Superforms, supercurrents, minimal manifolds and Riemannian geometry

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RESEARCH CONTRIBUTION

Superforms, supercurrents, minimal manifolds and Riemannian geometry

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Abstract

Supercurrents, as introduced by Lagerberg, were mainly motivated as a way to study tropical varieties. Here we will associate a supercurrent to any smooth submanifold of $\mathbb{R}^n$. Positive supercurrents resemble positive currents in complex analysis, but depend on a choice of scalar product on $\mathbb{R}^n$ and reflect the induced Riemannian structure on the submanifold. In this way we can use techniques from complex analysis to study real submanifolds. We illustrate the idea by giving area estimates of minimal manifolds and a monotonicity property of the mean curvature flow. We also use the formalism to give a relatively short proof of Weyl's tube formula.

1 Introduction

A *superform* on $\mathbb{R}^n$ is defined as a differential form on $\mathbb{C}^n$ whose coefficients do not depend on the imaginary part of the variable. The dual of the space of superforms (with coefficients compactly supported in $\mathbb{R}^n$ and with the usual topology from the theory of distributions), is the space of *supercurrents*. Superforms and supercurrents were introduced by Lagerberg (2012), as a way to study tropical varieties. A tropical variety in $\mathbb{R}^n$ defines a d-closed, positive, supercurrent of integration, and conversely any such supercurrent defines a tropical variety, given a condition on the dimension of the support. (See also the work of Babaee (2014), for related work using standard currents on $(\mathbb{C}^*)^n$ instead of supercurrents.)

Here we extend the 'superformalism' in a different direction by associating to any smooth (or piecewise smooth) submanifold, M, of $\mathbb{R}^n$ a supercurrent, $[M]_s$, with the aim to apply methods from complex analysis to real manifolds. These supercurrents are d-closed only if the manifold is a linear subspace, but $d[M]_s$ is given by an explicit formula involving the second fundamental form of M. As a result, it turns out that M is minimal if and only if

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$$[M]_s \wedge \beta^{m-1}/(m-1)!$$

is closed, where m is the dimension of M and β is the Euclidean Kähler form on $\mathbb{C}^n$ (Corollary 5.2). Thus, minimality is characterized by a rather simple linear equation, which suggests a generalization of minimal manifolds to minimal 'supercurrents'. This is of course similar to the use of (classical) currents and varifolds in the theory of minimal manifolds, but has the extra feature of a bidegree, as in complex analysis. With this we can imitate Lelong's method for positive closed currents to prove e. g. the monotonicity formula for minimal manifolds, and a volume estimate that generalizes a recent result of Brendle and Hung (2017) (Theorem 6.3). We also obtain a result on removable singularities for minimal manifolds along the lines of the El Mir–Skoda theorem from complex analysis (Theorem 7.2), and a formula for the variation of the volume under the mean curvature flow (Theorem 8.2). (I take the opportunity to thank Duong Phong for suggesting to apply the formalism to the mean curvature flow.)

After that we give an expression of the Riemann curvature tensor of M as a superform. This is basically a rewrite of Gauß's formula. We apply it in the last section, to give a rather short proof of Weyl's tube theorem, Weyl (1939). The proof is in essence the same as Weyl's proof, but we have included it, hoping to show that the superformalism is useful in computations.

Finally I would like to thank an anonymous referee for a very careful reading of my manuscript, spotting many errors and suggesting several improvements.

2 Preliminaries

We start by recalling the definitions and basic properties of superforms and supercurrents, mainly following Lagerberg (2012) (and Berndtsson (2006)), but with some modifications. Let E be an n-dimensional vector space over $\mathbb{R}$. Thus E can be identified with $\mathbb{R}^n$, but at some points it will be convenient not to fix a basis. We define the 'superspace' of E to be

$$E_s = E \oplus E = E_0 \oplus E_1,$$

where we use the subscripts to indicate the first or second summand. A superform on E is a differential form on E_s that is invariant under translation in the E_1-variable. If $x = (x_1, \ldots x_n)$ and $\xi = (\xi_1, \ldots \xi_n)$ are coordinates on E_0 and E_1 respectively a superform can then be written

$$a = \sum a_{I,J}(x)dx_I \wedge d\xi_J, \tag{2.1}$$

where the coefficients do not depend on ξ. We say that a superform a has bidegree (p, q) if the lengths of the multiindices in (2.1) satisfy $|I| = p$ and $|J| = q$. (Notice that this is not the same bigrading as in the complex case.) With these conventions, a superform of bidegree $(0, 0)$ can be identified with a function on E, since it does not depend on ξ. So, a 'superfunction' on E_s is a function on E.

We also equip $T^*(E_s) = T^*(E_0) \oplus T^*(E_1)$ with a complex structure J, such that J maps $T^*(E_0)$ to $T^*(E_1)$ and vice versa. (Here our definitions differ from Lagerberg's, who considers instead maps from $T^*(E_0)$ to $T^*(E_1)$ satisfying $J^2 = 1$.) In the sequel we will only consider bases of E_0 and E_1 such that $J(dx_i) = d\xi_i$, and therefore $J(d\xi_i) = -dx_i$. We then extend J to act on forms of arbitrary bidegree so that $J(a \wedge b) = J(a) \wedge J(b)$. When a is of bidegree $(p, 0)$ we sometimes write $J(a) = a^{\#}$.

The ordinary exterior derivative of a is

$$da = \sum da_{I,J} \wedge dx_I \wedge d\xi_J = \sum \frac{\partial a_{I,J}}{\partial x_k} dx_k \wedge dx_I \wedge d\xi_J,$$

and we also define

$$d^{\#}a := \sum d^{\#}a_{I,J} \wedge dx_I \wedge d\xi_J = \sum \frac{\partial a_{I,J}}{\partial x_k} d\xi_k \wedge dx_I \wedge d\xi_J.$$

E_s is thus just the complexification of E and $d^{\#}$ is just $d^c = i(\bar{\partial} - \partial)$, but we write $d^{\#}$ to emphasize that it acts only on superforms, i. e. forms not depending on ξ. Note that we also have

$$d^{\#}a = (-1)^k J d J(a),$$

where $k = p + q$ is the total degree of a.

We also suppose given a scalar product on E and extend it to E_s so that it is invariant under J and $E_0 \perp E_1$. Thus, if dx_i are orthonormal, $d\xi_i = dx_i^{\#}$ are orthonormal on E_1 and $dx_i, d\xi_i$ are orthonormal on E_s.

The main point in the construction of E_s is the definition of integrals. If a is a form of maximal bidegree (n, n), we write $a = a_0 dx \wedge d\xi$ with $dx = dx_1 \wedge \ldots dx_n$, $d\xi = dx^{\#}$ and put

$$\int_{E_s} a = \int_{E_0} a_0 dx \int_{E_1} d\xi.$$

The integral over E_0 here is well defined as soon as we have chosen an orientation of E_0, if we assume that a_0 has enough decrease at infinity to make the integral convergent. For the integral with respect to ξ we *define*

$$c_n \int d\xi = 1$$

if dx_i are orthonormal and oriented, where $c_n = (-1)^{n(n-1)/2}$. Except for the constant c_n, this is the Berezin integral (Berezin 1987). The reason for introducing the factor c_n is that we want the integral of

$$a = a_0 dx_1 \wedge d\xi_1 \wedge \cdots dx_n \wedge d\xi_n \tag{2.2}$$

to be positive if a_0 is positive in accordance with the complex case. If a is given by (2.2) and dx_i are orthonormal, we get

$$\int_{\mathbb{R}_s^n} a = c_n \int_{\mathbb{R}_s^n} a_0 dx \wedge d\xi = \int a_0 dx \, c_n \int d\xi = \int a_0 d\lambda(x), \qquad (2.3)$$

where $d\lambda$ is Lebesgue measure. Note that if we change orientation, the 'superintegral' remains the same. This follows since the integrals with respect to x and ξ both change sign, or directly from (2.3).

Let us briefly compare superintegration to classical integration over the complexification. The first problem with classical integration is of course that the classical integral over E_s would always be divergent since a_0 does not depend on ξ. This could be overcome by replacing E_1 by its quotient by a lattice, so that we would replace E_s by $E_0 \times T^n$, where T^n is a torus. The reason that does not work here is that we will later want to integrate forms of lower bidegree over linear subspaces, and these subspaces do not in general correspond to subtori, unless the subspace satisfies a rationality condition that we cannot assume to be satisfied.

We next list a few rules of computation for superintegrals. First

$$\int_{\mathbb{R}_s^n} a = \int_{\mathbb{R}_s^n} J(a)$$

since $J(a) = a$ if a is of bidegree (n, n). Furthermore, if a is compactly supported of bidegree $(n-1, n)$,

$$\int_{\mathbb{R}_s^n} da = 0.$$

This means that we have the usual formula for integration by parts

$$\int_{\mathbb{R}_s^n} da \wedge b = (-1)^{k+1} \int_{\mathbb{R}_s^n} a \wedge db$$

if at least one of the forms a and b is compactly supported. Similarly, using $d^\# = \pm J d J$ and that the superintegral is J-invariant, we also have

$$\int_{\mathbb{R}_s^n} d^\# a = 0.$$

This means that the same integration by parts formula as for d also holds for $d^\#$. From this one verifies, for example, that if ρ is a function and S is of bidegree $(n-1, n-1)$, then

$$\int_{\mathbb{R}_s^n} \rho dd^\# S = \int_{\mathbb{R}_s^n} dd^\# \rho \wedge S. \qquad (2.4)$$

We can also define superintegrals over submanifolds of $\mathbb{R}^n$ if the form we integrate is of bidegree (p, n), i. e. is of maximal degree in $d\xi$. For instance, if D is a smoothly bounded domain in $\mathbb{R}^n$ and $a = \alpha \wedge d\xi$ is of bidegree $(n-1, n)$, then

$$\int_{\partial D} a := \int_{\partial D} \alpha \int_{E_1} d\xi = c_n \int_{\partial D} \alpha.$$

It thus follows directly that Stokes' formula holds:

$$\int_{\partial D} a = \int_{\mathbb{R}^n_s} \chi_D da =: \int_{D_s} da.$$

'Superintegration' over a submanifold of forms not of full degree in $d\xi$ is a different matter that will be discussed later in Sect. 4, but as a preparation for that we next discuss the case of linear subspaces. Let F be a linear subspace of $E = E_0$ of dimension m. Then its complexification is $F_s = F \oplus J(F)$, so F defines a superspace which is a complex linear subspace of E_s. Restricting a superform of bidegree (m, m) on E_s to F_s, we thus have a definition of the integral of a over F_s,

$$\int_{F_s} a$$

as before.

Having defined superforms and their integrals, we now turn to supercurrents. The space of supercurrents of bidegree (p, q), or *bidimension* $(n - p, n - q)$, is the dual of the space of smooth, compactly supported (in x !), superforms of bidegree $(n - p, n - q)$. Here we use the classical notion of duals from the theory of distributions, and we say that a supercurrent is of order zero if it is continuous for the uniform topology on superforms. Note that a supercurrent of bidegree (n, n) is a (classical) current of top degree on E, since superforms of bidegree $(0, 0)$ are functions on E. In particular, a supercurrent of top degree and order zero is a measure on E (not on E_s). As usual, given coordinates, a supercurrent can be written

$$T = \sum T_{I,J} dx_I \wedge d\xi_J,\tag{2.5}$$

where $T_{I,J}$ are distributions on E_0. Let us explain this notation a bit more. We think of a distribution as having bidegree $(0, 0)$ (a generalized *function*), i. e. as acting on forms of top degree. Then (2.5) means that if a is a test form of complementary bidegree

$$T.\alpha = \sum T_{I,J} . (a_{K,L} dx_I \wedge d\xi_J \wedge dx_K \wedge d\xi_L).$$

In particular, if $T_{I,J}$ are locally integrable functions, then

$$T.\alpha = \int_{\mathbb{R}^n_s} T \wedge \alpha.$$

In practice, for us, the coefficients will mostly be at worst measures. (The exception is Sect. 8 where we discuss the mean curvature flow.) If μ is a measure, it acts on functions, and so should be regarded as a current of bidegree (n, n). We define the corresponding object of bidegree $(0, 0)$, $*\mu$, by

$$*\mu.\alpha_0 dx_1 \wedge d\xi_1 \wedge \cdots dx_n \wedge d\xi_n = \mu.\alpha_0.$$

As an example of this we consider superintegration over a linear subspace F. Choose orthonormal coordinates so that

$$F = \{x_{m+1} = \cdots x_n = 0\}$$

and write an (m, m)-form as

$$a = a_0 dx_1 \wedge d\xi_1 \cdots dx_m \wedge d\xi_m + \cdots$$

Then the restriction of a to F_s is just $a_0 dx_1 \wedge d\xi_1 \cdots dx_m \wedge d\xi_m$, and

$$\int_{F_s} a = \int_F a_0 d\lambda_F$$

as before. Unwinding definitions, this means that the supercurrent defined by superintegration over F is

$$[F]_s := (*d\lambda_F) dx_{m+1} \wedge d\xi_{m+1} \cdots dx_n \wedge d\xi_m$$
$$= c_{n-m}(*d\lambda_F) dx_{m+1} \cdots \wedge dx_n \wedge d\xi_{m+1} \cdots \wedge d\xi_n.$$

Since the standard current of integration on F as a subspace of E is

$$[F] = (*d\lambda_F) dx_{m+1} \cdots \wedge dx_n,$$

we see that

$$[F]_s = c_{n-m}[F] \wedge d\xi_{m+1} \cdots \wedge d\xi_n.$$

Note that $d\xi_i = dx_i^\#$ where dx_i for $i = m + 1, \ldots n$ is a frame for the conormal bundle of F. In the next section we shall use this to define superintegration over general smooth submanifolds of $E = \mathbb{R}^n$.

An important point to notice is that whereas the standard current of integration is well defined without any extra structure, the supercurrent $[F]_s$ depends on the choice of scalar product.

Let T be a supercurrent (or superform) of bidegree (p, p),

$$T = \sum_{|I|=|J|=p} T_{IJ} dx_I \wedge d\xi_J.$$

Then

$$J(T) = \sum_{|I|=|J|=p} T_{IJ}d\xi_I \wedge dx_J(-1)^p = \sum_{|I|=|J|=p} T_{IJ}dx_J \wedge d\xi_I(-1)^{p+p^2}$$

$$= \sum_{|I|=|J|=p} T_{IJ}dx_J \wedge d\xi_I.$$

We therefore say that T is *symmetric* if $J(T) = T$, or equivalently $T_{IJ} = T_{JI}$.

In analogy with the complex case we now say that a symmetric supercurrent T of bidimension (m, m) is (weakly) positive if

$$T.\alpha_1 \wedge \alpha_1^{\#} \wedge \cdots \alpha_m \wedge \alpha_m^{\#} \geq 0$$

for any choice of compactly supported $(1, 0)$-forms α_j. It is then easily verified that the supercurrent of a linear subspace is symmetric and positive, and it is also immediately clear that it is d-closed.

Similarly, we say that a superform α of bidegree $(n-m, n-m)$ is (weakly) positive if

$$\alpha \wedge \alpha_1 \wedge \alpha_1^{\#} \wedge \cdots \alpha_m \wedge \alpha_m^{\#} \geq 0 \tag{2.6}$$

at every point. This is clearly equivalent to saying that the expression in (2.6) is positive at any point when α_j are constant. When

$$\alpha = \sum \alpha_{jk}dx_j \wedge d\xi_k$$

is of bidegree $(1, 1)$, this means that the matrix of coefficients (α_{jk}) is positive semidefinite. At any point, such a form can be written

$$\alpha = \sum \alpha_j \wedge \alpha_j^{\#}$$

so we get

Proposition 2.1 *The wedge product between a positive (symmetric) $(1, 1)$-form and a positive (symmetric) form of bidegree (p, p) is again positive.*

By approximation, the same thing holds for the product of $(1, 1)$-forms with currents of bidegree (p, p).

We next introduce the analog of the Kähler form in $\mathbb{R}^n$. This is by definition

$$\beta := \sum dx_j \wedge d\xi_j = (1/2)dd^{\#}|x|^2.$$

Then, the volume form on a linear subspace F can be written

$$d\lambda_F = [F]_s \wedge \beta^m/m!$$

in good analogy with the complex case.

In the coming sections we will frequently use contraction with a 1-form in the computations. This is well defined, since we have induced scalar products on the space of all forms, and we define contraction as the dual of exterior multiplication. If $a = \sum a_j dx_j$ is a $(1, 0)$-form, then $a \lrcorner \beta = a^\# = J(a)$, and $a^\# \lrcorner \beta = -a = J(a^\#)$. Therefore,

$$a \lrcorner \beta = J(a)$$

for any 1-form a.

Similarly, contraction with a vector field

$$\vec{V} = \sum V_j \frac{\partial}{\partial x_j}$$

is well defined. Lowering indices, $\vec{V}$ corresponds to the $(1, 0)$-form $V = \sum V_j dx_j$, and $\vec{V}$ and V act in the same way by contraction. Hence, e. g. $\vec{V} \lrcorner \beta = V^\#$.

Finally, we will discuss how superforms transform under diffeomorphisms: If G is a (local) diffeomorphism on $E = \mathbb{R}^n$ and $\alpha = \sum \alpha_{I,J} dx_I \wedge d\xi_J$ we define the pull back of α under G as

$$G^*(\alpha) = \sum G^*(\alpha_{I,J} dx_I) \wedge d\xi_J, \tag{2.7}$$

i. e. the pull back is the standard pull back on the x-part of the form and the $d\xi_j$'s are invariant. This means that we extend G to a (local) diffeomorphism on E_s by leaving E_1 fixed, and then take the usual pullback. Then

$$\int_{\mathbb{R}^n_s} G^*(\alpha) = \int_{\mathbb{R}^n_s} \alpha$$

if α is of bidegree (n, n) and G is orientation preserving.

Let now $\vec{V}$ be a vector field on $\mathbb{R}^n$ and let G_t be its flow, or one parameter family of diffeomorphisms. The classical formula of Cartan for the Lie derivative, Warner (1983), says that if α is of bidegree $(p, 0)$ then

$$\frac{dG_t^*(\alpha)}{dt}|_{t=0} = d\left(\vec{V} \lrcorner \alpha\right) + \vec{V} \lrcorner d\alpha =: L_{\vec{V}} \alpha, \tag{2.8}$$

where $\vec{V} \lrcorner$ means contraction with the field $\vec{V}$. This formula holds also for forms of general bidegree. Indeed, this follows immediately from Cartan's formula on the superspace E_s, if we extend $\vec{V}$ to a vector field on E_s that has no component in the second factor.

3 Relation to convex functions and tropical varieties

It is clear from the last part of the previous section that if ϕ is a smooth function on $\mathbb{R}^n$, then ϕ is convex if and only if $dd^{\#}\phi$ is a positive superform. By approximation, it follows that a general, possibly not smooth, function ϕ is convex if and only if $dd^{\#}\phi$ is a positive supercurrent. Conversely, if

$$\alpha = \sum \alpha_{jk} dx_j \wedge d\xi_k$$

is a symmetric closed $(1, 1)$ current, we can always write $\alpha = dd^{\#}\phi$ for some distribution ϕ. This is because

$$\frac{\partial \alpha_{jk}}{\partial x_l} = \frac{\partial \alpha_{lk}}{\partial x_j}$$

so

$$\alpha_{jk} = \frac{\partial \phi_k}{\partial x_j}$$

for some ϕ_k. By symmetry

$$\frac{\partial \phi_k}{\partial x_j} = \frac{\partial \phi_j}{\partial x_k}$$

so $\phi_k = (\partial \phi / \partial x_k)$ for some ϕ. If α is moreover positive, ϕ must be convex, so the positive symmetric $(1, 1)$-currents are precisely the ones that can be written $dd^{\#}\phi$ for some convex ϕ.

If ϕ is a smooth convex function, we can define $(dd^{\#}\phi)^n/n!$ which equals

$$c_n \det \left(\frac{\partial^2 \phi}{\partial x_j \partial x_k} \right) dx \wedge d\xi.$$

The corresponding measure on $\mathbb{R}^n$ is the Monge–Ampére measure of ϕ

$$MA(\phi) = \det \left(\frac{\partial^2 \phi}{\partial x_j \partial x_k} \right) dx.$$

By the Bedford-Taylor theory, Bedford and Taylor (1976), this definition makes sense for general convex functions. Indeed, Bedford and Taylor define $(dd^c\phi)^n/n!$ for general locally bounded plurisubharmonic functions, hence in particular for convex functions on $\mathbb{R}^n$, considered as functions on $\mathbb{C}^n$ that do not depend on the imaginary part of the variable. This means that for a general convex function on $\mathbb{R}^n$,

$$\left(dd^{\#}\phi \right)^n /n! = MA(\phi),$$

where the Monge–Ampére measure in the right hand side is taken in the sense of Alexandrov. This follows, since the two sides coincide for smooth functions and are continuous under uniform convergence.

(To be quite honest, Bedford and Taylor (1976) define $(dd^c\phi)^n$ as a measure on $\mathbb{C}^n$ and prove continuity under decreasing sequences in Bedford and Taylor (1982), hence also for uniform convergence when the functions are continuous. Since

$$(dd^\#\phi)^n/n! = \pi_*(\chi(\xi)(dd^c\phi)^n/n!)),$$

if χ is a function of ξ with integral 1 and π is the projection from $\mathbb{C}^n$ to $\mathbb{R}^n$, existence and continuity of $(dd^\#\phi)^n$ follows.)

Let us now consider convex functions of the form

$$\phi(x) = \max_{i \in I} a^i \cdot x + b^i = \max_{i \in I} l_i(x),$$

where I is a finite set. We will call such functions, i. e. the maxima of a finite collection of affine functions, quasitropical polynomials, reserving the term tropical polynomials for such functions where all components of the (co)vectors a^i and the numbers b^i are integers, see Mikhalkin (2006). We may assume that all the a^i are different. Indeed, if $a^i = a^k$ and say $b^i > b^k$, then $l_i > l_k$ everywhere, and we get the same function if we omit l_k.

Let $E_j = \{\phi = l_j\}$. Then E_j is defined by a finite set of linear inequalities. Assume that one of these sets, E_k has empty interior. Then we can define

$$\phi^k = \max_{i \neq k} l_i$$

and get a new quasitropical polynomial which equals ϕ on a dense set. Hence, by continuity, $\phi_k = \phi$ everywhere, so we may as well omit l_k in the definition of ϕ, and can assume that all E_j have non empty interior. We therefore get a decomposition of $\mathbb{R}^n$ as a finite union of non degenerate but possibly unbounded polyhedra. We then also have that all the E_j are different, since if $a^j \cdot x + b^j = a^k \cdot x + b^k$ on an open set, $a^j = a^k$ which we have assumed is not the case.

It is clear that

$$d^\#\phi = \sum \chi_{E_j}(a^j)^\#,$$

where χ_{E_j} is the characteristic function of the polyhedron E_j. Since $d\chi_{E_j} = [\partial E_j]$ for some choice of orientation, we get

$$dd^\#\phi = \sum [F_l] \wedge v_l^\#$$

for some $(1, 0)$ (constant) covectors v_l, where F_l is an enumeration of the faces of the polyhedron E_j. These v_l must be normal to F_l, because $dd^\#\phi$ is symmetric (see the next section for this). Positivity of $dd^\#\phi$ implies that they point in the same direction as the normal to F_l determining the orientation.

In this way $dd^{\#}\phi$ describes the *tropical variety* defined by the faces F_l, endowed with the multiplicity vectors v_l. (Perhaps it would be more proper to talk of quasitropical variety since the multiplicity vectors are not necessarily integral.) The fact that $dd^{\#}\phi$ is closed is equivalent to the *balancing condition* in tropical geometry: At a point where several faces intersect, the sum of their multiplicity vectors vanish (see Lagerberg 2012). Conversely, Lagerberg shows that a positive closed supercurrent of bidegree $(1, 1)$ with support of dimension $n - 1$ (see Lagerberg 2012 for precise, and also more general, statements) equals $dd^{\#}\phi$ for some quasitropical polynomial ϕ.

4 Supercurrents associated to general submanifolds of $\mathbb{R}^n$

Let M be a smooth submanifold of $\mathbb{R}^n$ of dimension m. Given an orientation of M we get the current of integration of M. Let us first assume that M is a hypersurface, locally defined by an equation $\rho = 0$, where ρ is smooth and has nonvanishing gradient on M. Dividing by $|d\rho|$, we may assume that $|d\rho| = 1$ on M, and we let $n = d\rho$; it is a unit normal form on M. Now it is a familiar fact that the current of integration on M can be written

$$[M] = n * dS_M$$

where dS_M is the surface measure on M and the Hodge star indicates that we think of it as a current of degree zero. The choice of sign of n determines the orientation of M. From this formula we see in particular that if $[M] \wedge v^{\#}$ is symmetric, then v must be a multiple of n, hence normal to M (we used this at the end of the last section, with $M = F_j$). Moreover, if $[M] \wedge v^{\#}$ is positive, v is a positive multiple of n.

Now we define the supercurrent $[M]_s$ by

$$[M]_s := [M] \wedge n^{\#} = n \wedge n^{\#} * dS_M.$$

It is clearly positive and symmetric. More generally, if M has codimension p, it is locally defined by p equations $\rho_j = 0$, such that $d\rho_j$ are linearly independent on M. Replacing ρ_j by $\sum a_{jk}\rho_k =: \rho'_j$, for a suitable matrix of functions a_{jk}, we may assume that $n_j := d\rho_j$ are orthonormal on M. Then the currents of integration on M can be written

$$[M] = n_1 \wedge \cdots n_p * dS_M =: n * dS_M; \quad n = n_1 \wedge \cdots n_p.$$

The supercurrent associated to M is defined as

$$[M]_s := c_p n \wedge n^{\#} * dS_M.$$

The definition uses the forms n_j that are only locally defined, but it is easily verified that a different choice n'_j leads to the same supercurrent, since n_j and n'_j are related on M by an orthogonal transformation. If α is a superform on the ambient space,

$$\alpha \wedge [M]_s$$

kills all the components of α in the directions n_j and $n_j^{\#}$; we will interpret it as the restriction of α to M_s; the 'superspace' associated to M. Accordingly, we say that α vanishes on M_s if (and only if) $\alpha \wedge [M]_s = 0$

In general, $[M]_s$ is not closed, unless M is linear. When computing $d[M]_s$ we will have use for the $(1, 1)$-forms

$$F := dn^{\#}, \quad F_j = dn_j^{\#}.$$

These forms depend on the choice of defining function(s), but transform in a natural way when we change defining functions. When the codimension $p = 1$, the restriction of F to M is the second fundamental form of M. In higher codimension the F_j restricted to M are the components of the vector valued second fundamental form $\sum F_j \otimes \vec{n}_j$ with values in the normal bundle of M. (Recall that n_j form an orthonormal system for the space of normal $(1, 0)$ forms at each point.)

Again, we start with the case $p = 1$. Then

$$d[M]_s = d([M] \wedge n^{\#}) = -F \wedge [M],$$

since $[M]$ is d-closed. This expresses $d[M]_s$ in terms of $[M]$ but we want to write it in terms of $[M]_s$. Therefore we introduce the operator

$$\mathcal{F} = F \otimes n^{\#} \lrcorner$$

which acts on a superform or supercurrent by first contracting with $n^{\#}$ and then wedging with F. Notice that $\mathcal{F}$ is an antiderivation. Then

$$d[M]_s = \mathcal{F}[M]_s. \tag{4.1}$$

In a similar way, when $p > 1$ we let

$$\mathcal{F} = \sum_j F_j \otimes n_j^{\#} \lrcorner,$$

and still get $d[M]_s = \mathcal{F}[M]_s$. In the same way we get that

$$d^{\#}[M]_s = -\mathcal{F}^{\#}[M]_s, \tag{4.2}$$

where

$$\mathcal{F}^{\#} = \sum_j F_j \otimes n_j \lrcorner.$$

Because of the following lemma, we can also write $\mathcal{F} = \sum_j n_j^{\#} \lrcorner \otimes F_j$, i. e. we can first wedge with F_j and then contract.

Lemma 4.1 $n_j \rfloor F_j = 0 = n_j^\# \rfloor F_j$ on M_s, i. e. when wedged with $[M]_s$.

Proof We prove this when $p = 1$. This is only to simplify the index notation; in the proof n_j denotes the components of n, i. e. the partial derivatives of ρ with respect to x_j. Then

$$n \rfloor F = \sum n_j \rho_{jk} d\xi_k = 1/2 \sum \frac{\partial n_j^2}{\partial x_k} d\xi_k = 1/2 d^\# |n|^2.$$

Since the norm of n is constant on M, this vanishes on M_s. □

5 Minimal submanifolds

We start with the following computational proposition.

Proposition 5.1 *If M is a smooth m-dimensional submanifold of $\mathbb{R}^n$*

$$d\left([M]_s \wedge \beta^p\right) = \sum_j n_j^\# \rfloor \left(F_j \wedge [M]_s \wedge \beta^p\right).$$

Proof We have, since $d\beta = 0$,

$$d\left([M]_s \wedge \beta^p\right) = \sum F_j \wedge n_j^\# \rfloor [M]_s \wedge \beta^p.$$

Since, by Lemma 4.1, $n_j^\# \rfloor (F_j) \wedge [M]_s = 0$, this equals

$$\sum n_j^\# \rfloor (F_j \wedge [M]_s) \wedge \beta^p.$$

Finally, $n_j^\# \rfloor \beta = -n_j$, which vanishes when wedged with $[M]_s$. This gives the formula in the proposition. □

If $F = \sum F_{ij} dx_i \wedge d\xi_j$ is a $(1, 1)$-form,

$$F \wedge \beta^{n-1}/(n-1)! = tr(F)\beta^n/n!,$$

where $tr(F) = \sum F_{ii}$ is the trace of F. This implies that

$$F \wedge [M]_s \wedge \beta^{m-1}/(m-1)! = tr'(F)[M]_s \wedge \beta^m/m!,$$

where $tr'(F)$ is the trace of F's restriction to M_s, since wedging with $[M]_s$ kills all the components of F and β in the normal directions. The traces $tr'(F_j) =: H_j$ are the coefficients of the *mean curvature vector*

$$\vec{H} := \sum H_j \vec{n}_j.$$

(There seem to be different conventions as to the sign of the mean curvature vector. We follow here the convention in Colding et al. (2015), so that the mean curvature of a sphere points outwards.)

Applying Proposition 5.1 with $p = m - 1$ we get

$$d([M] \wedge \beta^{m-1}/(m-1)!) = \sum H_j n_j^\# \rfloor ([M]_s \wedge \beta^m/m!). \qquad (5.1)$$

Since n_j are linearly independent, this vanishes exactly when $\vec{H}$ vanishes, i. e. when M is a *minimal manifold*, so we have proved

Corollary 5.2 $[M]_s \wedge \beta^{m-1}$ *is a closed current if and only if M is a minimal submanifold.*

We are therefore led to the following

Definition A positive symmetric supercurrent T of bidimension (m, m) is minimal if

$$dT \wedge \beta^{m-1} = 0.$$

$\square$

It is clear that this property is conserved by regularisation, e.g. by convolution with an approximate identity. For a general positive supercurrent of bidimension (m, m), if it happens that there is a vector field $\vec{V} = \sum V_j \partial/\partial x_j$ such that

$$dT \wedge \beta^{m-1}/(m-1)! = \sum V_j d\xi_j \rfloor (T \wedge \beta^m/m!),$$

we say that $\vec{V}$ is a mean curvature vector for T. As we have seen, this is the case when $T = [M]_s$ is associated to a smooth manifold, but it certainly also holds when T is strictly positive and smooth, so that $T \wedge \beta^m/m!$ is a strictly positive volume form. In both these cases, $\vec{V}$ is uniquely determined, so we may speak of *the* mean curvature vector. In Sect. 8 we shall see that when $T = [M]_s$ is associated to a smooth manifold,

$$dd^\#(T \wedge \beta^{m-1}/(m-1)!)$$

can be computed in terms of the flow of the mean curvature field.

We next discuss briefly the Dirichlet form on a symmetric and positive supercurrent of bidimension (m, m):

$$D(u, u) := \int_T du \wedge d^\# u \wedge \beta^{m-1}/(m-1)! = T.du \wedge d^\# u \wedge \beta^{m-1}/(m-1)!, \quad (5.2)$$

if u is sufficiently smooth on $\mathbb{R}^n$. When $T = [M]_s$ is the supercurrent associated to an m-dimensional manifold, this is precisely the standard Dirichlet form

$$\int_M |d_M u|^2 dS_M, \qquad (5.3)$$

where $d_M u$ is the differential of u restricted to M, and $|d_M u|$ is the norm induced by the Euclidean metric on $\mathbb{R}^n$. Writing $S_T = T \wedge \beta^{m-1}/(m-1)!$ we have in general

$$D(u, u) = \int du \wedge d^\# u \wedge S_T = -\int u d(d^\# u \wedge S_T). \qquad (5.4)$$

This makes it natural to define the Laplacian on T by

$$\Delta_T u = d(d^\# u \wedge S_T). \qquad (5.5)$$

With this definition, $\Delta_T u$ is a measure on $\mathbb{R}^n$. If T is the supercurrent of a manifold, or more generally has a mean curvature vector, we can make this more explicit: First we compute

$$\Delta_T u = dd^\# u \wedge S_T - d^\# u \wedge dS_T. \qquad (5.6)$$

If T has a mean curvature vector $\mathbf{H} = \sum H_j \partial/\partial x_j$, and we let $H^\# = \sum H_j d\xi_j$ be the corresponding $(0, 1)$- form, we have with $\sigma = T \wedge \beta^m/m!$,

$$dS_T = H^\# \lrcorner \sigma. \qquad (5.7)$$

Hence

$$d^\# u \wedge dS_T = (H^\# \lrcorner d^\# u)\sigma = \mathbf{H}(u)\sigma. \qquad (5.8)$$

Therefore

$$\Delta_T u = dd^\# u \wedge S_T - \mathbf{H}(u)\sigma. \qquad (5.9)$$

If T is minimal, so that S_T is closed and $\mathbf{H}$ vanishes, we have

$$\Delta_T u = dd^\# u \wedge S_T = dd^\# u \wedge \beta^{m-1}/(m-1)! \wedge T \qquad (5.10)$$

just like in the Kähler case. In particular, the Laplacian has no first order terms and linear functions are harmonic. For minimal manifolds this is a well known property, cf. Colding and Minicozzi (1999). In general, for T not necessarily minimal, we get by applying (5.9) to a coordinate function, $u = x_j$, that the components of $\mathbf{H}$, $H_j = \mathbf{H}(x_j)$ are given by

$$-H_j\sigma = \Delta_T x_j. \qquad (5.11)$$

(In particular, the mean curvature vector field is uniquely determined.)

6 Volume estimates for minimal submanifolds

To prove volume estimates for minimal manifolds (or supercurrents), we will now follow the method of Lelong to prove such estimates in the complex setting. This requires one little twist since the minimal supercurrent T (e.g. $T = [M]_s$) is not closed itself; it is only $T \wedge \beta^{m-1}$ that is closed. This is taken care of by the following lemma.

Lemma 6.1 *Let for $\delta \geq 0$, $|x|_\delta := (|x|^2 + \delta)^{1/2}$, and let for $p > 0$*

$$E_{p,\delta} := \frac{-1}{p} |x|_\delta^{-p},$$

and for $p = 0$

$$E_{0,\delta} = \log |x|_\delta.$$

Then, if p is an integer,

$$dd^\# E_{p,\delta} \wedge \beta^{p+1} = (dd^\# |x|_\delta)^{p+2}.$$

Proof This is a direct computation and we will do it for $p > 0$, the case $p = 0$ being similar but simpler. First,

$$d^\# E_{p,\delta} = |x|_\delta^{-(p+2)} (1/2) d^\# |x|^2$$

and

$$dd^\# E_{p,\delta} = |x|_\delta^{-(p+2)} \left(\beta - (p+2) \frac{\gamma}{|x|_\delta^2} \right), \tag{6.1}$$

with $\gamma = d|x|^2 \wedge d^\# |x|^2 / 4$. On the other hand

$$dd^\# |x|_\delta = |x|_\delta^{-1} \left(\beta - \frac{\gamma}{|x|_\delta^2} \right).$$

Expanding by the binomial theorem we get

$$\left(dd^\# |x|_\delta \right)^{p+2} = |x|_\delta^{-(p+2)} \left(\beta^{p+2} - (p+2) \frac{\gamma}{|x|_\delta^2} \wedge \beta^{p+1} \right). \tag{6.2}$$

The lemma follows from (6.1) and (6.2). $\qquad \square$

The reason we consider $E_{p,\delta}$ instead of $E_{p,0}$ is just that we want our functions to be smooth across zero. The main conclusion we draw from the lemma is that

$$dd^\# E_{m-2,\delta} \wedge \beta^{m-1} \wedge T$$

is positive if T is positive of bidegree (m, m). This follows from Proposition (2.1) since $|x|_\delta$ is convex. We also remark that it follows from the proof of the proposition that

$$dd^\# E_{p,\delta} \wedge \beta^{m-1} \wedge T \geq 0 \tag{6.3}$$

if $0 \leq p \leq m - 2$.

Let T be a minimal current of bidimension (m, m), defined in a neighbourhood of the origin. Its mass in a ball of radius r centered at the origin is

$$\sigma(r) := \int_{|x| < r} T \wedge \beta^m / m!.$$

In the computations below we first assume that T is smooth. Then, writing $S = T \wedge \beta^{m-1} / m!$

$$\sigma(r) = \int_{|x|=r} (1/2) d^\# |x|^2 \wedge S = \int_{|x|=r} |x|_\delta^m d^\# E_{m-2,\delta} \wedge S$$

$$= (r^2 + \delta)^{m/2} \int_{|x| < r} (dd^\# |x|_\delta)^m \wedge T / m!, \tag{6.4}$$

since $dS = 0$. Since the integrand in the right hand side is nonnegative it follows that

$$(r^2 + \delta)^{-(m/2)} \sigma(r)$$

is (weakly) increasing. This holds for any $\delta > 0$, so $r^{-m} \sigma(r)$ is also increasing. By approximation with smooth forms, this holds also for general minimal currents, so we have proved the following generalization of the *monotonicity theorem* for minimal manifolds (see Colding and Minicozzi 1999).

Theorem 6.2 *Let T be a minimal (super)current of bidimension (m, m) defined in a neighbourhood of the origin. Then*

$$r^{-m} \int_{|x| < r} T \wedge \beta^m / m!$$

is nondecreasing.

When T is the supercurrent of a minimal manifold, this says that the area of the manifold inside a ball of radius r, divided by r^{-m} is nondecreasing. In analogy with the case of minimal manifolds we call

$$\gamma_T(0) := \lim_{r \to 0} r^{-m} \int_{|x| < r} T \wedge \beta^m / m!,$$

the density of T at the origin. When T is the supercurrent of a (smooth) minimal manifold it equals ω_m, the volume of m-dimensional unit ball. (The corresponding

limit for closed positive (m, m)-currents in complex analysis is the *Lelong number* of the current.)

Let us now look again at the formula

$$\int_{|x|<r} T \wedge \beta^m /m! = (r^2 + \delta)^{m/2} \int_{|x|<r} T \wedge (dd^\# |x|_\delta)^m /m!.$$

We proved this for T smooth, but by approximation it holds for general minimal supercurrents of bidimension (m, m). It means in particular that the integral in the right hand side is bounded as $\delta \to 0$, so there is a subsequence of δ's such that

$$T \wedge (dd^\# |x|_\delta)^m /m!$$

converges weakly to a measure, μ. Outside the origin, where $|x|$ is smooth, this measure must be

$$T \wedge (dd^\# |x|)^m /m!.$$

We also see that μ must have a point mass at the origin of size $\gamma_T(0)$, since the mass of μ in any ball centered at the origin with small radius r is

$$r^{-m} \int_{|x|<r} T \wedge \beta^m /m!.$$

Thus, every subsequence has the same limit

$$\mu = \chi_{x \neq 0} T \wedge (dd^\# |x|)^m /m! + \gamma_T(0)\delta_0.$$

By Lemma 6.1 we also have

$$\lim_{\delta \to 0} T \wedge dd^\# E_{m-2,\delta} \wedge \beta^{m-1} /m! = \mu,$$

which can be interpreted as saying that the 'Laplacian' of $E_{m-2,0}$ on T (the trace of $dd^\# E_{m-2,0}$) equals a point mass at the origin of size γ_T, plus a nonnegative contribution outside the origin. The contribution outside the origin vanishes when T is the supercurrent of an m-dimensional plane through the origin. This reflects the fact that $E_{m-2,0}$ is a fundamental solution of the Laplacian then; the crucial observation is that $E_{m-2,0}$ is always 'subharmonic on T'.

We shall next generalize the proof of Theorem 6.2 to a general domain, D. Then $|x|$ is not constant on the boundary of D, so instead we write $|x|^m = w(x)$ on the boundary , where w is a positive smooth function on the closure of D to be chosen later. Let T be a minimal supercurrent (which we tacitly take as smooth at first) in a neighbourhood of $\bar{D}$. Then the mass of T in D is

$$\int_D T \wedge \beta^m /m! = \int_{\partial D} d^\# |x|^2 /2 \wedge S/m! = \int_{\partial D} w d^\# E_{m-2} \wedge S/m!.$$

(Here we skip the part of the argument where we approximate $|x|$ by $|x|_\delta$ and we write E_{m-2} for $E_{m-2,0}$.) By Stokes' theorem this equals

$$\int_D w dd^\# E_{m-2} \wedge S/m! + \int_D dw \wedge d^\# E_{m-2} \wedge S/m! =: I + II$$

since S is closed. By what we have just seen, $I \geq w(0)\gamma_T(0)$. To see when II is positive we compute (for $m > 2$)

$$0 = (m-2)^{-1} \int_{\partial D} \left(\frac{1}{|x|^{m-2}} - \frac{1}{w^{1-2/m}} \right) d^\# w \wedge S$$

$$= \frac{1}{m-2} \int_D \left(\frac{1}{|x|^{m-2}} - \frac{1}{w^{1-2/m}} \right) dd^\# w \wedge S + m^{-1} \int_D \frac{1}{w^{2-2/m}} dw \wedge d^\# w \wedge S$$

$$- \int_D dE_{m-2} \wedge d^\# w \wedge S. \tag{6.5}$$

Since

$$d\phi \wedge d^\# \psi \wedge S = d\psi \wedge d^\# \phi \wedge S$$

when S is symmetric we find that

$$II = \frac{1}{m-2} \int_D \left(\frac{1}{|x|^{m-2}} - \frac{1}{w^{1-2/m}} \right) dd^\# w \wedge S/m!$$

$$+ m^{-1} \int_D \frac{1}{w^{2-2/m}} dw \wedge d^\# w \wedge S/m!.$$

It follows that $II \geq 0$ if $w \geq |x|^m$ in D and w is convex.

Theorem 6.3 *Let D be a smoothly bounded domain in $\mathbb{R}^n$. Let a be any point in D and let w be a convex function on $\bar{D}$ such that $w = |x - a|^m$ on the boundary of D. If T is a minimal supercurrent in D of bidimension (m, m), then its mass satisfies*

$$\int_D T \wedge \beta^m/m! \geq w(a)\gamma_T(a)$$

(where $\gamma_T(a)$ is the density of T at a).

(We shall see in the next section that the convexity assumption on w can be relaxed considerably.)

In the proof we may of course assume that $a = 0$. For $m > 2$ the theorem follows immediately from the argument above, since we can always assume that $w \geq |x|^m$, replacing it if necessary by $\max(w, |x|^m)$. The case $m = 2$ is similar; we just have to replace the boundary integral in (6.5) by

$$\int_{\partial D} \log \frac{w}{|x|^2} d^\# w \wedge S.$$

As a special case we get a recent result of Brendle and Hung (2017), which generalizes an older estimate of Alexander and Osserman for the case $m = 2$ (Alexander and Osserman 1975).

Corollary 6.4 *Let M be a minimal manifold of dimension m in the unit ball which contains the point a. Then the volume of M satisfies*

$$|M| \geq \omega_m (1 - |a|^2)^{m/2},$$

where ω_m is the volume of the m-dimensional unit ball.

Proof We have

$$|x - a|^2 = 1 + |a|^2 - 2a \cdot x =: v$$

on the boundary of the ball. Since v is convex (in fact linear), $w := v^{m/2}$ is also convex and the corollary follows directly from the theorem, since $\gamma_T(a) = \omega_m$ when T is the supercurrent of a (smooth) manifold. □

7 Removable singularities

The techniques of the previous section can also be used to give a variant of the El Mir–Skoda theorem on extension of positive closed currents (El Mir 1984; Skoda 1982) in the setting of minimal manifolds or minimal currents. It should be stressed that here we are not aiming to prove that a minimal surface with possible singularities on a small set is actually smooth, but rather that the minimal current defined by the surface outside the singularities extends as a minimal current. This is a much simpler problem than regularity, but the results are still useful as they will allow us to apply the calculus of supercurrents to manifolds with singularities on a set of sufficiently small dimension. We refer to Harvey and Lawson (1975) for a discussion of and very precise results on removable singularities in the classical setting.

We first define a locally integrable function u to be *m-subharmonic* if

$$dd^\# u \wedge \beta^{m-1}$$

is positive. This condition is preserved under regularization (convolution with an approximate identity) so we can approximate general m-subharmonic functions with smooth ones. If u is smooth and we fix a point we can make an orthogonal change of coordinates so that

$$dd^\# u = \sum \lambda_j dx_j \wedge d\xi_j$$

at that point. We also have that

$$\beta^{m-1}/(m-1)! = \sum_{|I|=m-1} dV_I,$$

where

$$dV_I = dx_{i_1} \wedge d\xi_{i_1} \wedge \cdots dx_{i_{m-1}} \wedge d\xi_{i_{m-1}},$$

for $I = (i_1, \ldots i_{m-1})$. Thus, at the chosen point,

$$dd^\# u \wedge \beta^{m-1}/(m-1)! = \sum_{|J|=m} \lambda_J dV_J, \qquad (7.1)$$

where $\lambda_J = \sum_{i \in J} \lambda_i$. In conclusion, a smooth function u is m-subharmonic if and only if the sum of any m-tuple of eigenvalues of $dd^\# u$ is nonnegative at any point. We also see from (7.1) that if u is smooth and m-subharmonic, then

$$dd^\# u \wedge \beta^{m-1} \wedge T \geq 0$$

if T is a positive supercurrent of bidegree $(n - m, n - m)$.

Lemma 6.1 implies that $E_{m-2,\delta}$ is m-subharmonic, and taking limits when $\delta \to 0$ the same thing holds for E_{m-2}. Therefore, any potential

$$u(x) := \int \frac{-1}{|x - y|^{m-2}} d\mu(y) \qquad (7.2)$$

for $m > 2$ and

$$u(x) := \int \log |x - y| d\mu(y)$$

when $m = 2$ is also m-subharmonic, if μ is a positive measure.

Kernels like E_{m-2} on $\mathbb{R}^n$ are called Riesz kernels, and have a well developed potential theory, following the lines of the more classical potential theory for the Newtonian kernel E_{n-2}, see Landkof (1972). Thus, we have a notion of capacity C_{m-2} associated to E_{m-2} and any set of sigma-finite $(m - 2)$-dimensional Hausdorff measure has capacity zero.

We also say that a set F is m-polar is there is an m-subharmonic function which is equal to $-\infty$ on F (and maybe elsewhere as well). By Landkof (1972), any compact set K with $C_{m-2}(K) = 0$ is m-polar, and moreover there is a measure μ supported on K whose potential equals $-\infty$ precisely on K. Therefore we get

Proposition 7.1 *Any compact set K of σ-finite $(m-2)$-dimensional Hausdorff measure is m-polar and there is a potential u of a measure supported on K which equals $-\infty$ on K.*

To illustrate this, notice that it is immediate that a discrete set of points is 2-polar, and that a submanifold of dimension $(m - 2)$ is m-polar in general. Indeed, it suffices to take μ equal to surface measure.

Theorem 7.2 *Let T be a minimal supercurrent of bidimension (m, m), defined in $B \setminus K$, where B is a ball and K and is compact in $\mathbb{R}^n$ with sigma-finite $(m - 2)$-dimensional Hausdorff measure. Assume that T has finite mass*

$$\int_{B \setminus K} T \wedge \beta^m < \infty.$$

Then the trivial extension of T, $\tilde{T} := \chi_{B \setminus K} T$, is a minimal supercurrent in B.

Proof We follow almost verbatim the proof of Proposition 11.1 in Berndtsson and Sibony (2002). Let u be a potential as in (7.2) with μ supported on K, which equals $-\infty$ on K. Thus, u is smooth outside K and m-subharmonic. Let $\chi(t)$ be an increasing smooth convex function, defined when $t \leq 0$, with $\chi(0) = 1$ and $\chi(t) = 0$ for $t \leq -1$. Put

$$u_k = \chi(u/k)$$

for $k = 1, 2, \ldots$. Then u_k are smooth, m-subharmonic, $0 \leq u_k < 1$, and u_k tend to 1 uniformly on compacts outside K. Moreover, all u_k vanish in a neighbourhood of K.

Let θ be a smooth function with compact support in B. Then, by integration by parts,

$$\int_B \theta T \wedge \beta^{m-1} \wedge dd^{\#} u_k^2 = \int_B u_k^2 dd^{\#} \theta \wedge T \wedge \beta^{m-1} \leq C \qquad (7.3)$$

where C is a fixed constant independent of k. Here we have used that $T \wedge \beta^{m-1}$ is closed outside of K, and that u_k vanishes near K. This implies that

$$\int_B \theta T \wedge \beta^{m-1} \wedge du_k \wedge d^{\#} u_k \leq C,$$

since

$$dd^{\#} u_k^2 = 2u_k dd^{\#} u_k + 2du_k \wedge d^{\#} u_k,$$

and $dd^{\#} u_k \wedge \beta^{m-1} \geq 0$ for u_k m-subharmonic.

Let $p(x)$ be a smooth function on $\mathbb{R}$, such that $p(x) = 1$ if $x > 1/2$ and $p(x) = 0$ if $x < 1/3$, and put $\chi_k = p(u_k)$. Then χ_k tends to $\chi_{B \setminus K}$ and we get if ψ is a smooth $(1, 0)$-form supported in B,

$$\int_B d\left(\chi_{B \setminus K} T \wedge \beta^{m-1}\right) \wedge \psi^{\#} = \lim_{k \to \infty} \int_B d\left(\chi_k T \wedge \beta^{m-1}\right) \wedge \psi^{\#}$$

$$= \lim_{k \to \infty} \int_B p'(u_k) du_k \wedge T \wedge \beta^{m-1} \wedge \psi^{\#}.$$

The integral

$$\int_{B\setminus K} \alpha \wedge \tau^{\#} \wedge T \wedge \beta^{m-1} =: (\alpha, \tau)$$

is a positive semidefinite bilinear form on the space of smooth $(1, 0)$-forms in B. By Cauchy's inequality we have, choosing $\alpha = du_k$ and $\tau = p'(u_k)\psi$ that

$$\int_B p'(u_k)du_k \wedge T \wedge \beta^{m-1} \wedge \psi^{\#}$$
$$\leq \sqrt{\int_{B\setminus K} du_k \wedge d^{\#}u_k \wedge T \wedge \beta^{m-1} \int_{B\setminus K} (p'(u_k))^2 \psi \wedge \psi^{\#} \wedge T \wedge \beta^{m-1}}.$$

(It does not matter if we integrate over B or $B \setminus K$ in the left hand side since du_k and $p'(u_k)$ vanish near K.) By (7.3), the first integral on the right hand side is bounded independently of k. The second integral in the right hand tends to zero by dominated convergence since $p'(u_k)$ tends to zero on $B \setminus K$. Hence

$$\lim_{k \to \infty} \int_B p'(u_k)du_k \wedge T \wedge \beta^{m-1} \wedge \psi^{\#} = 0.$$

Therefore $\chi_{B\setminus K} T \wedge \beta^{m-1}$ is d-closed, so we are done.   □

One consequence of this is that Corollary 6.4 also holds for minimal manifolds with singularities. More precisely, by the theorem, the extension $\tilde{T}$ has a density also at points in K. It is easy to see, by the monotonicity theorem, that

$$\gamma_{\tilde{T}}(a) \geq \limsup_{b \to a} \gamma_{\tilde{T}}(b).$$

If T is associated with a minimal manifold of dimension m that has singularities at the set E the density is ω_m (the volume of an m-dimensional unit ball) at all the regular points. Therefore it is at least ω_m at the singular points.

(A natural question in this context seems to be if minimal manifolds can be characterized within the class of minimal supercurrents by conditions on the density. For instance, if T is minimal and the density is constant on the support of T, is T then $c[M]_s$ for a smooth minimal manifold M ?)

We also note that the assumption that w be convex in Theorem 6.3 can be relaxed to m-subharmonic, since we just need that

$$dd^{\#}w \wedge S = dd^{\#}w \wedge T \wedge \beta^{m-1}$$

be nonnegative.

8 Variation of volume and the mean curvature flow

We have seen in formula (5.1), that if M is a smooth manifold of dimension m, then

$$d\left([M]_s \wedge \beta^{m-1}/(m-1)!\right) = \sum H_j n_j^{\#} \rfloor \left([M]_s \wedge \beta^m/m!\right),\qquad(8.1)$$

where H_j are the components of the mean curvature vector. In this section we will use the notation $\beta_m = \beta^m/m!$,

$$S = [M]_s \wedge \beta_{m-1},$$

and

$$\sigma = [M]_s \wedge \beta_m.$$

Then formula (8.1) says that

$$dS = H^{\#} \rfloor \sigma.\qquad(8.2)$$

Since S is J-invariant and of even degree it follows after a small computation that

$$d^{\#}S = JdS = -H \rfloor \sigma.\qquad(8.3)$$

Our main objective in this section is to give a formula for the derivative of the volume form

$$\sigma_M := [M]_s \wedge \beta^m/m!$$

when M moves under *the mean curvature flow*, see Colding et al. (2015), Smoczyk (1999). As in these references, we say that a family M_t of smooth submanifolds of $\mathbb{R}^n$ moves by the mean curvature flow if there is a vector field $\vec{H}$ on the ambient space such that $F_t(M_0) = M_t$, where F_t is the flow of $-\vec{H}$, which restricts to the mean curvature vector field of M_t on each M_t.

Again we point out that we use here the same conventions about the sign of the mean curvature vector as in Colding et al. (2015), which seems to be the opposite of the one in Smoczyk (1999). To make matters worse, the condition

$$F_t(M_0) = M_t,$$

means in terms of pullbacks of currents that

$$F_{-t}^*([M_0]) = [M_t],$$

so since we will work with pullbacks, we are dealing with the flow G_t of the vector field $+\vec{H}$ after all. Recalling Cartan's formula for the Lie derivative (cf. Sect. 2) we get

$$\frac{d}{dt}|_{t=0}[M_t] = L_{\vec{H}}[M] = d\left(\vec{H}\rfloor[M]\right) = d\left(H\rfloor[M]\right),\tag{8.4}$$

since $[M]$ is closed.

We first note that the next proposition follows directly from (8.3) and Cartan's formula.

Proposition 8.1

$$\frac{dF_{-t}^*(\sigma)}{dt}|_{t=0} = -dd^\# S.$$

We now let $\sigma_t = [M_t]_s \wedge \beta_m$; the volume form of M_t. (Note that this is not the same thing as $F_{-t}^*(\sigma)$ which has total mass independent of t, whereas the mass of M_t changes.) The next theorem is the main result of this section.

Theorem 8.2 *Let M_t be a family of smooth m-dimensional submanifolds of $\mathbb{R}^n$, moving under the mean curvature flow, with $M_0 = M$. Let*

$$\sigma_t = \sigma_{M_t} = [M_t]_s \wedge \beta_m$$

be their volume forms. Then

$$\frac{d\sigma_t}{dt}|_{t=0} = -|\vec{H}|^2\sigma - dd^\# S.\tag{8.5}$$

We will deduce the theorem from the proposition. Note that

$$F_{-t}^*(\sigma) = F_{-t}^*([M_0]_s) \wedge F_{-t}^*(\beta_m).$$

We now differentiate this with respect to t at $t = 0$ and use the proposition in the left hand side. The result is

$$-dd^\# S = \frac{d\sigma_t}{dt}|_{t=0} + [M_0]_s \wedge \frac{d}{dt}|_{t=0}F_{-t}^*(\beta_m).\tag{8.6}$$

By Cartan's formula

$$\frac{d}{dt}|_{t=0}F_{-t}^*(\beta_m) = d(H\rfloor\beta) \wedge \beta_{m-1} = dH^\# \wedge \beta_{m-1}.$$

Hence

$$[M_0]_s \wedge \frac{d}{dt}|_{t=0}F_{-t}^*(\beta_m) = (dH^\#) \wedge [M_0]_s \wedge \beta_{m-1} = d(H^\# \wedge S) + H^\# \wedge dS.$$

But

$$H^\# \wedge S = 0$$

since $H^\# = \sum H_j n_j^\#$ and S contains a factor $n^\# = n_1^\# \wedge \ldots n_p^\#$. Moreover, by (8.2),

$$H^\# \wedge dS = H^\# \wedge (H^\# \lrcorner \sigma) = |H|^2 \sigma.$$

Thus

$$[M_0]_s \wedge \frac{d}{dt}|_{t=0} F^*_{-t}(\beta_m) = |H|^2 \sigma,$$

and inserting this in (8.6) the theorem follows.

Integrating (8.5) against a smooth function ρ, and using (2.4), we obtain

Corollary 8.3 *Let M_t be a family of compact manifolds of dimension m, moving under the mean curvature flow. Then, if ρ is a sufficiently smooth function*

$$(d/dt)|_{t=0} \int \rho \sigma_t = -\int \rho |\vec{H}|^2 \sigma - \int dd^\# \rho \wedge S.$$

As a consequence, if ρ is nonnegative and convex (or, more generally, m-subharmonic), then

$$\int \rho \sigma_t$$

is decreasing.

To understand the intuitive meaning of the corollary, let first ρ be constant equal to 1. Then we get the well known fact that the volume of M decreases under the mean curvature flow. When ρ is nonconstant, convex and nonnegative we get an extra negative contribution to the derivative of the integral of ρ over M_t, so, roughly speaking, M_t tends to move towards where ρ is small. The next consequence is a more precise formulation of this.

Corollary 8.4 *Under the same hypotheses as in Theorem 8.2, assume also that $M = M_0$ lies in a convex open set Ω. Then M_t lies in Ω for $t > 0$ as long as the mean curvature flow exists.*

This follows since we may choose ρ to be zero in the convex set and strictly positive outside. Gerhard Huisken has kindly informed me that this is a well known property that follows from the parabolic maximum principle, using the formula for the mean curvature field in formula (5.11).

An open set is convex if it has a convex defining function, i.e. if there is a nonnegative convex function ρ, such that $\Omega = \{\rho < 1\}$. Analogously we could define m-convex domains as domains that possess an m-subharmonic defining function. (Recall that we have defined ρ to be m-subharmonic if $dd^\# \rho \wedge \beta_{m-1} \geq 0$.) Then the last corollary also holds if Ω is only m-convex, where m is the dimension of M. This condition becomes weaker when the dimension is large: For curves we need convexity, but for hypersurfaces $(n-1)$-convexity is enough.

Formula (8.5) reflects the fact that the mean curvature flow is described by a (non linear) parabolic equation. It says that the volume forms then flow by a (linear) parabolic equation. It seems to be a natural question if there is a corresponding equation for the supercurrents $[M_t]_s$ themselves, i. e. without taking traces as we have done here. Since the minimal surface equation becomes linear in the supercurrents formalism, this is perhaps not completely unrealistic.

9 General submanifolds and their Riemannian geometry

In this section we shall describe the Levi-Civita connection and Riemannian curvature of a submanifold M of $\mathbb{R}^n$ in terms of the 'superstructure'.

Let M be a smooth submanifold of $\mathbb{R}^n$ and let $[M]_s$ be its associated supercurrent. With the notation from Sect. 3 we have

$$d[M]_s = \mathcal{F}[M]_s, \quad d^{\#}[M]_s = -\mathcal{F}^{\#}[M]_s.$$

Motivated by these formulas we introduce the operators

$$D := d - \mathcal{F}, \quad D^{\#} : d^{\#} + \mathcal{F}^{\#}$$

and get

$$D[M]_s = D^{\#}[M]_s = 0.$$

Notice that D and $D^{\#}$ are antiderivations, so they satisfy the same rules of computation as d and $d^{\#}$. If a is a superform on the ambient space we get

$$D(a \wedge [M]_s) = (Da) \wedge [M]_s, \quad D^{\#}(a \wedge [M]_s) = (D^{\#}a) \wedge [M]_s.$$

In particular, if a vanishes on $[M]_s$, i.e. $a \wedge [M]_s = 0$, then Da and $D^{\#}a$ also vanish on $[M]_s$, which means that D and $D^{\#}$ are well defined as operators on forms on $[M]_s$.

Since $\mathcal{F}$ involves contraction with $n_j^{\#}$, it vanishes on any form of bidegree $(p, 0)$. Hence $D = d$ on such forms and we have just retrieved the fact that the exterior derivative is well defined on submanifolds. On the other hand, $\mathcal{F}^{\#}$ involves contraction with n_j and does not vanish on forms of bidegree $(p, 0)$, so $d^{\#}$ and $D^{\#}$ are different on $(p, 0)$ forms. In fact, $D^{\#}a$ can be thought of as the Levi-Civita connection acting on a. Indeed, $D^{\#}a$ is a form of bidegree $(p, 1)$, so it can be seen as a 1-form in ξ with values in the space of $(p, 0)$-forms in x. If $V = \sum V_j \partial/\partial x_j$ is a tangent vector to M, the derivative of a in the direction V is

$$V^{\#} \lrcorner D^{\#}a,$$

where $V^{\#} = \sum V_j \partial/\partial \xi_j$. It is not hard to verify that this is the Levi-Civita connection.

To compute its curvature we need to apply $D^\#$ twice. This is done in the following proposition which is just a way of writing Gauß's formula for the curvature in supernotation (see also Weyl (1939)).

Proposition 9.1 *The curvature tensor R on M is given by the symmetric $(2, 2)$ form*

$$R = (1/2) \sum F_j \wedge F_j,$$

in the sense that, if a is of bidegree $(1, 0)$

$$(D^\#)^2 a = a \lfloor (1/2) \sum F_j \wedge F_j.$$

If a is a form of bidegree (p, q) on M_s we can always extend it to a (p, q)-form on the ambient space. It is convenient to choose a special extension. We say that the extension is canonical if $n_j \lfloor a = n_j^\# \lfloor a = 0$ for $j = 1, \ldots p$. It is obvious that canonical extensions exist locally since $n_j = d\rho_j$ are linearly independent and can be completed to a basis for the $(1, 0)$ forms. We then just take any extension, write it in terms of the new basis, and throw away the terms that contain some n_j or $n_j^\#$. The resulting form coincides with a on M_s in the sense that its wedge product with $[M]_s$ equals $a \wedge [M]_s$.

If a is a canonical extension of a form on M_s, then $\mathcal{F}a = \mathcal{F}^\# a = 0$, so

$$D^\# a = d^\# a.$$

Thus

$$(D^\#)^2 a = \mathcal{F}^\# d^\# a = \sum F_j \wedge n_j \lfloor d^\# a.$$

To continue, we introduce the notation that if $F = \sum F_{ij} dx_i \wedge d\xi_j$ is a $(1, 1)$ form (on $\mathbb{R}_s^n$) and a is (p, q) form, then

$$F \cup a = - \sum F_{ij} d\xi_j \wedge (dx_i \lfloor a).$$

Notice that in the particular case when a has bidegree $(1, 0)$

$$F \cup a = -a \lfloor F.$$

Lemma 9.2 *If $F = d\theta^\#$ (where θ is $(1, 0)$), then we have the commutator formula*

$$[d^\#, \theta \lfloor] a := d^\# \theta \lfloor a + \theta \lfloor d^\# a = -F \cup a$$

where a is any superform (canonical or not).

Proof First we note that the commutator is a scalar operator, so that

$$[d^\#, \theta \lfloor] f a = f [d^\#, \theta \lfloor] a$$

if f is a function. Therefore we may assume that $a = dx_I \wedge d\xi_J$. Then $d^{\#}a = 0$ and

$$d^{\#}\theta \rfloor a = \sum \frac{\partial \theta_i}{\partial x_j} d\xi_j \wedge (dx_i \rfloor a),$$

which proves the formula. □

The lemma gives that

$$(D^{\#})^2 a = -\sum F_j \wedge F_j \cup a,$$

for a of any bidegree. Combined with the remark immediately before the lemma, this gives that

$$(D^{\#})^2 a = a \rfloor (1/2) \sum F_j \wedge F_j$$

if a is $(1, 0)$, which concludes the proof of the proposition.

From Proposition 9.1 we see that integrals like

$$\int_{[M]_s} \chi \left(\sum F_j \wedge F_j \right)^q \wedge \beta^{m-2q},$$

where χ is a function on M are intrinsic, i. e. they do not depend on the embedding of M in $\mathbb{R}^n$, since the Riemann curvature (and the metric) are intrinsic. (This is of course not true for similar integrals containing arbitrary combinations of F_j.) In the next section we shall give an illustration of this.

10 Weyl's tube formula

We will now use the formalism of the previous section to give a quick proof of Weyl's tube formula (Weyl 1922; Gray 1990). The proof is not really different from the original proof, but the formalism helps to organise the formulas.

Let M be a compact submanifold of $\mathbb{R}^n$. We will consider $T_r(M)$, the tube around M of width r, which when M is without boundary is defined as

$$T_r(M) = \left\{ x \in \mathbb{R}^n; d(x, M) < r \right\}.$$

When r is sufficiently small, $T_r(M)$ is by the tubular neighbourhood theorem diffeomorphic to a neighbourhood of the zero section of the normal bundle $N(M)$ of M,

$$\Delta(M, r) := \{ v \in N(M); |v| < r \}.$$

A point v in the normal bundle is a vector, normal to $T_p(M)$ where $p = \pi(v)$, π being the projection from $N(M)$ to M, and the diffeomorphism is

$$G : \Delta(M, r) \to \mathbb{R}^n, \quad G(v) = \pi(v) + v.$$

When M is a compact manifold *with* boundary, we *define* $T_r(M)$ to be the image of $\Delta(M, r)$ under this map. Weyl's tube formula expresses the volume of these tubes, $|T_r(M)|$, in terms of curvature integrals over the manifold.

Theorem 10.1 *Let M be a compact m-dimensional submanifold of $\mathbb{R}^n$ and let $T_r(M)$ be the tube of width r around M. Then, for r small*

$$|T_r(M)| = \sum_{2q \le m} c_{2q} r^{2q+p} \int_{[M]_s} R^q \wedge \beta^{m-2q}$$

for some constants c_{2q} that can be explicitly calculated, where $p = n - m$ is the codimension of M.

Thus the theorem says first that the volume of the tubes of width r is a polynomial in r for r small, and moreover and most remarkably that the coefficients of the polynomial are intrinsic. Thus, if we have two isometric embeddings of the Riemannian manifold M into $\mathbb{R}^n$, the tube volumes are the same.

For the proof, we have first

$$|T_r(M)| = \int_{[T_r(M)]_s} \left(\sum dx_k \wedge d\xi_k \right)^n /n!.$$

It is enough to prove the theorem locally; i. e. we may assume that the normal forms $n_j = d\rho_j$ of Sect. 4 are defined on all of M. Then the normal bundle is trivial and isometric to $M \times B_r(0)$, where $B_r(0)$ is the ball in $\mathbb{R}^n$ of radius r and center 0, via the orthonormal frame $\vec{n}_j$. Hence the diffeomorphism G described above can be written

$$G : M \times B_r(0) \to T_r(M),$$

with

$$G(y, t) = y + \sum t_j \vec{n}_j.$$

Pulling back by G as in Sect. 2 we get

$$|T_r(M)| = \int_{M \times B_r(0)} \left(\sum dy_k \wedge d\xi_k + \sum t_j dn_j^{\#} + \sum dt_j \wedge n_j^{\#} \right)^n /n!$$

$$= \int_{M \times B_r(0)} \left(\sum dy_k \wedge d\xi_k + \sum t_j F_j + \sum dt_j \wedge n_j^{\#} \right)^n /n!$$

Although $M \times B(0, r)$ is not strictly speaking a domain in $\mathbb{R}^n$, the integral here should be interpreted as an ordinary integral with respect to (y, t) and the Berezin integral

with respect to ξ. We expand the integrand by the trinomial theorem (since all three forms within the parenthesis are of degree 2 they commute), and get

$$|T_r(M)| = \sum_l c_{l,m} \int_{|t|<r} dt \int_{[M]_s} \beta^l \wedge \left(\sum t_j F_j\right)^{m-l}. \tag{10.1}$$

Here $\beta = \sum dy_k \wedge d\xi_k$, and we are using

$$[M] \wedge \left(\sum dt_j \wedge n_j^\#\right)^{n-m} = c[M]_s dt_1 .. \wedge dt_k$$

for some constant c. What remains is therefore to compute the form valued integral

$$\int_{|t|<r} \left(\sum t_j F_j\right)^{m-l} dt.$$

We first compute the integral

$$\int_{|t|<r} \left(\sum t_j a_j\right)^{m-l} dt, \tag{10.2}$$

where $a = (a_1, \ldots a_p)$ lies in $\mathbb{R}^p$. It clearly vanishes when $m - l$ is odd, and when $m - l = 2q$ it equals a constant times

$$|a|^{2q} r^{2q+p}.$$

This follows since the integral is rotational invariant and homogenous of degree $2q$ in a, and homogenous of order $2q + p$ in r. Hence, up to a constant, the integral in (10.2) equals

$$\left(\sum a_j^2\right)^q r^{2q+p} \tag{10.3}$$

From a different point view, we could have expanded the integrand in (10.2) and obtained a linear combination of monomials in a of degree $m - l$. The fact that the resulting homogenous polynomial in a is given by (10.3) is therefore an algebraic identity. This must also hold when a_j are not real numbers but lie in any commutative algebra over the reals, like in our case when $a_j = F_j$ are two-forms. Hence

$$\int_{|t|<r} \left(\sum t_j F_j\right)^{m-l} dt = c_q r^{2q+p} \left(\sum F_j^2\right)^q = c_q' r^{2q+p} R^q.$$

Inserting this into (10.1), the theorem follows.

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