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ABSTRACT

In this paper, we address the issue of the quaternionic Toledo invariant to study the character variety of two-dimensional complex hyperbolic uniform lattices into $SU(n, 2)$, $n \geq 4$. We construct four distinct representations to prove that the character variety contains at least seven distinct components. We also show the existence of holomorphic horizontal lift to various period domains of $SU(n, 2)$.

I. INTRODUCTION

After Weil's local rigidity theorem of uniform lattices in semisimple Lie groups, there have been many generalizations in different contexts. Due to Margulis' super-rigidity and Corelette's theorem, lattices in higher rank semisimple Lie groups and in quaternionic, octonionic hyperbolic groups are very rigid. Hence, it is only meaningful to study the representation variety of uniform lattices Γ in real and complex hyperbolic groups into different Lie groups G.

A local rigidity property of lattices of a four-dimensional real hyperbolic group in $Sp(2, 1)$ was studied in Ref. 10. Several studies apart from the surface group have been done for complex hyperbolic lattices Γ in various semisimple Lie groups G. In terms of maximal representations, Burger and Iozzi studied the representations of a lattice in $SU(1, p)$ with values in a Hermitian Lie group G.[4] Koziarz and Maubon[14] studied the similar representations in rank 2 Hermitian Lie groups. Pozzetti[16] dealt with maximal representations of complex hyperbolic lattices in $SU(m, n)$. On the other hand, García-Prada and Toledo[7] proved a global rigidity of complex hyperbolic lattices in quaternionic hyperbolic spaces. More precisely, they defined the Toledo invariant $c(\rho)$ of a complex hyperbolic lattice Γ under the representation $\rho : \Gamma \to PSp(m, 1)$ by

$$c(\rho) = \int_M f_\rho^* \omega \wedge \omega_0^{n-2},$$

where f_ρ is a descended map to $M = \Gamma\backslash SU(n, 1)/S(U(n) \times U(1))$ from a ρ-equivariant map from $H^n_{\mathbb{C}}$ to $H^m_{\mathbb{H}}$. Here, ω is the quaternionic Kähler form on $H^m_{\mathbb{H}}$, and ω_0 is the complex Kähler form on M. They showed that this invariant $c(\rho)$ satisfies Milnor–Wood inequality, and its maximum is achieved if and only if the representation stabilizes a copy of $H^n_{\mathbb{C}}$ inside $H^m_{\mathbb{H}}$. Such a statement on the Toledo invariant goes back to Toledo[19] where he proved that a maximal representation from a surface group into $SU(1, q)$ fixes a complex geodesic. Hernández[9] also studied maximal representations from a surface group into $SU(2, q)$ and showed that the image must stabilize a symmetric space associated with the group $SU(2, 2)$. These results are generalized to $SU(p, q)$ by Bradlow *et al.* in Ref. 3.

The first goal of this paper would be to prove a similar result in

$$\Gamma \subset SU(n, 1) \subset SU(m, 2)$$

using Toledo invariant

$$c(\rho) = \int_M f_\rho^* \omega^{\frac{n}{2}}$$

for n even, where ω is the quaternionic Kähler four-form on the associated symmetric space of $SU(m,2)$. This Toledo invariant is constant on each connected component of the character variety $\chi(\Gamma, SU(m,2))$. Hence, two representations with a distinct Toledo invariant must belong to the different components of the character variety.

As a starting point, we consider the simplest case

$$\Gamma \subset SU(2,1) \subset SU(n,2),$$

$n \geq 4$. This case is interesting because the symmetric space of $SU(n,2)$ has both Hermitian and quaternionic structures, and it is worth studying the interplay between them. In this paper, we focus on calculating the Toledo type invariant associated with the quaternionic four-form ω since several authors already pursued the research using powers of the Kähler form. This type of Toledo invariant using the quaternionic four-form seems relatively new and promising.

To this end, we will consider several different embeddings coming from the natural holomorphic, totally real, and symmetric square representations and obtain the following.

Theorem 1.1. *There are at least seven distinct connected components in $\chi(\Gamma, SU(n,2))$, where $\Gamma \subset SU(2,1)$ is a uniform lattice and $n \geq 4$.*

Here, the group $SU(n,2)$ acts on $\mathrm{Hom}(\Gamma, SU(n,2))$ via conjugation on the target group, and the character variety is defined by

$$\chi(\Gamma, SU(n,2)) = \mathrm{Hom}(\Gamma, SU(n,2))//SU(n,2)$$

in the sense of geometric invariant theory.

This is one of the first examples known in higher dimensional complex hyperbolic lattices. For different examples of character variety $\chi(\Gamma, SU(2,1))$, see Ref. 20. It is known in the surface group case that there are $6(g-1)+1$ distinct components in $\chi(\pi_1(S), PSU(2,1))$.[8,21] Indeed, in Ref. 8, a discrete faithful representation $\rho \in \chi(\Gamma, SU(2,1))$ is constructed such that on each component of $S\backslash\Sigma_0$, where Σ_0 is a set of disjoint simple closed geodesics, ρ stabilizes either a complex line or a totally real plane. Then, the Toledo invariants are maximal on the pieces contained in the complex line and are zero on the pieces contained in a totally real plane. Hence, one can realize any even integer between $\chi(S)$ and $-\chi(S)$. Xia finally showed that there are $6(g-1)+1$ distinct components in $\chi(\pi_1(S), PSU(2,1))$.[21]

To prove the global rigidity for $\rho \in \chi(\Gamma, G)$, the common technique known so far is to consider a holomorphic horizontal lifting of a ρ-equivariant map to a proper period domain (or twistor space) where one can apply complex geometry. It was successful in the case that García-Prada and Toledo considered in Ref. 7. However, in general, for a higher rank case, it is not known if there always exists a horizontal holomorphic lifting.

Theorem 1.2. *Consider the symmetric square representation ρ of $SU(2,1)$ in $SU(4,2)$ and in $SU(n,2)$, $n \geq 4$, via the inclusion $SU(4,2) \subseteq SU(n,2)$. Let $\iota : B \to \mathcal{X}$ be the totally geodesic map induced by the representation ρ, where $B = SU(2,1)/S(U(2)\times U(1))$ and $\mathcal{X} = SU(n,2)/S(U(n)\times U(2))$. Then, it lifts to a holomorphic horizontal map to the period domain $\mathcal{D}_2 = SU(n,2)/S(U(n-1)\times U(1)\times U(2))$.*

See Sec. III B for the definition of the symmetric square representation.

II. QUATERNIONIC STRUCTURE OF $\mathcal{X} = SU(n,2)/S(U(n)\times U(2))$ AND ITS PERIOD DOMAINS

A. Quaternionic Kähler manifolds

A Riemannian manifold M of real dimension $4n$ is quaternionic Kähler if its holonomy group is contained in $Sp(n)Sp(1)$. We denote by $\mathcal{P}_M$ the canonical $Sp(n)Sp(1)$-reduction of the principal bundle of orthogonal frames of M and by $\mathcal{E}_M$ the canonical three-dimensional parallel sub-bundle $\mathcal{P}_M \times_{Sp(n)Sp(1)} \mathbb{R}^3$ of $End(TM)$. Since the $Sp(n)Sp(1)$-module $\wedge^4(\mathbb{R}^{4n})^*$ admits a unique trivial submodule of rank 1, any quaternionic Kähler manifold M admits a nonzero closed four-form ω, canonical up to homothety. In Ref. 17, it is proved that the form ω (properly normalized) is the Chern–Weil form of the first Pontryagin class $p_1(\mathcal{E}_M) \in H^4(M,\mathbb{Z})$.

Let N be a smooth closed manifold and $\rho : \pi_1(N) \to G$ be a representation into a quaternionic Kähler group G, i.e., the associated symmetric space $X = G/K$ is a quaternionic Kähler noncompact irreducible symmetric space. Choose any ρ-equivariant smooth map $\phi : \tilde{N} \to X$ from the universal covering space $\tilde{N}$ to X. The pull-back $\phi^*\mathcal{E}_X$ descends to a bundle over N, still denoted as $\phi^*\mathcal{E}_X$. By the functoriality of characteristic classes, the four-form $\phi^*\omega$ represents the Pontryagin class $p_1(\phi^*\mathcal{E}_X) \in H^4(N,\mathbb{Z})$. As X is contractible, any two ρ-equivariant maps give rise to the same class depending only on ρ. Then, by the integrality of the Pontryagin class, the quaternionic Toledo invariant $c(\rho) = \int_N \phi^*\omega^{\frac{n}{2}}$, for even n, is constant on each connected component of the character variety.

B. Kähler and quaternionic structures of $SU(n,2)/S(U(n)\times U(2))$

Let $G = SU(p,q)$, $p \geq q$, be in its standard realization as linear transformations on $\mathbb{C}^{p+q} = \mathbb{C}^p \oplus \mathbb{C}^q$ preserving the indefinite Hermitian form of signature (p,q). Let $\mathcal{X}$ be the Hermitian symmetric space $\mathcal{X} = G/K$, where $K = S(U(p)\times U(q))$. We recall briefly[18] the Harish-Chandra realization (also due to Cartan in the classical groups) of the symmetric space $\mathcal{X}$ into $M_{p\times q}$, which might be useful in understanding various totally geodesic embeddings in our present paper. Fix $V_0^+ = \mathbb{C}^p$, $V_0^- = \mathbb{C}^q$, two orthogonal subspaces of $\mathbb{C}^{p+q}$, which are positive and negative definite, respectively, with respect to the Hermitian form $h^{\mathbb{C}}$. Fix orthonormal basis $\{e_1,\dots,e_p\}$, $\{e_{p+1},\dots,e_{p+q}\}$ of V_0^+, V_0^-, respectively. Then, G acts on the set $\mathcal{X}$ of q-dimensional negative definite subspaces. Any other q-dimensional negative definite subspace V^- is a graph of a unique linear map $A_{p\times q} = (z_{ij})$ from V_0^- so that

$$\sum_{i=1}^{p} e_i z_{ij} + e_{p+j},\ j = 1,\dots,q$$

form a basis of V^-. Hence, $\mathcal{X}$ is identified with

$$\mathcal{X} = \{Z \in M_{p\times q} : I_q - Z^t\bar{Z} > 0\}.$$

The center of a maximal compact subgroup K is parameterized by the center of $U(p)$, and it defines a complex Kähler structure. To be more precise, let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ be its Cartan decomposition, where $\mathfrak{k}$ is the Lie algebra of K, with $\mathfrak{p}$ consisting of matrices of the form

$$\begin{pmatrix} 0 & Z \\ Z^* & 0 \end{pmatrix}, Z \in M_{p\times q}.$$

The real tangent space at $o = eK$ of $\mathcal{X} = G/K$ is identified with $\mathfrak{p}$. The complex structure J on $T_o\mathcal{X}$ acts as

$$J\begin{pmatrix} 0 & Z \\ Z^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & iZ \\ -iZ^* & 0 \end{pmatrix}.$$

The Kähler metric on $T_o\mathcal{X}$ is

$$g_o(A,B) = 2\mathrm{Tr}(B^*A) = 4\mathrm{Re}\mathrm{Tr}(W^*Z) \text{ for } A = \begin{pmatrix} 0 & Z \\ Z^* & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & W \\ W^* & 0 \end{pmatrix}.$$

The corresponding Kähler form is

$$\Omega_o(X,Y) = g_o(JX,Y). \tag{2.1}$$

Now, let $G = SU(2n,2)$. The second factor $U(2)$ of K defines a quaternionic structure as follows. The holomorphic tangent space of $\mathcal{X}$ at o is identified with $\begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}$, where $Z \in M_{2n\times 2}$. The real tangent space will be parameterized by the holomorphic tangent space via $\begin{pmatrix} 0 & Z \\ Z^* & 0 \end{pmatrix} \leftrightarrow Z$. The adjoint action of $k \in K = S(U(2n)\times U(2))$ on $Z \in \mathfrak{p}$, written as a block diagonal matrix $k = \mathrm{diag}(A,D)$, is $Z \to AZD^{-1}$. Thus, the Lie algebra action of $\mathrm{diag}(A,D) \in \mathfrak{k}$ is $Z \to AZ - ZD$.

Now, the following three elements of the Lie algebra of $\mathfrak{su}(2) \subset \mathfrak{k} = \mathfrak{u}(2n) + \mathfrak{su}(2)$

$$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

act on the tangent space as the quaternionic multiplications by i, j, k. Indeed, we have

$$\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} : Z = (x,y) \mapsto (x,y)\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = (xi, -yi). \tag{2.2}$$

Similarly, we find the action by the other two elements. We can express the actions in the usual quaternionic algebra, so we identify a matrix $(x,y) \in M_{2n\times 2} = \mathbb{C}^{2n}\times\mathbb{C}^{2n}$ with a quaternionic vector $q \in \mathbb{H}^{2n}$, with $\mathbb{H} = \mathbb{C} + \mathbb{C}j$ being the quaternionic numbers, by

$$X = (x,y) \leftrightarrow q_X = (x_1 + y_1 j, x_2 + y_2 j, \dots, x_{2n} + y_{2n} j).$$

Hence, the previous matrix (2.2) is identified with the quaternionic vector,

$$(x_1 i + y_1(-i)j, x_2 i + y_2(-i)j, \ldots, x_{2n} i + y_{2n}(-i)j)$$
$$= (x_1 + y_1 j, x_2 + y_2 j, \ldots, x_{2n} + y_{2n} j)i = q_X i,$$

i.e., the adjoint action of $\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ on a Lie algebra is just the multiplication of $q_X \mapsto q_X i$ by i on the right. It is easy to check that the adjoint action of the other two elements corresponds to the multiplication by j and k on the right. When no confusion would arise, we shall just use the identification $X \to q_X$ as $q_X = X$.

The parallel closed nondegenerate quaternionic Kähler four-form at the origin is given by

$$\omega = \omega_i \wedge \omega_i + \omega_j \wedge \omega_j + \omega_k \wedge \omega_k, \tag{2.3}$$

where

$$\omega_u(X, Y) = \mathrm{Re}(q_Y^* \cdot q_X u), \quad u = i, j, k.$$

Here, we use $q^* \cdot p = \sum_{m=1}^{2n} \bar{q}_m p_m$, and $\mathrm{Re}\, x = x_0$ is the real part of a quaternionic number $x = x_0 + x_1 i + x_2 j + x_3 k$. In this paper, we fix and use the convention

$$\alpha \wedge \alpha(X_1, X_2, X_3, X_4)$$
$$= \alpha(X_1, X_2)\alpha(X_3, X_4) - \alpha(X_1, X_3)\alpha(X_2, X_4) + \alpha(X_1, X_4)\alpha(X_2, X_4).$$

The four-form ω is then K-invariant even though the individual ω_i, ω_j, and ω_k are not invariant. This is known, but for completeness, we sketch an elementary proof. An element $k = \mathrm{diag}(A, D)$ of $K = S(U(2n) \times U(2))$, $A \in U(2n)$, and $D \in U(2)$ acts on $\mathfrak{p} = M_{2n\times 2}$ as $kZ = AZD^{-1}$. We may take $D \in SU(2)$ and represent it by a unit quaternion d. Through our identification of $Z = (x, y) \mapsto q = x + yj$, the action of k is $kq = Aqd^{-1}$, where Aq is the complex matrix multiplication on quaternionic vectors q. The pull-back of ω_u is $k^*\omega_u = \omega_{d^{-1}ud}$ since

$$(k^*\omega_u)(p, q) = \mathrm{Re}((Aqd^{-1})^*(Apd^{-1})u) = \mathrm{Re}(dq^* A^* Apd^{-1}u)$$
$$= \omega_u(dpd^{-1}, dqd^{-1}) = \mathrm{Re}(q^* pd^{-1}ud) = \omega_{d^{-1}ud}(p, q).$$

Hence,

$$k^*\omega = \omega_{d^{-1}id} \wedge \omega_{d^{-1}id} + \omega_{d^{-1}jd} \wedge \omega_{d^{-1}jd} + \omega_{d^{-1}kd} \wedge \omega_{d^{-1}kd}$$

on $\mathbb{H}^{2n}$. However, the action $u = \alpha i + \beta j + \gamma k \to d^{-1}ud$ is orthogonal on $\mathrm{Im}\mathbb{H} = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$, $u \to \omega_u$ is linear, and two forms $\omega_u \wedge \omega_v = \omega_v \wedge \omega_u$ are commuting. Thus, the action preserves the symmetric tensor

$$i \otimes i + j \otimes j + k \otimes k = d^{-1}id \otimes d^{-1}id + d^{-1}jd \otimes d^{-1}jd + d^{-1}kd \otimes d^{-1}kd,$$

and therefore, also $\omega = \omega_i \wedge \omega_i + \omega_j \wedge \omega_j + \omega_k \wedge \omega_k$; namely, $k^*\omega = \omega$. This proves the invariance and hence the well-definedness of ω on G/K.

Then, it is easy to check that this ω and Ω_o^2, where Ω_o is the complex Kähler form on $\mathcal{X}$ defined above, are linearly independent on $H^4(M, \mathbb{R})$, where $M = \Gamma\backslash\mathcal{X}$.

C. Twistor space and period domain of the quaternionic structures of $SU(n,2)/S(U(n) \times U(2))$

We describe one twistor space and one period domain for the quaternionic structures of $G/K = SU(n, 2)/S(U(n) \times U(2))$.

For any Lie algebra $\mathfrak{s}$, we denote its complexification by $\mathfrak{s}^{\mathbb{C}}$. Let $\mathcal{D}_1 = SU(n, 2)/S(U(n) \times U(1) \times U(1))$ be a twistor space. We shall realize it as an open subset in a homogeneous flag manifold. Let $W = \mathbb{C}^{n+2}$, and let W^* be the dual space equipped with the G-invariant metric of signature $(n, 2)$. Denote $\{\epsilon_j\}$ in W^*, the dual basis of $\{E_j\}$. Let $\mathcal{D}_1^c$ be the set of orthogonal pairs (l, λ) in $\mathbb{P}(W) \times \mathbb{P}(W^*)$, i.e., satisfying $\epsilon(e) = 0$ for all $(e, \epsilon) \in l \times \lambda$. Then, $\mathcal{D}_1^c$ is a compact homogeneous space of $SU(n+2)$, $\mathcal{D}_1^c = SU(n+2)/S(U(n) \times U(1) \times U(1))$. As a homogeneous manifold of $SU(n, 2)$, $\mathcal{D}_1 = SU(n, 2)/S(U(n) \times U(1) \times U(1))$ can be realized as the open domain in $\mathcal{D}_1^c$ of (l, λ) such that l and λ are negative definite. Indeed, first, it is elementary to see that $SU(n, 2)$ acts transitively on the domain. Second, we need to check that a stabilizer of (l, λ) is $S(U(n) \times U(1) \times U(1))$. A stabilizer of the negative two-plane $l + (\ker \lambda)^{\perp}$ in W is $S(U(n) \times U(2))$, and a stabilizer in $S(U(n) \times U(2))$ of the pair $(l, \ker \lambda)$ of subspaces in W, equivalently the pair (l, λ) in $\mathbb{P}(W) \times \mathbb{P}(W^*)$, is exactly $U(1) \times U(1)$. Hence, as a differentiable manifold, $\mathcal{D}_1$ has such a realization.

Then, $\mathcal{D}_1 \subset \mathbb{P}(W) \times \mathbb{P}(W^*)$ is an open subset equipped with the corresponding complex structure.

In general, if a homogeneous manifold $G/(L \times U(1))$ has a $U(1)$ factor in the stabilizer, it inherits a complex structure as follows. Let $\mathfrak{u}(1) = \mathbb{R}iH_1$, and consider the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ under the action of

$$H_1 = \begin{pmatrix} 0 & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix}.$$

Set $\mathfrak{b}$ to be the Borel subalgebra consisting of zero and negative eigenspaces. The positive eigenspace $\mathfrak{n}^+$ constitutes the holomorphic tangent space for $G^{\mathbb{C}}/B$ at the base point $eB \in G^{\mathbb{C}}/B$, and further on the whole space $G^{\mathbb{C}}/B$, in particular for open set

$$G/(L \times U(1)) \subset G^{\mathbb{C}}/B.$$

We find the holomorphic tangent space of $\mathcal{D}_1$ in this context. To find a realization of the complex tangent space, we fix the pair $(\mathbb{C}E_{n+2}, \mathbb{C}\epsilon_{n+1})$ as a base point of $\mathcal{D}_1$. The space $\mathcal{D}_1$ is an open subset of the complex homogeneous space of $SL(n+2,\mathbb{C})/B$, where B is the Borel subgroup whose Lie algebra consists of elements in $\mathfrak{sl}(n+2,\mathbb{C})$ of the special form.

To justify this, note that B is equal to the stabilizer of $(\mathbb{C}E_{n+2}, \mathbb{C}\epsilon_{n+1})$. Hence, B should have the block matrix of the form

$$\begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ * & * & * \end{pmatrix},$$

the size of the matrix being $(n+1+1) \times (n+1+1)$. Alternatively, $\mathfrak{b}$ consists of non-positive root spaces of H_1, i.e., eigenspaces of $ad(H_1)$. Thus, holomorphic tangent space $\mathfrak{n}^+$ consists of elements of $\mathfrak{g}^{\mathbb{C}}$ of the form, the size of the matrix being the same as above,

$$\begin{pmatrix} 0 & 0 & * \\ * & 0 & * \\ 0 & 0 & 0 \end{pmatrix}.$$

We now consider another domain

$$\mathcal{D}_2 = SU(n,2)/S(U(n-1) \times U(1) \times U(2)).$$

More precisely, let $\{E_1,\dots,E_n; E_{n+1}, E_{n+2}\}$ be the standard basis of $\mathbb{C}^{n+2}$ as before, and

$$H_2 = \mathrm{diag}(1,\dots,1,-n,1,0,0) \in \mathfrak{k}^{\mathbb{C}},$$

and let $U(1) = \exp(i\mathbb{R}\mathbb{H}_2)$ be the corresponding subgroup of K. The centralizer of H_2 in K is then $U(n-1) \times U(1) \times U(2)$. Here, $U(n-1)$ stands for the unitary group of the subspace $\mathbb{C}^{n-1} := \langle E_1,\dots,E_{n-2},E_n\rangle$.

Now, the eigenspaces of positive eigenvalues of $\mathrm{ad}(H_2)$ constitute the holomorphic tangent space of $\mathcal{D}_2$,

$$\begin{pmatrix} 0 & * & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & * & 0 & 0 \end{pmatrix},$$

written in the block form of size $((n-2)+1+1+2) \times ((n-2)+1+1+2)$.

The compact homogeneous space $\mathcal{D}_2^c := SU(n+2)/S(U(n-1) \times U(1) \times U(2))$ is precisely the partial flag manifold of pairs (p_1,p_2) of subspaces $p_1 \subset p_2$ in $\mathbb{C}^{n+2}$ of dimensions 1and n, respectively. In particular, the map $(p_1,p_2) \mapsto p_2$ from $\mathcal{D}_2^c$ to the Grassmannian manifold $Gr_n(n+2)$ realizes $\mathcal{D}_2^c$ as the projectivization of the tautological bundle of $Gr_n(n+2)$.

D. Cohomology groups of period domains

Let $\mathcal{X}^c = SU(n+2)/S(U(n) \times U(2))$ be the compact dual of $\mathcal{X}$. Then, $\mathcal{X}^c$ can be realized as Grassmannian manifolds $Gr_2(n+2)$ of two planes in $\mathbb{C}^{n+2}$. Let $\mathcal{D}_1^c = SU(n+2)/S(U(n) \times U(1) \times U(1)), \mathcal{D}_2^c = SU(n+2)/S(U(n-1) \times U(1) \times U(2))$ be as above, and

$$\pi : \mathcal{D}_1^c, \mathcal{D}_2^c \to \mathcal{X}^c$$

be the natural fibrations. The homogeneous manifolds $\mathcal{D}_1^c$ and $\mathcal{D}_2^c$ are equipped with a G-invariant Kähler form $\hat{\Omega}$; see Ref. 1, Corollary 8.59, and Example 8.111.

Proposition 2.1.

(1) *The cohomology group $H^4(\mathcal{D}_1^c)$ is three dimensional and is generated by $\pi^*(\Omega^2), \pi^*(\omega), \hat{\Omega}^2$.*
(2) *Let $n \geq 3$. The cohomology group $H^4(\mathcal{D}_2^c)$ is four dimensional and is generated by $\pi^*(\Omega^2), \pi^*(\omega), \hat{\Omega}^2, \hat{\Omega} \wedge \pi^*(\Omega)$.*

Proof. The map π defines the twister space $\mathcal{D}_1^c$ as a $\mathbb{CP}^1 = S^2$-bundle over the Grassmannian manifold $\mathcal{X}^c$. We recall the Gysin complex (Ref. 2, Proposition 14.33) for the sphere covering $\pi : \mathcal{D}_1^c \mapsto \mathcal{X}^c$,

$$H^1(\mathcal{X}^c) \xrightarrow{\wedge e} H^4(\mathcal{X}^c) \xrightarrow{\pi^*} H^4(\mathcal{D}_1^c) \xrightarrow{\pi_*} H^2(\mathcal{X}^c) \xrightarrow{\wedge e} H^5(\mathcal{X}^c),$$

where $\wedge e$ is the multiplication by the Euler class e of the sphere bundle, π^* is the pull-back, and π_* is the integration along the fiber S^2. Now, $H^1(\mathcal{X}^c) = 0, H^5(\mathcal{X}^c) = 0, H^2(\mathcal{X}^c) = \mathbb{R}$, and $H^4(\mathcal{X}^c) = \mathbb{R}^2$ since the cohomology of $\mathcal{X}^c$ is known; see, e.g., Ref. 2, Proposition 23.1 for the computation of the cohomology in complex coefficients. Thus, the above sequence reduces to

$$0 \to \mathbb{R}^2 = H^4(\mathcal{X}^c) \xrightarrow{\pi^*} H^4(\mathcal{D}_1^c) \xrightarrow{\pi_*} \mathbb{R} = H^2(\mathcal{X}^c) \to 0,$$

from which we deduce that $H^4(\mathcal{D}^c) = \mathbb{R}^3$. It follows further that π^* is an injection. The square $\hat{\Omega}^2$ of the Kähler form is clearly not contained in $\pi^* H^4(\mathcal{X}^c)$ since its integration along the fibers are nonzero; thus, $H^4(\mathcal{D}^c)$ is generated by $\pi^*(\omega), \pi^*(\Omega^2)$, and $\hat{\Omega}^2$. This proves (1).

Note that the map π defines the space $\mathcal{D}_2^c$ as the projectivization $\mathbb{P}(L) \mapsto [L]$ of the tautological bundle $L \to [L]$ of the Grassmannian $\mathcal{X}^c$ of n dimensional subspaces $[L]$ in $\mathbb{C}^{n+2}$. The $(1, 1)$-form $\hat{\Omega}$ restricted to each fiber $\mathbb{P}(L)$ is the Chern class $c_1(\mathbb{P}(v))$ of the projective space. It follows from the Leray–Hirsch theorem [Ref. 2, (5.11) and (20.7)] or by Ref. 2, (20.8) that $H^4(\mathcal{D}_2^c)$ is of dimension 4 and is generated by the four forms as claimed. □

E. Pseudo-Riemannian metrics on period domains

Let $\mathcal{X} = SU(n, 2)/S(U(n) \times U(2))$ and $\mathcal{D} = SU(n, 2)/K'$, where K' is a subgroup of $K = S(U(n) \times U(2))$. The metric on $\mathcal{X}$ comes from the Killing form on $\mathfrak{g}$ whose tangent space at $o = eK$ is identified with $\mathfrak{p}$ according to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Hence, the metric on a period domain $\mathcal{D}$ comes from the Killing form on $\mathfrak{k}/\mathfrak{k}' \oplus \mathfrak{p}$, where $\mathfrak{k}'$ is the Lie algebra of K'. This metric is positive definite on the horizontal direction $\mathfrak{p}$, which coincides with the metric on $\mathcal{X}$, and negative definite on $\mathfrak{k}/\mathfrak{k}'$ along the fiber direction of the projection $\pi : \mathcal{D} \to \mathcal{X}$. If Ω is a Kähler form on $\mathcal{X}$ defined by such a metric, $\hat{\Omega}$ is a pseudo-Kähler form on $\mathcal{D}$, then on the horizontal direction of $T\mathcal{D}$, $\hat{\Omega}$ and $\pi^*\Omega$ coincide since the Kähler form is determined by the metric, as in Eq. (2.1). We normalize a quaternionic Kähler form ω on $\mathcal{X}$ so that its restriction to a copy of $H^2_{\mathbb{C}}$ in $\mathcal{X}$ is equal to Ω^2.

III. TOTALLY GEODESIC EMBEDDINGS OF THE COMPLEX HYPERBOLIC SPACE B IN $\mathcal{X}$ AND THEIR POSSIBLE HOLOMORPHIC LIFTINGS TO PERIOD DOMAINS

We consider several natural totally geodesic imbeddings of the complex ball B^m into the quaternionic symmetric spaces and consider the corresponding pull-back of the quaternionic four-forms and the Kähler forms. In Ref. 7, the authors study some holomorphic liftings of mappings from the complex hyperbolic ball to quaternionic hyperbolic ball to holomorphic mapping to the (pseudo-Hermitian) twistor space, which enable them to apply a variant of the Schwarz lemma and to prove rigidity theorems. Following a suggestion of Toledo, we shall study holomorphic liftings in our context.

A. Holomorphic and totally real embeddings

The complex hyperbolic space $H^n_{\mathbb{C}}$, i.e., the symmetric space $SU(n, 1)/S(U(n) \times U(1))$, will be realized as the unit ball B in $\mathbb{C}^n$, as in Sec. II B.

Recall also the normalization of the Kähler metric on B and on $\mathcal{X}$,

$$g_B(u, v) = 4\mathrm{Re}(\sum u_i \bar{v}_i),\ g_{\mathcal{X}}(u, v) = 4\mathrm{ReTr} v^* u,$$

where the real tangent spaces of B and $\mathcal{X}$ at $z = 0$ and $Z = 0$ are identified with $\mathbb{C}^n$ and $M_{2n\times 2}$; the respective Kähler forms are $\Omega_B(u, v) = g_B(iu, v)$ and $\Omega_{\mathcal{X}}(u, v) = g_{\mathcal{X}}(iu, v)$.

A natural holomorphic embedding of $H^n_{\mathbb{C}} = B = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \sum |z_i|^2 < 1\}$ into $\mathcal{X}$ is given by

$$\rho : (z_1, \ldots, z_n) \hookrightarrow Z = \begin{pmatrix} z_1 I_2 \\ z_2 I_2 \\ \cdots \\ z_n I_2 \end{pmatrix},$$

which gives rise to zero Toledo invariant of ω. The push-forward on holomorphic tangent vectors at $0 \in B$ is

$$\rho_* : x = (x_1, \dots, x_n) \in \mathbb{C}^n \mapsto X = (x_1, x_1 j, \dots, x_n, x_n j) \in \mathbb{H}^{2n},$$

where $x_l = a_l + ib_l \in \mathbb{C}$ and on which the forms ω_j and ω_k vanish. Thus,

$$\begin{aligned}\omega(X,Y,Z,W) = \omega_i \wedge \omega_i(X,Y,Z,W) &= \mathrm{Re}(\bar{Y}\cdot\bar{X}i)\mathrm{Re}(\bar{W}\cdot Zi) \\ &\quad - \mathrm{Re}(\bar{Z}\cdot Xi)\mathrm{Re}(\bar{W}\cdot Yi) + \mathrm{Re}(\bar{W}\cdot Xi)\mathrm{Re}(\bar{Z}\cdot Yi) \\ &= \mathrm{Re}(i\sum_{m=1}^{n}(x_m\bar{y}_m + \bar{x}_m y_m))\mathrm{Re}(i\sum_{m=1}^{n}(z_m\bar{w}_m + \bar{z}_m w_m)) \\ &\quad - \mathrm{Re}(i\sum_{m=1}^{n}(x_m\bar{z}_m + \bar{x}_m z_m))\mathrm{Re}(i\sum_{m=1}^{n}(y_m\bar{w}_m + \bar{y}_m w_m)) \\ &\quad + \mathrm{Re}(i\sum_{m=1}^{n}(x_m\bar{w}_m + \bar{x}_m w_m))\mathrm{Re}(i\sum_{m=1}^{n}(y_m\bar{z}_m + \bar{y}_m z_m)) = 0.\end{aligned}$$

However, when we write $X = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}$,

$$\Omega_o(X,Y) = g_o(JX,Y) = 4\mathrm{Re}\,\mathrm{Tr}(iB^*A) = 4\mathrm{Re}(2i\sum_{m=1}^{n} x_m\bar{y}_m).$$

Hence,

$$\Omega_o^2(X,Y,Z,W) = 4\Omega_B^2(x,y,z,w)$$

for tangent vectors $X = \rho_*(x), Y = \rho_*(y), Z = \rho_*(z), W = \rho_*(w)$ at the image of the natural holomorphic embedding of $H_{\mathbb{C}}^n = B$, where Ω_B is the Kähler form on B. In other words,

$$\rho^*\Omega_o^2 = 4\Omega_B^2, \quad \rho^*\omega = 0, \tag{3.1}$$

for the natural holomorphic embedding ρ of $H_{\mathbb{C}}^n$ into $\mathcal{X}$.

On the other hand, another natural embedding

$$\lambda : SU(n,1) \hookrightarrow Sp(n,1) \hookrightarrow SU(2n,2) \tag{3.2}$$

gives rise to a totally real embedding

$$\lambda : B \to \mathcal{X},\ (z_1,\dots,z_n) \hookrightarrow Z = \begin{pmatrix} \begin{pmatrix} z_1 & 0 \\ 0 & \bar{z}_1 \end{pmatrix} \\ \cdots \\ \begin{pmatrix} z_n & 0 \\ 0 & \bar{z}_n \end{pmatrix} \end{pmatrix} \tag{3.3}$$

whose Toledo invariant of ω is

$$\lambda_* : x = (x_1, \dots, x_n) \in \mathbb{C}^n \mapsto X = (x_1, \bar{x}_1 j, \dots, x_n, \bar{x}_n j) \in \mathbb{H}^{2n},$$

$$\begin{aligned}\omega(X,Y,Z,W) = \omega_i \wedge \omega_i(X,Y,Z,W) &= \mathrm{Re}(\bar{Y}\cdot\bar{X}i)\mathrm{Re}(\bar{W}\cdot Zi) \\ &\quad - \mathrm{Re}(\bar{Z}\cdot Xi)\mathrm{Re}(\bar{W}\cdot Yi) + \mathrm{Re}(\bar{W}\cdot Xi)\mathrm{Re}(\bar{Z}\cdot Yi) \\ &= \mathrm{Re}(2i\sum_{m=1}^{n}(x_m\bar{y}_m))\mathrm{Re}(2i\sum_{m=1}^{n}(z_m\bar{w}_m)) \\ &\quad - \mathrm{Re}(2i\sum_{m=1}^{n}(x_m\bar{z}_m))\mathrm{Re}(2i\sum_{m=1}^{n}(y_m\bar{w}_m)) \\ &\quad + \mathrm{Re}(2i\sum_{m=1}^{n}(x_m\bar{w}_m))\mathrm{Re}(2i\sum_{m=1}^{n}(y_m\bar{z}_m)) = \frac{1}{4}\Omega_B^2(x,y,z,w).\end{aligned}$$

Hence,

$$\lambda^* \omega = \frac{1}{4}\Omega_B^2. \tag{3.4}$$

Contrary to the $SU(1,1)$ case, this totally real embedding is locally rigid for $n > 1$; see Ref. 11.

1. Conjecture

These two particular embeddings suggest that the Toledo invariant of ω is maximal on totally real embedding and zero on holomorphic embedding. More precisely, if a representation attains a maximum Toledo invariant, then it should be conjugate to the totally real embedding above, and if the quaternionic Toledo invariant is zero, then it is conjugate to the holomorphic embedding.

2. Warning

If we identify the holomorphic tangent space of $\mathcal{X}$ with $\mathbb{H}^{2n}$ by $(x_1 + jy_1, \ldots, x_{2n} + jy_{2n})$, ω vanishes on totally real embedding and $16\omega = \Omega_o^2$ on holomorphic embedding. Hence, the convention determines which one has a maximal Toledo invariant. In Ref. 7, it seems that they use a different convention from ours. Nevertheless, we stick to our convention in this paper.

B. Symmetric square representation of $SU(2,1)$ and related four-forms

Denote $V = \mathbb{C}^{2+1} = \mathbb{C}^2 + \mathbb{C}_3$ the space $\mathbb{C}^3$ equipped with the Hermitian metric with signature $(2, 1)$ and $B = SU(2,1)/S(U(2) \times U(1))$, as in Sec. II B. Recall that it is also identified as the open domain in $\mathbb{P}^2$ of lines $\mathbb{C}(z \oplus e_3)$ with a negative metric, i.e., $|z| = |(z_1, z_2)| < 1$.

Let $W = V^2$ be the symmetric square of V. Then, W is equipped with the square of the Hermitian metric of V, and $W = \mathbb{C}^4 + \mathbb{C}^2 = ((\mathbb{C}^2)^2 + \mathbb{C}e_3^2) \oplus (\mathbb{C}^2 \odot e_3)$ is of signature $(4, 2)$. Here, $e_i \odot e_j = \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i)$. We fix an orthonormal basis $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ of W with

$$E_j = e_j^2, E_4 = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 + e_2 \otimes e_1) = \sqrt{2}e_1 \odot e_2,$$

$$E_{4+i} = \frac{1}{\sqrt{2}}(e_3 \otimes e_i + e_i \otimes e_3) = \sqrt{2}e_3 \odot e_i, j = 1, 2, 3, i = 1, 2.$$

The square of the defining representation of $H = SU(2,1)$ defines a representation

$$\iota : H \to G = SU(4,2),\ \ g \mapsto \otimes^2 g.$$

As in Sec. II B, the symmetric space $\mathcal{X}$ of $SU(4,2)$ will be realized as the open domain of Grassmannian manifold $Gr(2, W)$ of two-dimensional complex subspaces in W with a negative metric and is further identified with the space of 4×2 matrices Z with matrix norm $\|Z\| < 1$ under the identification

$$\{Zx \oplus x; x \in \mathbb{C}^2\} \mapsto Z.$$

Recall also the normalization of the Kähler metric on B and on $\mathcal{X}$,

$$g_B(u, v) = 4\mathrm{Re}(u_1\bar{v}_1 + u_2\bar{v}_2),\ g_{\mathcal{X}}(u, v) = 4\mathrm{ReTr}v^* u,$$

where the real tangent spaces of B and $\mathcal{X}$ at $z = 0$ and $Z = 0$ are identified with $\mathbb{C}^2$ and $M_{4\times 2}$; the respective Kähler forms are $\Omega_B(u, v) = g_B(iu, v)$ and $\Omega_{\mathcal{X}} = g_{\mathcal{X}}(iu, v)$.

The representation $\iota : H \to G$ induces a totally geodesic mapping (with the same notation) $\iota: B \to \mathcal{X}$. In terms of the above identification of B and X as submanifolds of projective and Grassmannian manifolds, the map ι is

$$\iota(l) = l \odot l^{\perp},$$

where $l^{\perp}$ is the orthogonal complement of l in V and $l \odot l^{\perp}$ is the subspace of vectors $u \otimes v + v \otimes u$, where $u \in l, v \in l^{\perp}$. We find now the map ι_* at $z = 0 \in B$.

Fixing the reference line $\mathbb{C}_3 \in \mathbb{P}^2$ and the plane $\mathbb{C}^2 \odot e_3 \in Gr(2, W)$ corresponding to the points $0 \in B$ and $0 \in \mathcal{X}$, the map ι is

$$\iota : \exp(tX) \cdot (\mathbb{C}_3) \mapsto (\exp(tX) \cdot (\mathbb{C}_3)) \odot (\exp(tX) \cdot (\mathbb{C}^2)),$$

where

$$X = \begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ \bar{a}_1 & \bar{a}_2 & 0 \end{pmatrix} \in \mathfrak{p}$$

and $\mathfrak{su}(2,1) = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition. Thus, $\iota_*(X) \in \mathfrak{p}' = M_{4\times 2}$, with $\mathfrak{su}(4,2) = \mathfrak{k}' + \mathfrak{p}'$ being the corresponding Cartan decomposition, is the linear transformation

$$\iota_*(X) : \mathbb{C}^2 \to \mathbb{C}^4,$$

$$\mathbb{C}\{E_5, E_6\} \mapsto \mathbb{C}\{(Xe_3) \odot e_1 + e_3 \odot (Xe_1), (Xe_3) \odot e_2 + e_3 \odot (Xe_2)\}.$$

Note that $Xe_1 = \bar{a}_1 e_3,\ Xe_2 = \bar{a}_2 e_3,\ Xe_3 = a_1 e_1 + a_2 e_2$ and

$$(Xe_3) \odot e_1 + e_3 \odot (Xe_1) = (a_1 e_1 + a_2 e_2) \odot e_1 + e_3 \odot \bar{a}_1 e_3$$

$$= a_1 e_1 \odot e_1 + a_2 e_2 \odot e_1 + \bar{a}_1 e_3 \odot e_3 = a_1 E_1 + \bar{a}_1 E_3 + \frac{a_2}{\sqrt{2}} E_4.$$

A similar calculation for the second factor shows that, under the basis $\{E_j\}$, $\iota_*(X)$ corresponds to the 4×2 matrix

$$\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \\ \bar{a}_1 & \bar{a}_2 \\ \frac{a_2}{\sqrt{2}} & \frac{a_1}{\sqrt{2}} \end{bmatrix} = T_a.$$

Taking the basis vectors $X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}$, we find the corresponding images in $\mathbb{H}^4$ under ι_*

$$\iota_*(X) = (1, 0, 1, \frac{j}{\sqrt{2}}),\ \ \iota_*(Y) = (i, 0, -i, \frac{k}{\sqrt{2}}),$$

$$\iota_*(Z) = (0, j, j, \frac{1}{\sqrt{2}}),\ \ \iota_*(W) = (0, k, -k, \frac{i}{\sqrt{2}})$$

and that

$$\omega(\iota_*(X), \iota_*(Y), \iota_*(Z), \iota_*(W)) = -\frac{3}{4}. \tag{3.5}$$

Namely,

$$\iota^* \omega = -\frac{3}{64} \Omega_B^2, \tag{3.6}$$

where Ω_B is the Kähler form on B. We can likewise compute $\iota^* \Omega^2$ and find

$$\iota^* \Omega^2 = \frac{1}{4} \Omega_B^2.$$

Now, there is a natural inclusion of $SU(4,2)$ as a subgroup of $SU(n,2)$, where $n \geq 4$, and we will also view ι as a homomorphism $\iota : SU(2,1) \to SU(4,2) \to SU(n,2)$.

C. Holomorphic lifting properties

As mentioned in the Introduction, it is of interest to know if a harmonic map $f : B \to \mathcal{X}$ can be lifted to a holomorphic map into a period domain[5] (or Griffiths-Schmid domain) $\mathcal{D}$, namely, a homogeneous complex manifold $\mathcal{D} = G/L$ with a G equivariant fibration $\pi : \mathcal{D} \to \mathcal{X} = G/K$. We give an elementary criterion below.

Proposition 3.1. Suppose that there exists a holomorphic lifting $\hat{f}$ of a harmonic map $f : B \to \mathcal{X}$ to a period domain $\mathcal{D}$. Then, $d^{(1,0)}f(v)$ is nilpotent for any $v \in \mathfrak{p}^+ = T_x^{(1,0)}(B)$.

Proof. Let $\hat{f} : B \to \mathcal{D}$ be a holomorphic lift of $f : B \to \mathcal{X}$. We can fix a reference point $x = 0$ and assume that $\hat{f}(0) = o = eL \in \mathcal{D} = G/L$. The holomorphic tangent space of $T_o(\mathcal{D})$ is given by the $\mathfrak{n}^+$-space as in Sec. II C and is a nilpotent algebra of $\mathfrak{g}$. Now, $f = \pi \circ \hat{f}$, and $f_*(0)(v) = \pi_*(o)(\hat{f}_*(0)(v))$, for $v \in T_0^{(1,0)}(B)$. However, $\hat{f}(0)(v) \in \mathfrak{n}^+$ since $\hat{f}$ is holomorphic, so $\hat{f}(0)(v) \in \mathfrak{n}^+$ is nilpotent, which implies that $f_*(0)(v)$ is nilpotent since π_* maps nilpotent elements to nilpotent elements where π is the quotient map $\mathcal{D} = G/L \to \mathcal{X} = G/K$. □

We find a holomorphic lift of the non-holomorphic map λ.

Lemma 3.2. The totally real imbedding $\lambda : B \to \mathcal{X}$ can be lifted to a holomorphic horizontal mapping into the period domain $\mathcal{D} = SU(2n, 2)/S(U(2n) \times U(1) \times U(1))$.

Proof. Let $\mathbb{C}^{2n+2} = \mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1} = \mathbb{C}_+^n \oplus \mathbb{C}_+^n \oplus \mathbb{C}_- \oplus \mathbb{C}_-$ be equipped with the Hermitian form $\langle , \rangle$ of signature $(2n, 2)$ with $\mathbb{C}^{n+1}$ being of signature $(n, 1)$ as before, with the sub-indices $\pm$ indicating the positivity or negativity of the form. We denote the standard basis as $\{e_1, \ldots, e_n, e_{n+1}\}$ for the first factor $\mathbb{C}^{n+1}$ and $\{f_1, \ldots, f_{n+1}\}$ for the second summand $\mathbb{C}^{n+1}$.

Then, according to the notations in Sec. II B, the space $\mathcal{X}$ is the set of pairs of $2n$-coordinates

$$(z_1, \ldots, z_n, w_1, \ldots, w_n), (z'_1, \ldots, z'_n, w'_1, \ldots, w'_n)$$

such that

$$z_1 e_1 + w_1 f_1 + \cdots + z_n e_n + w_n f_n + e_{n+1},$$

$$z'_1 e_1 + w'_1 f_1 + \cdots + z'_n e_n + w'_n f_n + f_{n+1}$$

represents a two-dimensional negative definite subspace.

Consider the flag manifolds $\mathcal{D}$ of pairs (p_1, p_2), where p_1 is an one-dimensional subspace with negative form $\langle , \rangle$ and p_2 is a $(2n+1)$-dimensional subspace with signature $(2n, 1)$ containing p_1. Then, $\mathcal{D}$ is a $G = SU(2n, 2)$-homogeneous manifold, and $\mathcal{D} = G/L$, $L = S(U(2n) \times U(1) \times U(1))$. The homogeneity can be proved by the elementary linear algebra. Fixing the point $p_0 = (p_1, p_2)$, $p_1 = \mathbb{C}e_{n+1}$, and $p_2 = \mathbb{C}^{2n+1} = \mathbb{C}_+^n \oplus p_1 \oplus \mathbb{C}_+^n \oplus 0$ as a reference, then the isotropic group of p in G is exactly L, proving the realization of $\mathcal{D}$. The complex structure on $\mathcal{D}$ is realized as an open subset of the flag manifold $\mathcal{D}^c = SU(2n+2)/S(U(2n) \times U(1) \times U(1)) = SL(2n+2, \mathbb{C})/P$ of all pairs (p_1, p_2) of one-dimensional subspaces p_1 in $(2n+1)$-dimensional subspaces p_2, considered as a homogeneous space of $SL(2n+2, \mathbb{C})$ with P being a Borel subgroup as the isotropic subgroup fixing the reference point $p_0 = (p_1, p_2) = (\mathbb{C}e_{n+1}, \mathbb{C}^{2n+1})$ above. The fibration $\pi : \mathcal{D} \to \mathcal{X}$ is then the map

$$(p_1, p_2) \mapsto p_1 \oplus p_2^\perp.$$

Clearly, $p_1 \oplus p_2^\perp$ is a two-dimensional subspace in $\mathbb{C}^{2n+2}$ of signature $(0, 2)$; namely, it is an element in $\mathcal{X}$, and this map is G equivariant. Now, we consider the map $\tilde{\lambda} : B \to \mathcal{D}$,

$$\tilde{\lambda}(z) = (p_1, p_2);\ \ p_1 = (z_1 e_1 + \cdots + z_n e_n + e_{n+1}), p_2 = (\bar{z}_1 f_1 + \cdots + \bar{z}_n f_n + f_{n+1})^\perp,$$

with the orthogonal complement being computed with respect to the fixed indefinite form. It follows immediately from the formula that $\tilde{\lambda}$ is holomorphic in z. To be more precise, complex coordinates near p_0 can be chosen as

$$(x, x', y, y') \in \mathbb{C}^n \times \mathbb{C}^{n+1} \times \mathbb{C}^n \times \mathbb{C}^n \mapsto (p_1, p_2), p_2 = p_1 \oplus q_2,$$

$$p_1 = \mathbb{C}(e_{n+1} \oplus (x_1 e_1 + \cdots + x_n e_n + x'_1 f_1 + \cdots + x'_{n+1} f_{n+1})),$$

$$q_2 = \mathrm{span}\{e_1 + y_1 e_{n+1}, \ldots, e_n + y_n e_{n+1};\quad f_1 + y'_1 f_{n+1}, \ldots, f_n + y'_n f_{n+1}\}.$$

In terms of these coordinates, the map $\tilde{\lambda}$ is

$$\lambda : z = (z_1, \ldots, z_n) \mapsto (x, x', y, y') = (z, 0, 0, z)$$

and is, indeed, holomorphic. We further have

$$\pi \circ \tilde{\lambda} : z \mapsto (p_1, p_2) \mapsto p_1 \oplus p_2^{\perp} = \begin{pmatrix} \begin{pmatrix} z_1 & 0 \\ 0 & \bar{z}_1 \end{pmatrix} \\ \cdots \\ \begin{pmatrix} z_n & 0 \\ 0 & \bar{z}_n \end{pmatrix} \end{pmatrix}.$$

This corresponds precisely to our map λ; namely, $\tilde{\lambda}$ is a lift of λ. □

We consider now the lifting property of ι.

Lemma 3.3. The above quadratic map $\iota : B \to \mathcal{X}$ does not lift to a holomorphic horizontal mapping into $\mathcal{D}_1 = SU(4,2)/S(U(4) \times U(1) \times U(1))$.

Proof. Suppose that F is a holomorphic horizontal lifting. The complexification of F_*, still denoted by F_*, maps $\mathfrak{b}^+$, the holomorphic tangent space of B to holomorphic tangent space $\mathfrak{n}^+$ [up to changing of the base point under SU(2)-action]. In particular, the image of $\mathfrak{b}^+$ under ι_* is contained in $\pi_*(\mathfrak{n}^+)$, where $\pi : \mathcal{D}_1 \to \mathcal{X}$ is the natural projection. In particular, $\iota_*(\mathfrak{b}^+)$ is a subspace of $\pi_*(\mathfrak{n}^+)$. Using the above formula for $\mathfrak{n}^+$, we find that elements in $\iota_*(\mathfrak{b}^+) \subset \pi_*(\mathfrak{n}^+)$ are of the form

$$\begin{pmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

However, our computations above show that for

$$\begin{aligned} S &= \begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \frac{1}{2}\left(\begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ \bar{a}_1 & \bar{a}_2 & 0 \end{pmatrix} - \sqrt{-1} \begin{pmatrix} 0 & 0 & ia_1 \\ 0 & 0 & ia_2 \\ -i\bar{a}_1 & -i\bar{a}_2 & 0 \end{pmatrix} \right) \in \mathfrak{b}^+, \end{aligned} \tag{3.7}$$

its image $\iota_*(S)$ is

$$\iota_*(S) = \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix},$$

where

$$U = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \\ 0 & 0 \\ \frac{a_2}{\sqrt{2}} & \frac{a_1}{\sqrt{2}} \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & a_1 & 0 \\ 0 & 0 & a_2 & 0 \end{pmatrix}.$$

This is a contradiction to the form of $\pi_*(\mathfrak{n}^+)$. □

We may construct similarly the twistor cover $SU(2m,2)/S(U(2m) \times U(1) \times U(1))$ of $\mathcal{X} = SU(2m,2)/S(U(2m) \times U(2))$ as above and consider the question of holomorphic lifting of maps from B to $\mathcal{X}$. The above proof leads to a simple necessary condition for the existence.

Corollary 3.4. Given a representation $\rho : \Gamma \subset SU(n,1) \to SU(2m,2)$, with a ρ-equivariant map f on the associated symmetric spaces $B = SU(n,1)/S(U(n) \times U(1))$, $\mathcal{X} = SU(2m,2)/S(U(2m) \times U(2))$, and a fixed base point $o = [K] \in SU(n,1)/S(U(n) \times U(1))$, let

$$Df_o \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & U \\ U^* & 0 \end{pmatrix}$$

be a differential map at the base point, where $X \in \mathbb{C}^n$, $U = (U_1, U_2) \in M_{2m\times 2}$. For f to have a holomorphic lift to the twistor space, every component of U_1 is an conjugate $\mathbb{C}$-linear in X, and every component of U_2 is a $\mathbb{C}$-linear in X. Here, we regard Df_o as a map from $\mathbb{C}^n$ to $M_{2m\times 2} = \mathbb{C}^{4m}$.

Proof. Note that Df_o is a real linear map between real tangent spaces T_oB and $T_{f(o)}\mathcal{X}$. For $X = (z_1, \ldots, z_n)$ and $z_i = x_i + iy_i$, let $X = (x_1, \ldots, x_n, y_1, \ldots, y_n) = (x, y)$, with the same notation, be the corresponding coordinates in $\mathbb{R}^{2n}$. Then, iX corresponds to

$$iX = (-y_1, \ldots, -y_n, x_1, \ldots, x_n) = (-y, x)$$

as usual. For f to lift to the holomorphic map to the twistor space, Eq. (3.7) should read

$$Df_o(X - \sqrt{-1}iX) = (U_1', U_2') = (0, U_2').$$

Hence, from $U_1' = 0$, we get

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} - \sqrt{-1}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{bmatrix} -y \\ x \end{bmatrix} = 0.$$

It is

$$\begin{pmatrix} Ax + By \\ Cx + Dy \end{pmatrix} + \begin{pmatrix} -Cy + Dx \\ -(-Ay + Bx) \end{pmatrix} = 0.$$

From this, we get

$$A = -D, \ B = C.$$

This exactly implies that every component function of U_1 is conjugate $\mathbb{C}$-linear in $X = (z_1, \ldots, z_n)$ variables. Using the equation for U^*, a similar calculation shows that every component function of U_2 must be $\mathbb{C}$-linear in z_i variables for f to have a holomorphic lift to the twistor space. □

We prove, however, that the map ι can be lifted to a holomorphic mapping to $\mathcal{D}_2 = SU(4, 2)/S(U(3) \times U(1) \times U(2))$.

Let f associate the triple $(S^2L^\perp, L^2, L \odot L^\perp)$ to a negative line L in $V = \mathbb{C}^{2+1}$. Then, $S^2L^\perp$ is a positive three-dimensional space in W, L^2 is a positive line in W, and $L \odot L^\perp$ is a negative plane in W. In the explicit coordinates, if $L = \mathbb{C}e_3$, then $L^\perp = \langle e_1, e_2\rangle$, and

$$S^2L^\perp = \langle e_1^2, e_2^2, e_1 \odot e_2\rangle = \langle E_1, E_2, E_4\rangle, \ \ L^2 = \langle e_3^2\rangle = \langle E_3\rangle,$$

$$L \odot L^\perp = \langle e_1 \odot e_3, e_2 \odot e_3\rangle = \langle E_5, E_6\rangle.$$

Hence, the stabilizers of $S^2L^\perp, L^2, L \odot L^\perp$ are $U(3), U(1)$ and $U(2)$, respectively. Therefore, $f : L \mapsto (S^2L^\perp, L^2, L \odot L^\perp)$ induces a map

$$f : B \to \mathcal{D}_2 = SU(4, 2)/S(U(3) \times U(1) \times U(2)).$$

Since

$$\iota(L) = (L \odot L^\perp, (L \odot L^\perp)^\perp),$$

$f(L) = ((S^2L^\perp, L^2), \iota(L))$ is a lifting of ι to $\mathcal{D}_2$.

We claim that f is holomorphic with respect to a complex structure on the period domain $\mathcal{D}_2$ introduced in Sec. II C.

Hence, the claim follows from the fact that the holomorphic tangent vector in B

$$S = \begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\left(\begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ \bar{a}_1 & \bar{a}_2 & 0 \end{pmatrix} - \sqrt{-1}\begin{pmatrix} 0 & 0 & ia_1 \\ 0 & 0 & ia_2 \\ -i\bar{a}_1 & -i\bar{a}_2 & 0 \end{pmatrix}\right) \in \mathfrak{b}^+$$

is mapped to $\iota_*(S)$,

$$\iota_*(S) = \begin{bmatrix} 0 & U \\ V & 0 \end{bmatrix}$$

as in the Proof of Lemma 3.1, where

$$U = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \\ 0 & 0 \\ \frac{a_2}{\sqrt{2}} & \frac{a_1}{\sqrt{2}} \end{pmatrix}, \; V = \begin{pmatrix} 0 & 0 & a_1 & 0 \\ 0 & 0 & a_2 & 0 \end{pmatrix}.$$

Here, we give another way to prove the liftability. Note that $\mathcal{D}_2$ can be identified with the open $SU(4,2)$ orbit in the homogeneous complex manifold $\hat{\mathcal{D}}$ of partial flags consisting of lines inside three-planes. The stabilizer of the partial flag is $S(U(3) \times U(1) \times U(2))$. There is an obvious holomorphic map F from $\mathbb{CP}^2$ to $\hat{\mathcal{D}}$, which associates the flag $l \odot l \subset l \odot \mathbb{C}^{2,1}$ to a line in $\mathbb{C}^{2,1}$. The restriction of this map to $H^2_{\mathbb{C}} \subset \mathbb{CP}^2$ is a holomorphic map. Furthermore, the projection from $\mathcal{D}_2$ to $\mathcal{X}$ is

$$l \odot l \subset l \odot \mathbb{C}^{2,1} \to l \odot l^{\perp},$$

and hence, $\iota = \pi \circ F$.

Now, we show the horizontality, i.e., the image lies in the form $L \odot L^{\perp}$. For any smooth curve in B, denote it by $L(t) = \langle v_0 + w(t) \rangle$, where $w(t) \subset v_0^{\perp}$, a differentiable family of lines, such that $w(0) = 0, w'(0) \in v_0^{\perp}$. Then, we can write $L(t)^{\perp} = \langle v(t) \rangle^{\perp}$, where $v(0) = v_0$, $v'(0) = w'(0) \in v_0^{\perp}$.

Since $L(t) \odot L(t)^{\perp}$ is already horizontal, it suffices to show the horizontality of $L(t)^2$ and $S^2(L(t)^{\perp})$. However,

$$L(t)^2 = \langle (v_0 + w(t)) \odot (v_0 + w(t)) \rangle = \langle v_0^2 + v_0 \odot w(t) + w(t)^2 \rangle.$$

Hence,

$$\frac{d}{dt}\Big|_{t=0} L(t)^2 = v_0 \odot w'(0) \in L(0) \odot L(0)^{\perp}.$$

Similar calculation shows that

$$\frac{d}{dt}\Big|_{t=0} S^2(L(t)^{\perp}) = \frac{d}{dt}\Big|_{t=0} \langle v(t)^{\perp} \odot v(t)^{\perp} \rangle$$

$$= \langle v'(0)^{\perp} \odot v_0^{\perp} \rangle \subset \langle v_0 \odot v_0^{\perp} \rangle \subset L(0) \odot L(0)^{\perp},$$

completing the proof.

IV. CHARACTER VARIETY $\chi(\Gamma, SU(n,2))$

Theorem 4.1. *There are at least seven distinct connected components in $\chi(\Gamma, SU(n,2))$, $n \geq 4$, where $\Gamma \subset SU(2,1)$ is a uniform lattice in $SU(2,1)$.*

Proof. We view $SU(4,2)$ as a subgroup of $SU(n,2)$. Consider first the holomorphic and totally real embeddings ρ and λ as in Sec. III A. Then, by Eqs. (3.1) and (3.4), $\rho^*\omega = 0, \lambda^*\omega = \frac{1}{4}\Omega_B^2$, whereas for the square representation ι, by Eq. (3.6), $\iota^*\omega = -\frac{3}{64}\Omega_B^2$. This implies that the quaternionic Toledo invariants are

$$\int_{\Gamma\backslash H^2_{\mathbb{C}}} \lambda^*\omega = \frac{1}{4}\int_{\Gamma\backslash H^2_{\mathbb{C}}} \Omega_B^2 = \frac{1}{4}\mathrm{vol}(\Gamma\backslash H^2_{\mathbb{C}}),$$

$$\int_{\Gamma\backslash H^2_{\mathbb{C}}} \iota^*\omega = -\frac{3}{64}\int_{\Gamma\backslash H^2_{\mathbb{C}}} \Omega_B^2 = -\frac{3}{64}\mathrm{vol}(\Gamma\backslash H^2_{\mathbb{C}}),$$

$$\int_{\Gamma\backslash H^2_{\mathbb{C}}} \rho^*\omega = 0,$$

respectively.

The last representation with a different Toledo invariant is given by the embedding $\phi : (z_1, \ldots, z_n) \to ((z_1, 0), \ldots, (z_n, 0))$. Then, the push-forward on holomorphic tangent vectors at $0 \in B$ is

$$\phi_* : x = (x_1, \ldots, x_n) \in \mathbb{C}^n \mapsto X = (x_1, 0, \ldots, x_n, 0) \in \mathbb{H}^{2n},$$

where $x_l = a_l + ib_l \in \mathbb{C}$ and on which the forms ω_j and ω_k vanish and

$$\omega(X, Y, Z, W) = \omega_i \wedge \omega_i(X, Y, Z, W)$$

$$= \mathrm{Re}(iX \cdot \bar{Y})\mathrm{Re}(iZ \cdot \bar{W}) - \mathrm{Re}(iX \cdot \bar{Z})\mathrm{Re}(iY \cdot \bar{W}) + \mathrm{Re}(iX \cdot \bar{W})\mathrm{Re}(iY \cdot \bar{Z})$$

$$= \mathrm{Re}(i\sum_{m=1}^{n} \bar{y}_m x_m)\mathrm{Re}(i\sum_{m=1}^{n} \bar{w}_m z_m) - \mathrm{Re}(i\sum_{m=1}^{n} \bar{z}_m x_m)\mathrm{Re}(i\sum_{m=1}^{n} \bar{w}_m y_m)$$

$$+\mathrm{Re}(i\sum_{m=1}^{n} \bar{w}_m x_m)\mathrm{Re}(i\sum_{m=1}^{n} \bar{z}_m y_m).$$

However,

$$\Omega_B^2(x, y, z, w) = 4\mathrm{Re}(i\sum_{m=1}^{n} \bar{y}_m x_m)4\mathrm{Re}(i\sum_{m=1}^{n} \bar{w}_m z_m)-$$

$$4\mathrm{Re}(i\sum_{m=1}^{n} \bar{z}_m x_m)4\mathrm{Re}(i\sum_{m=1}^{n} \bar{w}_m y_m) + 4\mathrm{Re}(i\sum_{m=1}^{n} \bar{w}_m x_m)4\mathrm{Re}(i\sum_{m=1}^{n} \bar{z}_m y_m)$$

$$= 16\omega(X, Y, Z, W).$$

Hence,

$$\phi^*\omega = \frac{1}{16}\Omega_B^2. \tag{4.1}$$

Since the quaternionic Toledo invariant is constant on each connected component, we get four different connect components.

If we take the complex conjugate of ϕ, we get the embedding

$$\bar{\phi} : (z_1, \ldots, z_n) \mapsto ((\bar{z}_1, 0), \ldots, (\bar{z}_n, 0)),$$

which entails

$$\omega(\bar{\phi}_*(x), \bar{\phi}_*(y), \bar{\phi}_*(z), \bar{\phi}_*(w)) = \mathrm{Re}(i\sum_{m=1}^{n} \bar{x}_m y_m)\mathrm{Re}(i\sum_{m=1}^{n} \bar{z}_m w_m)$$

$$-\mathrm{Re}(i\sum_{m=1}^{n} \bar{x}_m z_m)\mathrm{Re}(i\sum_{m=1}^{n} \bar{y}_m w_m) + \mathrm{Re}(i\sum_{i=1}^{n} \bar{x}_m w_m)\mathrm{Re}(i\sum_{m=1}^{n} \bar{y}_m z_m)$$

$$= -\frac{1}{16}\Omega_B^2(x, y, z, w).$$

This induces that

$$(\bar{\phi})^*\omega = -\frac{1}{16}\Omega_B^2.$$

The same calculation holds for the complex conjugates of λ and ι for Eqs. (3.4) and (3.6). By taking the complex conjugates of λ, ι and ϕ, we get seven components. This completes the proof. □

Remark 4.2. Note that the above proof works for a uniform lattice $\Gamma \subset SU(n, 1)$ *except the symmetric square representation. Hence, there are at least five distinct components in the character variety* $\chi(\Gamma, SU(2n, 2))$.

Note that for a lattice $\Gamma \subset SU(2, 1)$, the holomorphic embedding ρ corresponds to the diagonal embedding $\gamma \to (\gamma, \gamma) \in SU(2, 1) \times SU(2, 1) \subset SU(4, 2)$, and the totally real embedding corresponds to $\gamma \to (\gamma, \bar{\gamma})$, whereas the last example in the previous theorem corresponds to the embedding $\gamma \to (\gamma, id) \in SU(2, 1) \times SU(2, 1) \subset SU(4, 2)$.

In this direction, Toledo constructed the following examples.[20] There exist two complex hyperbolic surfaces $X = \Gamma\backslash H_{\mathbb{C}}^2$ and $Y = \Gamma'\backslash H_{\mathbb{C}}^2$ with a surjective holomorphic map $f : X \to Y$ with $0 < \deg(f) < \frac{\mathrm{vol}(X)}{\mathrm{vol}(Y)}$, which induces a group homomorphism $f_* : \Gamma \to \Gamma'$. See also Refs. 6

and 15 for the constructions of various subgroups $\Gamma' \subset \Gamma$ of a finite index. [The volumes vol(X) and vol(Y) can be further computed by using the Chern–Gauss–Bonnet theorem for orbifolds.] Consider the following representation:

$$\Gamma \xrightarrow{f_*} \Gamma' \xrightarrow{\phi} SU(4,2),$$

where ϕ is the restriction of the holomorphic embedding above. Then, the quaternionic Toledo invariant of this representation is

$$\int_X f^*(\phi^*\omega) = \int_X f^*(\frac{1}{16}\Omega_B^2) = \frac{1}{16}\deg(f)\mathrm{vol}(Y) < \frac{1}{16}\mathrm{vol}(X),$$

with $\frac{1}{16}$vol(X) being the smallest among the positive Toledo invariants in Theorem 4.1. We thus obtain an improvement of Theorem 4.1 in this case, viz.

Proposition 4.3. Let $\Gamma \subset \Gamma'$ be as above. There exist at least nine distinct components in $\chi(\Gamma, SU(4,2))$.

Some versions of local rigidity for the representations in some of the components above have been studied in Refs. 11 and 12.

V. MILNOR–WOOD INEQUALITY AND GLOBAL RIGIDITY FOR QUATERNIONIC TOLEDO INVARIANT

In this section, we show that if there exists a holomorphic horizontal lifting, then the Milnor–Wood type inequality holds with a quaternionic Kähler form. In this section, we normalize the metrics on $H^2_{\mathbb{C}}$ and on $\mathcal{X} = SU(2n,2)/S(U(2n)\times U(2))$ so that the holomorphic sectional curvatures are equal to -1.

Lemma 5.1. Let $\mathcal{D}$ be a period domain of $\mathcal{X}$ with a pseudo-Kähler metric such that it is negative definite on vertical directions and positive definite on horizontal directions. Its associated pseudo-Kähler form is $\hat{\Omega}$, which agrees with $\pi^*(\Omega)$ on the horizontal direction where the Kähler form on $\mathcal{X}$ is denoted as Ω. If $f: H^2_{\mathbb{C}} \to \mathcal{D}$ is a horizontal holomorphic map, then the Schwarz lemma holds, i.e., *$f^*(\hat{\Omega}) \le \Omega_B$, where Ω_B is the Kähler form on $H^2_{\mathbb{C}}$. Equality holds at every point if and only if f is a horizontal holomorphic geodesic embedding of $H^2_{\mathbb{C}}$ in $\mathcal{D}$.*

Proof. The proof is exactly the same as the one given in Theorem 3.3 in Ref. 7. The idea is as follows. First consider the case of a mapping from the hyperbolic plane $H^1_{\mathbb{C}}$, $f: H^1_{\mathbb{C}} \to \mathcal{D}$. If $f^*\hat{\Omega} = u\Omega_{B^1}$, then by the method of Sec. 2 of Chaps. I and III of Ref. 13, one can show that $u \le 1$. If equality holds at every point, then f is an isometric immersion. If M is the image and α is the second fundamental form, then since both holomorphic sectional curvatures are -1, one can show that $\alpha = 0$; consequently, f is a totally geodesic holomorphic embedding. For the $f: H^2_{\mathbb{C}} \to \mathcal{D}$ case, by considering all hyperbolic hyperbolic planes $H^1_{\mathbb{C}}$ in $H^2_{\mathbb{C}}$, one concludes that the second fundamental form vanishes, hence the totally geodesic embedding. ☐

Proposition 5.2. Let $M = \Gamma\backslash H^2_{\mathbb{C}}$. Suppose that $\rho: \Gamma \to SU(n,2)$ is a representation whose associated ρ-equivariant harmonic map $f: B \to \mathcal{X}$ lifts to a holomorphic horizontal map $\hat{f}$ to $\mathcal{D}$. Then, the Milnor–Wood type inequality holds. If equality holds, then it is a holomorphic embedding.

Proof. Since $H^4(M,\mathbb{R}) = \mathbb{R}$, the pull-backs of four-forms to M are all proportional to each other up to exact forms. Specifically,

$$f^*\omega = \hat{f}^*(\pi^*\omega) = c\hat{f}^*(\pi^*\Omega^2) + d\alpha = cf^*\Omega^2 + d\alpha.$$

A Kähler form $\hat{\Omega}$ of $\mathcal{D}$ agrees with $\pi^*(\Omega)$ on horizontal directions; hence, $\hat{f}^*(\hat{\Omega}) = \hat{f}^*(\pi^*\Omega) = f^*\Omega$. However, since $\hat{f}$ is holomorphic, by the Schwarz lemma,

$$f^*\Omega = \hat{f}^*(\pi^*\Omega) \le \Omega_B.$$

Hence,

$$\frac{1}{c}\int_M f^*\omega = \int_M f^*\Omega^2 \le \int_M \Omega_B^2 = vol(M).$$

Now, since we normalize ω so that its restriction to complex two-dimensional hyperbolic space is equal to Ω^2, we have $c = 1$ and $\hat{f}^*(\pi^*\omega) = \hat{f}^*(\pi^*\Omega^2) + d\alpha$, and consequently, the Milnor–Wood inequality

$$\int_M f^*\omega = \int_M f^*\Omega^2 \le vol(M).$$

Suppose that $\int_M f^*\omega = vol(M)$. Then, $f^*\Omega^2 = \Omega_B^2$ pointwise, which implies that f is a holomorphic embedding by the previous lemma. ☐

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