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Research Article

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A Finite Element Splitting Method for a Convection-Diffusion Problem

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Abstract: For a spatially periodic convection-diffusion problem, we analyze a time stepping method based on Lie splitting of a spatially semidiscrete finite element solution on time steps of length k, using the backward Euler method for the diffusion part and a stabilized explicit forward Euler approximation on $m \geq 1$ intervals of length k/m for the convection part. This complements earlier work on time splitting of the problem in a finite difference context.

Keywords: Convection-Diffusion Problem, Time Stepping, Backward Euler Method, Lie Splitting, Finite Elements, Lumped Mass Method

MSC 2010: 35K10, 65M15, 65M60

Dedicated to the memory of Alexander Andreevich Samarskii

1 Introduction

In this paper, we shall consider a numerical method for the solution of the convection-diffusion problem in the square $\Omega = (0,1) \times (0,1)$,

$$\frac{\partial U}{\partial t} = \nabla \cdot (a \nabla U) + b \cdot \nabla U + F \quad \text{in } \Omega, \quad \text{for } t \geq 0, \quad \text{with } u(0) = V, \tag{1.1}$$

with periodic boundary conditions, where, with $x = (x_1, x_2)$, the initial function $V = V(x)$, the positive definite 2×2 matrix $a = a(x) = (a_{ij}(x))$, the vector $b = b(x) = (b_1(x), b_2(x))$ and the forcing term $F = F(x,t)$ are 1-periodic in x_1 and x_2 and smooth. Our method is an explicit-implicit time stepping method based on Lie splitting of a spatially discrete finite element version of (1.1).

With $AU = -\nabla \cdot (a \nabla U)$ and $BU = b \cdot \nabla U$, the exact solution of (1.1) may be formally expressed as

$$U(t) = \mathcal{E}(t)V + \int_0^t \mathcal{E}(t-y)F(y)\,dy \quad \text{for } t \geq 0, \quad \text{where } \mathcal{E}(t) = e^{-t(A-B)}.$$

This representation in terms of the solution operator $\mathcal{E}(t)$ of the homogeneous case of (1.1) is the basis for our discretization method. Such a method was analyzed in [1] within the framework of finite differences, and here our purpose is to carry out the corresponding program with finite elements. We refer to [1] also for further references to work on splitting methods.

As a first step to define our finite element splitting method, we thus consider a spatially discrete finite element version of (1.1). Let $\mathcal{T}_h$ be a quasi-uniform family of triangulations of Ω, and let S_h be the periodic, continuous piecewise linear functions on $\mathcal{T}_h$. With $(\cdot\,,\cdot)$ the inner product in $L_2(\Omega)$, and

$$A(\psi, \chi) = (a \cdot \nabla\psi, \nabla\chi), \quad B(\psi, \chi) = (b \cdot \nabla\psi, \chi) \quad \text{for all } \psi, \chi \in S_h,$$

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the standard Galerkin spatially semidiscrete version of (1.1) is then to find $u(t) \in S_h$ for $t \geq 0$ such that

$$(u_t, \chi) + A(u, \chi) - B(u, \chi) = (F, \chi) \quad \text{for all } \chi \in S_h,\ t \geq 0, \quad \text{with } u(0) = v \approx V.$$

This equation may also be expressed in matrix form. With $\{\Phi_j\}_{j=1}^N$ the pyramid function basis for S_h, let

$$\mathcal{M} = (m_{ij}),\ m_{ij} = (\Phi_i, \Phi_j), \quad \mathcal{A} = (a_{ij}),\ a_{ij} = A(\Phi_i, \Phi_j), \quad \mathcal{B} = (b_{ij}),\ b_{ij} = B(\Phi_i, \Phi_j),$$
$$\mathbf{f} = (f_j)^T,\ f_j = (F, \Phi_j) \quad \text{and} \quad \mathbf{v} = (v_1, \dots, v_N)^T.$$

With $u_h(t) = \sum_{j=1}^N \mathbf{u}_j(t)\Phi_j$ and $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_N)^T$, we have

$$\mathcal{M}\mathbf{u}' + \mathcal{A}\mathbf{u} = \mathcal{B}\mathbf{u} + \mathbf{f} \quad \text{for } t \geq 0, \quad \text{with } \mathbf{u}(0) = \mathbf{v}. \tag{1.2}$$

For our purposes, it will be more convenient to use instead a semidiscrete method based on the lumped mass method, employing the approximate inner product on S_h defined by

$$(\psi, \chi)_h = \sum_{\tau \in \mathcal{T}_h} W_{\tau,h}(\psi\chi) \quad \text{with } W_{\tau,h}(f) = \frac{1}{3}|\tau| \sum_{j=1}^{3} f(P_{\tau,j}) \approx \int_\tau f\,dx,$$

where, for a triangle τ of the triangulation $\mathcal{T}_h$, $P_{\tau,j}$, $j = 1, 2, 3$, are its vertices and where $|\tau| = \text{area}(\tau)$; cf. [2, Chapter 15]. We shall then consider the semidiscrete problem, with $u(t) \in S_h$ for $t \geq 0$,

$$(u_t, \chi)_h + A(u, \chi) - B(u, \chi) = (F, \chi) \quad \text{for all } \chi \in S_h,\ t \geq 0, \quad u(0) = v. \tag{1.3}$$

Introducing the operators $A_h, B_h\colon S_h \to S_h$ by

$$(A_h\psi, \chi)_h = A(\psi, \chi), \quad (B_h\psi, \chi)_h = B(\psi, \chi) \quad \text{for all } \psi, \chi \in S_h,$$

equation (1.3) may also be written as

$$u_t + A_h u - B_h u = f \quad \text{for } t \geq 0, \quad \text{with } f = \widetilde{P}_h F, \quad u(0) = v, \tag{1.4}$$

where $\widetilde{P}_h\colon L_2 \to S_h$ is defined by $(\widetilde{P}_h F, \chi)_h = (F, \chi)$ for all $\chi \in S_h$. In matrix form, this means that the matrix $\mathcal{M}$ in (1.2) is replaced by the diagonal matrix $\mathcal{D} = (d_{ij})$, where $d_{ij} = (\Phi_i, \Phi_j)_h$. The solution of (1.4) satisfies

$$u(t) = E_h(t)v + \int_0^t E_h(t-y)f(y)\,dy, \quad \text{where } E_h(t) = e^{-t(A_h - B_h)}, \quad \text{for } t \geq 0.$$

We recall that $E_h(t)$, the solution operator of the homogeneous case of (1.4), is stable and that the solution of (1.4) satisfies an $O(h^2)$ error estimate for suitable v.

To define our basic finite element splitting method, let k be a time step and $t_n = nk$ for $n \geq 0$. On each time interval (t_{n-1}, t_n), we then introduce the Lie splitting $E_{h,k} = e^{kB_h}e^{-kA_h}$ of $E_h(k)$ and define a time discrete solution of (1.4) by

$$u^n = E_{h,k}u^{n-1} + kf^n \quad \text{for } n \geq 1, \quad \text{with } u^0 = v. \tag{1.5}$$

We note that the two factors of $E_{h,k}$ are associated with the hyperbolic and parabolic parts of equation (1.1),

For a computable discretization in space and time, we then need to replace e^{-kA_h} and e^{kB_h} by rational functions of A_h and B_h, respectively. For the parabolic part, we shall use the implicit backward Euler operator

$$Q_{h,k} = (I_h + kA_h)^{-1} \approx e^{-kA_h}, \tag{1.6}$$

where I_h is the identity operator on S_h. For the hyperbolic part, we define

$$H_{h,k} = I_h + kB_h - \gamma k^2 L_h \approx e^{kB_h},$$

where the operator L_h, defined by $(L_h\psi, \chi)_h = (\nabla\psi, \nabla\chi)$ for all $\psi, \chi \in S_h$, is added to secure stability, under a mesh-ratio condition for k/h. In matrix form, the application of $H_{h,k}$ takes the form

$$\mathcal{D}\mathbf{w} = (\mathcal{D} + k\mathcal{B} - \gamma k^2\mathcal{L})\mathbf{v}, \quad \text{where } \mathcal{L} = (l_{ij}), \quad l_{ij} = ((\nabla\Phi_i, \nabla\Phi_j)),$$

where the diagonal matrix on the left shows that $H_{h,k}$ is essentially explicit.

More generally, on each interval (t_{n-1}, t_n), we shall allow the use m steps of the hyperbolic time stepping operator with step $k_m = k/m$, i.e., instead of $H_{h,k}$, we apply $H^m_{h,k_m} \approx e^{kB_h}$. This will increase the accuracy and the bound for the mesh ratio but not significantly the cost of the computation since H_{h,k_m} is explicit. We consider thus the time discrete solution defined for $n \geq 1$ by

$$\widetilde{u}^n_m = \widetilde{E}_{h,k,m}\widetilde{u}^{n-1}_m + kf^n, \quad \text{where } \widetilde{E}_{h,k,m} = H^m_{h,k_m} Q_{h,k}, \quad \text{with } \widetilde{u}^0_m = v. \tag{1.7}$$

Our main result is then that, for v appropriate, the error satisfies

$$\|\widetilde{u}^n - U(t_n)\| \leq C'_T(U)h + C''_T(U)k + C'''_T(U)k_m \quad \text{for } t_n \leq T.$$

We remark that the first term in this error bound comes from the spatial discretization, the second from the splitting and the backward Euler discretization of the parabolic part, and the third from the approximation of the hyperbolic part. In [1], numerical illustrations of these partial errors were presented in the finite difference case, in the present context essentially corresponding to uniform triangulations, and we note that the spatial discretization is then of order $O(h^2)$. In [1], the choice of m to balance the last two terms was also discussed.

2 Splitting of the Semidiscrete Problem

In this section, we will show that the result u^n in (1.5) of Lie splitting of the spatially discrete problem (1.3) differs from the exact solution $U(t_n)$ of (1.1) by $O(h + k)$, under the appropriate regularity assumptions and choice of v.

For 1-periodic functions, we define $\|V\|_s = \|V\|_{H^s(\Omega)}$ for $s \geq 0$. Note that $\|AV\|_s \leq C\|V\|_{s+2}$, $\|Bv\|_s \leq C\|V\|_{s+1}$ and $\|V\|_s \leq C(\|A^{s/2}V\| + \|V\|)$, where $\|V\| = \|V\|_{L_2(\Omega)}$. Further, for (1.1), we recall the stability estimates

$$\|e^{-t(A-B)}V\|_s + \|e^{-tA}V\|_s + \|e^{tB}V\|_s \leq C_T\|V\|_s \quad \text{for } s \geq 0, \quad 0 \leq t \leq T, \tag{2.1}$$

and the smoothing property

$$\|\mathcal{E}(t)V\|_j \leq Ct^{-(s-j)/2}\|V\|_s \quad \text{for } 0 < s < j, \quad t > 0. \tag{2.2}$$

We begin the analysis with the following stability result for the operator $E_{h,k}$ in (1.5). Here and below, we shall use the norm $\|\chi\|_h = (\chi, \chi)_h^{1/2}$ on S_h, which is equivalent to $\|\chi\|$ or, more precisely, satisfies

$$\frac{1}{4}\|\chi\|_h \leq \|\chi\| \leq \|\chi\|_h.$$

Lemma 2.1. *With $\beta_0 = \frac{1}{2}\|\operatorname{div} b\|_{\mathcal{C}}$, we have, for $v \in S_h$ and $t \geq 0$,*

$$\|e^{-t(A_h-B_h)}v\|_h + \|e^{tB_h}v\|_h \leq e^{\beta_0 t}\|v\|_h \quad \text{and} \quad \|e^{-tA_h}v\|_h \leq \|v\|_h. \tag{2.3}$$

Further, for $E_{h,k} = e^{kB_h}e^{-kA_h}$, we have

$$\|E^n_{h,k}v\|_h \leq e^{\beta_0 t_n}\|v\|_h \quad \text{for } v \in S_h,\ n \geq 0. \tag{2.4}$$

Proof. Since, for $\chi \in S_h$,

$$|B(\chi, \chi)| = |(b \cdot \nabla\chi, \chi)| = |\tfrac{1}{2}(\operatorname{div} b\chi, \chi)| \leq \beta_0\|\chi\|^2 \leq \beta_0\|\chi\|_h^2, \tag{2.5}$$

we have, for the solution $u(t) = e^{-t(A_h-B_h)}v$ of (1.4) with $f = 0$,

$$(u_t, u)_h + A(u, u) = B(u, u) \leq \beta_0\|u\|_h^2 \quad \text{for } t \geq 0.$$

Hence $(d/dt)\|u\|_h \leq \beta_0\|u\|_h$, from which the first part of (2.3) follows. The other parts are shown analogously. For (2.4), we conclude

$$\|E^n_{h,k}v\|_h = \|(e^{kB_h}e^{-kA_h})^n v\|_h \leq e^{\beta_0 nk}\|v\|_h. \qquad \square$$

We next show an error estimate for one step of the Lie splitting of $\mathcal{E}(k)$.

Lemma 2.2. *We have, for* $\mathcal{E}_k = e^{kB}e^{-kA}$,

$$\|\mathcal{E}_k V - \mathcal{E}(k)V\| \le Ck^j\|V\|_{2j}, \quad j = 1, 2. \tag{2.6}$$

Proof. Setting $G(t) = e^{tB}e^{-tA} - e^{-t(A-B)}$ and noting that $G(0) = G'(0) = 0$, we have, by Taylor's formula,

$$\|G(k)V\| = \|(G(k) - G(0) - kG'(0))V\| \le \frac{1}{2}k^2 \sup_{s\le k}\|G''(s)V\|.$$

Here, for $s \le k$,

$$\|G''(s)V\| \le C \sum_{i_1+i_2=2} \|B^{i_1}e^{sB}A^{i_2}e^{-sA}v\| + \|(A-B)^2e^{-s(A-B)}V\| \le C\|V\|_4,$$

which by (2.1) shows (2.6) for $j = 2$. In the same way, the case $j = 1$ follows from $\|G'(s)V\| \le C\|V\|_2$ for $s \le k$. □

We shall need error bounds for the spatial discretizations of the parabolic and hyperbolic parts of $E_{h,k}$. Here and below, we shall use the Ritz projection $R_h\colon H^1 \to S_h$ defined by $\widehat{A}(R_hV - V, \chi) = 0$ for all $\chi \in S_h$, where $\widehat{A}(\psi,\chi) = A(\psi,\chi) + (\psi,\chi)$. Recall that

$$\|R_hV - V\|_s \le Ch^{j-s}\|V\|_j \quad \text{for } s = 0, 1, \quad s \le j \le 2. \tag{2.7}$$

We also recall that, for $\varepsilon_h(\psi,\chi) = (\psi,\chi)_h - (\psi,\chi)$,

$$|\varepsilon_h(\psi,\chi)| \le Ch^2\|\nabla\psi\|\|\nabla\chi\| \le Ch^j\|\psi\|_j\|\chi\| \quad \text{for all } \psi,\chi \in S_h, \quad j = 0, 1, \tag{2.8}$$

where, in the last step, we have used the inverse inequality

$$\|\chi\|_1 \le \nu h^{-1}\|\chi\| \quad \text{for } \chi \in S_h, \tag{2.9}$$

valid since the family $\{\mathcal{T}_h\}$ is quasi-uniform.

Lemma 2.3. *For any* $\varepsilon > 0$ *and for* $j = 0, 1$, *we have*

$$\|e^{-kA_h}R_hV - R_he^{-kA}V\| \le C_\varepsilon k^jh\|V\|_{s_j}, \quad \text{where } s_0 = 1 + \varepsilon,\ s_1 = 3, \tag{2.10}$$

$$\|e^{kB_h}R_hV - R_he^{kB}V\| \le Ckh^j\|V\|_{1+j}. \tag{2.11}$$

Proof. For (2.10), we set $W(t) = e^{-tA}V$ and $w(t) = e^{-tA_h}R_hV$. We want to bound $\theta(k)$, where $\theta = w - R_hW$. With $\rho = R_hW - W$, we have

$$(\theta_t,\chi)_h + A(\theta,\chi) = -(\rho_t,\chi) + (\rho,\chi) - \varepsilon_h(R_hW_t,\chi) \quad \text{for all } \chi \in S_h, \quad \text{for } t \ge 0.$$

Choosing $\chi = \theta$, we find, using (2.7) and (2.8), for $t \ge 0$,

$$\frac{d}{dt}\|\theta\|_h \le C(\|\rho_t\| + \|\rho\| + h\|R_hW_t\|_1) \le Ch\|W\|_3 \le Cht^{-\frac{1}{2}(3-s_j)}\|V\|_{s_j}.$$

Since $\theta(0) = 0$, this shows $\|\theta(k)\| \le Chk^{\frac{1}{2}(s_j-1)}\|V\|_{s_j}$ and thus (2.10).

For (2.11), we set $W(t) = e^{tB}V$ and $w(t) = e^{tB_h}R_hV$. In this case, again with $\theta = w - R_hW$, $\rho = R_hW - W$, we have

$$(\theta_t,\chi)_h = B(\theta,\chi) - (\rho_t,\chi) + B(\rho,\chi) - \varepsilon_h(R_hW_t,\chi) \quad \text{for all } \chi \in S_h, \quad \text{for } t \ge 0.$$

Choosing $\chi = \theta$ and using (2.5), (2.7) and (2.8), we find

$$\frac{d}{dt}\|\theta\|_h \le \beta_0\|\theta\| + C(\|\rho_t\| + \|\rho\|_1 + h^j\|R_hW_t\|_j) \le \beta_0\|\theta\|_h + Ch^j\|V\|_{1+j},$$

which implies (2.11) since $\theta(0) = 0$. □

This shows the following error estimate for the operator $E_{h,k}$.

Lemma 2.4. *For any $\varepsilon > 0$, we have, for $j = 0, 1$, with $s_0 = 1 + \varepsilon$, $s_1 = 3$,*

$$\|E_{h,k}R_h V - R_h \mathcal{E}(k)V\| \le C_\varepsilon k^j h\|V\|_{s_j} + Ck^{1+j}\|V\|_{2+2j}. \tag{2.12}$$

Proof. Recalling $\mathcal{E}_k = e^{kB}e^{-kA}$, we have

$$E_{h,k}R_h - R_h\mathcal{E}_k = e^{kB_h}(e^{-kA_h}R_h - R_h e^{-kA}) + (e^{kB_h}R_h - R_h e^{kB})e^{-kA}.$$

Hence, by Lemmas 2.1 and 2.3, $\|E_{h,k}R_h V - R_h\mathcal{E}_k V\|$ is bounded as in (2.12). By Lemma 2.2 and the boundedness of R_h, we also find $\|R_h(\mathcal{E}_k - \mathcal{E}(k))V\| \le Ck^{1+j}\|V\|_{2+2j}$, $j = 0, 1$. Together, these estimates show the lemma. □

We now show a global error estimate for the homogeneous equation.

Lemma 2.5. *We have, for any $\varepsilon > 0$, if $\|v - V\| \le Ch\|V\|_1$,*

$$\|E_{h,k}^n v - \mathcal{E}(t_n)V\| \le C'_{\varepsilon,T}h\|V\|_{1+\varepsilon} + C''_{\varepsilon,T}k\|V\|_{2+\varepsilon} \quad \textit{for } t_n \le T. \tag{2.13}$$

Proof. We find, using (2.4), and Lemma 2.4 with $j = 0$ for the first term on the right below and with $j = 1$ for the terms in the sum, and then the smoothing estimate (2.2),

$$\begin{aligned}
\|(E_{h,k}^n R_h - R_h\mathcal{E}(t_n))V\| &= \Big\|\sum_{j=0}^{n-1} E_{h,k}^{n-j-1}(E_{h,k}R_h - R_h\mathcal{E}(k))\mathcal{E}(t_j)V\Big\| \\
&\le C\|(E_{h,k}R_h - R_h\mathcal{E}(k))V\| + C\sum_{j=1}^{n-1}\|(E_{h,k}R_h - R_h\mathcal{E}(k))\mathcal{E}(t_j)V\| \\
&\le C_\varepsilon h\|V\|_{1+\varepsilon} + Ck\|V\|_2 + Ck\sum_{j=1}^{n-1}(h\|\mathcal{E}(t_j)V\|_3 + k\|\mathcal{E}(t_j)V\|_4) \\
&\le C_\varepsilon\Big(1 + k\sum_{j=1}^{n-1} t_j^{-1+\varepsilon/2}\Big)(h\|V\|_{1+\varepsilon} + k\|V\|_{2+\varepsilon}).
\end{aligned}$$

Since $\|E_{h,k}^n(v - R_h V)\| + \|(R_h - I)\mathcal{E}(t_n)V\| \le Ch\|V\|_1$, (2.13) follows. □

One possible choice for v with $\|v - V\| \le Ch\|V\|_1$ which will be used below is $v = \widetilde{P}_h V$. In fact, since

$$(\widetilde{P}_h V - P_h V, \chi)_h = -\varepsilon_h(P_h V, \chi),$$

we find $\|\widetilde{P}_h V - P_h V\|_h \le Ch\|P_h V\|_1 \le Ch\|V\|_1$, from which our claim follows.

We now show a complete error estimate for our basic splitting method.

Theorem 2.1. *Let u^n be the solution of (1.5) with $\|v - V\| \le Ch\|V\|_1$ and $U(t_n)$ that of (1.1). Then we have, for any $\varepsilon > 0$, $t_n \le T$,*

$$\|u^n - U(t_n)\| \le C'_{\varepsilon,T}hZ_{1+\varepsilon}(U, t_n) + C''_{\varepsilon,T}k\widehat{Z}_{2+\varepsilon}(U, t_n),$$

where

$$Z_s(U, t) = \|V\|_s + \int_0^t \|F(y)\|_s\,dy, \quad \widehat{Z}_s(U, t) = Z_s(U, t) + \int_0^t \|F'(y)\|\,dy.$$

Proof. In view of Lemma 2.5, it suffices to consider the case $v = 0$. Recalling that $f = \widetilde{P}_h F$, we may write the error $e^n = u^n - U(t_n)$ as

$$\begin{aligned}
e^n &= k\sum_{j=1}^n E_{h,k}^{n-j}f^j - \int_0^{t_n}\mathcal{E}(t_n - y)F(y)\,dy \\
&= k\sum_{j=1}^n (E_{h,k}^{n-j}\widetilde{P}_h - \mathcal{E}(t_{n-j}))F^j + \sum_{j=1}^n\Big(k\mathcal{E}(t_{n-j})F^j - \int_{I_j}\mathcal{E}(t_n - y)F(y)\,dy\Big) = J' + J'', \quad I_j = (t_{j-1}, t_j).
\end{aligned}$$

To bound J', we set $\mathcal{F}_{h,k,j} = E_{h,k}^j \widetilde{P}_h - \mathcal{E}(t_j)$ and write

$$k(E_{h,k}^{n-j}\widetilde{P}_h - \mathcal{E}(t_{n-j}))F^j = \int_{I_j} \mathcal{F}_{h,k,n-j}F(y)\,dy + \int_{I_J}\int_y^{t_j} \mathcal{F}_{h,k,n-j}F'(\sigma)\,d\sigma\,dy.$$

By Lemma 2.5, we find

$$\Big\|\sum_{j=1}^n \int_{I_j} \mathcal{F}_{h,k,n-j}F(y)\,dy\Big\| \le C_{\varepsilon,T}\int_0^{t_n}(h\|F(y)\|_{1+\varepsilon} + k\|F(y)\|_{2+\varepsilon})\,dy$$

and, since $\mathcal{F}_{h,k,j}$ is bounded,

$$\Big\|\sum_{j=1}^n \int_{I_j}\int_y^{t_j} \mathcal{F}_{h,k,n-j}F'(\sigma)\,d\sigma\,dy\Big\| \le C_{\varepsilon,T}k\int_0^{t_n}\|F'(y)\|\,dy.$$

Thus J' is bounded as claimed. Similarly, $J'' = \sum_{j=1}^n J_j''$, where

$$\begin{aligned} J_j'' = \int_{I_j}(\mathcal{E}(t_{n-j})F^j - \mathcal{E}(t_n - y)F(y))\,dy &= \int_{I_j}\int_y^{t_j} \frac{d}{d\sigma}(\mathcal{E}(t_n-\sigma)F(\sigma))\,d\sigma\,dy \\ &= \int_{I_j}\int_y^{t_j} \mathcal{E}(t_n-\sigma)((\mathcal{A}-\mathcal{B})F(\sigma) + F'(\sigma))\,d\sigma\,dy. \end{aligned}$$

Thus, using the stability of $\mathcal{E}(t)$,

$$\|J''\| \le \sum_{j=1}^n \|J_j''\| \le C_T k\int_0^{t_n}(\|F(\sigma)\|_2 + \|F'(\sigma)\|)\,d\sigma \quad \text{for } t_n \le T.$$

Adding these estimates completes the proof. □

3 Complete Discretization in Time and Space

We now turn to the analysis of the complete discretization using the operators $Q_{h,k}$ and H_{h,k_m}^m in (1.6) and (1.7) for the parabolic and hyperbolic factors of $E_{h,k} = e^{kB_h}e^{-kA_h}$. We note that, since

$$|(B_h\psi,\chi)_h| \le \|b\cdot\nabla\psi\|\|\chi\| \le \beta_1\|\nabla\psi\|\|\chi\| \quad \text{for } \psi,\chi\in S_h, \quad \text{where } \beta_1 = \sup_\Omega|b(x)|,$$

we have $\|B_h\psi\|_h \le \beta_1\|\nabla\psi\|$ for $\psi\in S_h$. Further, by (2.9), $\|L_h\psi\|_h \le \nu h^{-1}\|\nabla\psi\|$.

We shall first consider the stability of $\widetilde{E}_{h,k,m}$.

Lemma 3.1. *We have $\|Q_{h,k}v\|_h \le \|v\|_h$ for $v\in S_h$. Let $\beta_0 = \frac12\|\operatorname{div} b\|_{\mathcal{C}}$ and β_1 and ν as above. Then, if $\gamma > \beta_1^2$, we have*

$$\|H_{h,k_m}^m v\|_h \le (1+\beta_0 k)\|v\|_h \quad \textit{for } k_m/h \le \lambda_0 = \sqrt{(\gamma-\beta_1^2)/(\gamma\nu)^2}.$$

The time stepping operator $\widetilde{E}_{hk,m} = H_{h,k_m}^m Q_{h,k}$ is stable and

$$\|\widetilde{E}_{h,k,m}^n v\|_h \le e^{\beta_0 t_n}\|v\|_h \quad \textit{for } t_n \ge 0. \tag{3.1}$$

Proof. The first inequality is obvious by the definition of $Q_{h,k}$. For the hyperbolic part, we begin with $m = 1$. Setting $w^1 = H_{h,k}v$, $w^0 = v$, we have $\bar\partial_t w^1 + \gamma k L_h v = B_h v$, where $\bar\partial_t w^1 = (w^1 - w^0)/k$. Recalling (2.5), we have

$$(\bar\partial_t w^1, v)_h + \gamma k\|\nabla v\|^2 = B(v,v) \le \beta_0\|v\|^2 \le \beta_0\|v\|_h^2$$

or, writing $v = \frac{1}{2}(w^1 + v) - \frac{1}{2}(w^1 - v)$,

$$\tfrac{1}{2}(\|w^1\|_h^2 - \|v\|_h^2) - \tfrac{1}{2}\|w^1 - v\|_h^2 + \gamma k^2\|\nabla v\|^2 \le \beta_0 k\|v\|_h^2. \tag{3.2}$$

Here, since $w^1 - v = kB_h v - k^2\gamma L_h v$ and using (2.9), we have, for $k/h \le \lambda_0$,

$$\tfrac{1}{2}\|w^1 - v\|_h^2 \le k^2\|B_h v\|_h^2 + \gamma^2 k^4\|L_h v\|_h^2 \le k^2(\beta_1^2 + \gamma^2\nu^2\lambda_0^2)\|\nabla v\|^2 = \gamma k^2\|\nabla v\|^2.$$

Hence, by (3.2), $\|w^1\|_h^2 \le (1 + 2\beta_0 k)\|v\|_h^2$, which shows our claim for $m = 1$.

For $m > 1$, we have

$$\|H_{h,k_m}^m v\|_h \le (1 + \beta_0 k_m)^m\|v\|_h \le e^{\beta_0 k}\|v\|_h \quad \text{for } k_m/h \le \lambda_0, \tag{3.3}$$

and (3.1) now follows at once. □

For our analysis, we shall need norms on S_h which are analogues of $\|\cdot\|_s$. We introduce $\bar{A}_h\colon S_h \to S_h$ by $(\bar{A}_h\psi, \chi) = A(\psi, \chi)$ for all $\chi \in S_h$. Noting that $\bar{A}_h$ is positive semidefinite, we set $\widehat{A}_h = \bar{A}_h + I_h$ and define

$$\|\psi\|_{h,s} = \|\widehat{A}_h^{s/2}\psi\| = (\widehat{A}_h^s\psi, \psi)^{1/2} \quad \text{for all } \psi \in S_h, \quad \text{for } s \ge 0.$$

For $s = 1$, we have the obvious norm equivalence $c\|\psi\|_1 \le \|\psi\|_{h,1} \le C\|\psi\|_1$ for $\psi \in S_h$, with $c > 0$, and, by the inverse inequality (2.9),

$$\|\psi\|_{h,s} \le (\rho/h)^{s-j}\|\psi\|_{h,j} \quad \text{for } \psi \in S_h, \quad 0 \le j \le s. \tag{3.4}$$

We shall need the following lemma.

Lemma 3.2. *We have*

$$\|e^{tB_h}v\|_{h,2} \le C\|v\|_{h,2} \quad \text{for } v \in S_h, \quad 0 \le t \le Ch. \tag{3.5}$$

Proof. We first show

$$\|B_h\psi\|_{h,1} \le C\|\psi\|_{h,2} \quad \text{for } \psi \in S_h. \tag{3.6}$$

We see at once that $\widetilde{P}_h$ is bounded on L_2 and, using (2.9),

$$\|\widetilde{P}_h\psi\|_1 \le \|R_h\psi\|_1 + Ch^{-1}\|\widetilde{P}_h\psi - \psi\|_h + Ch^{-1}\|R_h\psi - \psi\| \le C\|\psi\|_1.$$

Inequality (3.6) may also be formulated as $\|B_h T_h\psi\|_1 \le C\|\psi\|$ for $\psi \in S_h$, where $T_h = \widehat{A}_h^{-1}$. To show this inequality, we have, with $T = (A + I)^{-1}$,

$$\begin{aligned}\|B_h T_h\psi\|_1 &\le \|\widetilde{P}_h(b\cdot\nabla T)\psi\|_1 + Ch^{-1}\|\widetilde{P}_h(b\cdot\nabla(T_h - T))\psi\|_h\\ &\le C\|\nabla T\psi\|_1 + Ch^{-1}\|(T_h - T)\psi\|_1 \le C\|T\psi\|_2 \le C\|\psi\|.\end{aligned}$$

With $w(t) = e^{tB_h}v$ and $(\psi, \chi)_{h,2} = (\widehat{A}_h\psi, \widehat{A}_h\chi)$ for $\psi, \chi \in S_h$, we have, using (3.4) and (3.6),

$$\frac{1}{2}\frac{d}{dt}\|w\|_{h,2}^2 = (w_t, w)_{h,2} = (B_h w, w)_{h,2} \le Ch^{-1}\|B_h w\|_{h,1}\|w\|_{h,2} \le Ch^{-1}\|w\|_{h,2}^2.$$

Hence, by integration, $\|w(t)\|_{h,2} \le Ce^{Cth^{-1}}\|v\|_{h,2}$, which shows (3.5). □

We have the following error estimates for one step of the parabolic and hyperbolic approximations.

Lemma 3.3. *We have, for $v \in S_h$,*

$$\|(Q_{h,k} - e^{-kA_h})v\| \le Ckh^j\|v\|_{h,2+j} + Ck^{1+j}\|v\|_{h,2+2j}, \quad j = 0, 1. \tag{3.7}$$

With $\beta_1, \gamma, \lambda_0$ as in Lemma 3.1, we have, for $k_m/h \le \lambda_0$,

$$\|(H_{h,k_m}^m - e^{kB_h})v\| \le Cm^{-1}k^2\|v\|_{h,2} \quad \text{for } v \in S_h. \tag{3.8}$$

Proof. We first note that, with $\bar{Q}_{h,k} = (I_h + k\bar{A}_h)^{-1}$,

$$\|(\bar{Q}_{h,k} - e^{-k\bar{A}_h})v\| \le Ck^{1+j}\|\bar{A}_h^{1+j}v\| \le Ck^{1+j}\|v\|_{h,2+2j} \quad \text{for all } v \in S_h.$$

Estimate (3.7) will now follow from

$$\|Q_{h,k}-\bar{Q}_{h,k})v\| + \|(e^{kA_h}-e^{-k\bar{A}_h})v\| \le Ckh^j\|v\|_{h,2+j} \quad \text{for all } v\in S_h. \tag{3.9}$$

Setting $w^1 = Q_{h,k}v$, $\bar{w}^1 = \bar{Q}_{h,k}v$, $w^0 = \bar{w}^0 = v$ and $\omega^j = w^j - \bar{w}^j$, we have

$$(\bar{\partial}_t\omega^1,\chi)_h + A(\omega^1,\chi) = \varepsilon_h(\bar{\partial}_t\bar{w}^1,\chi) \quad \text{for } \chi\in S_h$$

or, with $\chi=\omega^1$, since $\omega^0 = 0$, and using (2.8),

$$\|Q_{h,k}-\bar{Q}_{h,k})v\| = \|\omega^1\| \le Ckh^j\|\bar{\partial}_t\bar{w}^1\|_j \le Ckh^j\|\bar{A}_h\bar{w}^1\|_j \le Ckh^j\|v\|_{h,2+j}.$$

Now, let $w(t) = e^{-tA_h}v$, $\bar{w}(t) = e^{-t\bar{A}_h}v$, and set $\theta(t) = w(t)-\bar{w}(t)$. Then

$$(\theta_t,\chi)_h + A(\theta,\chi) = \varepsilon_h(\bar{w}_t,\chi) \quad \text{for } \chi\in S_h,\ t\ge 0, \quad \text{with } \theta(0)=0.$$

Setting $\chi=\theta$ and using (2.8), we obtain

$$\|\theta(k)\|_h \le Ch^j\int_0^k\|\bar{w}_t(s)\|_j\,ds \le Ch^j\int_0^k\|\bar{A}_h\bar{w}(s)\|_j\,ds \le Ckh^j\|v\|_{h,2+j},$$

which completes the proof of (3.9).

We turn to (3.8) and begin with $m=1$. We have

$$\|(H_{h,k}-e^{kB_h})v\| \le \|(I_h+kB_h-e^{kB_h})v\| + \gamma k^2\|L_hv\|$$

Using (3.6) and (3.5), we obtain

$$\|(H_{h,k}-e^{kB_h})v\| \le Ck^2\sup_{s\le k}\|B_h^2e^{sB_h}v\| \le Ck^2\|v\|_{h,2}.$$

To complete the proof, we show $\|L_hv\|\le C\|v\|_{h,2}$ or, equivalently, with T_h as in Lemma 3.2, $\|L_hT_hv\|\le C\|v\|$. For this, we use the Ritz projection defined by $(\nabla\widetilde{R}_hw,\nabla\chi)+(\widetilde{R}_hw,\chi) = (\nabla w,\nabla\chi)+(w,\chi)$ for all $\chi\in S_h$. We find $(L_h\widetilde{R}_hw,\chi)_h = (\nabla\widetilde{R}_hw,\nabla\chi) = (\nabla w,\nabla\chi)-(\widetilde{R}_hw-w,\chi) = -(\Delta w,\chi)-(\widetilde{R}_hw-w,\chi)$ and hence $\|L_h\widetilde{R}_hw\|\le C\|w\|_2$. Using also $\|L_h\chi\|\le Ch^{-2}\|\chi\|$, the proof is completed by

$$\|L_hT_hw\| \le \|L_h\widetilde{R}_hTw\| + Ch^{-2}(\|\widetilde{R}_hTw-Tw\| + \|T_hw-Tw\|) \le C\|w\|.$$

To show (3.8) for $m>1$, we write

$$H^m_{h,k_m}v - e^{kB_h}v = \sum_{j=0}^{m-1}H^{m-j-1}_{h,k_m}(H_{h,k_m}-e^{k_mB_h})e^{jk_mB_h}v. \tag{3.10}$$

By Lemma 3.2 with $t=jk_m\le k\le\lambda_0h$, we have $\|e^{jk_mB_h}v\|_{h,2}\le C\|v\|_{h,2}$. Using (3.10), (3.3) and the already proven case of (3.8), with k replaced by k_m, we find

$$\begin{aligned}\|H^m_{h,k_m}v - e^{kB_h}v\|_h &\le e^{\beta_0k}\sum_{j=0}^{m-1}\|(H_{h,k_m}-e^{k_mB_h})e^{jk_mB_h}v\|_h\\ &\le Ck_m^2\sum_{j=0}^{m-1}\|e^{jk_mB_h}v\|_{h,2} \le Cmk_m^2\|v\|_{h,2} = Cm^{-1}k^2\|v\|_{h,2}.\end{aligned}$$ □

As a result, we have the following error estimate for the completely discrete time stepping operator.

Lemma 3.4. *If $\gamma>\beta_1^2$ and $k_m/h\le\lambda_0$, we have, with $s_0 = 1+\varepsilon$, $s_1 = 3$,*

$$\|\widetilde{E}_{h,k,m}R_hV - R_h\mathcal{E}(k)V\| \le C'kh^j\|V\|_{s_j} + C''k^{1+j}\|V\|_{2+2j} + C'''m^{-1}k^2\|V\|_2 \quad \text{for } j=0,1.$$

Proof. In view of Lemma 2.4, it remains to bound $(\widetilde{E}_{h,k,m} - E_{h,k})R_h V$. We have, for $v \in S_h$, by Lemmas 3.1 and 3.3,

$$\begin{aligned}\|(\widetilde{E}_{h,k,m} - E_{h,k})v\| &\le \|H^m_{h,k_m}(Q_{h,k} - e^{-kA_h})v\| + \|(H^m_{h,k_m} - e^{kB_h})e^{-ktA_h}v\| \\ &\le C'kh^j\|v\|_{h,1+2j} + C''k^{1+j}\|v\|_{h,2+2j} + C'''m^{-1}k^2\|v\|_{h,2}.\end{aligned}$$

For $v = R_h V$, we find $\|v\|_{h,2} \le C\|V\|_2$, and since $\widehat{A}_h R_h = P_h \widehat{A}$,

$$\begin{aligned}\|v\|_{h,4} = \|\widehat{A}_h^2 R_h V\| = \|\widehat{A}_h P_h A V\| &\le \|\widehat{A}_h R_h A V\| + \|\widehat{A}_h (P_h - R_h)\widehat{A} V\| \\ &\le \|P_h \widehat{A}^2 V\| + Ch^{-2}h^2\|\widehat{A}V\|_2 \le C\|V\|_4,\end{aligned}$$

and similarly, $\|v\|_{h,3} \le C\|V\|_3$, which completes the proof of the lemma. □

This implies the following result for the homogeneous equation.

Lemma 3.5. *Let $\varepsilon > 0$ and $\gamma > \beta_1^2$. Then, if $\|v - V\| \le Ch\|V\|_1$, we have, for $k_m/h \le \lambda_0$, $t_n \le T$,*

$$\|\widetilde{E}^n_{h,k,m}v - \mathcal{E}(t_n)V\| \le C'_{\varepsilon,T}h\|V\|_{1+\varepsilon} + C''_{\varepsilon,T}k\|V\|_{2+\varepsilon} + C_T m^{-1}k\|V\|_{\varepsilon}.$$

Proof. Using the stability of $\widetilde{E}_{h,k,m}$ and Lemma 2.4 instead of Lemma 3.4, the result follows as that in Lemma 2.5. □

We are now ready to formulate our main result.

Theorem 3.1. *Assume that $\gamma > \beta_1^2$. Then, for the solutions $\widetilde{u}^n$ of (1.7) with $\|v - V\| \le h\|V\|_1$ and $U(t_n)$ of (1.1), we have, with $Z_s(U,t)$ and $\widehat{Z}_s(U,t)$ as in Theorem 2.1, for any $\varepsilon > 0$ and $k_m/h \le \lambda_0$,*

$$\|\widetilde{u}^n - U(t_n)\| \le C'_{\varepsilon,T}hZ_{1+\varepsilon}(U,t_n) + C''_{\varepsilon,T}k\widehat{Z}_{2+\varepsilon}(U,t_n) + C'''_{\varepsilon,T}m^{-1}kZ_{\varepsilon}(U,t_n).$$

Proof. The proof is analogous to that of Theorem 2.1, with $E_{h,k}$ replaced by $\widetilde{E}_{h,k,m}$, the stability property (2.4) by (3.1), and Lemma 2.5 by Lemma 3.5. □

References

[1] A. K. Pani, V. Thomeée and A. S. Vasudeva Murthy, A first order explicit-implicit splitting method for a convection-diffusion problem, *Comput. Methods Appl. Math.* (2020), DOI 10.1515/cmam-2020-0009.

[2] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, 2nd ed., Springer, Berlin, 2006.