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TENSOR DECOMPOSITION-BASED BEAMSPACE ESPRIT ALGORITHM FOR MULTIDIMENSIONAL HARMONIC RETRIEVAL

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ABSTRACT

Beamspace processing is an efficient and commonly used approach in harmonic retrieval (HR). In the beamspace, measurements are obtained by linearly transforming the sensing data, thereby achieving a compromise between estimation accuracy and system complexity. Meanwhile, the widespread use of multi-sensor technology in HR has highlighted the necessity to move from a matrix (two-way) to tensor (multi-way) analysis. In this paper, we propose a beamspace tensor-ESPRIT for multidimensional HR. In our algorithm, parameter estimation and association are achieved simultaneously.

Index Terms— Tensor, beamspace-ESPRIT, harmonic retrieval, CANDECOMP/PARAFAC decomposition.

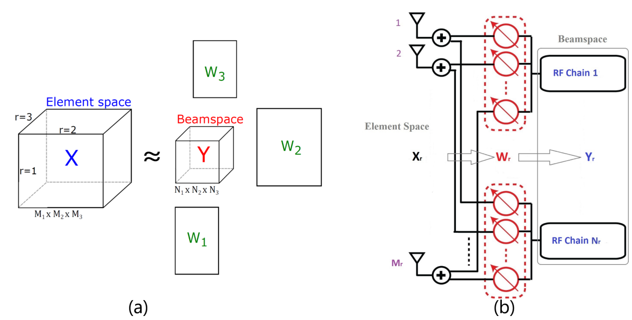

Fig. 1. Illustration of measurements in beamspace. (a) general case. (b) MIMO example with hybrid combining.

1. INTRODUCTION

Multidimensional (R-D) harmonic retrieval (HR) is required in numerous signal processing problems [1] and has been studied for many decades in a variety of fields, such as mobile communications [2], multiple-input multiple-output (MIMO) radar [3] and nuclear magnetic resonance spectroscopy [4]. A number of HR techniques are available in the literature [5]. Estimation of signal parameters via rotational invariance techniques (ESPRIT) [6] and its variants [7, 8] have become one of the popular search-free signal subspace-based parameter estimation techniques [9]. A tensor is a natural approach to describe R-D data for $R \geq 3$ [10]. CANDECOMP/PARAFAC (CP) and Tucker are two widely used tensor decomposition approaches [11]. Note that association of frequencies in each dimension is also important for multidimensional HR. In tensor-ESPRIT-type algorithms, the association is usually achieved by computing the eigenvalues of the estimated matrices jointly via a joint approximate eigen-decomposition [12] or a simultaneous Schur decomposition [13].

In some applications HR must be performed in beamspace, due to hardware constraints or achieving a compromise between estimation accuracy and system complexity. An important application is millimeter wave (mmWave) MIMO communications, where beamspace measurements naturally occur due to hybrid array architectures [14]. In beamspace HR, the sensing data are not available directly, but measurements are obtained by their linear transformation. Correspondingly, they are called beamspace measurements to distinguish between element space measurements in standard HR. As shown in Fig. 1, element space measurements of dimensions $M_1 \times M_2 \times M_3$ are linearly transformed to obtain beamspace measurements of dimensions $N_1 \times N_2 \times N_3$. Beamspace ESPRIT methods were proposed in [15] and recently applied to mmWave MIMO communications [16] for $R = 3$. A similar problem was addressed in [17], where a tensor decomposition-aided mmWave channel estimation algorithm was developed for $R = 3$ in a discrete Fourier transform (DFT) beamspace. However, the method requires a line search for each of the dimensions, so that the complexity may be too high for real-time applications [5].

A search-free R-D and general beamspace tensor-ESPRIT algorithm is proposed in [18] for mmWave MIMO channel estimation. In this paper, we extend the algorithm in [18] for multidimensional HR. We first evaluate the bias and root-mean square error (RMSE) performance for different beamspaces, followed by demonstrating the robustness of the method with respect to uncertainty in the number of sources, as well as the applicability for sources with partially distinct frequencies.

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2. PROBLEM FORMULATION

We use $(\cdot)^{\mathsf{H}}$, $(\cdot)^{\mathsf{T}}$ and $\mathbb{C}$ to denote Hermitian transpose, transpose and the set of complex numbers, respectively. We follow the tensor operations defined in [19]. The $(i_1, i_2, \cdots, i_R)$ entry of an R-D tensor $\boldsymbol{\mathcal{A}}$ is denoted as $a_{i_1,i_2,\cdots,i_R}$.

Element-Space Model We consider sequential transmissions, where for the kth snapshot, the *element-space* tensor $\boldsymbol{\mathcal{X}}_k$ has entries of the form:[1]

$$x_{m_1,\cdots,m_R,k} = \sum_{l=1}^{L} \gamma_l(k) \prod_{r=1}^{R} e^{j\omega_{r,l} \times m_r}, \tag{1}$$

where $m_r = 0, 1, \cdots, M_r - 1$, $r = 1, 2, \cdots, R$, $l = 1, 2, \cdots, L$. Here, M_r, R and L denote the number of sensors for the rth dimension, the number of dimensions and the number of R-D frequencies (i.e., number of signal paths or sources), respectively, $\gamma_l(k)$ represents the unknown complex amplitude of the lth frequency at the kth snapshot, while $\omega_{r,l} \in (-\pi, \pi)$ is the frequency in the rth dimension of the lth source. For multiple snapshots, the tensor dimension is (R+1). After obtaining the RL frequencies, parameter association is required to obtain the structured information of the measurements. The frequencies associated to the lth source are denoted as $\{\omega_{1,l}, \omega_{2,l}, \cdots, \omega_{R,l}\}$, where, $l = 1, 2, \cdots, L$. The number of sources L is assumed to be known (e.g., from an R-D source enumeration method [20]). The tensor $\boldsymbol{\mathcal{X}}_k \in \mathbb{C}^{M_1 \times M_2 \times \cdots \times M_R}$ can be expressed as

$$\boldsymbol{\mathcal{X}}_k = \sum_{l=1}^{L} \gamma_l \, \mathbf{a}_{1,l} \circ \mathbf{a}_{2,l} \circ \cdots \circ \mathbf{a}_{R,l}, \tag{2}$$

where $\mathbf{a}_{r,l} = \begin{bmatrix} e^{j\omega_{r,l}} & e^{j2\omega_{r,l}} & \cdots & e^{jM_r\omega_{r,l}} \end{bmatrix}^{\mathsf{T}}$ and $\circ$ denotes the vector outer product [11]. We also define:

$$\mathbf{A}_r = \begin{bmatrix} \mathbf{a}_{r,1} & \mathbf{a}_{r,2} & \cdots & \mathbf{a}_{r,L} \end{bmatrix} \in \mathbb{C}^{M_r \times L}. \tag{3}$$

Beamspace Model For *beamspace* measurements, after the r-mode product of $\boldsymbol{\mathcal{X}}_k$ with linear transformation matrix [11], the model (2) is modified to

$$\boldsymbol{\mathcal{Y}}_k = \sum_{l=1}^{L} \gamma_l \, \mathbf{b}_{1,l} \circ \mathbf{b}_{2,l} \circ \cdots \circ \mathbf{b}_{R,l}, \tag{4}$$

where the beamspace array manifold is defined as

$$\mathbf{B}_r = \begin{bmatrix} \mathbf{b}_{r,1} & \mathbf{b}_{r,2} & \cdots & \mathbf{b}_{r,L} \end{bmatrix} = \mathbf{W}_r^{\mathsf{H}} \mathbf{A}_r \in \mathbb{C}^{N_r \times L}. \tag{5}$$

Here $\mathbf{W}_r^{\mathsf{H}} = \begin{bmatrix} \mathbf{w}_{r,1} & \mathbf{w}_{r,2} & \cdots & \mathbf{w}_{r,M_r} \end{bmatrix} \in \mathbb{C}^{N_r \times M_r}$ is the linear transformation matrix. The transformation (5) can be interpreted as a matrix beamformer. The columns of transformation matrix $\mathbf{W}_r$ are usually chosen as beamformers to cover a sector of source locations, provided that *a priori* information about the true frequencies is available. The DFT beamforming matrix is one of the common transformation matrices, which is defined as

$$\mathbf{W}_r = \begin{bmatrix} \mathbf{a}_{r,1} & \mathbf{a}_{r,2} & \cdots & \mathbf{a}_{r,N_r} \end{bmatrix} \in \mathbb{C}^{M_r \times N_r}, \tag{6}$$

where the pointing frequencies $\omega_{r,n_r}, n_r = 1, 2, \cdots, N_r$, are spaced uniformly within the sector of interest [21]. The number N_r should be chosen properly, so that the beams cover most of the signal energy. Furthermore, $\mathbf{W}_r^{\mathsf{H}} \mathbf{W}_r = \mathbf{I}_{N_r}$ is required to maintain white in the beamspace output. This is achievable by whitening the non-orthogonal beams [22].

Our objective is to estimate $\omega_{r,l}$, for $r = 1, \cdots, R$ and $l = 1, \cdots, L$, from noisy measurements $\tilde{\boldsymbol{\mathcal{Y}}}_k = \boldsymbol{\mathcal{Y}}_k + \boldsymbol{\mathcal{V}}_k$, using the beamspace tensor-ESPRIT method and $\boldsymbol{\mathcal{V}}_k$ denotes the white Gaussian noise tensor.

3. R-D BEAMSPACE TENSOR-ESPRIT FOR HR

In the CP decomposition, a tensor is decomposed into a sum of rank-one component tensors,

$$\tilde{\boldsymbol{\mathcal{Y}}}_k = \sum_{l=1}^{L} \lambda_l \, \mathbf{u}_{1,l} \circ \mathbf{u}_{2,l} \circ \cdots \circ \mathbf{u}_{R,l}. \tag{7}$$

Both association and noise reduction are achieved simultaneously. ESPRIT algorithms utilize the shift invariant property:

$$\mathbf{J}_r^{(1)} \mathbf{A}_r = \mathbf{J}_r^{(2)} \mathbf{A}_r \boldsymbol{\Phi}_r, \tag{8}$$

where $\boldsymbol{\Phi}_r$ contains the frequencies of all sources in rth dimension, $\boldsymbol{\Phi}_r = \operatorname{diag}\begin{bmatrix} e^{-j\omega_{r,1}} & e^{-j\omega_{r,2}} & \cdots & e^{-j\omega_{r,L}} \end{bmatrix}$, $\mathbf{J}_r^{(1)} = \begin{bmatrix} \mathbf{I}_{N_r-1} & \mathbf{0}_{(N_r-1)\times 1} \end{bmatrix}$ and $\mathbf{J}_r^{(2)} = \begin{bmatrix} \mathbf{0}_{(N_r-1)\times 1} & \mathbf{I}_{N_r-1} \end{bmatrix}$ are selection matrices.

In beamspace, the row transformation $\mathbf{W}_r^{\mathsf{H}}$ alters the transitional invariance structure in the array manifold [15], and consequently $\mathbf{J}_r^{(1)} \mathbf{B}_r \neq \mathbf{J}_r^{(2)} \mathbf{B}_r \boldsymbol{\Phi}_r$. However, the shift invariance structure can be restored, if $\mathbf{W}_r$ has a similar structure. Suppose we are able to find a non-singular $N_r \times N_r$ matrix $\mathbf{F}_r$ that satisfies

$$\mathbf{J}_r^{(1)} \mathbf{W}_r = \mathbf{J}_r^{(2)} \mathbf{W}_r \mathbf{F}_r. \tag{9}$$

The least squares (LS) estimate of $\mathbf{F}_r$ is given by

$$\hat{\mathbf{F}}_r = \left(\mathbf{J}_r^{(2)} \mathbf{W}_r\right)^{\dagger} \mathbf{J}_r^{(1)} \mathbf{W}_r \tag{10}$$

where $\dagger$ denotes the pseudo-inverse.

Theorem 1. *Let $\mathbf{F}_r$ be defined as in* (9)*. If there exists a $\mathbf{Q}_r \in \mathbb{C}^{N_r \times N_r}$, such that*

$$\mathbf{Q}_r \mathbf{w}_{r,M_r} = \mathbf{0}_{N_r \times 1}, \ \textit{and} \ \mathbf{Q}_r \mathbf{F}_r^{\mathsf{H}} \mathbf{w}_{r,1} = \mathbf{0}_{N_r \times 1}, \tag{11}$$

then

$$\mathbf{Q}_r \mathbf{F}_r^{\mathsf{H}} \mathbf{B}_r = \mathbf{Q}_r \mathbf{B}_r \boldsymbol{\Phi}_r^{\mathsf{H}}. \tag{12}$$

[1]For notational compactness, noise is omitted in the equations

Proof. See Appendix A in [18]. □

It is worth mentioning that $\mathbf{Q}_r$ in (11) can be obtained by forming a projection matrix corresponding to the orthogonal subspace of $\mathcal{R}\{\mathbf{w}_{r,M_r}, \mathbf{F}_r^{\mathsf{H}}\mathbf{w}_{r,1}\}$. Then

$$\hat{\mathbf{Q}}_r = \mathbf{I}_{N_r} - \mathbf{w}_{r,M_r}\mathbf{w}_{r,M_r}^{\mathsf{H}} - \left(\mathbf{F}_r^{\mathsf{H}}\mathbf{w}_{r,1}\right)\left(\mathbf{F}_r^{\mathsf{H}}\mathbf{w}_{r,1}\right)^{\mathsf{H}}. \quad (13)$$

Let $\mathbf{U}_r = \begin{bmatrix}\mathbf{u}_{r,1} & \mathbf{u}_{r,2} & \cdots & \mathbf{u}_{r,L}\end{bmatrix}$, its columns span the signal subspace, and replace $\mathbf{B}_r$ by $\mathbf{B}_r = \mathbf{U}_r\mathbf{D}_r$, where $\mathbf{D}_r \in \mathbb{C}^{L\times L}$ is a non-singular matrix. Then (12) becomes

$$\mathbf{Q}_r\mathbf{F}_r^{\mathsf{H}}\mathbf{U}_r = \mathbf{Q}_r\mathbf{U}_r\mathbf{\Gamma}_r \quad (14)$$

where

$$\mathbf{\Gamma}_r = \mathbf{D}_r\mathbf{\Phi}_r^{\mathsf{H}}\mathbf{D}_r^{-1} \in \mathbb{C}^{L\times L}. \quad (15)$$

It is interesting to note that $\mathbf{D}_r$ is incorporated in $\mathbf{\Gamma}_r$, direct computing is not required. We estimate $\mathbf{\Gamma}_r$ via LS from (14):

$$\hat{\mathbf{\Gamma}}_r = \left(\mathbf{Q}_r\mathbf{U}_r\right)^{\dagger}\mathbf{Q}_r\mathbf{F}_r^{\mathsf{H}}\mathbf{U}_r. \quad (16)$$

From (15), eigenvalues of $\mathbf{\Gamma}_r$ are the diagonal elements of $\mathbf{\Phi}_r^{\mathsf{H}}$, so that the lth eigenvalue of $\mathbf{\Gamma}_r$ is given by $e^{j\omega_{r,l}}$. Note that the lth column of $\mathbf{U}_r$ corresponds to the same source. For the rth dimension, the L frequencies $\boldsymbol{\omega}_r = \{\omega_{r,1}, \omega_{r,2}, \cdots, \omega_{r,L}\}$ can be estimated jointly from the eigenvalues of $\mathbf{\Gamma}_r$. The proposed algorithm is summarized in Algorithm 1.

Algorithm 1: R-D Beamspace Tensor-ESPRIT for HR

Input: Measurements $\tilde{\boldsymbol{\mathcal{Y}}}_k$ and $\mathbf{W}_r, r = 1, 2, \cdots, R$.
Output: Frequencies $\boldsymbol{\omega}_l, l = 1, 2, \cdots, L$.
Estimate source number L [20] and
obtain $\mathbf{U}_r$ by taking CP decomposition on $\tilde{\boldsymbol{\mathcal{Y}}}_k$.
for $r = 1, 2,$ **to** R **do**
  Estimate $\hat{\mathbf{F}}_r$ from (10) and $\hat{\mathbf{Q}}_r$ from (13).
  Estimate $\hat{\boldsymbol{\omega}}_r$ by TLS-ESPRIT [15].
end
For the lth source $\boldsymbol{\omega}_l = \{\omega_{1,l}, \omega_{2,l}, \cdots, \omega_{R,l}\}$.

4. SIMULATION RESULTS

The performance of the proposed method is evaluated in terms of the bias and RMSE on estimated parameters:

$$\text{Bias} = \frac{1}{RL}\sum_{r=1}^{R}\sum_{l=1}^{L}\left(\omega_{r,l} - \hat{\omega}_{r,l}\right) \quad (17)$$

and RMSE $= \sqrt{\frac{1}{RL}\mathbb{E}_t\left\{\sum_{r=1}^{R}\sum_{l=1}^{L}\left(\omega_{r,l} - \hat{\omega}_{r,l}\right)^2\right\}}$, where $\hat{\omega}_{r,l}$ is an estimate of $\omega_{r,l}$, and $\mathbb{E}_t\{\cdot\}$ denotes the expected value based on 200 Monte-Carlo trials. The proposed method is compared with tensor-ESPRIT (T-ESPRIT) algorithm [7] in element space, as well as Cramér-Rao lower bound (CRLB). In the following simulations, a DFT beamforming matrix is considered. The unknown amplitude of the lth frequency at the kth snapshot, $\gamma_l(k)$, is drawn from $\mathcal{N}(0, 1)$. Signal-to-noise ratio (SNR) is defined as SNR $= \|\tilde{\boldsymbol{\mathcal{X}}}_k - \boldsymbol{\mathcal{V}}_k\|_F^2/\|\boldsymbol{\mathcal{V}}_k\|_F^2$, where $\|\cdot\|_F$ denotes the tensor Frobenius norm [11]. In element space, $M_1 = M_2 = M_3 = 8$ and the number of measurements is $K = 10$. The Matlab package Tensorlab [23] is utilized for tensor computation. In the first test, there are three sources and the frequencies to be estimated are:

$$r = 1 : (\omega_{1,1}, \omega_{1,2}, \omega_{1,3}) = (0.550\pi, 0.719\pi, 0.906\pi)$$
$$r = 2 : (\omega_{2,1}, \omega_{2,2}, \omega_{2,3}) = (0.410\pi, 0.777\pi, 0.276\pi)$$
$$r = 3 : (\omega_{3,1}, \omega_{3,2}, \omega_{3,3}) = (0.340\pi, 0.906\pi, 0.358\pi).$$

Fig. 2 shows the 3-D HR parameter estimation performance versus SNR using DFT beamforming matrix. RMSE performance under different SNR is shown in Fig. 2. Comparison element space T-ESPRIT is also included. For beamspace ESPRIT, the data dimensions are $N_1 \times N_2 \times N_3$, and different values of N_r are considered: $N_r = 4$, 6 and 8. We observe that both bias and RMSE are reduced by increasing the size of the matrix beamformer. The performance is close to element space ESPRIT when a larger N_r is used.

Sources with partially distinct frequencies are common phenomena in real applications. In the second test, the performance of the proposed method for partially distinct frequencies is evaluated. Now there are four sources, and the frequencies are same in at least one of the 3 dimensions. In the beamspace, $N_1 = N_2 = N_3 = 6$ and and SNR is 20 dB. The 3-D HR frequencies to be estimated are:

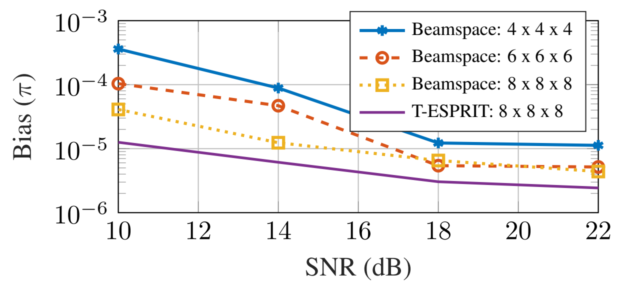
$$r = 1 : (\omega_{1,1}, \omega_{1,2}, \omega_{1,3}, \omega_{1,4}) = (\boxed{0.2\pi, 0.2\pi}, 0.6\pi, 0.8\pi)$$
$$r = 2 : (\omega_{2,1}, \omega_{2,2}, \omega_{2,3}, \omega_{2,4}) = (0.9\pi, \boxed{0.4\pi, 0.4\pi}, 0.6\pi)$$
$$r = 3 : (\omega_{3,1}, \omega_{3,2}, \omega_{3,3}, \omega_{3,4}) = (0.1\pi, 0.2\pi, \boxed{0.8\pi, 0.8\pi}).$$

As shown in Fig. 3, the four sources are estimated and associated correctly, even with partially distinct frequencies.

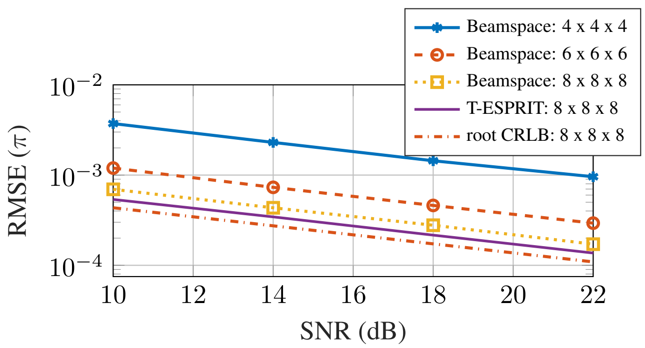
In the third test, different source numbers are considered. In the beamspace, $N_1 = N_2 = N_3 = 6$ and SNR is 10 dB. The 3-D HR frequencies to be estimated are:

$$r = 1 : (\omega_{1,1}, \omega_{1,2}, \omega_{1,3}) = (0.1\pi, 0.3\pi, 0.8\pi)$$
$$r = 2 : (\omega_{2,1}, \omega_{2,2}, \omega_{2,3}) = (0.9\pi, 0.4\pi, 0.2\pi)$$
$$r = 3 : (\omega_{3,1}, \omega_{3,2}, \omega_{3,3}) = (0.4\pi, 0.1\pi, 0.7\pi).$$

The number of sources is $L = 3$. The amplitudes of the first, second and third sources are 1, 0.85 and 0.75, respectively. As shown in Fig. 4, the 3-D parameters are associated, even with inaccurate source number information. Let $\hat{L}$ be the estimated number of sources. An interesting observation is that, if $\hat{L} \leq L$, then the first dominant $\hat{L}$ sources are identified, and the corresponding results are shown in Fig. 4(a)-Fig. 4(c). Otherwise, the first dominant L sources are still observable, but the extra fake sources are randomly distributed, as shown in Fig. 4(d).



(a) Bias.



(b) RMSE.

Fig. 2. Parameter estimation performance versus SNR for distinct frequencies.

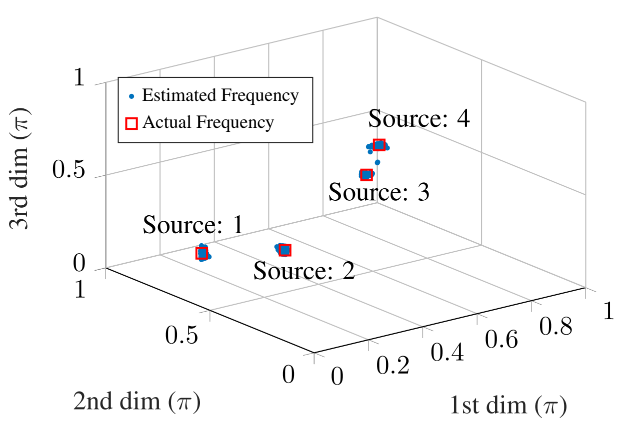
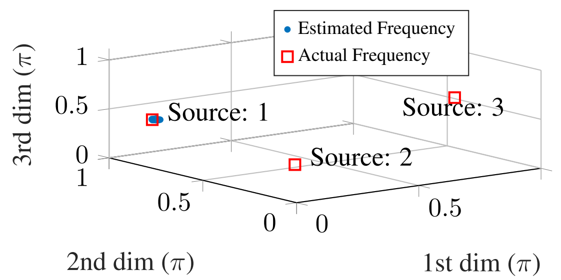
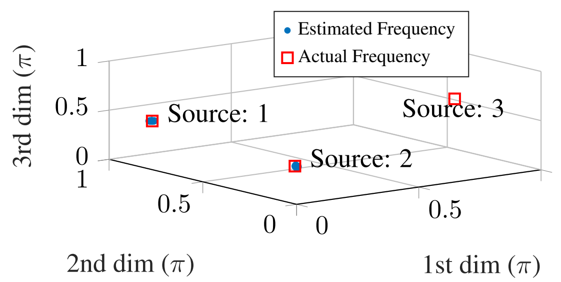


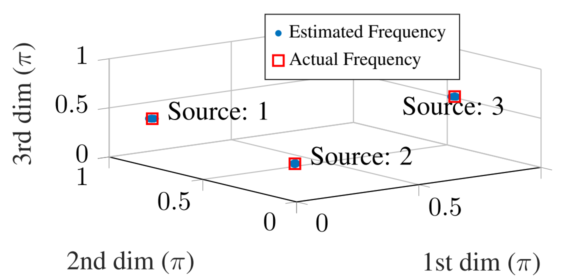
Fig. 3. Parameter estimation for partially distinct frequencies.



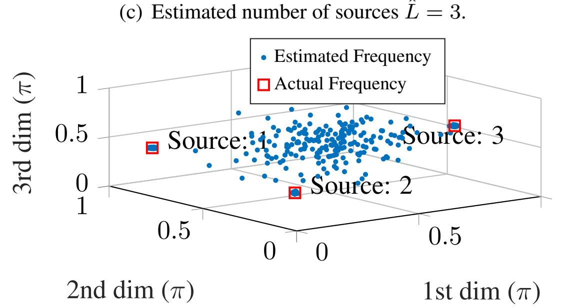
(a) Estimated number of sources $\hat{L} = 1$.



(b) Estimated number of sources $\hat{L} = 2$.



(c) Estimated number of sources $\hat{L} = 3$.



(d) Estimated number of sources $\hat{L} = 4$.

Fig. 4. Parameter association performance for the proposed method with inaccurate source number information.

5. CONCLUSION

A beamspace R-D tensor-ESPRIT algorithm is developed for multidimensional harmonic retrieval. Source parameters estimation and association are achieved simultaneously. Furthermore, the effect of errors in the estimated number of sources is investigated, as well as the applicability for sources with partially distinct frequencies is demonstrated.

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