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Unloaded prestressed shell formed from a closed surface unattached to any supports

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This version includes an addendum made 2019-11-07 following equation (29).

Abstract

In this paper we attempt to begin to answer the question, 'under what conditions can an unloaded shell formed of a closed surface unattached to any supports contain a state of membrane stress which can be induced by prestressing?'. We show that a sphere cannot be prestressed, but a torus can be.

Keywords: Unloaded prestressed shell, torus, force surface, moment surface.

1 Introduction

Shells carry loads through a combination of membrane and bending action [1], and both are required to resist buckling. However membrane action is much more efficient than bending action and so membrane action predominates if the shape of the shell and its supports permit it.

Inextensional deformation of a surface is deformation in which the surface is bent, but lengths on the surface do not change, and membrane action alone cannot resist inextensional deformation if the shape of the shell and its supports permit it.

The topic in differential geometry entitled the *rigidity of surfaces* [2, 3] is concerned with surfaces which can, at least in theory, carry loads by membrane action alone and inextensional deformation is not possible. If inextensional deformation is not possible then membrane action is possible for any load case and vice versa. Note that even if inextensional deformation is possible, then membrane action can still apply as long as the load does not 'activate' the mechanism. The form finding of masonry shells relies very much on this principal.

2 Static – kinematic analogy

The static – kinematic analogy or the static – geometric analogy [1] relates the equations of static equilibrium involving loads and stresses to the kinematic equations relating velocities to strain rates. Here we will limit ourselves to inextensional deformation and membrane action in an unloaded shells.

It may seem odd that we should be concerned with membrane action in an unloaded shell, but imagine that we have a model of an unloaded shell and we poke it with a finger. Ideally it will resist the poke by membrane action wherever we poke it and whatever the direction. The shell is unloaded, except under our finger and therefore what happens in an unloaded shell is of fundamental importance in resisting the load and transferring the force to the supports.

Table 1 lists the kinematic and static variables, and eqs. (1) to (8) give the two sets of analogous equations with kinematic quantities on the left and static quantities on the right. The equations apply to a surface $\mathbf{r} = \mathbf{r}\left(\theta^1, \theta^2\right)$ in which θ^1 and θ^2 are parameters or surface coordinates replacing the u and v which are often used in order to use the tensor notation for geometric, kinematic and static quantities [4]. The vector $\mathbf{n} = \mathbf{r}_{,1} \times \mathbf{r}_{,2}/||\mathbf{r}_{,1} \times \mathbf{r}_{,2}||$ is the standard unit normal of the surface in which the subscript $,\alpha$ means $\partial/\partial\theta^\alpha$.

Table 1: Kinematic and static quantities

Kinematic quantity	**Static quantity**
angular velocity, $\mathbf{w}$	force, $\mathbf{f}$
velocity, $\mathbf{v}$	complementary moment, $\mathbf{h}$
complementary velocity, $\mathbf{y}$	moment, $\mathbf{m}$
rate of bending tensor, $\mathbf{B}$	membrane stress tensor, $\mathbf{S}$

$$\mathbf{v} - \mathbf{y} = \mathbf{w} \times \mathbf{r} \qquad \mathbf{h} - \mathbf{m} = \mathbf{f} \times \mathbf{r} \tag{1}$$

$$d\mathbf{v} = \mathbf{w} \times d\mathbf{r} \qquad d\mathbf{q} = \mathbf{f} \times d\mathbf{r} \tag{2}$$

$$d\mathbf{y} = \mathbf{r} \times d\mathbf{w} \qquad d\mathbf{m} = \mathbf{r} \times d\mathbf{f} \tag{3}$$

$$\delta\mathbf{w} \times d\mathbf{r} = d\mathbf{w} \times \delta\mathbf{r} \qquad \delta\mathbf{f} \times d\mathbf{r} = d\mathbf{f} \times \delta\mathbf{r} \tag{4}$$

$$d\mathbf{w} = (d\mathbf{r} \times \mathbf{n}) \cdot \mathbf{B} \qquad d\mathbf{f} = (d\mathbf{r} \times \mathbf{n}) \cdot \mathbf{S} \tag{5}$$

$$\mathbf{B} = \mathbf{B}^{\mathrm{T}} \qquad \mathbf{S} = \mathbf{S}^{\mathrm{T}} \tag{6}$$

$$\mathbf{b} \cdot \cdot \mathbf{B} = 0 \qquad \mathbf{b} \cdot \cdot \mathbf{S} = 0 \tag{7}$$

$$(d\mathbf{r} \times \mathbf{n}) \cdot \delta\mathbf{B} = (\delta\mathbf{r} \times \mathbf{n}) \cdot d\mathbf{B} \qquad (d\mathbf{r} \times \mathbf{n}) \cdot \delta\mathbf{S} = (\delta\mathbf{r} \times \mathbf{n}) \cdot d\mathbf{S} \tag{8}$$

Thus we have a total of six additional surfaces, $\mathbf{w}$, $\mathbf{v}$, $\mathbf{y}$, $\mathbf{f}$, $\mathbf{h}$ and $\mathbf{m}$ parameterised by θ^1 and θ^2 and two surface tensors, $\mathbf{B}$ and $\mathbf{S}$ which arise because the unit normals to $\mathbf{w}$ and $\mathbf{f}$ are equal to $\mathbf{n}$.

We shall now explain how the equations arise, beginning withe the kinematic relationships.

2.1 Kinematic relationships

Imagine two adjacent points on a moving surface. At a given instant of time $d\mathbf{r}$ is the vector from one point to the other and $d\mathbf{v}$ is the difference in velocities between the two points. If the surface is inextensional then the rate of membrane strain is zero then

$$d\mathbf{v} \cdot d\mathbf{r} = 0 \tag{9}$$

everywhere on the surface, regardless of the direction of $d\mathbf{r}$. Therefore the the difference in velocities must be due solely to an angular velocity $d\mathbf{w}$, leading to the left hand eq. (2), which automatically satisfies eq. (9).

All the other equations follow from this. For example writing the left hand eq. (2) as

$$\mathbf{v}_{,\alpha} = \mathbf{w} \times \mathbf{r}_{,\alpha}, \tag{10}$$

we have

$$\mathbf{v}_{,\alpha\beta} = \mathbf{w}_{,\beta} \times \mathbf{r}_{,\alpha} + \mathbf{w} \times \mathbf{r}_{,\alpha\beta}, \tag{11}$$

so that

$$\mathbf{w}_{,\beta} \times \mathbf{r}_{,\alpha} = \mathbf{w}_{,\alpha} \times \mathbf{r}_{,\beta}. \tag{12}$$

Thus we arrive at the left hand eq. (4) in which $\delta\mathbf{r}$ is the vector from our first point to a third adjacent point.

The left hand eq. (7) is of interest since it tells us that the Gaussian curvature remains constant during in extensional deformation, which of course follows from Gauss' *Theorema Egregium*.

2.2 Static relationships

If an unloaded shell is in static equilibrium, then the total force and total moment about a fixed point must be zero for any region of the surface, see fig. 1. This applies even if there are bending moments in the surface. This gives rise to the surfaces $\mathbf{f}$ and $\mathbf{m}$ in which $d\mathbf{f}$ and $d\mathbf{m}$ are the force and moment crossing $d\mathbf{r}$. In the membrane theory we have the right hand eq. (3), which leads to all the other equations.

The right hand eqs. (7) and (8) are the equilibrium equations in the normal and tangential directions.

2.3 Solution of equations

There are many known solutions to the above equations, particularly for surfaces of revolution, hyperbolic paraboloids and minimal surfaces. However the general case is fraught with difficulties and in the following sections we will consider some approaches.

3 Pin jointed frames

Imagine a triangulated grid shell with a fine grid. It will behave very much in the same way as a continuous shell and in the membrane theory the bars can be pin jointed, although bending stiffness is of course required to resist buckling [5].

An icosahedron has 20 triangular faces, 30 edges and 12 vertices. Thus, if we construct a pin-jointed frame in the form of an icosahedron, the number of bars, $b = 30$, and the number of joints, $j = 12$. Thus $b = 3j - 6$, satisfying Maxwell's rule [6] for a statically determinate structure. The 3 in $3j$ is because we have 3 equations of equilibrium at each node and the -6 is

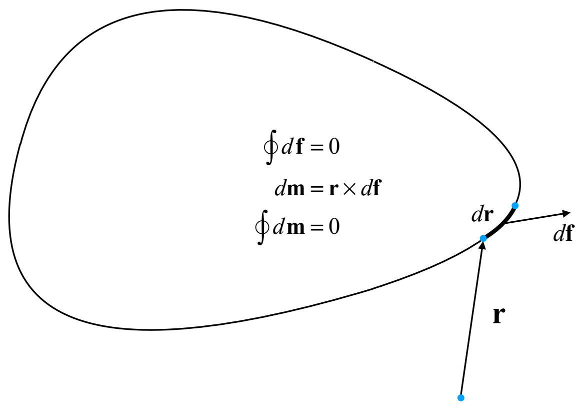

Figure 1: Portion of unloaded shell

because we need 6 restraints to stop an object moving or rotating in 3D. The rule clearly also applies to a tetrahedron for which $b = 6$ and $j = 4$.

If $b < 3j - 6$ we have a mechanism and if $b > 3j - 6$ we have a statically indeterminate structure which can be prestressed by shortening one of the bars.

Thus an icosahedron made from pin-jointed bars which is unloaded and not attached to supports is a rigid structure which cannot be prestressed. The same applies to one of Buckminster Fuller's geodesic domes in which the triangles are subdivided and projected onto a sphere using gnomonic projection since the subdivision adds 3 times as many bars as joints. Thus we would expect a sphere to be rigid and cannot be prestressed which is confirmed by the discussion in section 4.

Now let us apply the same reasoning to a torus. We can cover a torus with a triangular grid with no poles or other extraordinary points in which every joint is connected to 6 bars. Each bar is shared by 2 joints and therefore $b = 3j$. Thus we would expect a closed torus made from pin-jointed bars which is unloaded and not attached to supports to be a rigid structure which can be prestressed in 6 possible ways. However, now let us imagine that the axis of the torus is vertical and that our triangulation is such that we have two circles of horizontal triangles on the top and bottom of the torus, then our structure is a mechanism since there is no way to transmit a vertical force from the inside of the torus to the outside. Thus we have a mechanism, which using the logic in Calladine's classic paper [6], can be prestressed in more than 6 ways.

4 The rigidity of surfaces and the Cohn-Vossen theorem

A fully triangulated gridshell with a fine grid is very much like a continuous shell and as such the tensions or compressions in the bars are essentially the same as membrane stresses.

The Cohn-Vossen theorem [7, 8] tells us that a closed convex surface, that is a surface with positive Gaussian curvature, is rigid. Our argument about horizontal rings in the previous section tells us that a closed torus is not rigid. A torus has positive Gaussian curvature on the outside and negative Gaussian curvature inside. A hyperboloid of one sheet attached to a foundation, like a cooling tower, is a rigid surface in which loads can be transmitted to the ground along the straight line generators. A sphere with a dimple has both positive and negative Gaussian curvature and is not rigid for the same reason as for a torus. However it is not immediately clear whether an hourglass is rigid or not.

It follows that if a sphere or any closed convex surface is unloaded, then there can be no membrane stress in the surface and it cannot be prestressed.

5 Surface of revolution subject to torque about the axis

A general surface of revolution with the z-axis as rotation axis and the profile curve described in the xz-plane can be written

$$\mathbf{r} = r\mathbf{p} + z\hat{\mathbf{k}} \tag{13}$$

$$r = r\left(\theta^2\right) \tag{14}$$

$$z = z\left(\theta^2\right) \tag{15}$$

$$\mathbf{p} = \cos\theta^1\hat{\mathbf{i}} + \sin\theta^1\hat{\mathbf{j}} \tag{16}$$

in which $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are unit vectors in the direction of the Cartesian axes. Writing

$$\mathbf{q} = \frac{d\mathbf{p}}{d\theta^1} = -\sin\theta^1\hat{\mathbf{i}} + \cos\theta^1\hat{\mathbf{j}} \tag{17}$$

we have

$$\mathbf{r}_{,1} = r\mathbf{q} \tag{18}$$

$$\mathbf{r}_{,2} = r'\mathbf{p} + z'\hat{\mathbf{k}}. \tag{19}$$

The simplest state of stress in an unloaded surface of revolution is that corresponding to a pure torque of value T about the axis, in which case there is a membrane shear stress equal to $\frac{T}{2\pi r^2}$ because the circumference is $2\pi r$ and the lever arm is r. Clearly the stress tends to infinity at the axis, but for a torus the stress is finite.

The corresponding membrane stress tensor is

$$\mathbf{S} = \frac{\left(\mathbf{r}_{,1}\mathbf{r}_{,2} + \mathbf{r}_{,2}\mathbf{r}_{,1}\right)}{||\mathbf{r}_{,1}||\,||\mathbf{r}_{,2}||}\frac{T}{2\pi r^2} \tag{20}$$

and so using the equations in section 2,

$$\mathbf{f}_{,1} = -\frac{||\mathbf{r}_{,1}||\,\mathbf{r}_{,2}}{||\mathbf{r}_{,2}||}\cdot\mathbf{S} = -\frac{T}{2\pi r^2}\mathbf{r}_{,1} = -\frac{T}{2\pi r}\mathbf{q} \tag{21}$$

$$\mathbf{f}_{,2} = \frac{||\mathbf{r}_{,2}||\,\mathbf{r}_{,1}}{||\mathbf{r}_{,1}||}\cdot\mathbf{S} = \frac{T}{2\pi r^2}\mathbf{r}_{,2} = \frac{T}{2\pi r^2}\left(r'\mathbf{p} + z'\hat{\mathbf{k}}\right) \tag{22}$$

$$\mathbf{f} = \frac{T}{2\pi}\left(\frac{\mathbf{p}}{r} + \int\frac{z'}{r^2}d\theta^2\hat{\mathbf{k}}\right) \tag{23}$$

$$\mathbf{m}_{,1} = \mathbf{r}\times\mathbf{f}_{,1} = -\frac{T}{2\pi r}\left(r\mathbf{p} + z\hat{\mathbf{k}}\right)\times\mathbf{q} = -\frac{T}{2\pi r}\left(r\hat{\mathbf{k}} - z\mathbf{p}\right) \tag{24}$$

$$\mathbf{m}_{,2} = \mathbf{r}\times\mathbf{f}_{,2} = \frac{T}{2\pi r^2}\left(r\mathbf{p} + z\hat{\mathbf{k}}\right)\times\left(r'\mathbf{p} + z'\hat{\mathbf{k}}\right) = \frac{T}{2\pi r^2}\left(zr' - rz'\right)\mathbf{q} \tag{25}$$

$$\mathbf{m} = -\frac{T}{2\pi}\left(\frac{z}{r}\mathbf{q} + \theta^1\hat{\mathbf{k}}\right). \tag{26}$$

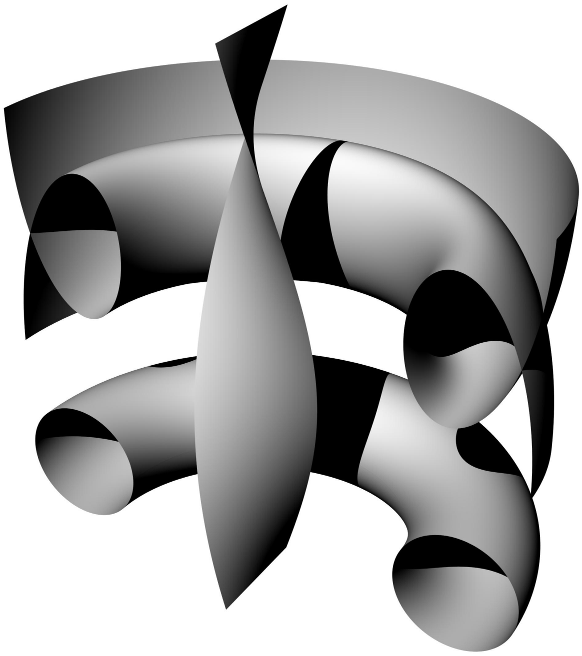

Figure 2: Half of complete torus (bottom) with the **f** (top) and **m** (down the middle) surfaces

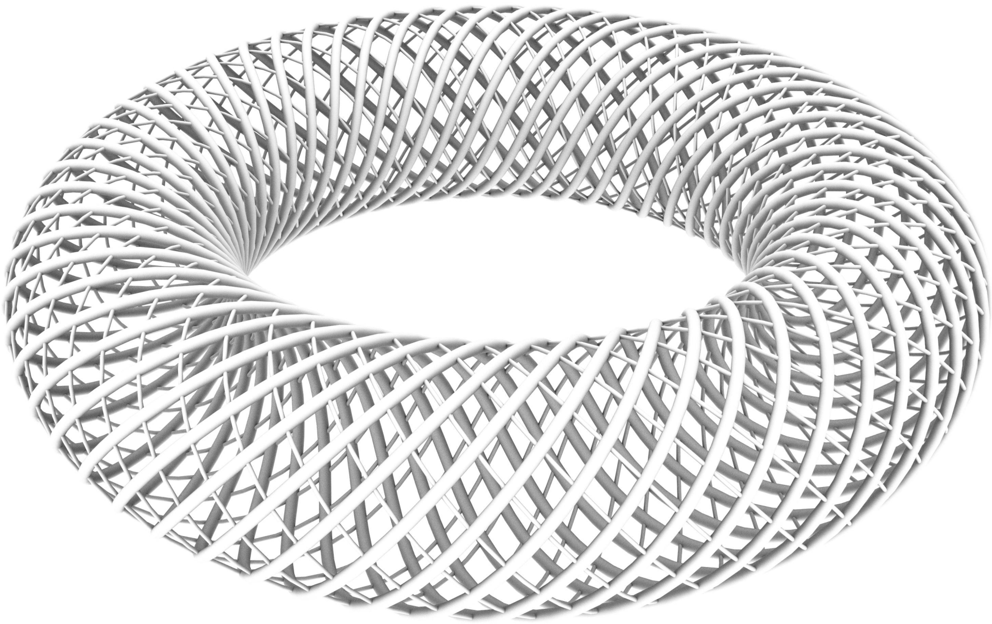

Figure 3: Prestressed torus with the thicker bars in compression and the thinner in tension

5.1 Torus

A torus is described by

$$r = A + B\cos\theta^2 \tag{27}$$

$$z = B\sin\theta^2 \tag{28}$$

in which A and B are constants. The integral

$$\begin{aligned}\int \frac{z'}{r^2} d\theta^2 &= \int \frac{B\cos\theta^2}{\left(A + B\cos\theta^2\right)^2} d\theta^2 \\ &= \frac{1}{B}\frac{\left(C^2-1\right)^2}{2C^3}\left(\left(1-C^2\right)\beta + \left(1+C^2\right)\cos\beta\sin\beta\right)\end{aligned} \tag{29}$$

where

$$C^2 = \frac{A+B}{A-B},$$

$$\tan\left(\frac{\theta^2}{2}\right) = C\tan\beta.$$

The torus together with the **f** and **m** surfaces are plotted in fig. 2. The shear stress can be produced by a combination of tension and compression elements as shown in fig. 3.

Note that if a cut is made through the torus there is a resultant force acting along the axis, very much like in a coil spring, and this explains why the **f** and **m** surfaces are not closed. Of course if we cut the torus in half with 2 cuts, as in fig. 2, there is no resultant force on each half.

6 Conclusion

The starting point for this research was to examine unloaded closed surfaces unattached to any supports which can be prestressed. A sphere cannot be prestressed but a torus can be and we have examined one particular stress state. However there must almost certainly be more, and this is the topic of further research.

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