Thesis for the Degree of Licentiate of Philosophy

Multipoint Okounkov bodies

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Göteborg, 2019

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Typeset with LaTeX.
Printed in Gothenburg, Sweden 2019

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Abstract

During the nineties, the field medallist Okounkov found a way to associate to an ample line bundle L over an $n-$complex dimensional projective manifold X a convex body in $\mathbb{R}^n$, now called Okounkov body $\Delta(L)$. The construction depends on the choice of a valuation centered at one point $p \in X$ and it works even if L is a big line bundle. In the last decade $\Delta(L)$ turned out to be an accurate simplified image of $(L \to X; p)$. Indeed it encodes important global invariants like the volume, $\mathrm{Vol}_X(L)$, and it can be a finer invariant of the Seshadri constant of L at p. Moreover it can be useful to approximates $L \to X$ through an $n-$complex dimensional torus-invariant domain equipped with the standard flat metric.
In this thesis I propose a generalization of the Okounkov bodies. Namely, starting from a big line bundle L over an $n-$complex dimensional projective manifold X, and from the choice of N valuations centered at N different points $p_1, \dots, p_N \in X$, I construct N *multipoint Okounkov bodies* $\Delta_1(L), \dots, \Delta_N(L) \subset \mathbb{R}^n$. They are a simpler copy of $(L \to X; p_1, \dots, p_N)$ since they forms a finer invariant of the volume $\mathrm{Vol}_X(L)$ and of the multipoint Seshadri constant of L at $p_1, \dots, p_N$. The latter in particular is related to several important conjectures in Algebraic Geometry, like the Nagata's conjecture which concerns the projective plane $\mathbb{P}^2$. Related to this, in the thesis there are further small results for surfaces.
Moreover the multipoint Okounkov bodies consent to define N torus-invariant domains in $\mathbb{C}^n$ which approximate simultaneously $L \to X$, i.e. they produce a perfect Kähler packing (the holomorphic analogue of the symplectic packing), and this leads to an interpretation of the multipoint Seshadri constant in terms of packings.
Finally in the toric case, in different situations, the multipoint Okounkov bodies can be recovered directly subdividing the polytope.

Keywords: Projective manifold, ample line bundle, Okounkov body, Seshadri constant, symplectic packings, Kähler geometry.

Acknowledgments

As PhD student of two different universities, I can count on the support of two advisors: David and Stefano. It is a pleasure to thank them because, in addition to their careful advisory assistance, they have been sympathetic regard all my little doubts and they have been very patient with all the issues of this joint PhD. I am also grateful to my examiner Robert to be always available when needed, and to Eleonora to have helped me to start this PhD and to represent the opponent of the Licentiate Seminar.
About Göteborg, I would like to thank the wonderful group of complex geometry and the others PhD students, professors, staff of Chalmers I met during these years: thank specially to Martin, Lukas, Mattias, Richard, Zakarias, Jakob, Jimmy, Hossein, Xu, Bo, Elizabeth, Mats, Håkan, Julia, Elin, Jimmy, Jonathan, Milo, Martin and Valentina. They kindly helped me to integrate, despite my bad English in the first year, and with some of them I spent jovial afterworks and other beautiful moments. Finally I would like to thank all my local crazy (and someone temporary) friends of these years. In particular Iris and Suraj.
About Rome, I know that I did not spend so much time at the department, but in those days there has always been the DenteTor group ready to talk about math and, mostly, about other things. Thanks to all its member, and especially to Gianluca, Antonio, Matteo, Josias and Daniele. Thanks also to the complex geometry group, with its PhD students, of the three universities in Rome. In addition, I would like to say thank to my friends of the athletic groups and to my old

friends: principally thank to Stefano, Alessio, Andrea, Marco, Fabio e Michele. They are, have been, and I am sure they will be, true friends. Finally I am and I will be immensely grateful to my girlfriend Martina and to my family: Mom, Dad and Alice (with Gino and my little niece Penelope). The former to have spent with me the biggest part of my life far from math, while my family to have always supported me in this adventure and to have steadily believed in me. I know how much long, weird and difficult have been my periods in Göteborg for all of them, so thank you again.

Antonio Trusiani
Göteborg, April 2019

Contents

Chapter 1

Introduction

Projective manifolds.

The common zero set of a family of homogeneous polynomial equations in $n+1$ variables defines a geometric locus in the (complex) projective space $\mathbb{P}^n$. And if this geometric locus is irreducible, then it is a *projective variety*, while if it is also smooth then it is a *projective manifold.* Let us be more precise.
Let $\mathbb{P}^n$ be defined as $\mathbb{C}^{n+1} \setminus \{0\}/\sim$ where the equivalence relation is $(z_0, \ldots, z_n) \sim (w_0, \ldots, w_n)$ if there exists $\lambda \in \mathbb{C}^*$ such that $(z_0, \ldots, z_n) = \lambda(w_0, \ldots, w_n)$. Namely the projective space $\mathbb{P}^n$ is the compactification of the $n-$complex space $\mathbb{C}^n$ adding all the points at infinity, and points of $\mathbb{P}^n$ are complex lines of $\mathbb{C}^{n+1}$ that we will denote as $[Z_0 : \cdots : Z_n]$. If now p is an homogeneous polynomial in $z_0, \ldots, z_n$ then its locus $p = 0$ descends to an equation on $\mathbb{P}^n$, therefore given a set of homogeneous polynomial equations it makes sense to look at the common zero set in $\mathbb{P}^n$ of these equations which is called projective algebraic set. Moreover we say that it is a projective variety if it is not an union of two proper projective algebraic sets, and we say that a projective variety is a projective manifold if it is smooth, i.e. if the Jacobian matrix of the first order partial derivatives of the polynomial defining the variety has constant rank (this is equivalent to ask that the tangent space is well defined for any point of the variety).

Therefore by what we have said until now, it is quite clear that a smooth projective variety is a *complex manifold*, and we say that a complex manifold is a projective manifold if it can be embedded into some projective space $\mathbb{P}^N$, i.e. if it can be seen as a smooth projective variety.

Line bundles.

A line bundle L over a complex manifold X is a complex manifold of $\dim X + 1$ with an holomorphic surjective map $\pi : L \to X$ such that $L_p \simeq \mathbb{C}$ for any $p \in X$, where $L_p := \pi^{-1}(p)$ is the fiber over p, and such that locally L looks like a product of the base times $\mathbb{C}$, i.e. there exist open subsets $\{U_j\}_{j\in J}$ of X such that $L_{|\pi^{-1}(U_j)} \simeq U_j \times \mathbb{C}$ for any $j \in J$. An (holomorphic) *section* of $L \to X$ is an holomorphic map $s : X \to L$ such that $\pi \circ s = \mathrm{Id}_X$. We observe that since any line bundle is locally trivial, it has a lot of local sections, while if X is compact $L \to X$ admits a non-zero section only if L twists, i.e. it is not a global product (by the maximum principle for holomorphic functions). Moreover, assuming X compact, the set of all sections $s : X \to L$ is denoted with $H^0(X, L)$ and it is a vector space of finite dimension over $\mathbb{C}$.
It is also possible to consider the tensor product $L_1 \otimes L_2$ of two line bundles L_1 and L_2, and if we consider the same line bundle L the tensor product corresponds to take multiples $L^{\otimes k}$ (it is often preferable to use the additive notation kL).
Taking as example the projective space $\mathbb{P}^n$, there is a line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ such that the fiber over $[Z_0 : \cdots : Z_n]$ is the dual of the complex line passing through $(Z_0, \dots, Z_n)$. And it is not hard to check that $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$ is isomorphic to the space of homogeneous polynomials of degree $k \in \mathbb{N}$ where by notation $\mathcal{O}_{\mathbb{P}^n}(k) := k\mathcal{O}_{\mathbb{P}^n}(1)$. Therefore $n! \dim_{\mathbb{C}} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))/k^n \to 1$ for $k \to \infty$.
Similarly for any line bundle L over a compact complex manifold X, it is possible to show that $\dim_{\mathbb{C}} H^0(X, kL) = C\frac{k^n}{n!} + O(k^n)$, and such value C is called the *volume* of L, $\mathrm{Vol}_X(L)$. A line bundle is called *big* if $\mathrm{Vol}_X(L) > 0$.

Positive metrics.

Given a line bundle L on a complex manifold X. an hermitian *metric* on L is a choice of a scalar product on each fiber L_p such that the family of these products varies smoothly over X. Locally, if $U_j \subset X$ is a trivializing open set (i.e. $L_{|\pi^{-1}(U_j)} \simeq U_j \times \mathbb{C}$), an hermitian metric h is realized by a smooth function $\phi_j : U_j \to \mathbb{R}$ such that $|f_j|_h^2 = e^{-\phi_j}$ where f_j is a non zero holomorphic local section for L over a neighbourhood of U_j. The *curvature* of an hermitian metric is a global $(1,1)-$form given locally as $dd^c\phi_j$ where $d^c := \frac{i}{4\pi}(\bar{\partial} - \partial)$ so that $dd^c = \frac{i}{2\pi}\partial\bar{\partial}$. If for any j the complex Hessian matrix $\left(\frac{\partial^2\phi_j}{\partial z_j \partial \bar{z}_k}\right)_{j,k}$ is positive definite, i.e. if ϕ_j is strictly *plurisubharmonic*, then h is said to be *positive* and the curvature is a *Kähler form*. By the well-known Kodaira embedding Theorem a line bundle L over a compact complex manifold X admits a positive hermitian metric if and only if L is *ample*, i.e. for $k \gg 0$ big enough kL can be realized as restriction of the line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ for an embedding $X \to \mathbb{P}^N$ (in particular X is projective).

Given a line bundle L on a projective manifold X, it can be showed that the curvatures of all hermitian metrics on L belongs to the same cohomology class $c_1(L) \in H^2(X,\mathbb{Z}) \cap H^{1,1}(X,\mathbb{C})$ (by the $\partial\bar{\partial}-$lemma), and L admits a positive hermitian metrics if and only if $c_1(L)$ belongs to the *Kähler cone* $\mathcal{K} \subset H^{1,1}(X,\mathbb{R})$. Recalling that $\alpha \in \mathcal{K}$ if it contains a Kähler form ω as representative, where a Kähler form is a $(1,1)-$form positive.

Finally we recall that any Kähler form defines a *Riemannian metric* on the $2n-$real dimensional manifold X as $g_\omega(u,v) := \omega(u, Jv)$ where $u, v \in T_pX$ are two tangent vectors and J is the *almost complex structure*. Therefore a Kähler manifold (i.e. a manifold X with a Kähler form ω on X) includes the Riemannian structure and the complex structure, hence it can be thought as a very rigid geometric object.

Toric manifolds.

A *toric manifold* of dimension n is a complex manifold that has an action $(\mathbb{C}^*)^n \curvearrowright X$ with a dense open orbit where $(\mathbb{C}^*)^n$ is the complex

$n-$dimensional torus. If X is a toric projective manifold, then it can be embedded in $\mathbb{P}^n$ for some N, thus X becomes the compactification of the image of $(\mathbb{C}^*)^n$ and the embedded copy of the action lifts to the *toric line bundle* L over X given by the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$.
The main feature of toric geometry is that there is a 1-1 correspondence between ample toric line bundles and (Delzant) polytopes P, where Delzant means that P is a lattice polytope (i.e. it is the convex hull of a finite number of points in $\mathbb{Z}^n$) and any vertex has exactly n edges starting from it. In particular given a (Delzant) polytope P it is possible to construct an ample toric line bundle L over a toric projective manifold X. More precisely for any $k \in \mathbb{N}$, the map $f_{kP} : (\mathbb{C}^*)^n \to \mathbb{P}^{N_k-1}$ defined as

$$f_{kP}(z) := [z^{\alpha_1}, \dots, z^{\alpha_{N_k}}]$$

where $\alpha_1, \dots, \alpha_{N_k}$ is an enumeration of all points in $kP \cap \mathbb{Z}^n$, is an embedding for $k \gg 1$ big enough. Thus X_P is defined as compactification and L_P as the pull-back of $\mathcal{O}_{\mathbb{P}^{N_1-1}}(1)$.
Finally we observe that if $L_P \to X_P$ is the ample toric line bundle associated to the polytope P, then

$$H^0(X_P, kL_P) \simeq \bigoplus_{\alpha \in kP \cap \mathbb{Z}^n} \langle z^\alpha \rangle,$$

i.e. there exists a basis $\{s_\alpha\}_{\alpha \in kP \cap \mathbb{Z}^n}$ of the vector space $H^0(X_P, kL_P)$ such that $s_\alpha / s_\beta = z^{\alpha - \beta}$ on the torus $(\mathbb{C}^*)^n$. In particular

$$n!\mathrm{Vol}_{\mathbb{R}^n}(P) = \mathrm{Vol}_X(L_P).$$

See [Ful93], [Cox11] for basic facts about toric varieties.

Seshadri constants.

Demailly in [Dem90] introduced a way to measure the positivity of an ample line bundle L over a projective manifold X at a point $p \in X$, the *Seshadri constant* of L at p, $\epsilon_S(L; p)$. Intuitively it measures the asymptotic order of the expansions at p that $R(X, L)$ can completely prescribe, i.e. if, for any $k \in \mathbb{N}$, $s_k \in \mathbb{N}$ is chosen so that the sections in $H^0(X, kL)$ can prescribe all the expansions of order less or equal to s_k but they do not prescribe all the expansions of order $s_k + 1$, then $\epsilon_S(L; p) = \lim_{k\to\infty} s_k/k$. It is also worth to recall that $\epsilon_S(L; p) = \sup\{t > 0 \,:\, \mu^*L - tE \,\text{is ample}\}$ where $\mu : \tilde{X} \to X$ is the *blow-up* at p and E is the exceptional divisor. Moreover there is another characterization of this local invariant: $\epsilon_S(L; p)$ is equal to the supremum of radii r such that there exists an holomorphic embedding of the ball of radius r into X and a metric on L that extends the standard metric given by the embedding.

Subsequently Nakamaye in [Nak03] introduced a generalization of the Seshadri constant for big line bundles, which can be interpreted as before, i.e. as asymptotic order of the expansions at the point prescribed by all sections.

Considering more points, say $p_1, \dots, p_N$, there is an analogue of the Seshadri constant called *multipoint Seshadri constant*, $\epsilon_S(L; p_1, \dots, p_N)$, which measures the positivity of L at the set $\{p_1, \dots, p_N\}$, i.e. considering the asymptotic order of the expansions at all points concurrently that $R(X, L)$ can completely prescribe. In the ample case $\epsilon_S(L; p_1, \dots, p_N) = \sup\{t > 0 \,:\, \mu^*L - t(E_1 + \cdots + E_N) \,\text{is ample}\}$ where $\mu : \tilde{X} \to X$ is the blow-up at $p_1, \dots, p_N$ and $E_1, \dots, E_N$ are the exceptional divisors.

Try to calculate the multipoint Seshadri constant is of considerable importance in Algebraic Geometry because it is connected to several desired conjectures, of which one of the most famous is the *Nagata's conjecture* ([Nag58]). The latter concerns the projective plane $\mathbb{P}^2$, the line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ and $N \geq 9$ different points in very general position, and it is equivalent to prove that $\epsilon_S(L; p_1, \dots, p_N) = 1/\sqrt{N}$, i.e. to show that the multipoint Seshadri constant is maximal.

Okounkov body.

Okounkov in [Oko96] and in [Oko03] associated to an ample line bundle L over an $n-$dimensional projective manifold X a convex body $\Delta(L) \subset \mathbb{R}^n$ called *Okounkov body*. Later Lazarsfeld-Mustaţă ([LM09]) and Kaveh-Khovanskii ([KKh12]) extended the definition for big line bundles.

The construction starts fixing a point $p \in X$ and an admissible flag centered at p or, equivalently, holomorphic coordinates on a trivializing open set U centered at p together with a non-zero holomorphic local section $t : U \to L$. Then any global section $s \in H^0(X, kL)$ locally writes as $s_{|U} = ft^k$ for $f \in \mathcal{O}_X(U)$, and the Okounkov body $\Delta(L)$ is defined as

$$\Delta(L) := \overline{\bigcup_{k\geq 1} \{\frac{\nu(s)}{k} \,:\, s \in H^0(X, kL) \setminus \{0\}\}}$$

where $\nu(s) := \min_{lex}\{\alpha \in \mathbb{N}^n \,:\, a_\alpha \neq 0 \,\text{where}\, f = \sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha\}$, i.e. ν is a valuation that associates to any section its leading term exponent at p with respect to the lexicographic order. It is not difficult to see that the construction does not depend on the local section t, but it depends on the choice of the point p and on the choice of the holomorphic coordinates on U. Anyway the following holds:

$$n!\mathrm{Vol}_{\mathbb{R}^n}\big(\Delta(L)\big) = \mathrm{Vol}_X(L),$$

namely although $\Delta(L)$ seems to depend on *local* elements, it is a finer invariant than the volume, which is an important *global* invariant of L. Furthermore using results in convex geometry (like the *Brunn-Minkoswki inequality*) it is easier to show significant theorems in complex geometry (like the log-concavity of the volume function on the big cone).

It is worth to observe that if L_P is an ample toric line bundle over a toric projective manifold X_P associated to a polytope P then $\Delta(L_P)$ is essentially equal to P.

The local aspect of the construction leads to the natural question if $\Delta(L)$ can encode also local properties of L. Küronya-Lozovanu proved in [KL15a] and in [KL17] that, considering the *degree-lexicographic* order in the definition, the more L is positive in p the more $\Delta(L)$ contains a bigger multiple of the unit $n-$simplex Σ_n. Indeed they showed that

$$\epsilon_S(L;p) = \sup\{t \geq 0 \,:\, t\Sigma_n \subset \Delta(L)\}$$

where $\epsilon_S(L;p)$ is the Seshadri constant of L at p.
Finally we recall that, in the ample case, if $\epsilon_S(L;p)$ is not maximal then the volume of a ball of radius $\epsilon_S(L;p)$ is less than the volume of L, by the characterization through the holomorphic embedding of the balls given previously. However Witt Nyström in [WN15] showed how a torus-invariant domain $D(L) \subset \mathbb{C}^n$, constructed from $\Delta(L)$, equipped with the standard flat metric, approximates (X, L), in the sense that for any relatively compact open set $U \subset D(L)$ there exists an holomorphic embedding $f : U \to X$ such that the standard metric extends to a metric on L and such that the volume of $D(L)$ is equal to the volume of L (similar results hold in the big case).

Main results of the paper.

By what we have said before, it is natural to ask if starting from N different points $p_1, \dots, p_N$ on a projective manifold X, and from a big line bundle L, it is possible to give a construction of N Okounkov bodies $\Delta_1(L), \dots, \Delta_N(L)$ that encodes global invariants as the volume, local invariants as the multipoint Seshadri constant $\epsilon_S(L; p_1, \dots, p_N)$ and that connects the multipoint Seshadri constant to the notion of *Kähler packing* defining N torus-invariant domains $D_1(L), \dots, D_N(L) \subset \mathbb{C}^n$ which approximate (X, L). These are the main results of the paper (Chapter 2). Let us be more precise.
Given $(L \to X; p_1, \dots, p_N)$ as before, we define

$$\Delta_j(L) := \overline{\bigcup_{k\geq 1} \big\{ \frac{\nu^{p_j}(s)}{k} \ : \ s \in V_{k,j} \big\}} \subset \mathbb{R}^n$$

where $V_{k,j} := \{s \in H^0(X, kL) \ : \ \nu^{p_j}(s) < \nu^{p_i}(s) \,\text{for any}\, i \neq j\}$, and $\nu^{p_1}, \dots, \nu^{p_N}$ are valuations defined as in the one-point case considering the leading term exponents at $p_1, \dots, p_N$ and $>$ is the lexicographic order (the valuations may also be more general).
Then we get, as Theorem A, that

$$n! \sum_{j=1}^{N} \mathrm{Vol}_{\mathbb{R}^n}\big(\Delta_j(L)\big) = \mathrm{Vol}_X(L),$$

and other small results on the variation of the volumes and on the slices of these *multipoint Okounkov bodies* (section 2.3).
Moreover, restricting for simplicity to the ample case (similar results hold in the big case), we also construct N torus-invariant domains $D_1(L), \dots, D_N(L) \subset \mathbb{C}^n$ that approximate (X, L) in the sense that for any family of relatively compact open subsets $U_1 \Subset D_1(L), \dots, U_N \Subset D_N(L)$ there exists an embedding $f : \bigsqcup_{j=1}^N U_j \to X$ such that the standard flat metrics extends to a metric on L and such that the sum of the volumes of these domains is equal to the volume of L (this is a perfect Kähler packing, Theorem C).
Furthermore, assuming that the multipoint Okounkov bodies $\Delta_j(L)$ are constructed considering the degree-lexicographic order, we obtain the following equality (Theorem B):

$$\epsilon_S(L; p_1, \dots, p_N) = \sup\{t \geq 0 \, : \, t\Sigma_n \subset \Delta_j(L)^{ess} \, \text{for any} \, j = 1, \dots, N\}$$

where $\Delta_j(L)^{ess} \subset \Delta_j(L)$ is the essential part of $\Delta_j(L)$ (introduced in [WN15] for the one-point case), and $\epsilon_S(L; p_1, \dots, p_N)$ is the multipoint Seshadri constant. As said previously, the multipoint Seshadri constant is connected to several conjectures like the Nagata's conjecture. In section 2.6.2 we interpret this conjecture in terms of the shape of the multipoint Okounkov bodies, and obtain some further small results for the particular case of surfaces.
As a consequence of these main theorems we also get that $\epsilon_S(L; p_1, \dots, p_N)$ is equal to the supremum of radii r such that there exists a Kähler packing of N balls of radius r into (X, L) (this result was proved in dimension 2 by Eckl in [Eckl17]).
Finally we remark that in the toric case, in many different situations, we can recover the multipoint Okounkov bodies directly subdividing the polytope (section 2.6.1).

Chapter 2

Multipoint Okounkov bodies

Abstract

Starting from the data of a big line bundle L on a projective manifold X with a choice of $N \geq 1$ different points on X we give a new construction of N Okounkov bodies that encodes important geometric features of $(L \to X, p_1, \dots, p_N)$ such as the volume of L, the (moving) multipoint Seshadri constant of L at $p_1, \dots, p_N$, and the possibility to construct Kähler packings centered at $p_1, \dots, p_N$.

2.1 Introduction

Okounkov in [Oko96] and [Oko03] found a way to associate a convex body $\Delta(L) \subset \mathbb{R}^n$ to a polarized manifold (X, L) where $n = \dim_{\mathbb{C}} X$. Namely,

$$\Delta(L) := \overline{\bigcup_{k \geq 1} \left\{ \frac{\nu^p(s)}{k} \ : \ s \in H^0(X, kL) \setminus \{0\} \right\}}$$

where $\nu^p(s)$ is the *leading term exponent* at p with respect to a total additive order on $\mathbb{Z}^n$ and holomorphic coordinates centered at $p \in X$ (see subsection 2.2.4). This convex body is now called *Okounkov body*. Okounkov's construction was inspired by toric geometry, indeed in the toric case, if L_P is a torus-invariant ample line bundle, $\Delta(L_P)$ is essentially equal to the polytope P.

The Okounkov body construction works in the more general setting of a big line bundle L, i.e. a line bundle such that $\mathrm{Vol}_X(L) := \limsup_{k\to\infty} \frac{n!}{k^n} \dim_{\mathbb{C}} H^0(X, kL) > 0$, as proved in [LM09], [KKh12] (see also [Bou14]) and it captures the volume of L, i.e.

$$\mathrm{Vol}_X(L) = n!\mathrm{Vol}_{\mathbb{R}^n}\big(\Delta(L)\big).$$

Moreover if $>$ is the lexicographical order then the $(n-1)-$volume of any not trivial *slice* of the Okounkov body is related to the *restricted volume* of $L - tY$ along Y where Y is a smooth irreducible divisor such that $Y_{|U_p} = \{z_1 = 0\}$.
Another invariant that can be encoded by the Okounkov body is the *(moving) Seshadri constant* $\epsilon_S(||L||; p)$ (see [Dem90] in the ample case, or [Nak03] for the extension to the big case). Indeed, as Küronya-Lozovanu showed in [KL15a], [KL17], if the Okounkov body is defined using the *deglex order*[1], then

$$\epsilon_S(\| L \|; p) = \max\big\{0, \sup\{t \geq 0 \, : \, t\Sigma_n \subset \Delta(L)\}\big\}$$

where Σ_n is the unit $n-$simplex.
As showed by Witt Nyström in [WN15], we can restrict to consider the *essential* Okounkov body $\Delta(L)^{ess}$ to get the same characterization of the moving Seshadri constant.
Recall that $\Delta(L)^{ess} := \bigcup_{k\geq 1} \Delta^k(L)^{ess}$, where $\Delta^k(L) = Conv(\{\frac{\nu(s)}{k} : s \in H^0(X, kL) \setminus \{0\}\})$ and the *essential* part of $\Delta^k(L)$ consists of its interior as subset of $\mathbb{R}^n_{\geq 0}$ with its natural induced topology.

Seshadri constants are also defined for a collection of different points. For a nef line bundle L, the *multipoint Seshadri constant of L at* $p_1, \dots, p_N$ is defined by

$$\epsilon_S(L; p_1, \dots, p_N) := \inf_C \frac{L \cdot C}{\sum_{j=1}^N \mathrm{mult}_{p_j} C}.$$

In this paper we introduce a multipoint version of the Okounkov body. More precisely, for a fixed big line bundle L on a projective manifold

[1] recall that $\alpha <_{deglex} \beta$ iff $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and $\alpha <_{lex} \beta$, where $<_{lex}$ is the lexicographical order

X of dimension n and $p_1, \ldots, p_N \in X$ different points, we construct N Okounkov bodies $\Delta_j(L) \subset \mathbb{R}^n$ for $j = 1, \ldots, N$:

Definition 2.1.1. *Let L be a big line bundle and let $>$ be a fixed total additive order.*

$$\Delta_j(L) := \overline{\bigcup_{k\geq 1} \left\{ \frac{\nu^{p_j}(s)}{k} \ : \ s \in V_{k,j} \right\}} \subset \mathbb{R}^n$$

*is called **multipoint Okounkov body** of L at p_j, where $V_{k,j} := \{s \in H^0(X, kL) \setminus \{0\} \ : \ \nu^{p_j}(s) < \nu^{p_i}(s) \, \text{for any} \, i \neq j\}$ for any $k \geq 0$.*

We observe that the multipoint Okounkov Body of L at p_j is obtained by considering all sections whose leading terms in p_j is strictly smaller than those at the other points.
They are convex compact sets in $\mathbb{R}^n$ but, unlike the one-point case, for $N \geq 2$ it can happen that some $\Delta_j(L)$ is empty or its interior is empty (Remark 2.3.7). The definition does not depend on the order of the points.
Our first theorem concerns the relationship between the multipoint Okounkov bodies and the volume of the line bundle:

Theorem A. [2] *Let L be a big line bundle. Then*

$$n! \sum_{j=1}^{N} \mathrm{Vol}_{\mathbb{R}^n}\big(\Delta_j(L)\big) = \mathrm{Vol}_X(L).$$

Furthermore, similar to the section §4 in [LM09], we show the existence of a open subset of the big cone containing $B_+(p_j)^C = \{\alpha \in N^1(X)_{\mathbb{R}} : p_j \notin \mathbb{B}_+(\alpha)\}$ over which $\Delta_j(\cdot)$ is a numerical invariant and can be extended continuously (see section §2.3.2).
Moreover if $>$ is the lexicographical order and $Y_1, \ldots, Y_N$ are smooth irreducible divisors such that $Y_{j|U_{p_j}} = \{z_{j,1} = 0\}$ we relate the fiber

[2] The theorem holds in the more general setting of a family of *faithful valuations* $\nu^{p_j} : \mathcal{O}_{X,p_j} \setminus \{0\} \to (\mathbb{Z}^n, >)$ respect to a fixed total additive order $>$ on $\mathbb{Z}^n$.

of $\Delta_j(L)$ to the restricted volume of $L - t\sum_{i=1}^N Y_i$ along Y_j (see section§2.3.3).

With this new construction it is possible to read the *moving* multipoint Seshadri constant (a natural generalization of the multipoint Seshadri constant to big line bundles, see section § 2.5) directly from the geometry of the multipoint Okounkov bodies:

Theorem B. *Let L be a big line bundle and let $>$ be the deglex order. Then*

$$\epsilon_S(||L||; p_1, \dots, p_N) = \max\big\{0, \xi(L; p_1, \dots, p_N)\big\}$$

where $\xi(L; p_1, \dots, p_N) := \sup\{t \geq 0 \,:\, t\Sigma_n \subset \Delta_j(L)^{ess} \, \text{for any} \, j = 1, \dots, N\}$

Next we recall another interpretation of the one point Seshadri constant: $\epsilon_S(L; p)$ is equal to the supremum of r such that there exists an holomorphic embedding $f : (B_r(0), \omega_{st}) \to (X, L)$ with the property that $f_*\omega_{st}$ extends to a Kähler form ω with cohomology class $c_1(L)$ (see Theorem 5.1.22 and Proposition 5.3.17. in [Laz04]). This result is consequence to a deeper work in symplectic geometry of McDuff-Polterovich ([MP94]), where they dealt the *symplectic packings problem* (in the same spirit, Biran in [Bir97] proved the symplectic analogoues on the Nagata's conjecture).
Successively Kaveh in [Kav16] showed how the one-point Okounkov body can be used to construct a sympletic packing. On the same line Witt Nyström in [WN15] introduced the torus-invariant domain $D(L) := \mu^{-1}\big(\Delta(L)^{ess}\big)$ (called *Okounkov domain*) for $\mu : \mathbb{C}^n \to \mathbb{R}^n$, $\mu(z_1, \dots, z_n) := (|z_1|^2, \dots, |z_n|^2)$, and showed how it approximates the manifold.

To get a similar characterization of the multipoint Seshadri constant, we introduce the following definition:

Definition 2.1.2. *We say that a finite family of $n-$dimensional Kähler manifolds $\{(M_j, \eta_j)\}_{j=1,\dots,N}$ packs into (X, L) for L ample line bundle on a $n-$dimensional projective manifold X if for any family of* *relatively compact open set $U_j \Subset M_j$ there is an holomorphic embedding $f : \bigsqcup_{j=1}^N U_j \to X$ and a Kähler form ω lying in $c_1(L)$ such that $f_*\eta_j = \omega_{|f(U_j)}$. If, in addition,*

$$\sum_{j=1}^N \int_{M_j} \eta_j^n = \int_X c_1(L)^n$$

then we say that $\{(M_j, \eta_j)\}_{j=1,\dots,N}$ packs perfectly into (X, L).

Following [WN15] we define the *multipoint Okounkov domains* as the torus-invariant domains of $\mathbb{C}^n$ given by $D_j(L) := \mu^{-1}\big(\Delta_j(L)^{ess}\big)$ and we prove the following

Theorem C. [3] *Let L be an ample line bundle. Then $\{(D_j(L), \omega_{st})\}_{j=1,\dots,N}$ packs perfectly into (X, L).*

Note that for big line bundles a similar theorem holds, given a slightly different definition of *packings* (see section 2.4.2).
As a consequence (Corollary 2.5.17), we get that, if $>$ is the deglex order,

$$\epsilon_S(||L||; p_1, \dots, p_N) = \max\big\{0, \sup\{r > 0 \,:\, B_r(0) \subset D_j(L)\, \forall j = 1, \dots, N\}\big\}.$$

We remark that this was known in dimension 2 by the work of Eckl ([Eckl17]).

Moving to particular cases, for toric manifolds we prove that, chosen torus-fixed points and the deglex order, the multipoint Okounkov bodies can be obtained *subdiving* the polytope (Theorem 2.6.4). If we consider all torus-invariant points the subdivision is of *type barycentric* (Corollary 2.6.6). As a consequence we get that the multipoint Seshadri constant of N torus-fixed points is in $\frac{1}{2}\mathbb{N}$ (Corollary 2.6.7). Finally in the surface case, we extend the result in [KLM12] showing, for the lexicographical order, the polyhedrality of $\Delta_j(L)$ for any

[3] the theorem holds even if ν^{p_j} is a family of faithful *quasi-monomial* valuations respect to the same linearly independent vectors $\vec{\lambda}_1, \dots, \vec{\lambda}_n \in \mathbb{N}^n$.

$j \in \{1, \dots, N\}$ such that $\Delta_j(L)^\circ \neq \emptyset$ (Theorem 2.6.9). And for $\mathcal{O}_{\mathbb{P}^2}(1)$ over $\mathbb{P}^2$ we completely characterize $\Delta_j(\mathcal{O}_{\mathbb{P}^2}(1))$ in function of $\epsilon_S(\mathcal{O}_{\mathbb{P}^2}(1); N)$ obtaining an explicit formula for the restricted volume of $\mu^*\mathcal{O}_{\mathbb{P}^2}(1) - t\mathbb{E}$ for $t \in \mathbb{Q}$ where $\mu : \tilde{X} \to X$ is the blow-up at N very general points and $\mathbb{E} := \sum_{j=1}^N E_j$ is the sum of the exceptional divisors (Theorem 2.6.14). As a consequence we independently get a result present in [DKMS15]: the ray $\mu^*\mathcal{O}_{\mathbb{P}^2}(1) - t\mathbb{E}$ meets at most two *Zariski chambers*.

2.1.1 Organization

Section 2.2 contains some preliminary facts on singular metrics, base loci of divisors and Okounkov bodies.

In section 2.3 we develop the theory of multipoint Okounkov bodies: the goal is to generalize some results in [LM09] for $N \geq 1$. We prove here Theorem A.

Section 2.4 is dedicated to showing Theorem C.

In section 2.5 we introduce the notion of *moving* multipoint Seshadri constants. Moreover we prove Theorem B, connecting the moving multipoint Seshadri constant in a more analytical language in the spirit of [Dem90], and prove the connection between the moving multipoint Seshadri constant and *Kähler packings*.

The last section 2.6 deals with the two aforementioned particular cases: toric manifolds and surfaces.

2.1.2 Related works

In addition to the already mentioned papers of Witt Nyström ([WN15]), Eckl ([Eckl17]), and Kürona-Lozovanu ([KL15a], [KL17]), during the preparation of this paper the work of Shin [Sh17] appeared as a preprint. Starting from the same data of a big divisor over a projective manifold of dimension n and the choice of r different points, he gave a construction of an *extended Okounkov Body* $\Delta_{Y_\cdot^1, \dots, Y_\cdot^r}(D) \subset \mathbb{R}^{rn}$ from a valuation associated to a family of admissible or infinitesimal flags $Y_\cdot^1, \dots, Y_\cdot^r$. In the ample case thanks to the Serre's vanishing Theorem, the multipoint Okounkov bodies can be recovered from the extended Okounkov body as projections after suitable subdivisions. Precisely, with the notation given in [Sh17], we get

$$F(\Delta_j(D)) = \pi_j\Big(\Delta_{Y_\cdot^1,\dots,Y_\cdot^r}(D)\cap H_{1,j}\cap\cdots\cap H_{j-1,j}\cap H_{j+1,j}\cap\cdots\cap H_{r,j}\Big)$$

where $\pi_j : \mathbb{R}^{rn}\to\mathbb{R}^n, \pi_j(\vec{x}_1,\dots,\vec{x}_r) := \vec{x}_j$, $H_{i,j} := \{(\vec{x}_1,\dots,\vec{x}_r)\in\mathbb{R}^{rn} : x_{i,1}\geq x_{j,1}\}$ and $F : \mathbb{R}^n\to\mathbb{R}^n, F(y_1,\dots,y_n) := (|y|, y_1,\dots,y_{n-1})$. Note that $x_{i,1}$ means the first component of the vector $\vec{x_i}$ while $|y| = y_1+\cdots+y_n$. The same equality holds if $L := \mathcal{O}_X(D)$ is big and $c_1(L)\in\operatorname{Supp}(\Gamma_j(X))^\circ$ (see section 2.3.2).

2.1.3 Acknowledgements

I want to thank David Witt Nyström and Stefano Trapani for proposing the project to me and for their suggestions and comments. It is also a pleasure to thank Bo Berndtsson for reviewing this article, Valentino Tosatti for his interesting comments and Christian Schultes for pointing out a mistake in the previous version.

2.2 Preliminaries

2.2.1 Singular metrics and (currents of) curvature

Let L be an holomorphic line bundle over a projective manifold X. A smooth (hermitian) metric φ is the collection of an open cover $\{U_j\}_j$ of X and of smooth functions $\varphi_j\in\mathcal{C}^\infty(U_j)$ such that on each not-empty intersection $U_i\cap U_j$ we have $\varphi_i=\varphi_j+\ln|g_{i,j}|^2$ where $g_{i,j}$ are the transition function defining the line bundle L. The curvature of a smooth metric φ is given on each open U_j by $dd^c\varphi_j$ where $d^c=\frac{i}{4\pi}(\bar\partial-\partial)$ so that $dd^c=\frac{i}{2\pi}\partial\bar\partial$. We observe that it is a global $(1,1)-$form on X, so for convenience we use the notation $dd^c\varphi$. The metric is called *positive* if $dd^c\varphi$ is a Kähler form, i.e. if the functions φ_j are strictly plurisubharmonic. By the well-known Kodaira Embedding Theorem, a line bundle admits a positive metric iff it is ample.
Demailly in [Dem90] introduced a weaker notion of metric: a (hermitian) *singular* metric φ is given by a collection of data as before

but with the weaker condition that $\varphi_j \in L^1_{loc}(U_j)$. If the functions φ_j are also plurisubharmonic, then we say that φ is a singular positive metric. Note that the $dd^c\varphi$ exists in the weak sense, indeed it is a closed positive $(1,1)-$current (we will call it the current of curvature of the metric φ). We say that $dd^c\varphi$ is a Kähler current if it dominates some Kähler form ω. By Proposition 4.2. in [Dem90] a line bundle is big iff it admits a singular positive metric whose current of curvature is a Kähler current.

In this paper we will often work with $\mathbb{R}-$line bundles, i.e. with formal linear combinations of line bundles. Moreover since we will work exclusively with projective manifolds, we will often consider an $\mathbb{R}-$line bundle as a class of $\mathbb{R}-$divisors modulo linear equivalence and its first Chern class as a class of $\mathbb{R}-$divisors modulo numerical equivalence.

2.2.2 Base loci

We recall here the construction of the *base loci* (see [ELMNP06]). Given a $\mathbb{Q}-$divisor D, let $\mathbb{B}(D) := \bigcap_{k\geq 1} \mathrm{Bs}(kD)$ be the *stable base locus* of D where $\mathrm{Bs}(kD)$ is the base locus of the linear system $|kD|$. The base loci $\mathbb{B}_+(D) := \bigcap_A \mathbb{B}(D-A)$ and $\mathbb{B}_-(D) := \bigcup_A \mathbb{B}(D+A)$, where A varies among all ample $\mathbb{Q}-$divisors, are called respectively *augmented* and *restricted base locus* of D. They do not change by multiplication of a positive integer and $\mathbb{B}_-(D) \subset \mathbb{B}(D) \subset \mathbb{B}_+(D)$. Moreover as described in the work of Nakamaye, [Nak03], the restricted and the augmented base loci are numerical invariants and can be considered as defined in the Neron-Severi space (for a real class it is enough to consider only ample $\mathbb{R}-$divisors A such that $D \pm A$ is a $\mathbb{Q}-$divisor). The stable base loci does not, see Example 1.1. in [ELMNP06], although by Proposition 1.2.6. in [ELMNP06] the subset where the augmented and restricted base loci are equal is open and dense in the Neron-Severi space $\mathrm{N}^1(X)_{\mathbb{R}}$.

Thanks to the numerical invariance of the restricted and augmented base loci, we will often talk of restricted and/or augmented base loci of a $\mathbb{R}-$line bundle L. Moreover the restricted base locus can be thought as a measure of the *nefness* since D is nef iff $\mathbb{B}_-(D) = \emptyset$, while the augmented base locus can be thought as a measure of a *ampleness* since D is ample iff $\mathbb{B}_+(D) = \emptyset$. Moreover $\mathbb{B}_-(D) = X$ iff D is not pseudoeffective while $\mathbb{B}_+(D) = X$ iff D is not big.

2.2.3 Additive Semigroups and their Okounkov bodies

We briefly recall some notions about the theory of the Okounkov bodies constructed from an additive semigroup (the main references are [KKh12] and [Bou14], see also [Kho93]).
Let $S \subset \mathbb{Z}^{n+1}$ be an additive subsemigroup not necessarily finitely generated. We denote by $C(S)$ the closed cone in $\mathbb{R}^{n+1}$ generated by S, i.e. the closure of the set of all linear combinations $\sum_i \lambda_i s_i$ with $\lambda_i \in \mathbb{R}_{\geq 0}$ and $s_i \in S$. In this paper we will work exclusively with semigroups S such that the pair $(S, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$ is *admissible*, i.e. $S \subset \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, or *strongly admissible*, i.e. it is admissible, $C(S)$ is strictly convex and it intersects the hyperplane $\mathbb{R}^n \times \{0\}$ only in the origin (for the general definition see section §1.2 in [KKh12]). We have fixed the usual order on $\mathbb{R}$ in the last coordinate. We recall that a closed convex cone C with *apex* the origin is called *strictly convex* iff the biggest linear subspace contained in C is the origin.

Definition 2.2.1. *Let $(S, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$ be an admissible pair. Then*

$$\Delta(S) := \pi\big(C(S) \cap \{\mathbb{R}^n \times \{1\}\}\big)$$

*is called **Okounkov convex set of** $(\mathbf{S}, \mathbb{R}^{\mathbf{n}} \times \mathbb{R}_{\geq \mathbf{0}})$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection to the first n coordinates. If $(S, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$ is strongly admissible, $\Delta(S)$ is also called **Okounkov body of** $(\mathbf{S}, \mathbb{R}^{\mathbf{n}} \times \mathbb{R}_{\geq \mathbf{0}})$.*

Remark 2.2.2. The fact that it is convex is immediate, and it is not hard to check that it is compact iff the pair is strongly admissible. Furthermore S generates a subgroup of $\mathbb{Z}^{n+1}$ of maximal rank iff $\Delta(S)$ has interior not-empty.

Defining $S^k := \{\alpha : (k\alpha, k) \in S\} \subset \mathbb{R}^n$ for $k \in \mathbb{N}$, we get

Proposition 2.2.3. *Let $(S, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$ be an admissible pair, then*

$$\Delta(S) = \overline{\bigcup_{k \geq 1} S^k}.$$

Moreover for any $K \subset \Delta(S)^\circ \subset \mathbb{R}^n$ compact subset, $K \subset Conv(S^k)$ for $k \gg 1$ divisible enough, where Conv denotes the closed convex hull. As a consequence

$$\Delta(S)^\circ = \bigcup_{k\geq 1} Conv(S^k)^\circ = \bigcup_{k\geq 1} Conv(S^{k!})^\circ$$

with $Conv(S^{k!})$ non-decreasing in k.

Proof. It is clear that $\Delta(S) \supset \overline{\bigcup_{k\geq 1} S^k}$. The reverse implication follows from Theorem 1.4. in [KKh12] if S is finitely generated, while in general we can approximate $\Delta(S)$ by Okounkov bodies of finitely generated subsemigroups of S. The second statement is the content of Lemma 2.3 in [WN14] if S is finitely generated, while the general case follows observing that $Conv(S^{k!})$ is non-decreasing in k by definition. $\square$

If a strong admissible pair $(S, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$ satisfies the further hypothesis $\Delta(S) \subset \mathbb{R}^n_{\geq 0}$ then let

$$\Delta(S)^{ess} := \bigcup_{k\geq 1} Conv(S^k)^{ess}$$

denote the *essential* Okounkov body where $Conv(S^k)^{ess}$ is the interior of $Conv(S^k)$ as subset of $\mathbb{R}^n_{\geq 0}$ with its induced topology. We note that if S is finitely generated then $\Delta(S)^{ess}$ coincides with the interior of $\Delta(S)$ as subset of $\mathbb{R}^n_{\geq 0}$, but in general they may be different.

Proposition 2.2.4. *Let $(S, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$ be a strongly admissible pair such that $\Delta(S) \subset \mathbb{R}^n_{\geq 0}$, and let $K \subset \Delta(S)^{ess} \subset \mathbb{R}^n$ be a compact subset of the Okounkov body of $(S, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$. Then there exists $k \gg 1$ divisible enough such that $K \subset Conv(S^k)^{ess}$. As a consequence*

$$\Delta(S)^{ess} = \bigcup_{k\geq 1} Conv(S^{k!})^{ess}$$

with $Conv(S^{k!})^{ess}$ non-decreasing in k. In particular $\Delta(S)^{ess}$ is an open convex set of $\mathbb{R}^n_{\geq 0}$.

Proof. We may assume that $\Delta(S)^{ess} \neq \emptyset$ otherwise it is trival. Therefore we know that the subgroup of $\mathbb{Z}^{n+1}$ generated by S has maximal rank. Then as in Proposition 2.2.3 it is enough to prove the proposition assuming S finitely generated. Thus we conclude similarly to Lemma 2.3 in [WN14] using Theorem 1.4. in [KKh12]. $\square$

We also recall the following important Theorem:

Theorem 2.2.5 ([Bou14], Théorème 1.12.; [KKh12], Theorem 1.14.). *Let $(S, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$ be a strongly admissible pair, then*

$$\mathrm{Vol}_{\mathbb{R}^n}(\Delta(S)) = \lim_{m\to\infty, m\in\mathbb{N}(S)} \frac{\#S^m}{m^n}.$$

where $\mathbb{N}(S) := \{m \in \mathbb{N} \,:\, S^m \neq \emptyset\}$ and the volume is respect to the Lebesgue measure.

Finally we need to introduce the *valuations*:

Definition 2.2.6. *Let V be an algebra over $\mathbb{C}$. A **valuation** from V to $\mathbb{Z}^n$ equipped with a total additive order $>$ is a map $\nu : V \setminus \{0\} \to (\mathbb{Z}^n, >)$ such that*

i) $\nu(f+g) \geq \min\{\nu(f), \nu(g)\}$ for any $f, g \in V \setminus \{0\}$ such that $f + g \neq 0$;

ii) $\nu(\lambda f) = \nu(f)$ for any $f \in V \setminus \{0\}$ and any $\mathbb{C} \ni \lambda \neq 0$;

iii) $\nu(fg) = \nu(f) + \nu(g)$ for any $f, g \in V \setminus \{0\}$.

Often ν is defined on the whole V adding $+\infty$ to the group $\mathbb{Z}^n$ and imposing $\nu(0) := +\infty$.
For any $\alpha \in \mathbb{Z}^n$ the α*−leaf* of the valuation is defined as the quotient of vector spaces

$$\hat{V}_\alpha := \frac{\{f \in V \setminus \{0\} \,:\, \nu(f) \geq \alpha\} \cup \{0\}}{\{f \in V \setminus \{0\} \,:\, \nu(f) > \alpha\} \cup \{0\}}.$$

A valuation is said to have *one-dimensional leaves* if the dimension of any leaf is at most 1.

Proposition 2.2.7 ([KKh12], Proposition 2.6)**.** *Let V be an algebra over $\mathbb{C}$, and let $\nu : V \setminus \{0\} \to (\mathbb{Z}^n, >)$ be a valuation with one-dimensional leaves. Then for any no trivial subspace $W \subset V$,*

$$\#\nu(W \setminus \{0\}) = \dim_{\mathbb{C}} W.$$

A valuation is said to be *faithful* if its image is the whole $\mathbb{Z}^n$. If a valuation is faithful then it has one-dimensional leaves (see Remark 2.26. in [Bou14]).

2.2.4 The Okounkov body associated to a line bundle

In this section we recall the construction and some known results of the Okounkov body associated to a line bundle L around a point $p \in X$ (see [LM09], [KKh12] and [Bou14]).

Consider the abelian group $\mathbb{Z}^n$ equipped with a total additive order $>$, let $\nu : \mathbb{C}(X) \setminus \{0\} \to (\mathbb{Z}^n, >)$ be a faithful valuation with *center* $p \in X$. We recall that $p \in X$ is the (unique) center of ν if $\mathcal{O}_{X,p} \subset \{f \in \mathbb{C}(X) : \nu(f) \geq 0\}$ and $\mathfrak{m}_{X,p} \subset \{f \in \mathbb{C}(X) : \nu(f) > 0\}$, and that the semigroup $\nu(\mathcal{O}_{X,p} \setminus \{0\})$ is well-ordered by the induced order (see §2 in [Bou14]).

Assume that $L_{|U}$ is trivialized by a non-zero local section t. Then any section $s \in H^0(X, kL)$ can be written locally as $s = ft^k$ with $f \in \mathcal{O}_X(U)$. Thus we define $\nu(s) := \nu(f)$, where we identify $\mathbb{C}(X)$ with the meromorphic function field and $\mathcal{O}_{X,p}$ with the stalk of $\mathcal{O}_X$ at p. We observe that $\nu(s)$ does not depend on the trivialization t since any other trivialization t' of $L_{|V}$ differs from t on $U \cap V$ by an unit $u \in \mathcal{O}_X(U \cap V)$. We define an additive semigroup associated to the valuation by

$$\Gamma := \{(\nu(s), k) \,:\, s \in H^0(X, kL) \setminus \{0\}, k \geq 0\} \subset \mathbb{Z}^n \times \mathbb{Z}.$$

We call the **Okounkov body**, $\Delta(L)$, the Okounkov convex set of $(\Gamma, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$ (see Definition 2.2.1), i.e.

$$\Delta(L) := \pi\big(C(\Gamma) \cap \{\mathbb{R}^n \times \{1\}\}\big)$$

where $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the projection to the first n coordinates. By Proposition 2.2.3 we have

$$\Delta(L) := Conv\Big(\Big\{\frac{\nu(s)}{k}\ :\ s \in H^0(X, kL) \setminus \{0\}, k \geq 1\Big\}\Big) =$$
$$= \overline{\bigcup_{k\geq 1} \Big\{\frac{\nu(s)}{k}\ :\ s \in H^0(X, kL) \setminus \{0\}\Big\}}$$

and we note that it is a convex set of $\mathbb{R}^n$ but it has interior non-empty iff Γ generates a subgroup of $\mathbb{Z}^{n+1}$ of maximal rank (Remark 2.2.2). Furthermore for a prime divisor $D \in \mathrm{Div}(X)$ we will denote $\nu(D) = \nu(f)$ for f any local equation for D near p, and the map $\nu : \mathrm{Div}(X) \to \mathbb{Z}^n$ extends to a $\mathbb{R}-$linear map from $\mathrm{Div}(X)_{\mathbb{R}}$.

Theorem 2.2.8 ([LM09], [KKh12]). *The following statements hold:*

i) $\Delta(L)$ *is a compact convex set lying in* $\mathbb{R}^n$*;*

ii) $n!\mathrm{Vol}_{\mathbb{R}^n}(\Delta(L)) = \mathrm{Vol}_X(L)$*, and in particular* L *is big iff* $\Delta(L)^\circ \neq \emptyset$*, i.e.* $\Delta(L)$ *is a convex body;*

iii) if L *is big then* $\Delta(L) = \overline{\nu\big(\{D \in \mathrm{Div}_{\geq 0}(X)_{\mathbb{R}}\ :\ D \equiv_{num} L\}\big)}$ *and, in particular, the Okounkov body depends only on the numerical class of the big line bundle.*

Quasi-monomial valuation Equip $\mathbb{Z}^n$ of a total additive order $>$, fix $\vec{\lambda}_1, \dots, \vec{\lambda}_n \in \mathbb{Z}^n$ linearly independent and fix local holomorphic coordinates $\{z_1, \dots, z_n\}$ around a fixed point p. Then we can define the *quasi-monomial* valuation[4] $\nu : \mathcal{O}_{X,p} \setminus \{0\} \to \mathbb{Z}^n$ by

$$\nu(f) := \min\{\sum_{i=1}^n \alpha_i \vec{\lambda}_i\ :\ a_\alpha \neq 0 \,\text{where locally around}\, p,\ f =_U \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha\}$$

[4]recall that there exists an unique valuation $\tilde{\nu} : \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^n$ that extends ν defined by $\tilde{\nu}(f/g) = \nu(f) - \nu(g)$, for a more general construction we refer to the section §2.3 in [Bou14]

where the minumun is taken respect to the $>$ order. Note that it is faithful iff $det(\vec{\lambda}_1; \dots; \vec{\lambda}_n) = \pm 1$.
For instance if we equip $\mathbb{Z}^n$ of the lexicographical order and we take $\vec{\lambda}_j = \vec{e}_j$ ($j-$th vector of the canonical base of $\mathbb{R}^n$) we get

$$\nu(f) := \min_{lex}\{\alpha \,:\, a_\alpha \neq 0 \,\text{where locally around}\, p,\ f =_U \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha\}.$$

This is the valuation associated to an admissible flag $X = Y_0 \supset Y_1 \supset \cdots \supset Y_n = \{p\}$, in the sense of [LM09][5], such that locally $Y_i := \{z_1 = \cdots = z_i = 0\}$.
A change of coordinates with the same local flag produces the same valuation, i.e. the valuation described depends uniquely on the local flag.
Note: *In the paper a valuation associated to an admissible flag $Y.$ will be the valuation constructed by the local procedure starting from local holomorphic coordinates as just described.*
On the other hand if we equip $\mathbb{Z}^n$ of the deglex order and we take $\vec{\lambda}_i = \vec{e}_i$, we get the valuation $\nu : \mathcal{O}_{X,p} \setminus \{0\} \to \mathbb{Z}^n$,

$$\nu(f) := \min_{deglex}\{\alpha \,:\, a_\alpha \neq 0 \,\text{where locally around}\, p,\ f =_U \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha\}.$$

This is the valuation associated to an *infinitesimal* flag $Y.$ in p: given a flag of subspaces $T_pX =: V_0 \supset V_1 \supset \cdots \supset V_{n-1} \supset V_n = \{0\}$ such that $\dim_{\mathbb{C}} V_i = n - i$, consider on $\tilde{X} := \mathrm{Bl}_p X$ the flag

$$\tilde{X} =: Y_0 \supset \mathbb{P}(T_pX) = \mathbb{P}(V_0) =: Y_1 \supset \cdots \supset \mathbb{P}(V_{n-1}) =: Y_n =: \{\tilde{p}\}.$$

Note that $Y.$ is an admissible flag around $\tilde{p}$ on the blow-up $\tilde{X}$. Indeed we recover the valuation on $\tilde{X}$ associated to this admissible flag considering $F \circ \nu$ where $F : (\mathbb{Z}^n, >_{deglex}) \to (\mathbb{Z}^n, >_{lex})$ is the order-preserving isomorphism $F(\alpha) := (|\alpha|, \alpha_1, \dots, \alpha_{n-1})$, i.e. considering the quasi-monomial valuation given by the lexicographical order and $\vec{\lambda}_i = \vec{e}_1 + \vec{e}_i$.

[5] Y_i smooth irreducible subvariety of X of codimension i such that Y_i is a Cartier divisor in Y_{i-1} for any $i = 1, \dots, n$.

Note: *In the paper a valuation associated to an infinitesimal flag Y. will be the valuation ν constructed by the local procedure starting from local holomorphic coordinates as just described, and in particular the total additive order on $\mathbb{Z}^n$ will be the deglex order in this case.*

2.2.5 A moment map associated to an $(S^1)^n$−action on a particular manifold

In this brief subsection we recall some facts regarding a moment map for an $(S^1)^n$−action on a symplectic manifold (X, ω) constructed from a convex hull of a finite set $\mathcal{A} \subset \mathbb{N}^n$ (see section § 3 in [WN15]).

Let $\mathcal{A} \subset \mathbb{N}^n$ be a finite set, let $\mu : \mathbb{C}^n \to \mathbb{R}^n$ be the map $\mu(z_1, \ldots, z_n) := (|z_1|^2, \ldots, |z_n|^2)$.
Then if $Conv(\mathcal{A})^{ess} \neq \emptyset$, we define

$$\mathcal{D}_{\mathcal{A}} := \mu^{-1}(Conv(\mathcal{A})^{ess}) = \mu^{-1}(Conv(\mathcal{A}))^{\circ}$$

where with $Conv(\mathcal{A})^{ess}$ we have indicated the interior of $Conv(\mathcal{A})$ respect to the induced topology on $\mathbb{R}^n_{\geq 0}$. Next we define $X_{\mathcal{A}}$ as the manifold that we get removing from $\mathbb{C}^n$ all submanifolds given by $\{z_{i_1} = \cdots = z_{i_r} = 0\}$ which do not intersect $\mathcal{D}_{\mathcal{A}}$. We equip the manifold with the form $\omega_{\mathcal{A}} := dd^c\phi_{\mathcal{A}}$ where

$$\phi_{\mathcal{A}}(z) := \ln\Big(\sum_{\alpha\in\mathcal{A}} |z^{\alpha}|^2\Big).$$

Clearly, by construction, $\omega_{\mathcal{A}}$ is an $(S^1)^n$−invariant Kähler form on $X_{\mathcal{A}}$, so in particular $(X_{\mathcal{A}}, \omega_{\mathcal{A}})$ can be thought as a symplectic manifold. Moreover, defining $f(w_1, \ldots, w_n) := (e^{w_1/2}, \ldots, e^{w_n/2})$, the function $u_{\mathcal{A}}(w) := \phi_{\mathcal{A}} \circ f(w)$ is plurisubharmonic and independent of the imaginary part y_i, and $f^*\omega_{\mathcal{A}} = dd^c u_{\mathcal{A}}$. Thus an easy calculation shows that

$$dd^c u_{\mathcal{A}} = \frac{1}{4\pi}\sum_{j,k=1}^{n} \frac{\partial^2 u_{\mathcal{A}}}{\partial x_k \partial x_j} dy_k \wedge dx_j$$

which implies

$$d\frac{\partial}{\partial x_k} u_{\mathcal{A}} = dd^c u_{\mathcal{A}}\big((4\pi)\frac{\partial}{\partial y_k}, \cdot\big).$$

Therefore, setting $H_k := \frac{\partial u_{\mathcal{A}}}{\partial x_k} \circ f^{-1}$, since $(f^{-1})_*\big(2\pi\frac{\partial}{\partial\theta_k}\big) = 4\pi\frac{\partial}{\partial y_k}$, we get

$$dH_k = \omega_{\mathcal{A}}(2\pi\frac{\partial}{\partial\theta_k}, \cdot).$$

Hence $\mu_{\mathcal{A}} = (H_1, \dots, H_n) = \nabla u_{\mathcal{A}} \circ f^{-1}$ is a moment map for the $(S^1)^n$-action on the symplectic manifold $(X_{\mathcal{A}}, \omega_{\mathcal{A}})$. Furthermore it is not hard to check that $\mu_{\mathcal{A}}((C^*)^n) = Conv(\mathcal{A})^\circ$, that $\mu_{\mathcal{A}}(X_A) = Conv(\mathcal{A})^{ess}$ and that for any $U \subset X_{\mathcal{A}}$, setting $f^{-1}(U) = V \times (i\mathbb{R}^n)$,

$$\int_U \omega_{\mathcal{A}}^n = \int_{V\times(i[0,4\pi])^n} (dd^c u_{\mathcal{A}})^n = n! \int_V \det(\mathrm{Hess}(u_{\mathcal{A}})) =$$
$$= n! \int_{\nabla u_{\mathcal{A}}(V)} dx = n!\mathrm{Vol}(\mu_{\mathcal{A}}(U)).$$

Finally we quote here an useful result:

Lemma 2.2.9 ([WN15], Lemma 3.1.). *Let U be a relatively compact open subset of $\mathcal{D}_{\mathcal{A}}$. Then there exists a smooth function $g : X_{\mathcal{A}} \to \mathbb{R}$ with compact support such that $\omega := \omega_{\mathcal{A}} + dd^c g$ is Kähler and $\omega = \omega_{st}$ over U.*

2.3 Multipoint Okounkov bodies

We fix an additive total order $>$ on $\mathbb{Z}^n$ and a family of faithful valuations $\nu^{p_j} : \mathbb{C}(X) \setminus \{0\} \to (\mathbb{Z}^n, >)$ centered at p_j, where recall that $p_1, \dots, p_N \in X$ are different points chosen on the $n-$dimensional projective manifold X and L is a line bundle on X.

Definition 2.3.1. *We define $V_{\cdot,j} \subset R(X, L)$ as*

$$V_{k,j} := \{s \in H^0(X, kL) \setminus \{0\} \, : \, \nu^{p_j}(s) < \nu^{p_i}(s) \, \text{for any} \, i \neq j\}.$$

Remark 2.3.2. They are disjoint graded subsemigroups since $\nu^{p_i}(s_1 \otimes s_2) = \nu^{p_i}(s_1) + \nu^{p_i}(s_2)$, but $\bigcup_{j=1}^N V_{k,j}$ may be strictly contained in $H^0(X, kL) \setminus \{0\}$ for some $k \geq 1$.

Clearly the properties of the valuation ν^{p_j} assure that

i) $\nu^{p_j}(s) = +\infty$ iff $s = 0$ (by extension $\nu^{p_j}(0) := +\infty$);

ii) for any $s \in V_{\cdot,j}$ and for any $0 \neq a \in \mathbb{C}$, $\nu^{p_j}(as) = \nu^{p_j}(s)$.

Thus we can define

$$\Gamma_j := \{(\nu^{p_j}(s), k) \, : \, s \in V_{k,j}, k \geq 0\} \subset \mathbb{Z}^n \times \mathbb{Z}.$$

Lemma 2.3.3. *Γ_j is an additive subsemigroup of $\mathbb{Z}^{n+1}$, and $(\Gamma_j, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$ is a strongly admissible pair.*

Proof. The first part is an immediate consequence of the definition, while the last part follows from the fact that Γ_j is a subsemigroup of $\Gamma_{p_j} := \{(\nu^{p_j}(s), k) \, : \, s \in H^0(X, kL) \setminus \{0\}, k \geq 0\}$ (see subsection 2.2.4). □

Definition 2.3.4. *We call $\Delta_j(L) := \Delta(\Gamma_j)$ the* ***multipoint Okounkov body*** *of L at p_j, i.e. $\Delta_j(L) = \overline{\bigcup_{k\geq 1} \frac{\nu^{p_j}(V_{k,j})}{k}}$ by Proposition 2.2.3.*

The multipoint Okounkov bodies depend on the choice of the faithful valuations $\nu^{p_1}, \dots, \nu^{p_N}$, but we will omit the dependence to simplify the notation.

Remark 2.3.5. If we fix local holomorphic coordinates $\{z_{j,1}, \cdots, z_{j,n}\}$ around p_j, we can consider any family of faithful quasi-monomial valuations with center $p_1, \dots, p_N$ (see paragraph § 2.2.4), where any ν^{p_j} is given by the same choice of a total additive order on $\mathbb{Z}^n$ and the choice of a family of $\mathbb{Z}-$linearly independent vectors $\vec{\lambda}_{1,j}, \dots, \vec{\lambda}_{n,j} \in \mathbb{Z}^n$ (they may be different). For instance we can choose those associated to the family of admissible flags $Y_{j,i} := \{z_{j,1} = \cdots = z_{j,i} = 0\}$ (with $\mathbb{Z}^n$ equipped of the lexicographical order) or those associated to the family of infinitesimal flags $Y_{\cdot}$ (with in this case $\mathbb{Z}^n$ equipped of the deglex order).

Lemma 2.3.6. *The followings statements hold:*

i) $\Delta_j(L)$ is a compact convex set contained in $\mathbb{R}^n$;

ii) if $p_j \notin \mathbb{B}_+(L)$ then $\Gamma_j(L)$ generates a subgroup of $\mathbb{Z}^{n+1}$ of maximal rank. In particular $\Delta_j(L)^\circ \neq \emptyset$.

Proof. The first point follows by construction (see Definition 2.2.1 and Remark 2.2.2).
For the second point, proceeding similarly to Lemma 2.2 in [LM09], let D be a big divisor such that $L = \mathcal{O}_X(D)$ and let A, B be two fixed ample divisors such that $D = A - B$. Since D is big there exists $\mathbb{N} \ni k \gg 1$ such that $kD - B$ is linearly equivalent to an effective divisor F.
Moreover, since by hypothesis $p_j \notin \mathbb{B}_+(L)$, by taking $k \gg 1$ big enough, we may assume that $p_j \notin \mathrm{Supp}(F)$ (see Corollary 1.6. in [ELMNP06]), thus F is described by a global section f that is an unity in $\mathcal{O}_{X,p_j}$. Then, possibly adding a very ample divisor to A and B we may suppose that there exist sections $s_0, s_1, \dots, s_n \in V_{1,j}(B)$ such that $\nu^{p_j}(s_0) = \vec{0}$ and $\nu^{p_j}(s_t) = \vec{\lambda}_t$ for any $t = 1, \dots, n$ where $\vec{\lambda}_1, \dots, \vec{\lambda}_n$ are $\mathbb{Z}-$linearly independent vectors in $\mathbb{Z}^n$ (remember that the valuations ν^{p_j} are faithful). Thus, since $s_i \otimes f \in V_{1,j}(kL)$ for any $i = 0, \dots, n$ and $\nu^{p_j}(f) = \vec{0}$, we get

$$(\vec{0}, k), (\vec{\lambda}_1, k), \dots, (\vec{\lambda}_n, k) \in \Gamma_j(L).$$

And, since $(k+1)D - F$ is linearly equivalent to A we may also assume that $(\vec{0}, k+1) \in \Gamma_j(L)$, which concludes the proof. $\square$

Remark 2.3.7. It is natural to ask if all the multipoint Okounkov bodies of a big line bundle L have not-empty interior or if they are all non-zero. But both questions have negative answers, as the following simple example shows.
Consider on $X = \mathrm{Bl}_q\mathbb{P}^2$ two points $p_1 \notin \mathrm{Supp}(E)$ and $p_2 \in \mathrm{Supp}(E)$ (E exceptional divisor), and consider the big line bundle $L := H + aE$ for $a > 1$. Clearly, if we consider the family of admissible flags given by any fixed holomorphic coordinates centered at p_1 and holomorphic coordinates $\{z_{1,2}, z_{2,2}\}$ centered at p_2 where locally $E = \{z_{1,2} = 0\}$, then for the family of valuations ν^{p_1}, ν^{p_2} associated we find $\Delta_2(L) = \emptyset$. Indeed for the theory of Okounkov bodies for surfaces (see section 6.2 in [LM09]) $\Delta_1(L) \subset \Delta^{p_1}(L) = \Sigma$ (where Σ is the standard $2-$simplex and $\Delta^{p_1}(L)$ the one-point Okounkov body) while $\Delta_2(L) \subset \Delta^{p_2}(L) = (a, 0) + \Sigma^{-1}$ ($\Sigma^{-1} = Conv(\vec{0}, \vec{e}_1, \vec{e}_1 + \vec{e}_2)$ inverted symplex), and the

conclusion follows by construction.
Actually, from Theorem A we get $\Delta_1(L) = \Sigma$.

2.3.1 Proof of Theorem A

The goal of this section is to prove Theorem A, whose formulation we recall:

Theorem A. *Let L be a big line bundle. Then*

$$n! \sum_{j=1}^{N} \mathrm{Vol}_{\mathbb{R}^n}(\Delta_j(L)) = \mathrm{Vol}_X(L)$$

We first introduce $W_{\cdot,j} \subset R(X, L)$ defined as

$$W_{k,j} := \{s \in H^0(X, kL) \setminus \{0\} \,:\, \nu^{p_j}(s) \leq \nu^{p_i}(s) \text{ if } 1 \leq i \leq j \text{ and } \nu^{p_j}(s) < \nu^{p_i}(s) \text{ if } j < i \leq N\}$$

and we set $\Gamma_{W,j} := \{(\nu^{p_j}(s), k) \,:\, s \in W_{k,j}, k \geq 0\}$. It is clear $W_{\cdot,j}$ are graded subsemigroups of $R(X, L)$ and that Lemma 2.3.3 holds for $\Gamma_{W,j}$. Moreover they are closely related to $V_{\cdot,j}$, and $\bigsqcup_{j=1}^{N} W_{k,j} = H^0(X, kL) \setminus \{0\}$ for any $k \geq 0$.

Lemma 2.3.8. *For every $k \geq 1$ we have that*

$$\sum_{j=1}^{N} \#\Gamma_{W,j}^k = h^0(X, kL),$$

where we recall that $\Gamma_{W,j}^k := \{\alpha \in \mathbb{R}^n : (k\alpha, k) \in \Gamma_{W,j}\}$.

Proof. We define a new valuation $\nu : \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^n \times \cdots \times \mathbb{Z}^n \simeq \mathbb{Z}^{Nn}$ given by $\nu(f) := (\nu^{p_1}(f), \ldots, \nu^{p_N}(f))$, where we put on $\mathbb{Z}^{Nn}$ the lexicographical order on the product of N total ordered abelian groups $\mathbb{Z}^n$, i.e.

$$(\lambda_1, \ldots, \lambda_N) < (\mu_1, \ldots, \mu_N) \quad \text{if} \quad \exists\, j \in \{1, \ldots, N\} \text{ s.t. } \lambda_i = \mu_i \,\forall i < j \text{ and } \lambda_j < \mu_j$$

Fix $k \in \mathbb{N}$. For every $j = 1, \dots, N$, let $\{\alpha_{j,1}, \dots, \alpha_{j,r_j}\} \in \Gamma^k_{W,j}$ be the set of all valuative points. Then let $s_{j,1}, \dots, s_{j,r_j} \in W_{k,j}$ be a set of sections such that $\nu^{p_j}(s_{j,l}) = \alpha_{j,l}$ for any $l = 1, \dots, r_j$.
We want to prove that $\{s_{1,1}, \dots, s_{N,r_N}\}$ is a base of $H^0(X, kL)$.
Let $\sum_{i=1}^r \mu_i s_i = 0$ be a linear relation in which $\mu_i \neq 0$, $s_i \in \{s_{1,1}, \dots, s_{N,r_N}\}$ for all $i = 1, \dots, r$ and $s_i \neq s_j$ if $i \neq j$. By construction we know that $\nu(s_1), \dots, \nu(s_r)$ are different points in $\mathbb{Z}^{Nn}$. Thus without loss of generality we can assume that $\nu(s_1) < \cdots < \nu(s_r)$, but the relation

$$s_1 = -\frac{1}{\mu_1} \sum_{i=2}^{N} \mu_i s_i$$

implies that $\nu(s_1) \geq \min\{\nu(s_j) : j = 2, \dots, r\}$ which is the contradiction. Hence $\{s_{1,1}, \dots, s_{N,r_N}\}$ is a system of linearly independent vectors, and to conclude the proof it is enough to show that it generates all $H^0(X, kL)$.
Let $t_0 \in H^0(X, kL) \backslash \{0\}$ be a section and set $\lambda_0 := (\lambda_{0,1}, \dots, \lambda_{0,N}) := \nu(t_0)$. By definition of $W_{\cdot,j}$ there exists an unique $j_0 \in 1, \dots, N$ such that $t_0 \in W_{k,j_0}$. Thus we know that $\lambda_{0,i} \geq \lambda_{0,j_0}$ if $1 \leq i \leq j_0$, and that $\lambda_{0,i} > \lambda_{0,j_0}$ if $j_0 < i \leq N$. And clearly there exists $l \in \{1, \dots, r_{j_0}\}$ such that $\lambda_{0,j_0} = \nu^{p_{j_0}}(s_{j_0,l})$, so we set $s_0 := s_{j_0,l}$. But

$$\dim\left(\frac{\{s \in H^0(X, kL) \setminus \{0\} : \nu^{p_{j_0}}(s) \geq \lambda_{0,j_0}\} \cup \{0\}}{\{s \in H^0(X, kL) \setminus \{0\} : \nu^{p_{j_0}}(s) > \lambda_{0,j_0}\} \cup \{0\}}\right) \leq 1,$$

since $\nu^{p_{j_0}}$ has one-dimensional leaves, therefore there exists a coefficient $a_0 \in \mathbb{C}$ such that $\nu^{p_{j_0}}(t_0 - a_0 s_0) > \lambda_{0,j_0}$. Thus if $t_0 = a_0 s_0$ we can conclude the proof, otherwise we set $t_1 := t_0 - a_0 s_0$ and $\lambda_1 := (\lambda_{1,1}, \dots, \lambda_{1,N}) := \nu(t_1)$, observing that $\min_j \lambda_{1,j} \geq \min_j \lambda_{0,j} = \lambda_{0,j_0}$ and that the inequality is strict if $t_1 \in W_{k,j_0}$. Iterating, we get $t_0, t_1, \dots, t_l \in H^0(X, kL) \setminus \{0\}$ such that $t_l := t_{l-1} - a_{l-1} s_{l-1} \in W_{k,j_l}$ for an unique $j_l \in \{1, \dots, N\}$ where $s_{l-1} \in \{s_{j_{l-1},1}, \dots, s_{j_{l-1},r_{l-1}}\}$ satisfies $\nu^{p_{j_{l-1}}}(t_{l-1}) = \nu^{p_{j_{l-1}}}(s_{l-1})$, and $\min_j \lambda_{l,j} \geq \min_j \lambda_{l-1,j}$ for $\nu(t_l) =: \lambda_l$. Therefore we get a sequence of valuative point λ_l such that $\min_j \lambda_{l,j} \geq \min_j \lambda_{l-1,j} \geq \cdots \geq \min_j \lambda_{0,j}$ where by construction there is at least one strict inequality if $l > N$. Hence we deduce that the iterative process will finish since that the set of valuative point of ν has finite cardinality as easy consequence of the finitess of the cardinality of $\Gamma_{W,j}^k$ for each $j = 1, \ldots, N$. $\square$

Proposition 2.3.9. *Let L be a big line bundle. Then $\Delta_j(mL) = m\Delta_j(L)$ and $\Delta_j^W(mL) = m\Delta_j^W(L)$ for any $m \in \mathbb{N}$ and for any $j = 1, \ldots, N$ where $\Delta_j^W(L)$ is the Okounkov body associated to the additive semigroup $\Gamma_{W,j}(L)$.*

Proof. The proof proceeds in the same way as the proof of Proposition 4.1.*ii* in [LM09], exploiting again the property of the total order on $\mathbb{Z}^n$.
We may assume $\Delta_j(L) \neq \emptyset$, otherwise it would be trivial, and we can choose $r, t \in \mathbb{N}$ such that $V_{r,j}, V_{tm-r,j} \neq \emptyset$, i.e. there exist sections $e \in V_{r,j}$ and $f \in V_{tm-r,j}$. Thus we get the inclusions

$$k\Gamma_j(mL)^k + \nu^{p_j}(e) + \nu^{p_j}(f) \subset (km+r)\Gamma_j(L)^{km+r} + \nu^{p_j}(f) \subset (k+t)\Gamma_j(mL)^{k+t}.$$

Letting $k \to \infty$, we find $\Delta_j(mL) \subset m\Delta_j(L) \subset \Delta_j(mL)$.
The same proof works for $\Delta_j^W(L)$. $\square$

Now we are ready to prove the Theorem A.

Proof of Theorem A. By Proposition 2.3.8 and Theorem 2.2.5 we get

$$n! \sum_{j=1}^{N} \mathrm{Vol}_{\mathbb{R}^n}(\Delta_j^W(L)) = \lim_{k \in \mathbb{N}(L), k \to \infty} \frac{n! \sum_{j=1}^{N} \#\Gamma_{W,j}^k}{k^n} =$$
$$= \lim_{k \in \mathbb{N}(L), k \to \infty} \frac{h^0(X, kL)}{k^n/n!} = \mathrm{Vol}_X(L). \quad (2.1)$$

Hence to conclude the proof it is sufficient to show that, for any $j = 1, \ldots, N$,

$$\Delta_j^W(L)^\circ = \Delta_j(L)^\circ,$$

and since $\Gamma_{V,j} \subset \Gamma_{W,j}$ we need only to prove that $\Delta_j^W(L)^\circ \subset \Delta_j(L)^\circ$.
Let A be a fixed ample line bundle A such that there exist $s_1, \ldots, s_N \in H^0(X, A)$ with $s_i \in V_{1,i}(A)$ and $\nu^{p_i}(s_i) = 0$. Thus we get $\Delta_j^W(mL - A) \subset \Delta_j(mL)$

for each $m \in \mathbb{N}$ and for any $j = 1, \dots, N$ since $s \otimes s_j^k \in V_{k,j}(mL)$ for any $s \in W_{k,j}(mL - A)$. Hence $\Delta_j^W(L - \frac{1}{m}A) \subset \Delta_j(L)$ by Proposition 2.3.9 enlarging the definition of multipoint Okounkov bodies to $\mathbb{Q}-$line bundles. Moreover as a consequence of (2.1) and of the continuity of the volume we get that $m \to \mathrm{Vol}_{\mathbb{R}^n}(\Delta_j^W(L - \frac{1}{m}A))$ is a continuous increasing function converging to $\mathrm{Vol}_{\mathbb{R}^n}(\Delta_j^W(L))$ since it is clear that $\Delta_j^W(L - \frac{1}{m}A) \subset \Delta_j^W(L - \frac{1}{l}A)$ if $l > m$ for any $j = 1, \dots, N$. Therefore we deduce that $\Delta_j^W(L)^\circ \subset \Delta_j(L)^\circ$ for any $j = 1, \dots, N$. $\square$

2.3.2 Variation of multipoint Okounkov bodies

Similarly to the section §4 in [LM09], we prove that for an open subset of the big cone the construction of the multipoint Okounkov Body is a cohomological construction, i.e. $\Delta_j(L)$ depends only from the first Chern class $c_1(L) \in \mathrm{N}^1(X)$ of the big line bundle L, where we have indicated with $\mathrm{N}^1(X)$ the Neron-Severi group. Recall that $\rho(X) := \dim \mathrm{N}^1(X)_{\mathbb{R}} < \infty$ where $\mathrm{N}^1(X)_{\mathbb{R}} := \mathrm{N}^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

Proposition 2.3.10. *Let L be a big line bundle. If $\Delta_j(L)^\circ \neq \emptyset$, then $\Delta_j(L)$ depends uniquely on the numerical class of the big line bundle L.*

Proof. Pick two big line bundles L, L' such that $L' = L + P$ for P numerically trivial and let A be a fixed ample line bundle. We observe that for any $m \in \mathbb{N}$ there exists $k_m \in \mathbb{N}$ and $s_m \in H^0(X, k_m m(P + \frac{1}{m}A))$ such that $s_m(p_i) \neq 0$ for any $i = 1, \dots, N$ since $P + \frac{1}{m}A$ is a ample $\mathbb{Q}-$line bundle. Hence we get $\Delta_j(L) \subset \Delta_j(L' + \frac{1}{m}A)$ by homogeneity (Proposition 2.3.9) since $s \otimes s_m^k \in V_{k,j}(k_m mL' + k_m A)$ for any section $s \in V_{k,j}(k_m mL)$. Then Theorem A, the continuity of the volume and the easy inclusion $\Delta_j(L' + \frac{1}{m}A) \subset \Delta_j(L' + \frac{1}{l}A)$ if $m > l$ imply that $m \to \mathrm{Vol}_{\mathbb{R}^n}(\Delta_j(L' + \frac{1}{m}A))$ is a continuous decreasing function converging to $\mathrm{Vol}_{\mathbb{R}^n}(\Delta_j(L'))$ as $m \to \infty$. Thus $\Delta_j(L) \subset \Delta_j(L')$ if $\Delta_j(L)^\circ \neq \emptyset$.
We conclude replacing L by $L + P$ and P by $-P$. $\square$

Setting $r := \rho(X)$ for simplicity, fix $L_1, \dots, L_r$ line bundle such that $\{c_1(L_1), \dots, c_1(L_r)\}$ is a $\mathbb{Z}-$basis of $\mathrm{N}^1(X)$: this lead to natural identifications $\mathrm{N}^1(X) \simeq \mathbb{Z}^r$, $\mathrm{N}^1(X)_{\mathbb{R}} \simeq \mathbb{R}^r$. Moreover by Lemma 4.6. in [LM09] we may choose $L_1, \dots, L_r$ such that the pseudoeffective cone is contained in in the positive orthant of $\mathbb{R}^r$.

Definition 2.3.11. *Letting*

$$\Gamma_j(X) := \Gamma_j(X; L_1, \dots, L_r) := \{(\nu^{p_j}(s), \vec{m}) \, : \, s \in V_{\vec{m},j}(L_1, \dots, L_r)) \backslash \{0\}, \\ \vec{m} \in \mathbb{N}^r\} \subset \mathbb{Z}^n \times \mathbb{N}^r$$

be the global multipoint semigroup of X at p_j with $p_1, \dots \hat{p_j}, \dots, p_N$ fixed *(it is an addittive subsemigroup of* $\mathbb{Z}^{n+r}$*) where* $V_{\vec{m},j}(L_1, \dots, L_r) := \{s \in H^0(X, \vec{m} \cdot (L_1, \dots, L_r)) \setminus \{0\} \, : \, \nu^{p_j}(s) < \nu^{p_i}(s) \text{ for any } i \neq j\}$*, we define*

$$\Delta_j(X) := C(\Gamma_j(X))$$

as the closed convex cone in $\mathbb{R}^{n+r}$ *generated by* $\Gamma_j(X)$*, and call it the* ***global multipoint Okounkov body*** *at* p_j*.*

Lemma 2.3.12. *The semigroup* $\Gamma_j(X)$ *generates a subgroup of* $\mathbb{Z}^{n+r}$ *of maximal rank.*

Proof. Since the cone $\mathrm{Amp}(X)$ is open non-empty set in $\mathrm{N}^1(X)_{\mathbb{R}}$ (we have indicated with $\mathrm{Amp}(X)$ the *ample* cone, see [Laz04]), we can fix $F_1, \dots, F_r$ ample line bundles that generate $\mathrm{N}^1(X)$ as free $\mathbb{Z}-$module. Moreover, by the assumptions done for $L_1, \dots, L_r$ we know that for every $i = 1, \dots, r$ there exists $\vec{a}_i$ such that $F_i = \vec{a}_i \cdot (L_1, \dots, L_r)$. Thus, for any $i = 1, \dots, r$, the graded semigroup $\Gamma_j(F_i)$ sits in $\Gamma_j(X)$ in a natural way and it generates a subgroup of $\mathbb{Z}^n \times \mathbb{Z} \cdot \vec{a}_i$ of maximal rank by point *ii*) in Lemma 2.3.6 since $\mathbb{B}_+(F_i) = \emptyset$. We conclude observing that $\vec{a}_1, \dots, \vec{a}_r$ span $\mathbb{Z}^r$. □

Next we need a further fact about additive semigroups and their cones. Let $\Gamma \subset \mathbb{Z}^n \times \mathbb{N}^r$ be an additive semigroup, and let $C(\Gamma) \subset \mathbb{R}^n \times \mathbb{R}^r$ be the closed convex cone generated by Γ. We call the *support* of Γ respect to the last r coordinates, $\mathrm{Supp}(\Gamma)$, the closed convex cone $C(\pi(\Gamma)) \subset \mathbb{R}^r$ where $\pi : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^r$ is the usual projection. Then, given $\vec{a} \in \mathbb{N}^r$, we set $\Gamma_{\mathbb{N}\vec{a}} := \Gamma \cap (\mathbb{Z}^n \times \mathbb{N}\vec{a})$ and denote by $C(\Gamma_{\mathbb{N}\vec{a}}) \subset \mathbb{R}^n \times \mathbb{R}\vec{a}$ the closed convex cone generated by $\Gamma_{\mathbb{N}\vec{a}}$ when we consider it as an additive semigroup of $\mathbb{Z}^n \times \mathbb{Z}\vec{a} \simeq \mathbb{Z}^{n+1}$.

Proposition 2.3.13 ([LM09],Proposition 4.9.)**.** *Assume that* Γ *generates a subgroup of finite index in* $\mathbb{Z}^n \times \mathbb{Z}^r$*, and let* $\vec{a} \in \mathbb{N}^r$ *be a vector lying in the interior of* $\mathrm{Supp}(\Gamma)$*. Then*

$$C(\Gamma_{\mathbb{N}\vec{a}}) = C(\Gamma) \cap (\mathbb{R}^n \times \mathbb{R}\vec{a})$$

Now we are ready to prove the main theorem of this section:

Theorem 2.3.14. *The global multipoint Okounkov body* $\Delta_j(X)$ *is characterized by the property that in the following diagram*

$$\begin{tikzcd}
\Delta_j(X) \arrow[dr] & \subset & \mathbb{R}^n \times \mathbb{R}^r \simeq \mathbb{R}^n \times \mathrm{N}^1(X)_{\mathbb{R}} \arrow[dl, "\mathrm{pr}_2"] \\
& \mathbb{R}^r \simeq \mathrm{N}^1(X)_{\mathbb{R}} &
\end{tikzcd}$$

the fiber of $\Delta_j(X)$ *over any cohomology class* $c_1(L)$ *of a big* $\mathbb{Q}-$*line bundle* L *such that* $c_1(L) \in \mathrm{Supp}(\Gamma_j(X))^{\circ}$ *is the multipoint Okounkov body associated to* L *at* p_j*, i.e* $\Delta_j(X) \cap \mathrm{pr}_2^{-1}(c_1(L)) = \Delta_j(L)$*.*

Remark 2.3.15. It is seems a bit unclear what $\mathrm{Supp}(\Gamma_j(X))^{\circ}$ is. By second point in Lemma 2.3.6, it contains the open convex set $B_+(p_j)^C$ where $B_+(p_j) := \{\alpha \in \mathrm{N}^1(X)_{\mathbb{R}} \,:\, p \in \mathbb{B}_+(\alpha)\}$ is closed respect to the metric topology on $N^1(X)_{\mathbb{R}}$ by Proposition 1.2. in [KL15a] and its complement is convex as easy consequence of Proposition 1.5. in [ELMNP06]. But in general $\mathrm{Supp}(\Gamma_j(X))^{\circ}$ may be bigger: for instance if $N = 1$ $\mathrm{Supp}(\Gamma_j(X)) = \overline{\mathrm{Eff}}(X)$, and it is not hard to construct an example with $p_1, p_2 \in \mathbb{B}_-(L)$ and $\Delta_j(L)^{\circ} \neq \emptyset$ for $j = 1, 2$.

Proof. For any vector $\vec{a} \in \mathbb{N}^r$ such that $L := \vec{a} \cdot (L_1, \dots, L_r)$ is a big line bundle in $\mathrm{Supp}(\Gamma_j(X))^{\circ}$, we get $\Gamma_j(X)_{\mathbb{N}\vec{a}} = \Gamma_j(L)$, and so the base of the cone $C(\Gamma_j(X)_{\mathbb{N}\vec{a}}) = C(\Gamma_j(L)) \subset \mathbb{R}^n \times \mathbb{R}\vec{a}$ is the multipoint Okounkov body $\Delta_j(L)$, i.e.

$$\Delta_j(L) = \pi\Big(C(\Gamma_j(X)_{\mathbb{N}\vec{a}}) \cap \big(\mathbb{R}^n \times \{1\}\big)\Big).$$

Then Proposition 2.3.13 implies that the right side of the last equality coincides with the fiber $\Delta_j(X)$ over $c_1(L)$. We conclude observing that both side of the request equality rescale linearly. $\square$

Corollary 2.3.16. *The function* $\mathrm{Vol}_{\mathbb{R}^n} : \mathrm{Supp}(\Gamma_j(X))^\circ \to \mathbb{R}_{>0}$, $c_1(L) \to \mathrm{Vol}_{\mathbb{R}^n}(\Delta_j(L))$ *is well-defined, continuous, homogeneous of degree n and log-concave, i.e.*

$$\mathrm{Vol}_{\mathbb{R}^n}(\Delta_j(L+L'))^{1/n} \geq \mathrm{Vol}_{\mathbb{R}^n}(\Delta_j(L))^{1/n} + \mathrm{Vol}_{\mathbb{R}^n}(\Delta_j(L'))^{1/n}$$

Proof. The fact that it is well-defined and its homogeneity follow immediately by Proposition 2.3.9, while the other statements follow from standard results in convex geometry, using the Brunn-Minkowski Theorem thanks to Theorem 2.3.14. □

Finally we note that the Theorem 2.3.14 allows us to describe the multipoint Okounkov bodies similarly to the Proposition 4.1. in [Bou14]:

Corollary 2.3.17. *If $L = \mathcal{O}_X(D)$ is a big line bundle such that $c_1(L) \in \mathrm{Supp}(\Gamma_j(X))^\circ$, then*

$$\Delta_j(L) = \overline{\nu^{p_j}\{D' \in \mathrm{Div}_{\geq 0}(X)_{\mathbb{R}} : D' \equiv_{num} D \,\mathrm{and}\, \nu^{p_j}(D') < \nu^{p_i}(D')\, \forall i \neq j\}}$$

where we have indicated with $\equiv_{num}$ the numerical equivalence. In particular every rational point in $\Delta_j(L)^\circ$ is valuative and if it contains a small $n-$symplex with valuative vertices then any rational point in the $n-$symplex is valuative.

Proof. The first part follows directly from Theorem 2.3.14 since $D' \equiv_{num} D$ iff $c_1(L) = c_1(\mathcal{O}_X(D'))$ by definition (considering the $\mathbb{R}-$line bundle $\mathcal{O}_X(D')$). While the second statement is a consequence of the multiplicative property of the valuation ν^{p_j}. □

2.3.3 Geometry of multipoint Okounkov bodies

To investigate the geometry of the multipoint Okounkov bodies we need to introduce the following important invariant:

Definition 2.3.18. *Let L be a line bundle, $V \subset X$ a subvariety of dimension d and* $H^0(X|V, kL) := \mathrm{Im}\Big(H^0(X, kL) \to H^0(V, kL_{|V})\Big)$. *Then the quantity*

$$\mathrm{Vol}_{X|V}(L) := \limsup_{k\to\infty} \frac{\dim H^0(X|V, kL)}{k^d/d!}$$

is called ***restricted volume of*** L ***along*** V.

We refer to [ELMNP09] and reference therein for the theory about this new object.
In the repeatedly quoted paper [LM09], given a valuation $\nu^p(s) = (\nu^p(s)_1, \ldots, \nu^p(s)_n)$ associated to an admissible flag $Y. = (Y_1, \ldots, Y_n)$ such that $Y_1 = E$ and a line bundle L such that $E \not\subset \mathbb{B}_+(L)$, the authors also defined the one-point Okounkov body of the graded linear sistem $H^0(X|E, kL) \subset H^0(E, kL_{|E})$ by

$$\Delta_{X|E}(L) := \Delta(\Gamma_{X|E})$$

with $\Gamma_{X|E} := \{(\nu^p(s)_2, \ldots, \nu^p(s)_n, k) \in \mathbb{N}^{n-1} \times \mathbb{N} : s \in H^0(X|E, kL) \setminus \{0\}, k \geq 1\}$ and they proved the following

Theorem 2.3.19 ([LM09], Theorem 4.24, Corollary 4.25)**.** *Let* $E \not\subset \mathbb{B}_+(L)$ *be a prime divisor with* L *big* $\mathbb{R}-$*line bundle and let* $Y.$ *be an admissible flag such that* $Y_1 =: E$*. Let* $C_{\max} := \sup\{\lambda \geq 0 \,:\, L - \lambda E \,\text{is big}\}$. *Then for any* $0 \leq t < C_{max}$

$$\Delta(L)_{x_1 \geq t} = \Delta(L - tE) + t\vec{e}_1$$
$$\Delta(L)_{x_1 = t} = \Delta_{X|E}(L - tE)$$

Moreover

i) $\mathrm{Vol}_{\mathbb{R}^{n-1}}(\Delta(L)_{x_1=t}) = \frac{1}{(n-1)!}\mathrm{Vol}_{X|E}(L - tE)$*;*

ii) $\mathrm{Vol}_X(L) - \mathrm{Vol}_X(L - tE) = n\int_0^t \mathrm{Vol}_{X|E}(L - \lambda E)d\lambda$*;*

In this section we suppose to have fixed a family of valuations ν^{p_j} associated to a family of admissible flags $Y. = (Y_{\cdot,1}, \ldots, Y_{\cdot,N})$ on a projective manifold X, centered respectively in $p_1, \ldots, p_N$ (see paragraph 2.2.4 and Remark 2.3.5). Given a big line bundle L, and prime divisors $E_1, \ldots, E_N$ where $E_j = Y_{1,j}$ for any $j = 1, \ldots, N$, we set

$$\mu(L; \mathbb{E}) := \sup\{t \geq 0 \,:\, L - t\mathbb{E} \,\text{is big}\}$$

where $\mathbb{E} := \sum_{i=1}^N E_i$, and

$$\mu(L; E_j) := \sup\{t \geq 0 \,:\, \Delta_j(L - t\mathbb{E})^\circ \neq \emptyset\}.$$

Theorem 2.3.20. *Let L a big $\mathbb{R}-$line bundle, ν^{p_j} a family of valuations associated to a family of admissible flags $Y.$ centered at $p_1,\dots,p_N$. Then, letting $(x_1,\dots,x_n)$ be fixed coordinates on $\mathbb{R}^n$, for any $j\in\{1,\dots,N\}$ such that $\Delta_j(L)^\circ\neq\emptyset$ the followings hold:*

i) $\Delta_j(L)_{x_1\geq t}=\Delta_j(L-t\mathbb{E})+t\vec{e}_1$ *for any* $0\leq t<\mu(L;E_j)$*, for any* $j=1,\dots,N$*;*

ii) $\Delta_j(L)_{x_1=t}=\Delta_{X|E_j}(L-t\mathbb{E})$ *for any* $0\leq t<\mu(L;\mathbb{E})$*, for any* $j=1,\dots,N$*;*

iii) $\mathrm{Vol}_{\mathbb{R}^{n-1}}(\Delta_j(L)_{x_j=t})=\frac{1}{(n-1)!}\mathrm{Vol}_{X|E_j}(L-t\mathbb{E})$ *for any* $0\leq t<\mu(L;\mathbb{E})$*, for any* $j=1,\dots,N$*, and in particular* $\mu(L;E_j)=\sup\{t\geq 0:E_j\not\subset\mathbb{B}_+(L-t\mathbb{E})\}$.

Moreover

iv) $\mathrm{Vol}_X(L)-\mathrm{Vol}_X(L-t\mathbb{E})=n\int_0^t\sum_{i=1}^N\mathrm{Vol}_{X|E_i}\Big(L-\lambda\mathbb{E}\Big)d\lambda$ *for any* $0\leq t<\mu(L;\mathbb{E})$.

Proof. The first point follows as in Proposition 4.1. in [LM09], noting that if L is a big line bundle and $0\leq t<\mu(L;E_j)$ integer then $\{s\in V_{k,j}(L):\nu^{p_j}(s)_1\geq kt\}\simeq V_{k,j}(L-t\mathbb{E})$ for any $k\geq 1$. Therefore $\Gamma_j(L)_{x_1\geq t}=\varphi_t(\Gamma_j(L-t\mathbb{E}))$ where $\varphi_t:\mathbb{N}^n\times\mathbb{N}\to\mathbb{N}^n\times\mathbb{N}$ is given by $\varphi_t(\vec{x},k):=(\vec{x}+tk\vec{e}_1,k)$. Passing to the cones we get $C(\Gamma_j(L)_{x_1\geq t})=\varphi_{t,\mathbb{R}}\big(C(\Gamma_j(L-t\mathbb{E}))\big)$ where $\varphi_{t,\mathbb{R}}$ is the linear map between vector spaces associated to φ_t. Hence, taking the base of the cones, the equality $\Delta_j(L)_{x_1\geq t}=\Delta_j(L-t\mathbb{E})+t\vec{e}_1$ follows. Finally, since both sides in $i)$ rescale linearly by Proposition 2.3.9, the equality holds for any L $\mathbb{Q}-$line bundle and $t\in\mathbb{Q}$. We conclude the proof of the first point by the continuity given by Theorem 2.3.14 since $0\leq t<\mu(L;E_j)$.
Let us show point $ii)$, assuming first L $\mathbb{Q}-$line bundle and $0\leq t<\mu(L;E_j)$ rational.

We consider the additive semigroups

$$\Gamma_{j,t}(L) = \{(\nu^{p_j}(s), k) \in \mathbb{N}^n \times \mathbb{N} : s \in V_{k,j}(L) \text{ and } \nu^{p_j}(s)_1 = kt\}$$
$$\Gamma_{X|E_j}(L - t\mathbb{E}) := \{(\nu^{p_j}(s)_2, \dots, \nu^{p_j}(s)_n, k) \in \mathbb{N}^{n-1} \times \mathbb{N} :$$
$$s \in H^0(X|E_j, k(L - t\mathbb{E})) \setminus \{0\}, k \geq 1\}$$

and, setting $\psi_t : \mathbb{N}^{n-1} \times \mathbb{N} \to \mathbb{N}^n \times \mathbb{N}$ as $\psi_t(\vec{x}, k) := (kt, \vec{x}, k)$, we easily get $\Gamma_{j,t}(L) \subset \psi_t\big(\Gamma_{X|E_j}(L - t\mathbb{E})\big)$. Thus passing to the cones we have

$$C(\Gamma_j(L))_{x_1=t} = C\big(\Gamma_{j,t}(L)\big) \subset \psi_{t,\mathbb{R}}\Big(C\big(\Gamma_{X|E_j}(L - t\mathbb{E})\big)\Big)$$

where the equality follows from Proposition $A.1$ in [LM09]. Hence $\Delta_j(L)_{x_1=t} \subset \Delta_{X|E_j}(L - t\mathbb{E})$ for any $0 \leq t < \mu(L; E_j)$ rational. Moreover it is trivial that the same inclusion holds for any $\mu(L; E_j) < t < \mu(L; \mathbb{E})$.
Next let $0 \leq t < \mu(L; \mathbb{E})$ fixed and let A be a fixed ample line bundle such that there exists $s_j \in V_{1,j}(A)$ with $\nu^{p_j}(s_j) = \vec{0}$ and $\nu^{p_i}(s_j)_1 > 0$ for any $i \neq j$. Thus since to any section $s \in H^0(X|E_j, k(L - t\mathbb{E})) \setminus \{0\}$ we can associate a section $\tilde{s} \in H^0(X, kL)$ with $\nu^{p_j}(\tilde{s}) = (kt, \nu^{p_j}(s)_2, \dots, \nu^{p_j}(s)_n)$ and $\nu^{p_i}(\tilde{s})_1 \geq kt$ for any $i \neq j$, we get that $\tilde{s}^m \otimes s_j^k \in V_{k,j}(mL + A)$ for any $m \in \mathbb{N}$. By homogeneity this implies

$$\frac{\nu^{p_j}(\tilde{s}^m \otimes s_j^k)}{mk} = \frac{\nu^{p_j}(\tilde{s})}{k} = \Big(t, \frac{\nu^{p_j}(s)}{k}\Big) =: x \in \Delta_j\Big(L + \frac{1}{m}A\Big)_{x_1=t}$$

for any $m \in \mathbb{N}$. Hence since $\Delta_j(L)^\circ \neq \emptyset$ we get $0 \leq t \leq \mu(L; E_j)$ and $x \in \Delta_j(L)_{x_1=t}$ by the continuity of Theorem 2.3.14.
Summarizing we have showed that both sides of $ii)$ are empty if $\mu(L; E_j) < t < \mu(L; \mathbb{E})$ and that they coincides for any rational $0 \leq t < \mu(L; E_j)$. Moreover since by Theorem 2.3.19 $\Delta_{X|E_j}(L - t\mathbb{E}) = \Delta(L - t\sum_{i=1, i\neq j}^N E_i)_{x_1=t}$ with respect to the valuation ν^{p_j}, we know that both sides vary continuously for $0 \leq t \leq \mu(L; E_j)$ if $\mu(L; E_j) < \mu(L; E_j)$ by Theorem 4.5. in [LM09], while they vary continuously for $0 \leq t < \mu(L; \mathbb{E})$ if $\mu(L; E_j) = \mu(L; \mathbb{E})$. Hence the second point follows by homogeneity (Proposition 2.3.9) and by continuity (Theorem 2.3.14).
The point $iii)$ is an immediate consequence of $ii)$ using Theorem

2.3.19.i) and Theorem A and C in [ELMNP09], while last the point follows by integration using Theorem A. $\square$

We observe that the Theorem may be helpful when we fix a big line bundle L and a family of valuations associated to a family of infinitesimal flags centered at $p_1, \ldots, p_N \notin \mathbb{B}_+(L)$. Indeed, similarly as stated in the paragraph § 2.2.4, componing with $F : \mathbb{R}^n \to \mathbb{R}^n$, $F(x) = (|x|, x_1, \ldots, x_{n-1})$, the Theorem holds and in particular, for any $j = 1, \ldots, N$, we get

i) $F\big(\Delta_j(L)\big)_{x_{j,1} \geq t} = \Delta_j\big(f^*L - t\mathbb{E}\big) + t\vec{e}_1$ for any $0 \leq t < \mu(f^*L; E_j)$;

ii) $F\big(\Delta_j(L)\big)_{x_{j,1} = t} = \Delta_{\tilde{X}|E_j}(f^*L - t\mathbb{E})$ for any $0 \leq t < \mu(f^*L; \mathbb{E})$;

iii) $\mathrm{Vol}_{\mathbb{R}^{n-1}}\big(F(\Delta_j(L))_{x_{j,1}=t}\big) = \frac{1}{(n-1)!}\mathrm{Vol}_{\tilde{X}|E_j}(f^*L - t\mathbb{E})$ for any $0 \leq t < \mu(f^*L; \mathbb{E})$;

where we have set $f : \tilde{X} \to X$ for the blow-up at $Z = \{p_1, \ldots, p_N\}$. Note that $\mathbb{E} = \sum_{j=1}^N E_j$ is the sum of the exceptional divisors given by the blow-up and that the multipoint Okounkov body on the right side in i) is calculated from the family of valutions $\{\tilde{\nu}^{\tilde{p}_j}\}_{j=1}^N$ (it is associated to the family of admissible flags on $\tilde{X}$ given by the family of infinitesimal flags on X).
In this setting the Theorem, describing the geometry of the multipoint Okounkov bodies, yields a new tool to study the *multipoint Seshadri constant* as stated in the Introduction (see Theorem B). And as application in the surfaces case we refer to the subsection 2.6.2.

2.4 Kähler Packings

Recalling the notation of the subsection § 2.2.3, the *essential* multipoint Okounkov body is defined as

$$\Delta_j(L)^{ess} := \bigcup_{k\geq 1} \Delta_j^k(L)^{ess} = \bigcup_{k\geq 1} \Delta_j^{k!}(L)^{ess}$$

where $\Delta_j^k(L)^{ess} := Conv(\Gamma_j^k)^{ess} = \frac{1}{k}Conv(\nu^{p_j}(V_{k,j}))^{ess}$ is the interior of $\Delta_j^k(L) := Conv(\Gamma_j^k)$ as subset of $\mathbb{R}^n_{\geq 0}$ with its induced topology.
Fix a family of local holomorphic coordinates $\{z_{j,1}, \dots, z_{j,n}\}$ for $j = 1, \dots, N$ respectively centered at $p_1, \dots, p_N$ and assume that the faithful valuations $\nu^{p_1}, \dots, \nu^{p_N}$ are quasi-monomial respect to the same additive total order $>$ on $\mathbb{Z}^n$ and respect to the same vectors $\vec{\lambda}_1, \dots, \vec{\lambda}_n \in \mathbb{N}$ (see Remark 2.3.5). Thus similarly to the Definition 2.7. in [WN15], we give the following

Definition 2.4.1. *For every $j = 1, \dots, N$ we define $D_j(L) := \mu^{-1}(\Delta_j(L)^{ess})$ and call it the* ***multipoint Okounkov domains****, where $\mu(w_1, \dots, w_n) := (|w_1|^2, \dots, |w_n|^2)$.*

Note that, as stated in the subsection 2.2.5, we get $n!\mathrm{Vol}_{\mathbb{R}^n}(\Delta_j(L)) = \mathrm{Vol}_{\mathbb{C}^n}(D_j(L))$ for any $j = 1, \dots, N$.
We will construct *Kähler packings* (see Definition 2.4.2 and 2.4.6) of the multipoint Okounkov domains with the standard metric into (X, L) for L big line bundle. We will first address the ample case and then we will generalize to the big case in subsection § 2.4.2.

2.4.1 Ample case

Definition 2.4.2. *We say that a finite family of $n-$dimensional Kähler manifolds $\{(M_j, \eta_j)\}_{j=1,\dots,N}$ packs into (X, L) for L ample if for every family of relatively compact open set $U_j \Subset M_j$ there is a holomorphic embedding $f : \bigsqcup_{j=1}^N U_j \to X$ and a Kähler form ω lying in $c_1(L)$ such that $f_*\eta_j = \omega_{|f(U_j)}$. If, in addition,*

$$\sum_{j=1}^N \int_{M_j} \eta_j^n = \int_X c_1(L)^n$$

then we say that $\{(M_j, \eta_j)\}_{j=1,\dots,N}$ packs perfectly into (X, L).

Letting $\mu : \mathbb{C}^n \to \mathbb{R}^n$ be the map $\mu(z_j) := (|z_{j,1}|^2, \dots, |z_{j,n}|^2)$ and letting

$$\mathcal{D}_{k,j} := \mu^{-1}(k\Delta_j^k(L))^\circ = \mu^{-1}(k\Delta_j^k(L)^{ess}),$$

we define $X_{k,j}$ like the manifold we get by removing from $\mathbb{C}^n$ all the submanifolds of the form $\{z_{j,i_1} = \cdots = z_{j,i_m} = 0\}$ which do not intersect $\mathcal{D}_{k,j}$.
Thus

$$\phi_{k,j} := \ln\Big(\sum_{\alpha_j \in \nu^{p_j}(V_{k,j})} |z_j^{\alpha_j}|^2\Big)$$

is a strictly plurisubharmonic function on $X_{k,j}$ and we denote by $\omega_{k,j} := dd^c\phi^{k,j}$ the Kähler form associated (recall that $dd^c = \frac{i}{2\pi}\partial\bar{\partial}$, see subsection 2.2.1). Note that we have set $z_j = (z_{j,1}, \ldots, z_{j,n})$ to simplify the notation.

Lemma 2.4.3 ([And13], Lemma 5.2.)**.** *For any finite set $\mathcal{A} \subset \mathbb{N}^m$ with a fixed additive total order $>$, there exists a $\gamma \in (\mathbb{N}_{>0})^m$ such that*

$$\alpha < \beta \quad \text{iff} \quad \alpha \cdot \gamma < \beta \cdot \gamma$$

for any $\alpha, \beta \in \mathcal{A}$.

Theorem 2.4.4. *If L is ample then for $k > 0$ big enough $\{(X_{k,j}, \omega_{k,j})\}_{j=1}^N$ packs into (X, kL).*

Using the idea of the Theorem A in [WN15] we want to construct a Kähler metric on kL such that locally around the points $p_1, \ldots, p_N$ approximates the metrics $\phi_{k,j}$ after a suitable *zoom*. We observe that for any $\gamma \in \mathbb{N}^n$ and any section $s \in H^0(X, kL)$ we have $s(\tau^\gamma z_j)/\tau^{\gamma\cdot\alpha_j} \sim z_j^{\alpha_j}$ for $\mathbb{R}_{>0} \ni \tau$ converging to zero. Therefore locally around p_j we have $\ln(\sum_{\alpha_j\in\nu^{p_j}(V_{k,j})} |\frac{s_{\alpha_j}(\tau^\gamma z_j)}{\tau^{\gamma\cdot\alpha_j}}|^2) \sim \phi_{k,j}$ where s_{α_j} are sections in $V_{k,j}$ with leading terms of their expansion at p_j equal to α_j. Thus the idea is to consider the metric on kL given by $\ln(\sum_{i=1}^N \sum_{\alpha_i\in\nu^{p_i}(V_{k,i})} |\frac{s_{\alpha_i}}{\tau^{\gamma\cdot\alpha_i}}|^2))$ and define an opportune factor γ such that this metric approximates the local plurisubharmonic functions around the points $p_1, \ldots, p_N$ after the uniform zoom τ^γ for τ small enough. This will be possible thanks to Lemma 2.4.3 and the definition of $V_{k,j}$. Finally a standard regularization argument will conclude the proof.

Proof. We assume that the local holomorphic coordinates $\{z_{j,1}, \ldots, z_{j,n}\}$ centered a p_j contains the unit ball $B_1 \subset \mathbb{C}^n$ for every $j = 1, \ldots, n$.

Set $\mathcal{A}_j := \nu^{p_j}(V_{k,j})$ and $\mathcal{B}_i^j := \nu^{p_i}(V_{k,j})$ for $i \neq j$ to simplify the notation, let k be large enough so that $\Delta_j^k(L)^{ess} \neq \emptyset$ for any $j = 1, \dots, N$ (by Lemma 2.3.6 and Proposition 2.2.4) and let $\{U_j\}_{j=1}^N$ be a family of relatively compact open set (respectively) in $\{X_{k,j}\}_{j=1}^N$. Pick $\gamma \in \mathbb{N}^n$ as in Lemma 2.4.3 for $\mathcal{S} = \bigcup_{j=1}^N \left(\mathcal{A}_j \cup \bigcup_{i \neq j} \mathcal{B}_i^j\right)$ ordering with the total additive order $>$ induced by the family of quasi-monomial valuations, i.e. $\alpha > \beta$ iff $\alpha \cdot \gamma > \beta \cdot \gamma$.
Next, for any $j = 1, \dots, N$, by construction we can choice a family of sections s_{α_j} in $V_{k,j}$, parametrized by $\mathcal{A}_j$, such that locally

$$s_{\alpha_j}(z_j) = z_j^{\alpha_j} + \sum_{\eta_j > \alpha_j} a_{j,\eta_j} z_j^{\eta_j}$$

$$s_{\alpha_j}(z_i) = a_{i,j} z_i^{\beta_i^j} + \sum_{\eta_i > \beta_i^j} a_{i,\eta_i} z_i^{\eta_i}$$

with $a_{i,j} \neq 0$ and $\alpha_j < \beta_i^j$ for any $i \neq j$.
Thus if we define, $z_j := (\tau^{\gamma_1} z_{j,1} \dots, \tau^{\gamma_n} z_{j,n})$ for $\tau \in \mathbb{R}_{\geq 0}$, τ^γ then we get for any $\alpha_j \in \mathcal{A}_j$

$$s_{\alpha_j}(\tau^\gamma z_j) = \tau^{\gamma \cdot \alpha_j}(z_j^{\alpha_j} + o(|\tau|)) \qquad \forall\, \tau^\gamma z_j \in B_1 \tag{2.2}$$

$$s_{\alpha_j}(\tau^\gamma z_i) = \tau^{\gamma \cdot \beta_i^j}(a_{i,j} z^{\beta_i^j} + o(|\tau|)) \qquad \forall\, \tau^\gamma z_i \in B_1 \tag{2.3}$$

Let, for any $j = 1, \dots, N$, $g_j : X_{k,j} \to [0,1]$ be a smooth function such that $g_j \equiv 0$ on U_j and $g_j \equiv 1$ on K_j^C for some smoothly bounded compact set K_j such that $U_j \Subset K_j \subset X_{k,j}$. Furthermore let U_j' be a relatively compact open set in $X_{k,j}$ such that $K_j \subset U_j'$.
Then pick $0 < \delta \ll 1$ such that $\phi_j := \phi_{k,j} - 4\delta g_j$ is still strictly plurisubharmonic for any $j = 1, \dots, N$.
Now we claim that for any j there is a real positive number $0 < \tau_j = \tau_j(\delta) \ll 1$

such that for every $0 < \tau \leq \tau_j$ the following statements hold:

$$\tau^\gamma z_j \in B_1 \qquad \forall\, z_j \in U_j'$$
$$\phi_j > \ln\Big(\sum_{i=1}^N \sum_{\alpha_i \in \mathcal{A}_i} |\frac{s_{\alpha_i}(\tau^\gamma z_j)}{\tau^{\gamma\cdot\alpha_i}}|^2\Big) - \delta \quad \text{on } U_j$$
$$\phi_j < \ln\Big(\sum_{i=1}^N \sum_{\alpha_i \in \mathcal{A}_i} |\frac{s_{\alpha_i}(\tau^\gamma z_j)}{\tau^{\gamma\cdot\alpha_i}}|^2\Big) - 3\delta \quad \text{near } \partial K_j$$

Indeed it is sufficient that each request is true for $\tau \in (0, a)$ with a positive real number. For the first request it is obvious, while the others follow from the equations (2.2) and (2.3) since $g_j \equiv 0$ on U_j and $g_j \equiv 1$ on K_j^C (recall that g_j is smooth and that $\gamma \cdot \alpha_i < \gamma \cdot \beta_i^j$ if $\alpha_i \in \mathcal{A}_i$ for any $j \neq i$).
So, since $p_1, \ldots, p_N$ are distinct points on X, we can choose $0 < \tau_k \ll 1$ such that the requests above hold for every $j = 1, \ldots, N$ and $W_j \cap W_i$ for $j \neq i$ where $W_j := \varphi_j^{-1}(\tau_k^\gamma U_j')$, where φ_j is the coordinate map giving the local holomorphic coordinates centered at p_j.
Next we define, for any $j = 1, \ldots, N$,

$$\phi_j' := \max_{reg}\left(\phi_j, \ln\Big(\sum_{i=1}^N \sum_{\alpha_i \in \mathcal{A}_i} |\frac{s_{\alpha_i}(\tau^\gamma z_j)}{\tau^{\gamma\cdot\alpha_i}}|^2\Big) - 2\delta\right)$$

where $\max_{reg}(x, y)$ is a smooth convex function such that $\max_{reg}(x, y) = \max(x, y)$ whenever $|x - y| > \delta$. Therefore, by construction, we observe that ϕ_j' is smooth and strictly plurisubharmonic on $X_{k,j}$, identically equal to $\ln\Big(\sum_{i=1}^N \sum_{\alpha_i \in \mathcal{A}_i} |\frac{s_{\alpha_i}(\tau^\gamma z_j)}{\tau^{\gamma\cdot\alpha_i}}|^2\Big) - 2\delta$ near ∂K_j and identically equal to $\phi_{k,j}$ on U_j. So

$$\omega_j := dd^c \phi_j'$$

is equal to $\omega_{k,j}$ on U_j.
Therefore since for $k > 0$ big enough $\ln\Big(\sum_{i=1}^N \sum_{\alpha_i \in \mathcal{A}_i} |\frac{s_{\alpha_i}}{\tau^{\gamma\cdot\alpha_i}}|^2\Big) - 2\delta$ extends as a positive hermitian metric of kL, with abuse of notation and unless restrict further τ, we get that $\{\omega_j\}_{j=1}^N$ extend to a Kähler

form ω such that

$$\omega_{f(U_j)} = f_*(\omega_{j|U_j}) = f_*\omega_{k,j}$$

where we are set $f : \bigsqcup_{j=1}^N U_j \to X, f_{|U_j} := \varphi_j^{-1} \circ \tau^\gamma$, the *uniform rescaling* for the embedding.
Since $\{U_j\}_{j=1}^N$ are arbitrary, this shows that $\{(X_{k,j}, \omega_{k,j})\}_{j=1}^N$ packs into (X, kL). $\square$

Theorem C (Ample Case)**.** *Let L be an ample line bundle. We have that $\{(D_j(L), \omega_{st})\}_{j=1}^N$ packs perfectly into (X, L).*

Proof. If $U_1, \dots, U_N$ are relatively compact open sets, respectively, in $D_j(L)$ then by Proposition 2.2.4 there exists $k > 0$ divisible enough such that U_j is compactly contained in $\mu^{-1}(Conv(\Delta_j^k(L))^\circ$ for any $j = 1, \dots, N$, i.e. $\sqrt{k}U_j \Subset \mathcal{D}_{k,j} \Subset X_{k,j}$ for any $j = 1, \dots, N$.
By Lemma 2.2.9 there exist smooth functions $g_j : X_{k,j} \to \mathbb{R}$ with support on relatively compact open sets $U_j' \supset \sqrt{k}U_j$ such that $\tilde{\omega}_j := \omega_{k,j} + dd^c g_j$ is Kähler and $\tilde{\omega}_j = \omega_{st}$ holds on $\sqrt{k}U_j$.
Furthermore, fixing relatively compact open sets $V_j \subset X_{k,j}$ such that $U_j' \Subset V_j$ for any $j = 1, \dots, N$, by Theorem 2.4.4 we can find a holomorphic embedding $f' : \bigsqcup_{j=1}^N V_j \to X$ and a Kähler form ω' in $c_1(kL)$ such that $\omega'_{|f'(V)} = f'_*\omega_{k,j}$ for any $j = 1, \dots, N$.
Next, let χ_j be smooth cut-off functions on X such that $\chi_j \equiv 1$ on $f'(U_j')$ and $\chi_j \equiv 0$ outside $\overline{f'(V_j)}$. Thus, since $f'(V_j) \cap f'(V_i) =$ for every $j \neq i$ and since $g_j \circ f'^{-1}_{|f'(V_j)}$ has compact support in $f'(U_j')$, the function $g = \sum_{j=1}^N \chi_j g_j \circ f'^{-1}$, extends to 0 outside $\bigcup_{j=1}^N \overline{f'(V_j)}$ and $g_{|f'(V_j)} = g_j \circ f'^{-1}_{|f'(V_j)}$.
Finally defining $f : \bigsqcup_{j=1}^N U_j \to X$ by $f_{|U_j}(z_j) := f'_{|\sqrt{k}U_j}(\sqrt{k}z_j)$, we get

$$(\omega' + dd^c g)_{|f(U_j)} = f'_*(\omega_{k,j} + dd^c g_j)_{|\sqrt{k}U_j} = k f_*\omega_{st|U_j}$$

by construction. Hence $\omega := \frac{1}{k}(\omega' + dd^c g)$ is a Kähler form with class $c_1(L)$ that satisfies the requests since by Theorem A

$$\sum_{j=1}^N \int_{D_j(L)} \omega_{st}^n = n! \sum_{j=1}^N \mathrm{Vol}_{\mathbb{R}^n}(\Delta_j(L)) = \mathrm{Vol}_X(L) = \int_X \omega^n.$$

$\square$

Remark 2.4.5. If the family of valuations fixed is associated to a family of admissible flags $Y_{j,i} = \{z_{j,1} = \cdots = z_{j,i} = 0\}$ then each associated embedding $f : \bigsqcup_{j=1}^{N} U_j \to X$ can be chosen so that

$$f^{-1}_{|f(U_j)}(Y_{j,i}) = \{z_{j,1} = \cdots = z_{j,i} = 0\}$$

In particular if $N = 1$ we recover the Theorem A in [WN15].

2.4.2 The big case

Definition 2.4.6. *If L is big, we say that a finite family of $n-$dimensional Kähler manifolds $\{(M_j, \eta_j)\}_{j=1,\dots,N}$ packs into (X, L) if for every family of relatively compact open set $U_j \Subset M_j$ there is a holomorphic embedding $f : \bigsqcup_{j=1}^{N} U_j \to X$ and there exist a kähler current with analytical singularities T lying in $c_1(L)$ such that $f_* \eta_j = T_{|f(U_j)}$. If, in addition,*

$$\sum_{j=1}^{N} \int_{M_j} \eta_j^n = \int_X c_1(L)^n$$

then we say that $\{(M_j, \eta_j)\}_{j=1,\dots,N}$ packs perfectly into (X, L).

Reasoning as in the previous section we prove the following

Theorem C (Big Case)**.** *Let L be a big line bundle. We have that $\{(D_j(L), \omega_{st})\}_{j=i_1,\dots,i_q}^{N}$ packs perfectly into (X, L) where $\Delta_l(L)^\circ = \emptyset$ if $l \notin \{i_1, \dots, i_q\}$ while $\Delta_{i_m}(L)^\circ \neq \emptyset$ for any $m = 1, \dots, q$.*

Proof. We can assume that $\{i_1, \dots, i_q\} = \{1, \dots, N\}$ since $\Delta_{i_m}(L)$ and $D_{i_m}(L)$ for $m = 1, \dots, q$ do not change removing the others points.
Thus letting $k \gg 0$ big enough such that $\Delta_j^k(L)^{ess} \neq \emptyset$ for any j (Proposition 2.2.4) we can proceed similarly to the Theorem 2.4.4 with the unique difference that $\ln\Big(\sum_{i=1}^{N} \sum_{\alpha_i \in \mathcal{A}_i} |\frac{s_{\alpha_i}}{\tau^{\gamma \cdot \alpha_i}}|^2\Big)$ extends to a positive singular hermitian metric, hence we get a (current of) curvature T that is a Kähler current with analytical singularities. Therefore, as

in the ample case, we can show that $\{(D_j(L), \omega_{st})\}_{j=1}^N$ packs perfectly into (X, L). $\square$

Remark 2.4.7. If the family of valuations fixed is associated to a family of admissible flags $Y_{j,i} = \{z_{j,1} = \cdots = z_{j,i} = 0\}$ then each associated embedding $f : \bigsqcup_{j=1}^N U_j \to X$ can be chosen so that

$$f^{-1}_{|f(U_j)}(Y_{j,i}) = \{z_{j,1} = \cdots = z_{j,i} = 0\}$$

In particular if $N = 1$ we recover the Theorem C in [WN15].

2.5 Local Positivity

2.5.1 Moving Multipoint Seshadri Constant

Definition 2.5.1. *Let L be a nef line bundle on X. The quantity*

$$\epsilon_S(L; p_1, \dots, p_N) := \inf \frac{L \cdot C}{\sum_{i=1}^N \mathrm{mult}_{p_i} C}$$

where the infimum is on all irreducible curve $C \subset X$ passing through at least one of the points $p_1, \dots, p_N$ is called the ***multipoint Seshadri constant at $\mathbf{p_1}, \dots, \mathbf{p_N}$ of $\mathbf{L}$****.*

This constant has played an important role in the last three decades and it is the natural extension of the Seshadri constant introduced by Demailly in [Dem90].
The following Lemma is well-known and its proof can be found for instance in [Laz04], [BDRH$^+$09]:

Lemma 2.5.2. *Let L be a nef line bundle on X. Then*

$$\begin{aligned}\epsilon_S(L; p_1, \dots, p_N) = \sup\{t \geq 0 \,:\, \mu^* L - t \sum_{i=1}^N E_i \text{ is nef}\} = \\ = \inf \Big(\frac{L^{\dim V} \cdot V}{\sum_{j=1}^N \mathrm{mult}_{p_j} V} \Big)^{\frac{1}{\dim V}}\end{aligned}$$

where $\mu : \tilde{X} \to X$ is the blow-up at $Z = \{p_1, \ldots, p_N\}$, E_i is the exceptional divisor above p_i and where the infimum on the right side is on all positive dimensional irreducible subvariety V containing at least one point among $p_1, \ldots, p_N$.

The Lemma just showed allows to extend the definition to nef $\mathbb{Q}-$line bundles by homogeneity and to nef $\mathbb{R}-$line bundles by continuity.
Here we describe a possible generalization of the multipoint Seshadri constant for big line bundles:

Definition 2.5.3. *Let L be a big $\mathbb{R}-$line bundle, we define the* ***moving multipoint Seshadri constant at $\mathbf{p_1, \ldots, p_N}$ of $\mathbf{L}$*** *as*

$$\epsilon_S(||L||; p_1, \ldots, p_N) := \sup_{f^*L = A+E} \epsilon_S(A; f^{-1}(p_1), \ldots, f^{-1}(p_N))$$

if $p_1, \ldots, p_N \notin \mathbb{B}_+(L)$ and $\epsilon_S(||L||; p_1, \ldots, p_N) := 0$ otherwise, where the supremum is taken over all modifications $f : Y \to X$ with Y smooth such that f is an isomorphism around $p_1, \ldots, p_N$ and over all decomposition $f^*L = A + E$ where A is an ample $\mathbb{Q}-$divisor and E is effective with $f^{-1}(p_j) \notin \mathrm{Supp}(E)$ for any $j = 1, \ldots, N$.

For $N = 1$, we retrieve the definition given in [ELMNP09].
The following properties can be showed similarly as for the one-point case and they are left to the reader:

Proposition 2.5.4. *Let L, L' be big $\mathbb{R}-$line bundles. Then*

i) $\epsilon_S(||L||; p_1, \ldots, p_N) \leq \left(\frac{\mathrm{Vol}_X(L)}{N}\right)^{1/n}$;

ii) *if* $c_1(L) = c_1(L')$ *then* $\epsilon_S(||L||; p_1, \ldots, p_N) = \epsilon_S(||L'||; p_1, \ldots, p_N)$;

iii) $\epsilon_S(||\lambda L||; p_1, \ldots, p_N) = \lambda \epsilon_S(||L||; p_1, \ldots, p_N)$ *for any* $\lambda \in \mathbb{R}_{>0}$;

iv) *if* $p_1, \ldots, p_N \notin \mathbb{B}_+(L) \cup \mathbb{B}_+(L')$ *then* $\epsilon_S(||L + L'||; p_1, \ldots, p_N) \geq \epsilon_S(||L||; p_1, \ldots, p_N) + \epsilon_S(||L'||; p_1, \ldots, p_N)$.

We check that the moving multipoint Seshadri constant is an effective generalization of the multipoint Seshadri constant:

Proposition 2.5.5. *Let L be a big and nef $\mathbb{Q}-$line bundle. Then*

$$\epsilon_S(||L||;p_1,\dots,p_N)=\epsilon_S(L;p_1,\dots,p_N)$$

Proof. By homogeneity we can assume L line bundle and $p_1,\dots,p_N\notin\mathbb{B}_+(L)$ since if $p_j\in\mathbb{B}_+(L)$ for some j then by Proposition 1.1. and Corollary 5.6. in [ELMNP09] there exist an irreducible positive dimensional component $V\subset\mathbb{B}_+(L), p_j\in V$ such that $L^{\dim V}\cdot V=0$ and Lemma 2.5.2 gives the equality.
Thus, fixed a modification $f:Y\to X$ as in the definition, we get

$$\frac{L\cdot C}{\sum_{i=1}^N \mathrm{mult}_{p_i}C}=\frac{f^*L\cdot\tilde{C}}{\sum_{i=1}^N \mathrm{mult}_{f^{-1}(p_i)}\tilde{C}}\geq\frac{A\cdot\tilde{C}}{\sum_{i=1}^N \mathrm{mult}_{f^{-1}(p_i)}\tilde{C}}$$

since $f^{-1}(p_1),\dots,f^{-1}(p_N)\notin\mathrm{Supp}(E)$ and $\epsilon_S(||L||;p_1,\dots,p_N)\leq\epsilon_S(L;p_1,\dots,p_N)$ follows.
For the reverse inequality, we can write $L=A+E$ with A ample $\mathbb{Q}-$line bundle and E effective such that $p_1,\dots,p_N\notin\mathrm{Supp}(E)$, and we note that $L=A_m+\frac{1}{m}E$ for any $m\in\mathbb{N}$ for the ample $\mathbb{Q}-$line bundle $A_m:=\frac{1}{m}A+(1-\frac{1}{m})L$. Thus $\epsilon_S(||L||;p_1,\dots,p_N)\geq\epsilon_S(A_m;p_1,\dots,p_N)$ and letting $m\to\infty$ the inequality requested follows from the continuity of $\epsilon_S(\cdot;p_1,\dots,p_N)$ in the nef cone. □

The following Proposition justifies the name given as generalization of the definition in [Nak03]:

Proposition 2.5.6. *If L is a big $\mathbb{Q}-$line bundle such that $p_1,\dots,p_N\notin\mathbb{B}(L)$ then*

$$\epsilon_S(||L||;p_1,\dots,p_N)=\lim_{k\to\infty}\frac{\epsilon_S(M_k;\mu_k^{-1}(p_1),\dots,\mu_k^{-1}(p_N))}{k}=$$
$$=\sup_{k\to\infty}\frac{\epsilon_S(M_k;\mu_k^{-1}(p_1),\dots,\mu_k^{-1}(p_N))}{k}$$

where $M_k:=\mu_k^*(kL)-E_k$ is the moving part *of $|mL|$ given by a resolution of the base ideal $\mathfrak{b}_k:=\mathfrak{b}(|kL|)$ (or set $M_k=0$ if $H^0(X,kL)=\{0\}$).*

Note that $\epsilon_S(M_k;\mu_k^{-1}(p_1),\ldots,\mu_k^{-1}(p_N)))$ does not depend on the resolution chosen and given k_1, k_2 divisible enough we may choose resolutions such that $M_{k_1+k_2} = M_{k_1} + M_{k_2} + E$ where E is an effective divisor with $p_1,\ldots,p_N \notin \mathrm{Supp}(E)$, so the existence of the limit in the definition follows from Proposition 2.5.4.iv).

Proof of Proposition 2.5.6. By homogeneity we can assume L big line bundle, $\mathbb{B}(L) = \mathrm{Bs}(|L|)$ and that the rational map $\varphi : X \setminus \mathrm{Bs}(|L|) \to \mathbb{P}^N$ associated to the linear system $|L|$ has image of dimension n.
Suppose first that there exist $j \in \{1,\ldots,N\}$, an integer $k_0 \geq 1$ such that $\mu_{k_0}^{-1}(p_j) \in \mathbb{B}_+(M_{k_0})$. Thus for any $\mathbb{N} \ni k \geq k_0$ we get $\mu_k^{-1}(p_j) \in \mathbb{B}_+(M_k)$. Then, since M_k is big and nef, there exists a subvariety V of dimension $d \geq 1$ such that $M_k^d \cdot V = 0$ and $V \ni \mu_k^{-1}(p_j)$ (Corollary 5.6. in [ELMNP09]), thus $\epsilon_S(M_k;\mu_k^{-1}(p_1),\ldots,\mu_k^{-1}(p_N)) = 0$ by Lemma 2.5.2 and the equality follows.
Therefore we may assume $\mu_k^{-1}(p_1),\ldots,\mu_k^{-1}(p_N) \notin \mathbb{B}_+(M_k)$ for any $k \geq 1$ and we can write $M_k = A+E$ with A ample and E effective with $\mu_k^{-1}(p_1),\ldots,\mu_k^{-1}(p_N) \notin \mathrm{Supp}(E)$. Clearly for any $m \in \mathbb{N}$, setting $A_m := \frac{1}{m}A + (1-\frac{1}{m})M_k$, the equality $M_k = A_m + \frac{1}{m}E$ holds. Hence, since by definition $\epsilon_S(||L||;p_1,\ldots,p_N) \geq \frac{1}{k}\epsilon_S(A_m;\mu_k^{-1}(p_1),\ldots,\mu_k^{-1}(p_N))$ for any $m \in \mathbb{N}$, we get $\epsilon_S(||L||;p_1,\ldots,p_N) \geq \frac{1}{k}\epsilon_S(M_k;\mu_k^{-1}(p_1),\ldots,\mu_k^{-1}(p_N))$ letting $m \to \infty$.
For the reverse inequality, let $f : Y \to X$ be a modification as in the definition of the moving multipoint Seshadri constant, i.e. $f^*L = A + E$ with A ample $\mathbb{Q}-$divisor and E effective divisor with $p_1,\ldots,p_N \notin \mathrm{Supp}(E)$, and let $k \gg 1$ big enough such that kA is very ample. Thus, unless taking a log resolution of the base locus of $f^*(kL)$ that is an isomorphism around $f^{-1}(p_1),\ldots,f^{-1}(p_N)$, we can suppose $f^*(kL) = M_k + E_k$ with $p_1,\ldots,p_N \notin \mathrm{Supp}(E_k)$ for E_k effective and M_k nef and big. Then, since kA is very ample, $M_k = kA + E'_k$ with E'_k effective and $E'_k \leq kE$. Hence we get $f^{-1}(p_1),\ldots,f^{-1}(p_N) \notin \mathrm{Supp}(E'_k)$ and $\frac{1}{k}\epsilon_S(M_k;f^{-1}(p_1),\ldots,f^{-1}(p_N)) \geq \epsilon_S(A;f^{-1}(p_1),\ldots,f^{-1}(p_N))$ by homogeneity. $\square$

Proposition 2.5.7. *Let L be a big $\mathbb{Q}-$line bundle. Then*

$$\epsilon_S(||L||;p_1,\dots,p_N) = \inf\left(\frac{\mathrm{Vol}_{X|V}(L)}{\sum_{j=1}^N \mathrm{mult}_{p_j} V}\right)^{1/\dim V}$$

where the infimum is over all positive dimensional irreducible subvarieties V containing at least one of the points $p_1,\dots,p_N$.

Proof. We may assume $p_1,\dots,p_N \notin \mathbb{B}_+(L)$ since otherwise the equality is a consequence of Corollary 5.9. in [ELMNP09]. Thus $V \not\subset \mathbb{B}_+(L)$ for any positive dimensional irreducible subvariety that pass through at least one of the points $p_1,\dots,p_N$, hence by Theorem 2.13. in [ELMNP09] it is sufficient to show that

$$\epsilon_S(||L||;p_1,\dots,p_N) = \inf\left(\frac{\parallel L^{\dim V}\cdot V\parallel}{\sum_{j=1}^N \mathrm{mult}_{p_j} V}\right)^{1/\dim V}$$

where the infimum is over all positive dimensional irreducible subvarieties V that contain at least one of the points $p_1,\dots,p_N$. We recall that the *asymptotic intersection number* is defined as

$$\parallel L^{\dim V}\cdot V\parallel := \lim_{k\to\infty}\frac{M_k^{\dim V}\cdot \tilde{V}_k}{k^{\dim V}} = \sup_k \frac{M_k^{\dim V}\cdot \tilde{V}_k}{k^{\dim V}}$$

where M_k is the moving part of $\mu_k^*(kL)$ as in Proposition 2.5.6 and $\tilde{V}_k$ is the proper trasform of V through μ_k (the last equality follows from the Remark 2.9. in [ELMNP09]).
Lemma 2.5.2 and Proposition 2.5.6 (M_k is nef) imply

$$\epsilon_S(||L||;p_1,\dots,p_N) = \sup_k \frac{\epsilon_S(M_k;\mu_k^{-1}(p_1),\dots,\mu_k^{-1}(p_N))}{k} =$$

$$= \sup_k \inf_V \frac{1}{k}\left(\frac{(M_k^{\dim V}\cdot\tilde{V}_k)}{\sum_{j=1}^N \mathrm{mult}_{p_j} V}\right)^{1/\dim V} \le \inf_V\left(\frac{\parallel L^{\dim V}\cdot V\parallel}{\sum_{j=1}^N \mathrm{mult}_{p_j} V}\right)^{1/\dim V}.$$

Vice versa by the approximate Zariski decomposition showed in [Tak06] (Theorem 3.1.) for any $0<\epsilon<1$ there exists a modification $f: Y_\epsilon \to X$

that is an isomorphism around $p_1, \ldots, p_N$, $f^*L = A_\epsilon + E_\epsilon$ where A_ϵ ample and E_ϵ effective with $f^{-1}(p_1), \ldots, f^{-1}(p_N) \notin \mathrm{Supp}(E_\epsilon)$, and

$$A_\epsilon^{\dim V} \cdot \tilde{V} \geq (1-\epsilon)^{\dim V} \parallel L^{\dim V} \cdot V \parallel$$

for any $V \not\subset \mathbb{B}_+(L)$ positive dimensional irreducible subvariety ($\tilde{V}$ proper trasform of V through f). Therefore, passing to the infimum over all positive dimensional irreducible subvariety that pass through at least one of the points $p_1, \ldots, p_N$ we get

$$\begin{aligned} \epsilon_S(||L||; p_1, \ldots, p_N) \geq \epsilon_S(A_\epsilon; f^{-1}(p_1), \ldots, f^{-1}(p_N)) \geq \\ \geq (1-\epsilon) \inf \left(\frac{\parallel L^{\dim V} \cdot V \parallel}{\sum_{j=1}^N \mathrm{mult}_{p_j} V} \right)^{1/\dim V} \end{aligned}$$

which concludes the proof. □

Theorem 2.5.8. *For any choice of different points $p_1, \ldots, p_N \in X$, the function $\mathrm{N}^1(X)_\mathbb{R} \ni L \to \epsilon_S(||L||; p_1, \ldots, p_N) \in \mathbb{R}$ is continuous.*

Proof. The homogeneity and the concavity described in Proposition 2.5.4 implies the locally uniform continuity of $\epsilon_S(||L||; p_1, \ldots, p_N)$ on the open convex subset $\left(\bigcup_{j=1}^N B_+(p_j)\right)^C$ (see Remark 2.3.15). Thus it is sufficient to show that $\lim_{L' \to L} \epsilon_S(||L'||; p_1, \ldots, p_N) = 0$ if $c_1(L) \in \bigcup_{j=1}^N B_+(p_j)$. But this is a consequence of the Proposition 2.5.7 using the continuity of the restricted volume described in the Theorem 5.2. in [ELMNP09]. □

To conclude the section we recall that for a line bundle L and for a integer $s \in \mathbb{Z}_{\geq 0}$, we say that L *generates $s-$jets at* $p_1, \ldots, p_N$ if the map

$$H^0(X, L) \twoheadrightarrow \bigoplus_{j=1}^N H^0(X, L \otimes \mathcal{O}_{X,p_j}/\mathfrak{m}_{p_j}^{s+1})$$

is surjective where we have set $\mathfrak{m}_{p_j}$ for the maximal ideal in $\mathcal{O}_{X,p_j}$. And we report the following last characterization of the moving multipoint Seshadri constant:

Proposition 2.5.9 ([Ito13], Lemma 3.10.). *Let L be a big line bundle. Then*

$$\epsilon_S(||L||;p_1,\ldots,p_N) = \sup_{k>0} \frac{s(kL;p_1,\ldots,p_N)}{k} = \lim_{k\to\infty} \frac{s(kL;p_1,\ldots,p_N)}{k}$$

where $s(kL;p_1,\ldots,p_N)$ is 0 if kL does not generate $s-$jets at $p_1,\ldots,p_N$ for any $s \in \mathbb{Z}_{\geq 0}$, otherwise it is the biggest non-negative integer such that kL generates the $s(kL;p_1,\ldots,p_N)-$jets at $p_1,\ldots,p_N$.

2.5.2 Proof of Theorem B

In the spirit of the aforementioned work of Demailly [Dem90], we want to describe the moving multipoint Seshadri constant $\epsilon(||L||;p_1,\ldots,p_N)$ in a more analytical language.

Definition 2.5.10. *We say that a singular metric φ of a line bundle L has* isolated logarithmic poles *at $p_1,\ldots,p_N$ of coefficient γ if* $\min\{\nu(\varphi,p_1),\ldots,\nu(\varphi,p_N)\} = \gamma$ *and φ is finite and continuous in a small punctured neighborhood $V_j \setminus \{p_j\}$ for every $j = 1,\ldots,N$. We have indicated with $\nu(\varphi,p_j)$ the* Lelong number *of φ at p_j,*

$$\nu(\varphi,p_j) := \liminf_{z\to x} \frac{\varphi_j(z)}{\ln|z-x|^2}$$

where φ_j is the local plurisubharmonic function defining φ around $p_j = x$.
We set $\gamma(L;p_1,\ldots,p_N) := \sup\{\gamma \in \mathbb{R}$: L has a positive singular metric with isolated logarithmic poles at $p_1,\ldots,p_N$ of coefficient $\gamma\}$

Note that for $N = 1$ we recover the definition given in [Dem90].

Proposition 2.5.11. *Let L be a big $\mathbb{Q}-$line bundle. Then*

$$\gamma(L;p_1,\ldots,p_N) = \epsilon_S(||L||;p_1,\ldots,p_N)$$

Proof. By homogeneity we can assume L to be a line bundle, and we fix a family of local holomorphic coordinates $\{z_{j,1},\ldots,z_{j,n}\}$ in open coordinated sets $U_1,\ldots,U_N$ centered respectively at $p_1,\ldots,p_N$.

Setting $z_j := (z_{j,1}, \dots, z_{j,N})$ and $s := s(kL; p_1, \dots, p_N)$ for $k \geq 1$ natural number, we can find holomorphic section f_α, parametrized by all $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^{Nn}$ such that $|\alpha_j| = s$ and $f_{\alpha|U_j} = z_j^{\alpha_j}$ for any $j = 1, \dots, N$. In other words, we can find holomorphic sections of kL whose jets at $p_1, \dots, p_N$ generates all possible combination of monomials of degree s around the points chosen. Thus the positive singular metric φ on L given by

$$\varphi := \frac{1}{k} \log \Big(\sum_\alpha |f_\alpha|^2 \Big)$$

has isolated logarithmic poles at $p_1, \dots, p_N$ of coefficient s/k. Hence $\gamma(L; p_1, \dots, p_N) \geq s(kL; p_1, \dots, p_N)/k$, and letting $k \to \infty$ Proposition 2.5.9 implies $\gamma(L; p_1, \dots, p_N) \geq \epsilon_S(||L||; p_1, \dots, p_N)$.

Vice versa, assuming $\gamma(L; p_1, \dots, p_N) > 0$, let $\{\gamma_t\}_{t \in \mathbb{N}} \subset \mathbb{Q}$ be an increasing sequence of rational numbers converging to $\gamma(L; p_1, \dots, p_N)$ and let $\{k_t\}_{t \in \mathbb{N}}$ be an increasing sequence of natural numbers such that $\{k_t \gamma_t\}_{t \in \mathbb{N}}$ converges to $+\infty$. Moreover let A be an ample line bundle such that $A - K_X$ is ample, and let $\omega = dd^c \phi$ be a Kähler form in the class $c_1(A - K_X)$.

Thus for any positive singular metric φ_t of L with isolated logarithmic poles at $p_1, \dots, p_N$ of coefficient $\geq \gamma_t$, $k_t \varphi_t + \phi$ is a positive singular metric of $k_t L + A - K_X$ with Kähler current $dd^c(k_t \varphi_t) + \omega$ as curvature and with isolated logarithmic poles at $p_1, \dots, p_N$ of coefficient $\geq k_t \gamma_t$. Therefore, for $t \gg 1$ big enough, $k_t L_t + A$ generates all $(k_t \gamma_t - n)-$jets at $p_1, \dots, p_N$ by Corollary 3.3. in [Dem90], and thanks to the Proposition 2.5.9 we obtain

$$\epsilon_S(||L + \frac{1}{k_t} A||; p_1, \dots, p_N) \geq \frac{k_t \gamma_t - n}{k_t} = \gamma_t - \frac{n}{k_t}.$$

Letting $t \to \infty$ we get $\epsilon_S(||L||; p_1, \dots, p_N) \geq \gamma(L; p_1, \dots, p_N)$ using the continuity of Theorem 2.5.8. $\square$

Remark 2.5.12. We observe that the same result cannot be true if we restrict to consider metric with logarithmic poles at $p_1, \dots, p_N$ not necessarily isolated. Indeed Demailly in [Dem93] showed that for any nef and big $\mathbb{Q}-$line bundle L over a projective manifold, for any different points $p_1, \dots, p_N$, and for any $\tau_1, \dots, \tau_N$ positive real numbers with $\sum_{j=1}^N \tau_j^n < (L^n)$ there exist a positive singular metric φ with logarithmic poles at any p_j of coefficient, respectively, τ_j.

From now until the end of the section we fix a family of valuations ν^{p_j} associated to a family of infinitesimal flags centered at $p_1, \dots, p_N$ and the multipoint Okounkov bodies $\Delta_j(L)$ constructed from ν^{p_j} (see paragraph 2.2.4 and 2.3.5).

Definition 2.5.13. *Let L be a big line bundle. We define*

$$\xi(L; p_1, \dots, p_N) := \sup\{\xi \geq 0 \text{ s.t. } \xi\Sigma_n \subset \Delta_j(L)^{ess} \text{ for every } j = 1, \dots, N\}.$$

Remark 2.5.14. By definition, we note that $\xi(L; p_1, \dots, p_N) = \sup\{r > 0 : B_r(0) \subset D_j(L) \text{ for any } j = 1, \dots, N\}$.

If $N = 1$ then $\Delta_1(L) = \Delta(L)$, and it is well-known that the maximum δ such that $\delta\Sigma_n$ fits into the Okounkov body, coincides with $\epsilon_S(||L||; p)$ (Theorem C in [KL17]). The next theorem recover and generalize this result for any N:

Theorem B. *Let L be a big $\mathbb{R}-$line bundle, then*

$$\max\big\{\xi(L; p_1, \dots, p_N), 0\big\} = \epsilon_S(||L||; p_1, \dots, p_N)$$

Proof. By the continuity given by Theorem 2.3.14 and Theorem 2.5.8 and by the homogeneity of both sides we can assume L big line bundle. Moreover we may also assume $\Delta_j(L)^\circ \neq \emptyset$ for any $j = 1, \dots, N$ since otherwise it is a consequence of point $ii)$ in Lemma 2.3.6.
Let $\{\lambda_m\}_{m\in\mathbb{N}} \subset \mathbb{Q}$ be an increasing sequence convergent to $\xi(L; p_1, \dots, p_N)$ (assuming that the latter is > 0). By Proposition 2.2.4, for any $m \in \mathbb{N}$ there exist $k_m \gg 1$ such that $\lambda_m\Sigma_n \subset \Delta_j^{k_m}(L)^{ess}$ for any $j = 1, \dots, N$. Therefore, chosen a set of section $\{s_{j,\alpha}\}_{j,\alpha} \subset H^0(X, k_mL)$ parametrized in a natural way by all valuative points in $\Delta_j^{k_m}(L)^{ess} \setminus \lambda_m\Sigma_n^{ess}$ for any $j = 1, \dots, N$ (i.e. $s_{j,\alpha} \in V_{k_m,j}$, $\nu^{p_j}(s_{j,\alpha}) = \alpha$ and $\alpha \notin \lambda_m\Sigma_n^{ess}$) the metric

$$\varphi_{k_m} := \frac{1}{k_m} \ln\Big(\sum_{j=1}^N \sum_\alpha |s_{j,\alpha}|^2\Big)$$

is a positive singular metric on L such that $\nu(\varphi_{k_m}, p_j) \geq \lambda_m$ while φ_{k_m} is continuos and finite on a punctured neighborhood $V_j \setminus \{p_j\}$ for any $j = 1, \ldots, N$ by Corollary 2.3.17. Hence letting $m \to \infty$, we get $\epsilon_S(||L||; p_1, \ldots, p_N) = \gamma(L; p_1, \ldots, p_N) \geq \xi(L; p_1, \ldots, p_N)$, where the equality is the content of Proposition 2.5.11.
On the other hand, letting $\{\lambda_m\}_{m\in\mathbb{N}} \subset \mathbb{Q}$ be a increasing sequence converging to $\epsilon_S(||L||; p_1, \ldots, p_N) > 0$, Proposition 2.5.9 implies that for any $m \in \mathbb{N}$ there exists $k_m \gg 0$ divisible enough such that $s(tk_mL; p_1, \ldots, p_N) \geq tk_m\lambda_m$ for any $t \geq 1$. Thus, since the family of valuation is associated to a family of infinitesimal flags, we get

$$\frac{\lceil tk_m\lambda_m \rceil}{tk_m}\Sigma_n \subset \Delta_j^{k_m}(L)^{ess} \subset \Delta_j(L)^{ess} \; \forall j = 1, \ldots, N \text{ and } \forall t \geq 1.$$

Hence $\lambda_m\Sigma_n \subset \Delta_j(L)^{ess}$ for any $j = 1, \ldots, N$, which concludes the proof. □

Remark 2.5.15. In the case L ample line bundle, to prove the inequality $\epsilon_S(L; p_1, \ldots, p_N) \geq \xi(L; p_1, \ldots, p_N)$ we could have used Theorem C. In fact it implies that $\{(B_{\xi(L;p_1,\ldots,p_N)}(0), \omega_{st})\}_{j=1}^N$ fits into (X, L), and so by symplectic blow-up procedure for Kähler manifold (see section §5.3. in [MP94], or Lemma 5.3.17. in [Laz04]) we deduce $\xi(L; p_1, \ldots, p_N) \leq \epsilon_S(L; p_1, \ldots, p_N)$.

Remark 2.5.16. The proof of the Theorem shows that $\xi(L; p_1, \ldots, p_N)$ is independent from the choice of the family of valuations given by a family of infinitesimal flags.

The following corollary extends Theorem 0.5 in [Eckl17] to all dimension (as Eckl claimed in his paper) and to big line bundles.

Corollary 2.5.17. *Let L be a big line bundle. Then*

$$\epsilon_S(||L||; p_1, \ldots, p_N) = \max\Big\{0, \sup\{r > 0 \,:\, B_r(0) \subset D_j(L) \; \forall j = 1, \ldots, N\}\Big\}$$

For $N = 1$ it is the content of Theorem 1.3. in [WN15].

2.6 Some particular cases

2.6.1 Projective toric manifolds

In this section $X = X_\Delta$ is a smooth projective toric variety associated to a fan Δ in $N_\mathbb{R} \simeq \mathbb{R}^n$, so that the torus $T_N := N \otimes_\mathbb{Z} \mathbb{C}^* \simeq (\mathbb{C}^*)^n$ acts on X ($N \simeq \mathbb{Z}^n$ denote a lattice of rank n with dual $M := \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})$, see [Ful93], [Cox11] for notation and basic fact about toric varieties). It is well-known that there is a correspondence between toric manifolds X polarized by T_N-invariant ample divisors D and lattice *polytopes* $P \subset M_\mathbb{R}$ of dimension n. Indeed to any such divisor $D = \sum_{\rho \in \Delta(1)} a_\rho D_\rho$ (where we indicate with $\Delta(k)$ the cones of dimension k) the polytope P_D is given by $P_D := \bigcap_{\rho \in \Delta(1)} \{m \in M_\mathbb{R} \, : \, \langle m, v_\rho \rangle \geq -a_\rho\}$ where v_ρ indicates the generator of $\rho \cap N$. Vice versa any such polytope P can be described as $P := \bigcap_{F \,\mathrm{facet}} \{m \in M_\mathbb{R} \, : \, \langle m, n_F \rangle \geq -a_F\}$ where a *facet* is a $1-$codimensional face of P and $n_F \in N$ is the unique primitive element that is normal to F and that point toward the interior of P. Thus the *normal fan associated to* P is $\Delta_P := \{\sigma_\mathcal{F} \, : \, \mathcal{F}\,\mathrm{face\,of}\, P\}$ where $\sigma_\mathcal{F}$ is the cone in $\mathrm{N}_\mathbb{R}$ generated by all normal elements n_F as above for any facet that contains the face $\mathcal{F}$. In particular vertices of P correspond to T_N-invariant points on the toric manifold X_P associated to Δ_P while facets of P correspond to T_N-invariant divisor on X_P. Finally the polarization is given by $D_P := \sum_{F \,\mathrm{facet}} a_F D_F$.

Thus, given an ample toric line bundle $L = \mathcal{O}_X(D)$ on a projective toric manifold X we can fix local holomorphic coordinates around a T_N-invariant point $p \in X$ (corresponding to a vertex $x_\sigma \in P$) such that $\{z_i = 0\} = D_{i|U_\sigma}$ for D_i T_N-invariant divisor and we can assume $D_{|U_\sigma} = 0$.

Proposition 2.6.1 ([LM09],Proposition 6.1.(i))**.** *In the setting as above, the equality*

$$\phi_{\mathbb{R}^n}(P_D) = \Delta(L)$$

holds, where $\phi_\mathbb{R}$ is the linear map associated to $\phi : M \to \mathbb{Z}^n, \phi(m) := (\langle m, v_1 \rangle, \dots, \langle m, v_n \rangle)$, for $v_i \in \Delta_{P_D}(1)$ generators of the ray associated *to D_i, and $\Delta(L)$ is the one-point Okounkov body associated to the admissible flag given by the local holomorphic coordinates chosen.*

Moreover we recall that it is possible to describe the positivity of the toric line bundle at a T_N-invariant point x_σ corresponding to a vertex in P directly from the polytope:

Lemma 2.6.2. (Lemma 4.2.1, [BDRH$^+$09]) *Let (X, L) be a toric polarized manifold, and let P be the associated polytope with vertices $x_{\sigma_1}, \dots, x_{\sigma_l}$. Then L generates $k-$jets at x_{σ_j} iff the length $|e_{j,i}|$ is bigger than k for any $i = 1, \dots, n$ where $e_{j,i}$ is the edge connecting x_{σ_j} to another vertex $x_{\sigma_{\tau(i)}}$.*

Remark 2.6.3. By assumption, we know that P is a *Delzant* polypote, i.e. there are exactly n edges originating from each vertex, and the first integer points on such edges form a lattice basis (for *integer* we mean a point belonging in M). Moreover fixed the first integer points on the edges starting from a vertex x_σ (i.e. fixed a basis for $M \simeq \mathbb{Z}^n$) we define the length of an edge starting from x_σ as the usual length in $\mathbb{R}^n$ observing that it is always an integer since the polytope is a *lattice* polytope.

Similarly to Proposition 2.6.1, chosen R T_N-invariants points corresponding to R vertices of the polytope P, we retrieve the multipoint Okounkov bodies of the corresponding R T_N-invariant points on X *directly* from the polytope:

Theorem 2.6.4. *Let (X, L) be a toric polarized manifold, and let P be the associated polytope with vertices $x_{\sigma_1}, \dots, x_{\sigma_l}$ corresponding, respectively, to the T_N-points $p_1, \dots, p_l$. Then for any choice of R different points ($R \leq l$) $p_{i_1}, \dots, p_{i_R}$ among $p_1, \dots, p_l$, there exist a subdivision of P into R polytopes (a priori not* lattice *polytopes) $P_1, \dots, P_R$ such that $\phi_{\mathbb{R}^n,j}(P_j) = \Delta_j(L)$ for a suitable choice of a family of valuations associated to infinitesimal (toric) flags centered at $p_{i_1}, \dots, p_{i_R}$, where $\phi_{\mathbb{R}^n,j}$ is the map given in the Proposition 2.6.1 (for the point x_{σ_j}).*

Proof. Unless reordering, we can assume that the T_N-invariants points $p_1, \dots, p_R$ correspond to the vertices $x_{\sigma_1}, \dots, x_{\sigma_R}$.

Next for any $j = 1, \dots, R$, after the identification $M \simeq \mathbb{Z}^n$ given by the choice of a lattice basis $m_{j,1}, \dots, m_{j,n}$ as explained in Remark 2.6.3, we retrieve the Okounkov Body $\Delta(L)$ at p_j associated to an infinitesimal flag given by the coordinates $\{z_{1,j}, \dots, z_{n,j}\}$ as explained in Proposition 2.6.1 composing with the map $\phi_{\mathbb{R}^n,j}$. Thus, by construction, we know that any valuative point lying in the diagonal face of the $n-$symplex $\delta\Sigma_n$ for $\delta \in \mathbb{Q}$ correspond to a section $s \in H^0(X, kL)$ such that $\mathrm{ord}_{p_j}(s) = k\delta$. Working directly on the polytope P, the diagonal face of the $n-$symplex $\delta\Sigma_n$ corresponds to the intersection of the polytope P with the hyperplane $H_{\delta,j}$ parallel to the hyperplane passing for $m_{1,j}, \dots, m_{n,j}$ and whose distance from the point x_{σ_j} is equal to δ (the *distance* is calculated from the identification $M \simeq \mathbb{Z}^n$). Therefore defining

$$P_j := \overline{\bigcup_{(\delta_1,\dots,\delta_n)\in\mathbb{Q}^n_{\geq 0}, \delta_j<\delta_i\ \forall i\neq j} H_{\delta_1,1} \cap \dots \cap H_{\delta_R,R} \cap P} =$$

$$= \overline{\bigcup_{(\delta_1,\dots,\delta_n)\in\mathbb{Q}^n_{\geq 0}, \delta_j\leq\delta_i\ \forall i\neq j} H_{\delta_1,1} \cap \dots \cap H_{\delta_R,R} \cap P}$$

we get by Proposition 2.2.3 $\phi_{\mathbb{R}^n,j}(P_j) = \Delta_j(L)$ since any valuative point in $H_{\delta_1,1} \cap \dots \cap H_{\delta_R,R} \cap P$ belongs to $\Delta_j(L)$ if $\delta_j < \delta_i$ for any $i \neq j$, while on the other hand any valuative point in $\Delta_j(L)$ belongs to $H_{\delta_1,1} \cap \dots \cap H_{\delta_R,R} \cap P$ for certain rational numbers $\delta_1, \dots, \delta_R$ such that $\delta_j \leq \delta_i$. □

Remark 2.6.5. As easy consequence, we get that for any polarized toric manifold (X, L) and for any choice of R T_N-invariants points $p_1, \dots, p_R$, the multipoint Okounkov bodies constructed from the infinitesimal flags as in the Theorem are polyhedral.

Corollary 2.6.6. *In the same setting of the Theorem 2.6.4, if $R = l$, then the subdivision is of type* barycenteric. *Namely, for any fixed vertex x_{σ_j}, if $F_1, \dots, F_n$ are the facets containing x_{σ_j} and $b_1, \dots, b_n$ are their respective barycenters, then the polytope P_j is the convex body defined by the intersection of P with the n hyperplanes $H_{O,j}$ passing through the baricenter O of P and the barycenters $b_1, \dots, b_{j-1}, b_{j+1}, \dots, b_n$.*

Finally we retrieve and extend Corollary 2.3. in [Eckl17] as consequence of Theorem 2.6.4 and Theorem B:

Corollary 2.6.7. *In the same setting of the Theorem 2.6.4, for any* $j = 1, \dots, R$*, let* $\epsilon_{S,j} := \min_{i=1,\dots,n}\{\delta_{j,i}\}$ *be the minimum among all the* reparametrized *length* $|e_{j,i}|$ *of the edges* $e_{j,i}$ *for* $i = 1, \dots, n$*, i.e.* $\delta_{j,i} := |e_{j,i}|$ *if* $e_{j,i}$ *connect* x_{σ_j} *to another point* x_{σ_i} *corresponding to a point* $p \notin \{p_1, \dots, p_R\}$*, while* $\delta_{j,i} := \frac{1}{2}|e_{j,i}|$ *if* $e_{j,i}$ *connect to a point* x_{σ_i} *corresponding to a point* $p \in \{p_1, \dots, p_R\}$*. Then*

$$\epsilon_S(L; p_1, \dots, p_R) = \min\{\epsilon_{S,j} : j = 1, \dots, R\}$$

In particular $\epsilon_S(L; p_1, \dots, p_R) \in \frac{1}{2}\mathbb{N}$*.*

2.6.2 Surfaces

When X has dimension 2, the following famous decomposition holds:

Theorem 2.6.8 (Zariski decomposition)**.** *Let* L *be a pseudoeffective* $\mathbb{Q}-$*line bundle on a surface* X*. Then there exist* $\mathbb{Q}-$*line bundles* P, N *such that*

i) $L = P + N$*;*

ii) P *is nef;*

iii) N *is effective;*

iv) $H^0(X, kP) \simeq H^0(X, kL)$ *for any* $k \geq 1$*;*

v) $P \cdot E = 0$ *for any* E *irreducible curves contained in* $\mathrm{Supp}(N)$*.*

Moreover we recall that by the main theorem of [BKS04] there exists a locally finite decomposition of the big cone into rational polyhedral subcones (*Zariski chambers*) such that in each interior of these subcones the negative part of the Zariski decomposition has constant support and the restricted and augmented base loci are equal (i.e. the divisors with cohomology classes in a interior of some Zariski chambers are *stable*, see [ELMNP06]).
Similarly to Theorem 6.4. in [LM09] and the first part of Theorem B in [KLM12] we describe the multipoint Okounkov bodies as follows:

Theorem 2.6.9. *Let L be a big line bundle over a surface X, let $p_1, \dots, p_N \in X$, and let ν^{p_j} a family of valuations associated to admissible flags centered at $p_1, \dots, p_N$ with $Y_{1,i} = C_{i|U_{p_i}}$ for irreducible curves C_i for $i = 1, \dots, N$. Then for any $j = 1, \dots, N$ such that $\Delta_j(L)^\circ \neq \emptyset$ there exist piecewise linear functions $\alpha_j, \beta_j : [t_{j,-}, t_{j,+}] \to \mathbb{R}_{\geq 0}$ for $0 \leq t_{j,-} = \inf\{t \geq 0 \,:\, C_j \not\subset \mathbb{B}_+(L - t\mathbb{G})\} < t_{j,+} = \sup\{t \geq 0 \,:\, C_j \not\subset \mathbb{B}_+(L - t\mathbb{G})\} \leq \mu(L; \mathbb{G}) := \sup\{t \geq 0 \,:\, L - t\mathbb{G} \text{ is big}\}$ where $\mathbb{G} = \sum_{j=1}^N C_j$, with α_j convex and β_j concave, $\alpha_j \leq \beta_j$, such that*

$$\Delta_j(L) = \{(t, y) \in \mathbb{R}^2 \,:\, t_{j,-} \leq t \leq t_{j,+} \text{ and } \alpha_j(t) \leq y \leq \beta_j(t)\}$$

In particular $\Delta_j(L)$ is polyhedral for any $j = 1, \dots, N$ such that $\Delta_j(L)^\circ \neq \emptyset$.

Proof. Fix $j \in \{1, \dots, N\}$ such that $\Delta_j(L)^\circ \neq \emptyset$ (it exists by Theorem A). By Theorem A and C in [ELMNP09], we know that $0 \leq t_{j,-} < t_{j,+} \leq \mu(L; \mathbb{G})$ and that $[t_{j,-}, t_{j,+}] \times \mathbb{R}_{\geq 0}$ is the smallest vertical strip containing $\Delta_j(L)$. Then by Theorem 2.3.20 and Lemma 6.3. in [LM09] we easily get $\Delta_j(L) = \{(t, y) \in \mathbb{R}^2 \,:\, t_{j,-} \leq t \leq t_{j,+} \text{ and } \alpha_j(t) \leq y \leq \beta_j(t)\}$ defining $\alpha_j(t) := \mathrm{ord}_{p_j}(N_{t|C_j})$ and $\beta_j(t) := \mathrm{ord}_{p_j}(N_{t|C_j}) + (P_t \cdot C_j)$ for $P_t + N_t$ Zariski decomposition of $L - t\mathbb{G}$ (N_t can be restricted to C_j since $\mathrm{Supp}(N_t) = \mathbb{B}_-(L - t\mathbb{G})$).
Then we proceed similarly to [KLM12] to show the polyhedrality of $\Delta_j(L)$, i.e. we set $L' := L - t_{j,+}\mathbb{G}$, $s = t_{j,+} - t$ and consider $L'_s := L' + s\mathbb{G} = L - t\mathbb{G}$ for $s \in [0, t_{j,+} - t_{j,-}]$. Thus the function $s \to N'_s$ is decreasing, i.e. $N'_{s'} - N'_s$ is effective for any $0 \leq s' < s \leq t_{j,+} - t_{j,-}$, where $L'_s = P'_s + N'_s$ is the Zariski decomposition of L'_s. Moreover, letting $F_1, \dots, F_r$ be the irreducible (negative) curves composing N'_0, we may assume (unless rearraging the F_i's) that the support of $N'_{t_{j,+} - t_{j,-}}$ consists of $F_{k+1}, \dots, F_r$ and that $0 =: s_0 < s_1 \leq \dots \leq s_k \leq t_{j,+} - t_{j,-} =: s_{k+1}$ where $s_i := \sup\{s \geq 0 \,:\, F_i \subset \mathbb{B}_-(L'_s) = \mathrm{Supp}(N'_s)\}$ for any $i = 1, \dots, k$.
So, by the continuity of the Zariski decomposition in the big cone, it is enough to show that N'_s is linear in any not-empty open interval (s_i, s_{i+1}) for $i \in \{0, \dots, k\}$. But the Zariski algorithm implies that N'_s is determined by $N'_s \cdot F_l = (L' + s\mathbb{G}) \cdot F_l$ for any $l = i + 1, \dots, r$,

and, since the intersection matrix of the curves $F_{i+1}, \ldots, F_r$ is non-degenerate, we know that there exist unique divisors A_i and B_i supported on $\cup_{l=i+1}^r F_l$ such that $A_i \cdot F_l = L' \cdot F_l$ and $B_i \cdot F_l = \mathbb{G} \cdot F_l$ for any $l = i+1, \ldots, r$. Hence $N'_s = A_i + sB_i$ for any $s \in (s_i, s_{i+1})$, which concludes the proof. $\square$

Remark 2.6.10. We observe that for any $j \in \{1, \ldots, N\}$ such that $\Delta_j(L)^\circ \neq \emptyset$ $\Delta_j(L) \cap [0, \mu(L; \mathbb{G}) - \epsilon] \times \mathbb{R}$ is *rational* polyhedral for any $0 < \epsilon < \mu(L; \mathbb{G})$ thanks to the proof and to the main theorem in [BKS04].

A particular case is when $p_1, \ldots, p_N \notin \mathbb{B}_+(L)$ and ν^{p_j} is a family of valuations associated to infinitesimal flags centered respectively at $p_1, \ldots, p_N$. Indeed in this case on the blow-up $\tilde{X} = \mathrm{Bl}_{\{p_1, \ldots, p_N\}} X$ we can consider the family of valuations $\tilde{\nu}^{\tilde{p}_j}$ associated to the admissible flags centered respectively at points $\tilde{p}_1, \ldots, \tilde{p}_N \in \tilde{X}$ (see paragraph §2.2.4). Observe that $\tilde{Y}_{1,j} = E_j$ are the exceptional divisors over the points.

Lemma 2.6.11. *In the setting just mentioned, we have $t_{j,-} = 0$ and $t_{j,+} = \mu(f^*L; \mathbb{E})$ where $\mathbb{E} = \sum_{i=1}^N E_i$ and $f : \tilde{X} \to X$ is the blow-up map.*

Proof. Theorem B easily implies $t_{j,-} = 0$ for any $j = 1, \ldots, N$ since $p_1, \ldots, p_N \notin \mathbb{B}_+(L)$ and $F(\Delta_j(L)) = \Delta_j(f^*L)$ for $F(x_1, x_2) = (x_1 + x_2, x_1)$. Next if there exists $j \in \{1, \ldots, N\}$ such that $t_{j,+} < \mu(f^*L; \mathbb{E})$, then by Theorem 2.3.20 and Theorem A and C in [ELMNP09] we get $\bar{t} := \sup\{t \geq 0 : E_j \not\subset \mathbb{B}_+(f^*L - t\mathbb{E})\} = \sup\{t \geq 0 : E_j \not\subset \mathbb{B}_-(f^*L - t\mathbb{E})\} < \mu(f^*L; \mathbb{E})$. Therefore setting $L_t := f^*L - t\mathbb{E} = P_t + N_t$ for the Zariski decomposition, we know that $E_j \in \mathrm{Supp}(N_t)$ iff $t > \bar{t}$ (see Proposition 1.2. in [KL15a]). But for any $\bar{t} < t < \mu(f^*L; \mathbb{E})$ we find out

$$0 = (L_t + t\mathbb{E}) \cdot E_j = L_t \cdot E_j + tE_j^2 < -t$$

where the first equality is justified by $P_t + N_t + t\mathbb{E} = f^*L$ while the inequality is a consequence of $L_t \cdot E_j < 0$ (since $E_j \in \mathrm{Supp}(N_t)$) and of $E_i \cdot E_j = \delta_{i,j}$. $\square$

About the Nagata's Conjecture: One of the version of the Nagata's conjecture says that for a choice of very general points $p_1, \dots, p_N \in \mathbb{P}^2$, for $N \geq 9$, the ample line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ has maximal multipoint Seshadri constant at $p_1, \dots, p_N$, i.e. $\epsilon_S(\mathcal{O}_{\mathbb{P}^2}(1); N) = 1/\sqrt{N}$ where to simplify the notation we did not indicate the points since they are very general. We can read it in the following way:

Conjecture 2.6.12 ([Nag58], Nagata's Conjecture)**.** *For $N \geq 9$ very general points in $\mathbb{P}^2$, let $\{\Delta_j(\mathcal{O}_{\mathbb{P}^2}(1))\}_{j=1}^N$ be the multipoint Okounkov bodies calculated from a family of valuations ν^{p_j} associated to a family of infinitesimal flags centered respectively at $p_1, \dots, p_N$. Then the following equivalent statements hold:*

i) $\epsilon_S(\mathcal{O}_{\mathbb{P}^2}(1); N) = 1/\sqrt{N}$*;*

ii) $\Delta_j(\mathcal{O}_{\mathbb{P}^2}(1)) = \frac{1}{\sqrt{N}}\Sigma_2$*, where Σ_2 is the standard $2-$symplex;*

iii) $D_j(\mathcal{O}_{\mathbb{P}^2}(1)) = B_{\frac{1}{\sqrt{N}}}(0)$*;*

Remark 2.6.13. It is well know that the conjecture holds if $N \geq 9$ is a perfect square. And a similar conjecture (called Biran-Nagata-Szemberg's conjecture) claims that for any ample line bundle L on a projective manifold of dimension n there exist $N_0 = N_0(X, L)$ big enough such that $\epsilon_S(L; N) = \sqrt[n]{\frac{L^n}{N}}$ for any $N \geq N_0$ very general points, i.e. it is maximal. This conjecture can be read through the multipoint Okounkov bodies as $\Delta_j(L) = \sqrt[n]{\frac{L^n}{N}}\Sigma_n$ for any $N \geq N_0$ very general points at X.

Theorem 2.6.14. *For $N \geq 9$ very general points in $\mathbb{P}^2$, there exists a family of valuations ν^{p_j} associated to a family of infinitesimal flags centered respectively at $p_1, \dots, p_N$ such that*

$$\Delta_j\Big(\mathcal{O}_{\mathbb{P}^2}(1)\Big) = \Big\{(x,y) \in \mathbb{R}^2 \,:\, 0 \leq x \leq \epsilon \text{ and } 0 \leq y \leq \frac{1}{N\epsilon}\Big(1-\frac{x}{\epsilon}\Big)\Big\} = $$
$$= Conv\big(\vec{0}, \epsilon\vec{e}_1, \frac{1}{N\epsilon}\vec{e}_2\big)$$

where $\epsilon := \epsilon_S(\mathcal{O}_{\mathbb{P}^2}(1); N)$. *In particular* $\mu(L, \mathbb{E}) = \frac{1}{N\epsilon}$ *and*

$$\mathrm{Vol}_{X|E_j}(f^*\mathcal{O}_{\mathbb{P}^2}(1) - t\mathbb{E})) = \begin{cases} t & \text{if} \quad 0 \leq t \leq \epsilon \\ \frac{\epsilon}{\frac{1}{N\epsilon} - \epsilon}\left(\frac{1}{N\epsilon} - t\right) & \text{if} \quad \epsilon \leq t \leq \frac{1}{N\epsilon} \end{cases}$$

where $f : X = \mathrm{Bl}_{\{p_1,\dots,p_N\}}\mathbb{P}^2 \to X$ *is the blow-up at* $Z = \{p_1, \dots, p_N\}$, $E_1, \dots, E_N$ *the exceptional divisors and* $\mathbb{E} = \sum_{j=1}^N E_j$.

Proof. If $\epsilon_S(\mathcal{O}_{\mathbb{P}^2}(1); N) = 1/\sqrt{N}$, i.e. maximal, then $\Delta_j(\mathcal{O}_{\mathbb{P}^2}(1)) = \frac{1}{\sqrt{N}}\Sigma_2$ as consequence of Theorem A and Theorem B. Thus we may assume $\epsilon_S(\mathcal{O}_{\mathbb{P}^2}(1); N) < 1/\sqrt{N}$, and we know that there exists $C = \gamma H - \sum_{j=1}^N m_j E_j$ sub-maximal curve, i.e. an irreducible curve such that $\epsilon_S(\mathcal{O}_{\mathbb{P}^2}(1); N) = \frac{\gamma}{M}$ where $M := \sum_{j=1}^N m_j$. Moreover, since the points are very general, for any cycle σ of lenght N there exists a curve $C_\sigma = \gamma H - \sum_{j=1}^N m_{\sigma(j)} E_j$, which implies $\mu(f^*\mathcal{O}_{\mathbb{P}^2}(1); \mathbb{E}) \geq \frac{M}{N\gamma} = \frac{1}{N\epsilon}$ since there exists a section $s \in H^0(\mathbb{P}^2, N\gamma)$ such that $\mathrm{ord}_{p_j}(s) = M$ for any j. Recall that $\mu(f^*\mathcal{O}_{\mathbb{P}^2}(1); \mathbb{E}) = \sup\{t \geq 0 \,:\, f^*\mathcal{O}_{\mathbb{P}^2}(1) - t\mathbb{E} \,\text{is big}\}$. Next for any $j = 1, \dots, N$ we can easily fix holomorphic coordinates $(z_{1,j}, z_{2,j})$ such that $\nu^{p_j}(s) = (0, M)$ with respect to the deglex order. So considering an ample line bundle A such that there exist sections $s_1, \dots, s_N \in H^0(X, A)$ with $\nu^{p_j}(s_j) = (0, 0)$ and $\nu^{p_i}(s_j) > 0$ for any $i \neq j$ and for any $j = 1, \dots, N$, we get $s^l \otimes s_j^{N\gamma} \in V_{N\gamma, j}(lL + A)$, i.e. $(0, \frac{M}{N\gamma}) \in \Delta_j(L + \frac{1}{l}A)$ by homogeneity (Proposition 2.3.9) for any $l \in \mathbb{N}$ and any $j = 1, \dots, N$. Hence by Theorem 2.3.14 we get $(0, \frac{M}{N\gamma}) \in \Delta_j(L)$ for any $j = 1, \dots, N$.
Finally since by Theorem B we know that $\epsilon_S(\mathcal{O}_{\mathbb{P}^2}(1); N)\Sigma_2 \subset \Delta_j(L)$ for any $j = 1 \dots, N$, Theorem A and the convexity imply that the multipoint Okounkov bodies have necessarily the shape requested. □

Corollary 2.6.15. *The ray* $f^*\mathcal{O}_{\mathbb{P}^2}(1) - t\mathbb{E}$ *meet at most two Zariski chambers.*

This result was already showed in Proposition 2.5. of [DKMS15].

Remark 2.6.16. We recall that Biran in [Bir97] gave an homological criterion to check if a 4−dimensional symplectic manifold admits a full symplectic packings by N equal balls for large N, showing that $(\mathbb{P}^2, \omega_{FS})$ admits a full symplectic packings for $N \geq 9$. Moreover it is well-known that for any $N \leq 9$ the supremum r such that $\{(B_r(0), \omega_{st})\}_{j=1}^{N}$ packs into $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ coincides with the supremum r such that $(\mathbb{P}^2, \omega_{FS})$ admits a symplectic packings of N balls of radius r (called *Gromov width*), therefore by Theorem C and Corollary 2.5.17 the Nagata's conjecture is true iff the Gromov width of N balls on $(\mathbb{P}^2, \omega_{FS})$ coincides with the multipoint Seshadri constant of $\mathcal{O}_{\mathbb{P}^2}(1)$ at N very general points.

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