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LIMIT THEOREMS FOR MULTI-TYPE GENERAL BRANCHING PROCESSES WITH POPULATION DEPENDENCE

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Abstract

A general multi-type population model is considered, where individuals live and reproduce according to their age and type, but also under the influence of the size and composition of the entire population. We describe the dynamics of the population as a measure-valued process and obtain its asymptotics as the population grows with the environmental carrying capacity. Thus, a deterministic approximation is given, in the form of a law of large numbers, as well as a central limit theorem. This general framework is then adapted to model sexual reproduction, with a special section on serial monogamic mating systems.

Keywords: Age and type structure dependent population processes; size dependent reproduction; carrying capacity; law of large numbers; central limit theorem; diffusion approximation

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1. Introduction

Classical stochastic population dynamics (branching processes of various degrees of generality) assume independently acting individuals, usually in stable circumstances, whereas deterministic approaches, while paying attention to the feedback loop between a population and its environment, tend to sweep dependence among individuals under the carpet. In a series of papers, branching processes with population size dependence have been studied, first in the discrete-time Galton–Watson case [25, 26], and more recently also for single-type general processes [9, 12, 13, 20, 21]. The papers [9] and [12] can be viewed as single-type companion papers to the present work; [12] investigated the law of large numbers and [9] the central limit theorem. A first approach to general multi-type processes, comprising sexual reproduction under a specific model, was made in [22], where the law of large numbers was discussed. The papers [13] and [20] looked at how long a population lives before extinction.

In this paper, we introduce a general model where an individual is characterised by its age and type. Reproduction and death may depend on these as well as on the environment (the size

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and composition of the whole population). This is achieved by describing the population as a measure-valued process, as done in [9]. The dependence on the composition of the population could be, for instance, on the sizes of subpopulations of different types or ages. The resulting general model has a high level of flexibility and can be adapted to the dynamics of quite diverse biological populations. Such complex models are beginning to be used in cancer modelling (e.g. [4]).

In an ecological community, species interact; reproduction and death depend on the composition of the community. This complex structure can be approached in our set-up. In particular, in predator–prey models, species can be viewed as types, and in food webs (e.g. [33]), many complex interactions occur, where age, sex, and other types are important.

Another example is the modelling of cell proliferation and differentiation, which has been studied via multi-type branching processes (e.g. [15], [32]). One particular phenomenon is that of the formation of oligodendrocytes: progenitor cells dividing into new progenitor cells, differentiating into oligodendrocytes, or dying. Here a detailed description of the cell population and its evolution can be given, for high cell densities.

This paper starts with model set-up and the statement of limiting results for a general multi-type population model, in Section 2. Section 3 gives a special case of sexual reproduction, where couple formation is considered in a (serial) monogamy system. In nature such systems can have quite varying forms, from season-long couples to lifelong couples, which persist until the death of one of the partners. The approach can be adapted to different strategies of couple formation. Proofs for Section 2 will be given in Section 4 and proofs for Section 3 in Section 5.

Naively speaking, our object of study is the process describing how the number of individuals $S_t^K(B \times A)$, with ages in an interval A and types in a set B, evolves as time $t \geqslant 0$ passes, for large $K > 0$. Here, K is historically the habitat carrying capacity, more generally interpretable as a system size parameter. The process is supposed to start from a measure S_0^K, the total mass of which is of the order of K, and the evolution of S_t^K is governed by birth and death intensities that depend on individual age and type, population structure, and K. K is typically large and in this paper is made to increase to infinity.

Throughout this paper, we will use the letter c, with or without subscript, to denote generic constants, independent of K. The proofs in this paper are compressed for the requirement of publication; a longer version with more detailed proofs can be found on the arXiv [8].

2. Multi-type population structure dynamics

2.1. The model set-up

The population model is defined as in [9], but with some additional characteristics. It builds upon the Ulam–Harris family space, with type inherited from mother to child, as described in [19]. An individual $x = x_1 x_2 \cdots x_n$ is thought of as the x_nth child of $\cdots$ of the x_2th child of the x_1th ancestor, and

$$I := \bigcup_{n=1}^{\infty} \mathbb{N}^n$$

denotes the set of possible individuals. In particular, xj denotes the jth child of individual x.

Each individual x born into the population is characterised by its type, κ_x, and birth time, τ_x, which is recursively defined as the birth time of its mother plus her age at the bearing of the individual; cf. [14, 18, 19]. Let λ_x denote the lifespan of x and write $\sigma_x = \tau_x + \lambda_x$ for the death time. Then at each time $\tau_x \leqslant t < \tau_x + \lambda_x$ the individual will be in *state*

$s_x(t) = (\kappa_x, t - \tau_x) \in \mathbb{K} \times \mathbb{A} =: \mathbb{S}$. We write $\mathbb{K}$ for the set of possible types, assumed to be finite, and $\mathbb{A}$ for the set of possible ages, assumed to be a bounded interval $[0, \omega]$ with $\omega < \infty$ denoting the maximal age (following classical demographic notation), which in the present case will be defined in Section 2.2.

The composition of the population at time t can be represented by the measure

$$S_t(di, dv) = \sum_{x \in I} \mathbf{1}_{\tau_x \leqslant t < \sigma_x} \delta_{(\kappa_x, t - \tau_x)}(di, dv), \tag{1}$$

where δ_s denotes the Dirac measure at s, assigning unit mass to s. Thus, S_t is a measure with unit mass at the state of each individual that is alive at time t. In this section and later we allow ourselves to suppress writing the dependence upon carrying capacity K and often also upon time t. We shall denote the set of finite nonnegative measures on $\mathbb{S}$, with its weak topology, by $\mathcal{M}(\mathbb{S})$, or $\mathcal{M}$ for short. Thus $S_t \in \mathcal{M}$ for each t. Here $\mathbb{S}$ has the product topology of discrete and Euclidean topology, of course.

The initial population S_0 is assumed to be finite and deterministic. Suppose that bearing and death times are stochastically given by type-, age-, and population-dependent rates. An individual of state s in a population composition S gives birth at rate $b_S(s)$ until it dies, with the death intensity being $h_S(s)$. At each birth event, a random number $\check{\xi}^i_S(s)$ of offspring of type $i \in \mathbb{K}$ is generated, with distribution depending on i, s, and S. Similarly, a random number $\hat{\xi}^i_S(s)$ of offspring of type i may be born at the death of the mother (splitting) with distribution depending on i, s, and S. We reiterate that the suffix S represents the population composition, which could include population size and other aspects of the population structure. Alternatively, the reproduction process could have been given as an integer-valued random measure on $\mathbb{S}$, as in [19], disintegrated into a stream of events and a random mass at each event. The variables $\check{\xi}^i_S(s)$ then have the Palm distribution, given a birth event at $s \in \mathbb{S}$ [17, 24]. Those pertaining to the same mother but at different s are also assumed independent.

Remark 1. Indeed, the birth rate pertaining to each individual x with $\tau_x < \infty$ gives rise to a point process of bearings by x at ages $0 < \alpha_x^1 < \alpha_x^2 < \ldots < \lambda_x$. (For simplicity, we disregard the possibility of immediate bearing at birth [18]. Births at death (splitting) are handled separately in this paper.) The random variable

$$\check{\xi}^{x,i}_{S_{(\tau_x + \alpha_x^j)-}}\left(\kappa_x, \alpha_x^j\right),$$

which stands for the number of type i children borne by x at time $\tau_x + \alpha_x^j$, then follows the distribution of $\check{\xi}^i_S(s)$, with $S = S_{(\tau_x + \alpha_x^j)-}$ and $s = (\kappa_x, \alpha_x^j)$. A corresponding remark is valid for the number of children generated at death. We shall not enter into the awkward details of this but refer to the construction in [19]. At the individual level, the construction there is, however, more general than the present not only through a richer type space but also since earlier reproduction history may influence the propensity to give birth.

The population model can be described in either of two ways: through the generator of the process (as in [20] and [22]), or through the evolution equation of the process (as in [9]). Here we give the second approach; see Section 4.1. Before giving the dynamic equation, we clarify the concepts of differentiation and integration on $\mathbb{S}$. Derivatives of a function on $\mathbb{S}$ refer to the derivatives with respect to the second, continuous variable, i.e. age. In particular, for $f : \mathbb{S} \to \mathbb{R}$ and $s = (i, v)$, we write $f^{(j)}$ to mean

$$f^{(j)}(s) = f^{(j)}(i, v) = \partial_v^j f(i, v),$$

where ∂_v^j denotes the jth derivative with respect to the variable v. We also use f' for $f^{(1)}$. If μ is a Borel (positive or signed) measure on $\mathbb{S}$, then for $f : \mathbb{S} \to \mathbb{R}$, we write

$$(f, \mu) = \int_{\mathbb{S}} f(s)\mu(ds) = \int_{\mathbb{K}\times\mathbb{A}} f(i, v)\mu(di \times dv) = \sum_{i\in\mathbb{K}} \int_{\mathbb{A}} f(i, v)\mu(\{i\} \times dv).$$

For nonnegative integers j, we write $C^j(\mathbb{S})$ for the space of functions on $\mathbb{S}$ with continuous derivatives (with respect to the age variable) up to order j. Since we only consider a compact domain $\mathbb{S}$, functions in $C^j(\mathbb{S})$ are bounded, and so are the j derivatives. We can define the norm

$$\|f\|_{C^j(\mathbb{S})} = \max_{0\leqslant\iota\leqslant j} \sup_{s\in\mathbb{S}} |f^{(\iota)}(s)| = \max_{0\leqslant\iota\leqslant j} \sup_{i\in\mathbb{K}, v\in\mathbb{A}} |f^{(\iota)}(i, v)|,$$

and will use $\|\cdot\|_{C^0}$ and $\|\cdot\|_\infty$ interchangeably.

We write $\check{m}_S^i(s) = \mathbb{E}[\check{\xi}_S^i(s)|S]$, $i \in \mathbb{K}$, for the expectation of the number of type i progeny at the birth event in question, given population size and composition, and the individual's state at that time. Note that the expectation is that of a random variable having the specified conditional distribution and that the conditional covariances of the number of children borne by the same mother at two different bearing events vanish. Similarly, we write

$$\check{\gamma}_S^{i_1 i_2}(s) = \mathbb{E}\big[\check{\xi}_S^{i_1}(s)\check{\xi}_S^{i_2}(s)|S\big], \qquad i_1, i_2 \in \mathbb{K},$$

and define $\hat{m}_S^i(s)$ and $\hat{\gamma}_S^{i_1 i_2}(s)$ for $\hat{\xi}$ in a similar vein. Then the mean intensity of births of an individual of state s at time t is

$$\sum_i \check{m}_{S_t}^i(s)b_{S_t}(s) + \sum_i \hat{m}_{S_t}^i(s)h_{S_t}(s).$$

The dynamic equation can be obtained in the form of a semimartingale, similar to that given in [9] and [22]. For $f \in C^1(\mathbb{S})$,

$$(f, S_t) = (f, S_0) + \int_0^t (L_{S_u} f, S_u)du + M_t^f, \tag{2}$$

where

$$L_S f = f' - h_S f + \sum_{i\in\mathbb{K}} f(i, 0)\Big(b_S \check{m}_S^i + h_S \hat{m}_S^i\Big)$$

and M_t^f is a locally square-integrable martingale with predictable quadratic variation

$$\begin{aligned}\langle M^f\rangle_t = \int_0^t \Bigg(\sum_{i_1\in\mathbb{K}} \sum_{i_2\in\mathbb{K}} f(i_1, 0)f(i_2, 0)\Big(b_{S_u}\check{\gamma}_{S_u}^{i_1 i_2} + h_{S_u}\hat{\gamma}_{S_u}^{i_1 i_2}\Big)& \\ + h_{S_u} f^2 - 2\sum_{i\in\mathbb{K}} f(i, 0)h_{S_u}\hat{m}_{S_u}^i f, S_u\Bigg)du.&\end{aligned}$$

For simplicity of notation, we shall write

$$n^i = b\check{m}^i + h\hat{m}^i \quad \text{and} \quad w^{i_1 i_2} = b\check{\gamma}^{i_1 i_2} + h\hat{\gamma}^{i_1 i_2}$$

from here onwards.

The differential equation for a specific characteristic of the population can be obtained by choosing an appropriate test function f (and F). For example, if $f=1$, the result is the population size; taking $f(i,v)=v$ yields the sum of the ages of the population. It is also possible to count individuals of a certain type; for instance, taking $f(i,v)=\mathbf{1}_{i=1}$, we have the size of the subpopulation of type 1, and similarly the average age of individuals of a type can be obtained.

Remark 2. As is done in [9], the above results can be extended to test functions on $\mathbb{S}\times\mathbb{T}$, where $\mathbb{T}$ is a time interval $[0,T]$. Consider test functions $f(s,t)\equiv f(i,v,t)$ on $\mathbb{S}\times\mathbb{T}$. We shall write $f_t(s)$ to mean $f(s,t)$ and use the two forms of notation interchangeably. For such f, $\partial_1 f$ refers to the derivative with respect to the variable s (or v in this case), and $\partial_2 f$ refers to the derivative with respect to t. For $f\in C^{1,1}(\mathbb{S}\times\mathbb{T})$,

$$(f_t, S_t)=(f_0, S_0)+\int_0^t \Big(\partial_1 f_u+\partial_2 f_u - f_u h_{S_u}+\sum_{i\in\mathbb{K}} f_u(i,0)n^i_{S_u}, S_u\Big)du + M^f_t,$$

where M^f_t is a martingale with the predictable quadratic variation

$$\langle M^f\rangle_t=\int_0^t \Bigg(\sum_{i_1\in\mathbb{K}}\sum_{i_2\in\mathbb{K}} f_u(i_1,0)f_u(i_2,0)w^{i_1 i_2}_{S_u}+h_{S_u}f_u^2 \\ -2\sum_{i\in\mathbb{K}} f_u(i,0)h_{S_u}\hat{m}^i_{S_u}f_u, S_u\Bigg)du. \tag{3}$$

Remark 3. Using the same argument as in [9, Section 5.2], we can show that, for $f\in C^0(\mathbb{S})$, $M^f_t=(f,M_t)$ with measure

$$\begin{aligned}M_t(ds)=&\sum_{i\in\mathbb{K}}\delta_{(i,0)}(ds)\Big(\check{B}^i((0,t])-\int_0^t (b_{S_u}\check{m}^i_{S_u}, S_u)du\Big)\\ &+\sum_{i\in\mathbb{K}}\delta_{(i,0)}(ds)\Big(\hat{B}^i((0,t])-\int_0^t (h_{S_u}\hat{m}^i_{S_u}, S_u)du\Big)\\ &-\Big(\sum_{x\in I}\delta_{(\kappa_x,\lambda_x)}(ds)\mathbf{1}_{\sigma_x\leqslant t}-\int_0^t\sum_{x\in I}\delta_{(\kappa_x,u-\tau_x)}(ds)h_{S_u}(s)\mathbf{1}_{\tau_x\leqslant u<\sigma_x}du\Big).\end{aligned}$$

Considering $\int_0^t \varphi(u)d(g,M_u)$ for $g\in C^0(\mathbb{S})$ and $\varphi\in C^0(\mathbb{T})$, and using the monotone class theorem (e.g. [6, I.22.1]), we can then show that $\int_0^t f(\cdot,u), d\tilde{M}_u)$ exists for $f\in C^{0,0}(\mathbb{S}\times\mathbb{T})$ and is a martingale with predictable quadratic variation (3).

2.1.1 *Applications: sexual reproduction.* A simple application of the stochastic process introduced above, where individual life can be influenced by many factors, such as individual type and age, population size, and population structure, is to model sexual reproduction. Let $\mathbb{K}=\{1,2\}\equiv\{♀,♂\}$ with type 1 (denoted by ♀) representing females (the reproducing type) and type 2 (denoted by ♂), representing males, so that $b_S(♂,v)=0$ for any $S\in\mathcal{M}(\mathbb{S})$ and $v\in\mathbb{A}$; also, for $m=\check{m},\hat{m}$, $\gamma=\check{\gamma},\hat{\gamma}$, and $i,i_1,i_2=♀,♂$, $m^i_S(♂,v)=\gamma^{i_1,i_2}_S(♂,v)=0$. Write $\hat{\mathsf{m}}^i_S(v)=\hat{m}^i_S(♀,v)$, $\mathsf{n}^i_S(v)=n^i_S(♀,v)$, and $\mathsf{w}^{i_1 i_2}_S(v)=w^{i_1 i_2}_S(♀,v)$. Taking $f(i,v)=f_♀(v)\mathbf{1}_{i=♀}$ in (2) and writing $S^♀(dv)=S(\{♀\},dv)$ gives the age structure of the female subpopulation:

$$(f_♀, S^♀_t)=(f_♀, S^♀_0)+\int_0^t (f'_♀ - h_{S_u}(♀,\cdot)f_♀+f_♀(0)\mathsf{n}^♀_{S_u}, S^♀_u)du+M^♀_t$$

with

$$\langle M^{♀}\rangle_t = \int_0^t \big(f_{♀}(0)^2 \mathsf{w}_{S_u}^{♀♀} + h_{S_u}(♀, \cdot)f_{♀}^2 - 2f_{♀}(0)h_{S_u}(♀, \cdot)\hat{\mathsf{m}}_{S_u}^{♀} f_{♀}, S_u^{♀}\big)du;$$

whereas taking $f(i, v) = f_{♂}(v)\mathbf{1}_{i=♂}$ in (2) and writing $S^{♂}(dv) = S(\{♂\}, dv)$ gives the dynamics of the male subpopulation:

$$(f_{♂}, S_t^{♂}) = (f_{♂}, S_0^{♂}) + \int_0^t \big(f_{♂}' - h_{S_u}(♂, \cdot)f_{♂}, S_u^{♂}\big)du + f_{♂}(0)\int_0^t \big(\mathsf{n}_{S_u}^{♂}, S_u^{♀}\big)du + M_t^{♂}$$

with

$$\langle M^{♂}\rangle_t = f_{♂}(0)^2 \int_0^t \big(\mathsf{w}_{S_u}^{♂♂}, S_u^{♀}\big)du + \int_0^t \big(h_{S_u}(♂, \cdot)f_{♂}^2, S_u^{♂}\big)du.$$

This approach is different from the so-called bisexual branching process, where individuals mate to form couples, which then reproduce as in a Galton–Watson-type process in discrete time. Bisexual branching processes were introduced by Daley [5] and have been studied by many others (cf. [30] and references therein). A (pseudo-)continuous-time bisexual branching process was given in [31], with mating supposed to occur at the events of a point process and individuals borne by couples having independent and identically distributed lifespans. As opposed to that, our model is individual-based, and females reproduce with an intensity which may depend on the availability of males, among other things. Sexual reproduction without couple formation, as in fish, as well as asexual reproduction, can be easily handled within this framework. Couple formation and sexual reproduction with mating, as in many higher animals, can be captured, to some extent, by careful choice of rates and offspring distribution.

2.2. The limit theorems

We consider a family of population processes as above, indexed by some parameter $K \geqslant 1$, which, as mentioned, arises from the notion of carrying capacity of the habitat and can often be interpreted as some size where a population tends neither to increase nor to decrease systematically. However, it need not play such a role, and can be viewed as just a natural system scaling parameter, since we consider starting populations whose size is of the order of K. The purpose is to establish the asymptotic behaviour as K increases, under the assumption that the dynamics of the population depends on K through the reproduction parameters. For notational consistency with limits, we write the parameters in the form $q^K_{S/K}(s)$, for $q = b, h, m, \gamma$. The conditions stated for m and γ are to be satisfied by all $\check{m}^i$, $\hat{m}^i$, $\check{\gamma}^{i_1 i_2}$, and $\hat{\gamma}^{i_1, i_2}$ for any $i, i_1, i_2 \in \mathbb{K}$.

We consider a finite time interval $\mathbb{T} = [0, T]$. Thus, to obtain a bounded age space $\mathbb{A}$, it suffices to assume that the age of the oldest individual over the family of processes at time 0, denoted by a^*, is finite:

$$a^* := \sup_{K\geqslant 1} \sup_{i\in\mathbb{K}} \big(\inf\{v > 0 : S_0^K(i, (v, \infty)) = 0\}\big) < \infty.$$

Then the age of the oldest individual in the population at time t cannot exceed $t + a^*$, and we can take $\mathbb{A} = [0, \omega]$ with $\omega = T + a^*$ as the age space, and $\mathbb{S} = \mathbb{K} \times \mathbb{A}$ as the individual state space.

Specifically, we establish the law of large numbers and the central limit theorem associated with our age and type structure process. For a simple case of sexual reproduction (cf. Section 2.1.1) the law of large numbers was stated in [22], without proof. Here, we give and prove results for more general cases. The law of large numbers is stated in Section 2.2.1 with proof in Section 4.2, and the central limit theorem is stated in Section 2.2.3 with proof in Section 4.3.

2.2.1 *Law of large numbers.* Denote by $\bar{S}_t^K = S_t^K/K$ the *density* of the population. 'Density' here refers to the population size as compared to the carrying capacity. For any $f \in C^1(\mathbb{S})$ and $t \in \mathbb{T}$,

$$(f, \bar{S}_t^K) = (f, \bar{S}_0^K) + \int_0^t \left(L^K_{\bar{S}_u^K} f, \bar{S}_u^K \right) du + \frac{1}{K} M_t^{f,K}, \tag{4}$$

where

$$L_S^K f = f' - h_S^K f + \sum_{i \in \mathbb{K}} f(i, 0) n_S^{i,K} \tag{5}$$

and $M_t^{f,K}$ is a locally square-integrable martingale with predictable quadratic variation

$$\langle M^{f,K} \rangle_t = \int_0^t \left(\sum_{i_1 \in \mathbb{K}} \sum_{i_2 \in \mathbb{K}} f(i_1, 0) f(i_2, 0) w^{i_1 i_2, K}_{\bar{S}_u^K} + h^K_{\bar{S}_u^K} f^2 - 2 \sum_{i \in \mathbb{K}} f(i, 0) h^K_{\bar{S}_u^K} \hat{m}^{i,K}_{\bar{S}_u^K} f, S_u^K \right) du.$$

The next theorem gives the law of large numbers when the population process is *demographically smooth*, i.e. satisfies the following conditions:

- (C0) The model parameters b, h, m, and γ are uniformly bounded; i.e., for $q = b, h, m, \gamma$, $\sup q_\mu^K(s) < \infty$, where the supremum is taken over $s \in \mathbb{S}$, $K \geqslant 1$, and $\mu \in \mathcal{M}(\mathbb{S})$.
- (C1) The model parameters b, h, and m are normed uniformly Lipschitz in the following sense: for $q = b, h, m$, there exists $c > 0$ such that for all $K \geqslant 1$, $\|q_\mu^K - q_\nu^K\|_\infty \leqslant c\|\mu - \nu\|$, with $\|\mu\| := \sup_{\|f\|_\infty \leqslant 1, f \text{ continuous}} |(f, \mu)|$.
- (C2) For $q = b, h, m$, the sequence q^K converges (pointwise in μ and uniformly in s) to $q_\mu^\infty(s) := \lim_{K\to\infty} q_\mu^K(s)$.
- (C3) $\bar{S}_0^K$ converges weakly to $\bar{S}_0$ and $\sup_K (1, \bar{S}_0^K) < \infty$; that is, the process stabilises initially.

Theorem 1. *Under the smooth demography conditions (C0)–(C3), the measure-valued process $\bar{S}^K$ converges weakly in the Skorokhod space $\mathbb{D}(\mathbb{T}, \mathcal{M}(\mathbb{S}))$, as $K \to \infty$, to a deterministic measure-valued process $\bar{S}$ satisfying*

$$(f, \bar{S}_t) = (f, \bar{S}_0) + \int_0^t \left(L^\infty_{\bar{S}_u} f, \bar{S}_u \right) du, \tag{6}$$

where

$$L_S^\infty f = f' - h_S^\infty f + \sum_{i \in \mathbb{K}} f(i, 0) n_S^{i,\infty} \tag{7}$$

for any $f \in C^1(\mathbb{S})$ and $t \in \mathbb{T}$.

2.2.2 *Relevant spaces and embeddings.* Before stating the central limit theorem, we introduce the following Sobolev spaces on $\mathbb{S}$ and the corresponding duals, where the convergence will take place. The compactness of the space $\mathbb{S}$ allows us to work with classical Sobolev spaces, instead of the weighted Sobolev spaces as is done by other scholars (e.g. [3, 28, 29]).

For $j \in \mathbb{N}_0$, let $W^j(\mathbb{S})$ be the closure of $C^\infty(\mathbb{S})$ with respect to the norm

$$\|f\|_{W^j(\mathbb{S})} = \left(\sum_{\iota=0}^{j} \sum_{i\in\mathbb{K}} \int_{\mathbb{A}} \left(f^{(\iota)}(i, v)\right)^2 dv \right)^{1/2},$$

where the $f^{(\iota)}$ are the (weak) derivatives of f. In other words, $W^j(\mathbb{S})$ is the set of functions f on $\mathbb{S}$ such that f and its weak derivatives up to order j have a finite $L^2(\mathbb{S})$ norm

$$\|f\|_{L^2(\mathbb{S})} = \left(\sum_{i\in\mathbb{K}} \int_{\mathbb{A}} f(i, v)^2 dv \right)^{1/2}.$$

The space $W^j(\mathbb{S})$ is a Hilbert space. As in [9], we have the following embeddings:

$$C^j(\mathbb{S}) \hookrightarrow W^j(\mathbb{S}), \qquad W^{j+1}(\mathbb{S}) \hookrightarrow C^j(\mathbb{S}), \quad \text{and} \quad W^{j+1}(\mathbb{S}) \underset{H.S.}{\hookrightarrow} W^j(\mathbb{S}),$$

where $H.S.$ stands for Hilbert–Schmidt embedding.

In particular, for the dual spaces $C^{-j}(\mathbb{S})$ and $W^{-j}(\mathbb{S})$ of $C^j(\mathbb{S})$ and $W^j(\mathbb{S})$ respectively, we have

$$C^{-0}(\mathbb{S}) \hookrightarrow C^{-1}(\mathbb{S}) \hookrightarrow W^{-2}(\mathbb{S}) \hookrightarrow W^{-3}(\mathbb{S}) \underset{H.S.}{\hookrightarrow} W^{-4}(\mathbb{S}).$$

The Hilbert–Schmidt embedding plays an important role, since with this, a ball in the smaller space (W^{-3}) is precompact in the larger space (W^{-4}). This fact renders it possible to prove the coordinate tightness of the fluctuation process. The other embeddings are required because of the derivative in the definition of the operator L_S^K.

For ease of notation, we will suppress writing $(\mathbb{S})$ for the spaces: we write W^j and C^j to mean $W^j(\mathbb{S})$ and $C^j(\mathbb{S})$. Further, L^∞ denotes the set of all bounded measurable functions on $\mathbb{S} = \mathbb{K} \times \mathbb{A}$ with its natural product sigma-algebra $\mathcal{B}(\mathbb{S})$.

2.2.3 *Central limit theorem.* Let $Z^K = \sqrt{K}\left(\bar{S}^K - \bar{S}\right)$. Then, for any $f \in C^1$ and $t \in \mathbb{T}$,

$$\left(f, Z_t^K\right) = \left(f, Z_0^K\right) + \sqrt{K} \int_0^t \left(L_{\bar{S}_u^K}^K f - L_{\bar{S}_u}^\infty f, \bar{S}_u \right) du + \int_0^t \left(L_{\bar{S}_u^K}^K f, Z_u^K \right) du + \tilde{M}_t^{f,K}, \tag{8}$$

where L_S^K and L_S^∞ are defined as in (5) and (7), and $\tilde{M}_t^{f,K}$ is a square-integrable martingale with predictable quadratic variation

$$\left\langle \tilde{M}^{f,K} \right\rangle_t = \int_0^t \left(\sum_{i_1\in\mathbb{K}} \sum_{i_2\in\mathbb{K}} f(i_1, 0) f(i_2, 0) w_{\bar{S}_u^K}^{i_1 i_2, K} + h_{\bar{S}_u^K}^K f^2 - 2 \sum_{i\in\mathbb{K}} f(i, 0) h_{\bar{S}_u^K}^K \hat{m}_{\bar{S}_u^K}^{i,K} f, \bar{S}_u^K \right) du.$$

In addition to (C0)–(C3), we impose the following assumptions:

(A0) The conditions (C1) and (C2) hold also for $q = \gamma$.

(A1) $\Xi := \sup_{i,s,S,K} \check{\xi}_S^{i,K}(s) \vee \sup_{i,s,S,K} \hat{\xi}_S^{i,K}(s)$ is square-integrable.

(A2) The reproduction parameters $b_S^K(s)$, $h_S^K(s)$, and $m_S^K(s)$ and their limits (in the sense of (C2)) are in the space C^4 (in the argument s) with convergence in C^4. Moreover, $\sqrt{K} \sup_\mu \|q_\mu^K - q_\mu^\infty\|_\infty \to 0$ as $K \to \infty$, $\sup_{K,\mu} \|q_\mu^K\|_{C^3} < \infty$, and $\sup_\mu \|q_\mu^\infty\|_{C^4} < \infty$, for $q = b, h, m$.

(A3) The limiting parameters (seen as functions of S) are Fréchet differentiable at every S. That is, for $q = b, h, m$, for every μ, there exists a continuous linear operator $\partial_S q^\infty_\mu : W^{-4} \to L^\infty$ such that

$$\lim_{\|\nu\|_{W^{-4}} \to 0} \frac{1}{\|\nu\|_{W^{-4}}} \|q^\infty_{\mu+\nu} - q^\infty_\mu - \partial_S q^\infty_\mu(\nu)\|_\infty = 0.$$

Moreover, $\sup_\mu \|\partial_S q^\infty_\mu\|_{\mathbb{L}^{-4}} \leqslant c$, where $\mathbb{L}^{-4} = L(W^{-4}, L^\infty)$ denotes the space of continuous linear mappings from W^{-4} to L^∞.

(A4) Z^K_0 converges to Z_0 in W^{-4} and $\sup_K \|Z^K_0\|_{W^{-2}} < \infty$.

The next theorem states that under these conditions the fluctuation process Z^K converges.

Theorem 2. *Under the assumptions (C0)–(C3) and (A0)–(A4), as $K \to \infty$, the process $(Z^K_t)_{t\in\mathbb{T}}$ converges weakly in $\mathbb{D}(\mathbb{T}, W^{-4})$ to $(Z_t)_{t\in\mathbb{T}}$ satisfying, for $f \in W^4$ and $t \in \mathbb{T}$,*

$$\begin{aligned}(f, Z_t) =& (f, Z_0) + \int_0^t \Big(- \partial_S h^\infty_{\bar{S}_u}(Z_u) f + \sum_{i\in\mathbb{K}} f(i, 0) \partial_S n^{i,\infty}_{\bar{S}_u}(Z_u), \bar{S}_u \Big) du \\ &+ \int_0^t \Big(f' - h^\infty_{\bar{S}_u} f + \sum_{i\in\mathbb{K}} f(i, 0) n^{i,\infty}_{\bar{S}_u}, Z_u \Big) du + \tilde{M}^{f,\infty}_t, \end{aligned} \tag{9}$$

where $\tilde{M}^{f,\infty}_t$ is a continuous Gaussian martingale with predictable quadratic variation

$$\big\langle \tilde{M}^{f,\infty} \big\rangle_t = \int_0^t \Big(\sum_{i_1\in\mathbb{K}} \sum_{i_2\in\mathbb{K}} f(i_1, 0) f(i_2, 0) w^{i_1 i_2,\infty}_{\bar{S}_u} + h^\infty_{\bar{S}_u} f^2 - 2 \sum_{i\in\mathbb{K}} f(i, 0) h^\infty_{\bar{S}_u} \hat{m}^{i,\infty}_{\bar{S}_u} f, \bar{S}_u \Big) du.$$

For each t, the limit Z_t takes value in W^{-4}. Under certain conditions, its expectation corresponds to a signed measure. This is made precise in the following proposition.

Proposition 1. *Suppose that $\partial_S h^\infty_\mu(B)(s)$ is of the form $(g^h(\mu, s, \cdot), B)$ where $g^h(S, s, \cdot) \in W^4$ with $\sup_{S,s} \|g^h(S, s, \cdot)\|_{W^4} < \infty$, and similarly, $\partial_S n^{i,\infty}_\mu(B)(s)$ is of the form $(g^{n,i}(\mu, s, \cdot), B)$ where $g^{n,i}(S, s, \cdot) \in W^4$ with $\sup_{S,s} \|g^{n,i}(S, s, \cdot)\|_{W^4} < \infty$. Then for each t, $\nu_t : W^4 \ni f \mapsto \mathbb{E}[(f, Z_t)]$ defines a signed measure on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$.*

We give the proofs in Section 4: that of Theorem 1 in Section 4.2, that of Theorem 2 in Section 4.3, and that of Proposition 1 in Section 4.4.

Loosely speaking, the law of large numbers gives the first-order approximation to the population when it is large; the central limit theorem gives the second-order approximation. Quantities of interest, such as the population size or the number of individuals of a certain type within a certain age interval, can be computed simply by choosing the appropriate test functions. In specific cases, where explicit model parameters are available, more information can be obtained and inferences can be made from observations.

2.2.4 *Applications: sexual reproduction, revisited.* We continue from Section 2.1.1, revising the notation slightly, writing parameters in the form $q^K_{\bar{S}}(s)$. Suppose that all the conditions (C0)–(C3) and (A0)–(A4) are satisfied, so that the law of large numbers and the central limit theorem hold. For example, q may take the form $q\big(i, v, (1, \bar{S}), (g, \bar{S})\big)$ with $g(i, v) = \mathbf{1}_{i=♀}$. Then the limiting parameters have the same form, and the Fréchet derivatives are

$$\partial_S q^\infty_\mu(B)(i, v) = \partial_3 q\big(i, v, (1, \mu), (g, \mu)\big)(1, B) + \partial_4 q\big(i, v, (1, \mu), (g, \mu)\big)(g, B).$$

In this case, the conditions on the parameters are satisfied if $\sup \partial_2^j q(i, x, y, z) < \infty$ for $j = 0, 1, \ldots, 4$, $\sup \partial_3 q(i, x, y, z) < \infty$, and $\sup \partial_4 q(i, x, y, z) < \infty$, where the supremum is taken over all i, x, y, z.

The limiting population density (6) can be decomposed into two equations, for the limiting female and male measures:

$$(f_♀, \bar{S}_t^♀) = (f_♀, \bar{S}_0^♀) + \int_0^t \big(f_♀' - h_{\bar{S}_u}^\infty(♀, \cdot) f_♀ + f_♀(0)\mathsf{n}_{\bar{S}_u}^{♀,\infty}, \bar{S}_u^♀\big) du;$$
$$(f_♂, \bar{S}_t^♂) = (f_♂, \bar{S}_0^♂) + \int_0^t \big(f_♂' - h_{\bar{S}_u}^\infty(♂, \cdot) f_♂, \bar{S}_u^♂\big) du + f_♂(0) \int_0^t \big(\mathsf{n}_{\bar{S}_u}^{♂,\infty}, \bar{S}_u^♀\big) du.$$

In the case where these measures have densities (with respect to Lebesgue measure), namely $s^♀(v, t)$ for $\bar{S}_t^♀$ and $s^♂(v, t)$ for $\bar{S}_t^♂$, the densities resemble the McKendrick–von Foerster equations, as pointed out in [22]: for $\circ = ♀, ♂$,

$$\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial v}\Big) s^\circ(v, t) = -h_{\bar{S}_t}^\infty(\circ, v) s^\circ(v, t), \qquad s^\circ(0, t) = \int_0^{t+a^*} \mathsf{n}_{\bar{S}_t}^{\circ,\infty}(v) s^♀(v, t) dv.$$

The upper limit in these integrals is due to $s^♀(v, t) = 0$ for $v > t + a^*$.

The fluctuation limit (9) can be decomposed as follows, with $Z^♀(dv) = Z(\{♀\}, dv)$ and $Z^♂(dv) = Z(\{♂\}, dv)$:

$$\begin{aligned}(f_♀, Z_t^♀) = (f_♀, Z_0^♀) &+ \int_0^t \big(-\partial_S h_{\bar{S}_u}^\infty(Z_u)(♀, \cdot) f_♀ + f_♀(0)\partial_S \mathsf{n}_{\bar{S}_u}^{♀,\infty}(Z_u), \bar{S}_u^♀\big) du \\ &+ \int_0^t \big(f_♀' - h_{\bar{S}_u}^\infty(♀, \cdot) f_♀ + f_♀(0)\mathsf{n}_{\bar{S}_u}^{♀,\infty}, Z_u^♀\big) du + \tilde{M}_t^♀,\end{aligned}$$

where

$$\langle \tilde{M}^♀ \rangle_t = \int_0^t \big(f_♀(0)^2 \mathsf{w}_{\bar{S}_u}^{♀♀,\infty} + h_{\bar{S}_u}^\infty(♀, \cdot) f_♀^2 - 2 f_♀(0) h_{\bar{S}_u}^\infty(♀, \cdot) \hat{\mathsf{m}}_{\bar{S}_u}^{♀,\infty} f_♀, \bar{S}_u^♀\big) du;$$

and

$$\begin{aligned}(f_♂, Z_t^♂) = (f_♂, Z_0^♂) &+ \int_0^t \big(-\partial_S h_{\bar{S}_u}^\infty(Z_u)(♂, \cdot) f_♂, \bar{S}_u^♂\big) du + f_♂(0) \int_0^t \big(\partial_S \mathsf{n}_{\bar{S}_u}^{♂,\infty}(Z_u), \bar{S}_u^♀\big) du \\ &+ \int_0^t \big(f_♂' - h_{\bar{S}_u}^\infty(♂, \cdot) f_♂, Z_u^♂\big) du + f_♂(0) \int_0^t \big(\mathsf{n}_{\bar{S}_u}^{♂,\infty}, Z_u^♀\big) du + \tilde{M}_t^♂,\end{aligned}$$

where

$$\langle \tilde{M}^♂ \rangle_t = \int_0^t \big(f_♂(0)^2 \mathsf{w}_{\bar{S}_u}^{♂♂,\infty}, \bar{S}_u^♀\big) du + \int_0^t \big(h_{\bar{S}_u}^\infty(♂, \cdot) f_♂^2, \bar{S}_u^♂\big) du.$$

The McKendrick–von Foerster equations in their turn may take many special forms in different situations, dependent upon the varying fertilisation, reproduction, and survival patterns, as well as the role of the carrying-capacity-type parameter. It is, however, important that such modelling be done in close relationship to the biological circumstances at hand. In this broad context, we content ourselves with the remark that typically female fertility will increase with the abundance of males, while mortality on the whole will tend to increase with competition and, thus, population size. We shall not enter into the intricate details of this, such as different forms of Allee effects, about which there is a rich literature, e.g. [7] and [11].

3. Sexual reproduction in the serial monogamy mating system

Our framework has been applied directly to model sexual reproduction in Sections 2.1.1 and 2.2.4. However, in those sections mating was not taken into account. In this section, we adapt our framework to model a population with serial monogamy. In other words, a female and a male form a couple either for life or for some period of time, such as a breeding season.

3.1. The model

Although we do not restrict the model to human populations, for convenience of terminology we refer to the formation of a couple as 'marriage'. Consider a population consisting of individuals of three types; type 1 (denoted by ♀) refers to single females, type 2 (denoted by ♂) refers to single males, and type 3 (denoted by ⚤) refers to 'married' couples. An individual from type ♀ and an individual from type ♂ can form a couple (get married) and becomes a type ⚤. A couple lasts for a period of random length interrupted by the death of one of the mates, at which point the survivor becomes single and available for mating again.

A type ⚤ gives birth at random times to random numbers of type ♀ and type ♂ offspring. We assume also the possibility that a type ♀ may give birth. The lifetime of each individual is random.

Let $\boldsymbol{♀}$ (resp. $\boldsymbol{♂}$) denote the set of all females (resp. males), single or married. Let $c_{x,y}$ denote the time at which individuals x and y become a couple, c_x^j the time of the jth 'marriage' of individual x, and d_x^j the time at which x becomes a 'widow' from its jth marriage. To accommodate the three types in the population, females, males and couples, the notion of age has to be changed. For couples, it will consist of two quantities, the ages of the two individuals in the couple. For a single female or male individual, we assign an additional index taking value ∞. Then the age structures at time t of the three types are given by

$$S_t^{♀}(di, dv, dw) = \sum_{x\in \boldsymbol{♀}} \mathbf{1}_{\tau_x \leqslant t \leqslant \sigma_x} \big(\mathbf{1}_{t\leqslant c_x^1} + \mathbf{1}_{d_x^1 \leqslant t \leqslant c_x^2} + \mathbf{1}_{d_x^2 \leqslant t \leqslant c_x^3} + \cdots \big) \delta_{(♀, t-\tau_x, \infty)},$$

$$S_t^{♂}(di, dv, dw) = \sum_{y\in \boldsymbol{♂}} \mathbf{1}_{\tau_y \leqslant t \leqslant \sigma_y} \big(\mathbf{1}_{t\leqslant c_y^1} + \mathbf{1}_{d_y^1 \leqslant t \leqslant c_y^2} + \mathbf{1}_{d_y^2 \leqslant t \leqslant c_y^3} + \cdots \big) \delta_{(♂, \infty, t-\tau_y)},$$

$$S_t^{⚤}(di, dv, dw) = \sum_{x\in \boldsymbol{♀}} \sum_{y\in \boldsymbol{♂}} \mathbf{1}_{\tau_x \leqslant t \leqslant \sigma_x} \mathbf{1}_{\tau_y \leqslant t \leqslant \sigma_y} \mathbf{1}_{c_{x,y}\leqslant t} \delta_{(⚤, t-\tau_x, t-\tau_y)},$$

and the structure of the entire population is $S_t(di, dv, dw) = \sum_{i=♀,♂,⚤} S_t^{i}(di, dv, dw)$.

Let $b_S(v, w)$ denote the bearing rate of a couple with a female at age v and a male at age w, when the population composition is S. At each birth, the number of females and the number of males born have the distributions $\xi_S^{♀}(v, w)$ and $\xi_S^{♂}(v, w)$ respectively. Write $m_S^i(v, w) = \mathbb{E}[\xi_S^i(v, w)]$ and $\gamma_S^{ij}(v, w) = \mathbb{E}[\xi_S^i(v, w)\xi_S^j(v, w)]$. These quantities with $w = \infty$ correspond to those of single females.

Let $\rho_S(v, w)$ be the rate at which a single female of age v and a single male of age w marry each other. In other words, for $x \in \boldsymbol{♀}$ and $y \in \boldsymbol{♂}$, with $d_x^0 := 0 =: d_y^0$,

$$\mathbf{1}_{c_{x,y}\leqslant t} - \int_0^{t\wedge c_{x,y}} \rho_{S_u}(u-\tau_x, u-\tau_y) \mathbf{1}_{\tau_x\leqslant u<\sigma_x} \mathbf{1}_{\tau_y \leqslant u<\sigma_y} \sum_{j\geqslant 0} \mathbf{1}_{d_x^j \leqslant u \leqslant c_x^{j+1}} \sum_{k\geqslant 0} \mathbf{1}_{d_y^k \leqslant u \leqslant c_y^{k+1}} du$$

is a martingale by compensation. Suppose that marriage lasts for a random length of time and breaks at rate $h_S^{⚤}(v, w)$, dependent upon the ages of the mates as well as the population composition.

We write $h_S^{♀}(v, w)$ (resp. $h_S^{♂}(v, w)$) for the death rate of a female (resp. male) when she (resp. he) is at age v (resp. w) and her (resp. his) partner is at age w (resp. v), when the population composition is S. For single females and males, the death rates are $h_S^{♀}(v, \infty)$ and $h_S^{♂}(\infty, w)$, respectively.

Consider test functions $f : \mathbb{K} \times \mathbb{A} \cup \{\infty\} \times \mathbb{A} \cup \{\infty\} \to \mathbb{R}$, with $\mathbb{K} = \{♀, ♂, ⚤\}$, such that $f(⚤, v, w) = f_{⚤}(v, w)$, $f(♀, v, w) = f(♀, v, \infty) = f_{♀}(v)$ and $f(♂, v, w) = f(♂, \infty, w) = f_{♂}(w)$, for any $v, w \in \mathbb{A}$, for some functions $f_{⚤}$, $f_{♀}$, and $f_{♂}$. For compactness of notation, we denote simple additions or subtractions of the functions by subscripts. For instance, $f_{♀-⚤}(v, w)$ means $f(♀, v, \infty) - f(⚤, v, w)$, which arises on widowing when a male partner dies, and $f_{⚤-♀-♂}(v, w)$ means $f(⚤, v, w) - f(♀, v, \infty) - f(♂, \infty, w)$, which arises in the event of biological coupling. Assume also that $f(i, v, w)$ is differentiable with respect to v and w, and $\partial_w f(♀, v, \infty) = \partial_v f(♂, \infty, w) = 0$.

If μ and ν are measures of the form $\mu(di, dv, dw) = \delta_{♀}(di)\tilde{\mu}(dv)\delta_{\infty}(dw)$ and $\nu(di, dv, dw) = \delta_{♂}(di)\delta_{\infty}(dv)\tilde{\nu}(dw)$, then for a function f on $\mathbb{A} \cup \{\infty\} \times \mathbb{A} \cup \{\infty\}$, we write $(f, \mu \otimes \nu)$ to mean $\int\int f(v, w)\tilde{\mu}(dv)\tilde{\nu}(dw) \equiv ((f(\bullet, *), \mu)_\bullet, \nu)_*$. We also write $\sum_\circ$ to mean $\sum_{\circ=♀,♂}$.

Then S has the following semimartingale representation (see Section 5.1):

$$
\begin{aligned}
(f, S_t) = (f, S_0) &+ \int_0^t \big(\partial_v f + \partial_w f, S_u\big) du + \int_0^t \big(\textstyle\sum_\circ f_\circ(0) m_{S_u}^{\circ} b_{S_u}, S_u^{⚤} + S_u^{♀}\big) du \\
&- \int_0^t \textstyle\sum_\circ (f_\circ h_{S_u}^{\circ}, S_u^{\circ}) du + \int_0^t \big(f_{♂-⚤} h_{S_u}^{♀} + f_{♀-⚤} h_{S_u}^{♂} + f_{♀+♂-⚤} h_{S_u}^{⚤}, S_u^{⚤}\big) du \\
&+ \int_0^t \big(f_{⚤-♀-♂} \rho_{S_u}, S_u^{♀} \otimes S_u^{♂}\big) du + M_t^f,
\end{aligned}
\tag{10}
$$

where M^f is a martingale with predictable quadratic variation

$$
\begin{aligned}
\big\langle M^f \big\rangle_t = &\int_0^t \big(\big[f_{♀}^2(0)\gamma_{S_u}^{♀♀} + f_{♂}^2(0)\gamma_{S_u}^{♂♂} + 2 f_{♀}(0) f_{♂}(0)\gamma_{S_u}^{♀♂}\big] b_{S_u}, S_u^{⚤} + S_u^{♀}\big) du \\
&+ \int_0^t \Big\{\textstyle\sum_\circ (f_\circ^2 h_{S_u}^{\circ}, S_u^{\circ}) + (f_{♂-⚤}^2 h_{S_u}^{♀} + f_{♀-⚤}^2 h_{S_u}^{♂} + f_{♀+♂-⚤}^2 h_{S_u}^{⚤}, S_u^{⚤})\Big\} du \\
&+ \int_0^t \big(f_{⚤-♀-♂}^2 \rho_{S_u}, S_u^{♀} \otimes S_u^{♂}\big) du.
\end{aligned}
$$

Populations, while having complex evolution, follow simple rules, such as those arising from births, deaths and marriages. Our model captures these. The mathematical model, while tedious, contains relatively straightforward terms. For example, the third term on the right-hand side of (10) corresponds to a birth (♀ or ♂) by a couple (⚤) or a single female (♀); the fourth term is due to the death of a single female (♀) or a single male (♂); the fifth term comes from widowing (⚤ becomes ♂ or ♀) and from separation or divorce of a couple (⚤ becomes ♀ + ♂); the sixth term is due to marriage (♀ + ♂ becomes ⚤). These can be read off by looking at the indexing of S, indicating the 'source' and the reproduction parameters, together with the indexing of f, indicating how the f values are changing. For instance, $(f_{\male-\text{⚥}} h^{\female}_{S_u}, S_u^{\text{⚥}})$ accounts for the death of a female (with rate $h^{\female}$) from a couple ($S^{\text{⚥}}$), and measures the impact such an event has on the metric $f_{\male} - f_{\text{⚥}}$.

Considering a family of the population processes indexed by $K \geqslant 1$ and writing the reproduction parameters in the form $q^K_{S/K}(\cdot)$, we obtain the asymptotics as $K \to \infty$.

3.2. Law of large numbers

Let $\bar{S}^K_t = S^K_t/K$. Then the scaled process satisfies

$$
\begin{aligned}
(f, \bar{S}^K_t) = {} & (f, \bar{S}^K_0) + \int_0^t \big(\partial_v f + \partial_w f, \bar{S}^K_u\big) du + \int_0^t \Big(\textstyle\sum_\circ f_\circ(0) m^{\circ,K}_{\bar{S}^K_u} b^K_{\bar{S}^K_u}, \bar{S}^{\text{⚥},K}_u + \bar{S}^{\female,K}_u\Big) du \\
& - \int_0^t \textstyle\sum_\circ (f_\circ h^{\circ,K}_{\bar{S}^K_u}, \bar{S}^{\circ,K}_u) du + \displaystyle\int_0^t \Big(f_{\male-\text{⚥}} h^{\female,K}_{\bar{S}^K_u} + f_{\female-\text{⚥}} h^{\male,K}_{\bar{S}^K_u} + f_{\female+\male-\text{⚥}} h^{\text{⚥},K}_{\bar{S}^K_u}, \bar{S}^{\text{⚥},K}_u\Big) du \\
& + \int_0^t \big(f_{\text{⚥}-\female-\male} K\rho^K_{\bar{S}^K_u}, \bar{S}^{\female,K}_u \otimes \bar{S}^{\male,K}_u\big) du + \bar{M}^{f,K}_t,
\end{aligned}
$$

with

$$
\begin{aligned}
\Big\langle \bar{M}^{f,K} \Big\rangle_t = {} & \frac{1}{K} \int_0^t \Big\{ \Big(\big[f_{\female}^2(0) \gamma^{\female\female,K}_{\bar{S}^K_u} + f_{\male}^2(0) \gamma^{\male\male,K}_{\bar{S}^K_u} + 2 f_{\female}(0) f_{\male}(0) \gamma^{\female\male,K}_{\bar{S}^K_u}\big] b^K_{\bar{S}^K_u}, \bar{S}^{\text{⚥},K}_u + \bar{S}^{\female,K}_u\Big) \\
& + \textstyle\sum_\circ (f_\circ^2 h^{\circ,K}_{\bar{S}^K_u}, \bar{S}^{\circ,K}_u) + \big(f^2_{\male-\text{⚥}} h^{\female,K}_{\bar{S}^K_u} + f^2_{\female-\text{⚥}} h^{\male,K}_{\bar{S}^K_u} + f^2_{\female+\male-\text{⚥}} h^{\text{⚥},K}_{\bar{S}^K_u}, \bar{S}^{\text{⚥},K}_u\big) \\
& + \big(f^2_{\text{⚥}-\female-\male} K\rho^K_{\bar{S}^K_u}, \bar{S}^{\female,K}_u \otimes \bar{S}^{\male,K}_u\big) \Big\} du.
\end{aligned}
$$

For the convergence of the sequence of processes $\bar{S}^K$, we need conditions like (C0)–(C3), with conditions on ρ as well:

(C0′) In addition to (C0), $K\rho^K$ is bounded.

(C1′) (C1) holds also for q^K being $K\rho^K$.

(C2′) (C2) holds, and $\lim_{K\to\infty} K\rho^K_\mu =: \rho^\infty_\mu$.

(C3′) Same as (C3).

Theorem 3. *Under the smooth demography conditions (C0')–(C3'), the scaled process* $\bar{S}^K$ *converges weakly in the Skorokhod space* $\mathbb{D}(\mathbb{T}, \mathcal{M}(\mathbb{S}))$*, as* $K \to \infty$*, to a deterministic measure-valued process* $\bar{S}$ *satisfying*

$$
\begin{aligned}
(f, \bar{S}_t) = {} & (f, \bar{S}_0) + \int_0^t \big(\partial_v f + \partial_w f, \bar{S}_u\big) du + \int_0^t \Big(\textstyle\sum_\circ f_\circ(0) m^{\circ,\infty}_{\bar{S}_u} b^\infty_{\bar{S}_u}, \bar{S}^{\text{⚥}}_u + \bar{S}^{\female}_u\Big) du \\
& - \int_0^t \textstyle\sum_\circ (f_\circ h^{\circ,\infty}_{\bar{S}_u}, \bar{S}^{\circ}_u) du + \displaystyle\int_0^t \big(f_{\male-\text{⚥}} h^{\female,\infty}_{\bar{S}_u} + f_{\female-\text{⚥}} h^{\male,\infty}_{\bar{S}_u} + f_{\female+\male-\text{⚥}} h^{\text{⚥},\infty}_{\bar{S}_u}, \bar{S}^{\text{⚥}}_u\big) du \\
& + \int_0^t \big(f_{\text{⚥}-\female-\male} \rho^\infty_{\bar{S}_u}, \bar{S}^{\female}_u \otimes \bar{S}^{\male}_u\big) du. \qquad (11)
\end{aligned}
$$

From this, we obtain a system of partial differential equations for the age densities of the three subpopulations. Suppose $a^{\text{♀}}(v,t)$, $a^{\text{♂}}(w,t)$, and $a^{\text{⚤}}(v,w,t)$ are the densities (in v and w) of $\bar{S}^i_t$, $i=\text{♀},\text{♂},\text{⚤}$, respectively. Then the densities satisfy the following:

$$
\begin{aligned}
&\partial_t a^{\text{♀}}(v,t)+\partial_v a^{\text{♀}}(v,t)=-h^{\text{♀},\infty}_{\bar{S}_t}(v,\infty)a^{\text{♀}}(v,t)\\
&\qquad+\int\big(h^{\text{♂},\infty}_{\bar{S}_t}(v,w)+h^{\text{⚤},\infty}_{\bar{S}_t}(v,w)\big)a^{\text{⚤}}(v,w,t)dw-\int\rho^{\infty}_{\bar{S}_t}(v,w)a^{\text{♂}}(w,t)dw\,a^{\text{♀}}(v,t),\\
&\partial_t a^{\text{♂}}(w,t)+\partial_w a^{\text{♂}}(w,t)=-h^{\text{♂},\infty}_{\bar{S}_t}(\infty,w)a^{\text{♂}}(w,t)\\
&\qquad+\int\big(h^{\text{♀},\infty}_{\bar{S}_t}(v,w)+h^{\text{⚤},\infty}_{\bar{S}_t}(v,w)\big)a^{\text{⚤}}(v,w,t)dv-\int\rho^{\infty}_{\bar{S}_t}(v,w)a^{\text{♀}}(v,t)dv\,a^{\text{♂}}(w,t),\\
&\partial_t a^{\text{⚤}}(v,w,t)+\partial_v a^{\text{⚤}}(v,w,t)+\partial_w a^{\text{⚤}}(v,w,t)=\\
&\qquad-\big(h^{\text{♀},\infty}_{\bar{S}_t}(v,w)+h^{\text{♂},\infty}_{\bar{S}_t}(v,w)+h^{\text{⚤},\infty}_{\bar{S}_t}(v,w)\big)a^{\text{⚤}}(v,w,t)+\rho^{\infty}_{\bar{S}_t}(v,w)a^{\text{♀}}(v,t)a^{\text{♂}}(w,t),
\end{aligned}
$$

with boundary conditions $a^{\text{⚤}}(0,w,t)=0$, $a^{\text{⚤}}(v,0,t)=0$, and, for $\circ=\text{♀},\text{♂}$,

$$
a^{\circ}(0,t)=\int\int m^{\circ,\infty}_{\bar{S}_t}(v,w)b^{\infty}_{\bar{S}_t}(v,w)a^{\text{⚤}}(v,w,t)dvdw+\int m^{\circ,\infty}_{\bar{S}_t}(v,\infty)b^{\infty}_{\bar{S}_t}(v,\infty)a^{\text{♀}}(v,t)dv.
$$

This is comparable to the system given by Fredrickson in [10].

3.3. Central limit theorem

Let $Z^K=\sqrt{K}(\bar{S}^K-\bar{S})$. Then, for any $f\in C^1$ and $t\in\mathbb{T}$,

$$
\begin{aligned}
&(f,Z^K_t)=(f,Z^K_0)+\int_0^t\big(\partial_v f+\partial_w f,Z^K_u\big)du\\
&+\int_0^t\sqrt{K}\Big(\textstyle\sum_\circ f_\circ(0)\big[m^{\circ,K}_{\bar{S}^K_u}b^K_{\bar{S}^K_u}-m^{\circ,\infty}_{\bar{S}_u}b^{\infty}_{\bar{S}_u}\big],\bar{S}^{\text{⚤}}_u+\bar{S}^{\text{♀}}_u\Big)du\\
&+\int_0^t\Big(\textstyle\sum_\circ f_\circ(0)m^{\circ,K}_{\bar{S}^K_u}b^K_{\bar{S}^K_u},Z^{\text{⚤},K}_u+Z^{\text{♀},K}_u\Big)du\\
&-\int_0^t\sqrt{K}\textstyle\sum_\circ\big(f_\circ\big[h^{\circ,K}_{\bar{S}^K_u}-h^{\circ,\infty}_{\bar{S}_u}\big],\bar{S}^{\circ}_u\big)du-\int_0^t\sum_\circ\big(f_\circ h^{\circ,K}_{\bar{S}^K_u},Z^{\circ,K}_u\big)du\\
&+\int_0^t\sqrt{K}\Big(f_{\text{♂}-\text{⚤}}\big[h^{\text{♀},K}_{\bar{S}^K_u}-h^{\text{♀},\infty}_{\bar{S}_u}\big]+f_{\text{♀}-\text{⚤}}\big[h^{\text{♂},K}_{\bar{S}^K_u}-h^{\text{♂},\infty}_{\bar{S}_u}\big]+f_{\text{♀}+\text{♂}-\text{⚤}}\big[h^{\text{⚤},K}_{\bar{S}^K_u}-h^{\text{⚤},\infty}_{\bar{S}_u}\big],\bar{S}^{\text{⚤}}_u\Big)du\\
&+\int_0^t\Big(f_{\text{♂}-\text{⚤}}h^{\text{♀},K}_{\bar{S}^K_u}+f_{\text{♀}-\text{⚤}}h^{\text{♂},K}_{\bar{S}^K_u}+f_{\text{♀}+\text{♂}-\text{⚤}}h^{\text{⚤},K}_{\bar{S}^K_u},Z^{\text{⚤},K}_u\Big)du\\
&+\int_0^t\sqrt{K}\Big(f_{\text{⚤}-\text{♀}-\text{♂}}\big[K\rho^K_{\bar{S}^K_u}-\rho^{\infty}_{\bar{S}_u}\big],\bar{S}^{\text{♀}}_u\otimes\bar{S}^{\text{♂}}_u\Big)du+\int_0^t\Big(f_{\text{⚤}-\text{♀}-\text{♂}}K\rho^K_{\bar{S}^K_u},Z^{\text{♀},K}_u\otimes\bar{S}^{\text{♂}}_u\Big)du\\
&+\int_0^t\Big(f_{\text{⚤}-\text{♀}-\text{♂}}K\rho^K_{\bar{S}^K_u},\bar{S}^{\text{♀}}_u\otimes Z^{\text{♂},K}_u\Big)du+\tilde{M}^{f,K}_t,
\end{aligned}
\tag{12}
$$

where $\tilde{M}_t^{f,K}$ is a square-integrable martingale with predictable quadratic variation

$$\left\langle \tilde{M}^{f,K} \right\rangle_t = \int_0^t \left(\left[f_♀^2(0)\gamma_{\bar{S}_u^K}^{♀♀,K} + f_♂^2(0)\gamma_{\bar{S}_u^K}^{♂♂,K} + 2f_♀(0)f_♂(0)\gamma_{\bar{S}_u^K}^{♀♂,K} \right] b_{\bar{S}_u^K}^K, \bar{S}_u^{⚥,K} + \bar{S}_u^{♀,K} \right) du$$
$$+ \int_0^t \left\{ \sum\nolimits_○ \left(f_○^2 h_{\bar{S}_u^K}^{○,K}, \bar{S}_u^{○,K} \right) + \left(f_{♂-⚥}^2 h_{\bar{S}_u^K}^{♀,K} + f_{♀-⚥}^2 h_{\bar{S}_u^K}^{♂,K} + f_{♀+♂-⚥}^2 h_{\bar{S}_u^K}^{⚥,K}, \bar{S}_u^{⚥,K} \right) \right\} du$$
$$+ \int_0^t \left(f_{⚥-♀-♂}^2 K\rho_{\bar{S}_u^K}^K, \bar{S}_u^{♀,K} \otimes \bar{S}_u^{♂,K} \right) du.$$

For the central limit theorem, we need to include the additional parameter ρ in conditions (A0)–(A4):

(A0′) Same as (A0).

(A1′) Same as (A1).

(A2′) In addition to (A2), the same holds for q^K being $K\rho^K$ and q^∞ being ρ^∞.

(A3′) (A3) holds for $q = b, h, m, \rho$.

(A4′) Same as (A4).

Theorem 4. *Under the assumptions (C0′)–(C3′) and (A0′)–(A4′), the process $(Z_t^K)_{t\in\mathbb{T}}$ converges weakly in $\mathbb{D}(\mathbb{T}, W^{-4})$, as $K \to \infty$, to the process $(Z_t)_{t\in\mathbb{T}}$ that satisfies, for $f \in W^4$ and $t \in \mathbb{T}$,*

$$\begin{aligned}
(f, Z_t) &= (f, Z_0) + \int_0^t \left(\partial_v f + \partial_w f, Z_u \right) du + \int_0^t \left(\sum\nolimits_○ f_○(0) \partial_S \left(m_{\bar{S}_u}^{○,\infty} b_{\bar{S}_u}^{\infty} \right)(Z_u), \bar{S}_u^{⚥} + \bar{S}_u^{♀} \right) du \\
&+ \int_0^t \left(\sum\nolimits_○ f_○(0) m_{\bar{S}_u}^{○,\infty} b_{\bar{S}_u}^{\infty}, Z_u^{⚥} + Z_u^{♀} \right) du - \int_0^t \sum\nolimits_○ \left(f_○ \partial_S h_{\bar{S}_u}^{○,\infty}(Z_u), \bar{S}_u^{○} \right) du - \int_0^t \sum\nolimits_○ \left(f_○ h_{\bar{S}_u}^{○,\infty}, Z_u^{○} \right) du \\
&+ \int_0^t \left(f_{♂-⚥} \partial_S h_{\bar{S}_u}^{♀,\infty}(Z_u) + f_{♀-⚥} \partial_S h_{\bar{S}_u}^{♂,\infty}(Z_u) + f_{♀+♂-⚥} \partial_S h_{\bar{S}_u}^{⚥,\infty}(Z_u), \bar{S}_u^{⚥} \right) du \\
&+ \int_0^t \left(f_{♂-⚥} h_{\bar{S}_u}^{♀,\infty} + f_{♀-⚥} h_{\bar{S}_u}^{♂,\infty} + f_{♀+♂-⚥} h_{\bar{S}_u}^{⚥,\infty}, Z_u^{⚥} \right) du \\
&+ \int_0^t \left(f_{⚥-♀-♂} \partial_S \rho_{\bar{S}_u}^{\infty}(Z_u), \bar{S}_u^{♀} \otimes \bar{S}_u^{♂} \right) du \\
&+ \int_0^t \left(f_{⚥-♀-♂} \rho_{\bar{S}_u}^{\infty}, Z_u^{♀} \otimes \bar{S}_u^{♂} \right) du + \int_0^t \left(f_{⚥-♀-♂} \rho_{\bar{S}_u}^{\infty}, \bar{S}_u^{♀} \otimes Z_u^{♂} \right) du + \tilde{M}_t^{f,\infty},
\end{aligned} \tag{13}$$

where $\tilde{M}^{f,\infty}$ is a continuous Gaussian martingale with predictable quadratic variation

$$\left\langle \tilde{M}^{f,\infty} \right\rangle_t = \int_0^t \left(\left[f_♀^2(0)\gamma_{\bar{S}_u}^{♀♀,\infty} + f_♂^2(0)\gamma_{\bar{S}_u}^{♂♂,\infty} + 2f_♀(0)f_♂(0)\gamma_{\bar{S}_u}^{♀♂,\infty} \right] b_{\bar{S}_u}^{\infty}, \bar{S}_u^{⚥} + \bar{S}_u^{♀} \right) du$$
$$+ \int_0^t \left\{ \sum\nolimits_○ \left(f_○^2 h_{\bar{S}_u}^{○,\infty}, \bar{S}_u^{○} \right) + \left(f_{♂-⚥}^2 h_{\bar{S}_u}^{♀,\infty} + f_{♀-⚥}^2 h_{\bar{S}_u}^{♂,\infty} + f_{♀+♂-⚥}^2 h_{\bar{S}_u}^{⚥,\infty}, \bar{S}_u^{⚥} \right) \right\} du$$
$$+ \int_0^t \left(f_{⚥-♀-♂}^2 \rho_{\bar{S}_u}^{\infty}, \bar{S}_u^{♀} \otimes \bar{S}_u^{♂} \right) du.$$

4. Proofs for Section 2

The proofs are similar in spirit to those in [9].

4.1. Proofs on the model set-up

Recall the representation of S_t in (1). We can write $(f, S_t) = \sum_{i\in\mathbb{K}} (f(i, \cdot), S_t^{(i)})$, with

$$S_t^{(i)}(dv) = S_t(\{i\}, dv) = \sum_{x\in I} \mathbf{1}_{\tau_x \leqslant t < \sigma_x} \mathbf{1}_{\kappa_x = i} \delta_{t-\tau_x}(dv)$$

being the age structure of the subpopulation of type i individuals. Let

$$B^i(t) = \sum_{x\in I} \mathbf{1}_{\kappa_x = i} \mathbf{1}_{\tau_x \leqslant t}$$

be the number of individuals of type i born by time t, including those that were alive before time 0, and write $B^i((0, t]) = \int_0^t B^i(du) = B^i(t) - B^i(0)$. Let

$$D((i, v), t) = \sum_{x\in I} \mathbf{1}_{\kappa_x = i} \mathbf{1}_{\lambda_x \leqslant v} \mathbf{1}_{\sigma_x \leqslant t}$$

be the number of individuals of type i who died by time t and whose lifespan was not greater than v. Using the single-type equation in [9, Proposition 1] and summing over the types, we obtain, for $f \in C^1(\mathbb{S})$,

$$(f, S_t) = (f, S_0) + \int_0^t (f', S_u)du + \sum_{i\in\mathbb{K}} f(i, 0)B^i((0, t]) - \int_{\mathbb{S}\times[0,t]} f(s)D(ds, du). \tag{14}$$

We then obtain (2) by compensating the last two terms in (14).

For each i,

$$M_{\check{B}^i, f}(t) := f(i, 0)\big(\check{B}^i((0, t]) - \int_0^t (b_{S_u} \check{m}^i_{S_u}, S_u)du\big)$$

is a martingale with predictable quadratic variation

$$\langle M_{\check{B}^i, f}\rangle_t = f^2(i, 0) \int_0^t (b_{S_u} \check{\gamma}^{ii}_{S_u}, S_u)du,$$

with $\check{B}$ denoting the number of individuals born through bearings of mothers by time t. We sum over i to obtain the compensated process

$$M_{\check{B}, f}(t) := \sum_{i\in\mathbb{K}} f(i, 0)\big(\check{B}^i((0, t]) - \int_0^t \big(b_{S_u} \check{m}^i_{S_u}, S_u\big)du\big)$$

as a martingale with the predictable quadratic variation

$$\langle M_{\check{B}, f}\rangle_t = \sum_{i_1\in\mathbb{K}} \sum_{i_2\in\mathbb{K}} f(i_1, 0)f(i_2, 0) \int_0^t \big(b_{S_u} \check{\gamma}^{i_1 i_2}_{S_u}, S_u\big)du. \tag{15}$$

Note that

$$\big[M_{\check{B}^{i_1}, f}, M_{\check{B}^{i_2}, f}\big]_t = \sum_{u\leqslant t} \Delta M_{\check{B}^{i_1}, f}(u) \Delta M_{\check{B}^{i_2}, f}(u).$$

For $i_1 \neq i_2$, the product of jumps

$$\Delta M_{\check{B}^{i_1},f}(u)\Delta M_{\check{B}^{i_2},f}(u)$$

is nonzero precisely when there are births of both types, in which case it is equal to

$$\sum_{x\in I} f(i_1, 0)f(i_2, 0)\mathbf{1}_{\sigma_x>u}\Bigg(\sum_{j\in\mathbb{N}} \mathbf{1}_{\tau_{xj}=u}\mathbf{1}_{\kappa_{xj}=i_1}\Bigg)\Bigg(\sum_{j\in\mathbb{N}} \mathbf{1}_{\tau_{xj}=u}\mathbf{1}_{\kappa_{xj}=i_2}\Bigg).$$

Thus,

$$\langle M_{\check{B}^{i_1},f}, M_{\check{B}^{i_2},f}\rangle_t = f(i_1, 0)f(i_2, 0)\int_0^t \big(b_{S_u}\check{\gamma}_{S_u}^{i_1,i_2}, S_u\big)du$$

is a compensator of $\big[M_{\check{B}^{i_1},f}, M_{\check{B}^{i_2},f}\big]_t$, and since

$$\langle M_{\check{B},f}\rangle_t = \sum_{i_1\in\mathbb{K}}\sum_{i_2\in\mathbb{K}} \langle M_{\check{B}^{i_1},f}, M_{\check{B}^{i_2},f}\rangle_t,$$

(15) follows. In a similar manner, with $\hat{B}$ denoting the number of individuals generated through splitting by time t,

$$M_{\hat{B},f}(t) := \sum_{i\in\mathbb{K}} f(i, 0)\Big(\hat{B}^i((0, t]) - \int_0^t \big(h_{S_u}\hat{m}^i_{S_u}, S_u\big)du\Big)$$

is a martingale with

$$\langle M_{\hat{B},f}\rangle_t = \sum_{i_1\in\mathbb{K}}\sum_{i_2\in\mathbb{K}} f(i_1, 0)f(i_2, 0)\int_0^t \big(h_{S_u}\hat{\gamma}_{S_u}^{i_1 i_2}, S_u\big)du,$$

and $M_{D,f}(t) := \int_{\mathbb{S}\times[0,t]} f(s)D(ds, du) - \int_0^t (h_{S_u}f, S_u)du$ is a martingale with

$$\langle M_{D,f}\rangle_t = \int_0^t \big(h_{S_u}f^2, S_u\big)du.$$

Now, let $M_t^f = M_{\check{B},f}(t) + M_{\hat{B},f}(t) - M_{D,f}(t)$. Analysing cross-terms and adding them together, we have, since $\langle M_{\check{B},f}, M_{\hat{B},f}\rangle_t = 0$ and $\langle M_{\check{B},f}, M_{D,f}\rangle_t = 0$, that

$$\begin{aligned}\langle M^f\rangle_t &= \langle M_{\check{B},f}\rangle_t + \langle M_{\hat{B},f}\rangle_t + \langle M_{D,f}\rangle_t - 2\langle M_{\hat{B},f}, M_{D,f}\rangle_t \\ &= \int_0^t \Bigg(\sum_{i_1\in\mathbb{K}}\sum_{i_2\in\mathbb{K}} f(i_1, 0)f(i_2, 0)w_{S_u}^{i_1 i_2} + h_{S_u}f^2 - 2\sum_{i\in\mathbb{K}} f(i, 0)h_{S_u}\hat{m}^i_{S_u}f, S_u\Bigg)du.\end{aligned}$$

In fact, it can further be shown that, for f, g on $\mathbb{S}$,

$$\begin{aligned}\langle M^f, M^g\rangle_t = \int_0^t \Bigg(&\sum_{i_1\in\mathbb{K}}\sum_{i_2\in\mathbb{K}} f(i_1, 0)g(i_2, 0)w_{S_u}^{i_1 i_2} + h_{S_u}fg \\ &- \sum_{i\in\mathbb{K}} \big(f(i, 0)g + g(i, 0)f\big)h_{S_u}\hat{m}^i_{S_u}, S_u\Bigg)du.\end{aligned}$$

4.2. Proof of the law of large numbers

This is done by checking the tightness of the scaled sequence $\bar{S}^K$ and the uniqueness of the limiting process. By Jakubowski [23, Theorem 4.6], $\{\bar{S}^K\}$ is tight in $\mathbb{D}(\mathbb{T}, \mathcal{M})$ if the following hold:

(J1) For each $\eta > 0$, there exists a compact set $\mathcal{C}_\eta \in \mathcal{M}$ such that

$$\liminf_{K\to\infty} \mathbb{P}\big(\bar{S}_t^K \in \mathcal{C}_\eta \ \forall t \in \mathbb{T}\big) > 1 - \eta.$$

(J2) For each $f \in C^1$, $\{(f, \bar{S}^K)\}$ is tight in $\mathbb{D}(\mathbb{T}, \mathbb{R})$.

In terms of the semimartingale decomposition $(f, \bar{S}_t^K) = \bar{V}_t^{f,K} + \bar{M}_t^{f,K}$, (J2) reduces to the following Aldous–Rebolledo criteria:

(J2a) For each $t \in \mathbb{T}$, $(f, \bar{S}_t^K)$ is tight; that is, for each $\epsilon > 0$, there exists $\delta > 0$ such that for all K, $\mathbb{P}\big(|(f, \bar{S}_t^K)| > \delta\big) < \epsilon$.

(J2b) For each $\epsilon_1, \epsilon_2 > 0$, there exist $\delta > 0$ and $K_0 \geqslant 1$ such that for every sequence of stopping times $\tau^K \leqslant T$,

$$\sup_{K>K_0} \sup_{\zeta<\delta} \mathbb{P}\Big(|\bar{V}^{f,K}_{(\tau^K+\zeta)\wedge T} - \bar{V}^{f,K}_{\tau^K}| > \epsilon_1\Big) < \epsilon_2, \tag{i}$$

$$\sup_{K>K_0} \sup_{\zeta<\delta} \mathbb{P}\Big(\big|\langle \bar{M}^{f,K}\rangle_{(\tau^K+\zeta)\wedge T} - \langle \bar{M}^{f,K}\rangle_{\tau^K}\big| > \epsilon_1\Big) < \epsilon_2. \tag{ii}$$

4.2.1 *Preliminary estimates.* Recall the operator L_S^K defined in (5). Let

$$\hat{L}_S^K f = -h_S^K f + \sum_{i\in\mathbb{K}} f(i, 0) n_S^{i,K}, \tag{16}$$

so that $L_S^K f = f' + \hat{L}_S^K f$. Define also the operator Π_S^K such that

$$\Pi_S^K f = \sum_{i_1\in\mathbb{K}} \sum_{i_2\in\mathbb{K}} f(i_1, 0) f(i_2, 0) w_S^{i_1 i_2, K} + h_S^K f^2 - 2\sum_{i\in\mathbb{K}} f(i, 0) h_S^K \hat{m}_S^{i,K} f,$$

so that

$$\langle M^{f,K}\rangle_t = \int_0^t \big(\Pi^K_{S_u^K} f, S_u^K\big) du.$$

Proposition 2. *Suppose (C0) holds. Then, for any $f \in C^0$,*

$$|\Pi_S^K f| \leqslant c\|f\|_{C^0}^2 \quad \text{and} \quad |\hat{L}_S^K f| \leqslant c\|f\|_{C^0},$$

and for any $f \in C^1$,

$$|L_S^K f| \leqslant c\|f\|_{C^1}.$$

In particular, $|\Pi_S^K 1| \leqslant c$ and $|L_S^K 1| = |\hat{L}_S^K 1| \leqslant c$.

Proof. This is due to the boundedness of the parameters and the definition of $\|f\|_{C^j}$. □

Proposition 3. *Suppose (C0) and (C3) hold. Then*

$$\mathbb{E}\big[\big(1, \bar{S}_t^K\big)\big] \leqslant \big(1, \bar{S}_0^K\big) e^{ct} \tag{17}$$

and

$$\sup_{K\geqslant 1}\mathbb{E}\Big[\sup_{t\leqslant T}\big(1,\bar{S}_t^K\big)\Big]<\infty. \tag{18}$$

Proof. From (4) and Proposition 2,

$$\big(1,\bar{S}_t^K\big)\leqslant(1,\bar{S}_0^K)+c_1\int_0^t(1,\bar{S}_u^K)du+\frac{1}{K}M_t^{1,K}. \tag{19}$$

Taking the expectation and applying Gronwall's inequality, we establish the first statement.

For the second statement, from (19), taking the supremum and applying (17), followed by Doob's inequality and Proposition 2, we arrive at

$$\mathbb{E}\Big[\sup_{t\leqslant T}\big(1,\bar{S}_t^K\big)\Big]\leqslant(1,\bar{S}_0^K)(1+c_1e^{c_1T}T)+\frac{1}{\sqrt{K}}c_2(1,\bar{S}_0^K)^{1/2}e^{c_3T}T^{1/2},$$

which is bounded in K by (C3). □

4.2.2 *Tightness of* $\bar{S}^K$.

Proposition 4. *Assume (C0) and (C3). Then the sequence* $\bar{S}^K$ *is tight in* $\mathbb{D}(\mathbb{T},\mathcal{M})$.

Proof. First, we show that (J1) holds. From Markov's inequality and (18), we have $\mathbb{P}\big(\sup_{t\leqslant T}\big(1,\bar{S}_t^K\big)>\delta\big)\leqslant\frac{c}{\delta}$ and the existence of δ_η such that $\mathbb{P}(\sup_{t\leqslant T}\big(1,\bar{S}_t^K\big)>\delta_\eta)\leqslant\eta$. Let $\mathcal{C}(\delta)=\{\mu\in\mathcal{M}:(1,\mu)\leqslant\delta\}$. Then $\mathcal{C}(\delta)$ is compact and for any $\eta>0$, $\liminf_{K\to\infty}\mathbb{P}\big(\bar{S}_t^K\in\mathcal{C}(\delta_\eta)\;\forall t\in\mathbb{T}\big)>1-\eta$.

Next, we show that (J2a) and (J2b) hold. The condition (J2a) is immediate from Markov's inequality and by using (17). For (J2b), note that Proposition 2 gives

$$|\bar{V}^{f,K}_{(\tau^K+\zeta)\wedge T}-\bar{V}^{f,K}_{\tau^K}|\leqslant c_1\|f\|_{C^1}\int_0^\zeta(1,\bar{S}^K_{(\tau^K+u)\wedge T})du\leqslant c_1\|f\|_{C^1}\delta\sup_{t\leqslant T}(1,\bar{S}_t^K),$$

and with the same trick,

$$\big|\big\langle\bar{M}^{f,K}\big\rangle_{(\tau^K+\zeta)\wedge T}-\big\langle\bar{M}^{f,K}\big\rangle_{\tau^K}\big|\leqslant c_2\|f\|^2_{C^1}\delta\frac{1}{K}\sup_{t\leqslant T}(1,\bar{S}_t^K).$$

Taking the expectation in each and using (18), we can establish (J2b)(i) and (J2b)(ii) with Markov's inequality. □

Remark 4. From the proof of Proposition 4, we can see that the martingale $\bar{M}^{f,K}=\frac{1}{K}M^{f,K}$ is tight for any $f\in C^1$. Moreover, it converges to 0 as K tends to infinity, since the predictable quadratic variation vanishes.

4.2.3 *Convergence of* $\bar{S}^K$ *and the limiting process.* Tightness implies the existence of a subsequence that converges. We now identify the limit of $\bar{S}^K$ and show the uniqueness of the limiting process.

Proposition 5. *Suppose (C0)–(C3) hold. Every limit point* $\mathcal{S}$ *of the sequence* $\bar{S}^K$ *satisfies the equation*

$$(f,\mathcal{S}_t)=(f,\mathcal{S}_0)+\int_0^t\big(L^\infty_{\mathcal{S}_u}f,\mathcal{S}_u\big)du \tag{20}$$

for any $f\in C^1$ *and* $t\in\mathbb{T}$*, where* $L^\infty_{\mathcal{S}}f$ *is as defined in* (7).

Proof. From Remark 4, the martingale sequence $\bar{M}^{f,K}$ vanishes as K tends to infinity. Given this together with the convergence of $\bar{S}_0^K$ by (C3), it remains to show the convergence of

$$\int_0^t (L^K_{\bar{S}^K_u} f, \bar{S}^K_u)du$$

to

$$\int_0^t (L^\infty_{\mathcal{S}_u} f, \mathcal{S}_u)du.$$

Note that, by (C1) and (C2),

$$\|h^K_{\bar{S}^K_u} - h^\infty_{\mathcal{S}_u}\|_\infty \leqslant \|h^K_{\bar{S}^K_u} - h^K_{\mathcal{S}_u}\|_\infty + \|h^K_{\mathcal{S}_u} - h^\infty_{\mathcal{S}_u}\|_\infty \to 0,$$

and similarly for the other model parameters. Thus, $\|L^K_{\bar{S}^K_u} f - L^\infty_{\mathcal{S}_u} f\|_\infty \to 0$ and

$$\int_0^t \left|(L^K_{\bar{S}^K_u} f - L^\infty_{\mathcal{S}_u} f, \bar{S}^K_u)\right| du \leqslant \int_0^t \left|\left|L^K_{\bar{S}^K_u} f - L^\infty_{\mathcal{S}_u} f\right|\right|_\infty (1, \bar{S}^K_u)du \to 0$$

by the dominated convergence theorem and Proposition 3. Note also that, for any $f \in C^1$, $|L^\infty_S f| \leqslant \|f'\|_\infty + c_1\|f\|_\infty \leqslant c_2\|f\|_{C^1}$. Thus,

$$\int_0^t \left|(L^\infty_{\mathcal{S}_u} f, \bar{S}^K_u - \mathcal{S}_u)\right| du \leqslant \int_0^t \|L^\infty_{\mathcal{S}_u} f\|_\infty \|\bar{S}^K_u - \mathcal{S}_u\| du \leqslant c_2\|f\|_{C^1} \int_0^t \|\bar{S}^K_u - \mathcal{S}_u\| du$$

vanishes as $K \to \infty$. Hence,

$$\begin{aligned}\left| \int_0^t (L^K_{\bar{S}^K_u} f, \bar{S}^K_u)du - \int_0^t (L^\infty_{\mathcal{S}_u} f, \mathcal{S}_u)du \right| \\ \leqslant \int_0^t \left(\left|(L^K_{\bar{S}^K_u} f - L^\infty_{\mathcal{S}_u} f, \bar{S}^K_u)\right| + \left|(L^\infty_{\mathcal{S}_u} f, \bar{S}^K_u - \mathcal{S}_u)\right| \right) du\end{aligned}$$

converges to zero. □

It remains to show the uniqueness of the solution to (20). To do this, we introduce an alternative representation to (20).

Proposition 6. *For $\phi \in C^1$, define the shift operator $\hat{\Theta}_r$ such that for each $r \in \mathbb{T}$ and $s = (i, v)$, $v \in [0, \omega - r]$, $\hat{\Theta}_r \phi(s) = \hat{\Theta}_r \phi(i, v) = \phi(i, v + r)$ and $\hat{\Theta}_r \phi \in C^0$ with $\|\hat{\Theta}_r \phi\|_{C^0} \leqslant c\|\phi\|_{C^0}$. (This is possible by reflecting ϕ about $v = \omega - r$, in which case $\|\hat{\Theta}_r \phi\|_{C^0} \leqslant \|\phi\|_{C^0}$.) Then (20) is equivalent to the following: for $\phi \in C^1$,*

$$(\phi, \mathcal{S}_t) = (\hat{\Theta}_t \phi, \mathcal{S}_0) + \int_0^t \left(-h^\infty_{\mathcal{S}_u} \hat{\Theta}_{t-u}\phi + \sum_{i \in \mathbb{K}} \hat{\Theta}_{t-u}\phi(i, 0) n^{i,\infty}_{\mathcal{S}_u}, \mathcal{S}_u \right) du. \tag{21}$$

Proof. This can be shown by writing an equation for $(g(\cdot, t), \mathcal{S}_t)$ with g of the form $g(s, t) = f(s)\varphi(t)$, where f and φ are functions in $C^1(\mathbb{S})$ and $C^1(\mathbb{T})$ respectively. We then apply the monotone class theorem to extend it to $g \in C^{1,1}(\mathbb{S} \times \mathbb{T})$. Next, fixing $t \in \mathbb{T}$ and $\phi \in C^1(\mathbb{S})$, we take $g(s, u) = \hat{\Theta}_{t-u}\phi(s) = \phi(i, v + t - u)$ for $s \in \mathbb{S}$ and $u \in [0, t]$; (21) follows.

From (21), we can also recover (20) by noting that $\hat{\Theta}_{t-u}\phi(s) = \phi(s) + \int_u^t \hat{\Theta}_{w-u}\phi'(s)dw$ and applying Fubini's theorem. □

Now, we can show the uniqueness of the solution to (20) by showing the uniqueness of the solution to (21).

Proposition 7. *If $\mathcal{S}^1$ and $\mathcal{S}^2$ both satisfy (21) with $\mathcal{S}^1_0 = \mathcal{S}^2_0$, then $\mathcal{S}^1 = \mathcal{S}^2$.*

Proof. Let

$$\hat{L}^{\infty}_S f = -h^{\infty}_S f + \sum_{i\in\mathbb{K}} f(i, 0) n^{i,\infty}_S,$$

so that (21) becomes

$$(\phi, \mathcal{S}_t) = (\hat{\Theta}_t\phi, \mathcal{S}_0) + \int_0^t \big(\hat{L}^{\infty}_{\mathcal{S}_u}\hat{\Theta}_{t-u}\phi, \mathcal{S}_u\big)du.$$

We have, for $\phi \in C^1$,

$$\begin{aligned}|(\phi, \mathcal{S}^1_t - \mathcal{S}^2_t)| &\leqslant \int_0^t \big|\big(\hat{L}^{\infty}_{\mathcal{S}^1_u}\hat{\Theta}_{t-u}\phi - \hat{L}^{\infty}_{\mathcal{S}^2_u}\hat{\Theta}_{t-u}\phi, \mathcal{S}^1_u\big)\big| + \big|\big(\hat{L}^{\infty}_{\mathcal{S}^2_u}\hat{\Theta}_{t-u}\phi, \mathcal{S}^1_u - \mathcal{S}^2_u\big)\big|du \\ &\leqslant \int_0^t \|\hat{L}^{\infty}_{\mathcal{S}^1_u}\hat{\Theta}_{t-u}\phi - \hat{L}^{\infty}_{\mathcal{S}^2_u}\hat{\Theta}_{t-u}\phi\|_\infty(1, \mathcal{S}^1_u) + \|\hat{L}^{\infty}_{\mathcal{S}^2_u}\hat{\Theta}_{t-u}\phi\|_\infty\|\mathcal{S}^1_u - \mathcal{S}^2_u\|du.\end{aligned}$$

Now, by (C1),

$$\|h^{\infty}_\mu - h^{\infty}_\nu\|_\infty \leqslant \|h^{\infty}_\mu - h^K_\mu\|_\infty + c_1\|\mu - \nu\| + \|h^K_\nu - h^{\infty}_\nu\|_\infty;$$

taking the limit as $K \to \infty$ on both sides, we obtain $\|h^{\infty}_\mu - h^{\infty}_\nu\|_\infty \leqslant c_1\|\mu - \nu\|$. A similar argument applies for other model parameters. Thus, $\|\hat{L}^{\infty}_\mu f - \hat{L}^{\infty}_\nu f\|_\infty \leqslant c_2\|f\|_\infty\|\mu - \nu\|$. We also have by (C0) and (C2) that $\|\hat{L}^{\infty}_\mu f\|_\infty \leqslant c_3\|f\|_\infty$. Finally, since $\|\hat{\Theta}_r\phi\|_{C^0} \leqslant c\|\phi\|_{C^0}$ for any $r \in \mathbb{T}$ and $(1, \mathcal{S}^1_u) \leqslant (1, \mathcal{S}^1_0)e^{c_6 u}$ by Gronwall's inequality, we have

$$\begin{aligned}|(\phi, \mathcal{S}^1_t - \mathcal{S}^2_t)| &\leqslant \int_0^t \big(c_4\|\phi\|_{C^0}\|\mathcal{S}^1_u - \mathcal{S}^2_u\|(1, \mathcal{S}^1_u) + c_5\|\phi\|_{C^0}\|\mathcal{S}^1_u - \mathcal{S}^2_u\|\big)du \\ &\leqslant c_7\|\phi\|_{C^0}\big(1 + (1, \mathcal{S}^1_0)e^{c_6 T}\big)\int_0^t \|\mathcal{S}^1_u - \mathcal{S}^2_u\|du. \qquad (22)\end{aligned}$$

Note that for any $\phi \in C^0$, there exists a sequence $\{\phi_n\}$ such that $\phi_n \in C^1$ and ϕ_n converges uniformly to ϕ. Thus, (22) holds for any $\phi \in C^0$, and $\|\mathcal{S}^1_t - \mathcal{S}^2_t\| = 0$ by Gronwall's inequality. □

Putting all this together, we have established the convergence of $\bar{S}^K$ as stated in Theorem 1.

4.3. Proof of the central limit theorem

This is again done in the classical way, by showing the tightness of Z^K and the uniqueness of the limiting process. Z^K is tight in $\mathbb{D}(\mathbb{T}, W^{-4})$ if the following conditions are satisfied:

(T1) For every $t \in \mathbb{T}$, $(Z^K_t)_{K\geqslant 1}$ is tight in W^{-4}; that is, for every $\epsilon > 0$, there exists a compact set $\mathcal{C}_\epsilon$ such that $\mathbb{P}(Z^K_t \in \mathcal{C}_\epsilon) > 1 - \epsilon$ for all $K \geqslant 1$, or equivalently, $\mathbb{P}(Z^K_t \notin \mathcal{C}_\epsilon) \leqslant \epsilon$ for all $K \geqslant 1$.

(T2) Suppose $Z_t^K = \tilde{V}_t^K + \tilde{M}_t^K$. For each $\epsilon_1, \epsilon_2 > 0$, there exist $\delta > 0$ and $K_0 \geqslant 1$ such that for every sequence of stopping times $\tau^K \leqslant T$,

(T2a) $\sup_{K>K_0} \sup_{\zeta<\delta} \mathbb{P}(\|\tilde{V}^K_{(\tau^K+\zeta)\wedge T} - \tilde{V}^K_{\tau^K}\|_{W^{-4}} > \epsilon_1) < \epsilon_2$, and

(T2b) $\sup_{K>K_0} \sup_{\zeta<\delta} \mathbb{P}\big(\big|\langle\!\langle \tilde{M}^K \rangle\!\rangle_{(\tau^K+\zeta)\wedge T} - \langle\!\langle \tilde{M}^K \rangle\!\rangle_{\tau^K}\big| > \epsilon_1\big) < \epsilon_2$,

where $\langle\!\langle \tilde{M}^K \rangle\!\rangle$ is defined so that

$$\big(\|\tilde{M}_t^K\|_{W^{-4}}^2 - \langle\!\langle \tilde{M}^K \rangle\!\rangle_t\big)_{t\geqslant 0}$$

is a martingale.

4.3.1 *Alternative representation.* We start by giving an alternative representation of Z_t^K, which is used in establishing the convergence of Z^K.

Remark 5. As in [9], we take $\Theta_r\phi$ as the function on $\mathbb{S}$ such that for each $r \in \mathbb{T}$ and $s = (i, v)$, $v \in [0, \omega - r]$, $\Theta_r\phi(s) = \Theta_r\phi(i, v) = \phi(i, v + r)$ and, whenever $\phi \in W^j$, then $\Theta_r\phi \in W^j$ with

$$\|\Theta_r\phi\|_{W^j} \leqslant c\|\phi\|_{W^j}. \tag{23}$$

The existence of such a function was proved in [9].

Equation (8) can be extended to test functions dependent on t, that is, for $f \in C^{1,1}(\mathbb{S} \times \mathbb{T})$. Then, for fixed t, by taking $f(s, u) = \Theta_{t-u}\phi(s)$ for $u \leqslant t$, we have the following equation.

Proposition 8. *For $\phi \in C^1$ and $t \in \mathbb{T}$,*

$$\begin{aligned}(\phi, Z_t^K) &= (\Theta_t\phi, Z_0^K)\\ &\quad + \sqrt{K}\int_0^t \Big(-\Big(h^K_{\bar{S}^K_u} - h^\infty_{\bar{S}_u}\Big)\Theta_{t-u}\phi + \sum_{i\in\mathbb{K}} \Theta_{t-u}\phi(i, 0)\Big(n^{i,K}_{\bar{S}^K_u} - n^{i,\infty}_{\bar{S}_u}\Big), \bar{S}_u\Big)du\\ &\quad + \int_0^t \Big(-h^K_{\bar{S}^K_u}\Theta_{t-u}\phi + \sum_{i\in\mathbb{K}} \Theta_{t-u}\phi(i, 0)n^{i,K}_{\bar{S}^K_u}, Z_u^K\Big)du + \int_0^t \big(\Theta_{t-u}\phi, d\tilde{M}_u^K\big),\end{aligned} \tag{24}$$

where $\tilde{M} = \frac{1}{\sqrt{K}}M$, and M is the measure defined in Remark 3.

Proof. This can be proved by the monotone class theorem in a similar fashion to Proposition 6; see also [9, Section 5.2]. □

4.3.2 *Preliminary estimates.* Recall the operator $\hat{L}_S^K$ defined in (16) and its limit $\hat{L}_S^\infty$.

Proposition 9. *Assume (A2) and (A3). Then, for any $f \in W^j$, $j \in \mathbb{N}$, and $t \in \mathbb{T}$,*

$$\Big|\Big|\sqrt{K}\Big(\hat{L}^K_{\bar{S}^K_t} - \hat{L}^\infty_{\bar{S}_t}\Big)f\Big|\Big|_\infty \leqslant c\big(1 + \|Z_t^K\|_{W^{-4}}\big)\|f\|_{W^j}. \tag{25}$$

Proof. The triangle inequality and the assumptions give a bound of $c(1 + \|Z_t^K\|_{W^{-4}})$ on

$$\sqrt{K}\|h^K_{\bar{S}^K_t} - h^\infty_{\bar{S}_t}\|_\infty$$

and

$$\sqrt{K}\|n^{i,K}_{\bar{S}^K_t} - n^{i,\infty}_{\bar{S}_t}\|_\infty$$

for any $i \in \mathbb{K}$; (25) then follows immediately. See also [9, Propositions 16–17]. □

Notice that the operator L_S^K maps a function from W^j to W^{j-1}, because of the derivative f'. We shall write $\mathcal{L}^{j,j'} = L(W^j, W^{j'})$ for the space of linear operators from W^j to $W^{j'}$. Then we have the following results.

Proposition 10. *Suppose (A2) holds. Then we have*

$$\sup_{K,S} \|\hat{L}_S^K\|_{\mathcal{L}^{j,j}} \leqslant c, \quad j \leqslant 3; \tag{i}$$

$$\sup_{K,S} \|L_S^K\|_{\mathcal{L}^{j,j-1}} \leqslant c, \quad 2 \leqslant j \leqslant 4. \tag{ii}$$

Proof. First, note that if $f \in C^j$ and $g \in W^j$, then $\|fg\|_{W^j} \leqslant c\|f\|_{C^j}\|g\|_{W^j}$. The triangle inequality then yields

$$\|\hat{L}^K_{\bar{S}_t^K} f\|_{W^j} \leqslant \|h^K_{\bar{S}_t^K}\|_{C^j}\|f\|_{W^j} + \|f\|_\infty \sum_{i \in \mathbb{K}} \|n^{i,K}_{\bar{S}_t^K}\|_{W^j} \leqslant c_1\|f\|_{W^j}$$

by (A2) and embedding. Thus, $\|\hat{L}^K_{\bar{S}_t^K}\|_{\mathcal{L}^{j,j}} \leqslant c_1$, and (i) follows. For (ii),

$$\|L_S^K f\|_{W^{j-1}} = \|f'\|_{W^{j-1}} + \|\hat{L}_S^K f\|_{W^{j-1}} \leqslant \|f\|_{W^j} + c_1\|f\|_{W^{j-1}} \leqslant c_2\|f\|_{W^j},$$

by (i) and again embedding. □

Now, define the operator Λ_t^K as

$$\Lambda_t^K f = \sqrt{K}\big(\big(\hat{L}^K_{\bar{S}_t^K} - \hat{L}^\infty_{\bar{S}_t}\big)f, \bar{S}_t\big) + \big(L^K_{\bar{S}_t^K} f, Z_t^K\big).$$

Then Propositions 9 and 10 imply the following.

Corollary 1. *Suppose that (A2) and (A3) hold. Let $2 \leqslant j \leqslant 4$. For $t \in \mathbb{T}$,*

$$\|\Lambda_t^K\|_{W^{-j}} \leqslant c_1\big(1 + (1, \bar{S}_0)e^{c_2 t}\big)\big(1 + \|Z_t^K\|_{W^{-(j-1)}}\big).$$

Proof. By Propositions 9 and 10(ii), we have

$$\begin{aligned}
|\Lambda_t^K f| &\leqslant \big(|\sqrt{K}\big(\hat{L}^K_{\bar{S}_t^K} - \hat{L}^\infty_{\bar{S}_t}\big)f|, \bar{S}_t\big) + \big|\big|L^K_{\bar{S}_t^K} f\big|\big|_{W^{j-1}}\|Z_t^K\|_{W^{-(j-1)}} \\
&\leqslant c_1\big(1 + \|Z_t^K\|_{W^{-4}}\big)\|f\|_{W^j}(1, \bar{S}_t) + c_2\|f\|_{W^{j-1}}\|Z_t^K\|_{W^{-(j-1)}} \\
&\leqslant c_3\big(1 + \|Z_t^K\|_{W^{-(j-1)}}\big)\|f\|_{W^j}(1, \bar{S}_0)e^{c_4 t} + c_2\|f\|_{W^j}\|Z_t^K\|_{W^{-(j-1)}},
\end{aligned}$$

where the last inequality is due to embeddings and the bound (by Gronwall's inequality)

$$(1, \bar{S}_t) \leqslant (1, \bar{S}_0)e^{ct}. \tag{26}$$

Rearranging and noting that $x \leqslant 1 + x$ for any x completes the proof. □

Now, define the operator Γ_t^K as

$$\begin{aligned}
\Gamma_t^K f &= \big(\Pi^K_{\bar{S}_t^K} f, \bar{S}_t^K\big) \\
&= \left(\sum_{i_1 \in \mathbb{K}} \sum_{i_2 \in \mathbb{K}} f(i_1, 0)f(i_2, 0)w^{i_1 i_2, K}_{\bar{S}_t^K} + h^K_{\bar{S}_t^K} f^2 - 2\sum_{i \in \mathbb{K}} f(i, 0)h^K_{\bar{S}_t^K}\hat{m}^{i,K}_{\bar{S}_t^K} f, \bar{S}_t^K\right),
\end{aligned} \tag{27}$$

and let $(p_l^j)_{l \geqslant 1}$ be a complete orthonormal basis of W^j.

Proposition 11. *Let $j \in \mathbb{N}$. Suppose (C0) holds. Then, for $t \in \mathbb{T}$,*

$$\Big| \sum_{l \geqslant 1} \Gamma_t^K p_l^j \Big| \leqslant c(1, \bar{S}_t^K), \tag{28}$$

and for $u \leqslant t$,

$$\Big| \sum_{l \geqslant 1} \Gamma_u^K \Theta_{t-u} p_l^j \Big| \leqslant c(1, \bar{S}_u^K). \tag{29}$$

Proof. First, note that

$$\sup_{s_1 \in \mathbb{S}} \sup_{s_2 \in \mathbb{S}} \Big| \sum_{l \geqslant 1} p_l^j(s_1) p_l^j(s_2) \Big| \leqslant c. \tag{30}$$

This can be seen by considering the operator $\mathcal{H}_s : f \mapsto f(s)$ on W^j. The Riesz representation theorem and Parseval's identity yield $\|\mathcal{H}_s\|_{W^{-j}}^2 = \sum_{l \geqslant 1} (p_l^j(s))^2$ on the one hand, and $|\mathcal{H}_s f| = |f(s)| \leqslant \|f\|_{C^0} \leqslant c_1 \|f\|_{W^j}$ gives $\|\mathcal{H}_s\|_{W^{-j}} \leqslant c_1$ on the other hand. Thus, $\sum_{l \geqslant 1} (p_l^j(s))^2 \leqslant c_2$ for all $s \in \mathbb{S}$. Now, for any $s_1 \in \mathbb{S}$ and $s_2 \in \mathbb{S}$,

$$\Big| \sum_{l \geqslant 1} p_l^j(s_1) p_l^j(s_2) \Big|^2 \leqslant c_3 \Big\| \sum_{l \geqslant 1} p_l^j(s_1) p_l^j \Big\|_{W^j}^2 = c_3 \sum_{l \geqslant 1} (p_l^j(s_1))^2 \leqslant c_4.$$

Thus, (30) follows. By this and (C0), (28) follows immediately from (27).

For (29), observe that by (C0),

$$\begin{aligned} \Big| \sum_{l \geqslant 1} \Gamma_u^K \Theta_{t-u} p_l^j \Big| \leqslant c_5 \Bigg(& \sum_{i_1 \in \mathbb{K}} \sum_{i_2 \in \mathbb{K}} \Big| \sum_{l \geqslant 1} \Theta_{t-u} p_l^j(i_1, 0) \Theta_{t-u} p_l^j(i_2, 0) \Big| \\ & + \Big| \sum_{l \geqslant 1} (\Theta_{t-u} p_l^j)^2 \Big| + \sum_{i \in \mathbb{K}} \Big| \sum_{l \geqslant 1} \Theta_{t-u} p_l^j(i, 0) \Theta_{t-u} p_l^j \Big|, \bar{S}_u^K \Bigg), \end{aligned}$$

and

$$\begin{aligned} & \sup_{s_1 \in \mathbb{K} \times [0, u+a^*]} \sup_{s_2 \in \mathbb{K} \times [0, u+a^*]} \Big| \sum_{l \geqslant 1} \Theta_{t-u} p_l^j(s_1) \Theta_{t-u} p_l^j(s_2) \Big| \\ & = \sup_{i_1 \in \mathbb{K}, w_1 \in [0, u+a^*]} \sup_{i_2 \in \mathbb{K}, w_2 \in [0, u+a^*]} \Big| \sum_{l \geqslant 1} p_l^j(i_1, w_1 + t - u) p_l^j(i_2, w_2 + t - u) \Big| \\ & \leqslant \sup_{s_1 \in \mathbb{K} \times [t-u, t+a^*]} \sup_{s_2 \in \mathbb{K} \times [t-u, t+a^*]} \Big| \sum_{l \geqslant 1} p_l^j(s_1) p_l^j(s_2) \Big|, \end{aligned}$$

which is finite by (30). Hence, (29) follows. □

4.3.3 *Bounds on the norm of Z^K.*

Proposition 12.

$$\sup_{t \leqslant T} \sup_{K \geqslant 1} \mathbb{E}\big[\|Z_t^K\|_{W^{-2}} \big] < \infty.$$

Proof. For this, we use the alternative representation of Z^K as given in Proposition 8 with $\phi \in W^2$. Applying Propositions 9 and 10, then (26) and (23), we arrive at

$$|(\phi, Z_t^K)| \leqslant c_3\|\phi\|_{W^2}\Big\{\|Z_0^K\|_{W^{-2}} + (1, \bar{S}_0)e^{c_4 t}t + \big(1 + (1, \bar{S}_0)e^{c_4 t}\big)\int_0^t \|Z_u^K\|_{W^{-2}}du + \Big\|\int_0^t \Theta_{t-u}^* d\tilde{M}_u^K\Big\|_{W^{-2}}\Big\},$$

where we write $\int_0^t \Theta_{t-u}^* d\tilde{M}_u^K$ for the operator such that

$$\Big(f, \int_0^t \Theta_{t-u}^* d\tilde{M}_u^K\Big) = \int_0^t (\Theta_{t-u}f, d\tilde{M}_u^K).$$

This gives an expression for $\|Z_t^K\|_{W^{-2}}$. Furthermore, the Riesz representation theorem and Parseval's identity yield, for $r \leqslant t$,

$$\mathbb{E}\Big[\Big\|\int_0^r \Theta_{t-u}^* d\tilde{M}_u^K\Big\|_{W^{-2}}^2\Big] = \mathbb{E}\Big[\sum_{l\geqslant 1}\Big(\int_0^r (\Theta_{t-u}p_l^2, d\tilde{M}_u^K)\Big)^2\Big] = \sum_{l\geqslant 1}\mathbb{E}\Big[\int_0^r \Gamma_u^K \Theta_{t-u}p_l^2 du\Big],$$

which is bounded by $c_5(1, \bar{S}_0^K)e^{c_6 r}r$ thanks to (29) and (17). Finally, using Gronwall's inequality, with (C3) and (A4), we establish the boundedness of $\mathbb{E}\big[\|Z_t^K\|_{W^{-2}}\big]$. □

Proposition 13.

$$\sup_{K\geqslant 1}\mathbb{E}\Big[\sup_{t\leqslant T}\|Z_t^K\|_{W^{-3}}\Big] < \infty.$$

Proof. This relies on the bound in Proposition 12. From (8), we have for $f \in W^3$ that

$$|(f, Z_t^K)| \leqslant \|f\|_{W^3}\|Z_0^K\|_{W^{-3}} + c_1\|f\|_{W^3}(1, \bar{S}_0)e^{c_2 t}\int_0^t (1 + \|Z_u^K\|_{W^{-2}})du + c_3\|f\|_{W^3}\int_0^t \|Z_u^K\|_{W^{-2}}du + \|f\|_{W^3}\|\tilde{M}_t^K\|_{W^{-3}},$$

by Propositions 9 and 10, (26), and embedding. This gives a bound on $\sup_{t\leqslant T}\|Z_t^K\|_{W^{-3}}$.

By the Riesz representation theorem and Parseval's identity again, now along with Doob's inequality as well as (28) and (17), we have

$$\mathbb{E}\Big[\sup_{t\leqslant T}\|\tilde{M}_t^K\|_{W^{-3}}^2\Big] = \mathbb{E}\Big[\sup_{t\leqslant T}\sum_{l\geqslant 1}\Big(\tilde{M}_t^{p_l^3, K}\Big)^2\Big] \leqslant c_5(1, \bar{S}_0^K)e^{c_6 T}T. \tag{31}$$

(C3), (A4), and Proposition 12 then complete the proof. □

4.3.4 *Tightness of* Z^K.

Proposition 14. *Both the sequences* Z^K *and* $\tilde{M}^K$ *are tight in* $\mathbb{D}(\mathbb{T}, W^{-4})$.

Proof. (T1) follows from Proposition 12. Let

$$B_{W^{-2}}(R) = \{\mu \in W^{-2} : \|\mu\|_{W^{-2}} \leqslant R\}.$$

Since W^{-2} is Hilbert–Schmidt-embedded in W^{-4}, $B_{W^{-2}}(R)$ is compact in W^{-4}. From Proposition 12, for any $\epsilon > 0$, there exists R such that $\mathbb{P}(Z_t^K \notin B_{W^{-2}}(R)) \leqslant \epsilon$ for all $K \geqslant 1$, since

$$\mathbb{P}(Z_t^K \notin B_{W^{-2}}(R)) = \mathbb{P}(\|Z_t^K\|_{W^{-2}} > R) \leqslant \frac{1}{R}\mathbb{E}\big[\|Z_t^K\|_{W^{-2}}\big].$$

For (T2a) and (T2b), we check the stronger conditions established in [9, Theorem 11]: there exists a $K_0 \geqslant 1$ such that

(T2a′) $\sup_{K\geqslant K_0} \mathbb{E}\big[\sup_{t\leqslant T} \|\Lambda_t^K\|_{W^{-j}}\big] \leqslant c_T$, and

(T2b′) $\sup_{K\geqslant K_0} \mathbb{E}\big[\sup_{t\leqslant T} |\sum_{l\geqslant 1} \Gamma_t^K p_l^j|\big] \leqslant c_T$,

where $(p_l^j)_{l\geqslant 1}$ is a complete orthonormal basis of W^j.

From Corollary 1, we obtain

$$\mathbb{E}\Big[\sup_{t\leqslant T} \|\Lambda_t^K\|_{W^{-4}}\Big] \leqslant c_1\big(1+(1, \bar{S}_0)e^{c_2 T}\big)\bigg(1+\mathbb{E}\Big[\sup_{t\leqslant T}\|Z_t^K\|_{W^{-3}}\Big]\bigg),$$

and (T2a′) holds as a result of Proposition 13; (T2b′) is a result of (28) and (18).

The tightness of $\tilde{M}^K$ follows from (31) and (T2b′). □

We can further show that Z^K and $\tilde{M}^K$ are C-tight; that is, the two sequences are tight and all limit points of the sequences are continuous.

Proposition 15. *Both the sequences Z^K and $\tilde{M}^K$ are C-tight; all limit points are elements of $\mathbb{C}(\mathbb{T}, W^{-4})$.*

Proof. As in [9], for C-tightness of Z^K, we show (see e.g. [16, Proposition VI 3.26(iii)]) that, for all $u \in \mathbb{T}$ and $\epsilon > 0$,

$$\lim_{K\to\infty} \mathbb{P}\bigg(\sup_{t\leqslant u} \|\Delta Z_t^K\|_{W^{-4}} > \epsilon\bigg) = 0.$$

Note that Z^K jumps when S^K jumps, which occurs when there is a birth or a death. We have $\|\Delta Z_t^K\|_{W^{-4}} \leqslant \frac{c}{\sqrt{K}}(1+\Xi)$, since for $f \in W^4$, by (A1),

$$\begin{aligned}|(f, \Delta Z_t^K)| &\leqslant \frac{1}{\sqrt{K}}\bigg(\sum_{i\in\mathbb{K}} \sup_{s\in\mathbb{S}} \big|\check{\xi}^{i,K}_{\bar{S}_t^K}(s) f(i, 0)\big| + \sum_{i\in\mathbb{K}} \sup_{s\in\mathbb{S}} \big|\hat{\xi}^{i,K}_{\bar{S}_t^K}(s) f(i, 0) - f(s)\big|\bigg)\\ &\leqslant \frac{c}{\sqrt{K}}\|f\|_{W^4}(1+\Xi).\end{aligned}$$

Hence,

$$\mathbb{P}\big(\sup_{t\leqslant u} \|\Delta Z_t^K\|_{W^{-4}} > \epsilon\big) \leqslant \frac{1}{\epsilon}\mathbb{E}\big[\sup_{t\leqslant u}\|\Delta Z_t^K\|_{W^{-4}}\big] \leqslant \frac{1}{\epsilon}\frac{c}{\sqrt{K}}\big(1+\mathbb{E}[\Xi]\big)$$

converges to zero as K tends to infinity.

Observing that Z^K and $\tilde{M}^K$ have the same discontinuities, i.e. $\Delta Z_t^K = \Delta\tilde{M}_t^K$, we conclude that $\tilde{M}^K$ also satisfies the conditions of being C-tight. □

4.3.5 *Convergence of $\tilde{M}^K$ and Z^K.* We now give the final steps in establishing the convergence of Z^K.

Proposition 16. *The sequence $\tilde{M}^K$ converges weakly to $\tilde{M}^\infty$ such that for any $f \in W^4$, $\tilde{M}_t^{f,\infty} \equiv (f, \tilde{M}_t^\infty)$, $t \in \mathbb{T}$, is a continuous Gaussian martingale with*

$$\big\langle \tilde{M}^{f,\infty}\big\rangle_t = \int_0^t \bigg(\sum_{i_1\in\mathbb{K}}\sum_{i_2\in\mathbb{K}} f(i_1, 0)f(i_2, 0)w_{\bar{S}_u}^{i_1 i_2,\infty} + h_{\bar{S}_u}^\infty f^2 - 2\sum_{i\in\mathbb{K}} f(i, 0)h_{\bar{S}_u}^\infty \hat{m}_{\bar{S}_u}^{i,\infty} f, \bar{S}_u\bigg) du. \tag{32}$$

Proof. Similarly to [9], this can be achieved by showing that $\tilde{M}^{f,K}$ converges to a continuous Gaussian martingale $\tilde{M}^{f,\infty}$ with predictable quadratic variation (32). In view of the tightness of $\tilde{M}^K$, $\tilde{M}^K$ converges to $\tilde{M}^\infty$. □

Proposition 17. *The limiting process $\mathcal{Z}$ of the sequence Z^K satisfies (9) for any $f \in W^4$ and $t \in \mathbb{T}$.*

Proof. Every limit point $\mathcal{Z}$ of the sequence Z^K satisfies, for $\phi \in W^4$ and $t \in \mathbb{T}$,

$$(\phi, \mathcal{Z}_t) = (\Theta_t\phi, \mathcal{Z}_0) + \int_0^t \Big(- \partial_S h^\infty_{\bar{S}_u}(\mathcal{Z}_u)\Theta_{t-u}\phi + \sum_{i\in\mathbb{K}} \Theta_{t-u}\phi(i,0)\partial_S n^{i,\infty}_{\bar{S}_u}(\mathcal{Z}_u), \bar{S}_u\Big) du$$
$$+ \int_0^t \Big(- h^\infty_{\bar{S}_u}\Theta_{t-u}\phi + \sum_{i\in\mathbb{K}} \Theta_{t-u}\phi(i,0) n^{i,\infty}_{\bar{S}_u}, \mathcal{Z}_u\Big) du + \int_0^t (\Theta_{t-u}\phi, d\tilde{M}^\infty_u). \tag{33}$$

This can be established by showing the convergence of each and every term of (24). Furthermore, by showing that if $\mathcal{Z}^1$ and $\mathcal{Z}^2$ both are solutions to (33) with $\mathcal{Z}^1_0 = \mathcal{Z}^2_0$, then $\|\mathcal{Z}^1_t - \mathcal{Z}^2_t\|_{W^{-4}} = 0$ for $t \in \mathbb{T}$, we have the uniqueness. Lastly, we note that (33) is equivalent to (9), which follows from Fubini's theorem and the fact that $\Theta_{t-u}\phi(s) = \phi(s) + \int_u^t \Theta_{w-u}\phi'(s)dw$. □

We have thus proved the central limit theorem (Theorem 2).

4.4. Proof of Proposition 1

Proof. Using the representation (33) and noting that $\mathbb{E}\big[\int_0^t (\Theta_{t-u}\phi, d\tilde{M}^\infty_u)\big] = 0$, with $\nu_t : f \mapsto \mathbb{E}[(f, Z_t)]$ we have for $\phi \in W^4$ that

$$(\phi, \nu_t) = (\Theta_t\phi, \nu_0) + \int_0^t \Big(- (g^h_{\bar{S}_u,*}, \nu_u)\Theta_{t-u}\phi + \sum_{i\in\mathbb{K}} \Theta_{t-u}\phi(i,0)(g^{n,i}_{\bar{S}_u,*}, \nu_u), \bar{S}_u\Big)_* du$$
$$+ \int_0^t \Big(- h^\infty_{\bar{S}_u}\Theta_{t-u}\phi + \sum_{i\in\mathbb{K}} \Theta_{t-u}\phi(i,0) n^{i,\infty}_{\bar{S}_u}, \nu_u\Big) du. \tag{34}$$

Thus, under the assumptions in the statement and (A2),

$$\|\nu_t\|_{W^{-4}} \leqslant \|\nu_0\|_{W^{-4}} + c_1\big(1 + (1, \bar{S}_0)e^{c_2 t}\big) \int_0^t \|\nu_u\|_{W^{-4}} du.$$

Gronwall's inequality then gives

$$\|\nu_t\|_{W^{-4}} \leqslant \|\nu_0\|_{W^{-4}} e^{c_1\left(1+(1,\bar{S}_0)e^{c_2 T}\right)T}.$$

Now, let $(\phi_m)_m$ be a sequence of functions in C^∞ converging to $\phi \in C^0$. By the dominated convergence theorem, each term in (34) with ϕ_m converges. Thus, (34) holds for $\phi \in C^0$. Moreover, ν_t is a bounded linear operator. Therefore, $\nu_t \in C^{-0}$; in other words, ν_t defines a signed measure. □

5. Proofs for Section 3

5.1. Semimartingale representation of the serial monogamy mating system

To obtain an equation for the population structure in the serial monogamy mating system, we need to define a few more terms. Let $R(v, w, t)$ be the number of marriages by time t between females at ages not greater than v and males at ages not greater than w. Let $Q^{-\text{♀}}(v, w, t)$ (resp. $Q^{-\text{♂}}(v, w, t)$) be the number of cases by time t in which female (resp. male) partners have died at ages not greater than v when their male (resp. female) partners were at ages not greater than w, and let $Q^{-\text{⚤}}(v, w, t)$ count the number of events by time t in which couples have separated while both of the mates were alive and with ages not greater than v and w. We also let $D^{\text{♀}}(v, t)$ (resp. $D^{\text{♂}}(v, t)$) be the number of single females (resp. males) who have died by time t at ages not greater that v, and we let $B^{\text{♀}}(t)$ (resp. $B^{\text{♂}}(t)$) be the number of females (resp. males) born by time t. Then, for test functions f as specified in Section 3.1, we have

$$\begin{aligned}(f_{\text{♀}}, S^{\text{♀}}_t) = (f_{\text{♀}}, S^{\text{♀}}_0) &+ \int_0^t (\partial_v f_{\text{♀}}, S^{\text{♀}}_u)du + f_{\text{♀}}(0)B^{\text{♀}}((0, t]) - \int_{\mathbb{A}\times[0,t]} f_{\text{♀}}(v)D^{\text{♀}}(dv, du)\\ &+ \int_{\mathbb{A}\times\mathbb{A}\times[0,t]} f_{\text{♀}}(v)Q^{-\text{♂}}(dv, dw, du) + \int_{\mathbb{A}\times\mathbb{A}\times[0,t]} f_{\text{♀}}(v)Q^{-\text{⚤}}(dv, dw, du)\\ &- \int_{\mathbb{A}\times\mathbb{A}\times[0,t]} f_{\text{♀}}(v)R(dv, dw, du),\end{aligned}$$

$$\begin{aligned}(f_{\text{♂}}, S^{\text{♂}}_t) = (f_{\text{♂}}, S^{\text{♂}}_0) &+ \int_0^t (\partial_w f_{\text{♂}}, S^{\text{♂}}_u)du + f_{\text{♂}}(0)B^{\text{♂}}((0, t]) - \int_{\mathbb{A}\times[0,t]} f_{\text{♂}}(w)D^{\text{♂}}(dw, du)\\ &+ \int_{\mathbb{A}\times\mathbb{A}\times[0,t]} f_{\text{♂}}(w)Q^{-\text{♀}}(dv, dw, du) + \int_{\mathbb{A}\times\mathbb{A}\times[0,t]} f_{\text{♂}}(w)Q^{-\text{⚤}}(dv, dw, du)\\ &- \int_{\mathbb{A}\times\mathbb{A}\times[0,t]} f_{\text{♂}}(w)R(dv, dw, du),\end{aligned}$$

$$\begin{aligned}(f_{\text{⚤}}, S^{\text{⚤}}_t) = (f_{\text{⚤}}, S^{\text{⚤}}_0) &+ \int_0^t (\partial_v f_{\text{⚤}} + \partial_w f_{\text{⚤}}, S^{\text{⚤}}_u)du - \int_{\mathbb{A}\times\mathbb{A}\times[0,t]} f_{\text{⚤}}(v, w)Q^{-\text{♀}}(dv, dw, du)\\ &- \int_{\mathbb{A}\times\mathbb{A}\times[0,t]} f_{\text{⚤}}(v, w)Q^{-\text{♂}}(dv, dw, du) - \int_{\mathbb{A}\times\mathbb{A}\times[0,t]} f_{\text{⚤}}(v, w)Q^{-\text{⚤}}(dv, dw, du)\\ &+ \int_{\mathbb{A}\times\mathbb{A}\times[0,t]} f_{\text{⚤}}(v, w)R(dv, dw, du).\end{aligned}$$

Compensating the birth, death, and marriage terms, we obtain a semimartingale representation for each of the three types, which can be combined and written as (10). In particular, the following processes are local martingales:

$$B^i((0, t]) - \int_0^t (bm^i, S^{\text{⚤}}_u + S^{\text{♀}}_u)du,$$

$$\int_{\mathbb{A}\times[0,t]} g(v)D^{\mathbf{i}}(dv, du) - \int_0^t (gh^i, S^i_u)du,$$

$$\int_{\mathbb{A}\times\mathbb{A}\times[0,t]} g(v, w)R(dv, dw, du) - \int_0^t (g\rho, S^{\text{♀}}_u \otimes S^{\text{♂}}_u)du,$$

$$\int_{\mathbb{A}\times\mathbb{A}\times[0,t]} g(v, w)Q^{-i}(dv, dw, du) - \int_0^t (gh^i, S^{\text{⚤}}_u)du.$$

The predictable quadratic variation of the martingale can also be obtained in the usual way as in Section 4.1.

5.2. Total population size in the serial monogamy mating system

Before we proceed to establish the convergence of $\{\bar{S}_t^K\}$, we first consider the total population size, by taking the test function $f(i, v, w) = \mathbf{1}_{♀}(i) + \mathbf{1}_{♂}(i) + 2\mathbf{1}_{⚤}(i)$. This gives a simple dynamics equation that can be easily analysed and controlled, the results of which will be useful in establishing the tightness of $\{\bar{S}_t^K\}$.

With a slight abuse of notation, write $X_t = (\mathbf{1}_{♀} + \mathbf{1}_{♂} + 2\mathbf{1}_{⚤}, S_t)$ and $\bar{X}_t^K = X_t^K/K$. Note that we have

$$\bar{X}_t^K = \bar{X}_0^K + \int_0^t \left(\textstyle\sum_{\circ} m^{\circ,K}_{\bar{S}_u^K} b^K_{\bar{S}_u^K}, \bar{S}_u^{⚤,K} + \bar{S}_u^{♀,K}\right) du \\ - \int_0^t \left\{\textstyle\sum_{\circ}(h^{\circ,K}_{\bar{S}_u^K}, \bar{S}_u^{\circ,K}) + (\textstyle\sum_{\circ} h^{\circ,K}_{\bar{S}_u^K}, \bar{S}_u^{⚤,K})\right\} du + \bar{M}_t^{X,K}, \tag{35}$$

with

$$\langle \bar{M}^{X,K}\rangle_t = \int_0^t \frac{1}{K}\left(\left[\gamma^{♀♀,K}_{\bar{S}_u^K} + \gamma^{♂♂,K}_{\bar{S}_u^K} + 2\gamma^{♀♂,K}_{\bar{S}_u^K}\right] b^K_{\bar{S}_u^K}, \bar{S}_u^{⚤,K} + \bar{S}_u^{♀,K}\right) du \\ + \int_0^t \frac{1}{K}\left\{\textstyle\sum_{\circ}(h^{\circ,K}_{\bar{S}_u^K}, \bar{S}_u^{\circ,K}) + (\textstyle\sum_{\circ} h^{\circ,K}_{\bar{S}_u^K}, \bar{S}_u^{⚤,K})\right\} du.$$

Since the reproduction parameters are bounded, by Gronwall's inequality, we have $\mathbb{E}[\bar{X}_t^K] \leqslant \bar{X}_0^K e^{ct}$ and thus $\sup_K \mathbb{E}[\bar{X}_t^K] < \infty$. In fact, we can further show that

$$\sup_K \mathbb{E}\Big[\sup_{t\in\mathbb{T}} \bar{X}_t^K\Big] < \infty. \tag{36}$$

From (35), noticing that $(1, \bar{S}_u^K) \leqslant \bar{X}_u^K$, we have

$$\mathbb{E}\Big[\sup_{t\in\mathbb{T}} \bar{X}_t^K\Big] \leqslant \bar{X}_0^K + c_1 \int_0^T \mathbb{E}[\bar{X}_u^K] du + \mathbb{E}\Big[\sup_{t\in\mathbb{T}} \bar{M}_t^{X,K}\Big].$$

Dealing with the last term by Jensen's and Doob's inequalities, together with the bound on $\mathbb{E}[\bar{X}_u^K]$, we obtain (36).

In a similar way, we can bound $\mathbb{E}[(\bar{X}_t^K)^2]$ and show that $\sup_K \mathbb{E}\Big[\sup_{t\in\mathbb{T}} (\bar{X}_t^K)^2\Big] < \infty$.

5.3. Proof of the law of large numbers (serial monogamy mating system)

Proposition 18. *The sequence $\{\bar{S}_t^K\}$ is tight in $\mathbb{D}(\mathbb{T}, \mathcal{M})$.*

Proof. The tightness is obtained by establishing (J1) and (J2). By Markov's inequality and (36), for each η, there exists δ_η such that $\mathbb{P}(\sup_{t\in\mathbb{T}} (1, \bar{S}_t^K) > \delta_\eta) \leqslant \eta$. Therefore (J1) holds with the compact set $C(\delta_\eta) = \{\mu \in \mathcal{M} : (1, \mu) \leqslant \delta_\eta\}$.

For (J2a), we have from the semimartingale representation of $(f, \bar{S}_t^K)$ that

$$\mathbb{E}\big[(f, \bar{S}_t^K)\big] \leqslant c_1 \|f\|_{C^{1,1}} \left\{(1, \bar{S}_0^K) + \int_0^t \left(\mathbb{E}\big[(1, \bar{S}_u^K)\big] + \mathbb{E}\big[(1, \bar{S}_u^K)^2\big]\right) du + \mathbb{E}\big[|\bar{M}_t^{f,K}|\big]\right\}$$

and

$$\mathbb{E}\big[|\bar{M}_t^{f,K}|\big]^2 \leqslant \mathbb{E}\big[\langle \bar{M}^{f,K}\rangle_t\big] \leqslant c_2 \|f\|^2_{C^{1,1}} \frac{1}{K} \int_0^t \left(\mathbb{E}\big[(1, \bar{S}_u^K)\big] + \mathbb{E}\big[(1, \bar{S}_u^K)^2\big]\right) du.$$

Notice also that $\mathbb{E}\big[(1, \bar{S}_u^K)\big] + \mathbb{E}\big[(1, \bar{S}_u^K)^2\big] \leqslant \mathbb{E}\big[\bar{X}_u^K\big] + \mathbb{E}\big[(\bar{X}_u^K)^2\big] \leqslant c_T$ from the previous section. (J2a) then follows by Markov's inequality.

Now, let $\zeta \leqslant \delta$ and let τ^K be a sequence of stopping times bounded by T. As in Section 4.2 we define $\bar{V}$ by the decomposition $(f, \bar{S}_t^K) = \bar{V}_t^{f,K} + \bar{M}_t^{f,K}$. We have

$$\mathbb{E}\big[|\bar{V}^{f,K}_{(\tau^K+\zeta)\wedge T} - \bar{V}^{f,K}_{\tau^K}|\big] \leqslant c_3 \|f\|_{C^{1,1}} \mathbb{E}\bigg[\int_{\tau^K}^{(\tau^K+\zeta)\wedge T} \big((1, \bar{S}_u^K) + (1, \bar{S}_u^K)^2\big)du\bigg],$$

and in a similar fashion as in the proof of Proposition 4,

$$\begin{aligned}\mathbb{E}\bigg[\int_{\tau^K}^{(\tau^K+\zeta)\wedge T} \big((1, \bar{S}_u^K) + (1, \bar{S}_u^K)^2\big)du\bigg] &\leqslant \int_0^{\zeta} \Big(\mathbb{E}\Big[\sup_{r\leqslant T}(1, \bar{S}_r^K)\Big] + \mathbb{E}\Big[\sup_{r\leqslant T}(1, \bar{S}_r^K)^2\Big]\Big)du \\ &\leqslant \int_0^{\zeta} \Big(\mathbb{E}\Big[\sup_{r\leqslant T}\bar{X}_r^K\Big] + \mathbb{E}\Big[\sup_{r\leqslant T}(\bar{X}_r^K)^2\Big]\Big)du \leqslant c_T\delta,\end{aligned}$$

where the last inequality follows from the results in the previous section. Similarly,

$$\begin{aligned}&\mathbb{E}\big[|\langle\bar{M}^{f,K}\rangle_{(\tau^K+\zeta)\wedge T} - \langle\bar{M}^{f,K}\rangle_{\tau^K}|\big] \\ &\quad\leqslant c_4\|f\|^2_{C^{1,1}}\frac{1}{K}\mathbb{E}\bigg[\int_{\tau^K}^{(\tau^K+\zeta)\wedge T}\big((1, \bar{S}_u^K) + (1, \bar{S}_u^K)^2\big)du\bigg] \leqslant c'_T\delta\frac{1}{K}\|f\|^2_{C^{1,1}}.\end{aligned}$$

(J2b) thus follows by Markov's inequality. □

We see also that the martingale $\bar{M}^{f,K}$ converges to 0, since its predictable quadratic variation vanishes.

Proof of Theorem 3—law of large numbers. Suppose that $\mathcal{S}$ is a limit point of $\bar{S}^K$. Note that for any bounded function f and reproduction parameter $q = h^{♀}, h^{♂}, h^{⚤}, bm^{♀}, bm^{♂}$,

$$\begin{aligned}|(fq^K_{\bar{S}_u^K}, \bar{S}_u^K) - (fq^{\infty}_{\mathcal{S}_u}, \mathcal{S}_u)| &\leqslant |(f(q^K_{\bar{S}_u^K} - q^{\infty}_{\mathcal{S}_u}), \bar{S}_u^K)| + |(fq^{\infty}_{\mathcal{S}_u}, \bar{S}_u^K - \mathcal{S}_u)| \\ &\leqslant \|f\|_\infty\|q^K_{\bar{S}_u^K} - q^{\infty}_{\mathcal{S}_u}\|_\infty(1, \bar{S}_u^K) + \|f\|_\infty\|q^{\infty}_{\mathcal{S}_u}\|_\infty\|\bar{S}_u^K - \mathcal{S}_u\| \qquad (37)\end{aligned}$$

converges to zero as in the proof of Proposition 5, thanks to (C1′) and (C2′). Similarly,

$$\begin{aligned}&\big|\big(gK\rho^K_{\bar{S}_u^K}, \bar{S}_u^{♀,K}\otimes\bar{S}_u^{♂,K}\big) - \big(g\rho^{\infty}_{\mathcal{S}_u}, \mathcal{S}_u^{♀}\otimes\mathcal{S}_u^{♂}\big)\big| \\ &\quad\leqslant \big|\big(gK\rho^K_{\bar{S}_u^K}, \bar{S}_u^{♀,K}\otimes\bar{S}_u^{♂,K}\big) - \big(g\rho^{\infty}_{\mathcal{S}_u}, \bar{S}_u^{♀,K}\otimes\bar{S}_u^{♂,K}\big)\big| \\ &\qquad + \big|\big(g\rho^{\infty}_{\mathcal{S}_u}, \bar{S}_u^{♀,K}\otimes\bar{S}_u^{♂,K}\big) - \big(g\rho^{\infty}_{\mathcal{S}_u}, \mathcal{S}_u^{♀}\otimes\bar{S}_u^{♂,K}\big)\big| \\ &\qquad + \big|\big(g\rho^{\infty}_{\mathcal{S}_u}, \mathcal{S}_u^{♀}\otimes\bar{S}_u^{♂,K}\big) - \big(g\rho^{\infty}_{\mathcal{S}_u}, \mathcal{S}_u^{♀}\otimes\mathcal{S}_u^{♂}\big)\big| \\ &\quad\leqslant \|g\|_\infty\|K\rho^K_{\bar{S}_u^K} - \rho^{\infty}_{\mathcal{S}_u}\|_\infty(1, \bar{S}_u^K)^2 + \|g\|_\infty\|\rho^{\infty}_{\mathcal{S}_u}\|_\infty\|\bar{S}_u^K - \mathcal{S}_u\|(1, \bar{S}_u^K) \\ &\qquad + \|g\|_\infty\|\rho^{\infty}_{\mathcal{S}_u}\|_\infty(1, \mathcal{S}_u)\|\bar{S}_u^K - \mathcal{S}_u\|,\end{aligned}$$

which can be shown to vanish as $K \to \infty$. Thus the limit point of $\bar{S}^K$ satisfies (11).

It remains to show that the limit is unique. This is done by considering (11) with test functions depending also on time, $f(i, v, w, t)$. Now, fix $t\in\mathbb{T}$, and for $u\leqslant t$, take f of the form $f(i, v, w, u)=\phi(i, v+t-u, w+t-u)=\hat{\Theta}_{t-u}\phi(i, v, w)$ as in Proposition 6. Then $\mathcal{S}$ is shown to satisfy the following equation:

$$\begin{aligned}(\phi, \mathcal{S}_t) &= (\hat{\Theta}_t\phi, \mathcal{S}_0)+\int_0^t\Big(\textstyle\sum_\circ\hat{\Theta}_{t-u}\phi_\circ(0)m^{\circ,\infty}_{\mathcal{S}_u}b^{\infty}_{\mathcal{S}_u}, \mathcal{S}^{⚥}_u+\mathcal{S}^{♀}_u\Big)du-\int_0^t\textstyle\sum_\circ\big(\hat{\Theta}_{t-u}\phi_\circ h^{\circ,\infty}_{\mathcal{S}_u}, \mathcal{S}^{\circ}_u\big)du\\ &+\int_0^t\Big(\hat{\Theta}_{t-u}\phi_{♂-⚥}h^{♀,\infty}_{\mathcal{S}_u}+\hat{\Theta}_{t-u}\phi_{♀-⚥}h^{♂,\infty}_{\mathcal{S}_u}+\hat{\Theta}_{t-u}\phi_{♀+♂-⚥}h^{⚥,\infty}_{\mathcal{S}_u}, \mathcal{S}^{⚥}_u\Big)du\\ &+\int_0^t\Big(\hat{\Theta}_{t-u}\phi_{⚥-♀-♂}\rho^{\infty}_{\mathcal{S}_u}, \mathcal{S}^{♀}_u\otimes\mathcal{S}^{♂}_u\Big)du. \end{aligned}\tag{38}$$

The uniqueness of the limit is then proved by considering processes $\mathcal{S}^1$ and $\mathcal{S}^2$ that are both solutions to (38) with the same initial point $\mathcal{S}^1_0=\mathcal{S}^2_0$, and showing that $|(\phi, \mathcal{S}^1_t-\mathcal{S}^2_t)|$ is bounded by $c_T\|\phi\|_\infty\|\mathcal{S}^1_u-\mathcal{S}^2_u\|$; then $\mathcal{S}^1=\mathcal{S}^2$ by Gronwall's inequality. □

5.4. Proof of the central limit theorem (serial monogamy mating system)

The mechanism in proving the central limit theorem in this serial monogamy mating system would be slightly different from that in Section 4.3 because of the iterative bracket coming from the coupling. To overcome the issue arising from the coupling, we need the stopping times

$$\tau^K_N=\inf\{t\in\mathbb{T}:\bar{X}^K_t>N\}.\tag{39}$$

As before, we obtain an alternative representation equation for Z^K. This is done by extending (12) to accommodate a test function depending on time, and then, with fixed t, taking $f(i, v, w, u)=\phi(i, v+t-u, w+t-u)=\Theta_{t-u}\phi(i, v, w)$ for $u\leqslant t$ and $\phi\in W^{-4}$ as in 5. We then have

$$\begin{aligned}(\phi, Z^K_t) &= (\Theta_t\phi, Z^K_0)+\int_0^t\sqrt{K}\Big(\textstyle\sum_\circ\Theta_{t-u}\phi_\circ(0)[m^{\circ,K}_{\bar{S}^K_u}b^K_{\bar{S}^K_u}-m^{\circ,\infty}_{\bar{S}_u}b^{\infty}_{\bar{S}_u}], \bar{S}^{⚥}_u+\bar{S}^{♀}_u\Big)du\\ &+\int_0^t\Big(\textstyle\sum_\circ\Theta_{t-u}\phi_\circ(0)m^{\circ,K}_{\bar{S}^K_u}b^K_{\bar{S}^K_u}, Z^{⚥,K}_u+Z^{♀,K}_u\Big)du\\ &-\int_0^t\sqrt{K}\textstyle\sum_\circ\big(\Theta_{t-u}\phi_\circ[h^{\circ,K}_{\bar{S}^K_u}-h^{\circ,\infty}_{\bar{S}_u}], \bar{S}^{\circ}_u\big)du-\int_0^t\textstyle\sum_\circ\big(\Theta_{t-u}\phi_\circ h^{\circ,K}_{\bar{S}^K_u}, Z^{\circ,K}_u\big)du\\ &+\int_0^t\sqrt{K}\Big(\Theta_{t-u}\phi_{♂-⚥}[h^{♀,K}_{\bar{S}^K_u}-h^{♀,\infty}_{\bar{S}_u}]+\Theta_{t-u}\phi_{♀-⚥}[h^{♂,K}_{\bar{S}^K_u}-h^{♂,\infty}_{\bar{S}_u}]\\ &+\Theta_{t-u}\phi_{♀+♂-⚥}[h^{⚥,K}_{\bar{S}^K_u}-h^{⚥,\infty}_{\bar{S}_u}], \bar{S}^{⚥}_u\Big)du\\ &+\int_0^t\Big(\Theta_{t-u}\phi_{♂-⚥}h^{♀,K}_{\bar{S}^K_u}+\Theta_{t-u}\phi_{♀-⚥}h^{♂,K}_{\bar{S}^K_u}+\Theta_{t-u}\phi_{♀+♂-⚥}h^{⚥,K}_{\bar{S}^K_u}, Z^{⚥,K}_u\Big)du\\ &+\int_0^t\sqrt{K}\Big(\Theta_{t-u}\phi_{⚥-♀-♂}[K\rho^K_{\bar{S}^K_u}-\rho^{\infty}_{\bar{S}_u}], \bar{S}^{♀}_u\otimes\bar{S}^{♂}_u\Big)du\\ &+\int_0^t\Big(\Theta_{t-u}\phi_{⚥-♀-♂}K\rho^K_{\bar{S}^K_u}, Z^{♀,K}_u\otimes\bar{S}^{♂}_u\Big)du\\ &+\int_0^t\Big(\Theta_{t-u}\phi_{⚥-♀-♂}K\rho^K_{\bar{S}^K_u}, \bar{S}^{♀}_u\otimes Z^{♂,K}_u\Big)du+\int_0^t\big(\Theta_{t-u}\phi, d\tilde{M}^K_u\big), \end{aligned}\tag{40}$$

where $\tilde{M}$ is the measure such that $(f, \tilde{M}_t^K) = \tilde{M}_t^{f,K}$. This representation helps establish the next result, which is in turn used in proving the tightness of Z^K.

Proposition 19. *For any $\epsilon > 0$, there exists $R > 0$ such that for all K,*

$$\mathbb{P}(\|Z_t^K\|_{W^{-2}} > R) \leqslant \epsilon.$$

Proof. Let $N > \sup_{t\in\mathbb{T}} \bar{X}_t^\infty$, where $\bar{X}_t^\infty = (\mathbf{1}_{♀} + \mathbf{1}_{♂} + 2\mathbf{1}_{⚥}, \bar{S}_t)$. With τ_N^K as in (39),

$$\begin{aligned}\mathbb{P}(\|Z_t^K\|_{W^{-2}} > R) &= \mathbb{P}(\|Z_t^K\|_{W^{-2}} > R,\, \tau_N^K > T) + \mathbb{P}(\|Z_t^K\|_{W^{-2}} > R,\, \tau_N^K \leqslant T)\\ &\leqslant \frac{1}{R}\mathbb{E}[\|Z_t^K\|_{W^{-2}}\mathbf{1}_{\tau_N^K > T}] + \mathbb{P}(\tau_N^K \leqslant T).\end{aligned} \tag{41}$$

Observe that

$$\mathbb{P}(\tau_N^K \leqslant T) = \mathbb{P}\Big(\sup_{t\leqslant T} \bar{X}_t^K > N\Big) \leqslant \frac{1}{N}\mathbb{E}\Big[\sup_{t\leqslant T} \bar{X}_t^K\Big];$$

therefore, by (36), for any ϵ, there exists N such that $\mathbb{P}(\tau_N^K \leqslant T) \leqslant \epsilon/2$ for all K.

For the first term in (41), we proceed from the representation in (40). For $\phi \in W^2$, as $\|\Theta_r\phi\|_{W^2} \leqslant c\|\phi\|_{W^2}$ for $r \in \mathbb{T}$, we have the following:

- $|(\Theta_t\phi, Z_0^K)| \leqslant \|\Theta_t\phi\|_{W^2}\|Z_0^K\|_{W^{-2}} \leqslant c\|\phi\|_{W^2}\|Z_0^K\|_{W^{-2}}$,
- $\sqrt{K}\big|\big(\Theta_{t-u}\phi(q^K_{\bar{S}_u^K} - q^\infty_{\bar{S}_u}), \bar{S}_u\big)\big| \leqslant \|\Theta_t\phi\|_\infty\|\sqrt{K}(q^K_{\bar{S}_u^K} - q^\infty_{\bar{S}_u})\|_\infty(1, \bar{S}_u)$
$\leqslant c\|\phi\|_{W^2}(1 + \|Z_u^K\|_{W^{-4}}) \leqslant c'\|\phi\|_{W^2}(1 + \|Z_u^K\|_{W^{-2}})$,
- $\big|\big(\Theta_{t-u}\phi q^K_{\bar{S}_u^K}, Z_u^K\big)\big| \leqslant \|\Theta_{t-u}\phi\|_{W^2}\|q^K_{\bar{S}_u^K}\|_{C^2}\|Z_u^K\|_{W^{-2}} \leqslant c\|\phi\|_{W^2}\|Z_u^K\|_{W^{-2}}$,
- $\sqrt{K}\big|\big(\Theta_{t-u}\phi_{⚥-♀-♂}(K\rho^K_{\bar{S}_u^K} - \rho^\infty_{\bar{S}_u}), \bar{S}_u^{♀} \otimes \bar{S}_u^{♂}\big)\big| \leqslant c\|\phi\|_{W^2}(1 + \|Z_u^K\|_{W^{-4}})(1, \bar{S}_u)^2$
$\leqslant c'\|\phi\|_{W^2}(1 + \|Z_u^K\|_{W^{-2}})|$,
- $\big|\big(\Theta_{t-u}\phi_{⚥-♀-♂}K\rho^K_{\bar{S}_u^K}, Z_u^{♀,K} \otimes \bar{S}_u^{♂}\big)\big| \leqslant c\|\Theta_t\phi\|_{W^2}\|K\rho^K_{\bar{S}_u^K}\|_{C^2}\|Z_u^K\|_{W^{-2}}(1, \bar{S}_u)$
$\leqslant c'\|\phi\|_{W^2}\|Z_u^K\|_{W^{-2}}$,
- $\big|\big(\Theta_{t-u}\phi_{⚥-♀-♂}K\rho^K_{\bar{S}_u^K}, \bar{S}_u^{♀,K} \otimes Z_u^{♂,K}\big)\big| = \big|\big(\Theta_{t-u}\phi_{⚥-♀-♂}K\rho^K_{\bar{S}_u^K}, Z_u^{♂,K} \otimes \bar{S}_u^{♀,K}\big)\big|$
$\leqslant c\|\phi\|_{W^2}\|Z_u^K\|_{W^{-2}}(1, \bar{S}_u^K)$,
- $\int_0^t (\Theta_{t-u}\phi, d\tilde{M}_u^K) \leqslant \|\phi\|_{W^2}\big|\big|\int_0^t \Theta_{t-u}^* d\tilde{M}_u^K\big|\big|_{W^{-2}}$,

where $\int_0^t \Theta_{t-u}^* d\tilde{M}_u^K$ is defined by

$$\Big(f, \int_0^t \Theta_{t-u}^* d\tilde{M}_u^K\Big) = \int_0^t (\Theta_{t-u}f, d\tilde{M}_u^K).$$

Therefore,

$$\|Z_t^K\|_{W^{-2}} \leqslant c\bigg(\|Z_0^K\|_{W^{-2}} + \int_0^t \Big(1 + \|Z_u^K\|_{W^{-2}}\big(1 + (1, \bar{S}_u^K)\big)\Big)du\bigg) + \bigg|\bigg|\int_0^t \Theta_{t-u}^* d\tilde{M}_u^K\bigg|\bigg|_{W^{-2}}.$$

Multiplying this by $\mathbf{1}_{\tau_N^K>T}$ and taking expectations, we obtain

$$\mathbb{E}\Big[\|Z_t^K\|_{W^{-2}}\mathbf{1}_{\tau_N^K>T}\Big] \leqslant c_2\bigg(\|Z_0^K\|_{W^{-2}} + t + (1+N)\int_0^t \mathbb{E}\Big[\|Z_u^K\|_{W^{-2}}\mathbf{1}_{\tau_N^K>T}\Big]du\bigg) + \mathbb{E}\bigg[\Big|\Big|\int_0^t \Theta_{t-u}^* d\tilde{M}_u^K\Big|\Big|_{W^{-2}}\bigg],$$

as $(1,\bar{S}_u^K)\leqslant \bar{X}_u^K$, which is less than N on the set $\{\tau_N^K>T\}$. The martingale term is also bounded, since, using (30) and the results in Section 5.2, for $r\leqslant t$ we have

$$\mathbb{E}\bigg[\Big|\Big|\int_0^r \Theta_{t-u}^* d\tilde{M}_u^K\Big|\Big|_{W^{-2}}^2\bigg] = \mathbb{E}\bigg[\sum_{l\geqslant 1}\Big(\int_0^r (\Theta_{t-u}p_l^2, d\tilde{M}_u^K)\Big)^2\bigg]$$
$$\leqslant \sum_{l\geqslant 1}\mathbb{E}\bigg[\Big\langle\int_0^\cdot (\Theta_{t-u}p_l^2, d\tilde{M}_u^K)\Big\rangle_r\bigg] \leqslant \mathbb{E}\bigg[c\int_0^r \big((1,\bar{S}_u^K)+(1,\bar{S}_u^K)^2\big)du\bigg]\leqslant c_T$$

(recall that$(p_l^j)_{l\geqslant 1}$ denote a complete orthonormal basis of W^j). Hence,

$$\mathbb{E}[\|Z_t^K\|_{W^{-2}}\mathbf{1}_{\tau_N^K>T}] \leqslant c_3\bigg(\|Z_0^K\|_{W^{-2}} + T + (1+N)\int_0^t \mathbb{E}[\|Z_u^K\|_{W^{-2}}\mathbf{1}_{\tau_N^K>T}]du + 1\bigg),$$

and by Gronwall's inequality,

$$\mathbb{E}[\|Z_t^K\|_{W^{-2}}\mathbf{1}_{\tau_N^K>T}] \leqslant c_3\big(\|Z_0^K\|_{W^{-2}} + T + 1\big)e^{c_3(1+N)T} \leqslant c_T(N). \tag{42}$$

Thus, for any ϵ and N, there exists R such that

$$\frac{1}{R}\mathbb{E}[\|Z_t^K\|_{W^{-2}}\mathbf{1}_{\tau_N^K>T}] \leqslant \epsilon/2$$

for all K. □

The assertion then follows.

Proposition 20. *Let $N>\sup_{t\in\mathbb{T}}\bar{X}_t^\infty$, where $\bar{X}_t^\infty = (\mathbf{1}_♀ + \mathbf{1}_♂ + 2\mathbf{1}_{⚤}, \bar{S}_t)$, and let τ_N^K be as in* (39). *Then*

$$\mathbb{E}\Big[\sup_{t\in\mathbb{T}}\|Z_t^K\|_{W^{-3}}\mathbf{1}_{\tau_N^K>T}\Big] \leqslant c_T(N).$$

Proof. Let $f\in W^3$. From (12), take the absolute value and bound each term on the right-hand side, similarly as in Proposition 19. With the inclusions of the spaces $W^{-2}\hookrightarrow W^{-3}\hookrightarrow W^{-4}$ and the boundedness of the model parameters, we have

$$|(f, Z_t^K)| \leqslant \|f\|_{W^3}\bigg(\|Z_0^K\|_{W^{-3}} + c_1\int_0^t \big(1+\|Z_u^K\|_{W^{-2}}\big(1+(1,\bar{S}_u^K)\big)\big)du + \|\tilde{M}_t^K\|_{W^{-3}}\bigg).$$

This gives a bound on $\|Z_t^K\|_{W^{-3}}$, and it follows that

$$\mathbb{E}\Big[\sup_{t\in\mathbb{T}}\|Z_t^K\|_{W^{-3}}\mathbf{1}_{\tau_N^K>T}\Big] \leqslant \|Z_0^K\|_{W^{-3}} + c_1\bigg(T+(1+N)\int_0^T \mathbb{E}[\|Z_u^K\|_{W^{-2}}\mathbf{1}_{\tau_N^K>T}]du\bigg) + \mathbb{E}\Big[\sup_{t\in\mathbb{T}}\|\tilde{M}_t^K\|_{W^{-3}}\Big].$$

The assertion then follows using (42) and the fact that

$$\mathbb{E}\Big[\sup_{t\in\mathbb{T}}\|\tilde{M}_t^K\|_{W^{-3}}\Big]=\mathbb{E}\Big[\sup_{t\in\mathbb{T}}\sum_{l\geqslant 1}(\tilde{M}_t^{p_l^3,K})^2\Big]\leqslant\sum_{l\geqslant 1}\mathbb{E}\Big[\sup_{t\in\mathbb{T}}(\tilde{M}_t^{p_l^3,K})^2\Big]$$
$$\leqslant 4\sum_{l\geqslant 1}\mathbb{E}\Big[\langle\tilde{M}^{p_l^3,K}\rangle_T\Big]\leqslant c\int_0^T\mathbb{E}[(1,\bar{S}_u^K)+(1,\bar{S}_u^K)^2]du\leqslant c_T. \qquad \square$$

Proposition 21. *The sequence Z^K is tight in $\mathbb{D}(\mathbb{T}, W^{-4})$.*

Proof. To prove tightness, we show (T1) and (T2). In a similar way as in Proposition 14, (T1) follows from Proposition 19.

For (T2a), observe that

$$\mathbb{P}\big(\|\tilde{V}^K_{(\tau^K+\zeta)\wedge T}-\tilde{V}^K_{\tau^K}\|_{W^{-4}}>\epsilon_1\big)\leqslant\mathbb{P}\big(\|\tilde{V}^K_{(\tau^K+\zeta)\wedge T}-\tilde{V}^K_{\tau^K}\|_{W^{-4}}>\epsilon_1,\,\tau_N^K>T\big)+\mathbb{P}(\tau_N^K\leqslant T).$$

As shown in the proof of Proposition 19, for any ϵ_2, there exists N such that $\mathbb{P}(\tau_N^K\leqslant T)<\epsilon_2/2$. Using similar techniques as before, we can also show that

$$\mathbb{E}\big[\|\tilde{V}^K_{(\tau^K+\zeta)\wedge T}-\tilde{V}^K_{\tau^K}\|_{W^{-4}}\mathbf{1}_{\tau_N^K>T}\big]\leqslant c\bigg(\zeta+(1+N)\mathbb{E}\bigg[\int_0^\zeta\|Z^K_{(\tau^K+u)\wedge T}\|_{W^{-3}}du\mathbf{1}_{\tau_N^K>T}\bigg]\bigg)$$
$$\leqslant c\bigg(\zeta+(1+N)\int_0^\zeta\mathbb{E}\Big[\sup_{u\leqslant T}\|Z_u^K\|_{W^{-3}}\mathbf{1}_{\tau_N^K>T}\Big]du\bigg),$$

which is bounded by $c_T(N)\delta$ thanks to Proposition 20. Therefore, by Markov's inequality,

$$\mathbb{P}(\|\tilde{V}^K_{(\tau^K+\zeta)\wedge T}-\tilde{V}^K_{\tau^K}\|_{W^{-4}}>\epsilon_1,\,\tau_N^K>T)\leqslant\frac{c_T(N)\delta}{\epsilon_1},$$

and for any ϵ_1, ϵ_2, and N, there exists δ such that the probability does not exceed $\epsilon_2/2$. (T2a) thus follows.

Now,

$$\|\tilde{M}_t^K\|^2_{W^{-4}}=\sum_{l\geqslant 1}(\tilde{M}_t^{p_l^4,K})^2$$

and

$$\langle\langle\tilde{M}^K\rangle\rangle_t=\sum_{l\geqslant 1}\langle\tilde{M}^{p_l^4,K}\rangle_t,$$

so

$$\mathbb{E}[|\langle\langle\tilde{M}^K\rangle\rangle_{(\tau^K+\zeta)\wedge T}-\langle\langle\tilde{M}^K\rangle\rangle_{\tau^K}|]=\mathbb{E}\bigg[\bigg|\sum_{l\geqslant 1}\langle\tilde{M}^{p_l^4,K}\rangle_{(\tau^K+\zeta)\wedge T}-\sum_{l\geqslant 1}\langle\tilde{M}^{p_l^4,K}\rangle_{\tau^K}\bigg|\bigg]$$
$$\leqslant\mathbb{E}\bigg[c\int_{\tau^K}^{(\tau^K+\zeta)\wedge T}((1,\bar{S}_u^K)+(1,\bar{S}_u^K)^2)du\bigg]\leqslant c\delta\bigg(\mathbb{E}\Big[\sup_{u\leqslant T}(1,\bar{S}_u^K)\Big]+\mathbb{E}\Big[\sup_{u\leqslant T}(1,\bar{S}_u^K)^2\Big]\bigg),$$

which is bounded by $c_T\delta$. Markov's inequality then gives (T2b). $\square$

Proposition 22. *Z^K and $\tilde{M}^K$ are C-tight; that is, all limit points of Z^K and $\tilde{M}^K$ are in $\mathbb{C}(\mathbb{T}, W^{-4})$.*

Proof. Note that Z^K jumps when S^K jumps, which happens when there is an event associated with birth, death, or marriage. Let $f \in W^4$. Then, assuming that the number of offspring is bounded by Ξ which has finite mean and variance,

$$
\begin{aligned}
&|(f, \Delta Z_t^K)| = \frac{1}{\sqrt{K}} |(f, S_t^K - S_{t-}^K)| \\
&\leqslant \frac{1}{\sqrt{K}} \Big(|f(1, 0, \infty)| \Xi + |f(2, \infty, 0)| \Xi + \sup_v |f(1, v, \infty)| + \sup_w |f(2, \infty, w)| + \sup_{v,w} |f(3, v, w)| \Big) \\
&\leqslant \frac{1}{\sqrt{K}} c \| f \|_\infty (1 + \Xi),
\end{aligned}
$$

and $\mathbb{E}\big[\sup_{t \leqslant u} \| \Delta Z_t^K \|_{W^{-4}} \big] \leqslant \frac{c}{\sqrt{K}} (1 + \mathbb{E}[\Xi])$. It follows that, for any u and $\epsilon > 0$,

$$
\mathbb{P}\bigg(\sup_{t \leqslant u} \| \Delta Z_t^K \|_{W^{-4}} > \epsilon \bigg) \leqslant \frac{c}{\epsilon \sqrt{K}} (1 + \mathbb{E}[\Xi]),
$$

which converges to zero as K tends to infinity. This, together with the tightness of Z^K, shows that the sequence is C-tight (see e.g. [16, Proposition VI.3.26]).

Similarly, $\tilde{M}^K$ is C-tight, since $\tilde{M}^K$ has the same jumps as Z^K. $\square$

Proposition 23. *$\tilde{M}^K$ converges in $\mathbb{D}(\mathbb{T}, W^{-4})$ to $\tilde{M}^\infty$, defined so that for each $f \in W^4$, $(f, \tilde{M}^\infty) = \tilde{M}^{f,\infty}$ is a martingale with predictable quadratic variation* (32).

Proof. See proof of Proposition 16. $\square$

Finally, Theorem 4 can be proved.

Proof of Theorem 4. Every limit point $\mathcal{Z}$ of Z^K satisfies, for $\phi \in W^4$,

$$
\begin{aligned}
(\phi, \mathcal{Z}_t) &= (\Theta_t \phi, \mathcal{Z}_0) + \int_0^t \Big(\textstyle\sum_\circ \Theta_{t-u} \phi_\circ(0) \partial_S (m_{\bar{S}_u}^{\circ,\infty} b_{\bar{S}_u}^{\infty})(\mathcal{Z}_u), \bar{S}_u^{\text{⚥}} + \bar{S}_u^{\text{♀}} \Big) du \\
&+ \int_0^t \Big(\textstyle\sum_\circ \Theta_{t-u} \phi_\circ(0) m_{\bar{S}_u}^{\circ,\infty} b_{\bar{S}_u}^{\infty}, \mathcal{Z}_u^{\text{⚥}} + \mathcal{Z}_u^{\text{♀}} \Big) du \\
&- \int_0^t \textstyle\sum_\circ \big(\Theta_{t-u} \phi_\circ \partial_S h_{\bar{S}_u}^{\circ,\infty}(\mathcal{Z}_u), \bar{S}_u^\circ \big) du - \int_0^t \textstyle\sum_\circ \big(\Theta_{t-u} \phi_\circ h_{\bar{S}_u}^{\circ,\infty}, \mathcal{Z}_u^\circ \big) du \\
&+ \int_0^t \Big(\Theta_{t-u} \phi_{\text{♂}-\text{⚥}} \partial_S h_{\bar{S}_u}^{\text{♀},\infty}(\mathcal{Z}_u) + \Theta_{t-u} \phi_{\text{♀}-\text{⚥}} \partial_S h_{\bar{S}_u}^{\text{♂},\infty}(\mathcal{Z}_u) \\
&+ \Theta_{t-u} \phi_{\text{♀}+\text{♂}-\text{⚥}} \partial_S h_{\bar{S}_u}^{\text{⚥},\infty}(\mathcal{Z}_u), \bar{S}_u^{\text{⚥}} \Big) du \\
&+ \int_0^t \Big(\Theta_{t-u} \phi_{\text{♂}-\text{⚥}} h_{\bar{S}_u}^{\text{♀},\infty} + \Theta_{t-u} \phi_{\text{♀}-\text{⚥}} h_{\bar{S}_u}^{\text{♂},\infty} + \Theta_{t-u} \phi_{\text{♀}+\text{♂}-\text{⚥}} h_{\bar{S}_u}^{\text{⚥},\infty}, \mathcal{Z}_u^{\text{⚥}} \Big) du \\
&+ \int_0^t \Big(\Theta_{t-u} \phi_{\text{⚥}-\text{♀}-\text{♂}} \partial_S \rho_{\bar{S}_u}^{\infty}(\mathcal{Z}_u), \bar{S}_u^{\text{♀}} \otimes \bar{S}_u^{\text{♂}} \Big) du + \int_0^t \Big(\Theta_{t-u} \phi_{\text{⚥}-\text{♀}-\text{♂}} \rho_{\bar{S}_u}^{\infty}, \mathcal{Z}_u^{\text{♀}} \otimes \bar{S}_u^{\text{♂}} \Big) du \\
&+ \int_0^t \Big(\Theta_{t-u} \phi_{\text{⚥}-\text{♀}-\text{♂}} \rho_{\bar{S}_u}^{\infty}, \bar{S}_u^{\text{♀}} \otimes \mathcal{Z}_u^{\text{♂}} \Big) du + \int_0^t (\Theta_{t-u} \phi, d\tilde{M}_u^\infty).
\end{aligned}
\tag{43}
$$

The proof is completed by showing the uniqueness of the solution to (43) and proving that (43) is equivalent to (13). $\square$

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References

[1] Adams, R. A. and Fournier, J. J. F. (2003). *Sobolev Spaces*, 2nd edn. Elsevier, Amsterdam.
[2] Asmussen, S. (1980). On some two-sex population models. *Ann. Prob.* **8**, 727–744.
[3] Borde-Boussion, A.-M. (1990). Stochastic demographic models: age of a population. *Stoch. Process. Appl.* **35**, 279–291.
[4] Clément, F., Robin, F. and Yvinec, R. (2019). Analysis and calibration of a linear model for structured cell populations with unidirectional motion: application to the morphogenesis of ovarian follicles. *SIAM J. Appl. Math.* **79**, 207–229.
[5] Daley, D. J. (1968). Extinction conditions for certain bisexual Galton–Watson branching processes. *Z. Wahrscheinlichkeitsth.* **9**, 315–322.
[6] Dellacherie, C. and Meyer, P. A. (1978). *Probabilities and Potential*. North-Holland, Amsterdam.
[7] Engen, S., Lande, R. and Sæther, B.-E. (2003). Demographic stochasticity and Allee effects in populations with two sexes. *Ecology* **84**, 2378–2386.
[8] Fan, J. Y., Hamza, K., Jagers, P. and Klebaner, F. C. (2019). Limit theorems for multi-type general branching processes with population dependence. Preprint. Available at https://arXiv.org/abs/1903.04747v2.
[9] Fan, J. Y., Hamza, K., Jagers, P. and Klebaner, F. C. (2020). Convergence of the age structure of general schemes of population processes. *Bernoulli* **26**, 893–926.
[10] Fredrickson, A. G. (1971). A mathematical theory of age structure in sexual populations: random mating and monogamous marriage models. *Math. Biosci.* **10**, 117–143.
[11] Gillespie, J. H. (2004). *Population Genetics: A Concise Guide*, 2nd edn. Johns Hopkins University Press.
[12] Hamza, K., Jagers, P. and Klebaner, F. C. (2013). The age structure of population-dependent general branching processes in environments with a high carrying capacity. *Proc. Steklov Inst. Math.* **282**, 90–105.
[13] Hamza, K., Jagers, P. and Klebaner, F. C. (2016). On the establishment, persistence and inevitable extinction of populations. *J. Math. Biol.* **72**, 797–820.
[14] Harris, T. E. (1963). *The Theory of Branching Processes*. Springer, Berlin, Heidelberg.
[15] Hyrien, O., Mayer-Pröschel, M., Noble, M. and Yakovlev, A. (2005). A stochastic model to analyze clonal data on multi-type cell populations. *Biometrics* **61**, 199–207.
[16] Jacod, J. and Shiryaev, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd edn. Springer, Berlin, Heidelberg.
[17] Jagers, P. (1974). Aspects of random measures and point processes. In *Advances in Probability Theory* 3, eds P. Ney and S. Port, Marcel Dekker, New York, pp. 179-239.
[18] Jagers, P. (1975). *Branching Processes with Biological Applications*. John Wiley, London, New York.
[19] Jagers, P. (1989). General branching processes as Markov fields. *Stoch. Process. Appl.* **32**, 183–212.
[20] Jagers, P. and Klebaner, F. C. (2000). Population-size-dependent and age-dependent branching processes. *Stoch. Process. Appl.* **87**, 235–254.
[21] Jagers, P. and Klebaner, F. C. (2011). Population-size-dependent, age-structured branching processes linger around their carrying capacity. *J. Appl. Prob.* **48**, 249–260.
[22] Jagers, P. and Klebaner, F. C. (2016). From size to age and type structure dependent branching: a first step to sexual reproduction in general population processes. In *Branching Processes and Their Applications* (Lecture Notes Statist. **219**), Springer, Cham, pp. 137–148.
[23] Jakubowski, A. (1986). On the Skorokhod topology. *Ann. Inst. H. Poincaré Prob. Statist.* **22**, 263–285.
[24] Kallenberg, O. (2017). *Random Measures, Theory and Applications*. Springer, Cham.
[25] Klebaner, F. C. (1984). Geometric rate of growth in population size dependent branching processes. *J. Appl. Prob.* **21**, 40–49.
[26] Klebaner, F. C. (1989). Geometric growth in near-supercritical population size dependent multitype Galton–Watson processes. *Ann. Prob.* **17**, 1466–1477.
[27] Klebaner, F. C. (1994). Asymptotic behaviour of Markov population processes with asymptotically linear rate of change. *J. Appl. Prob.* **31**, 614–625.
[28] Méléard, S. (1998). Convergence of the fluctuations for interacting diffusions with jumps associated with Boltzmann equations. *Stoch. Stoch. Reports* **63**, 195–225.

[29] MÉTIVIER, M. (1987). Weak convergence of measure valued processes using Sobolev-imbedding techniques. In *Stochastic Partial Differential Equations and Applications* (Lecture Notes Math. **1236**), Springer, Berlin, pp. 172–183.

[30] MOLINA, M., MOTA, M. AND RAMOS, A. (2006). On L^{α}-convergence ($1 \leqslant \alpha \leqslant 2$) for a bisexual branching process with population-size dependent mating. *Bernoulli* **12**, 457–468.

[31] MOLINA, M. AND YANEV, N. (2003). Continuous time bisexual branching processes. *C. R. Acad. Bulg. Sci.* **56**, 5–10.

[32] NORDON, R. E., KO, K.-H., ODELL, R. AND SCHROEDER, T. (2011). Multi-type branching models to describe cell differentiation programs. *J. Theoret. Biol.* **277**, 7–18.

[33] ROSSBERG, A. G. (2013). *Food Webs and Biodiversity: Foundations, Models, Data*. John Wiley, New York.