\nu \to L^q_\nu$ is bounded by Proposition 6. Therefore Lemma 10 yields

$$\|H^\nu_{f_1}(g)\|^q_{L^q_\nu} \le \|f_1 g\|_{L^q_\nu} + \|P_\nu(f_1 g)\|_{L^q_\nu} \lesssim \|f_1 g\|_{L^q_\nu} \lesssim \|g\|_{A^p_\omega} \quad g \in H^\infty.$$

It follows that $H^\nu_f, H^\nu_{\overline{f}} : A^p_\omega \to L^q_\nu$ are bounded. □

6 Anti-analytic symbols

Recall that the space $\mathcal{B}_{d\gamma}$ consists of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}_{d\gamma}} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|)\gamma(z) + |f(0)| < \infty,$$

where $\gamma(z) = \frac{\widehat{\nu}(z)^{\frac{1}{q}}(1-|z|)^{\frac{1}{q}}}{\widehat{\omega}(z)^{\frac{1}{p}}(1-|z|)^{\frac{1}{p}}}$ for all $z \in \mathbb{D}$.

Proposition 16 *Let $1 < p \le q < \infty$, $\omega, \nu \in \mathcal{D}$ and $r \ge r_0$, where $r_0 = r_0(\nu) > 0$ is that of Theorem 11(i). Then* $\mathrm{BMO}(\Delta) \cap \mathcal{H}(\mathbb{D}) = \mathrm{BMO}(\Delta)_{\omega,\nu,p,q,r} \cap \mathcal{H}(\mathbb{D}) = \mathcal{B}_{d\gamma}$.

Proof Let first $f \in \mathcal{B}_{d\gamma}$. By Theorem 11(iv) to deduce $f \in \mathrm{BMO}(\Delta)$ it is enough to prove

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|)^{c+1}\gamma(z)^q}{\widehat{\nu}(z)} \int_{\mathbb{D}} |f(\zeta)-f(z)|^q \frac{(1-|\zeta|^2)^\sigma}{|1-z\overline{\zeta}|^{2+c+\sigma}} \nu(\zeta)\, dA(\zeta) < \infty \tag{6.1}$$

for some $\sigma > 0$ and

$$c > 2\frac{q}{p}(\beta(\omega)+1) + \sigma + \max\{2\beta(\nu), \gamma(\nu)\}. \tag{6.2}$$

Since $f \in \mathcal{H}(\mathbb{D})$, the function $(f(\zeta)-f(z))(1-\zeta\overline{z})^{-\frac{2+c+\sigma}{q}}$ is an analytic function in ζ for each $z \in \mathbb{D}$. Therefore Lemma 5 shows that (6.1) is equivalent to

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|)^{c+1}\gamma(z)^q}{\widehat{\nu}(z)} \int_{\mathbb{D}} |f(\zeta)-f(z)|^q \frac{(1-|\zeta|^2)^{\sigma-1}}{|1-z\overline{\zeta}|^{2+c+\sigma}} \widehat{\nu}(\zeta)\, dA(\zeta) < \infty. \tag{6.3}$$

Further, Lemma A(ii) yields

$$\begin{aligned}
&\frac{(1-|z|)^{c+1}\gamma(z)^q}{\widehat{\nu}(z)} \int_{\mathbb{D}} |f(\zeta)-f(z)|^q \frac{(1-|\zeta|^2)^{\sigma-1}}{|1-z\overline{\zeta}|^{2+c+\sigma}} \widehat{\nu}(\zeta)\, dA(\zeta)\\
&\lesssim (1-|z|)^{c+1}\gamma(z)^q \int_{\mathbb{D}\setminus D(0,|z|)} |f(\zeta)-f(z)|^q \frac{(1-|\zeta|^2)^{\sigma-1}}{|1-z\overline{\zeta}|^{2+c+\sigma}}\, dA(\zeta)\\
&\quad + (1-|z|)^{c+1-\beta(\nu)}\gamma(z)^q \int_{D(0,|z|)} |f(\zeta)-f(z)|^q \frac{(1-|\zeta|^2)^{\sigma+\beta(\nu)-1}}{|1-z\overline{\zeta}|^{2+c+\sigma}}\, dA(\zeta)\\
&\le (1-|z|)^{c+1}\gamma(z)^q \int_{\mathbb{D}} |f(\zeta)-f(z)|^q \frac{(1-|\zeta|^2)^{\sigma-1}}{|1-z\overline{\zeta}|^{2+c+\sigma}}\, dA(\zeta)\\
&\quad + (1-|z|)^{c+1-\beta(\nu)}\gamma(z)^q \int_{\mathbb{D}} |f(\zeta)-f(z)|^q \frac{(1-|\zeta|^2)^{\sigma+\beta(\nu)-1}}{|1-z\overline{\zeta}|^{2+c+\sigma}}\, dA(\zeta)\\
&= I_1(z) + I_2(z), \quad z \in \mathbb{D}.
\end{aligned}$$

Fix $\sigma > \max\left\{0, 1 - \frac{q}{p}(1+\alpha(\omega)) + q\beta(\nu)\right\}$ and c satisfying (6.2). Then

$$c > \max\left\{\beta(\nu) - 1, -2 + \beta(\nu) + \frac{q}{p}(1+\beta(\omega)) - q\alpha(\nu)\right\}.$$

Therefore, [11, Lemma 7] together with Lemmas A(ii) and B gives

$$\begin{aligned}
I_1(z) &\lesssim (1-|z|)^{c+1}\gamma(z)^q \int_{\mathbb{D}} |f'(\zeta)|^q \frac{(1-|\zeta|^2)^{\sigma+q-1}}{|1-z\overline{\zeta}|^{2+c+\sigma}}\,dA(\zeta)\\
&\lesssim \|f\|^q_{\mathcal{B}_{d\gamma}} (1-|z|)^{c+1}\gamma(z)^q \int_{\mathbb{D}} \gamma(\zeta)^{-q} \frac{(1-|\zeta|^2)^{\sigma-1}}{|1-z\overline{\zeta}|^{2+c+\sigma}}\,dA(\zeta)\\
&\asymp \|f\|^q_{\mathcal{B}_{d\gamma}} (1-|z|)^{c+1}\gamma(z)^q \int_0^1 \frac{\widehat{\omega}(s)^{\frac{q}{p}}}{\widehat{\nu}(s)} \frac{(1-s)^{\frac{q}{p}+\sigma-2}}{(1-s|z|)^{1+\sigma+c}}\,ds\\
&\lesssim \|f\|^q_{\mathcal{B}_{d\gamma}} (1-|z|)^{c+1}\gamma(z)^q \frac{\widehat{\omega}(z)^{\frac{q}{p}}(1-|z|)^{\alpha(\nu)}}{(1-|z|)^{\frac{q}{p}\beta(\omega)}\widehat{\nu}(z)} \int_0^{|z|} \frac{ds}{(1-s)^{3+c-\frac{q}{p}-\frac{q}{p}\beta(\omega)+\alpha(\nu)}}\\
&\quad + \|f\|^q_{\mathcal{B}_{d\gamma}} (1-|z|)^{-\sigma}\gamma(z)^q \frac{\widehat{\omega}(z)^{\frac{q}{p}}(1-|z|)^{\beta(\nu)}}{(1-|z|)^{\frac{q}{p}\alpha(\omega)}\widehat{\nu}(z)} \int_{|z|}^1 (1-s)^{\frac{q}{p}+\sigma-2+\frac{q}{p}\alpha(\omega)-\beta(\nu)}\,ds\\
&\lesssim \|f\|^q_{\mathcal{B}_{d\gamma}} (1-|z|)^{-1+\frac{q}{p}}\gamma(z)^q \frac{\widehat{\omega}(z)^{\frac{q}{p}}}{\widehat{\nu}(z)} \asymp \|f\|^q_{\mathcal{B}_{d\gamma}} < \infty, \quad z\in\mathbb{D},
\end{aligned}$$

and

$$\begin{aligned}
I_2(z) &\lesssim (1-|z|)^{c+1-\beta(\nu)}\gamma(z)^q \int_{\mathbb{D}} |f'(\zeta)|^q \frac{(1-|\zeta|^2)^{\sigma+\beta(\nu)+q-1}}{|1-z\overline{\zeta}|^{2+c+\sigma}}\,dA(\zeta)\\
&\lesssim \|f\|^q_{\mathcal{B}_{d\gamma}} (1-|z|)^{c+1-\beta(\nu)}\gamma(z)^q \int_{\mathbb{D}} \gamma(\zeta)^{-q} \frac{(1-|\zeta|^2)^{\beta(\nu)+\sigma-1}}{|1-z\overline{\zeta}|^{2+c+\sigma}}\,dA(\zeta)\\
&\asymp \|f\|^q_{\mathcal{B}_{d\gamma}} (1-|z|)^{c+1-\beta(\nu)}\gamma(z)^q \int_0^1 \frac{\widehat{\omega}(s)^{\frac{q}{p}}}{\widehat{\nu}(s)} \frac{(1-s)^{\beta(\nu)+\frac{q}{p}+\sigma-2}}{(1-s|z|)^{1+\sigma+c}}\,ds\\
&\lesssim \|f\|^q_{\mathcal{B}_{d\gamma}} (1-|z|)^{c+1-\beta(\nu)}\gamma(z)^q \frac{\widehat{\omega}(z)^{\frac{q}{p}}(1-|z|)^{\alpha(\nu)}}{(1-|z|)^{\frac{q}{p}\beta(\omega)}\widehat{\nu}(z)^q} \int_0^{|z|} \frac{ds}{(1-s)^{3+c-\beta(\nu)-\frac{q}{p}-\frac{q}{p}\beta(\omega)+\alpha(\nu)}}\\
&\quad + \|f\|^q_{\mathcal{B}_{d\gamma}} (1-|z|)^{-\sigma-\beta(\nu)}\gamma(z)^q \frac{\widehat{\omega}(z)^{\frac{q}{p}}(1-|z|)^{\beta(\nu)}}{(1-|z|)^{\frac{q}{p}\alpha(\omega)}\widehat{\nu}(z)} \int_{|z|}^1 (1-s)^{\beta(\nu)+\frac{q}{p}+\sigma-2+\frac{q}{p}\alpha(\omega)-\beta(\nu)}\,ds\\
&\lesssim \|f\|^q_{\mathcal{B}_{d\gamma}} (1-|z|)^{-1+\frac{q}{p}}\gamma(z)^q \frac{\widehat{\omega}(z)^{\frac{q}{p}}}{\widehat{\nu}(z)^q} \asymp \|f\|^q_{\mathcal{B}_{d\gamma}} < \infty, \quad z\in\mathbb{D}.
\end{aligned}$$

By combining these estimates we deduce $f \in \mathrm{BMO}(\Delta)$, and thus $\mathcal{B}_{d\gamma} \subset \mathcal{H}(\mathbb{D}) \cap \mathrm{BMO}(\Delta)$.

Assume now that $f \in \mathcal{H}(\mathbb{D}) \cap \mathrm{BMO}(\Delta)$. Then (6.3) holds for some $\sigma > 1$ and c satisfying (6.2). Therefore (3.6) implies

$$\begin{aligned}
\infty &> \sup \frac{(1-|z|)^{c+1}\gamma(z)^q}{\widehat{\nu}(z)} \int_{\mathbb{D}} |f(\zeta)-f(z)|^q \frac{(1-|\zeta|^2)^{\sigma-1}}{|1-z\overline{\zeta}|^{2+c+\sigma}}\widehat{\nu}(\zeta)dA(\zeta)\\
&\gtrsim \frac{\gamma(z)^q}{(1-|z|)^2\widehat{\nu}(z)} \int_{\Delta(z,r)} |f(\zeta)-f(z)|^q \widehat{\nu}(\zeta)\,dA(\zeta)\\
&\asymp \frac{\gamma(z)^q}{|\Delta(z,r)|} \int_{\Delta(z,r)} |f(\zeta)-f(z)|^q\,dA(\zeta), \quad z\in\mathbb{D}.
\end{aligned}$$

By arguing as in [12, 1653–1654] we deduce $\mathcal{H}(\mathbb{D}) \cap \mathrm{BMO}(\Delta) \subset \mathcal{B}_{d\gamma}$. $\square$

The space $\mathcal{B}_{d\gamma}$ consists of constant functions only if $\limsup_{|z|\to 1^-}((1-|z|)\gamma(|z|))^{-1} = 0$. Moreover, $\mathcal{B}_{d\gamma}$ is a subset of the disc algebra if $((1-x)\gamma(x))^{-1} \in L^1(0,1)$, and $\mathcal{B}_{d\gamma}$ coincides with a Bloch-type space if γ is decreasing.

Proof of Theorem 2 Since $f \in A^1_\nu$, the operator $H^\nu_{\overline{f}}$ is densely defined. If $H^\nu_{\overline{f}} : A^p_\omega \to L^q_\nu$ is bounded, choosing $g \equiv 1$ it follows that $f \in A^q_\nu$, and therefore $f \in \mathcal{B}_{d\gamma}$ by Theorem 1 and Proposition 16.

Conversely, assume $f \in \mathcal{B}_{d\gamma}$. Since $\nu \in B_q$ is radial, Proposition 6 implies $\nu \in \mathcal{D}$. Therefore Lemmas A(ii) and B yield

$$
\begin{aligned}
\|f\|^q_{A^q_\nu} &\lesssim \int_{\mathbb{D}} \left(\int_0^{|z|} \left| f'\left(s\frac{z}{|z|}\right)\right| ds \right)^q \nu(z)\,dA(z) + |f(0)|^q \\
&\lesssim \|f\|^q_{\mathcal{B}_{d\gamma}} \left(1 + \int_0^1 \left(\int_0^t \frac{ds}{(1-s)\gamma(s)} \right)^q \nu(t)\,dt \right) \\
&\lesssim \|f\|^q_{\mathcal{B}_{d\gamma}} \left(1 + \int_0^1 \left(\int_0^t \frac{\widehat{\omega}(s)^{\frac{1}{p}}}{\widehat{\nu}(s)^{\frac{1}{q}}(1-s)^{1+\frac{1}{q}-\frac{1}{p}}}\,ds \right)^q \nu(t)\,dt \right) \\
&\lesssim \|f\|^q_{\mathcal{B}_{d\gamma}} \left(1 + \int_0^1 \frac{\widehat{\omega}(t)^{\frac{q}{p}}}{\widehat{\nu}(t)(1-t)^{\frac{q\alpha(\omega)}{p}-\beta(\nu)}} \left(\int_0^t \frac{ds}{(1-s)^{1+\frac{1+\beta(\nu)}{q}-\frac{1+\alpha(\omega)}{p}}} \right)^q \nu(t)\,dt \right)
\end{aligned}
$$

for all $f \in \mathcal{H}(\mathbb{D})$. If $\frac{1+\beta(\nu)}{q} - \frac{1+\alpha(\omega)}{p} > 0$, Lemma 3(ii) gives

$$
\begin{aligned}
\|f\|^q_{A^q_\nu} &\lesssim \|f\|^q_{\mathcal{B}_{d\gamma}} \left(1 + \int_0^1 \frac{\widehat{\omega}(t)^{\frac{q}{p}}(1-t)^{\frac{q}{p}-1}}{\widehat{\nu}(t)} \nu(t)\,dt \right) \\
&\lesssim \|f\|^q_{\mathcal{B}_{d\gamma}} \left(1 + \widehat{\omega}(0)^{\frac{q}{p}-1} \int_0^1 \frac{\widehat{\omega}(t)\nu(t)}{\widehat{\nu}(t)}\,dt \right) \lesssim \|f\|^q_{\mathcal{B}_{d\gamma}}.
\end{aligned}
$$

If $\frac{1+\beta(\nu)}{q} - \frac{1+\alpha(\omega)}{p} = 0$, then Lemmas B and 3(ii) yield

$$
\begin{aligned}
\|f\|^q_{A^q_\nu} &\lesssim \|f\|^q_{\mathcal{B}_{d\gamma}} \left(1 + \int_0^1 \frac{\widehat{\omega}(t)^{\frac{q}{p}}(1-t)^{\frac{q}{p}-1}}{\widehat{\nu}(t)} \log\frac{e}{1-t}\,\nu(t)\,dt \right) \\
&\lesssim \|f\|^q_{\mathcal{B}_{d\gamma}} \widehat{\omega}(0)^{\frac{q}{p}} \int_0^1 \frac{(1-t)^{\alpha(\omega)\frac{q}{p}+\frac{q}{p}-1}}{\widehat{\nu}(t)} \log\frac{e}{1-t}\,\nu(t)\,dt \\
&\lesssim \|f\|^q_{\mathcal{B}_{d\gamma}} \int_0^1 \frac{(1-t)^{\alpha(\omega)\frac{q}{2p}+\frac{q}{p}-1}}{\widehat{\nu}(t)} \nu(t)\,dt \lesssim \|f\|^q_{\mathcal{B}_{d\gamma}}.
\end{aligned}
$$

Finally, if $\frac{1+\beta(\nu)}{q} - \frac{1+\alpha(\omega)}{p} < 0$, then Lemma 3(ii) gives

$$
\|f\|^q_{A^q_\nu} \lesssim \|f\|^q_{\mathcal{B}_{d\gamma}} \left(1 + \int_0^1 \frac{\widehat{\omega}(t)^{\frac{q}{p}}(1-t)^{\frac{q}{p}-1}}{\widehat{\nu}(t)} \nu(t)\,dt \right) \lesssim \|f\|^q_{\mathcal{B}_{d\gamma}}.
$$

Therefore $f \in A^q_\nu$, and thus $\mathcal{B}_{d\gamma} \subset A^q_\nu$. This together with Theorem 1 and Proposition 16 finishes the proof. $\square$

Acknowledgements Open access funding provided by University of Gothenburg.

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