Thesis for the Degree of Licentiate of Philosophy

Classification of classical twists of the standard Lie bialgebra structure on a loop algebra

Stepan Maximov

CHALMERS | GÖTEBORGS UNIVERSITET

Department of Mathematical Sciences
Chalmers University of Technology and University of Gothenburg

Gothenburg, Sweden 2021

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Stepan Maximov

Department of Mathematical Sciences
Chalmers University of Technology and University of Gothenburg
SE-412 96 Gothenburg
Sweden

Chalmers digital printing
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Abstract

This licentiate thesis is based on the work "Classification of classical twists of the standard Lie bialgebra structure on a loop algebra" by R. Abedin and the author of this thesis.

The standard Lie bialgebra structure on an affine Kac-Moody algebra induces a Lie bialgebra structure on the underlying loop algebra and its parabolic subalgebras. We study classical twists of the induced Lie bialgebra structures and obtain their full classification in terms of Belavin-Drinfeld quadruples up to a natural notion of equivalence.

To obtain this classification we first show that the induced Lie bialgebra structures are determined by certain solutions of the classical Yang-Baxter equation (CYBE) with two parameters. Then, using the algebro-geometric theory of the CYBE, based on torsion free coherent sheaves, we reduce the problem to the well-known classification of trigonometric solutions given by A. Belavin and V. Drinfeld.

The classification of twists in the case of parabolic subalgebras allows us to answer recently posed open questions regarding the so-called quasi-trigonometric solutions of the CYBE.

Keywords: Lie bialgebra, loop algebra, classical twist, Yang-Baxter equation, Manin triple, Belavin-Drinfeld quadruple, geometric r-matrix.

Acknowledgements

I am deeply grateful to my supervisor Alexander Stolin for suggesting the problem uniting so many different fascinating areas of mathematics and physics, for patient guidance, continuous support and for sharing invaluable experience. I would like to thank my colleague Raschid Abedin for an amazing collaboration and profound and visionary discussions. Last but not least, I would like to thank my family, friends and colleagues for their support, understanding and motivation.

Contents

1 Pre-introduction

The paper on which this licentiate is based on contains a quite detailed description of its main players: Lie bialgebras, Manin triples and loop algebras. This section is devoted to an overview of important "trivial" results from the theory of Lie bialgebras and motivation of the problem considered in the paper as well as some historical remarks on its development. The underlying paper will be often referred to as "our work".

1.1 The classical road to Lie bialgebras

Lie bialgebras originate together with Poisson-Lie groups in mathematical physics as a part of a vast research program launched by L. Fadeev in 1970's. This process can be roughly described with the following diagram

$$\begin{tikzcd}
\text{QISM} \arrow[r] \arrow[d, dotted, leftrightarrow] & \text{Quantum groups} \arrow[d, dotted, leftrightarrow] \\
\text{ISM} \arrow[r] & \begin{array}{c}\text{Poisson-Lie groups}\\ \text{Lie bialgebras}\end{array}
\end{tikzcd}$$

where the arrows going up and down can be read as "quantization" and "quasi-classical limit" respectively.

Let us briefly comment on the vertices of this diagram. The Inverse Scattering Method (ISM) is a tool for solving integrable models. It was invented in 1960's during the course of investigation of the KdV equation. Its quantum mechanical version – Quantum Inverse Scattering Method (QISM) – is one of the main achievements of the above-mentioned research program. The development of QISM gave rise to many interesting constructions and notions.

One of such notions is the notion of a classical r-matrix. It was introduced by E. Sklyanin in [12] during the study of Hamiltonian structures associated with integrable systems solvable by ISM. It led V. Drinfeld to the concept of a Poisson-Lie group [6]. More detailed: Let G be a connected Lie group with the Lie algebra $\mathfrak{g}$. A classical r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$ defines a Poisson bivector $P := r^\lambda - r^\rho$, where $r^\lambda(g) := g \cdot r$ and $r^\rho(g) := r \cdot g$. Equivalently, it defines a Poisson bracket on G, called the Sklyanin bracket (or quadratic bracket), by

$$\{\varphi, \psi\} := P(\varphi, \psi). \tag{1.1}$$

When G is a matrix group, we can rewrite (1.1) using tensor notations:

$$\{L \overset{\otimes}{,} L\} := [r, L \otimes L]. \tag{1.2}$$

The form (1.2) is most common in physics books. Let us now denote by λ_x and ρ_x the left and right multiplication respectively by an element $x \in G$. The Sklyanin bracket satisfies the following multiplicative property:

$$\{\varphi, \psi\}(xy) = \{\varphi \circ \lambda_x, \psi \circ \lambda_x\}(y) + \{\varphi \circ \rho_y, \psi \circ \rho_y\}(x).$$

In other words, the multiplication in G is a Poisson map or, equivalentely, G is a Poisson-Lie group. This is precisely the remarkable "multiplicative property of monodromy matrices of difference equations describing lattice integrable systems" that motivated V. Drinfeld to introduce the notion of a Poisson-Lie group. The Lie algebra of a Poisson-Lie group is a Lie bialgebra and it was introduced in the same work [6].

On the other hand, classical r-matrices are the quasi-classical counterparts of quantum R-matrices – one of they key ingridients in QISM. The possibility to quantize r-matrices and obtain quantum R-matrices served as a motivating factor for the well-known classification of r-matrices by A. Belavin and V. Drinfeld [2]. This classification also plays a crucial role in the paper on which this licentiate is based on.

Another interesting notion that arose during the development of QISM is the notion of a quantum group. It is an abstract generalization of constructions that emerged in the depths of QISM. One classical example of such a construction is the $\mathbb{C}[\![h]\!]$-algebra $U_h(\mathfrak{sl}(2))$ generated by elements E, F, H subject to the following relations:

$$[H,E] = 2E, \quad [H,F] = 2F, \quad [E,F] = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}},$$

where $h \in \mathbb{C}^*$. This algebra appeared in the works by P. Kulish, N. Reshetikhin and E. Sklyanin [9, 13]. It evidently defines a deformation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, i.e. letting $h \to 0$ we obtain the universal enveloping algebra $U(\mathfrak{sl}(2,\mathbb{C}))$. V. Drinfeld noticed that such constructions are Hopf algebras and developed the algebraic framework for studying them. It was presented at ICM-86 in the famous talk "Qunatum groups" [7]. Within this framework Lie bialgebras appear as quasi-classical limits of quantized universal enveloping algebras. For example, the quasi-classical limit of the algebra $U_h(\mathfrak{sl}(2))$ above is the Lie bialgebra $(\mathfrak{sl}(2,\mathbb{C}), \delta)$ with

$$\delta(H) = 0, \quad \delta(E) = E \wedge H, \quad \delta(F) = F \wedge E.$$

Now we present the basic theory of Lie bialgebras and explain in more details their relations to some of the other objects mentioned above.

1.2 Lie bialgebras

Let k be a field of characteristic 0, $\mathfrak{g}$ be a Lie algebra (not necessarily finite-dimensional) over k and $\delta\colon \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ be a linear map. We say that $(\mathfrak{g}, \delta)$ is a Lie bialgebra if

1. The restriction of the dual map $\delta^\vee\colon (\mathfrak{g}\otimes\mathfrak{g})^\vee \longrightarrow \mathfrak{g}^\vee$ to $\mathfrak{g}^\vee \otimes \mathfrak{g}^\vee$ defines a Lie bracket on $\mathfrak{g}^\vee$;

2. δ is a 1-cocycle of $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$, i.e.

$$\delta([x,y]) = x \cdot \delta(y) - y \cdot \delta(x) \qquad \forall x, y \in \mathfrak{g},$$

where $x \cdot (a \otimes b) := [x,a] \otimes b + a \otimes [x,b]$.

Equivalently, a Lie bialgebra is a vector space with both a Lie algebra and a Lie coalgebra structures such that the compatibility condition 2 holds.

Example 1.1. Any Lie algebra $\mathfrak{g}$ can be endowed with the trivial Lie bialgebra structure $\delta = 0$. ◊

Let $\{x_i\}$ be a basis for a Lie algebra $\mathfrak{g}$. Consider an arbitrary linear function

$$\delta\colon \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}.$$

It is completely determined by the constants $d_k^{ij} \in \mathbb{C}$ in the expression $\delta(x_k) = d_k^{ij} x_i \otimes x_j$.[1] Conditions 1 and 2 in the definition above can be expressed in terms of the constants d_k^{ij} and the structure constants for $\mathfrak{g}$. More precisely, let $[x_i, x_j] := a_{ij}^k x_k$. Then δ defines a Lie bialgebra structure on $\mathfrak{g}$ if and only if the following conditions hold

1. $\forall i, j, k \qquad d_k^{ii} = 0 \;\text{ and }\; d_k^{ij} + d_k^{ji} = 0;$
2. $\forall i, j, k, m \qquad d_m^{ir} d_r^{jk} + d_m^{jr} d_r^{ki} + d_m^{kr} d_r^{ij} = 0;$
3. $\forall i, j \qquad a_{ij}^k d_k^{\ell r} = d_j^{kr} a_{ik}^{\ell} - d_i^{kr} a_{jk}^{\ell} + d_j^{\ell k} a_{ik}^{r} - d_i^{\ell k} a_{jk}^{r}.$

The first two conditions mean that $\delta^\vee$ is a Lie bracket on $\mathfrak{g}^\vee$ and the last condition is equivalent to the compatibility condition 2 above.

Example 1.2. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and

$$e := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

be its standard basis. In this case it is possible to solve the equations above using a computer. By doing so we get the following result

$$\begin{aligned} \delta(e) &= c_1 e \wedge f + c_2 e \wedge h, \\ \delta(f) &= c_3 e \wedge f + c_2 f \wedge h, \\ \delta(h) &= c_3 e \wedge h - c_1 f \wedge h, \end{aligned}$$

where $c_1, c_2, c_3 \in \mathbb{C}$. In other words, for any choice of constants c_1, c_2 and c_3 we have a Lie bialgebra structure on $\mathfrak{sl}(2, \mathbb{C})$. Let us write $\{e^*, f^*, h^*\}$ for the dual basis in $\mathfrak{sl}(2, \mathbb{C})^\vee$. Then the corresponding Lie bracket structure on $\mathfrak{sl}(2, \mathbb{C})^\vee$ is described by

$$\begin{aligned} [e^*, f^*] &= c_1 e^* + c_3 f^*, \\ [e^*, h^*] &= c_2 e^* + c_3 h^*, \\ [f^*, h^*] &= c_2 f^* - c_1 h^*. \end{aligned}$$

If we put $c_1 = c_2 = c_3 = 0$ we get the trivial Lie bialgebra structure from the previous example. The case $c_2 = 1, c_1 = c_3 = 0$ corresponds to the Lie bialgebra structure mentioned at the end of the previous section. ♢

Condition 2 in the definition of a Lie bialgebra can be written in a symmetric form using adjoint and coadjoint representations for $\mathfrak{g}$ and $\mathfrak{g}^\vee$:

$$\langle \mathrm{ad}_f g, \mathrm{ad}_x y \rangle + \langle \mathrm{ad}_x^* f, \mathrm{ad}_g^* y \rangle - \langle \mathrm{ad}_x^* g, \mathrm{ad}_f^* y \rangle + \langle \mathrm{ad}_y^* g, \mathrm{ad}_f^* x \rangle - \langle \mathrm{ad}_y^* f, \mathrm{ad}_g^* x \rangle = 0, \tag{1.3}$$

Here $x, y \in \mathfrak{g}$, $f, g \in \mathfrak{g}^\vee$ and $\langle -, - \rangle$ stands for the standard pairing between a vector space and its dual. If $\mathfrak{g}$ is a finite-dimensional Lie algebra with the bracket $\mu\colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$,

[1] Here we use the Einstein summation notation $\sum_{i,j} d_k^{ij} x_i \otimes x_j = d_k^{ij} x_i \otimes x_j$.

then (1.3) immediately implies that $(\mathfrak{g}, \delta)$ is a Lie bialgebra if and only if $(\mathfrak{g}^\vee, \mu^\vee)$ is a Lie bialgebra.

A morphism between two Lie bialgebras $(\mathfrak{g}, \delta)$ and $(\mathfrak{g}', \delta')$ is a Lie algebra homomorphism $\phi\colon \mathfrak{g} \longrightarrow \mathfrak{g}'$ such that

$$\delta'\phi = (\phi \otimes \phi)\delta.$$

Remark 1.3. In Example 1.2 we have constructed infinitely many different Lie bialgebra structures on $\mathfrak{sl}(2, \mathbb{C})$. However, it follows from works [14] and [15] by A. Stolin that there are only three Lie bialgebra structures on $\mathfrak{sl}(2, \mathbb{C})$ up to isomorphism, namely

(i) $\delta = 0$;

(ii) $\delta(e) = \frac{1}{2} e \wedge h, \qquad \delta(f) = \frac{1}{2} f \wedge h, \qquad \delta(h) = 0$;

(iii) $\delta(e) = 0, \qquad \delta(f) = e \wedge f, \qquad \delta(h) = e \wedge h.$

The second structure is usually called the standard Lie bialgebra structure on $\mathfrak{sl}(2, \mathbb{C})$. ◊

1.3 Coboundary Lie bialgebras

The easiest way to get a 1-cocycle $\delta\colon \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ is to take a 1-coboundary $dr \in B^1(\mathfrak{g}, \mathfrak{g} \otimes \mathfrak{g})$ for some $r \in \mathfrak{g} \otimes \mathfrak{g}$. This leads to the natural question: which properties should $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfy in order for $dr^\vee$ to define a Lie algebra structure on $\mathfrak{g}^\vee$. More precisely,

(i) $dr(x)$ must lie in $\bigwedge^2 \mathfrak{g}$ for any $x \in \mathfrak{g}$ by the skew-symmetry of a Lie bracket;

(ii) The restriction of $dr^\vee$ to $\mathfrak{g}^\vee \otimes \mathfrak{g}^\vee$ must satisfy the Jacobi identity.

Let $\tau\colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ be the map given by $\tau(x \otimes y) := y \otimes x$. For an element $r = x_i \otimes y^i \in \mathfrak{g} \otimes \mathfrak{g}$ we define the linear mapping

$$\underline{r}\colon \mathfrak{g}^\vee \longrightarrow \mathfrak{g},$$

by setting $\underline{r}(f) := f(x_i)y^i$. The equivalent formulations of the first property are provided by the following lemma.

Lemma 1.4. *Let $r = a + s \in \mathfrak{g} \otimes \mathfrak{g}$, where a and s are the skew-symmetric and symmetric parts of r respectively. The following conditions are equivalent:*

1. $\forall x \in \mathfrak{g} \quad dr(x) \in \bigwedge^2 \mathfrak{g}$;
2. $ds = 0$ *(i.e. s is ad-invariant);*
3. $\forall x \in \mathfrak{g} \quad \mathrm{ad}_x \circ \underline{s} = \underline{s} \circ \mathrm{ad}^*_x$.

To reformulate the Jacobi identity for $dr^\vee$ in terms of r let us first consider the case when r is skew-symmetric. In this case all the conditions of Lemma 1.4 are trivially satisfied because $s = 0$. We define the algebraic Schouten bracket $[\![r, r]\!]$ of a skew-symmetric tensor $r \in \mathfrak{g} \otimes \mathfrak{g}$ as the unique element in $\bigwedge^3 \mathfrak{g}$ satisfying the relation

$$\langle f \otimes g \otimes h, [\![r, r]\!]\rangle = -2 \circlearrowleft \langle f, [\underline{r}g, \underline{r}h]\rangle \qquad \forall f, g, h \in \mathfrak{g}^\vee,$$

where $\circlearrowleft$ means the summation over the circular permutations of f, g and h. Such an element always exists and if $r = x_i \otimes y^i$, we can write it explicitly

$$[\![r, r]\!] = -2([y^i, y^j] \otimes x_i \otimes x_j + x_j \otimes [y^i, y^j] \otimes x_i + x_i \otimes x_j \otimes [y^i, y^j]).$$

A direct computation yields the identity

$$\langle f \otimes g \otimes h, d([\![r, r]\!])(x)\rangle = 2 \circlearrowleft \langle \delta^\vee(\delta^\vee(f \otimes g) \otimes h), x\rangle \qquad \forall f, g, h \in \mathfrak{g}^\vee,$$

which in its turn implies the following result.

Proposition 1.5. *Let r be a skew-symmetric tensor in $\mathfrak{g} \otimes \mathfrak{g}$. Then $dr^\vee$ satisfies the Jacobi identity if and only if $[\![r, r]\!]$ is* ad*-invariant.*

The condition of Proposition 1.5 is called the generalized Yang-Baxter equation. It is clear that $[\![r, r]\!] = 0$ is a sufficient condition for $[\![r, r]\!]$ to be ad-invariant. Skew-symmetric tensors $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfying $[\![r, r]\!] = 0$ are called triangular r-matrices.

Now let us return to the general case, i.e. $r = a + s$, where a and s are the skew-symmetric and symmetric parts of r respectively. Combining all observations from the preceding discussion we obtain the following statement.

Corollary 1.6. *An element $r = a + s \in \mathfrak{g} \otimes \mathfrak{g}$ defines a Lie bialgebra structure on $\mathfrak{g}$ if and only if both s and $[\![a, a]\!]$ are* ad*-invariant.*

Elements of $\mathfrak{g} \otimes \mathfrak{g}$ satisfying the condition of Corollary 1.6 are called (classical) r-matrices. Therefore triangular r-matrices are automatically classical r-matrices. Let G be a connected Lie group with the Lie algebra $\mathfrak{g}$. One can show that r is a classical r-matrix if and only if the Sklyanin bracket (1.1) is a multiplicative Poisson bracket on G.

Now we analyse the invariance of the Schouten bracket $[\![a, a]\!]$ even further. For any element $r \in \mathfrak{g} \otimes \mathfrak{g}$ we define a skew-symmetric bilinear map $\underline{\langle r, r\rangle} \colon \mathfrak{g}^\vee \otimes \mathfrak{g}^\vee \longrightarrow \mathfrak{g}$ by letting

$$\underline{\langle r, r\rangle}(f, g) := [\underline{r}f, \underline{r}g] - (\underline{r} \circ \delta^\vee)(f, g).$$

We write $\langle r, r\rangle$ for the unique element in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ such that

$$\langle f \otimes g \otimes h, \langle r, r\rangle\rangle = \langle h, \underline{\langle r, r\rangle}(f, g)\rangle.$$

Again, if $r = x_i \otimes y^i$ the element can be written out explicitly

$$\langle r, r\rangle = x_i \otimes x_j \otimes [y^i, y^j] - [x_i, x_j] \otimes y^j \otimes y^i - x_j \otimes [x_i, y^j] \otimes y^i. \tag{1.4}$$

Note that if r is skew-symmetric, then $\langle r, r\rangle = -\frac{1}{2}[\![r, r]\!]$.

Theorem 1.7.

1. *If s is a symmetric* ad*-invariant element in $\mathfrak{g}\otimes\mathfrak{g}$, then $\langle s, s\rangle \in \bigwedge^3 \mathfrak{g}$ is* ad*-invariant;*
2. *If $r = a+s \in \mathfrak{g}\otimes\mathfrak{g}$, where a is skew-symmetric and s is symmetric and* ad*-invariant, then $\langle r, r\rangle \in \bigwedge^3 \mathfrak{g}$ and $\langle r, r\rangle = \langle a, a\rangle + \langle s, s\rangle$;*
3. *Let $r = a + s$ be as in 2. Then $\langle r, r\rangle = 0$ is a sufficient condition for $[\![a, a]\!]$ to be* ad*-invariant.*

The condition $\langle r, r\rangle = 0$ is called the classical Yang-Baxter equation (CYBE). Elements $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfying the CYBE are called quasi-triangular r-matrices. Therefore, skew-symmetric quasi-triangular r-matrices are triangular (and hence classical).

Example 1.8. Three structures on $\mathfrak{sl}(2, \mathbb{C})$ from Remark 1.3 are given by the following classical r-matrices

(i) $r = 0$;

(ii) $r = \frac{1}{4} h \otimes h + e \otimes f$;

(iii) $r = \frac{1}{2}(e \otimes h - h \otimes e)$.

The first and the last r-matrices are triangular. The last one is quasi-triangular with the ad-invariant symmetric part $s = \frac{1}{4} h \otimes h + \frac{1}{2}(e \otimes f + f \otimes e)$. ◊

A natural question that may arise after seeing Example 1.8 is which Lie bialgebra structures on a Lie algebra $\mathfrak{g}$ are coboundary structures? In case when $\mathfrak{g}$ is finite-dimensional and semi-simple this question has the following beautiful answer.

Theorem 1.9. *Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra over a field of characteristic* 0*. Then any Lie bialgebra structure on $\mathfrak{g}$ is given by a classical r-matrix. Moreover, any Lie bialgebra structure on a simple Lie algebra over an algebraically closed field of characteristic* 0 *is given by a quasi-triangular r-matrix.*

The notation $\langle r, r\rangle = 0$ for the classical Yang-Baxter equation is not standard. To obtain the standard notation we introduce three different embeddings of $\mathfrak{g} \otimes \mathfrak{g}$ into the triple tensor product $U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$:

$$\begin{aligned}(-)^{12} &\colon x \otimes y \longmapsto x \otimes y \otimes 1,\\ (-)^{13} &\colon x \otimes y \longmapsto x \otimes 1 \otimes y,\\ (-)^{23} &\colon x \otimes y \longmapsto 1 \otimes x \otimes y.\end{aligned}$$

Then the standard form of the classical Yang-Baxter equation reads

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0. \tag{1.5}$$

Remark 1.10. Finding solutions to (1.5) is a computationally hard problem: if $\mathfrak{g}$ has dimension n, then (1.5) amounts to solving n^3 quadratic equations in n^2 variables. However, in some particular cases it is possible to classify the solutions up to some notion of equivalence.

By Theorem 1.9 any Lie bialgebra structure on a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ is given by a quasi-triangular r-matrix $r = a + s$, with ad-invariant symmetric part s. Solutions with $s \neq 0$ were classified by A. Belavin and V. Drinfeld [2]. The skew-symmetric case is less friendly. It was proven by A. Stolin [16] that the classification of skew-symmetric solutions is equivalent to the classification of quasi-Frobenius subalgebras of $\mathfrak{g}$. This problem is known to be "representation wild". So there is no hope to get a full classification of such solutions even for a simple Lie algebra $\mathfrak{g}$. ◊

1.4 The classical Yang-Baxter equation with parameters

Another form of the classical Yang-Baxter equation which also arises in the study of integrable models in pretty much the same way as (1.5) is

$$[r^{12}(z_1-z_2), r^{13}(z_1-z_3)]+[r^{12}(z_1-z_2), r^{23}(z_2-z_3)]+[r^{13}(z_1-z_3), r^{23}(z_2-z_3)] = 0. \quad (1.6)$$

Here r is a meromorphic function $\mathbb{C} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$. As in the case of the constant CYBE the classification of solutions to (1.6) exists only under additional assumptions on both $\mathfrak{g}$ and r. More precisely, let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$. The Killing form κ on $\mathfrak{g}$ induces the isomorphism of vector spaces

$$\psi\colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}),$$
$$x \otimes y \longmapsto \kappa(-, y)x.$$

A function $r\colon \mathbb{C} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ is called non-degenerate if the endomorphism $\psi(r(u))$ is invertible for some $u \in \mathbb{C}$. We call two meromorphic functions $r_1, r_2\colon \mathbb{C} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ equivalent if there exists a holomorphic function $\varphi\colon \mathbb{C} \longrightarrow \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ such that

$$r_2(x-y) = (\varphi(x) \otimes \varphi(y))r_1(x-y) \qquad \forall x, y \in \mathbb{C}.$$

It is easy to see that if r_1 solves (1.6), then so does r_2. Within this setting we have the following famous trichotomy result.

Theorem 1.11 (A. Belavin, V. Drinfeld [2]). *Let $r\colon \mathbb{C} \longrightarrow \mathfrak{g}\otimes\mathfrak{g}$ be a non-degenerate meromorphic solution of (1.6). Then r is skew-symmetric, i.e. $r(z) = -\tau(r(-z))$, its poles form a lattice $\Gamma \subset \mathbb{C}$ and exactly one of the following three cases occurs:*

1. $\mathrm{rank}(\Gamma) = 2$. *Then r is called an elliptic solution;*
2. $\mathrm{rank}(\Gamma) = 1$, *there exists a rational function $f\colon \mathbb{C} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ and a constant $\lambda \in \mathbb{C}$ such that the function $z \longmapsto f(e^{\lambda z})$ is equivalent to r. In this case r is called trigonometric;*
3. $\mathrm{rank}(\Gamma) = 0$, *there exists a rational function $f\colon \mathbb{C} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ equivalent to r. Solutions of this type are called rational.*

Example 1.12. Examples for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$:

(i) Elliptic solution of R. Baxter

$$r(z) = \frac{\mathrm{cn}(z)}{\mathrm{sn}(z)} h \otimes h + \frac{1+\mathrm{dn}(z)}{\mathrm{sn}(z)}(e \otimes f + f \otimes e) + \frac{1-\mathrm{dn}(z)}{\mathrm{sn}(z)}(e \otimes e + f \otimes f);$$

(ii) Trigonometric solution of R. Baxter

$$r(z) = \frac{\cot(z)}{2} h \otimes h + \frac{1}{\sin(z)}(e \otimes f + f \otimes e);$$

(iii) Rational solution of C. Yang

$$r(z) = \frac{1}{z}\left(\frac{1}{2} h \otimes h + e \otimes f + f \otimes e\right).$$

$\Diamond$

A full classification of elliptic and trigonometric solutions was carried out in the same work [2]. It turned out that elliptic solutions are possible only in the case $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$. It was shown with the help of representation theory that their classification reduces to the classification of doubles (Γ, ε), where $\Gamma \subset \mathbb{C}$ is a two-dimensional lattice and ε is a primitive n-th root of unity. More precisely, any elliptic solution up to the above-mentioned equivalence is given by a triple (Γ, d, n), with $\gcd(n, d) = 1$. Considering holomorphic change of variables as an equivalence of solutions we get the classification of elliptic solutions by triples (ω, d, n), where $\omega \in \mathbb{C}$ with $\Im(\omega) \neq 0$.

To state the classification of trigonometric solutions we need to introduce the notion of an admissible triple. Let us fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \dotplus \mathfrak{h} \dotplus \mathfrak{n}_+$, a Chevalley basis $\{x_i^\pm, h_i\}$ for $\mathfrak{g}$ and an automorphism $\widetilde{\nu}$ of the corresponding Dynkin diagram. Then $\widetilde{\nu}$ induces an outer automorphism ν of $\mathfrak{g}$ by

$$\nu(x_i^\pm) = x_{\widetilde{\nu}(i)}^\pm, \quad \nu(h_i) = h_{\widetilde{\nu}(i)}.$$

The coset of ν in $\mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})/\mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ contains a special automorphism σ_1 of $\mathfrak{g}$ called the Coxeter automorphism. It is the automorphism $\sigma \in \nu\mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ of minimal order such that

$$\mathfrak{h}_0 := \{x \in \mathfrak{g} \mid \sigma(x) = x\} \subseteq \mathfrak{h}$$

is an abelian Lie algebra. Let m be the order of σ_1 and ε be the m-th primitive root of unity $e^{2\pi i/m}$. Define the following subspaces of $\mathfrak{g}$:

$$\mathfrak{g}_k := \{x \in \mathfrak{g} \mid \sigma_1(x) = \varepsilon^k x\}, \quad \mathfrak{g}_k^\alpha := \{x \in \mathfrak{g}_k \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}_0\},$$

where $k \in \{0, 1, \dots, m-1\}$. Set $\Pi^{\sigma_1} := \{\alpha \in \mathfrak{h}_0^\vee \mid \mathfrak{g}_{(\alpha,1)} \neq 0\}$. A Belavin-Drinfeld (BD) triple is a triple $(\Gamma_1, \Gamma_2, \gamma)$, where Γ_1 and Γ_2 are proper subsets of Π^{σ_1} and $\gamma \colon \Gamma_1 \longrightarrow \Gamma_2$ is a bijection such that

(i) $\kappa(\gamma(\alpha), \gamma(\beta)) = \kappa(\alpha, \beta)$ for all $\alpha, \beta \in \Gamma_1$;

(ii) For any $\alpha \in \Gamma_1$ there exists a positive integer k such that $\gamma^k(\alpha) \notin \Gamma_1$.

Fix an BD triple $(\Gamma_1, \Gamma_2, \gamma)$. Let $\mathfrak{S}^{\Gamma_i}$, $i = 1, 2$ be the subalgebras of $\mathfrak{g}$ generated by subspaces $\mathfrak{g}_1^\alpha$ and $\mathfrak{g}_{-1}^\alpha$ with $\alpha \in \Gamma_i$. The bijection γ induces an isomorphism $\theta_\gamma \colon \mathfrak{S}^{\Gamma_1} \longrightarrow \mathfrak{S}^{\Gamma_2}$ which we then extend by 0 to the whole $\mathfrak{g}$. The second condition of a BD triple guarantees that θ_γ is a nilpotent endomorphism of $\mathfrak{g}$. Set $\theta := \sum_1^\infty \theta_\gamma^k$. Let $C \in \mathfrak{g} \otimes \mathfrak{g}$ be the quadratic Casimir element. We denote its projections on $\mathfrak{g}_k \otimes \mathfrak{g}_{-k}$ by C_k. Assume $t_0 \in \mathfrak{h}_0 \wedge \mathfrak{h}_0$ is a tensor satisfying the condition

$$(\gamma(\alpha) \otimes 1 + 1 \otimes \alpha)(t_0 + C_0/2) = 0 \qquad \forall \alpha \in \Gamma_1. \tag{1.7}$$

Then the function

$$X(z) = \frac{C_0}{2} + t_0 + \frac{1}{e^z - 1} \sum_{k=0}^{m-1} e^{kz/m} C_k - \sum_{k=1}^{m-1} e^{kz/m} (\theta \otimes 1) C_k + \sum_{k=1}^{m-1} e^{-kz/m} (1 \otimes \theta) C_{-k} \tag{1.8}$$

is a trigonometric solution of the CYBE. Moreover, any trigonometric solution is equivalent to one of the form (1.8). Therefore, all trigonometric solutions are parametrized by Dynkin diagram automorphisms ν, BD triples $(\Gamma_1, \Gamma_2, \gamma)$ and tensors $t_0 \in \mathfrak{h}_0 \wedge \mathfrak{h}_0$. The datum $(\Gamma_1, \Gamma_2, \gamma, t_0)$ is called a BD quadruple.

Similar to the elliptic case, we can extend the notion of equivalence and make solutions (1.8) corresponding to different t_0 equivalent. In that case we have only finitely many trigonometric r-matrices for any simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$.

Remark 1.13. The condition (1.7) is a system of linear equations which is consistent for any BD triple $(\Gamma_1, \Gamma_2, \gamma)$. The dimension of its solution space is $\ell(\ell-1)/2$, where $\ell = |\Pi^{\sigma_1} \setminus \Gamma_1|$. Therefore, this classification also simplifies the calculation of r-matrices. $\Diamond$

Remark 1.14. While constructing the solution (1.8) one can get the feeling that there is a loop algebra lurking in the background. And this is indeed the case. Let $\mathfrak{g}_{k+m} := \mathfrak{g}_k$, then the sum

$$\mathfrak{L}^{\sigma_1} := \bigoplus_{k\in\mathbb{Z}} z^k \mathfrak{g}_k, \tag{1.9}$$

is the loop algebra corresponding to the Coxeter automorphism σ_1. The set Π^{σ_1} above is the simple root system of $\mathfrak{L}^{\sigma_1}$. It is in bijection with the affine Dynkin diagram of $\mathfrak{L}^{\sigma_1}$. More will be said about this in our work. The construction of σ-trigonometric solutions introduced there generalizes the construction (1.8). $\Diamond$

The classification of rational solutions is impossible in the sense that it contains a "representation wild" subproblem. However, there is a structure theory of rational solutions developed by A. Stolin [16, 17]. The first step in that theory is to consider the CYBE with two parameters instead

$$[r^{12}(z_1,z_2), r^{13}(z_1,z_3)] + [r^{12}(z_1,z_2), r^{23}(z_2,z_3)] + [r^{13}(z_1,z_3), r^{23}(z_2,z_3)] = 0. \tag{1.10}$$

Here r is a meromorphic function $\mathbb{C}^2 \longrightarrow \mathfrak{g}\otimes\mathfrak{g}$. Similarly to the one-parameter case, a function r is called non-degenerate if the endomorphism $\psi(r(x,y))$ is invertible for some $x, y \in \mathbb{C}$.

There is no known analogy for the trichotomy result in the two-parameter case. However, there is a way to locally "reduce" solutions with two parameters to solutions with only one parameter.

Theorem 1.15 (A. Belavin, V. Drinfeld [3]). *Let $r(x,y)$ be a non-degenerate solution to* (1.10)*. Then there exists an open neighbourhood $V \subseteq \mathbb{C}$ of* 0*, a holomorphic function $\varphi\colon V \longrightarrow \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ and a non-constant holomorphic function $f\colon V \longrightarrow \mathbb{C}$ such that*

$$(\varphi(v_1)\otimes\varphi(v_2))r(f(v_1), f(v_2))$$

depends on the difference $v_1 - v_2$.

Examining the proof of the theorem, one can conclude that any non-degenerate solution to (1.10) in the form

$$r(x,y) = \underbrace{\frac{C}{x-y}}_{=:r_0} + p(x,y), \tag{1.11}$$

where $C \in \mathfrak{g}\otimes\mathfrak{g}$ is the quadratic Casimir element and p is a polynomial in $\mathfrak{g}[x]\otimes\mathfrak{g}[y]$, is globally holomorphically equivalent to a rational solution. More precisely, there exists a holomorphic function $\varphi\colon \mathbb{C} \longrightarrow \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ and a rational solution $X\colon \mathbb{C} \longrightarrow \mathfrak{g}\otimes\mathfrak{g}$ of (1.6) such that

$$X(x-y) = (\varphi(x)\otimes\varphi(y))r(x,y) \qquad \forall x, y \in \mathbb{C}, x \neq y. \tag{1.12}$$

By Theorem 1.11 rational solutions with one-parameter are automatically skew-symmetric. Therefore (1.12) implies that $p(x,y) = -\tau(p(y,x))$. In other words, polynomial p is skew-symmetric.

Remark 1.16. The requirement on (1.11) to be non-degenerate is redundant. Indeed, the endomorphism $\psi((x-y)(r_0(x,y)+p(x,y))) = \mathrm{id}_{\mathfrak{g}} + \psi((x-y)p(x,y))$ is invertible in $x=y$. Therefore, by continuity $\psi(r_0(x,y)+p(x,y))$ is invertible for some $x \neq y$. $\Diamond$

Based on this equivalence, the second step in the theory by A. Stolin is to try to classify all solutions to (1.10) in the form (1.11) with skew-symmetric p. From now on such solutions will also be called rational. Note that holomorphic equivalences are no longer useful for the purpose of classification of rational solutions, because $(\varphi(x) \otimes \varphi(y))(r_0(x,y)+p(x,y))$ with a holomorphic φ may not be a rational solution anymore. This leads to the following notion of equivalence: two rational solutions r_1 and r_2 are called polynomially equivalent if there is an element

$$\varphi \in \mathrm{Aut}_{\mathbb{C}[z]-\mathrm{LieAlg}}(\mathfrak{g}[z]) \cong \{f\colon \mathbb{C} \longrightarrow \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g}) \mid f \text{ is regular}\}$$

such that

$$r_2(x,y) = (\varphi(x) \otimes \varphi(y))r_1(x,y) \qquad \forall x,y \in \mathbb{C}, x \neq y. \tag{1.13}$$

One can check by a direct computation that such an equivalence preserve the form r_0+p.

Let us fix a rational solution r_0+p. Consider the Lie algebra $\mathfrak{g}(\!(z^{-1})\!) := \mathfrak{g} \otimes \mathbb{C}(\!(z^{-1})\!)$ whose elements are Laurent polynomials of the form $\sum_{-\infty}^{N} a_k z^k$, $a_k \in \mathfrak{g}$ where $N \in \mathbb{Z}$. It can be equipped with the following non-degenerate symmetric invariant bilinear form

$$B\left(\sum_{-\infty}^{N} a_k z^k, \sum_{-\infty}^{M} b_k z^k\right) := \sum_{i+j=-1} \kappa(a_i, b_j),$$

where κ is the Killing form on $\mathfrak{g}$. Using this form we can associate with $p = f_i \otimes g^i$ a linear mapping

$$P := B(g^i, -)f_i\colon z^{-1}\mathfrak{g}[\![z^{-1}]\!] \longrightarrow \mathfrak{g}[z],$$

and a subspace

$$W := \{Pf - f \mid f \in z^{-1}\mathfrak{g}[\![z^{-1}]\!]\} \subseteq \mathfrak{g}(\!(z^{-1})\!).$$

Since p is not an arbitrary element in $\mathfrak{g}[x] \otimes \mathfrak{g}[y]$ the corresponding W possesses some specific properties. The converse turns out to be true as well.

Theorem 1.17 (A. Stolin [16, 17]). *There is a one-to-one correspondence between rational solutions r_0+p and Lagrangian Lie subalgebras $W \subseteq \mathfrak{g}(\!(z^{-1})\!)$ such that*

1. $\mathfrak{g}[z] \cap W = 0$;
2. $\mathfrak{g}[z] \dot{+} W = \mathfrak{g}(\!(z^{-1})\!)$;
3. $z^{-N}\mathfrak{g}[\![z^{-1}]\!] \subseteq W$ *for some* $N > 0$.

Moreover, for two rational solutions r_1 and r_2 the relation

$$r_2(x,y) = (\varphi(x) \otimes \varphi(y))r_1(x,y),$$

where $\varphi \in \mathrm{Aut}_{\mathbb{C}[z]-\mathrm{LieAlg}}(\mathfrak{g}[z])$, is equivalent to $W_2 = \varphi W_1$.

The third condition on W in Theorem 1.17 says that W is a so-called order in $\mathfrak{g}(\!(z^{-1})\!)$. Any order W is contained in a maximal order M. Applying polynomial equivalences one can make M to be one of the special maximal orders $\mathbb{O}_{\alpha_i}$, labeled by the vertices of the extended (untwisted) Dynkin diagram for $\mathfrak{g}$. Furthermore, for some particular roots α_i one can reduce the classification of orders $W \subseteq \mathbb{O}_{\alpha_i}$ to the classification of pairs (L, F), where L is a subalgebra of $\mathfrak{g}$ and F is a 2-cocycle on it subject to some specific conditions.

The following result shows that this method is also applicable for finding skew-symmetric solutions of the constant CYBE.

Lemma 1.18. *Let t be a skew-symmetric tensor in $\mathfrak{g} \otimes \mathfrak{g}$. Then $r_0 + t$ solves* (1.10) *if and only if t solves* (1.6)*.*

In particular, this leads to the classification of quasi-Frobenius subalgebras of $\mathfrak{g}$ mentioned in Remark 1.10.

Remark 1.19. Triples $(\mathfrak{g}(\!(z^{-1})\!), \mathfrak{g}[z], W)$ from Theorem 1.17 are called Manin triples. In Section 2 of our work we give a generalization of the theorem in the framework of Lie bialgebras for an arbitrary Manin triple (L, L_+, L_-). In Section 4 we also use the theory of maximal orders in the case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. $\diamondsuit$

1.5 Lie bialgebras (again)

We have seen that an r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$ with the ad-invariant symmetric part defines a Lie bialgebra structure on $\mathfrak{g}$. There is a similar result for solutions of the CYBE with parameters. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$. Then any non-degenerate meromorphic function $r \colon \mathbb{C}^2 \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ solving the two-parameter CYBE (1.10) induces a Lie bialgebra structure on an appropriate Lie algebra. This Lie algebra can be different from $\mathfrak{g}$ and it is not unique in general. For example, given an r-matrix r we can present it (probably after an appropriate change of variables) in a sufficiently small neighbourhood of 0 as

$$r(x, y) = \frac{C}{x - y} + h(x, y),$$

where h is a skew-symmetric holomorphic function (see [3]). Its Taylor series expansion at $y = 0$ is

$$r(x, y) = \sum_{k \geq 0} a_k(x) y^k = \sum_{k \geq 0} \sum_{i=1}^{n} (a_{k,\mu}(x) \otimes I_\mu) y^k \in (\mathfrak{g}(\!(x)\!) \otimes \mathfrak{g})[\![y]\!],$$

where $\{I_\mu\}_1^n$ is a basis for $\mathfrak{g}$. Then the Lie algebra

$$L_r := \operatorname{span}_{\mathbb{C}}\{a_{k,\mu} \mid k \geq 0, n \geq \mu \geq 1\} \subseteq \mathfrak{g}(\!(x)\!)$$

has a Lie bialgebra structure given by

$$\delta(a_{k,\mu})(x_1, x_2) := [a_{k,\mu}(x_1) \otimes 1 + 1 \otimes a_{k,\mu}(x_2), r(x_1, x_2)]. \tag{1.14}$$

Although in most cases the Lie algebra L_r is not interesting, the existence of such a universal construction is theoretically important and is used for example in the geometric theory of the CYBE [1, 4]

Remark 1.20. Formula (1.14) is a natural generalization of $\delta = dr$ from the constant "degenerate" case. Lie bialgebra structures obtained from r-matrices with parameters using formula (1.14) are called "pseudoquasitriangular". This construction also shows that two-parameter r-matrices are more natural when it comes to defining Lie bialgebra structures. It is unclear how to define a Lie bialgebra structure using a one-parameter solution $X(z)$ without viewing it as a two-parameter solution $r(x, y) = X(x - y)$. $\Diamond$

One can check directly or use the Manin triple approach to see that rational solutions in two variables $r(x, y) = C/(x - y) + p(x, y)$ discussed above define Lie bialgebra structures on $\mathfrak{g}[z]$, $\mathfrak{g}[\![z]\!]$ and $\mathfrak{g}[z, z^{-1}]$ using exactly the same formula

$$\delta(f)(x, y) := [f(x) \otimes 1 + 1 \otimes f(y), r(x, y)].$$

In [10] F.Montaner, A. Stolin and E. Zelmanov proved that all other Lie bialgebra structures on $\mathfrak{g}[\![z]\!]$ arise from the r-matrices of the following types

$$r(x, y) = 0, \quad r(x, y) = \frac{yC}{x - y} + p(x, y), \quad r(x, y) = \frac{xyC}{x - y} + p(x, y),$$

where $p \in \mathfrak{g}[x] \otimes \mathfrak{g}[y]$. In other words, we have four so-called twisting classes of Lie bialgebra structures on $\mathfrak{g}[\![z]\!]$. Therefore, we can classify all Lie bialgebra structures on $\mathfrak{g}[\![z]\!]$ by classifying Lie bialgebra structures within each of these four twisting classes, which in its turn is equivalent to the classification of r-matrices of certain types. This is exactly how all Lie bialgebra structures on $\mathfrak{g}[\![z]\!]$ were classified in [10].

Solutions of the form $xyC/(x - y) + p(x, y)$ are called quasi-rational. As in the case with rational solutions, the theory of maximal orders reduces their description to Lagrangian subalgebras $W \subseteq \mathfrak{g}(\!(z^{-1})\!) \times (\mathfrak{g} \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2))$ and in some special cases to the "subalgebra-cocycle" pairs (L, F) (see [18]).

Solutions of the CYBE in the form $yC/(x-y)+p(x, y)$ are called quasi-trigonometric. The name is motivated by the fact that there is a holomorphic function $\varphi\colon \mathbb{C} \longrightarrow \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ and a trigonometric solution $X\colon \mathbb{C} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ such that

$$(\varphi(x) \otimes \varphi(y)) \left(\frac{yC}{x - y} + p(x, y) \right) = Y(x/y), \quad \text{and} \quad Y(e^{z_1}/e^{z_2}) = X(z_1 - z_2). \tag{1.15}$$

Quasi-trigonometric r-matrices were fully classified by I. Pop and A. Stolin in [11] using the classification of Manin triples by P. Delorme [5]. The classification eventually reduces to BD quadruples mentioned in the previous section. This led to the natural questions: what is the relation between a quasi-trigonometric solution and a trigonometric solution (1.8) corresponding to the same BD quadruple? Can a quasi-trigonometric solution be written in a form similar to (1.8)?

These two questions and the following observation can be considered as a starting point for our work. Quasi-trigonometric r-matrices define Lie bialgebra structures on $\mathfrak{g}[z, z^{-1}]$ and trigonometric solutions (1.8) define Lie bialgebra structures on $\mathfrak{L}^{\sigma_1}$ (see Remark 1.14). Both these Lie algebras are loop algebras. By understanding the relation between different loop algebras, in particular $\mathfrak{g}[z, z^{-1}]$ and $\mathfrak{L}^{\sigma_1}$, we get a description of the relation between the corresponding r-matrices.

In more detail, a loop algebra $\mathfrak{L}^{\sigma}$, where σ is a finite-order automorphism of $\mathfrak{g}$, is the quotient $[\mathfrak{K}(A), \mathfrak{K}(A)]/Z(\mathfrak{K}(A))$ of the derived algebra of an affine Kac-Moody algebra

$\mathfrak{K}(A)$ by its center. Any affine Kac-Moody algebra possesses the standard Lie bialgebra structure described in [7]. Therefore, loop algebras being quotients of affine Kac-Moody algebras inherit that Lie bialgebra structure. Using the procedure called twisting, one can "twist" that structure and obtain new Lie bialgebra structures on $\mathfrak{L}^\sigma$. We show that all Lie bialgebra structures on $\mathfrak{L}^\sigma$ obtained in this way are defined by so-called σ-trigonometric r-matrices. Letting $\sigma = \mathrm{id}$ we get quasi-trigonometric r-matrices and setting $\sigma = \sigma_1$ we get trigonometric solutions (1.8). The theory of loop algebras tells us precisely how the relation between $\mathfrak{L}^\sigma$ and $\mathfrak{L}^{\sigma'}$ looks like: it is a composition of a conjugation and an operation known as regrading. It turns out that these operations can be performed at the level of σ-trigonometric r-matrices. In particular, this answers the above posed questions about quasi-trigonometric solutions. This idea is diagrammed in Figure 1.

$$\begin{tikzcd}
\mathfrak{L}^\sigma \arrow[r, leftrightarrow, "\substack{\text{regrading}\\ \text{conjugation}}"] \arrow[d, leftrightarrow, dotted] \arrow[dr, phantom, "\Downarrow"] & \mathfrak{L}^{\sigma'} \arrow[d, leftrightarrow, dotted] \\
r^\sigma \arrow[r, leftrightarrow] & r^{\sigma'}
\end{tikzcd}$$

Figure 1

Furthermore, as we already know σ_1-trigonometric r-matrices are classified in terms of BD quadruples. The geometric theory of CYBE [1, 4] allows to transfer that classification to Lie bialgebra structures on $\mathfrak{L}^\sigma$ for an arbitrary finite-order automorphism $\sigma \in \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$. More precisely, consider a Lie bialgebra structure δ^σ on $\mathfrak{L}^\sigma$ obtained by twisting the standard structure. Let r^σ be the σ-trigonometric r-matrix defining δ^σ. We prove that there exists a trigonometric solution $X\colon \mathbb{C} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ and a holomorphic function $\varphi_1\colon \mathbb{C} \longrightarrow \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ such that

$$(\varphi_1(x) \otimes \varphi_1(y))r^\sigma(e^{u/|\sigma|}, e^{v/|\sigma|}) = X(u - v).$$

Since any trigonometric solution is holomorphically equivalent to a solution of the form (1.8) for an appropriate BD quadruple Q, there must be a holomorphic function $\varphi_2\colon \mathbb{C} \longrightarrow \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ such that

$$(\varphi_2(u) \otimes \varphi_2(v))X(u - v) = r_Q^{\sigma_1}(e^{u/|\sigma|}, e^{v/|\sigma|}).$$

Now we can "regrade" and "conjugate" the r-matrix $r_Q^{\sigma_1}$ back to σ. This yields a holomorphic function $\varphi\colon \mathbb{C} \longrightarrow \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ such that

$$(\varphi(u) \otimes \varphi(v))r^\sigma(e^{u/|\sigma|}, e^{v/|\sigma|}) = r_Q^\sigma(e^{u/|\sigma|}, e^{v/|\sigma|}). \tag{1.16}$$

The geometric theory then says that the holomorphic equivalence (1.16) is actually regular, i.e. $\varphi \in \mathrm{Aut}_{\mathbb{C}[z^{|\sigma|}, z^{-|\sigma|}]-\mathrm{LieAlg}}(\mathfrak{L}^\sigma)$. Moreover, since φ is regular the corresponding Lie bialgebra structures must be equivalent, i.e.

$$\delta_Q^\sigma \varphi = (\varphi \otimes \varphi)\delta^\sigma.$$

In other words, all twisted versions of the standard Lie bialgebra structure on $\mathfrak{L}^\sigma$ are given (up to a regular equivalence of $\mathfrak{L}^\sigma$) by BD quadruples Q. We also describe the situations when two Lie bialgebra structures δ_Q^σ and $\delta_{Q'}^\sigma$ on $\mathfrak{L}^\sigma$ given by different BD quadruples are equivalent obtaining in this way a complete classification of the twists of the standard Lie bialgebra structures on $\mathfrak{L}^\sigma$. This process is presented schematically in Figure 2.

$$\begin{tikzcd}
 & (\mathfrak{L}^\sigma, \delta^\sigma) \arrow[rr, leftrightarrow, "\text{regular}"] \arrow[dl, dotted, leftrightarrow] & \arrow[d, Leftrightarrow] & (\mathfrak{L}^\sigma, \delta_Q^\sigma) \arrow[dr, dotted, leftrightarrow] & \\
r^\sigma(x,y) \arrow[rrrr, leftrightarrow, "\text{regular}"] \arrow[dr, leftrightarrow, "\text{holomorphic}"] & & {} & & r_Q^\sigma(x,y) \arrow[dl, leftrightarrow, "\text{holomorphic}"'] \\
 & X(x-y) \arrow[rr, leftrightarrow, "\text{holomorphic}"'] & \arrow[u, Rightarrow, "\text{geometry}"] & r_Q^{\sigma_1}(x,y) &
\end{tikzcd}$$

Figure 2

Our approach, among other things, provides an alternative proof for the classification [11] and reveals some previously unknown properties of quasi-trigonometric solutions. For example, the polynomial part $p(x,y) \in \mathfrak{g}[x] \otimes \mathfrak{g}[y]$ of a quasi-trigonometric solution (up to a polynomial equivalence) is of the form $p_1(x)+p_2(y)$, where $\deg(p_1) = \deg(p_2) \leq 1$. More results are stated in our paper.

1.6 Further plans

We have mentioned that all Lie bialgebra structures on $\mathfrak{g}[\![z]\!]$ can be separated into four subfamilies/twisting classes corresponding to four different types of r-matrices. Restricting (some of) these Lie bialgebra structures to $\mathfrak{g}[z]$ we observe some kind of a foliation: in some sense $\mathfrak{g}[z]$ has less symmetry than $\mathfrak{g}[\![z]\!]$ and, as a consequence, some equivalent Lie bialgebras on $\mathfrak{g}[\![z]\!]$ become non-equivalent after the restriction. As a result one gets seven twisting classes of Lie bialgebra structures on $\mathfrak{g}[z]$ (see [10] for details).

We think there is a way to extrapolate the methods and results in [10] to $\mathfrak{g}(\!(z)\!)$: determine the twisting classes of Lie bialgebra structures on $\mathfrak{g}(\!(z)\!)$ and then restrict them to $\mathfrak{g}[z,z^{-1}]$. Since this case is more "symmetric" we hope to get less twisting classes and an easier classification.

P. Etingof and D. Kazhdan showed in [8] that any Lie bialgebra or more precisely its universal enveloping algebra can be quantized. Therefore, quantizations of obtained Lie bialgebra structures is another possible direction of further research. It might lead to interesting infinite-dimensional examples of non-(co)commutative Hopf algebras. Another circumstance making this direction interesting is that loop algebras $\mathfrak{L}^\sigma$ actually arise as algebras for so-called loop groups appearing in theoretical physics.

One can also try to apply the geometric theory of the CYBE to rational solutions with two parameters in a similar way we applied it to σ-trigonometric r-matrices. By Theorem 1.15 any rational solution $r(x,y) = C/(x-y) + p(x,y)$ is holomorphically equivalent to a rational solution $X(x-y)$ in one variable. By Theorem 1.11 this one-parameter solution is holomorphically equivalent to $C/(x-y)+q(x-y)$ where $q \in \mathfrak{g}[z]$. If the composition of these holomorphic equivalences is regular (polynomial), then we get an enhancement of the theory of maximal orders for rational solutions.

References

[1] R. Abedin and I. Burban. *Algebraic geometry of Lie bialgebras defined by solutions of the classical Yang-Baxter equation.*

[2] A. Belavin and V. Drinfeld. *Solutions of the classical Yang-Baxter equation for simple Lie algebras.* In: Funct. Anal. Appl. 16, no.3 (1983).

[3] A. Belavin and V. Drinfeld. *The classical Yang-Baxter equation for simple Lie algebras.* In: Funct. Anal. Appl. 17, no.3 (1983).

[4] I. Burban and L. Galinat. *Torsion Free Sheaves on Weierstrass Cubic Curves and the Classical Yang–Baxter Equation.* In: Commun. Math. Phys. (2018).

[5] Patrick Delorme. *Classification des triples de Manin pour les algèbres de Lie réductives complexes: Avec un appendice de Guillaume Macey.* In: Journal of Algebra 246.1 (2001), pp. 97–174.

[6] V. Drinfeld. *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang–Baxter equations.* In: Dokl. Akad. Nauk SSSR, 268:2 (1983).

[7] V. Drinfeld. *Quantum groups.* In: Journal of Soviet Mathematics 41, 898–915 (1983).

[8] Pavel Etingof and David Kazhdan. *Quantization of Lie bialgebras. I.* In: Selecta Math. (N.S.) 2.1 (1996), pp. 1–41.

[9] P. Kulish and N. Reshetikhin. *Quantum linear problem for the sine-Gordon equation and higher representations.* In: Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 101 (1981), pp. 101–110, 207.

[10] F. Montaner, A. Stolin, and E. Zelmanov. *Classification of Lie bialgebras over current algebras.* In: Sel. Math. New. Ser. (2010).

[11] I. Pop and A. Stolin. *Lagrangian Subalgebras and Quasi-trigonometric r-Matrices.* In: Lett. Math. Phys. 85.2 (2008), pp. 249–262.

[12] E. Sklyanin. *On complete integrability of the Landau–Lifschitz equation.* In: LOMI-79-3 (1979).

[13] E. Sklyanin. *Some algebraic structures connected with the Yang-Baxter equation.* In: Funktsional. Anal. i Prilozhen. 16.4 (1982), pp. 27–34, 96.

[14] A Stolin. *Some remarks on Lie bialgebra structures on simple complex Lie algebras.* In: Comm. Algebra 27.9 (1999), pp. 4289–4302.

[15] A. Stolin. *Constant solutions of Yang-Baxter equation for $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$.* In: Mathematica Scandinavica 69 (1991), pp. 81–88.

[16] A. Stolin. *On rational solutions of Yang-Baxter equation for $\mathfrak{sl}(n)$.* In: Mathematica Scandinavica (1991).

[17] A. Stolin. *On rational solutions of Yang-Baxter equations. Maximal orders in loop algebra.* In: Comm. Math. Phys. (1991).

[18] A. Stolin and J. Yermolova-Magnusson. *The 4th structure.* In: Czechoslovak J. Phys. 56.10-11 (2006), pp. 1293–1297.

Classification of classical twists of the standard Lie bialgebra structure on a loop algebra

Raschid Abedin[1] and Stepan Maximov[2]

[1] *Institut für Mathematik, Universität Paderborn, 33098 Paderborn, Germany*
[2] *Department of Mathematical Sciences, Chalmers University of Technology and the University of Gothenburg, 412 96 Gothenburg, Sweden*

Abstract

The standard Lie bialgebra structure on an affine Kac-Moody algebra induces a Lie bialgebra structure on the underlying loop algebra and its parabolic subalgebras. In this paper we classify all classical twists of the induced Lie bialgebra structures in terms of Belavin-Drinfeld quadruples up to a natural notion of equivalence. To obtain this classification we first show that the induced Lie bialgebra structures are defined by certain solutions of the classical Yang-Baxter equation (CYBE) with two parameters. Then, using the algebro-geometric theory of CYBE, based on torsion free coherent sheaves, we reduce the problem to the well-known classification of trigonometric solutions given by Belavin and Drinfeld. The classification of twists in the case of parabolic subalgebras allows us to answer recently posed open questions regarding the so-called quasi-trigonometric solutions of CYBE.

1 Introduction

A Lie bialgebra is a pair (L, δ) consisting of a Lie algebra L and a linear map $\delta \colon L \longrightarrow L \otimes L$, called Lie cobracket, inducing a compatible Lie algebra structure on the dual space $L^\vee$. This notion originated in [11] as the infinitesimal counterpart of a Poisson Lie group. Shortly after, in [12, 13], Lie bialgebras were described as quasi-classical limits of certain quantum groups and received a fundamental role in the quantum group theory.

Having a Lie bialgebra structure δ on a Lie algebra L we can obtain new Lie bialgebra structures using a procedure called twisting. More precisely, let t be a skew-symmetric tensor in $L \otimes L$ satisfying

$$\mathrm{CYB}(t) = \mathrm{Alt}((\delta \otimes 1)t),$$

where

$$\mathrm{CYB}(t) := [t^{12}, t^{13}] + [t^{12}, t^{23}] + [t^{13}, t^{23}],$$
$$\mathrm{Alt}(x_1 \otimes x_2 \otimes x_3) := x_1 \otimes x_2 \otimes x_3 + x_2 \otimes x_3 \otimes x_1 + x_3 \otimes x_1 \otimes x_2$$

and, for example, $[(a \otimes b)^{12}, (c \otimes d)^{23}] := a \otimes [b, c] \otimes d$. Then the linear map $\delta_t := \delta + dt$ is a Lie bialgebra structure on L. Such a tensor t is called a classical twist.

The most important example of a Lie bialgebra structure is the standard structure δ on a symmetrizable Kac-Moody algebra $\mathfrak{K} := \mathfrak{K}(A)$ introduced in [12]. In the case when the Cartan matrix A is of finite type or, equivalently, when $\mathfrak{K}$ is a finite-dimensional semi-simple Lie algebra, the standard structure δ and all its twisted versions δ_t are known to be quasi-triangular, i.e. they are of the form dr for some $r \in \mathfrak{K} \otimes \mathfrak{K}$ satisfying the classical Yang-Baxter equation $\mathrm{CYB}(r) = 0$.

When the matrix A is of affine type, the standard structure on $\mathfrak{K}$ induces a Lie bialgebra structure on $[\mathfrak{K}, \mathfrak{K}]/Z(\mathfrak{K})$, where $Z(\mathfrak{K})$ is the center of $\mathfrak{K}$, which we will also call standard. The latter Lie algebra is known (see [23]) to be isomorphic to the loop algebra $\mathfrak{L}^\sigma$ over a simple finite-dimensional Lie algebra $\mathfrak{g}$ corresponding to an automorphism $\sigma \in \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ of finite order m. It has the following explicit description

$$\mathfrak{L}^\sigma = \left\{ f \in \mathfrak{g}[z, z^{-1}] \mid f(\varepsilon_\sigma z) = \sigma(f(z)) \right\}, \quad \varepsilon_\sigma := \exp(2\pi i/m).$$

We denote the induced standard Lie bialgebra structure on $\mathfrak{L}^\sigma$ by δ_0^σ and call its twists $\delta_t^\sigma := \delta_0^\sigma + dt$ twisted standard structures. These Lie bialgebra structures are not quasi-triangular, but pseudoquasitriangular as is shown in Theorem 3.3, i.e. they are defined by meromorphic functions $r \colon \mathbb{C}^2 \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$, also known as r-matrices, satisfying the two-parametric classical Yang-Baxter equation (CYBE)

$$\begin{aligned} \mathrm{CYB}(r)(x_1, x_2, x_3) := [r^{12}(x_1, x_2), r^{13}(x_1, x_3)] + [r^{12}(x_1, x_2), r^{23}(x_2, x_3)] \\ + [r^{13}(x_1, x_3), r^{23}(x_2, x_3)] = 0. \end{aligned}$$

For example, the trigonometric r-matrix given by a Belavin-Drinfeld (BD) quadruple Q, corresponding to an outer automorphism $\nu \in \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ (see [3]), gives rise to a twisted standard structure δ_Q^σ on $\mathfrak{L}^\sigma$ for any finite order automorophism σ whose coset is conjugate to $\nu \mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$. It turns out that any r-matrix defining a twisted standard bialgebra structure on $\mathfrak{L}^\sigma$ is globally holomorphically equivalent to a trigonometric solution in the sense of the Belavin-Drinfeld classification (see Theorem 3.4). We refer to such r-matrices as σ-trigonometric.

We call two twisted standard structures δ_t^σ and δ_s^σ (regularly) equivalent if there is a function

$$\begin{aligned} \phi \in & \mathrm{Aut}_{\mathbb{C}[z^m, z^{-m}]-\mathrm{LieAlg}}(\mathfrak{L}^\sigma) \\ & \cong \left\{ f \colon \mathbb{C}^* \longrightarrow \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g}) \mid f \text{ is regular and } f(\varepsilon_\sigma z) = \sigma f(z) \sigma^{-1} \right\}, \end{aligned}$$

called a regular equivalence, such that $\delta_t^\sigma \phi = (\phi \otimes \phi)\delta_s^\sigma$. The main result of this paper is the classification of twisted standard structures up to regular equivalence. The classification is obtained by reducing our problem to the classification of trigonometric r-matrices up to holomorphic equivalence given in [3]. To deal with the difference between the notions of equivalence we use the geometric formalism of CYBE presented in [7]. More precisely, one of the key results in [7] is that certain coherent sheaves of Lie algebras on Weierstraß cubic curves give rise to so-called geometric r-matrices, satisfying a geometric version of CYBE. In Section 5 we prove the following extension property:

Theorem A. *A formal equivalence of geometric r-matrices at the smooth point at infinity of the Weierstraß cubic curve gives rise to an isomorphism of the corresponding sheaves of Lie algebras.*

It is shown in [1] that all σ-trigonometric r-matrices arise as geometric r-matrices from coherent sheaves of Lie algebras on the nodal Weierstraß cubic with section $\mathfrak{L}^\sigma$ on the set of smooth points. Since holomorphic equivalences are formal, this result and Theorem A give the desired classification:

Theorem B. *For any twisted standard structure δ_t^σ there is a regular equivalence ϕ of $\mathfrak{L}^\sigma$ and a BD quadruple $Q = (\Gamma_1, \Gamma_2, \gamma, t_\mathfrak{h})$ such that*

$$\delta_t^\sigma \phi = (\phi \otimes \phi)\delta_Q^\sigma.$$

Furthermore, if $Q' = (\Gamma_1', \Gamma_2', \gamma', t_\mathfrak{h}')$ is another BD quadruple, then the twisted bialgebra structures δ_Q^σ and $\delta_{Q'}^\sigma$ are regularly equivalent if and only if there is an automorphism ϑ of the Dynkin diagram of $\mathfrak{L}^\sigma$ such that $\vartheta(\Gamma_i) = \Gamma_i'$ for $i = 1, 2$, $\vartheta\gamma\vartheta^{-1} = \gamma'$ and $(\vartheta \otimes \vartheta)t_\mathfrak{h} = t_\mathfrak{h}'$.

Let Π^σ be the set of simple roots of $\mathfrak{L}^\sigma$, $S \subsetneq \Pi$ and $\mathfrak{p}_+^S \subseteq \mathfrak{L}^\sigma$ be the corresponding parabolic subalgebra (see Section 2.2). The standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$ restricts to a Lie bialgebra structure on the parabolic subalgebra $\mathfrak{p}_+^S$. We refer to this Lie bialgebra structure as the restricted standard structure.

In the special case $\sigma = \mathrm{id}$ and $S = \Pi^{\mathrm{id}} \setminus \{\widetilde{\alpha}_0\}$, where $\widetilde{\alpha}_0$ is the affine root of $\mathfrak{L}^{\mathrm{id}} = \mathfrak{g}[z, z^{-1}]$, the classical twists of the restricted standard structure are in one-to-one correspondence with so-called quasi-trigonometric solutions of CYBE. Such r-matrices were studied and classified in terms of BD quadruples in [26, 32]. We use this classification in Section 4.3 to demonstrate the first part of Theorem B in the special case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. This connection to quasi-trigonometric solutions serves as a motivation for our study of restricted standard structures.

We discover that Theorem B also gives a full classification of classical twists of restricted Lie bialgera structures. More formally, for any classical twist $t \in \mathfrak{p}_+^S \otimes \mathfrak{p}_+^S$ the structure of $\mathfrak{L}^\sigma$ guarantees that the regular equivalence between δ_t^σ and some δ_Q^σ, given by Theorem B, can be chosen to fix the parabolic subalgebra $\mathfrak{p}_+^S$. The following theorem summarizes this observation.

Theorem C. *For any classical twist $t \in \mathfrak{p}_+^S \otimes \mathfrak{p}_+^S$ of the standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$ there exists a regular equivalence ϕ that restricts to an automorphism of $\mathfrak{p}_+^S$ and a BD quadruple $Q = (\Gamma_1, \Gamma_2, \gamma, t_\mathfrak{h})$ such that*

$$\Gamma_1 \subseteq S \quad \textit{and} \quad \delta_t^\sigma \phi = (\phi \otimes \phi)\delta_Q^\sigma.$$

Let $Q' = (\Gamma_1', \Gamma_2', \gamma', t_\mathfrak{h}')$, $\Gamma_1' \subseteq S$, be another BD quadruple. A regular equivalence between twisted standard structures δ_Q^σ and $\delta_{Q'}^\sigma$ restricts to an automorphism of $\mathfrak{p}_+^S$ if and only if the induced Dynkin diagram automorphism ϑ preserves S, i.e. $\vartheta(S) = S$.

The theorems stated above provide us with a list of interesting consequences:

- Letting $\sigma = \mathrm{id}$ and $S = \Pi^{\mathrm{id}} \setminus \{\widetilde{\alpha}_0\}$ in Theorem C we obtain an alternative proof of the classification of all quasi-trigonometric solutions [26, 32];

- The necessary and sufficient condition to have an id-trigonometric r-matrix which is not regularly equivalent to a quasi-trigonometric one is the existence of a BD qudruple $(\Gamma_1, \Gamma_2, \gamma, t_\mathfrak{h})$ such that for any automorphism ϑ of the extended Dynkin diagram of $\mathfrak{g}$ we have $\widetilde{\alpha}_0 \in \vartheta(\Gamma_1)$. Analyzing Dynkin diagrams, we conclude that any id-trigonometric r-matrix is regularly equivalent to a quasi-trigonometric one if and only if $\mathfrak{g}$ is of type A_n, C_n, B_{2-4} or D_{4-10};

- We have mentioned that any σ-trigonometric r-matrix is holomorphically equivalent to a trigonometric one in the sense of the Belavin-Drinfeld classification. Combining the structure theory of $\mathfrak{L}^\sigma$ (Section 2.2) and Theorem B we can improve that result and get more control over that equivalence. More precisely, let ν be an outer automorphism of $\mathfrak{g}$, σ be a finite order automorphism of $\mathfrak{g}$ whose coset is conjugate to $\nu\mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ and r_t^σ be the σ-trigonometric r-matrix defining a twisted standard Lie bialgebra structure δ_t^σ on $\mathfrak{L}^\sigma$. Applying to r_t^σ the regular equivalence, given by Theorem B, and regrading to the principle grading, i.e. grading corresponding to the Coxeter automorphism $\sigma_{(\mathbf{1};|\nu|)}$, we obtain a trigonometric r-matrix X depending on the quotient of its parameters:

$$r_t^\sigma(x,y) \xmapsto{\text{regular eq.}} r_Q^\sigma(x,y) \xmapsto{\text{regrading}} r_Q^{\sigma_{(\mathbf{1};|\nu|)}}(x,y) = X(x/y);$$

- We answer questions one and two posed at the end of [8] concerning an explicit formula for the quasi-trigonometric solution given by a BD quadruple Q and its connection with the trigonometric solution described by the same quadruple Q (see [3]).

In [29] Montaner, Stolin and Zelmanov classified all Lie bialgebra structures on $\mathfrak{g}[z]$ by classifying classical twists within each of four possible Drinfeld double algebras. A main point in their argument is the aforementioned classification of quasi-trigonometric solutions [32] or, equivalentely, the classification of classical twists within one of the doubles. From this perspective, our work is a natural step towards the classification of all Lie bialgebra structures on $\mathfrak{L}^{\mathrm{id}} = \mathfrak{g}[z,z^{-1}]$ or, more generally, $\mathfrak{L}^\sigma$.

Acknowledgements. The authors are thankful to I. Burban and A. Stolin for introduction to the topic and fruitful discussions as well as to E. Karolinsky for useful comments. R.A. also acknowledges the support from the DFG project Bu-1866/5-1.

2 Preliminaries

In this section we give a brief review of the theory of Lie bialgebras and loop algebras as well as set up notation and terminology used throughout the paper. Most of the presented results on Lie bialgebras can be found in [10, 14] and [27]. A detailed exposition of the theory of loop algebras can be found in [9, 23] and [20, Section X.5].

2.1 Lie bialgebras, Manin triples and twisting

A *Lie coalgebra* is a pair (L,δ) consisting of a vector space L over a field k of characteristic zero and a linear map $\delta\colon L \longrightarrow L\otimes L$, called *Lie cobracket*, such that for all $x\in L$

$$\delta(x) + \tau\delta(x) = 0 \quad\text{and}\quad \mathrm{Alt}((\delta\otimes 1)\delta(x)) = 0, \tag{2.1}$$

where $\tau(x_1\otimes x_2) \coloneqq x_2\otimes x_1$ and $\mathrm{Alt}(x_1\otimes x_2\otimes x_3) \coloneqq x_1\otimes x_2\otimes x_3 + x_2\otimes x_3\otimes x_1 + x_3\otimes x_1\otimes x_2$. These conditions guarantee that the restriction of the dual map $\delta^\vee\colon\ (L\otimes L)^\vee \longrightarrow L^\vee$ to $L^\vee\otimes L^\vee$ defines a Lie algebra structure. A *morphism* between two Lie coalgebras (L,δ) and (L',δ') is a linear map $\phi\colon L\longrightarrow L'$ such that

$$(\phi\otimes\phi)\delta = \delta'\phi. \tag{2.2}$$

A *Lie bialgebra* is a triple[1] $(L, [-,-], \delta)$ such that $(L, [-,-])$ is a Lie algebra, (L, δ) is a Lie coalgebra and the following compatibility condition holds

$$\delta([x,y]) = x \cdot \delta(y) - y \cdot \delta(x) \qquad \forall x, y \in L, \tag{2.3}$$

where $x \cdot (y_1 \otimes y_2) \coloneqq [x, y_1] \otimes y_2 + y_1 \otimes [x, y_2]$. In other words, δ is a 1-cocycle of L with values in $L \otimes L$. A linear map between two Lie bialgebras is a *Lie bialgebra morphism* if it is a morphism of both Lie algebra and Lie coalgebra structures.

Lie bialgebras are closely related to *Manin triples*, i.e. triples (L, L_+, L_-), where L is a Lie algebra equipped with an invariant non-degenerate symmetric bilinear form B and $L_\pm$ are isotropic subalgebras of L with respect to that form, such that $L = L_+ \dot{+} L_-$.[2] The definition immediately implies that $L_\pm$ are Lagrangian subalgebras of L which are paired non-degenerately by B. We say that two Manin triples (L, L_+, L_-) and $\left(L', L'_+, L'_-\right)$ are *isomorphic* if there is a Lie algebra isomorphism $\phi \colon L \longrightarrow L'$ such that

$$\phi(L_\pm) = L'_\pm \text{ and } B(x,y) = B(\phi(x), \phi(y)) \text{ for all } x, y \in L. \tag{2.4}$$

Every Lie bialgebra (L, δ) gives rise to the Manin triple $(L \dot{+} L^\vee, L, L^\vee)$ with the canonical bilinear form B given by

$$B(x + f, y + g) \coloneqq f(y) + g(x) \qquad \forall x, y \in L,\ \forall f, g \in L^\vee, \tag{2.5}$$

and the Lie algebra structure on $L \dot{+} L^\vee$ defined by

$$[x, f] \coloneqq \mathrm{ad}^*_x f + (f \otimes 1)(\delta(x)) \qquad \forall x \in L,\ \forall f \in L^\vee, \tag{2.6}$$

where $\mathrm{ad}^*_x \coloneqq -\mathrm{ad}^\vee_x$ is the coadjoint action.

Remark 2.1. The Lie algebra structure (2.6) is the unique Lie algebra structure on $L \dot{+} L^\vee$ making the canonical form B invariant and L, $L^\vee$ into Lagrangian subalgebras. The space $L \dot{+} L^\vee$ equipped with this particular Lie algebra structure is called the *classical double* of (L, δ). ♢

The converse statement is not true, i.e. not every Manin triple (L, L_+, L_-) induces a Lie bialgebra structure on L_+. However, this is the case when the dual map $[-,-]^\vee \colon L_-^\vee \longrightarrow (L_- \otimes L_-)^\vee$ of the Lie bracket on L_- restricts to a map $\delta \colon L_+ \longrightarrow L_+ \otimes L_+$, where we use the injection $L_+ \longrightarrow L_-^\vee$ induced by B. This condition can be equivalently formulated in the following way: there is a linear map $\delta \colon L_+ \to L_+ \otimes L_+$ such that

$$B(\delta(x), y \otimes z) = B(x, [y,z]) \qquad \forall x \in L_+,\ \forall y, z \in L_-. \tag{2.7}$$

When this condition is satisfied, we say that the Manin triple (L, L_+, L_-) *defines* the Lie bialgebra (L_+, δ).

Remark 2.2. Let ϕ be an isomorphism between two Manin triples $M = (L, L_+, L_-)$ and $M' = \left(L', L'_+, L'_-\right)$. If M defines a Lie bialgebra structure (L_+, δ), then M' also defines a Lie bialgebra structure (L'_+, δ') and $\phi|_{L_+} \colon (L_+, \delta) \longrightarrow (L'_+, \delta')$ is a Lie bialgebra isomorphism. ♢

[1] For convenience, the notation $[-,-]$ for the Lie bracket on L will be omitted from the triple.

[2] We write $A \dot{+} B$ (or $A \oplus B$) meaning the direct sum of A and B as vector spaces (modules), but not as Lie algebras. The latter is denoted by $A \times B$.

Remark 2.3. Generally, there may exist many non-isomorphic Manin triples defining the same Lie bialgebra structure. However, in the finite-dimensional case the condition (2.7) holds automatically and the correspondence between Manin triples and Lie bialgebras described above is one-to-one. $\diamondsuit$

Having a Lie bialgebra structure, we can produce a new bialgebra structure by means of a procedure called *twisting*. Let (L, δ) be a Lie bialgebra and $t \in L \otimes L$ be a skew-symmetric tensor satisfying the identity

$$\mathrm{CYB}(t) = \mathrm{Alt}\left((\delta \otimes 1)t\right), \tag{2.8}$$

where $\mathrm{CYB}(t) := [t^{12}, t^{13}] + [t^{12}, t^{23}] + [t^{13}, t^{23}]$. Then the linear map $\delta_t := \delta + dt$, where $dt(x) := x \cdot t$ for all $x \in L$, defines a new Lie bialgebra structure on L. The skew-symmetric tensor t is called a *classical twist* of δ.

It was implicitly shown in [26, 32, 33, 34] that the problem of classification of classical twists of some particular Lie bialgebra structures can be reduced to the classification of Lagrangian Lie subalgebras. In the following theorem we summarize and generalize these ideas.

Theorem 2.4. *Let (L_+, δ) be a Lie bialgebra defined by the Manin triple (L, L_+, L_-). Then there are the following one-to-one correspondences:*

Classical twists of δ, i.e.
skew-symmetric tensors $t \in L_+ \otimes L_+$
satisfying $\mathrm{CYB}(t) = \mathrm{Alt}\left((\delta \otimes 1)t\right)$

Lagrangian Lie subalgebras $L_t \subseteq L$
complementary to L_+ and
commensurable with L_-, i.e.
$\dim(L_t + L_-)/(L_t \cap L_-) < \infty$

$\longleftrightarrow$

Linear maps $T \colon L_- \longrightarrow L_+$ such that
$\dim(\mathrm{im}(T)) < \infty$ *and for all $w_1, w_2, w_3 \in L_-$*
holds $B(Tw_1, w_2) + B(w_1, Tw_2) = 0$ and
$B\left([Tw_1 - w_1, Tw_2 - w_2], Tw_3 - w_3\right) = 0$

Proof. Let $t = x_i \otimes y^i \in L_+ \otimes L_+$ be a classical twist[3] of δ. Define the linear map $T \colon L_- \longrightarrow L_+$ and the subspace $L_t \subseteq L$ by

$$T := B(y^i, -)x_i \quad \text{and} \quad L_t := \{Tw - w \mid w \in L_-\}. \tag{2.9}$$

We now show that they meet the requirements of the theorem. The conditions $\dim(\mathrm{im}(T)) < \infty$ and $L_+ \dotplus L_t = L$ hold by definition. For all $w_1, w_2 \in L_-$ we have

$$\begin{aligned} B\left(Tw_1 - w_1, Tw_2 - w_2\right) &= -B(Tw_1, w_2) - B(w_1, Tw_2) \\ &= -B(y^i, w_1)B(x_i, w_2) - B(y^i, w_2)B(x_i, w_1). \end{aligned} \tag{2.10}$$

Therefore, the skew-symmetry of t is equivalent to the skew-symmetry of T and to L_t being a Lagrangian subspace. To prove the commensurability of L_t and L_- we note that $\ker(T) = L_t \cap L_-$ and hence

$$\dim\left(L_-/\left(L_t \cap L_-\right)\right) = \dim\left(\mathrm{im}(T)\right). \tag{2.11}$$

[3] We use the Einstein summation convention: $x_i \otimes y^i = \sum_i x_i \otimes y^i$.

This shows that $L_t \cap L_-$ has finite codimension inside L_-. The commensurability now follows from the fact that L_- has codimension at most $\dim(\mathrm{im}(T))$ inside $L_t + L_-$. Finally, the last condition follows from the identity

$$B\left(w_1 \otimes w_2 \otimes w_3, \mathrm{CYB}(t) - \mathrm{Alt}\left((\delta \otimes 1)\, t\right)\right) = -B\left([Tw_1 - w_1, Tw_2 - w_2], Tw_3 - w_3\right), \tag{2.12}$$

where $w_1, w_2, w_3 \in L_-$. This identity is obtained by repeating the argument in the proof of [25, Theorem 7] within our framework.

Conversely, given a Lagrangian Lie subalgebra $L' \subseteq L$, satisfying the conditions of the theorem, we define the linear map $T\colon L_- \longrightarrow L_+$ in the following way: any $w \in L_-$ can be uniquely written as $w_+ + w'$, for some $w_+ \in L_+$ and $w' \in L'$; We let $T(w) := w_+$. Then $L' = \{Tw - w \mid w \in L_-\}$ and the commensurability of L' and L_- implies that the rank of T is finite. The other two conditions on T hold because of the relations (2.10) and the Lagrangian property of L'. To construct the classical twist $t \in L_+ \otimes L_+$ we note that B gives a non-degenerate pairing between the finite-dimensional spaces $L_-/\ker(T)$ and $\mathrm{im}(T)$. Let $\{Tw_i\}_{i=1}^n$ be a basis for $\mathrm{im}(T)$ and $\left\{v^i + \ker(T)\right\}_{i=1}^n$ be its dual basis for $L_-/\ker(T)$. Then

$$B(w_k, -Tv^i)Tw_i = B(Tw_k, v^i)Tw_i = Tw_k, \tag{2.13}$$

for all $k \in \{1, \dots, n\}$. Since T is completely determined by its action on $\{w_i\}_{i=1}^n$, we have the equality $T = -B(Tv^i, -)Tw_i$. We define $t := -Tw_i \otimes Tv^i$. The identities (2.12) and (2.10) guarantee that t meets the desired requirements and $L' = L_t$. ■

Remark 2.5. It follows that if (L_+, δ) is a Lie bialgebra defined by the Manin triple (L, L_+, L_-) and t is a classical twist of δ, then the twisted Lie bialgebra $(L_+, \delta + dt)$ is defined by the Manin triple (L, L_+, L_t). Equivalently,

$$B\left(\delta(x) + x \cdot t, (Tw_1 - w_1) \otimes (Tw_2 - w_2)\right) = B\left(x, [Tw_1 - w_1, Tw_2 - w_2]\right), \tag{2.14}$$

for all $x \in L_+$ and $w_1, w_2 \in L_-$. ◊

2.2 Loop algebras

Let $\mathfrak{g}$ be a fixed finite-dimensional simple Lie algebra over $\mathbb{C}$ and σ be an automorphism of $\mathfrak{g}$ of finite order $|\sigma| \in \mathbb{Z}_+$. The eigenvalues of σ are $\varepsilon_\sigma^k := e^{2\pi i k/|\sigma|}$, $k \in \mathbb{Z}$, and we have the following $\mathbb{Z}/|\sigma|\mathbb{Z}$-gradation of $\mathfrak{g}$

$$\mathfrak{g} = \bigoplus_{k=0}^{|\sigma|-1} \mathfrak{g}_k^\sigma, \tag{2.15}$$

where $\mathfrak{g}_k^\sigma$ is the eigenspace of σ corresponding to the eigenvalue ε_σ^k. The tensor product

$$\mathfrak{L} := \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}] = \bigoplus_{k \in \mathbb{Z}} z^k \mathfrak{g} = \left\{f\colon \mathbb{C}^* \longrightarrow \mathfrak{g} \mid f \text{ is regular}\right\}, \tag{2.16}$$

equipped with the bracket described by $[z^i x, z^j y] := z^{i+j}[x, y]$, for all $x, y \in \mathfrak{g}$ and $i, j \in \mathbb{Z}$, is a $\mathbb{Z}$-graded Lie algebra over $\mathbb{C}$. *The loop algebra* $\mathfrak{L}^\sigma$ over $\mathfrak{g}$ is the $\mathbb{Z}$-graded Lie subalgebra of $\mathfrak{L}$ defined by

$$\mathfrak{L}^\sigma := \bigoplus_{k \in \mathbb{Z}} z^k \mathfrak{g}_k^\sigma = \left\{f \in \mathfrak{L} \mid \sigma(f(z)) = f(\varepsilon_\sigma z)\right\}, \tag{2.17}$$

where $\mathfrak{g}^\sigma_{k+\ell|\sigma|} = \mathfrak{g}^\sigma_k$ for all $\ell \in \mathbb{Z}$. It possesses an invariant non-degenerate symmetric bilinear form B, which is given by

$$B(f,g) := \mathrm{res}_{z=0}\left[\frac{1}{z}\kappa(f(z),g(z))\right] \qquad \forall f,g \in \mathfrak{L}, \tag{2.18}$$

where κ stands for the Killing form on $\mathfrak{g}$.

Remark 2.6. We can extend σ to an automorphism on $\mathfrak{L}$ by $\sigma(z^k x) := (z/\varepsilon_\sigma)^k\sigma(x)$. Then $\mathfrak{L}^\sigma$ can be viewed as the Lie subalgebra of $\mathfrak{L}$ consisting of fixed points of the extended action of σ on $\mathfrak{L}$. In particular, we have the identity $\mathfrak{L} = \mathfrak{L}^{\mathrm{id}}$. This motivates our choice of notation. $\Diamond$

2.2.1 Structure theory (outer automorphism case)

The classification of all finite order automorphisms of $\mathfrak{g}$, explained in [22, 20, 23], gives the following relation at the level of loop algebras: for any finite order automorphism σ there is an automorphism ν of $\mathfrak{g}$, induced by an automorphism of the corresponding Dynkin diagram, such that $\mathfrak{L}^\sigma \cong \mathfrak{L}^\nu$. Therefore, we first describe the structure of $\mathfrak{L}^\nu$ and then explain how regrading of $\mathfrak{L}^\nu$ carries over the structure theory to $\mathfrak{L}^\sigma$.

Let $\mathfrak{g} = \mathfrak{n}'_- \dotplus \mathfrak{h}' \dotplus \mathfrak{n}'_+$ be a triangular decomposition of $\mathfrak{g}$ and $\widetilde{\nu}$ be an automorphism of the corresponding Dynkin diagram. The induced outer automorphism ν of $\mathfrak{g}$ is described explicitly by

$$\nu(x_i^\pm) = x^\pm_{\widetilde{\nu}(i)}, \quad \nu(h_i) = h_{\widetilde{\nu}(i)}, \tag{2.19}$$

where $\{x_i^-, h_i, x_i^+\}$ is a fixed set of standard Chevalley generators for $\mathfrak{g}$. The order of such an automorphism is necessarily 1, 2 or 3. The subalgebra $\mathfrak{g}^\nu_0$ turns out to be simple with the following triangular decomposition

$$\mathfrak{g}^\nu_0 = \underbrace{(\mathfrak{g}^\nu_0 \cap \mathfrak{n}'_-)}_{=:\,\mathfrak{n}_-} \dotplus \underbrace{(\mathfrak{g}^\nu_0 \cap \mathfrak{h}')}_{=:\,\mathfrak{h}} \dotplus \underbrace{(\mathfrak{g}^\nu_0 \cap \mathfrak{n}'_+)}_{=:\,\mathfrak{n}_+}. \tag{2.20}$$

Moreover, when $|\nu| = 2$ or 3 the subspace $\mathfrak{g}^\nu_1$ is an irreducible $\mathfrak{g}^\nu_0$-module. In the case $|\nu| = 3$ it is isomorphic (as a module) to $\mathfrak{g}^\nu_2 = \mathfrak{g}^\nu_{-1}$.

Remark 2.7. For any automorphism ρ of $\mathfrak{g}$ we have a natural $\mathbb{Z}$-graded Lie algebra isomorphism $\mathfrak{L}^\sigma \cong \mathfrak{L}^{\rho\sigma\rho^{-1}}$ given by $z^k x \longmapsto z^k\rho(x)$. Since the automorphism ν is defined by its order up to conjugation, this result implies that $\mathfrak{L}^\nu$ is also determined by the order of the automorphism ν. $\Diamond$

A pair (α, k), where $\alpha \in \mathfrak{h}^\vee$ and $k \in \mathbb{Z}$, is called a *root* if the joint eigenspace

$$\mathfrak{g}^\nu_{(\alpha,k)} = \{x \in \mathfrak{g}^\nu_k \mid [h,x] = \alpha(h)x \ \forall h \in \mathfrak{h}\} \tag{2.21}$$

is non-zero. Let Φ be the set of all roots and Φ_k be the set of roots of the form (α, k). The triangular decomposition (2.20) of $\mathfrak{g}^\nu_0$ gives rise to the polarization $\Phi_0 = \Phi_0^- \cup \{(0,0)\} \cup \Phi_0^+$. For convenience we introduce two more subsets of roots:

$$\begin{aligned} \Phi^+ &:= \Phi_0^+ \cup \{(\alpha,k) \in \Phi \mid k > 0\}, \\ \Phi^- &:= \Phi_0^- \cup \{(\alpha,k) \in \Phi \mid k < 0\}. \end{aligned} \tag{2.22}$$

The elements of Φ^+ and Φ^- are called *positive* and *negative roots* respectively. It is clear that $\Phi = \Phi^- \cup \{(0,0)\} \cup \Phi^+$ and $-\Phi^+ = \Phi^-$. Denoting $z^k \mathfrak{g}^\nu_{(\alpha,k)}$ by $\mathfrak{L}^\nu_{(\alpha,k)}$ we get the root space decomposition

$$\mathfrak{L}^\nu = \bigoplus_{(\alpha,k)\in\Phi} \mathfrak{L}^\nu_{(\alpha,k)}, \tag{2.23}$$

where $\dim(\mathfrak{L}^\nu_{(\alpha,k)}) = 1$ if $\alpha \neq 0$ and $\mathfrak{L}^\nu_{(0,0)} = \mathfrak{h}$. The form B pairs the spaces $\mathfrak{L}^\nu_{(\alpha,k_1)}$ and $\mathfrak{L}^\nu_{(\beta,k_2)}$ non-degenerately if $(\alpha,k_1)+(\beta,k_2) = (0,0)$; otherwise $B(\mathfrak{L}^\nu_{(\alpha,k_1)}, \mathfrak{L}^\nu_{(\beta,k_2)}) = 0$. Defining

$$\mathfrak{N}_\pm \coloneqq \bigoplus_{(\alpha,k)\in\Phi^\pm} \mathfrak{L}^\nu_{(\alpha,k)}, \tag{2.24}$$

we obtain analogues of a triangular decomposition and Borel subalgebras for $\mathfrak{L}^\nu$, namely

$$\mathfrak{L}^\nu = \mathfrak{N}_- \dotplus \mathfrak{h} \dotplus \mathfrak{N}_+ \quad \text{and} \quad \mathfrak{B}_\pm \coloneqq \mathfrak{h} \dotplus \mathfrak{N}_\pm. \tag{2.25}$$

Let $\{\alpha_1, \dots, \alpha_n\}$ be a set of simple roots of $\mathfrak{g}^\nu_0$ with respect to (2.20) and α_0 be the corresponding minimal root. For any root α we write $\alpha^\vee$ for the unique element in $\mathfrak{h}$ such that $B(\alpha^\vee, -) = \alpha(-)$. The set

$$\Pi \coloneqq \{\underbrace{(\alpha_0,1)}_{=:\widetilde{\alpha}_0}, \underbrace{(\alpha_1,0)}_{=:\widetilde{\alpha}_1}, \dots, \underbrace{(\alpha_n,0)}_{=:\widetilde{\alpha}_n}\}. \tag{2.26}$$

is called *the simple root system* of $\mathfrak{L}^\nu$. It satisfies the following properties:

1. Any $(\alpha,k) \in \Phi$ can be uniquely written in the form $(\alpha,k) = \sum_{i=0}^n c_i \widetilde{\alpha}_i$, where $c_i \in \mathbb{Z}$. If the root (α,k) is positive (negative), then the coefficients c_i in its decomposition are all non-negative (non-positive);

2. The matrix $A \coloneqq (a_{ij})$, where

$$a_{ij} \coloneqq 2\frac{B(\alpha_i^\vee, \alpha_j^\vee)}{B(\alpha_j^\vee, \alpha_j^\vee)} \in \mathbb{Z} \qquad i,j \in \{0,1,\dots,n\}, \tag{2.27}$$

   is a generalized Cartan matrix of affine type. We call it *the affine matrix associated to* $\mathfrak{L}^\nu$. The Dynkin diagram corresponding to A is called *the Dynkin diagram of* $\mathfrak{L}^\nu$.

Let $\Lambda_0 \coloneqq \{X_i^-, H_i, X_i^+\}_{i=1}^n$ be the set of standard Chevalley generators for $\mathfrak{g}^\nu_0$ with respect to the choice of simple roots we made earlier. Take two elements $X_0^\pm \in \mathfrak{L}_{(\pm\alpha_0,\pm1)}$ such that

$$\left[X_0^+, X_0^-\right] = \frac{\alpha_0^\vee}{B(\alpha_0^\vee, \alpha_0^\vee)} =: H_0. \tag{2.28}$$

By [20, Lemma X.5.8] the set $\Lambda \coloneqq \Lambda_0 \cup \{X_0^-, H_0, X_0^+\}$ generates the whole Lie algebra $\mathfrak{L}^\nu$. For any $S \subsetneq \Pi$ we denote by $\mathfrak{S}^S$ the semi-simple subalgebra of $\mathfrak{L}^\nu$ generated by $\{X_i^-, H_i, X_i^+\}_{\widetilde{\alpha}_i\in S}$ with the induced triangular decomposition $\mathfrak{S}^S = \mathfrak{N}_-^S \dotplus \mathfrak{h}^S \dotplus \mathfrak{N}_+^S$. The subalgebras $\mathfrak{p}_\pm^S \coloneqq \mathfrak{B}_\pm \dotplus \mathfrak{N}_\mp^S$ are the analogues for the parabolic subalgebras in the theory of semi-simple Lie algebras.

2.2.2 Classification of finite order automorphisms and regrading

We now explain the regrading procedure that makes it possible to transfer all the preceding results of this section to $\mathfrak{L}^\sigma$ for an arbitrary finite order automorphism σ. Let $s = (s_0, s_1, \ldots, s_n)$ be a sequence of non-negative integers with at least one non-zero element. Using the properties of the simple root system (2.26) we can write

$$(0, |\nu|) = |\nu| \sum_{i=0}^{n} a_i \widetilde{\alpha}_i \tag{2.29}$$

for some unique positive integers a_i. We define a positive integer $m := |\nu| \sum_{i=0}^{n} a_i s_i$. The following results were proven in [20, Theorem X.5.15]:

1. The set $\{X_j^+(1)\}_{j=0}^n$ generates the Lie algebra $\mathfrak{g}$ and the relations

$$\sigma_{(s;|\nu|)}(X_j^+(1)) := e^{2\pi i s_j/m} X_j^+(1) \qquad 0 \leq j \leq n \tag{2.30}$$

   define a unique automorphism $\sigma_{(s;|\nu|)}$ of $\mathfrak{g}$ of order m such that $\mathfrak{L}^\nu \cong \mathfrak{L}^{\sigma_{(s;|\nu|)}}$. In particular, $\nu = \sigma_{((1,0,\ldots,0);|\nu|)}$;

2. Up to conjugation any finite order automorphism σ of $\mathfrak{g}$ arise in this way.

It follows immediately that for any finite order automorphism σ of $\mathfrak{g}$ there is an automorphism $\sigma_{(s;|\nu|)}$ and an outer automorphism ν of $\mathfrak{g}$ such that

$$\mathfrak{L}^\nu \xrightarrow[G^s]{\sim} \mathfrak{L}^{\sigma_{(s;|\nu|)}} \xrightarrow{\sim} \mathfrak{L}^\sigma, \tag{2.31}$$

where the second isomorphism, given by conjugation, is described in Remark 2.7. The automorphism $\sigma_{(s;|\nu|)}$ is called *the automorphism of type* $(s; |\nu|)$. Note that the conjugacy class of the coset $\sigma_{(s;|\nu|)}\mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ is represented by ν.

Now we describe the first isomorphism in the chain (2.31). Define *the s-height* $\mathrm{ht}_s(\alpha, k)$ of a root $(\alpha, k) \in \Phi$ in the following way: decompose (α, k) with respect to the simple root system Π, i.e. $(\alpha, k) = \sum_{i=0}^{n} c_i \widetilde{\alpha}_i$ and set

$$\mathrm{ht}_s(\alpha, k) := \sum_{i=0}^{n} c_i s_i. \tag{2.32}$$

We introduce a new $\mathbb{Z}$-grading on $\mathfrak{L}^\nu$, called *$\mathbb{Z}$-grading of type s*, by declaring $\deg(f) = 0$ for $f \in \mathfrak{h}$ and $\deg(f) = \mathrm{ht}_s(\alpha, k)$ for $f \in \mathfrak{L}^\nu_{(\alpha,k)}$. The isomorphism $G^s \colon \mathfrak{L}^\nu \longrightarrow \mathfrak{L}^{\sigma_{(s;|\nu|)}}$, called *regrading*, is given by

$$G^s(z^k x) := z^{\mathrm{ht}_s(\alpha,k)} x \qquad \forall z^k x \in \mathfrak{L}^\nu_{(\alpha,k)}. \tag{2.33}$$

If $\mathfrak{L}^\nu$ is equipped with the grading of type s and $\mathfrak{L}^{\sigma_{(s;|\nu|)}}$ is equipped with the natural grading given by the powers of z, then G^s is a graded isomorphism. We write $G_s^{s'}$ for the resulting regrading $G^{s'} \circ (G^s)^{-1} \colon \mathfrak{L}^{\sigma_{(s;|\nu|)}} \longrightarrow \mathfrak{L}^{\sigma_{(s';|\nu|)}}$.

Remark 2.8. The grading given by $s = \mathbf{1} = (1, 1, \ldots, 1)$ is called *the principle grading* and the corresponding automorphism $\sigma_{(\mathbf{1};|\nu|)}$ is *the Coxeter automorphism* of the pair $(\mathfrak{g}, \nu)$. ◊

2.2.3 Structure theory (general case)

We finish the discussion of loop algebras by pushing the structure theory for $\mathfrak{L}^\nu$ to $\mathfrak{L}^\sigma$ through the chain of isomorphisms (2.31). We do it gradually, starting with the case $\sigma = \sigma_{(s;|\nu|)}$, $s = (s_0, s_1, \ldots, s_n)$. Let Φ and Π, as before, be the set of all roots and the simple root system of $\mathfrak{L}^\nu$. From the definition of regrading it is clear that $G^s(\mathfrak{h}) = \mathfrak{h}$. This allows us to define the joint eigenspaces $\mathfrak{g}^\sigma_{(\alpha,\ell)}$, $\alpha \in \mathfrak{h}^\vee$, $\ell \in \mathbb{Z}$, using the exact same formula (2.21) and call (α, ℓ) a root of $\mathfrak{L}^\sigma$ if $\mathfrak{g}^\sigma_{(\alpha,\ell)} \neq 0$. Using regrading we can describe the root spaces $\mathfrak{L}^\sigma_{(\alpha,\ell)} := z^\ell \mathfrak{g}^\sigma_{(\alpha,\ell)}$ of $\mathfrak{L}^\sigma$ in terms of the root spaces of $\mathfrak{L}^\nu$, namely

$$G^s\left(\mathfrak{L}^\nu_{(\alpha,k)}\right) = \mathfrak{L}^\sigma_{(\alpha,\mathrm{ht}_s(\alpha,k))} \qquad \forall (\alpha, k) \in \Phi. \tag{2.34}$$

This gives a bijection between roots of $\mathfrak{L}^\nu$ and $\mathfrak{L}^\sigma$. More precisely, let Φ_σ be the set of all roots of $\mathfrak{L}^\sigma$, then

$$\Phi_\sigma = \{(\alpha, \mathrm{ht}_s(\alpha, k)) \mid (\alpha, k) \in \Phi\}. \tag{2.35}$$

The subset $\Pi^\sigma := \{(\alpha_0, s_0), (\alpha_1, s_1), \ldots, (\alpha_n, s_n)\} \subseteq \Phi_\sigma$ is said to be the simple root system of $\mathfrak{L}^\sigma$. We again adopt the notation $\widetilde{\alpha_i}$ for the simple root (α_i, s_i). By definition of ht_s the root spaces $\mathfrak{L}^\sigma_{(\alpha,\ell_1)}$ and $\mathfrak{L}^\sigma_{(\beta,\ell_2)}$ are paired by the form B non-degenratly if $(\alpha, \ell_1) + (\beta, \ell_2) = (0, 0)$; otherwise $B(\mathfrak{L}^\sigma_{(\alpha,\ell_1)}, \mathfrak{L}^\sigma_{(\beta,\ell_2)}) = 0$. It is evident from (2.34) that the subspaces $\mathfrak{N}_\pm, \mathfrak{B}_\pm \subseteq \mathfrak{L}^\nu$ are fixed under regrading and thus we can unambiguously use the same notations for them considered as subspaces of $\mathfrak{L}^\sigma$. Applying regrading to the set of generators $\Lambda = \{X_i^-, H_i, X_i^+\}_{i=0}^n$ of $\mathfrak{L}^\nu$ we obtain the set

$$\Lambda^\sigma := \left\{z^{-s_i} X_i^-(1), H_i, z^{s_i} X_i^+(1)\right\}_{i=0}^n \tag{2.36}$$

of generators of $\mathfrak{L}^\sigma$. When $S \subsetneq \Pi^\sigma$ we use the same notation $\mathfrak{S}^S$ to denote the semi-simple subalgebra of $\mathfrak{L}^\sigma$ generated by $\{z^{-s_i} X_i^-(1), H_i, z^{s_i} X_i^+(1)\}_{\widetilde{\alpha}_i \in S}$ with the induced triangular decomposition $\mathfrak{S}^S = \mathfrak{N}_-^S \dotplus \mathfrak{h}^S \dotplus \mathfrak{N}_+^S$. The corresponding parabolic subalgebras of $\mathfrak{L}^\sigma$ are defined using the same formulas, namely $\mathfrak{p}_\pm^S := \mathfrak{B}_\pm \dotplus \mathfrak{N}_\mp^S$. We also define

$$\mathfrak{n}_\pm^\sigma := \bigoplus_{(\alpha,0)\in\Phi_\sigma^\pm} \mathfrak{L}^\sigma_{(\alpha,0)} = \mathfrak{g}_0^\sigma \cap \mathfrak{N}_\pm. \tag{2.37}$$

This gives the triangular decomposition $\mathfrak{g}_0^\sigma = \mathfrak{n}_-^\sigma \dotplus \mathfrak{h} \dotplus \mathfrak{n}_+^\sigma$.

Finally, we consider the case $\sigma = \rho\sigma_{(s;|\nu|)}\rho^{-1}$ for some $\rho \in \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$. We denote the natural isomorphism

$$\mathfrak{L}^{\sigma_{(s;|\nu|)}} \longrightarrow \mathfrak{L}^\sigma, \qquad z^k x \longmapsto z^k \rho(x) \tag{2.38}$$

with the same letter ρ. The roots of $\mathfrak{L}^\sigma$ with respect to the action of the Cartan subalgebra $\rho(\mathfrak{h})$ are of the form $(\alpha\rho^{-1}, \ell)$, where (α, ℓ) is a root of $\mathfrak{L}^{\sigma_{(s;|\nu|)}}$, and the root spaces are described by

$$\mathfrak{L}^\sigma_{(\alpha\rho^{-1},\ell)} = \rho\left(\mathfrak{L}^{\sigma_{(s;|\nu|)}}_{(\alpha,\ell)}\right). \tag{2.39}$$

The set of all roots is again denoted by Φ_σ, and its subset

$$\Pi^\sigma := \{(\alpha_0\rho^{-1}, s_0), (\alpha_1\rho^{-1}, s_1), \ldots, (\alpha_n\rho^{-1}, s_n)\} \tag{2.40}$$

is called the simple root system of $\mathfrak{L}^\sigma$. Applying ρ to the generators (2.36) of $\mathfrak{L}^{\sigma_{(s;|\nu|)}}$ we get the set

$$\Lambda^\sigma := \left\{z^{-s_i}\rho(X_i^-(1)), \rho(H_i), z^{s_i}\rho(X_i^+(1))\right\}_{i=0}^n \tag{2.41}$$

of generators of $\mathfrak{L}^\sigma$. Later, when there is no ambiguity, the same notations $X_i^\pm$ and H_i are used to denote the elements of generating sets (2.41) and (2.36). Combining (2.39) with (2.34) we define

$$\begin{aligned}
\mathfrak{n}_\pm^\sigma &:= \rho(\mathfrak{n}_\pm^{\sigma_{(s;|\nu|)}}) = \bigoplus_{(\alpha,0)\in\Phi^\pm} \mathfrak{L}^\sigma_{(\alpha\rho^{-1},\mathrm{ht}_s(\alpha,k))},\\
\mathfrak{N}_\pm^\sigma &:= \rho(\mathfrak{N}_\pm) = \bigoplus_{(\alpha,k)\in\Phi^\pm} \mathfrak{L}^\sigma_{(\alpha\rho^{-1},\mathrm{ht}_s(\alpha,k))},\\
\mathfrak{B}_\pm^\sigma &:= \rho(\mathfrak{B}_\pm) = \mathfrak{N}_\pm^\sigma \dotplus \rho(\mathfrak{h}).
\end{aligned} \tag{2.42}$$

where Φ, as before, is the set of all roots of $\mathfrak{L}^\nu$. Note that this notation is in consistence with the one defined earlier.

Remark 2.9. Let $\sigma = \rho\sigma_{(s;|\nu|)}\rho^{-1}$ and A be the affine matrix associated to $\mathfrak{L}^\sigma$, defined in a way similar to (2.27). Then A coincides with the affine Cartan matrix of $\mathfrak{L}^\nu$ and so does the Dynkin diagram of $\mathfrak{L}^\sigma$. ♢

2.2.4 Connection to Kac-Moody algebras

As the structure theory developed in the preceding subsections suggests, the notion of a loop algebra is closely related to the notion of an affine Kac-Moody algebra. More precisely, let A be an affine matrix of type $X_N^{(m)}$, $\mathfrak{g}$ be the simple finite-dimensional Lie algebra of type X_N and ν be an automorphism of $\mathfrak{g}$ induced by an automorphism of the corresponding Dynkin diagram with $|\nu| = m$. Then A is the Cartan matrix of $\mathfrak{L}^\nu$ and the affine Kac-Moody algebra $\mathfrak{K}(A)$ is isomorphic to

$$\mathfrak{L}^\nu \dotplus \mathbb{C}c \dotplus \mathbb{C}d, \tag{2.43}$$

where $\mathbb{C}c$ is the one-dimensional center of $\mathfrak{K}(A)$, d is the additional derivation element that acts on $\mathfrak{L}^\nu$ as $z\frac{d}{dz}$ and the Lie bracket is described by

$$[z^k x, z^\ell y] = z^{k+\ell}[x,y] + kB(z^k x, z^\ell y)c \qquad \forall z^k x, z^\ell y \in \mathfrak{L}. \tag{2.44}$$

Consequently $\mathfrak{L}^\nu \cong [\mathfrak{K}(A), \mathfrak{K}(A)]/\mathbb{C}c$ and the form (2.18) on $\mathfrak{L}^\nu$ extends to a standard bilinear form on $\mathfrak{K}(A)$ in the sense of [23, Section 2].

3 The standard Lie bialgebra structure on $\mathfrak{L}^\sigma$ and its twists

Let $\mathfrak{K}(A)$ be a symmetrizable Kac-Moody algebra with a fixed invariant non-degenerate symmetric bilinear form B. Then it possesses a Lie bialgebra structure δ_0, called the standard Lie bialgebra structure on $\mathfrak{K}(A)$, given by

$$\delta_0(H_i) = 0, \qquad \delta_0(D_i) = 0, \qquad \delta_0(X_i^\pm) = \frac{B(\alpha_i^\vee, \alpha_i^\vee)}{2} H_i \wedge X_i^\pm, \tag{3.1}$$

where $\{X_i^-, H_i, X_i^+\} \cup \{D_i\}$ is a set of standard generators for $\mathfrak{K}(A)$ (see [13, Example 3.2] and [10, Example 1.3.8]). We can immediately see that δ_0 induces a Lie bialgebra structure on

$$[\mathfrak{K}(A), \mathfrak{K}(A)]/Z(\mathfrak{K}(A)), \tag{3.2}$$

where $Z(\mathfrak{K}(A))$ is the center of $\mathfrak{K}(A)$. In particular, when A is an affine matrix and B is the form mentioned in Subsection 2.2.4 we get a Lie bialgebra structure δ_0^ν on $\mathfrak{L}^\nu$. Applying the methods described in Section 2.2 we induce a Lie bialgebra structure δ_0^σ, called *the standard Lie bialgebra structure*, on $\mathfrak{L}^\sigma$ for any finite order automorphism σ. Its twisted versions δ_t^σ are called *twisted standard structures.*

3.1 Pseudoquasitriangular structure

We want to prove that δ_t^σ is a pseudoquasitriangular Lie bialgebra structure, i.e. it is defined by an r-matrix. We restrict our attention to a special case $\sigma = \sigma_{(s;|\nu|)}$. The general result will then follow from the natural isomorphism mentioned in Remark 2.7.

Let C_k^σ be the projection of the Casimir element $C = \sum_{k=0}^{|\sigma|-1} C_k^\sigma \in \mathfrak{g} \otimes \mathfrak{g}$ on the eigenspace $\mathfrak{g}_k^\sigma \otimes \mathfrak{g}_{-k}^\sigma$. The triangular decomposition $\mathfrak{g}_0^\sigma = \mathfrak{n}_-^\sigma \dotplus \mathfrak{h} \dotplus \mathfrak{n}_+^\sigma$ leads to the splitting $C_0^\sigma = C_-^\sigma + C_\mathfrak{h} + C_+^\sigma$, where $C_\pm^\sigma \in \mathfrak{n}_\pm^\sigma \otimes \mathfrak{n}_\mp^\sigma$ and $C_\mathfrak{h} \in \mathfrak{h} \otimes \mathfrak{h}$. We introduce a rational function $r_0^\sigma \colon \mathbb{C}^2 \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ defined by

$$r_0^\sigma(x,y) \coloneqq \frac{C_\mathfrak{h}}{2} + C_-^\sigma + \frac{1}{(x/y)^{|\sigma|} - 1} \sum_{k=0}^{|\sigma|-1} \left(\frac{x}{y}\right)^k C_k^\sigma. \tag{3.3}$$

Remark 3.1. Formula (3.3) can be seen as a generalization of well-known r-matrices. P. Kulish introduced $r_0^{\sigma(\mathbf{1};1)}$ in [28]. More generally $r_0^{\sigma(\mathbf{1};|\nu|)}$ was introduced in [3] by A. Belavin and V.Drinfeld, which they later, in [5], called the simplest trigonometric solution. M. Jimbo used r_0^ν in [21] and the formula for r_0^{id} appears in the recent works [26, 32] and [8] under the name "quasi-trigonometric r-matrix". $\Diamond$

The statement in [3, Lemma 6.22] suggests the following holomorphic relations between functions defined by (3.3).

Lemma 3.2. *Let $\sigma, \sigma' \in \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ be two automorphisms of types $(s;|\nu|)$ and $(s';|\nu|)$ respectively, where $s = (s_0, s_1, \dots, s_n)$ and $s' = (s_0', s_1', \dots, s_n')$. Then*

1. *The equations $\alpha_i(\mu) = s_i'/|\sigma'| - s_i/|\sigma|$, $i \in \{0, 1, \dots, n\}$, define a unique element $\mu \in \mathfrak{h}$ such that*

$$e^{u\,\mathrm{ad}(\mu)} f\left(e^{u/|\sigma|}\right) = \left(G_s^{s'} f\right)\left(e^{u/|\sigma'|}\right) \qquad \forall f \in \mathfrak{L}^\sigma,\ \forall u \in \mathbb{C}; \tag{3.4}$$

2. *For all $u, v \in \mathbb{C}$, $u - v \notin 2\pi i\mathbb{Z}$ the functions r_0^σ and $r_0^{\sigma'}$ satisfy the relation*

$$\left(e^{u\,\mathrm{ad}(\mu)} \otimes e^{v\,\mathrm{ad}(\mu)}\right) r_0^\sigma\left(e^{u/|\sigma|}, e^{v/|\sigma|}\right) = r_0^{\sigma'}\left(e^{u/|\sigma'|}, e^{v/|\sigma'|}\right). \tag{3.5}$$

Proof. Using the formulas

$$|\sigma| = |\nu| \sum_{i=0}^n a_i s_i \quad \text{and} \quad |\sigma'| = |\nu| \sum_{i=0}^n a_i s_i', \tag{3.6}$$

we can easily deduce that the equations $\alpha_i(\mu) = s_i'/|\sigma'| - s_i/|\sigma|$ are consistent and define a unique element $\mu \in \mathfrak{h}$. Let $f = X_i^{\pm}$, $i \in \{0, 1, \dots, n\}$. Then for all $u \in \mathbb{C}$ we have

$$\begin{aligned} e^{u\,\mathrm{ad}(\mu)} X_i^{\pm}\left(e^{u/|\sigma|}\right) &= e^{\pm u s_i/|\sigma|} \sum_{k \geq 0} \frac{u^k}{k!} \left(\pm \frac{s_i'}{|\sigma'|} \mp \frac{s_i}{|\sigma|}\right)^k X_i^{\pm}(1) \\ &= e^{\pm u s_i'/|\sigma'|} X_i^{\pm}(1) = \left(G_s^{s'} X_i^{\pm}\right)\left(e^{u/|\sigma'|}\right). \end{aligned} \tag{3.7}$$

Since $\mathfrak{L}^\sigma$ is generated by $X_i^{\pm}$, identity (3.7) proves the first statement. To verify the second statement we choose a basis $\{b^i_{(\alpha,k)}\}$ for each $\mathfrak{L}^\sigma_{(\alpha,k)}$ such that

$$B\left(b^i_{(\alpha,k)}, b^j_{(-\alpha,-k)}\right) = \delta_{ij} \qquad \forall (\alpha, k) \in \Phi_\sigma. \tag{3.8}$$

Setting $n_{(\alpha,k)} \coloneqq \dim(\mathfrak{L}^\sigma_{(\alpha,k)})$ we can write

$$\left(\frac{y}{x}\right)^{-k+\ell|\sigma|} C_k^\sigma = \sum_{\substack{(\alpha,k)\in\Phi_\sigma^+ \\ 1\leq i\leq n_{(\alpha,k)}}} b^i_{(-\alpha,k-\ell|\sigma|)}(x) \otimes b^i_{(\alpha,-k+\ell|\sigma|)}(y) \qquad \forall x, y \in \mathbb{C}^*, \tag{3.9}$$

where Φ_σ^+ stands for the set of positive roots of $\mathfrak{L}^\sigma$. Then the Taylor series of r_0^σ in $y = 0$ for a fixed x is

$$r_0^\sigma(x, y) = \frac{C_{\mathfrak{h}}}{2} + \sum_{\substack{(\alpha,k)\in\Phi_\sigma^+ \\ 1\leq i\leq n_{(\alpha,k)}}} b^i_{(-\alpha,-k)}(x) \otimes b^i_{(\alpha,k)}(y). \tag{3.10}$$

It converges absolutely in $|y| < |x|$ allowing us to perform the following calculation

$$\begin{aligned} &\left(e^{u\,\mathrm{ad}(\mu)} \otimes e^{v\,\mathrm{ad}(\mu)}\right) r_0^\sigma\left(e^{u/|\sigma|}, e^{v/|\sigma|}\right) \\ &\qquad = \frac{C_{\mathfrak{h}}}{2} + \sum_{\substack{(\alpha,k)\in\Phi_\sigma^+ \\ 1\leq i\leq n_{(\alpha,k)}}} e^{u\,\mathrm{ad}(\mu)} b^i_{(-\alpha,-k)}\left(e^{u/|\sigma|}\right) \otimes e^{v\,\mathrm{ad}(\mu)} b^i_{(\alpha,k)}\left(e^{v/|\sigma|}\right) \\ &\qquad = \frac{C_{\mathfrak{h}}}{2} + \sum_{\substack{(\alpha,k)\in\Phi_\sigma^+ \\ 1\leq i\leq n_{(\alpha,k)}}} \left(G_s^{s'} b^i_{(-\alpha,-k)}\right)\left(e^{u/|\sigma'|}\right) \otimes \left(G_s^{s'} b^i_{(\alpha,k)}\right)\left(e^{v/|\sigma'|}\right) \\ &\qquad = r_0^{\sigma'}\left(e^{u/|\sigma'|}, e^{v/|\sigma'|}\right), \end{aligned}$$

for $|e^{v/|\sigma|}| < |e^{u/|\sigma|}|$ or, equivalently, $|e^{v/|\sigma'|}| < |e^{u/|\sigma'|}|$. Equality (3.5) now follows by the identity theorem for holomorphic functions of several variables (see [17]). ∎

Having this result at hand we can obtain the desired pseudoquasitriangularity for twisted standard structures δ_t^σ. Let us call a meromorphic function $r : \mathbb{C}^2 \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ *skew-symmetric* if $r(x, y) + \tau(r(y, x)) = 0$.

Theorem 3.3. *Let $\sigma \in \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ be a finite order automorphism and $t \in \mathfrak{L}^\sigma \otimes \mathfrak{L}^\sigma$. Then $r_t^\sigma \coloneqq r_0^\sigma + t$ is a skew-symmetric solution of the CYBE if and only if t is a classical twist of δ_0^σ. Moreover, if t is a classical twist of δ_0^σ, then the following relation holds:* [4]

$$\delta_t^\sigma(f)(x, y) = [f(x) \otimes 1 + 1 \otimes f(y), r_t^\sigma(x, y)] \qquad \forall f \in \mathfrak{L}^\sigma,\ \forall x, y \in \mathbb{C}^*. \tag{3.11}$$

[4] We define $(f \otimes g)(x, y) \coloneqq f(x) \otimes g(y)$ for any $x, y \in \mathbb{C}^*$ and $f, g \in \mathfrak{L}^\sigma$.

Proof. First, assume that $\sigma = \sigma_{(\mathbf{1};|\nu|)}$ and $t = 0$. In this case [3, Proposition 6.1] implies that r_0^σ is a skew-symmetric solution of the CYBE and (3.11) follows immediately from comparing [3, Equations (6.4) and (6.5)] with the projections of the defining relations (3.1) to $\mathfrak{L}^\sigma$. Secondly, applying Lemma 3.2 we get the statement for an arbitrary finite order automorphism σ and $t = 0$. Finally, since r_0^σ is skew-symmetric, the skew-symmetry of t is equivalent to the skew-symmetry of r_t^σ and a straightforward computation gives the equality

$$\mathrm{CYB}(r_t^\sigma) = \mathrm{CYB}(r_0^\sigma) + \mathrm{CYB}(t) - \mathrm{Alt}((\delta_0^\sigma \otimes 1)t) = \mathrm{CYB}(t) - \mathrm{Alt}((\delta_0^\sigma \otimes 1)t),$$

which completes the proof. ■

We finish this subsection by relating r-matrices of the form r_t^σ to trigonometric r-matrices in the sense of the Belavin-Drinfeld classification [3].

Theorem 3.4. *Let t be a classical twist of the standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$ and $r_t^\sigma = r_0^\sigma + t$ be the corresponding r-matrix. Then there exists a holomorphic function $\varphi\colon \mathbb{C} \longrightarrow \mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ and a trigonometric r-matrix $X\colon \mathbb{C} \longrightarrow \mathfrak{g}\otimes\mathfrak{g}$ such that*

$$X(u-v) = (\varphi(u)^{-1} \otimes \varphi(v)^{-1}) r_t^\sigma(e^{u/|\sigma|}, e^{v/|\sigma|}). \tag{3.12}$$

Proof. Let

$$r(x,y) := r_t^\sigma(x,y) = \frac{1}{(x/y)^{|\sigma|}-1}\widetilde{C}\,(x/y) + g(x,y), \tag{3.13}$$

where $\widetilde{C}(z) := \sum_{k=0}^{|\sigma|-1} z^k C_k^\sigma$. Following the arguments in [4] and [26, Theorem 11.3] we rewrite the CYBE for r in the form

$$[r^{12}(x,y), r^{13}(x,z)] + [r^{12}(x,y) + r^{13}(x,z), g^{23}(y,z)] \\ + \frac{1}{(y/z)^{|\sigma|}-1}[r^{12}(x,y) + r^{13}(x,z), \widetilde{C}^{23}(y/z)] = 0.$$

Calculating the limit $y \to z$ using L'Hospital's rule we obtain

$$[r^{12}(x,z), r^{13}(x,z)] + [r^{12}(x,z) + r^{13}(x,z), g^{23}(z,z) + |\sigma|^{-1}(\widetilde{C}'(1))^{23}] \\ + \frac{z}{|\sigma|}[\partial_z r^{12}(x,z), \widetilde{C}^{23}(y/z)] = 0.$$

Applying the function $1 \otimes L\colon \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \longrightarrow \mathfrak{g}\otimes\mathfrak{g}$, where $L(a\otimes b) := [a,b]$, we get the equality

$$[r(x,z), r(x,z)] + [r(x,z), 1\otimes f(z)] + \frac{z}{|\sigma|}\partial_z r(x,z) = 0, \tag{3.14}$$

where $f(z) := L(g(z,z) + |\sigma|^{-1}\widetilde{C}'(1))$ and $[a\otimes b, c\otimes d] := [a,c]\otimes[b,d]$. Similarly, letting $x \to y$ in the CYBE for r and then applying $L \otimes 1$ we obtain the identity

$$[r(y,z), r(y,z)] - [r(y,z), f(y)\otimes 1] - \frac{y}{|\sigma|}\partial_y r(y,z) = 0. \tag{3.15}$$

Subtracting (3.14) from (3.15) and setting $x = y = e^{u/|\sigma|}$ and $z = e^{v/|\sigma|}$ we get

$$\partial_u r(e^{u/|\sigma|}, e^{v/|\sigma|}) + \partial_v r(e^{u/|\sigma|}, e^{v/|\sigma|}) = [h(u)\otimes 1 + 1\otimes h(v), r(e^{u/|\sigma|}, e^{v/|\sigma|})], \tag{3.16}$$

for $h(u) := f(e^{u/|\sigma|})$. Since h is holomorphic on $\mathbb{C}$, we can find a holomorphic function $\varphi\colon \mathbb{C} \longrightarrow \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ such that $\varphi'(z) = \mathrm{ad}(h(z))\varphi(z)$ and $\varphi(0) = \mathrm{id}_{\mathfrak{g}}$ (see [26, Proof of Theorem 11.3]). The connected component of $\mathrm{id}_{\mathfrak{g}}$ in the group $\mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ is exactly the inner automorphisms of $\mathfrak{g}$ and thus $\varphi\colon \mathbb{C} \longrightarrow \mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$. Finally, the relation (3.16) implies that the r-matrix

$$\widetilde{X}(u,v) := (\varphi(u)^{-1} \otimes \varphi(v)^{-1}) r(e^{u/|\sigma|}, e^{v/|\sigma|}) \tag{3.17}$$

satisfies the equation $\partial_u \widetilde{X}(u,v) + \partial_v \widetilde{X}(u,v) = 0$. Therefore, we can define $X(u-v) := \widetilde{X}(u-v, 0) = \widetilde{X}(u,v)$. The set of poles of X is $2\pi i\mathbb{Z}$ and hence it is a trigonometric r-matrix. ■

From now on r-matrices of the form $r_t^\sigma = r_0^\sigma + t$, where σ is a finite order automorphism of $\mathfrak{g}$ and t is a classical twists of δ_0^σ, are called *σ-trigonometric*.

3.2 Manin triple structure

The standard Lie bialgebra stucture on an affine Kac-Moody algebra (3.1) can be defined using the standard Manin triple (see [13, Example 3.2] and [10, Example 1.3.8]). Restricting that triple to $\mathfrak{L}^\sigma$ we get a Manin triple defining the standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$. More precisely, δ_0^σ is defined by the Manin triple

$$(\mathfrak{L}^\sigma \times \mathfrak{L}^\sigma, \Delta, W_0)\,, \tag{3.18}$$

where Δ is the image of the diagonal embedding of $\mathfrak{L}^\sigma$ into $\mathfrak{L}^\sigma \times \mathfrak{L}^\sigma$ and W_0 is defined by

$$W_0 := \left\{(f,g) \in \mathfrak{B}_+^\sigma \times \mathfrak{B}_-^\sigma \mid f + g \in \mathfrak{N}_+^\sigma \dotplus \mathfrak{N}_-^\sigma\right\}. \tag{3.19}$$

The form $\mathcal{B}$ on $\mathfrak{L}^\sigma \times \mathfrak{L}^\sigma$ is given by

$$\mathcal{B}\left((f_1, f_2), (g_1, g_2)\right) := B(f_1, g_1) - B(f_2, g_2) \qquad \forall f_1, f_2, g_1, g_2 \in \mathfrak{L}^\sigma, \tag{3.20}$$

where B is the form (2.18). From Theorem 2.4 we know that classical twists t of δ_0^σ are in one-to-one correspondence with Lagrangian subalgebras $W_t \subseteq \mathfrak{L}^\sigma \times \mathfrak{L}^\sigma$ complementary to Δ and commensurable with W_0. We now describe the construction of such subalgebras using σ-trigonometric r-matrices.

Let $\psi\colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathrm{End}_{\mathbb{C}-\mathrm{Vect}}(\mathfrak{g})$ and $\Psi\colon \mathfrak{L}^\sigma \otimes \mathfrak{L}^\sigma \longrightarrow \mathrm{End}_{\mathbb{C}-\mathrm{Vect}}(\mathfrak{L}^\sigma)$ be the natural maps given by $a \otimes b \longmapsto \kappa(b, -)a$ and $a \otimes b \longmapsto B(b, -)a$ respectively. Then we have the following useful identity

$$\mathrm{res}_{y=0}\left[\frac{1}{y}\psi(P(z,y))(f(y))\right] = \Psi(P)(f)(z) \qquad \forall P \in \mathfrak{L}^\sigma \otimes \mathfrak{L}^\sigma,\ \forall f \in \mathfrak{L}^\sigma,\ \forall z \in \mathbb{C}^*. \tag{3.21}$$

Theorem 3.5. *Let t be a classical twist of the standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$ and $r_t = r_0^\sigma + t$ be the corresponding σ-trigonometric r-matrix. Denote by $\pi_{\mathfrak{h}}$ and $\pi_\pm$ the projections of $\mathfrak{L}^\sigma$ onto $\mathfrak{h}$ and $\mathfrak{N}_\pm^\sigma$ respectively. Then the linear map $R_t := \pi_{\mathfrak{h}}/2 + \pi_- + \Psi(t)$ satisfies the relation*

$$\mathrm{res}_{y=0}\left[\frac{1}{y}\psi(r_t(z,y))(f(y))\right] = R_t(f)(z) \qquad \forall f \in \mathfrak{L}^\sigma,\ \forall z \in \mathbb{C}^*, \tag{3.22}$$

and the Lagrangian subalgebra W_t, corresponding to t, can be described in the following way

$$W_t = \left\{\left((R_t - 1)\,f, R_t f\right) \mid f \in \mathfrak{L}^\sigma\right\}. \tag{3.23}$$

Proof. We prove the theorem for $\sigma = \sigma_{(s;|\nu|)}$ and $t = 0$. The general result then follows by linearity and equation (3.21). Writing $r_0^\sigma(z,y)$ as series (3.10) and applying ψ we get

$$\psi(r_0^\sigma(z,y))(f(y)) = \frac{\psi(C_\mathfrak{h})(f(y))}{2} + \sum_{\substack{(\alpha,k)\in\Phi_\sigma^+ \\ 1\le i\le n_{(\alpha,k)}}} \psi\left(b^i_{(-\alpha,-k)}(z)\otimes b^i_{(\alpha,k)}(y)\right)\left(f(y)\right).$$

The absolute convergence of the series in the annulus $\epsilon < |y| < |z|$ for any $\epsilon \in \mathbb{R}_+$ allows the componentwise calculation of the residue, i.e.

$$\operatorname{res}_{y=0}\left[\frac{1}{y}\psi(r_t(z,y))(f(y))\right] = \frac{\pi_\mathfrak{h}(f(y))}{2} + \sum_{(\alpha,k)\in\Phi_\sigma^+} \pi_{(-\alpha,-k)}(f(y)) = R_0(f)(z),$$

where $\pi_{(\alpha,k)}$ is the projection of $\mathfrak{L}^\sigma$ onto $\mathfrak{L}^\sigma_{(\alpha,k)}$.

For the second statement let us take an arbitrary $(w_1,w_2) \in W_0$. The relation $\pi_\mathfrak{h}(w_1) = -\pi_\mathfrak{h}(w_2)$ implies $(w_1,w_2) = ((R_0-1)(w_2-w_1), R_0(w_2-w_1))$. The desired result now follows from the fact that $(w_1,w_2) \longmapsto w_2 - w_1$ is an isomorphism between W_0 and $\mathfrak{L}^\sigma$. ■

Remark 3.6. For later sections it is convenient to define another, more geometric, Manin triple defining the standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$. Define $m := |\sigma|$, $O^\sigma := \mathbb{C}[z^m, z^{-m}]$ and $\widehat{O}^\sigma_\pm := \mathbb{C}((z^{\pm m}))$.[5] The Lie algebra $\mathfrak{L}^\sigma$ is naturally an O^σ-module and hence we can extend it to $\widehat{\mathfrak{L}}^\sigma_\pm = \mathfrak{L}^\sigma \otimes_{O^\sigma} \widehat{O}^\sigma_\pm$. Equip the product Lie algebra $\widehat{\mathfrak{L}}^\sigma_+ \times \widehat{\mathfrak{L}}^\sigma_-$ with the following bilinear form

$$\mathcal{B}((f_1,f_2),(g_1,g_2)) := \operatorname{res}_{z=0}\left[\frac{1}{z}\kappa(f_1,g_1)\right] - \operatorname{res}_{z=0}\left[\frac{1}{z}\kappa(f_2,g_2)\right], \tag{3.24}$$

where $\kappa(\sum_i a_i z^i, \sum_j b_j z^j) := \sum_{i,j}\kappa(a_i,b_j)z^{i+j}$ and $\operatorname{res}_{z=0}$ reads off the coefficient of z^{-1}. The restriction of this form to $\mathfrak{L}^\sigma \times \mathfrak{L}^\sigma$ is the form (3.20) defined earlier. Consider the subset

$$\widehat{W}_0 = \left\{(f,g) \in \widehat{\mathfrak{B}}^\sigma_+ \times \widehat{\mathfrak{B}}^\sigma_- \mid \widehat{\pi}^+_\mathfrak{h}(f) = -\widehat{\pi}^-_\mathfrak{h}(g)\right\} \subseteq \widehat{\mathfrak{L}}^\sigma_+ \times \widehat{\mathfrak{L}}^\sigma_-, \tag{3.25}$$

where $\widehat{\mathfrak{N}}^\sigma_\pm$ stands for the completion of $\mathfrak{N}^\sigma_\pm$ with respect to the ideal $(z^{\pm m}) \subseteq \mathbb{C}[z^{\pm m}]$, $\widehat{\mathfrak{B}}^\sigma_\pm := \mathfrak{h} \dotplus \widehat{\mathfrak{N}}^\sigma_\pm$ and $\widehat{\pi}^\pm_\mathfrak{h} \colon \widehat{\mathfrak{L}}^\sigma_\pm \longrightarrow \mathfrak{h}$ are the canonical projections . Then $\widehat{W}_0$ is a Lagrangian subalgebra complementary to the diagonal embedding Δ of $\mathfrak{L}^\sigma$ into $\widehat{\mathfrak{L}}^\sigma_+ \times \widehat{\mathfrak{L}}^\sigma_-$. Since the Lie bracket on W_0 is the restriction of the Lie bracket on $\widehat{W}_0$, the Manin triple

$$(\widehat{\mathfrak{L}}^\sigma_+ \times \widehat{\mathfrak{L}}^\sigma_-, \Delta, \widehat{W}_0) \tag{3.26}$$

also defines the standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$.

The geometric nature of this Manin triple is revealed in [1]: the sheaves used for construction of σ-trigonometric r-matrices can be viewed as formal gluing of twisted versions of $\widehat{W}_0$ with $\mathfrak{L}^\sigma \cong \Delta$ over the nodal Weierstraß cubic. ♢

[5] The notation $\mathbb{C}((u))$ is used to denote the ring of Laurent series of the form $\sum_{k=N}^\infty a_k u^k$, where $a_k \in \mathbb{C}$ and $N \in \mathbb{Z}$.

3.3 Regular equivalence

Let us fix a finite order automorphism σ of $\mathfrak{g}$. We now turn to defining the notion of equivalence for twisted standard bialgebra structures on $\mathfrak{L}^\sigma$ which is compatible with the corresponding pseudoquasitriangular and Manin triple structures. In other words, we want equivalences of Lie bialgebras to induce equivalences of the corresponding Manin triples and trigonometric r-matrices and vice versa. We stress that the notion of holomorphic equivalence used in the Belavin-Drinfeld classification [3] is unsuitable for our purpose, because in general it does not provide isomorphisms of loop algebras.

In the spirit of [33, 34] we define a *regular equivalence* on the loop algebra $\mathfrak{L}^\sigma$ to be a regular function $\phi\colon \mathbb{C}^* \longrightarrow \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ preserving the quasi-periodicity of $\mathfrak{L}^\sigma$, i.e.

$$\phi(\varepsilon_\sigma z) = \sigma\phi(z)\sigma^{-1}, \tag{3.27}$$

where $\varepsilon_\sigma = e^{2\pi i/|\sigma|}$. Recalling that $O^\sigma = \mathbb{C}[z^{|\sigma|}, z^{-|\sigma|}]$, we can equivalently define a regular equivalence on $\mathfrak{L}^\sigma$ to be an element of $\mathrm{Aut}_{O^\sigma-\mathrm{LieAlg}}(\mathfrak{L}^\sigma)$. The equivalence between these two definitions is given by $\phi(f)(z) \coloneqq \phi(z)f(z)$

By definition (2.17) the space $\mathfrak{L}^\sigma \otimes \mathfrak{L}^\sigma$ can be viewed as the space of regular functions $T\colon \mathbb{C}^* \times \mathbb{C}^* \longrightarrow \mathfrak{g}\otimes\mathfrak{g}$ such that $(1\otimes\sigma)T(x,y) = T(x, \varepsilon_\sigma y)$ and $(\sigma\otimes 1)T(x,y) = T(\varepsilon_\sigma x, y)$. It is straightforward to check that if such a function T vanishes along the diagonal, i.e. $T(z,z) = 0$ for all $z \in \mathbb{C}^*$, then it is divisible by $(x/y)^{|\sigma|} - 1$. Applying this observation to the function

$$(\phi(x)\otimes\phi(y))\sum_{k=0}^{|\sigma|-1}\left(\frac{x}{y}\right)^k C_k^\sigma - \sum_{k=0}^{|\sigma|-1}\left(\frac{x}{y}\right)^k C_k^\sigma, \tag{3.28}$$

where $\phi \in \mathrm{Aut}_{O^\sigma-\mathrm{LieAlg}}(\mathfrak{L}^\sigma)$, we see that $(\phi(x)\otimes\phi(y))r_t^\sigma(x,y) = r_0^\sigma(x,y) + s(x,y)$ for some classical twist s, i.e. it is again a σ-trigonometric r-matrix.

The following theorem demonstrates that the notion of a regular equivalence meets all our needs.

Theorem 3.7. *Let ϕ be a regular equivalence on $\mathfrak{L}^\sigma$ and $s, t \in \mathfrak{L}^\sigma\otimes\mathfrak{L}^\sigma$ be two classical twists of the standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$. The following are equivalent:*

1. $r_t^\sigma(x,y) = (\phi(x)\otimes\phi(y))r_s^\sigma(x,y)$ *for all* $x, y \in \mathbb{C}^*$, $x^{|\sigma|} \neq y^{|\sigma|}$;
2. $\delta_t^\sigma\phi = (\phi\otimes\phi)\delta_s^\sigma$;
3. $W_t = (\phi\times\phi)W_s$.

Proof. *"1. $\Longrightarrow$ 3.":* If $r_t(x,y) = (\phi(x)\otimes\phi(y))r_s(x,y)$ for all $x, y \in \mathbb{C}^*$, $x^{|\sigma|} \neq y^{|\sigma|}$, then (3.22) implies $R_t = \phi R_s \phi^*$. Since the adjoint of $\phi(z)$ with respect to the Killing form is $\phi(z)^{-1}$, we have $\phi^* = \phi^{-1}$. The formula (3.23) applied to both W_t and W_s gives

$$(\phi\times\phi)W_s = \left\{\left(\phi(R_s - 1)\phi^{-1}(\phi f), \phi R_s\phi^{-1}(\phi f)\right) \mid f \in \mathfrak{L}^\sigma\right\} = W_t. \tag{3.29}$$

"3. $\Longrightarrow$ 2.": Assuming $W_t = (\phi\times\phi)W_s$, we can easily see that $\phi\times\phi$ is an isomorphism of Manin tiples $(\mathfrak{L}^\sigma\times\mathfrak{L}^\sigma, \Delta, W_t)$ and $(\mathfrak{L}^\sigma\times\mathfrak{L}^\sigma, \Delta, W_s)$. Identifying Δ with $\mathfrak{L}^\sigma$ and applying Remark 2.2 we immediately get the desired isomorphism $\phi\colon (\mathfrak{L}^\sigma, \delta_t) \longrightarrow (\mathfrak{L}^\sigma, \delta_s)$.

"2. $\Longrightarrow$ 1.": Since $\mathfrak{L}^\sigma$ has no non-trivial finite-dimensional ideals (see [23, Lemma 8.6]), the only element in $\mathfrak{L}^\sigma \otimes \mathfrak{L}^\sigma$ invariant under the adjoint action of $\mathfrak{L}^\sigma$ is 0. Applying this result to the equality

$$\begin{aligned}[\phi(f)(x) \otimes 1 + 1 \otimes \phi(f)(y), r_t(x,y)] &= \delta_t \phi(f)(x,y) = (\phi(x) \otimes \phi(y))\delta_s(f)(x,y) \\ &= (\phi(x) \otimes \phi(y)) \left[f(x) \otimes 1 + 1 \otimes f(y), r_s(x,y)\right] \\ &= \left[\phi(f)(x) \otimes 1 + 1 \otimes \phi(f)(y), (\phi(x) \otimes \phi(y)) r_s(x,y)\right],\end{aligned}$$

where $f \in \mathfrak{L}^\sigma$ and $x, y \in \mathbb{C}^*, x^{|\sigma|} \neq y^{|\sigma|}$, we get the last implication. ■

We say that two twisted standard bialgebra structures or σ-trigonometric r-matrices are *regularly equivalent* if one of the equivalent conditions in Theorem 3.7 holds.

4 The main classification theorem and its consequences

Before stating the main classification theorem we recall the notion of a Belavin-Drinfeld quadruple for an arbitrary finite order automorphism σ, defined in [3], and then associate it with a classical twist of the standard Lie bialgebra structure δ_0^σ.

We start with the case $\sigma = \sigma_{(s;|\nu|)}$. Let $\Pi^\sigma, \Lambda^\sigma$ and Φ_σ be as at the end of Section 2.2. A Belavin-Drinfeld (BD) quadruple is a quadruple $Q = (\Gamma_1, \Gamma_2, \gamma, t_\mathfrak{h})$, where Γ_1 and Γ_2 are proper subsets of the simple root system Π^σ, $\gamma \colon \Gamma_1 \longrightarrow \Gamma_2$ is a bijection and $t_\mathfrak{h} \in \mathfrak{h} \wedge \mathfrak{h}$ such that

1. $B\left(\alpha^\vee_{\gamma(i)}, \alpha^\vee_{\gamma(j)}\right) = B\left(\alpha^\vee_i, \alpha^\vee_j\right)$ for all $\widetilde{\alpha}_i, \widetilde{\alpha}_j \in \Gamma_1$, where $\widetilde{\alpha}_{\gamma(i)} \coloneqq \gamma(\widetilde{\alpha}_i)$;
2. For any $\widetilde{\alpha}_i \in \Gamma_1$ there is a positive integer k such that $\gamma^k(\widetilde{\alpha}_i) \notin \Gamma_1$;
3. $(\alpha_{\gamma(i)} \otimes 1 + 1 \otimes \alpha_i)(t_\mathfrak{h} + C_\mathfrak{h}/2) = 0$ for all $\widetilde{\alpha}_i \in \Gamma_1$.

The bijection γ induces an isomorphism $\theta_\gamma \colon \mathfrak{S}^{\Gamma_1} \longrightarrow \mathfrak{S}^{\Gamma_2}$, $\theta_\gamma(z^{\pm s_i} X_i^\pm(1)) \coloneqq z^{\pm s_{\gamma(i)}} X^\pm_{\gamma(i)}(1)$, which we extend by 0 to the whole $\mathfrak{L}^\sigma$. Let $\Phi_1 \subseteq \Phi_\sigma$ be the subset of roots that can be written as linear combinations of elements in Γ_1. For each $\widetilde{\alpha} \in \Phi_1$ we choose an element $b_{\widetilde{\alpha}} \in \mathfrak{L}^\sigma_{\widetilde{\alpha}}$ such that $B(b_{\widetilde{\alpha}}, b_{-\widetilde{\alpha}}) = 1$ and construct the following skew-symmetric tensor

$$t_Q^\sigma \coloneqq t_\mathfrak{h} + \sum_{\widetilde{\alpha} \in \Phi_1^+} \sum_{j=1}^\infty b_{-\widetilde{\alpha}} \wedge \theta_\gamma^j(b_{\widetilde{\alpha}}) \in \mathfrak{L}^\sigma \otimes \mathfrak{L}^\sigma, \tag{4.1}$$

where $\Phi_1^+ = \Phi_1 \cap \Phi_\sigma^+$ and the second sum has only finitely many non-zero terms since θ_γ is nilpotent by condition 2.

We write $r_Q^\sigma, \delta_Q^\sigma, R_Q$ and W_Q instead of $r^\sigma_{t_Q^\sigma}, \delta^\sigma_{t_Q^\sigma}, R_{t_Q^\sigma}$ and $W_{t_Q^\sigma}$ respectively. In the case $s = (1, \dots, 1)$ the functions r_Q^σ and R_Q as well as the Cayley transform of R_Q were studied in details in [3]. Using regrading and Lemma 3.2 we derive the following statements:

- r_t^σ is a skew-symmetric solution of CYBE. Hence Theorem 3.3 implies that t_Q^σ is a classical twist of δ_0^σ;
- The inhomogeneous system of linear equations constraining $t_\mathfrak{h}$ is consistent. The dimension of its solution space is $\ell(\ell-1)/2$, where $\ell = |\Pi^\sigma \setminus \Gamma_1|$;

- Setting $\theta^{\pm}_{\gamma^{\pm 1}} \coloneqq \theta_{\gamma^{\pm 1}}|_{\mathfrak{N}_{\pm}}$, we have
$$R_Q = \theta^{+}_{\gamma}(\theta^{+}_{\gamma} - \pi_+)^{-1} + (\psi(t_{\mathfrak{h}}) + \mathrm{id}_{\mathfrak{h}}/2) + (\pi_- - \theta^{-}_{\gamma^{-1}})^{-1}; \tag{4.2}$$
- Let $\mathfrak{h}_1 \coloneqq \mathrm{im}(\psi(t_{\mathfrak{h}}) - \mathrm{id}_{\mathfrak{h}}/2)$ and $\mathfrak{h}_2 \coloneqq \mathrm{im}(\psi(t_{\mathfrak{h}}) + \mathrm{id}_{\mathfrak{h}}/2)$. The Cayley transform of R_Q is the triple (C^1_Q, C^2_Q, θ_Q), where
$$\begin{aligned} C^1_Q &\coloneqq \mathrm{im}(R_Q - \mathrm{id}) = \mathfrak{N}_+ \dotplus \mathfrak{h}_1 \dotplus \mathfrak{N}^{\Gamma_1}_-, \\ C^2_Q &\coloneqq \mathrm{im}(R_Q) = \mathfrak{N}^{\Gamma_2}_+ \dotplus \mathfrak{h}_2 \dotplus \mathfrak{N}_-, \end{aligned} \tag{4.3}$$
and θ_Q is the unique gluing of θ_γ with the natural isomorphism
$$\begin{aligned} \phi\colon \frac{\mathrm{im}(\psi(t_{\mathfrak{h}}) - \mathrm{id}_{\mathfrak{h}}/2)}{\ker(\psi(t_{\mathfrak{h}}) + \mathrm{id}_{\mathfrak{h}}/2)} &\longrightarrow \frac{\mathrm{im}(\psi(t_{\mathfrak{h}}) + \mathrm{id}_{\mathfrak{h}}/2)}{\ker(\psi(t_{\mathfrak{h}}) - \mathrm{id}_{\mathfrak{h}}/2)}, \\ [(\psi(t_{\mathfrak{h}}) - \mathrm{id}_{\mathfrak{h}}/2)(h)] &\longmapsto [(\psi(t_{\mathfrak{h}}) + \mathrm{id}_{\mathfrak{h}}/2)(h)], \end{aligned} \tag{4.4}$$
which coincides with θ_γ on the intersection of the domains. The subalgebra W_Q is then given by
$$W_Q = \left\{(x,y) \in C^1_Q \times C^2_Q \mid \theta_Q([x]) = [y]\right\}. \tag{4.5}$$

Conjugating σ by $\rho \in \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ we extend all statements and constructions given above to an arbitrary finite order automorphism of $\mathfrak{g}$.

Theorem 4.1 (The main classification theorem). *For any classical twist t of the standard Lie bialgebra structure δ^σ_0 on $\mathfrak{L}^\sigma$ there is a regular equivalence ϕ of $\mathfrak{L}^\sigma$ and a BD quadruple $Q = (\Gamma_1, \Gamma_2, \gamma, t_{\mathfrak{h}})$ such that*
$$\delta^\sigma_t \phi = (\phi \otimes \phi)\delta^\sigma_Q. \tag{4.6}$$
Furthermore, if $Q' = (\Gamma'_1, \Gamma'_2, \gamma', t'_{\mathfrak{h}})$ is another BD quadruple, the twisted bialgebra structures δ^σ_Q and $\delta^\sigma_{Q'}$ are regularly equivalent if and only if there is an automorphism ϑ of the Dynkin diagram of $\mathfrak{L}^\sigma$ such that $\vartheta(\Gamma_i) = \Gamma'_i$ for $i = 1,2$, $\vartheta\gamma\vartheta^{-1} = \gamma'$ and $(\vartheta \otimes \vartheta)t_{\mathfrak{h}} = t'_{\mathfrak{h}}$, which we denote by $\vartheta(Q) = Q'$.

We put off the proof of the theorem to Section 5. The rest of this section is devoted to various consequences of Theorem 4.1 and to the proof of its first part in the special case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and $\sigma = \mathrm{id}$.

4.1 Classification of twists for parabolic subalgebras

To simplify the notation we again assume $\sigma = \sigma_{(s;|\nu|)}$. The following results can be stated for an arbitrary finite order automorphism by applying conjugation.

Let $S \subsetneq \Lambda^\sigma$ be a proper subset of standard generators of $\mathfrak{L}^\sigma$. It is easy to see that the standard Lie bialgebra structure δ^σ_0 restricts to both $\mathfrak{S}^S$ and $\mathfrak{p}^S_\pm$. Such induced Lie bialgebra structures can be defined using modifications of the Manin triple (3.18). For example, the Lie bialgebra structure $(\mathfrak{p}^S_+, \delta^\sigma_0|_{\mathfrak{p}^S_+})$ is defined by the Manin triple
$$\left((\mathfrak{S}^S + \mathfrak{h}) \times \mathfrak{L}^\sigma, \Delta^S, W^S_0\right), \tag{4.7}$$
where $W^S_0 = W_0 \cap ((\mathfrak{S}^S + \mathfrak{h}) \times \mathfrak{L}^\sigma)$ and $\Delta^S = \{(\pi_S(f), f) \mid f \in \mathfrak{p}^S_+\}$ for the canonical projection $\pi_S\colon \mathfrak{p}^S_+ \longrightarrow (\mathfrak{S}^S + \mathfrak{h}) = \mathfrak{p}^S_+/(\mathfrak{p}^S_+)^\perp$. The following theorem gives a classification of classical twists of the restricted Lie bialgebra structure $\delta^\sigma_0|_{\mathfrak{p}^S_+}$ or, equivalently, classical twists of δ^σ_0 contained in $\mathfrak{p}^S_+ \otimes \mathfrak{p}^S_+$.

Theorem 4.2 (The classification theorem for parabolic subalgebras). *For any classical twist $t \in \mathfrak{p}_+^S \otimes \mathfrak{p}_+^S$ of the standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$ there exists a regular equivalence ϕ that restricts to an automorphism of $\mathfrak{p}_+^S$ and a BD quadruple $Q = (\Gamma_1, \Gamma_2, \gamma, t_\mathfrak{h})$ such that*

$$\Gamma_1 \subseteq S \quad \text{and} \quad \delta_t^\sigma \phi = (\phi \otimes \phi)\delta_Q^\sigma. \tag{4.8}$$

Let $Q' = (\Gamma_1', \Gamma_2', \gamma', t_\mathfrak{h}')$, $\Gamma_1' \subseteq S$, be another BD quadruple. A regular equivalence between twisted standard structures δ_Q^σ and $\delta_{Q'}^\sigma$ restricts to an automorphism of $\mathfrak{p}_+^S$ if and only if the induced Dynkin diagram automorphism ϑ preserves S, i.e. $\vartheta(S) = S$.

Remark 4.3. For a BD quadruple $Q = (\Gamma_1, \Gamma_2, \gamma, t_\mathfrak{h})$ with $\Gamma_1 \subseteq S$, formula (4.1) directly implies that $t_Q \in \mathfrak{p}_+^S \otimes \mathfrak{p}_+^S$. In particular t_Q is a twist of $\delta_0^\sigma|_{\mathfrak{p}_+^S}$. $\Diamond$

The proof of Theorem 4.2 is based on the following three structural results for $\mathfrak{L}^\sigma$.

Lemma 4.4.

1. *A subalgebra $\mathfrak{a}$ of $\mathfrak{L}^\sigma$ containing a coisotropic subalgebra $\mathfrak{h}_1$ of $\mathfrak{h}$ satisfies $[\mathfrak{h}, \mathfrak{a}] \subseteq \mathfrak{a}$;*
2. *A subalgebra $\mathfrak{p}$ of $\mathfrak{L}^\sigma$ containing $\mathfrak{B}_\pm$ is of the form $\mathfrak{p}_\pm^{S'}$ for some $S' \subseteq \Pi^\sigma$;*
3. *A mapping $\phi \in \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{L}^\sigma)$ fixing $\mathfrak{B}_+$ or $\mathfrak{B}_-$ induces an automorphism of the Dynkin diagram of $\mathfrak{L}^\sigma$.*

Proof. 1.: We can write

$$\mathfrak{L}^\sigma = \bigoplus_{\alpha' \in \mathfrak{h}_1^\vee} \mathfrak{L}^\sigma_{\alpha'} = \bigoplus_{\alpha' \in \mathfrak{h}_1^\vee} \bigoplus_{\substack{\alpha \in \mathfrak{h}^\vee \\ \alpha|_{\mathfrak{h}_1} = \alpha'}} \mathfrak{L}^\sigma_\alpha, \tag{4.9}$$

where $\mathfrak{L}^\sigma_\alpha = \{f \in \mathfrak{L}^\sigma \mid [h,f] = \alpha(h)f \ \forall h \in \mathfrak{h}\} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{L}^\sigma_{(\alpha,k)}$ for any $\alpha \in \mathfrak{h}^\vee$ and similarly $\mathfrak{L}^\sigma_{\alpha'} = \{f \in \mathfrak{L}^\sigma \mid [h,f] = \alpha'(h)f \ \forall h \in \mathfrak{h}_1\}$ for any $\alpha' \in \mathfrak{h}_1^\vee$. Assume there are distinct $\alpha_1, \alpha_2 \in \mathfrak{h}^\vee$ such that $\mathfrak{L}^\sigma_{\alpha_1}, \mathfrak{L}^\sigma_{\alpha_2} \neq 0$ and $(\alpha_1 - \alpha_2)|_{\mathfrak{h}_1} = 0$. Since $\mathfrak{h}_1$ is coisotropic inside $\mathfrak{h}$, we have $(\alpha_1 - \alpha_2)^\vee \in \mathfrak{h}_1^\perp \subseteq \mathfrak{h}_1$. From [20, Lemma X.5.6] it follows that $\alpha_1 - \alpha_2 = 0$ which, in its turn, implies that for any $\alpha' \in \mathfrak{h}_1^\vee$ there exists a unique weight $\alpha \in \mathfrak{h}^\vee$ such that $\mathfrak{L}^\sigma_{\alpha'} = \mathfrak{L}^\sigma_\alpha$. This observation combined with $[\mathfrak{h}_1, \mathfrak{a}] \subseteq \mathfrak{a}$ allows us to write (see [23, Proposition 1.5])

$$\mathfrak{a} = \bigoplus_{\alpha' \in \mathfrak{h}_1^\vee} \mathfrak{L}^\sigma_{\alpha'} \cap \mathfrak{a} = \bigoplus_{\alpha \in \mathfrak{h}^\vee} \mathfrak{L}^\sigma_\alpha \cap \mathfrak{a}, \tag{4.10}$$

implying $[\mathfrak{h}, \mathfrak{a}] \subseteq \mathfrak{a}$.

2.: Without loss of generality assume that $\mathfrak{p}$ contains $\mathfrak{B}_+$. The inclusion $\mathfrak{h} \subseteq \mathfrak{p}$ and [23, Proposition 1.5] imply that

$$\mathfrak{p} = \bigoplus_{\alpha \in \mathfrak{h}^\vee} \mathfrak{L}^\sigma_\alpha \cap \mathfrak{p}. \tag{4.11}$$

Take $X \in \mathfrak{L}^\sigma_{-\alpha} \cap \mathfrak{p} \cap \mathfrak{N}_-$ for some $\alpha \neq 0$. Let j be the maximal non-negative integer such that the $\mathfrak{L}_{(-\alpha,-j)}$-component of X is non-zero. The structure theory of $\mathfrak{L}^\sigma$ implies $\dim(\mathfrak{L}_{(-\alpha,-j)}) = 1$. Assume that $(-\alpha, -j+k)$ is a root for some positive integer k. Decomposing the difference of $(-\alpha, -j)$ and $(-\alpha, -j+k)$ into the sum of simple roots we get a relation of the form $\sum_{i=0}^n c_i\widetilde{\alpha}_i = (0,k)$. Then the identity $\sum_{i=0}^n c_i\alpha_i = 0$ and (2.29) imply that k is an integer multiple of $\sum_{i=0}^n a_i s_i$. Using [23, Theorem 5.6.b)] we see that $(0,k)$ is a root. Applying [20, Lemma X.5.5'.(iii)] iteratively we see that

$$\bigoplus_{k\geq 0} \mathfrak{L}_{(-\alpha,-j+k)} \subseteq \mathfrak{p}. \tag{4.12}$$

Following the proof of [24, Lemma 1.5] we now show that $\mathfrak{p} = \mathfrak{p}_+^{S'}$, where

$$S' = \{(\alpha_i, s_i) \in \Pi^\sigma \mid \mathfrak{L}_{(-\alpha_i,-s_i)} \subseteq \mathfrak{p}\}. \tag{4.13}$$

Assume the claim is false. Let $(-\gamma,-\ell) \notin \operatorname{span}_{\mathbb{Z}}(S')$ be a negative root of maximal height such that there exists an element $Y \in \mathfrak{L}^\sigma_{-\gamma} \cap \mathfrak{p} \cap \mathfrak{N}_-$ with a non-zero $\mathfrak{L}_{(-\gamma,-\ell)}$-component $Y_{-\ell}$. Then there exists $(\alpha_j, s_j) \in \Pi^\sigma \setminus S'$ such that $[X_j^+, Y_{-\ell}] \neq 0$ and $(-\gamma+\alpha_j, -\ell+s_j) \in \operatorname{span}_{\mathbb{Z}}(S')$, where X_j^+ is the standard generator of $\mathfrak{L}^\sigma$. Note that equation (4.12) implies $\gamma \neq \alpha_j$. By the structure theory of loop algebras we can find $Z \in \mathfrak{L}^\sigma_{(\gamma-\alpha_j,\ell-s_j)} \subseteq \mathfrak{B}_+ \subseteq \mathfrak{p}$ such that

$$B([X_j^+, Y_{-\ell}], Z) \neq 0. \tag{4.14}$$

The invariance of the form B then gives $0 \neq [Y_{-\ell}, Z] \in \mathfrak{L}^\sigma_{(-\alpha_j,-s_j)}$. Applying formula (4.12) to $X = [Y,Z] \in \mathfrak{p}$ we get $(\alpha_j, s_j) \in S'$ contradicting our choice of (α_j, s_j).

3.: Assume that $\phi(\mathfrak{B}_+) = \mathfrak{B}_+$. Since $\mathfrak{N}_+ = [\mathfrak{B}_+, \mathfrak{B}_+]$, we see that ϕ also fixes $\mathfrak{N}_+$ and $\mathfrak{h}$. By [24, Lemma 1.29] the automorphism ϕ maps Π^σ to a root basis. But since ϕ fixes $\mathfrak{N}_+$, this root basis cosists of positive roots and the only root basis in the set of positive roots is Π^σ. Hence $\phi(\Pi^\sigma) = \Pi^\sigma$ and thus ϕ induces an automorphism ϑ of the Dynkin diagram of $\mathfrak{L}^\sigma$. ■

Proof of Theorem 4.2. We prove the statement for $\sigma = \sigma_{(s;|\nu|)}$. The general result it obtained using conjugation. By Theorem 4.1 there is a regular equivalence ϕ_1 on $\mathfrak{L}^\sigma$ and a BD quadruple $Q' = (\Gamma_1', \Gamma_2', \gamma', t_{\mathfrak{h}}')$ such that $(\phi_1 \times \phi_1)W_t = W_{Q'}$. Since $t \in \mathfrak{p}_+^S \otimes \mathfrak{p}_+^S$ we have $W_t \subseteq \mathfrak{p}_+^S \times \mathfrak{L}^\sigma$. Let $\mathfrak{h}_1 \subseteq \mathfrak{h}$ be the image of $\psi(t_{\mathfrak{h}}) - \mathrm{id}_{\mathfrak{h}}/2$. Since $t_{\mathfrak{h}}$ is skew-symmetric, this is easily seen to be a coisotropic subspace of $\mathfrak{h}$. Then

$$C_Q^1 = \mathfrak{N}_+ \dotplus \mathfrak{h}_1 \dotplus \mathfrak{N}_-^{\Gamma_1} \subseteq \phi_1(\mathfrak{p}_+^S) \tag{4.15}$$

and, in particular, we have the inclusion $\mathfrak{h}_1 \subseteq \phi_1(\mathfrak{p}_+^S)$. By the first part of Lemma 4.4 we have

$$[\mathfrak{h}, \phi_1(\mathfrak{p}_+^S)] \subseteq \phi_1(\mathfrak{p}_+^S). \tag{4.16}$$

Since $\mathfrak{p}_+^S$ is self-normalizing, $\phi_1(\mathfrak{p}_+^S)$ is self-normalizing as well. Therefore we get $\mathfrak{h} \subseteq \phi_1(\mathfrak{p}_+^S)$ and consequently $\mathfrak{B}_+ \subseteq \phi_1(\mathfrak{p}_+^S)$. Then the second statement of Lemma 4.4 shows that $\phi_1(\mathfrak{p}_+^S) = \mathfrak{p}_+^{S'}$ for some $S' \subsetneq \Pi^\sigma$. The inclusion 4.15 implies that $\Gamma_1 \subseteq S'$.

Define $\mathfrak{B}' := \phi_1^{-1}(\mathfrak{B}_+)$. The subalgebra $\mathfrak{B}'/\mathfrak{p}_+^{S,\perp}$, being the preimage of the Borel subalgebra $\mathfrak{B}_+/\mathfrak{p}_+^{S',\perp}$ of $\mathfrak{S}^{S'} + \mathfrak{h} = \mathfrak{p}_+^{S'}/\mathfrak{p}_+^{S',\perp}$ under ϕ_1, is a Borel subalgebra of $\mathfrak{S}^S + \mathfrak{h} = \mathfrak{p}_+^S/\mathfrak{p}_+^{S,\perp}$. Therefore, by the conjugacy theorem for Borel subalgebras, there exists an inner automorphism ϕ_2 of $\mathfrak{S}^S + \mathfrak{h}$ mapping $\mathfrak{B}'/\mathfrak{p}_+^{S,\perp}$ to $\mathfrak{B}_+/\mathfrak{p}_+^{S,\perp}$. It can be seen from

[20, Lemma X.5.5] that ad_x is nilpotent on $\mathfrak{L}^\sigma$ for any $x \in \mathfrak{L}^\sigma_{(\alpha,k)}$ and $\alpha \neq 0$. Combining this result with the equality

$$\mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{S}^S + \mathfrak{h}) = \langle e^{\mathrm{ad}_x} \mid x \in \mathfrak{L}^\sigma_{(\alpha,k)}, (\alpha, k) \in \Phi_\sigma \cap \mathrm{span}_{\mathbb{Z}}(S), \alpha \neq 0\rangle$$

(see [6, §3.2]), we can view ϕ_2 as a regular equivalence on $\mathfrak{L}^\sigma$ that restricts to an automorphism of $\mathfrak{p}^S_+$ and maps $\mathfrak{B}'$ to $\mathfrak{B}_+$. The composition $\phi_2\phi_1^{-1}$ is then an automorphism of $\mathfrak{L}^\sigma$ mapping $\mathfrak{p}^{S'}_+$ to $\mathfrak{p}^S_+$ and fixing the Borel subalgebra $\mathfrak{B}_+$. The third part of Lemma 4.4 implies that $\phi_2\phi_1^{-1}$ induces an automorphism ϑ of the Dynkin diagram of $\mathfrak{L}^\sigma$ such that $\vartheta(S') = S$. Applying the second part of Theorem 4.1 to ϑ we obtain a regular equivalence ϕ_3 such that $(\phi_3 \times \phi_3)W_{Q'} = W_{Q:=\vartheta(Q')}$. The composition $\phi := \phi_3\phi_1$ and the quadruple Q satisfy all the requirements of the theorem. ∎

4.2 Quasi-trigonometric solutions of CYBE

Letting $\sigma = \mathrm{id}$ and $S = \Pi \setminus \{(\alpha_0, 1)\}$ the corresponding parabolic subalgebra $\mathfrak{p}^S_+$ becomes $\mathfrak{g}[z]$. The solutions to CYBE of the form $r_t = r_0 + t$, where $t \in \mathfrak{g}[z]^{\otimes 2}$, are called *quasi-trigonometric*. Two quasi-trigonometric solutions r_t and r_s are called *polynomially equivalent* if there exists a $\phi \in \mathrm{Aut}_{\mathbb{C}[z]-\mathrm{LieAlg}}(\mathfrak{g}[z])$ such that

$$r_s(x, y) = (\phi(x) \otimes \phi(y))r_t(x, y) \qquad \forall x, y \in \mathbb{C}^*,\ x \neq y. \tag{4.17}$$

Therefore, a polynomial equivalence is a regular equivalence that restricts to an automorphism of $\mathfrak{g}[z]$. Quasi-trigonometric r-matrices were introduced and classified up to polynomial equivalence and choice of a maximal order in [26, 32]. More precisely, it was shown that quasi-trigonometric solutions are in one-to-one correspondence with certain Lagrangian subalgebras of $\mathfrak{g} \times \mathfrak{g}((z^{-1}))$. Embedding the Lagrangian subalgebra, corresponding to a quasi-trigonometric solution r, into some maximal order of $\mathfrak{g} \times \mathfrak{g}((z^{-1}))$, the authors of [26, 32] obtained a unique quasi-trigonometric solution r_Q (given by a BD quadruple Q) polynomially equivalent to r. In this setting we get the following results:

- The classification theorem for parabolic subalgebras 4.2 together with Theorem 3.3 gives a new proof of the above-mentioned classification of quasi-trigonometric r-matrices;
- In general, a maximal order in which one can embed the Lagrangian subalgebra correspoding to a quasi-trigonometric solution r is not unique. Choosing two different maximal orders we get two different BD quadruples Q and Q' and two polynomially equivalent quasi-trigonometric r-matrices r_Q and $r_{Q'}$. By Theorem 4.2 this equivalence induces an automorphism ϑ of the Dynking diagram of $\mathfrak{L}$ that fixes the minimal root, i.e. $\vartheta(\widetilde{\alpha}_0) = \widetilde{\alpha}_0$. Therefore, any quasi-trigonometric solution is polynomially equivalent to exactly one quasi-trigonometric r-matrix r_Q, for some BD quadruple Q, if and only if $\mathfrak{g}$ is of type A_1, B_n, C_n, F_4, G_2 or E_8.

  We note that there exist regularly equivalent quasi-trigonometric solutions r_Q and $r_{Q'}$ which are not polynomially equivalent (see Figure 1). Therefore regular equivalence is strictly weaker than polynomial one;
- It was shown in [26] that for any quasi-trigonometric r-matrix r there exists a holomorphic function $\phi\colon \mathbb{C} \longrightarrow \mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ such that

  $$(\phi(x)^{-1} \otimes \phi(y)^{-1})r(x, y) = X(x/y),$$

where X is a trigonometric solution in the Belavin-Drinfeld classification [3]. Combining Lemma 3.2 and Theorem 4.2 we get a general version of this statement with more control over the holomorphic equivalence. Precisely, the trigonometric r-matrix $r_Q^{\sigma_{(\mathbf{1};|\nu|)}}$, by definition, always depends on the quotient of its parameters; in order to obtain it from a σ-trigonometric r-matrix r_t^σ, where the coset of σ is conjugate to $\nu\mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$, it is enough to apply a regular equivalence composed with the regrading to the principal grading:

$$r_t^\sigma(x,y) \xmapsto{\text{regular eq.}} r_Q^\sigma(x,y) \xmapsto{\text{regrading}} r_Q^{\sigma_{(\mathbf{1};|\nu|)}}(x,y) = X(x/y);$$

- Conjecture 1 in [8] is justified: Combining (3.3) with (4.1) we get the explicit formula for a quasi-trigonmetric solution r_Q given by a BD quadruple Q, namely

$$\begin{aligned} r_Q(x,y) &= \frac{yC}{x-y} + \frac{C_{\mathfrak{h}}}{2} + C_- + t_{\mathfrak{h}} + \sum_{\widetilde{\alpha}\in\Phi_1^+}\sum_{j=1}^{\infty} b_{-\widetilde{\alpha}} \wedge \theta_\gamma^j(b_{\widetilde{\alpha}}) \\ &= -\frac{1}{2}\left(\frac{y+x}{y-x}C + \sum_{\widetilde{\alpha}\in\Phi^+} b_{\widetilde{\alpha}} \wedge b_{-\widetilde{\alpha}} - t_{\mathfrak{h}} + \sum_{\widetilde{\alpha}\in\Phi_1^+}\sum_{j=1}^{\infty}\theta_\gamma^j(b_{\widetilde{\alpha}}) \wedge b_{-\widetilde{\alpha}}\right). \end{aligned} \tag{4.18}$$

  This formula (up to a sign) coincides with the one conjectured by Burban, Galinat and Stolin in [8];

- Question 2 in [8] is answered: Let Q be a BD quadruple and $h := |\sigma_{(\mathbf{1};1)}|$. The relation between the quasi-trigonmetric solution (4.18) and the trigonometric solution

$$\begin{aligned} X(u-v) = t_{\mathfrak{h}} + \frac{C_{\mathfrak{h}}}{2} + \frac{1}{e^{u-v}-1}\sum_{k=0}^{h-1} e^{\frac{k(u-v)}{h}} C_k^{\sigma_{(\mathbf{1};1)}} \\ + \sum_{\substack{\widetilde{\alpha}\in\Phi_1^+\\ \widetilde{\alpha}=(\alpha,k)}}\sum_{j=1}^{\infty} e^{\frac{k(u-v)}{h}}\theta_\gamma^j(b_{\widetilde{\alpha}})(1)\otimes b_{-\widetilde{\alpha}}(1) \\ - \sum_{\substack{\widetilde{\alpha}\in\Phi_1^+\\ \widetilde{\alpha}=(\alpha,k)}}\sum_{j=1}^{\infty} e^{\frac{k(v-u)}{h}} b_{-\widetilde{\alpha}}(1)\otimes\theta_\gamma^j(b_{\widetilde{\alpha}})(1), \end{aligned} \tag{4.19}$$

  given by the same quadruple Q (see [3]), is described by regrading from id to the Coxeter automorphism $\sigma_{(\mathbf{1};1)}$ using Lemma 3.2. More precisely,

$$\left(e^{u\,\mathrm{ad}(\mu)} \otimes e^{v\,\mathrm{ad}(\mu)}\right) r_Q(e^u, e^v) = r_Q^{\sigma_{(\mathbf{1};1)}}(e^{u/h}, e^{v/h}) = X(u-v). \tag{4.20}$$

4.3 Special case $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ and $\sigma = \mathrm{id}$

The classification of classical twists of the standard Lie bialgebra structure $\delta_0 := \delta_0^{\mathrm{id}}$ on $\mathfrak{L} = \mathfrak{g}[z,z^{-1}]$ with $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ can be done without heavy geometric machinery. More precisely, using the theory of maximal orders developed in [33], we can show that

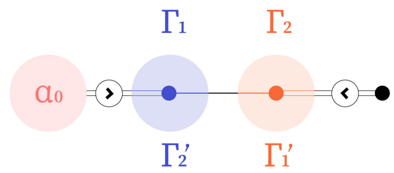

Figure 1: $\Gamma_1, \Gamma_1', \Gamma_2$ and Γ_2' leading to regularly but not polynomially equivalent r-matrices r_Q and $r_{Q'}$.

the equivalence classes of twisted standard bialgebra structures on $\mathfrak{L}$ are in one-to-one correspondence with the equivalence classes of quasi-trigonometric r-matrices, which were classified in [26] in terms of BD quadruples.

The following lemma explains the way in which orders emerge in our work.

Lemma 4.5. *Classical twists t of the standard Lie bialgebra structure δ_0 are in one-to-one correspondence with Lagrangian Lie subalgebras $\widehat{W}_t \subseteq \mathfrak{g}((z)) \times \mathfrak{g}((z^{-1}))$ satisfying the conditions:*

1. $\Delta \dotplus \widehat{W}_t = \mathfrak{g}((z)) \times \mathfrak{g}((z^{-1}))$;
2. *There are non-negative integers N and M such that*
$$z^N \mathfrak{g}[\![z]\!] \subseteq \pi_1 \widehat{W}_t \subseteq z^{-N}\mathfrak{g}[\![z]\!],$$
$$z^{-M}\mathfrak{g}[\![z^{-1}]\!] \subseteq \pi_2 \widehat{W}_t \subseteq z^{M}\mathfrak{g}[\![z^{-1}]\!],$$
where π_1 and π_2 are the projections of $\mathfrak{g}((z)) \times \mathfrak{g}((z^{-1}))$ onto its components $\mathfrak{g}((z))$ and $\mathfrak{g}((z^{-1}))$ respectively.

Proof. By Remark 3.6 the standard Lie bialgebra structure δ_0 on $\mathfrak{L}$ is defined by the Manin triple
$$(\mathfrak{g}((z)) \times \mathfrak{g}((z^{-1})), \Delta, \widehat{W}_0). \tag{4.21}$$
Therefore, in view of Theorem 2.4 and its proof, it is enough to show that condition *2.* corresponds to the commensurability condition on $\widehat{W}_t$ and $\widehat{W}_0$, or equivalently, to finite dimensionality of the image of the map $T = \psi(t) \colon \widehat{W}_0 \longrightarrow \Delta$. The latter correspondence is justified by the following chain of arguments: the condition $\dim(\mathrm{im}(T)) < \infty$ is equivalent to the inclusion
$$\mathrm{im}(T) \subseteq \left\{ \left(\sum_{k=-N}^{M} a_k z^k, \sum_{k=-N}^{M} a_k z^k \right) \middle| \, a_k \in \mathfrak{g} \right\}, \tag{4.22}$$
for some non-negative integers N and M, which in its turn is equivalent to
$$\begin{aligned} &\pi_1 \widehat{W}_t \subseteq z^{-N}\mathfrak{g}[\![z]\!], \\ &\pi_2 \widehat{W}_t \subseteq z^{M}\mathfrak{g}[\![z^{-1}]\!]; \end{aligned} \tag{4.23}$$
Since $\widehat{W}_t$ is Lagrangian, inclusions (4.23) are equivalent to condition *2.* of the theorem. ■

A subalgebra $W \subseteq \mathfrak{g}((u))$ is called an *order* if there is a non-negative integer N such that

$$u^N \mathfrak{g}[\![u]\!] \subseteq W \subseteq u^{-N}\mathfrak{g}[\![u]\!]. \tag{4.24}$$

Therefore, condition *2.* of Lemma 4.5 means that the projections $\pi_1\widehat{W}_t$ and $\pi_2\widehat{W}_t$ are orders.

The following two results from [33] play the key role in the classification of classical twists of δ_0.

Theorem 4.6. *For any order W in $\mathfrak{sl}(n,\mathbb{C}((u^{-1})))$ there is a matrix $A \in \mathrm{GL}(n,\mathbb{C}((u^{-1})))$ such that $W \subseteq A^{-1}\mathfrak{sl}(n,\mathbb{C}[\![u^{-1}]\!])A$. In particular, any maximal order must be of the form $A^{-1}\mathfrak{sl}(n,\mathbb{C}[\![u^{-1}]\!])A$ for some $A \in \mathrm{GL}(n,\mathbb{C}((u^{-1})))$.*

Lemma 4.7 (Sauvage Lemma). *The diagonal matrices* $\mathrm{diag}(u^{m_1},\dots,u^{m_n})$*, where $m_k \in \mathbb{Z}$ and $m_1 \le \ldots \le m_n$, represent all double cosets in*

$$\mathrm{GL}(n,\mathbb{C}[\![u^{-1}]\!])\backslash \mathrm{GL}(n,\mathbb{C}((u^{-1})))/\mathrm{GL}(n,\mathbb{C}[u]).$$

Let $W \subseteq \mathfrak{g}((z)) \times \mathfrak{g}((z^{-1}))$ be a Lagrangian subalgebra satisfying the conditions of Lemma 4.5. Since the projections $\pi_1 W \subseteq \mathfrak{g}((z))$ and $\pi_2 W \subseteq \mathfrak{g}((z^{-1}))$ are orders, by Theorem 4.6 there are matrices $A_\pm \in GL(n,\mathbb{C}((z^{\pm 1})))$ such that

$$\begin{aligned} \pi_1 W &\subseteq A_+^{-1}\mathfrak{sl}(n,\mathbb{C}[\![z]\!])A_+, \\ \pi_2 W &\subseteq A_-^{-1}\mathfrak{sl}(n,\mathbb{C}[\![z^{-1}]\!])A_-. \end{aligned} \tag{4.25}$$

By Sauvage Lemma 4.7 we can find matrices

$$\begin{aligned} P_\pm &\in \mathrm{GL}(n,\mathbb{C}[\![z^{\pm 1}]\!]), \\ d_\pm &\in \mathrm{GL}(n,\mathbb{C}[z,z^{-1}]), \\ Q_\pm &\in \mathrm{GL}(n,\mathbb{C}[z^{\mp 1}]), \end{aligned} \tag{4.26}$$

where $d_\pm$ are diagonal, such that

$$\begin{aligned} \pi_1 W &\subseteq Q_+^{-1}d_+^{-1}P_+^{-1}\mathfrak{sl}(n,\mathbb{C}[\![z]\!])P_+d_+Q_+, \\ \pi_2 W &\subseteq Q_-^{-1}d_-^{-1}P_-^{-1}\mathfrak{sl}(n,\mathbb{C}[\![z^{-1}]\!])P_-d_-Q_-. \end{aligned} \tag{4.27}$$

Taking the product and using the fact that $P_\pm^{-1}\mathfrak{sl}(n,\mathbb{C}[\![z^{\pm 1}]\!])P_\pm = \mathfrak{sl}(n,\mathbb{C}[\![z^{\pm 1}]\!])$ we obtain the inclusion

$$W \subseteq \left(Q_+^{-1}d_+^{-1}\mathfrak{sl}(n,\mathbb{C}[\![z]\!])d_+Q_+\right) \times \left(Q_-^{-1}d_-^{-1}\mathfrak{sl}(n,\mathbb{C}[\![z^{-1}]\!])d_-Q_-\right). \tag{4.28}$$

Note that the componentwise conjugation by Q_+ or d_+ is a regular equivalence. Applying these conjugations we get

$$\begin{aligned} \widetilde{W} := d_+Q_+WQ_+^{-1}d_+^{-1} &\subseteq \mathfrak{sl}(n,\mathbb{C}[\![z]\!]) \times \left(d_+Q_+Q_-^{-1}d_-^{-1}\mathfrak{sl}(n,\mathbb{C}[\![z^{-1}]\!])d_-Q_-Q_+^{-1}d_+^{-1}\right) \\ &\subseteq \mathfrak{sl}(n,\mathbb{C}[\![z]\!]) \times \mathfrak{sl}(n,\mathbb{C}((z^{-1}))) \end{aligned} \tag{4.29}$$

By Theorem 3.3 the classification problem of classical twists of the standard Lie bialgebra structure δ_0 is equivalent to the classification problem of id-trigonometric r-matrices. The following lemma reduces the question even further to quasi-trigonometric r-matrices.

Lemma 4.8. *Any* id*-trigonometric r-matrix is regularly equivalent to a quasi-trigonometric one.*

Proof. Let $r_t = r_0 + t$ be an id-trigonometric r-matrix, where t is a classical twist of δ_0, and W_t be the corresponding Lagrangian subalgebra of $\mathfrak{g}((z)) \times \mathfrak{g}((z^{-1}))$. By the argument preceding the lemma there is a regular equivalence $\phi \in \mathrm{Aut}_{\mathbb{C}[z,z^{-1}]-\mathrm{LieAlg}}(\mathfrak{g}[z, z^{-1}])$ such that

$$W_s := (\phi \times \phi)W_t \subseteq \mathfrak{sl}(n, \mathbb{C}[\![z]\!]) \times \mathfrak{sl}(n, \mathbb{C}((z^{-1}))). \tag{4.30}$$

for some classical twist s of δ_0. We now show that r_s is quasi-trigonometric, or equivalently, that $s \in \mathfrak{g}[z]^{\otimes 2} \cong \mathfrak{g}^{\otimes 2}[x, y]$. Let $\{I_\alpha\}_{\alpha=1}^n$ be an orthonormal basis for $\mathfrak{sl}(n, \mathbb{C})$. Then we can write

$$s = s_{ij}^{\alpha\beta} I_\alpha x^i \otimes I_\beta y^j. \tag{4.31}$$

Assume that $s_{k\ell}^{\alpha'\beta'} \neq 0$ for some $\alpha', \beta' \in \{1, \ldots, n\}$ and $k, \ell \in \mathbb{Z}$ such that at least one of the indices k or ℓ is strictly negative, i.e. the tensor s contains a negative power of z in one of its components. Since s is skew-symmetric we may assume without loss of generality that $k < 0$. Then

$$\pi_1\left(s_{ij}^{\alpha\beta} B\left((I_\beta z^j, I_\beta z^j), (I_{\beta'} z^{-\ell}, 0)\right)(I_\alpha z^i, I_\alpha z^i)\right) = s_{i\ell}^{\alpha\beta'} I_\alpha z^i \tag{4.32}$$

where the sum in the right-hand side contains z^k, $k < 0$. However, by (4.30) the projection $\pi_1(W_s)$ is contained in $\mathfrak{sl}(n, \mathbb{C}[\![z]\!])$ and hence cannot contain negative powers of z. This contradiction shows that both components of s are polynomials in z. ■

Quasi-trigonometric r-matrices over $\mathfrak{sl}(n, \mathbb{C})$ were classified (up to regular equivalence) in [26] using BD qudruples we introduced at the beginning of this section. One can show that if we lift the Lagrangian subalgebra $W \subseteq \mathfrak{g} \times \mathfrak{g}((z^{-1}))$, constructed from a BD quadruple Q in [26], to $\mathfrak{g}((z)) \times \mathfrak{g}((z^{-1}))$ we get precisely the Lagrangian subalgebra $\widehat{W}_Q$ determined by the relation

$$(\mathfrak{L} \times \mathfrak{L}) \cap \widehat{W}_Q = W_Q, \tag{4.33}$$

where W_Q is given by (4.5). By Lemma 4.5 the Lagrangian subalgebra $\widehat{W}_Q$ uniquely determines the classical twist t_Q. This gives the classification of classical twists and proves the first part of Theorem 4.1 in the special case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$.

Remark 4.9. The statement of Lemma 4.8 is not surprising. Its general version can be deduced from Theorems 4.1 and 4.2. Precisely, for any finite-dimensional simple Lie algebra $\mathfrak{g}$ an id-trigonometric solution $r_Q = r_0 + t_Q$, given by a BD qudruple $Q = (\Gamma_1, \Gamma_2, \gamma, t_\mathfrak{h})$, is regularly equivalent to a quasi-trigonometric one if and only if there is an automorphism ϑ of the Dynkin diagram of $\mathfrak{L}$ such that $\widetilde{\alpha}_0 \notin \vartheta(\Gamma_1)$. It is easy to check that this condition is always satisfied for Dynkin diagrams of types $A_n^{(1)}, C_n^{(1)}, B_{2-4}^{(1)}$ and $D_{4-10}^{(1)}$. Therefore, in these cases any id-trigonmetric solution is regularly equivalent to a quasi-trigonometric one. In other cases it is always possible to find a BD quadruple Q, such that r_Q is not equivalent to a quasi-trigonometric r-matrix (see Figure 2). ◊

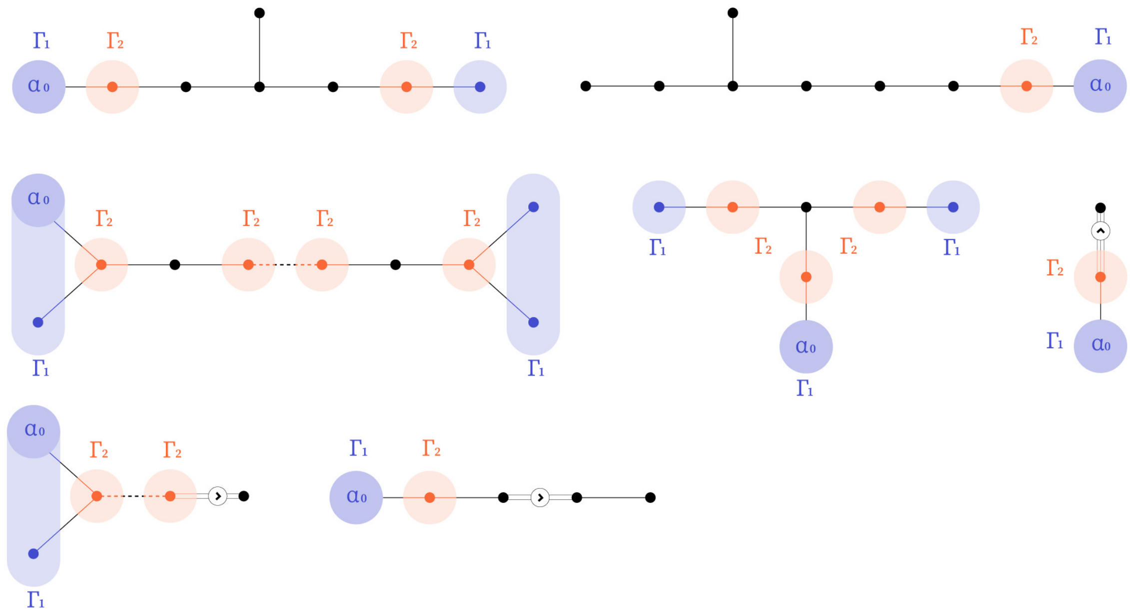

Figure 2: Examples of Γ_1 and Γ_2 giving rise to id-trigonometric solutions not equivalent to quasi-trigonometric ones. The dashed lines mean any number $m \geq 1$ of vertices.

5 Algebro-geometric proof of the main classification theorem

In this section we give a brief summary of the results in [7], prove the extension property for formal equivalences between geometric r-matrices (see Theorem 5.5) and, finally, combining this property with the results in [1] on geometrization of σ-trigonometric r-matrices we verify Theorem 4.1.

5.1 Survey on the geometric theory of the CYBE

Let E be an irreducible projective curve of arithmetic genus 1. Then E is a *Weierstraß cubic*, i.e. there are parameters $g_2, g_3 \in \mathbb{C}$ such that E is the projective closure of $E_\circ = \mathrm{V}(y^2 - 4x^3 + g_2x + g_3) \subseteq \mathbb{P}^2_{(w:x:y)}$ by a smooth point p at infinity. E is singular if and only if $g_2^3 = 27g_3^2$ and an elliptic curve otherwise. In the singular case it has a unique singular point s, which is a simple cusp if $g_2 = 0 = g_3$ and a simple node otherwise. Let $\breve{E}$ be the set of smooth points of E. Fix a non-zero section $\omega \in \Gamma(E, \Omega_E) \cong \mathbb{C}$, where Ω_E is the dualising sheaf. We view ω as a global regular 1-form in the Rosenlicht sense (see e.g. [2, Section II.6]).

We consider now a coherent sheaf $\mathcal{A}$ of Lie algebras on E such that

(i) $\mathrm{H}^0(E, \mathcal{A}) = 0 = \mathrm{H}^1(E, \mathcal{A})$ and

(ii) $\breve{\mathcal{A}} = \mathcal{A}|_{\breve{E}}$ is *weakly* $\mathfrak{g}$*-locally free*, i.e. $\mathcal{A}|_p \cong \mathfrak{g}$ as Lie algebras for all $p \in \breve{E}$.

Property (i) gives that $\mathcal{A}$ is torsion free and property (ii) ensures that the rational envelope A_K of the sheaf $\mathcal{A}$ is a simple Lie algebra over the field K of rational functions on E. Together these properties give the existence of a distinguished section, called *geometric r-matrix*, $\rho \in \Gamma(\breve{E} \times \breve{E}, \breve{\mathcal{A}} \boxtimes \breve{\mathcal{A}}(D))$, where $D = \{(x,x) \in \breve{E} \times \breve{E} \colon x \in \breve{E}\}$ is the *diagonal divisor.* This section satisfies a geometric version of a generalised CYBE, although, if E is singular, it lacks skew-symmetry in general, which prevents it to solve the CYBE. Thus we demand one more property of $\mathcal{A}$ in this case, which ensures skew-symmetry.

If E is singular with a singularity s, we can consider the invariant non-degenerate $\mathbb{C}$-bilinear form

$$B_s^\omega \colon A_K \times A_K \longrightarrow K \xrightarrow{\mathrm{res}_s^\omega} \mathbb{C}, \tag{5.1}$$

where the first map is the Killing form of A_K over K and $\mathrm{res}_s^\omega(f) = \mathrm{res}_s(f\omega)$ is the residue taken in the Rosenlicht sense.

(iii) $\mathcal{A}_s \subset A_K$ is isotropic, i.e. $B_s^\omega(\mathcal{A}_s, \mathcal{A}_s) = 0$.

Now the main statement of the geometric approach to the CYBE is the following.

Theorem 5.1 ([7, Theorem 4.3]). *The geometric r-matrix ρ is a non-degenerate and skew symmetric (both meant in an appropriate geometric manner) solution of a geometric version of the CYBE.* ■

We want to describe ρ as a series, which can be thought of as a Taylor expansion in the second coordinate at the smooth point p at infinity. To do so, let us switch from the sheaf theoretic setting to one localised at the formal neighbourhood of p. There is a unique element u inside the $\mathfrak{m}_p$-adic completion $\widehat{O}_p$ of the local ring $(\mathcal{O}_{E,p}, \mathfrak{m}_p)$, such that $u(p) = 0$ and $\widehat{\omega}_p = du$. We can identify $\widehat{O}_p$ with $\mathbb{C}[\![u]\!]$. Thus the field of fractions $\widehat{Q}_p$ can be identified with $\mathbb{C}(\!(u)\!)$. Consequently, we may view $O = \Gamma(E_\circ, \mathcal{O}_E)$ as a subalgebra of $\widehat{Q}_p = \mathbb{C}(\!(u)\!)$.

Since $\mathfrak{g}$ is simple, Whitehead's lemma implies that $\mathrm{H}^2(\mathfrak{g},\mathfrak{g}) = 0$ and hence all formal deformations of $\mathfrak{g}$ are trivial (see e.g. [19, Section A.8]). Thus $\widehat{\mathcal{A}}_p$, which can be understood as a formal deformation of $\mathfrak{g}$ by Property (ii) of $\mathcal{A}$, is trivial as a formal deformation, i.e. there exists an $\widehat{O}_p = \mathbb{C}[\![u]\!]$-linear isomorphism $\xi \colon \widehat{\mathcal{A}}_p \longrightarrow \mathfrak{g}[\![u]\!]$, called *formal trivialisation*, of Lie algebras such that the induced isomorphism

$$\mathfrak{g} \cong \widehat{\mathcal{A}}_p / \mathfrak{m}_p \widehat{\mathcal{A}}_p \longrightarrow \mathcal{A}|_p \cong \mathfrak{g} \tag{5.2}$$

is the identity. We obtain an induced Lie algebra isomorphism $Q(\widehat{\mathcal{A}}_p) = \widehat{\mathcal{A}}_p \otimes_{\widehat{Q}_p} \widehat{Q}_p \longrightarrow \mathfrak{g}(\!(u)\!)$ via the $\mathbb{C}(\!(u)\!)$-linear extension of ξ, which we denote by the same symbol. We write the image of $\Gamma(E_\circ, \mathcal{A}) \subseteq Q(\widehat{\mathcal{A}}_p)$ under ξ by $\mathfrak{g}(\rho) \subseteq \mathfrak{g}(\!(u)\!)$.

Note that $\mathfrak{g}(\!(u)\!)$ is equipped with the invariant non-degenerate $\mathbb{C}$-bilinear form

$$B_p(f,g) \coloneqq \mathrm{res}_0 \left[\kappa(f,g)du\right] \qquad f, g \in \mathfrak{g}(\!(u)\!). \tag{5.3}$$

Theorem 5.2 ([7, Proposition 6.1 & Theorem 6.4]). *$(\mathfrak{g}(\!(u)\!), \mathfrak{g}(\rho), \mathfrak{g}[\![u]\!])$ is a Manin triple and the Taylor expansion in the second coordinate with respect to u in p gives an injection*

$$\Gamma(\breve{E} \times \breve{E}, \breve{\mathcal{A}} \boxtimes \breve{\mathcal{A}}(D)) \longrightarrow (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!] \tag{5.4}$$

which maps ρ to $\sum_{k=0}^\infty \sum_{\ell=1}^n f_{k\ell} \otimes y^k b_\ell$, where $\{b_\ell\}$ is a basis of $\mathfrak{g}$ and $\{f_{k\ell}\}$ is the basis of $\mathfrak{g}(\rho) \subseteq \mathfrak{g}(\!(u)\!)$, uniquely determined by $B_p(f_{k\ell}, u^{k'} b_{\ell'}) = \delta_{k\ell}\delta_{k'\ell'}$. ■

Remark 5.3. Let us clarify what we mean by Taylor expansion in the second coordinate. Let $P_k = \mathrm{Spec}(\widehat{O}_p/\mathfrak{m}_p^k\widehat{O}_p)$ and $\iota_k\colon P_k \longrightarrow E$ be the injection, mapping the closed point of P_k to p. Then we can consider the pull-back with respect to $\mathrm{id}_{\breve{E}\setminus\{p\}} \times \iota_k$ to obtain the morphism

$$\Gamma(\breve{E}\times\breve{E}, \breve{\mathcal{A}}\boxtimes\breve{\mathcal{A}}(D)) \longrightarrow \Gamma(\breve{E}\setminus\{p\}\times P_k, \breve{\mathcal{A}}\boxtimes\breve{\mathcal{A}}(D)) \cong \Gamma(\breve{E}\setminus\{p\}, \breve{\mathcal{A}})\otimes\widehat{\mathcal{A}}_p/\mathfrak{m}_p^k\widehat{\mathcal{A}}_p,$$

where we have used that $(\breve{E}\setminus\{p\}\times P_k)\cap D) = \emptyset$ and applied the Künneth isomorphism. Mapping $\Gamma(\breve{E}\setminus\{p\}, \breve{\mathcal{A}})$ via ξ to $\mathfrak{g}(\!(u)\!)$, using $\widehat{\mathcal{A}}_p/\mathfrak{m}_p^k\widehat{\mathcal{A}}_p \cong \mathfrak{g}[u]/u^k\mathfrak{g}[u]$ and applying the projective limit with respect to k, yields the desired injection $\Gamma(\breve{E}\times\breve{E}, \breve{\mathcal{A}}\boxtimes\breve{\mathcal{A}}(D)) \longrightarrow (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$. ◊

The theorem suggests, that the geometric r-matrix ρ actually determines $\mathcal{A}$ completely. Our next goal is to formalise this idea. The construction we present is known in other situations, see e.g. [30]. The algebras O and $\mathfrak{g}(\rho)$ inherit the ascending filtrations from the natural filtrations of $\mathbb{C}(\!(u)\!)$ and $\mathfrak{g}(\!(u)\!)$, namely we have $O_j \coloneqq O\cap u^{-j}\mathbb{C}[\![u]\!]$, $\mathfrak{g}(\rho)_j \coloneqq \mathfrak{g}(\rho)\cap u^{-j}\mathfrak{g}[\![u]\!]$ and

$$\ldots = 0 = O_{-1}\subseteq \mathbb{C} = O_0 \subseteq O_1 \subseteq \ldots, \quad \ldots = 0 = \mathfrak{g}(\rho)_0 \subseteq \mathfrak{g}(\rho)_1 \subseteq \mathfrak{g}(\rho)_2 \subseteq \ldots, \tag{5.5}$$

such that $O_jO_k \subseteq O_{k+j}, O_j\mathfrak{g}(\rho)_k \subseteq \mathfrak{g}(\rho)_{j+k}$ and $[\mathfrak{g}(\rho)_j, \mathfrak{g}(\rho)_k] \subseteq \mathfrak{g}(\rho)_{j+k}$. Therefore, we can consider the associated graded objects[6] $\mathrm{gr}(O)$ and $\mathrm{gr}(\mathfrak{g}(\rho))$, given by

$$\mathrm{gr}(O) \coloneqq \bigoplus_{j=0}^{\infty} O_j \quad\text{and}\quad \mathrm{gr}(\mathfrak{g}(\rho)) \coloneqq \bigoplus_{j=0}^{\infty}\mathfrak{g}(\rho)_j. \tag{5.6}$$

Note that $\mathrm{gr}(\mathfrak{g}(\rho))$ is a graded Lie algebra over the graded $\mathbb{C}$-algebra $\mathrm{gr}(O)$. Let us denote by $\mathrm{gr}(\mathfrak{g}(\rho))^\sim$ the associated quasi-coherent sheaf of Lie algebras on $\mathrm{Proj}(\mathrm{gr}(O))$ (see e.g. [18, Section II.5]).

Lemma 5.4. *We have $E = \mathrm{Proj}(\mathrm{gr}(O))$ and the formal trivialisation ξ induces an isomorphism $\mathcal{A} \longrightarrow \mathrm{gr}(\mathfrak{g}(\rho))^\sim$ of sheaves of Lie algebras, which we again denote by ξ.*

Proof. We can view $E_\circ = \mathrm{Spec}(O)$ as an affine open subscheme of $\mathrm{Proj}(\mathrm{gr}(O))$ by identifying any prime ideal $\mathfrak{p}$ of O (which inherits a natural filtration) with $\mathrm{gr}(\mathfrak{p})$. Under this identification we have $\Gamma(E_\circ, \mathrm{gr}(\mathfrak{g}(\rho))^\sim) = \mathfrak{g}(\rho)$, which is most easily seen by using the definitions in [18, Section II.5]. In paticular, $\xi\colon \Gamma(E_\circ, \mathcal{A}) \longrightarrow \mathfrak{g}(\rho) = \Gamma(E_\circ, \mathrm{gr}(\mathfrak{g}(\rho))^\sim)$ is an isomorphism of Lie algebras over O.

By [31, Proposition 3] for any coherent sheaf $\mathcal{F}$ on an open neighbourhood U of p the sequence

$$0 \longrightarrow \Gamma(U, \mathcal{F}) \longrightarrow \Gamma(U\setminus\{p\}, \mathcal{F}) \oplus \hat{\mathcal{F}}_p \longrightarrow \hat{\mathcal{F}}_p \otimes_{\widehat{O}_p} \widehat{Q}_p \tag{5.7}$$

is exact. This implies $\Gamma(U, \mathcal{F}) = \Gamma(U\setminus\{p\}, \mathcal{F})\cap\hat{\mathcal{F}}_p$.

Thus, for any $a \in O_j\setminus O_{j-1}$ we have that $\mathrm{D}(a)\cup\{p\}$ is an affine (see [16, Proposition 5]) open neighbourhood of p in E with the coordinate ring $O[a^{-1}]\cap\mathbb{C}[\![u]\!] = (\mathrm{gr}(O)[a^{-1}])_0$, where $a \in \mathrm{gr}(O)$ is taken to have degree j. Thus we have a natural identification of affine

[6]These should not be confused with the graded objects associated to modules with a descending filtration.

schemes $E \supseteq \mathrm{D}(a) \cup \{p\} = \mathrm{D}_+(a) \subseteq \mathrm{Proj}(\mathrm{gr}(O))$ and we see that $E \subseteq \mathrm{Proj}(\mathrm{gr}(O))$. Since now E is a projective curve identified with an open subset in the projective curve $\mathrm{Proj}(\mathrm{gr}(O))$, [16, Proposition 1] implies the equality $E = \mathrm{Proj}(\mathrm{gr}(O))$.

Finally, by definition we have $\Gamma(\mathrm{D}_+(a), \mathrm{gr}(\mathfrak{g}(\rho))^\sim) = \Gamma(\mathrm{D}(a) \cup \{p\}, \mathrm{gr}(\mathfrak{g}(\rho))^\sim) = (\mathfrak{g}(\rho)[a^{-1}])_0 = \mathfrak{g}(\rho)[a^{-1}] \cap \mathfrak{g}[\![u]\!]$ and hence we obtain the isomorphism

$$\xi\colon \Gamma(\mathrm{D}(a) \cup \{p\}, \mathcal{A}) = \Gamma(\mathrm{D}(a), \mathcal{A}) \cap \hat{\mathcal{A}}_p \longrightarrow \mathfrak{g}(\rho)[a^{-1}] \cap \mathfrak{g}[\![u]\!] = \Gamma(\mathrm{D}(a) \cup \{p\}, \mathrm{gr}(\mathfrak{g}(\rho))^\sim).$$

This ends the proof, because $E = \mathrm{Proj}(\mathrm{gr}(O)) = E_\circ \cup \mathrm{D}_+(a)$. ■

5.2 Extension property of formal local equivalences

Now let us consider two coherent sheaves of Lie algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ on E satisfying the conditions (i) - (iii) of Section 5.1 and denote by ρ_1 and ρ_2 the corresponding geometric r-matrices. Fix formal trivialisations ξ_i of $\mathcal{A}_i$ at p and consider the corresponding isomorphisms $\xi_i\colon \mathcal{A}_i \longrightarrow \mathrm{gr}(\mathfrak{g}(\rho_i))^\sim$ for $i = 1, 2$, where $\mathfrak{g}(\rho_i) = \mathrm{Span}_{\mathbb{C}}(\{f_{k\ell}^{(i)}\}) \subseteq \mathfrak{g}(\!(u)\!)$ is the image of $\Gamma(E_\circ, \mathcal{A}_i)$ and

$$\sum_{k=0}^{\infty}\sum_{\ell=1}^{n} f_{k\ell}^{(i)} \otimes y^k b_\ell \in (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!] \tag{5.8}$$

are the Taylor expansions of ρ_i described in Theorem 5.2. We are now in a position to show that any formal equivalence of ρ_1 and ρ_2 at p extends to a global isomorphism of the corresponding sheaves.

Theorem 5.5. *Let $\phi\colon \mathfrak{g}[\![u]\!] \longrightarrow \mathfrak{g}[\![u]\!]$ be a $\mathbb{C}[\![u]\!]$-linear automorphism of Lie algebras such that*

$$\sum_{k=0}^{\infty}\sum_{\ell=1}^{n} \phi(f_{k\ell}^{(1)}) \otimes \phi(y^k b_\ell) = \sum_{k=0}^{\infty}\sum_{\ell=1}^{n} f_{k\ell}^{(2)} \otimes y^k b_\ell, \tag{5.9}$$

where we consider the $\mathbb{C}(\!(u)\!)$-linear expansion of ϕ in the first tensor factor. Then there is an isomorphism $\psi\colon \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ of coherent sheaves of Lie algebras such that $\xi_2\hat{\psi}_p\xi_1^{-1} = \phi$ and $(\psi \boxtimes \psi)\rho_1 = \rho_2$, where we consider the linear extension with respect to the rational functions on $\breve{E} \times \breve{E}$.

Proof. Write $\phi = \sum_{j=0}^{\infty} u^j \phi_j \in \mathrm{End}(\mathfrak{g})[\![u]\!]$. Then

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{\ell=1}^{n} \phi(f_{k\ell}^{(1)}) \otimes \phi(y^k b_\ell) &= \sum_{k=0}^{\infty}\sum_{\ell=1}^{n}\sum_{j=0}^{k} \phi(f_{(k-j)\ell}^{(1)}) \otimes y^k \phi_j(b_\ell) \\ &= \sum_{k=0}^{\infty}\sum_{\ell'=1}^{n} \left(\sum_{\ell=1}^{n}\sum_{j=0}^{k} a_{\ell'\ell}^{j} \phi(f_{(k-j)\ell}^{(1)})\right) \otimes y^k b_{\ell'},\end{aligned} \tag{5.10}$$

where $\phi_j(b_\ell) = \sum_{\ell'=1}^{n} a_{\ell'\ell}^{j} b_{\ell'}$. This shows that $\mathfrak{g}(\rho_2) \subseteq \phi(\mathfrak{g}(\rho_1))$ by comparing coefficients in (5.9). Since $\mathfrak{g}[\![u]\!] \dotplus \mathfrak{g}(\rho_2) = \mathfrak{g}[\![u]\!] \dotplus \phi(\mathfrak{g}(\rho_1))$, we see that $\mathfrak{g}(\rho_2) = \phi(\mathfrak{g}(\rho_1))$.

Clearly the $\mathbb{C}(\!(u)\!)$-linear extension of ϕ preserves the filtration of $\mathfrak{g}(\!(u)\!)$ and hence induces a graded isomorphism of Lie algebras $\phi\colon \mathrm{gr}(\mathfrak{g}(\rho_1)) \longrightarrow \mathrm{gr}(\mathfrak{g}(\rho_2))$. Since the procedure $(\cdot)^\sim$ of associating a quasi-coherent sheaf to a graded module on $E = \mathrm{Proj}(\mathrm{gr}(O))$ is functorial, we get an isomorphism $\phi^\sim\colon \mathrm{gr}(\mathfrak{g}(\rho_1))^\sim \longrightarrow \mathrm{gr}(\mathfrak{g}(\rho_2))^\sim$. Thus we can define $\psi := \xi_2^{-1}\phi^\sim\xi_1\colon \mathcal{A}_1 \longrightarrow \mathcal{A}_2$. Applying [15, Lemma 1.10] we see that $(\psi \boxtimes \psi)\rho_1 = \rho_2$. ■

Remark 5.6. Since the induced isomorphisms in equation (5.2) of the formal trivialisations ξ_i $(i = 1, 2)$ give the identity, we have $\phi_0 = \psi|_p$. $\Diamond$

5.3 Proof of the main classification theorem

Fix an automorphism $\sigma \in \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ of finite order m and an outer automorphism ν from the coset $\sigma\mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$. Let $t \in \mathfrak{L}^\sigma \otimes \mathfrak{L}^\sigma$ be a classical twist of the standard Lie bialgebra structure δ_0^σ on $\mathfrak{L}^\sigma$. In view of Theorem 3.7, to prove the first part of Theorem 4.1 we need to show that there exists a regular equivalence $\phi \in \mathrm{Aut}_{O^\sigma-\mathrm{LieAlg}}(\mathfrak{L}^\sigma)$, taking values in $\mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$, and a BD quadruple Q such that

$$(\phi(x) \otimes \phi(y))r_t^\sigma(x, y) = r_Q^\sigma(x, y) \tag{5.11}$$

for all $x, y \in \mathbb{C}^*, x^m \neq y^m$. Combining the results of Section 3.1 and [3], we get the following statement.

Lemma 5.7. *There exists a holomorphic function $\phi\colon \mathbb{C} \longrightarrow \mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ such that*

$$(\phi(u) \otimes \phi(v))r_t^\sigma(e^{u/m}, e^{v/m}) = r_Q^\sigma(e^{u/m}, e^{v/m}). \tag{5.12}$$

Proof. By Theorem 3.4 and its proof there exists a holomorphic function $\phi_1\colon \mathbb{C} \longrightarrow \mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ and a trigonometric (in the sense of the Belavin-Drinfeld classification) r-matrix X such that

$$\phi_1(0) = \mathrm{id}_\mathfrak{g} \quad \text{and} \quad X(u - v) = (\phi_1(u) \otimes \phi_1(y))r_t^\sigma(e^{u/m}, e^{v/m}). \tag{5.13}$$

Furthermore, it is shown in [3] that there is a holomorphic function $\phi_2\colon \mathbb{C} \longrightarrow \mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ such that

$$\phi_2(0) = \mathrm{id}_\mathfrak{g} \quad \text{and} \quad (\phi_2(u) \otimes \phi_2(v))X(u - v) = r_Q^{\sigma_{(\mathbf{1},\mathrm{ord}(\nu))}}(e^{u/h}, e^{v/h}), \tag{5.14}$$

where $h := |\sigma_{(\mathbf{1},\mathrm{ord}(\nu))}|$. Combining these results and applying the regrading scheme from Lemma 3.2 we get the desired holomorphic function. $\blacksquare$

Our next goal is to apply Theorem 5.5 to this holomorphic equivalence to obtain a regular one. Therefore, we need sheaves which give rise to the σ-trigonometric r-matrices from (5.12). These were constructed in [1].

Theorem 5.8 ([1, Theorem 6.9]). *Let t be a classical twist of δ_0^σ, E be a nodal Weierstraß cubic with global nonvanishing 1-form $\omega = d(z^m)/z^m$ under the identification $\breve{E} = \mathrm{Spec}(\mathbb{C}[z^m, z^{-m}])$. Then there exists a coherent sheaf of Lie algebras $\mathcal{A}_t$ on E, satisfying properties (i)-(iii) of Section 5.1, such that*

1. $\Gamma(\breve{E}, \mathcal{A}_t) = \mathfrak{L}^\sigma$ *and*
2. *the isomorphism*

$$\Gamma(\breve{E} \otimes \breve{E}, \breve{\mathcal{A}}_t \boxtimes \breve{\mathcal{A}}_t(D)) \cong \left(\frac{1}{(x/y)^m - 1}\right) \mathfrak{L}^\sigma \otimes \mathfrak{L}^\sigma \tag{5.15}$$

maps the geometric r-matrix ρ_t of $\mathcal{A}$ to r_t^σ. $\blacksquare$

We may identify the smooth point at infinity with $1 \in \mathbb{C}^* = \breve{E} = \operatorname{Spec}(\mathbb{C}[z^m, z^{-m}])$. The algebra homomorphism $\mathbb{C}[z^m, z^{-m}] \longrightarrow \mathbb{C}[\![u]\!]$ given by

$$z^m \longmapsto e^u = \sum_{k=0}^{\infty} \frac{u^n}{n!} \tag{5.16}$$

induces an identification $\widehat{\mathcal{O}}_{E,p} = \mathbb{C}[\![u]\!]$ in such a way that $u(1) = \log(1) = 0$ and $\widehat{\omega}_p = e^{-u}d(e^u) = du$. Thus u is the formal local parameter used in the setting of Theorem 5.2 and onwards. Now the $\mathbb{C}[z^m, z^{-m}]$-$\mathbb{C}[\![u]\!]$-equivariant Lie algebra morphism $\mathfrak{L}^\sigma \longrightarrow \mathfrak{g}[\![u]\!]$ given by $f(z) \longmapsto f(e^{u/m})$ induces a formal trivialization $\xi\colon \widehat{\mathcal{A}}_p \longrightarrow \mathfrak{g}[\![u]\!]$. With this choice, we can interpret the series expansion of ρ_t, described in Remark 5.3, as the Taylor series of $r_t^\sigma(e^{u/m}, e^{v/m})$ at $v = 0$. Therefore, we can apply Theorem 5.5 to the Taylor expansion of (5.12) at $v = 0$ to obtain an isomorphism $\psi\colon \mathcal{A}_t \longrightarrow \mathcal{A}_Q$, satisfying $(\psi \boxtimes \psi)\rho_t = \rho_Q$, where $\mathcal{A}_Q := \mathcal{A}_{t_Q}$ and $\rho_Q := \rho_{t_Q}$.

We obtain a regular equivalence $\psi\colon \mathfrak{L}^\sigma = \Gamma(\breve{E}, \mathcal{A}_t) \longrightarrow \Gamma(\breve{E}, \mathcal{A}_Q) = \mathfrak{L}^\sigma$ by applying $\Gamma(\breve{E}, -)$. Note that by Remark 5.6 and Lemma 5.7 the function ψ actually takes values in $\operatorname{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$. Using the commutative diagram

$$\begin{tikzcd} \Gamma(\breve{E} \times \breve{E}, \breve{\mathcal{A}} \boxtimes \breve{\mathcal{A}}) \arrow[r, "\psi\boxtimes\psi"] \arrow[d, "\cong"'] & \Gamma(\breve{E} \times \breve{E}, \breve{\mathcal{A}} \boxtimes \breve{\mathcal{A}}) \arrow[d, "\cong"] \\ \mathfrak{L}^\sigma \otimes \mathfrak{L}^\sigma \arrow[r, "\psi\otimes\psi"'] & \mathfrak{L}^\sigma \otimes \mathfrak{L}^\sigma \end{tikzcd}, \tag{5.17}$$

where the vertical isomorphisms are given by the Künneth theorem, we obtain the desired identity

$$(\psi(x) \otimes \psi(y)) r_t^\sigma(x, y) = r_Q^\sigma(x, y). \tag{5.18}$$

We finish the proof of the main theorem by explaining when two BD quadruples give rise to equivalent twisted standard structures.

Lemma 5.9. *Let $Q = (\Gamma_1, \Gamma_2, \gamma, t_\mathfrak{h})$ and $Q' = (\Gamma_1', \Gamma_2', \gamma', t_\mathfrak{h}')$ be two BD quadruples. Then δ_Q^σ and $\delta_{Q'}^\sigma$ are regularly equivalent if and only if there exists an automorphism ϑ of the Dynkin diagram of $\mathfrak{L}^\sigma$ such that $\vartheta(Q) = Q'$.*

Proof. To simplify the notations we assume $\sigma = \sigma_{(s,|\nu|)}$ for $s = (s_0, \dots, s_n)$. The general result follows from Remark 2.7.

" $\Longrightarrow$ ": Let ϕ be a regular equivalence between δ_Q^σ and $\delta_{Q'}^\sigma$. The identity

$$(\phi(x) \otimes \phi(y)) r_Q^\sigma(x, y) = r_{Q'}^\sigma(x, y), \tag{5.19}$$

given by Theorem 3.7, and the equation (3.22) imply $\phi R_Q \phi^{-1} = R_{Q'}$. If GE_0 and GE_0' are the generalized eigenspaces of R_Q and $R_{Q'}$ respectively, corresponding to the common eigenvalue 0, then $\phi(GE_0) = GE_0'$. Since both θ_γ^+ and $\theta_{\gamma'}^+$ are nilpotent, the identity (4.2) implies the equality of normalizers $\mathrm{N}_{\mathfrak{L}^\sigma}(GE_0) = \mathrm{N}_{\mathfrak{L}^\sigma}(GE_0') = \mathfrak{B}_+$. Therefore, ϕ is an automorphism of $\mathfrak{L}^\sigma$ fixing the Borel subalgebra $\mathfrak{B}_+$. From the third part of Lemma 4.4, we see that ϕ induces an automorphism ϑ of the Dynkin diagram of $\mathfrak{L}^\sigma$. The identity $\vartheta(Q) = Q'$ follows from Theorem 3.7 and formula (4.5).

" $\Longleftarrow$ ": Let ϑ be a Dynkin diagram automorphism, such that $\vartheta(Q) = Q'$. We want to show that ϑ defines a regular equivalence ϕ on $\mathfrak{L}^\sigma$, such that $\phi(\mathfrak{L}^\sigma_{(\alpha,k)}) = \mathfrak{L}^\sigma_{\vartheta(\alpha,k)}$ for any root (α, k) of $\mathfrak{L}^\sigma$. Let ϕ' be the automorphism defined by $\phi'(z^{\pm s_i} X_i^{\pm}(1)) \coloneqq z^{\pm s_{\vartheta(i)}} X^{\pm}_{\vartheta(i)}(1)$. Then it maps the root spaces onto each other in the desired way. We now adjust ϕ' to be $\mathbb{C}[z^m, z^{-m}]$-linear. By [23, Lemma 8.6] we have the equality $\phi'((z^m - 1)\mathfrak{L}^\sigma) = ((z/a)^m - 1)\mathfrak{L}^\sigma$ for some $a \in \mathbb{C}^*$. Let μ_a be the automorphism of $\mathfrak{L}^\sigma$ given by $\mu_a(f(z)) \coloneqq f(az)$. Note that it preserves the root spaces of $\mathfrak{L}^\sigma$. Define $\phi \coloneqq \mu_a \phi'$, then $\phi((z^m - 1)\mathfrak{L}^\sigma) = (z^m - 1)\mathfrak{L}^\sigma$ and thus

$$\phi(z^{\pm s_i + m} X_i^{\pm}(1)) = z^m \phi(z^{\pm s_i} X_i^{\pm}(1)) \tag{5.20}$$

implying the $\mathbb{C}[z^m, z^{-m}]$-linearity. ∎

References

[1] R. Abedin and I. Burban. *Algebraic geometry of Lie bialgebras defined by solutions of the classical Yang-Baxter equation.*

[2] W. Barth, K. Hulek, C. Peters, and A. van de Ven. *Compact Complex Surfaces.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. Springer Berlin Heidelberg, (2015).

[3] A. Belavin and V. Drinfeld. *Solutions of the classical Yang-Baxter equation for simple Lie algebras.* Funct. Anal. Appl. 16, No. 3 (1983).

[4] A. Belavin and V. Drinfeld. *The classical Yang-Baxter equation for simple Lie algebras.* Funct. Anal. Appl. 17, No. 3 (1983).

[5] A. Belavin and V. Drinfeld. *Triangle Equations and Simple Lie Algebras.* Soviet scientific reviews: Mathematical physics reviews. Harwood Academic Publishers, (1984).

[6] N. Bourbaki. *Lie Groups and Lie Algebras: Chapters 7-9.* Elements of mathematics. Springer-Verlag Berlin Heidelberg, (2005).

[7] I. Burban and L. Galinat. *Torsion Free Sheaves on Weierstrass Cubic Curves and the Classical Yang–Baxter Equation.* Commun. Math. Phys. (2018).

[8] I. Burban, L. Galinat, and A Stolin. *Simple vector bundles on a nodal Weierstrass cubic and quasi-trigonometric solutions of CYBE.* Journal of Physics A: Mathematical and Theoretical (2017).

[9] R. Carter. *Lie Algebras of Finite and Affine Type.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, (2005).

[10] V. Chari and A. Pressley. *A Guide to Quantum Groups.* Cambridge University Press, (1995).

[11] V. Drinfeld. *Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang–Baxter equations.* Dokl. Akad. Nauk SSSR, 268:2 (1983).

[12] V. Drinfeld. *Hopf algebras and the quantum Yang-Baxter equation.* Sov.Math.Dokl (1985).

[13] V. Drinfeld. *Quantum groups.* Journal of Soviet Mathematics 41, 898–915 (1983).

[14] P. Etingof and O. Schiffmann. *Lectures on Quantum Groups (Second Edition).* Lectures in Mathematical Physics, (2002).

[15] L. Galinat. *Algebro-Geometric Aspects of the Classical Yang-Baxter Equation.* PhD thesis. Universität zu Köln, (2015).

[16] J. Goodman. *Affine open subsets of algebraic varieties and ample divisors.* Annals of Mathematics, Jan. 1969, Second Series, Vol. 89, No 1, 160-183 (1967).

[17] R. Gunning and H. Rossi. *Analytic functions of several complex variables.* Prentice-Hall, (1965).

[18] R. Hartshorne. *Algebraic Geometry.* Graduate Texts in Mathematics. Springer, (1977).

[19] M. Hazewinkel and M. Gerstenhaber. *Deformation Theory of Algebras and Structures and Applications*. Springer Netherlands, (1988).

[20] S. Helgason. *Differential Geometry, Lie Groups, and Symmetric Spaces*. Pure and Applied Mathematics. Elsevier Science, (1978).

[21] M. Jimbo. *Quantum R matrix for the generalized Toda system*. Commun. Math. Phys. (1986).

[22] V. Kac. *Automorphisms of finite order of semisimple Lie algebras*. Funct. Anal. Appl. (1969).

[23] V. Kac. *Infinite dimensional Lie algebras (3rd Edition)*. Cambridge University Press, (1990).

[24] V. Kac and S. Wang. *On Automorphisms of Kac-Moody Algebras and Groups*. Advances in Mathematics 92, 129-195 (1992).

[25] E. Karolinsky and A. Stolin. *Classical Dynamical r-Matrices, Poisson Homogeneous Spaces, and Lagrangian Subalgebras*. Lett. Math. Phys. (2002).

[26] S. Khoroshkin, I. Pop, M. Samsonov, A. Stolin, and V. Tolstoy. *On Some Lie Bialgebra Structures on Polynomial Algebras and their Quantization*. Communications in Mathematical Physics (2008).

[27] Y. Kosmann-Schwarzbach. *Lie bialgebras, Poisson Lie groups and dressing transformations. Integrability of Nonlinear Systems*. Lecture Notes in Phys., Vol. 495, 104 – 170. Springer Berlin Heidelberg, (1997).

[28] P. Kulish. *Quantum difference nonlinear Schrödinger equation*. Lett. Math. Phys. (1981).

[29] F. Montaner, A. Stolin, and E. Zelmanov. *Classification of Lie bialgebras over current algebras*. Sel. Math. New. Ser. (2010).

[30] D. Mumford. *An algebro-geometric construction of commuting operators and of solutions to the Toda lattice equation, Korteweg deVries equation and related nonlinear equation*. Proc. of the Intl. Symp. Algebraic Geometry, Kyoto, (1978).

[31] A. Parshin. *Integrable systems and local fields*. Communications in Algebra (2001).

[32] I. Pop and A. Stolin. *Lagrangian Subalgebras and Quasi-trigonometric r-Matrices*. Lett. Math. Phys., Vol. 85, No. 2, 249 – 262 (2008).

[33] A. Stolin. *On rational solutions of Yang-Baxter equation for* $\mathfrak{sl}(n)$. Mathematica Scandinavica (1991).

[34] A. Stolin. *On rational solutions of Yang-Baxter equations. Maximal orders in loop algebra*. Comm. Math. Phys. (1991).