Quasi-invariant Hermite Polynomials and Lassalle-Nekrasov Correspondence

Downloaded from: https://research.chalmers.se, 2026-09-10 05:00 UTC

Citation for the original published paper (version of record):

Feigin, M., Hallnäs, M., Veselov, A. (2021). Quasi-invariant Hermite Polynomials and Lassalle-Nekrasov Correspondence. Communications in Mathematical Physics, 386(1): 107-141. http://dx.doi.org/10.1007/s00220-021-04036-8

N.B. When citing this work, cite the original published paper.

research.chalmers.se offers the possibility of retrieving research publications produced at Chalmers University of Technology. It covers all kind of research output: articles, dissertations, conference papers, reports etc. since 2004. research.chalmers.se is administrated and maintained by Chalmers Library

(article starts on next page)

Commun. Math. Phys.
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-021-04036-8

Communications in
Mathematical Physics

Quasi–invariant Hermite Polynomials and Lassalle–Nekrasov Correspondence

Misha V. Feigin[1,4], **Martin A. Hallnäs**[2], **Alexander P. Veselov**[3,4,5]

[1] School of Mathematics and Statistics, University of Glasgow, University Place, Glasgow G12 8QQ, UK.
E-mail: misha.feigin@glasgow.ac.uk
[2] Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, 412 96 Gothenburg, Sweden.
E-mail: hallnas@chalmers.se
[3] Department of Mathematical Sciences, Loughborough University, Leicestershire LE11 3TU, UK.
E-mail: A.P.Veselov@lboro.ac.uk
[4] Faculty of Mechanics and Mathematics, Moscow State University, Moscow, Russia
[5] Steklov Mathematical Institute, Moscow, Russia

Received: 25 September 2020 / Accepted: 5 February 2021
© The Author(s) 2021

Abstract: Lassalle and Nekrasov discovered in the 1990s a surprising correspondence between the rational Calogero–Moser system with a harmonic term and its trigonometric version. We present a conceptual explanation of this correspondence using the rational Cherednik algebra and establish its quasi-invariant extension. More specifically, we consider configurations $\mathcal{A}$ of real hyperplanes with multiplicities admitting the rational Baker–Akhiezer function and use this to introduce a new class of non-symmetric polynomials, which we call $\mathcal{A}$-Hermite polynomials. These polynomials form a linear basis in the space of $\mathcal{A}$-quasi-invariants, which is an eigenbasis for the corresponding generalised rational Calogero–Moser operator with harmonic term. In the case of the Coxeter configuration of type A_N this leads to a quasi-invariant version of the Lassalle–Nekrasov correspondence and its higher order analogues.

1. Introduction

In 1971 Calogero [Cal71] studied the quantum system describing N particles on the line pairwise interacting with the rational potential

$$U_R(x_1,\dots,x_N)=\sum_{1\le i<j\le N}\frac{\gamma}{(x_i-x_j)^2}+\omega^2x^2,\quad x^2=\sum_{i=1}^N x_i^2. \tag{1}$$

Almost at the same time Sutherland [Sut71] considered the quantum system of N particles on a circle with trigonometric interaction

$$U_T(x_1,\dots,x_N)=\sum_{1\le i<j\le N}\frac{\gamma a^2}{\sin^2 a(x_i-x_j)}, \tag{2}$$

The work of the first (M.V.F.) and third (A.P.V.) authors (Sects. 4, 6, 7, 8) was supported by the Russian Science Foundation Grant No. 20-11-20214. The second author (M.A.H.) was supported by the Swedish Research Council (Project ID 2018-04291).

Published online: 15 March 2021

which in the limit $a \to 0$ turns into the Calogero system without the harmonic term.

At the classical level both of these systems were studied in 1975 by Moser in a very influential paper [Mos75], who proved their integrability in Liouville sense by finding their Lax representations. The classical system (1) with nonzero ω was studied independently by Adler and Perelomov, see references in [Adl77] and [OP76A]. Olshanetsky and Perelomov made another substantial contribution, in particular, by introducing the generalisations related to arbitrary root systems [OP76A,OP83].

Note that the dynamics in the rational and trigonometric cases are very different. In the rational case (1) the system is isochronous: all solutions are periodic with the same period $T = 2\pi/\omega$, while in the trigonometric case the motion is much more complicated [OP76B].

So it came as a surprise when in 1997 Nekrasov [Nek97] discovered that the systems (both in the classical and the quantum case) are essentially equivalent. More precisely, he showed that there is a symplectomorphism from the phase space of the rational system onto the open (positive) part of the phase space of the trigonometric system. Under this equivalence integrals are mapped to integrals with the Hamiltonian of the rational system being mapped to the momentum of the trigonometric system (which gives another proof of isochronicity of the rational system). To derive this map Nekrasov used the Hamiltonian reduction, following the ideas of the work by Kazhdan, Kostant and Sternberg [KKS78].

This explains an earlier construction of Lassalle [Las91] of multivariable Hermite polynomials from Jack polynomials, which can be interpreted as a correspondence between the eigenfunctions of the two quantum systems with potentials (1) and (2) (see also Baker and Forrester's paper [BF97] which, in particular, contains further unpublished results due to Lassalle). For these reasons we call this equivalence the *Lassalle–Nekrasov correspondence*.

One of the aims of this paper is to give a new explanation of the Lassalle–Nekrasov correspondence in the quantum case using the rational Cherednik algebra and extend it for the special parameter values $\gamma = 2m(m+1)$, $m \in \mathbb{Z}_{\geq 0}$, from the symmetric to the much wider quasi-invariant setting.

We start with a more general quantum system with the Hamiltonian

$$\mathcal{H}_{\mathcal{A}} = -\Delta + \sum_{\alpha \in \mathcal{A}} \frac{m_\alpha (m_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2} + x^2, \tag{3}$$

where $x^2 = \sum_{i=1}^N x_i^2 = (x, x)$ and $\mathcal{A}$ is any configuration of vectors in the Euclidean space $\mathbb{R}^N$ with multiplicities $m_\alpha \in \mathbb{Z}_{\geq 0}$ admitting the so-called rational Baker–Akhiezer (BA) function $\phi(x, \lambda)$. Any Coxeter configuration belongs to this class, but there are also non-symmetric configurations (see [CFV99] and the next section).

Following [CV90,CFV99,FV02] we define an algebra of *quasi-invariant* polynomials $\mathcal{Q}_{\mathcal{A}}$ for any such configuration $\mathcal{A}$. It consists of the polynomials $q(x)$, $x \in \mathbb{R}^N$, which are invariant up to order $2m_\alpha$ with respect to orthogonal reflection about the hyperplane $(\alpha, x) = 0$ for all $\alpha \in \mathcal{A}$, or equivalently, for each $\alpha \in \mathcal{A}$, the odd normal derivatives $\partial_\alpha^{2s-1} q(x) = (\alpha, \frac{\partial}{\partial x})^{2s-1} q(x)$ vanish on the hyperplane $(\alpha, x) = 0$ for $s = 1, 2, \ldots, m_\alpha$.

Our starting observation is that after multiplication by a Gaussian factor the rational BA function can be considered as the generating function of a new interesting class of quasi-invariant polynomials, which we call $\mathcal{A}$-*Hermite polynomials*. In the symmetric case multivariable generalisations of Hermite polynomials were introduced and studied

by Lassalle [Las91], with important subsequent contributions including works by Baker and Forrester [BF97], van Diejen [vDie97] and Rösler [Ros98].

More precisely, choosing a homogeneous basis of quasi-invariants q^i, $i \in \mathbb{Z}_{\geq 0}$, we define the $\mathcal{A}$-Hermite polynomials $H_{q_i}(x)$ by the generating function

$$\phi(x,\lambda)\exp(-\lambda^2/2) = \sum_{i=0}^{\infty} H_{q_i}(x) q^i(\lambda), \quad x,\lambda \in \mathbb{R}^N. \tag{4}$$

The polynomial $H_{q_i}(x)$ is a non-homogeneous polynomial with the highest degree term $q_i(x)$, where collection of polynomials q_j, $j \in \mathbb{Z}_{\geq 0}$, forms the dual basis in the space of quasi-invariants (see details in Sect. 4). Expansion (4) is a far-reaching generalisation of a well-known generating function expansion (see e.g. [Sze39]) for the classical Hermite polynomials:

$$\exp(\lambda x - \lambda^2/2) = \sum_{n=0}^{\infty} H_n(x)\frac{\lambda^n}{n!},$$

corresponding to $m = 0$ and $N = 1$.

More generally, we define the *Hermitisation map* $\chi_H \colon \mathcal{Q}_{\mathcal{A}} \to \mathcal{Q}_{\mathcal{A}}$ sending q to H_q, which can be given explicitly by

$$H_q(x) = (-1)^{\deg q}\exp(x^2/2) L_q \exp(-x^2/2)$$

or, alternatively,

$$H_q(x) = e^{-L/2} q(x),$$

where $L = \Delta - \sum_{\alpha\in\mathcal{A}} \frac{2m_\alpha}{(\alpha,x)}\partial_\alpha$ is a (conjugated) rational Calogero–Moser (CM) Hamiltonian and L_q is its quantum integral with highest order term $q(\partial)$ (see Sect. 4). These relations generalise properties of the classical Hermite polynomials (see formulae (9), (10) below). Lassalle's generalised Hermite polynomials [Las91] correspond to the Coxeter configuration $\mathcal{A}$ of type A_{N-1} and Jack symmetric polynomials q.

The corresponding functions $\Psi_q(x) = A_m(x)^{-1} H_q(x) e^{-x^2/2}$, where $A_m(x) = \prod_{\alpha\in\mathcal{A}}(x,\alpha)^{m_\alpha}$, are (formal) eigenfunctions of the generalised CM Hamiltonian (3) with harmonic term.

In Sect. 5 we study in more detail the case of the Coxeter configuration $\mathcal{A}$ of type A_{N-1} with all roots of multiplicity m. In particular, we extend the Lassalle–Nekrasov correspondence to the quasi-invariants. More precisely, we show that the map χ_H intertwines the action on $\mathcal{Q}_{\mathcal{A}}$ of quantum integrals $\mathcal{L}_p$, $p \in \mathbb{R}[\xi_1,\dots,\xi_N]^{S_N}$, of the trigonometric CM system in exponential coordinates and the corresponding quantum integrals $\mathcal{L}_p^{\mathcal{H}}$ of the rational CM system with harmonic term, so that the diagram

$$\begin{array}{ccc} \mathcal{Q}_{\mathcal{A}} & \xrightarrow{\chi_H} & \mathcal{Q}_{\mathcal{A}} \\ {\scriptstyle \mathcal{L}_p}\downarrow & & \downarrow{\scriptstyle \mathcal{L}_p^H} \\ \mathcal{Q}_{\mathcal{A}} & \xrightarrow{\chi_H} & \mathcal{Q}_{\mathcal{A}} \end{array}$$

is commutative. The original Lassalle–Nekrasov correspondence is the restriction of this construction to the subspace of symmetric polynomials $\mathbb{R}[x_1,\dots,x_N]^{S_N} \subset \mathcal{Q}_{\mathcal{A}}$.

In Sect. 6 we provide a more conceptional explanation of the Lassalle–Nekrasov correspondence by showing that it can be obtained via an automorphism of the corresponding rational Cherednik algebra.

By considering more general automorphisms in Sect. 7, we establish a higher order analogue of the Lassalle–Nekrasov correspondence in which the trigonometric side of the correspondence is unchanged and the momentum operator corresponds to the rational operator $K = E - L_p$, where $E = \sum x_i \partial_i$ is the Euler operator and $p = \sum \xi_i^{l+1}$, $l \in \mathbb{N}$. In particular, the operator K has a complete set of commuting quantum integrals.

Note that in the case $l = 1$, which corresponds to the usual Lassalle–Nekrasov correspondence, the operator K is conjugated to the rational CM operator with harmonic term. In the simplest one-dimensional case with multiplicity 0 and arbitrary l its polynomial eigenfunctions are known as Gould–Hopper polynomials [GH62], which were studied in various relations, in particular, in [Cha11, DLMT96, VL13]. We show that our generalisations of these polynomials satisfy certain bispectrality properties (cf. Duistermaat and Grünbaum [DG86]).

In Sect. 8 we extend the Lassalle–Nekrasov correspondence to the deformed Calogero–Moser systems describing the interaction of two types of particles [CFV98, SV04].

2. Baker–Akhiezer Function Related to Configurations of Hyperplanes

Let $\mathcal{A}$ be a finite collection of non-collinear vectors $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ with multiplicities $m_\alpha \in \mathbb{N}$. Consider the corresponding configuration of hyperplanes with multiplicities (which we also denote as $\mathcal{A}$) defined by

$$A_m(x) := \prod_{\alpha \in \mathcal{A}} (x, \alpha)^{m_\alpha} = 0.$$

The (rational) *Baker–Akhiezer (BA) function* $\phi(x, \lambda), x, \lambda \in \mathbb{C}^N$ associated to $\mathcal{A}$ satisfies the following two conditions [CV90]:

(I) $\phi(x, \lambda)$ is of the form $\phi(x, \lambda) = P(x, \lambda) e^{(x,\lambda)}$ with $P(x, \lambda)$ a polynomial in x with highest degree term equal to $A_m(x) A_m(\lambda)$;

(II) for all $\alpha \in \mathcal{A}$, we have the vanishing conditions

$$\partial_\alpha \phi(x, \lambda) = \partial_\alpha^3 \phi(x, \lambda) = \dots = \partial_\alpha^{2m_\alpha - 1} \phi(x, \lambda) \equiv 0 \ \text{ when } \ (x, \alpha) = 0,$$

where $\partial_\alpha = (\alpha, \partial/\partial x)$ is the normal derivative in direction α.

The existence of ϕ puts a strong restriction on $\mathcal{A}$. The known cases besides the Coxeter arrangements with invariant multiplicities include their deformed versions $\mathcal{A}_{N-1}(p)$ and $\mathcal{C}_N(r, s)$ as well as some special 2-dimensional configurations, see [CFV99, FJ14].

It is known that if the BA function ϕ exists, then it is unique and symmetric with respect to x and λ: $\phi(x, \lambda) = \phi(\lambda, x)$, see [CFV99].

The key fact [CV90, CFV99, VSC93] is that if the configuration $\mathcal{A}$ admits the BA function $\phi(x, \lambda)$, then there is a homomorphism $\chi^{\mathcal{A}}$ mapping a quasi-invariant $q \in \mathcal{Q}_{\mathcal{A}}$ to a differential operator $L_q(x, \frac{\partial}{\partial x})$ such that

$$L_q \phi(x, \lambda) = q(\lambda) \phi(x, \lambda). \tag{5}$$

In particular, ϕ satisfies the eigenvalue equation

$$L\phi = \lambda^2 \phi, \quad \lambda^2 = \sum_{i=1}^N \lambda_i^2, \tag{6}$$

for the CM operator

$$L = \Delta - \sum_{\alpha\in\mathcal{A}} \frac{2m_\alpha}{(\alpha, x)}\partial_\alpha, \quad \Delta = \sum_{i=1}^{N} \partial_i^2. \tag{7}$$

If $\mathcal{A}$ is a Coxeter system, then L is gauge-equivalent to the generalised CM operator from Olshanetsky–Perelomov [OP83] in the special case of integer parameters.

In the general case the BA function $\phi(x, \lambda)$ can be given in terms of the operator L by Berest's formula [Ber98]

$$\phi(x, \lambda) = \left(2^{|m|}|m|!\right)^{-1}\left(L - \lambda^2\right)^{|m|}\left(A_m(x)^2 e^{(\lambda,x)}\right),$$

where $|m| = \sum_{\alpha\in\mathcal{A}} m_\alpha$, see Theorem 3.1 in [CFV99].

In the Coxeter case Etingof and Ginzburg [EG02A] observed that the function $\phi(x, \lambda)$ has the expansion

$$\phi(x, \lambda) = \sum_{i=0}^{\infty} q_i(x)q^i(\lambda), \tag{8}$$

where q^i, $i \in \mathbb{Z}_{\geq 0}$, is a homogeneous basis of quasi-invariants, and q_i is the dual basis with respect to a natural bilinear form $\langle\cdot, \cdot\rangle_m$ (see formulae (27)–(28) in Sect. 4). The same statement and proof hold true in the general case provided that $\phi(0, 0) \neq 0$, which is satisfied in all the known cases (see [FV03,FHV13]).

Multiplication of the left-hand side of formula (8) by the Gaussian factor $\exp(-\lambda^2/2)$ remarkably leads to a very interesting quasi-invariant version of Lassalle's multivariable Hermite polynomials, see formula (4) above.

In the next section we study these polynomials in the simplest one-dimensional case, leaving the general case for Sect. 4.

3. *m*-Hermite Polynomials on the Line

The classical Hermite polynomials $H_n(x)$ (in the "probabilistic" convention) are the monic orthogonal polynomials with Gaussian weight $w(x) = e^{-x^2/2}$ (see e.g. [Sze39]). They satisfy the recurrence relation

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$$

and the differential equation

$$(-\frac{d^2}{dx^2} + x\frac{d}{dx})H_n = nH_n.$$

The corresponding functions $\psi_n(x) = e^{-x^2/4}H_n(x)$ are the eigenfunctions of the quantum harmonic oscillator

$$(-\frac{d^2}{dx^2} + \frac{x^2}{4})\psi_n = (n + \frac{1}{2})\psi_n.$$

The Hermite polynomials can be given by the formulas

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}, \tag{9}$$

$$H_n(x) = e^{-\frac{1}{2}\frac{d^2}{dx^2}} x^n. \tag{10}$$

Here are the first few of them:

$$H_0(x) = 1,\ H_1(x) = x,\ H_2(x) = x^2 - 1,\ H_3(x) = x^3 - 3x,$$
$$H_4(x) = x^4 - 6x^2 + 3,\ H_5(x) = x^5 - 10x^3 + 15x.$$

It will be important for us that Hermite polynomials can also be defined by the following generating series [Sze39]:

$$e^{kx-k^2/2} = \sum_{n=0}^{\infty} H_n(x) \frac{k^n}{n!}. \tag{11}$$

Let us define now their m-version, where $m \in \mathbb{Z}_{\geq 0}$. In our geometric setting they correspond to the configuration $\mathcal{A}$ consisting of the point 0 taken with multiplicity m. Consider the corresponding Schrödinger operator

$$\mathcal{H}_m = -\frac{d^2}{dx^2} + \frac{x^2}{4} + \frac{m(m+1)}{x^2},$$

which is the simplest case of a CM operator with harmonic term. It is well known that it is the result of Darboux transformations applied to the harmonic oscillator $\mathcal{H}_0$ at the first m odd levels.

Namely, let $W_{1,3,\dots,2m-1}$ be the Wronskian of the first m odd Hermite polynomials

$$W_{1,3,\dots,2m-1} = W[H_1(x), H_3(x), \dots, H_{2m-1}(x)].$$

Then it is easy to check that

$$W_{1,3,\dots,2m-1} = W[x, x^3, \dots, x^{2m-1}] = \prod_{j=1}^{m-1} (m-j)!\, 2^{\frac{m(m-1)}{2}} x^{\frac{m(m+1)}{2}}.$$

Introduce the notation

$$W_{1,3,\dots,2m-1}[f] := W[x, x^3, \dots, x^{2m-1}, f] = W[H_1, H_3, \dots, H_{2m-1}, f]$$

and the differential operator D_m by

$$D_m(f) = \frac{W_{1,3,\dots,2m-1}[f e^{x^2/4}]}{W_{1,3,\dots,2m-1}} e^{-x^2/4}.$$

Then one can check that the following intertwining relation holds:

$$(\mathcal{H}_m + m)\, D_m = D_m \mathcal{H}_0. \tag{12}$$

Define now the *m-Hermite polynomials* on the line by

$$H_n^{(m)}(x) = \frac{1}{\prod_{j=1}^{m-1}(m-j)!} (2x)^{-\frac{m(m-1)}{2}} W_{1,3,\dots,2m-1}[H_n]. \tag{13}$$

From the intertwining relation (12) it follows that

$$\psi_n^{(m)} := D_m(\psi_n) = x^{-m} H_n^{(m)} e^{-x^2/4} \tag{14}$$

is an eigenfunction of the CM operator $\mathcal{H}_m$:

$$\mathcal{H}_m \psi_n^{(m)} = (n - m + \frac{1}{2})\psi_n^{(m)}. \tag{15}$$

The operator (7) in this case has the form

$$L_m = \frac{d^2}{dx^2} - \frac{2m}{x}\frac{d}{dx}.$$

The rational BA function, satisfying the equation

$$L_m \phi_m(x, \lambda) = \lambda^2 \phi_m(x, \lambda),$$

can be written as

$$\phi_m(x, \lambda) = \frac{x^m W_{1,3,\dots,2m-1}[e^{\lambda x}]}{W_{1,3,\dots,2m-1}},$$

(see [CFV99]), or, more explicitly,

$$\phi_m(x, \lambda) = (x\frac{d}{dx} - 2m + 1)(x\frac{d}{dx} - 2m + 3)\dots(x\frac{d}{dx} - 1)e^{\lambda x}. \tag{16}$$

Theorem 1. *The m-Hermite polynomials have the generating function*

$$\phi_m(x, k)e^{-k^2/2} = \sum_{n=0}^{\infty} H_n^{(m)}(x)\frac{k^n}{n!}.$$

Proof. Consider the operator Φ_m given by $\Phi_m[f] = \frac{x^m W_{1,3,\dots,2m-1}[f]}{W_{1,3,\dots,2m-1}}$. The statement follows by applying the operator Φ_m to the equality (11). □

More properties of these polynomials are given by the following

Proposition 1. *The m-Hermite polynomials $H_n^{(m)}$ have the following properties:*

1. $H_n^{(m)}(-x) = (-1)^n H_n^{(m)}(x)$,
2. $H_1^{(m)} = H_3^{(m)} = \dots = H_{2m-1}^{(m)} = 0$,
3. $\mathcal{L}_m H_n^{(m)} = n H_n^{(m)}$, *where* $\mathcal{L}_m = -\frac{d^2}{dx^2} + x\frac{d}{dx} + \frac{2m}{x}\frac{d}{dx}$,
4. $H_n^{(m)}(x) = c_{m,n} e^{-L_m/2} x^n$, *where* $c_{m,n} = \prod_{k=1}^{m}(n - 2k + 1)$,
5. $H_n^{(m)} = c_{m,n} p_n^{(m)}(x)$, *where* $p_n^{(m)}(x) \in \mathbb{Z}[x]$ *is a monic polynomial of degree n,*
6. $\langle H_n^{(m)}(x) \colon n \in \mathbb{Z}_{\geq 0}\rangle = Q_m$, *where* $Q_m = \langle 1, x^2, x^4, \dots, x^{2m}, x^{2m+1}, \dots\rangle$ *is the corresponding algebra of quasi-invariants.*

Proof. Properties 1 and 2 follow from the definition of m-Hermite polynomials (13). Property 3 follows from formulas (14), (15) and the relation

$$\mathcal{L}_m = x^m e^{x^2/4}(\mathcal{H}_m + m - \frac{1}{2})x^{-m}e^{-x^2/4}.$$

It follows from more general Propositions 4, 8 below that $H_n^{(m)}$ is proportional to $e^{-L_m/2}x^n$. It is easy to see from Theorem 1 that the coefficient of proportionality equals $c_{m,n}$, which proves property 4.

Consider a monic polynomial $p(x)$ of degree n, it has the form

$$p(x) = \sum_{i=0}^{n} a_i x^{n-i}$$

for some $a_i \in \mathbb{C}$ with $a_0 = 1$. We have to show that the equality

$$\mathcal{L}_m p = np \tag{17}$$

implies that $a_i \in \mathbb{Z}$ for all i. It is easy to see that $a_i = 0$ if i is odd. Also the equality (17) implies the following recurrence relation for any even i, $0 \le i \le n$:

$$(n-i)(n-i-1-2m)a_i = -(i+2)a_{i+2}, \tag{18}$$

where we put $a_{n+1} = a_{n+2} = 0$. By applying relation (18) iteratively we get for $i = 2k$, $k \in \mathbb{Z}$, that

$$\prod_{j=1}^{k}(n-2(j-1)) \prod_{j=1}^{k}(n-2m-1-2(j-1)) = (-1)^k 2^k k! a_{2k}. \tag{19}$$

Depending on the parity of n one of the two products in the left-hand side of equality (19) is the product of k successive even integers. If n is even then the first product divided by $2^k k!$ equals $\binom{n/2}{k}$. If n is odd then the second product divided by $2^k k!$ equals $\binom{(n-2m-1)/2}{k}$, where $\binom{x}{k} := x(x-1)\dots(x-k+1)/k!$ is integer for all integer x. This implies that $a_i \in \mathbb{Z}$ for all i.

As to the final property the quasi-invariance of $H_n^{(m)}$ follows from Theorem 1 and property (II) of the BA function. The statement that these polynomials generate the space of quasi-invariants Q_m follows from property 5. □

We note that integrality property 5 also follows from more general results in [BHSS18].

Remark 1. Arithmetical properties of Hermite polynomials were studied by Schur in remarkable papers [Sch29,Sch31]. In particular, he proved that all polynomials H_{2k} and $x^{-1}H_{2k-1}$ (with the exception of $H_2(x) = x^2 - 1$) are irreducible over $\mathbb{Q}$. It would be interesting to investigate similar problems for our m-Hermite polynomials.

We also note that combinatorial and representation theoretic interpretations of the coefficients of general Wronskians of Hermite polynomials were obtained recently in [BDS20].

Here are the first few examples of m-Hermite polynomials for $m = 1$:

$$H_0^{(1)} = -1,\ H_1^{(1)} = 0,\ H_2^{(1)} = x^2 + 1,\ H_3^{(1)} = 2x^3,\ H_4^{(1)} = 3(x^4 - 2x^2 - 1),$$
$$H_5^{(1)} = 4x^3(x^2 - 5),\ H_6^{(1)} = 5(x^6 - 9x^4 + 9x^2 + 3),\ H_7^{(1)} = 6x^3(x^4 - 14x^2 + 35),$$
$$H_8^{(1)} = 7(x^8 - 20x^6 + 90x^4 - 60x^2 - 15),\ H_9^{(1)} = 8x^3(x^6 - 27x^4 + 189x^2 - 315),$$
$$H_{10}^{(1)} = 9(x^{10} - 35x^8 + 350x^6 - 1050x^4 + 525x^2 + 105).$$

Note that m-Hermite polynomials $H_{2k+1}^{(m)}$, $k \in \mathbb{Z}_{\geq 0}$, are odd and divisible by x^{2m+1}. Indeed, they are quasi-invariant and x^{2m+1} is the odd quasi-invariant of the lowest degree.

Let us define the monic polynomials $E_k^{(m)}(x)$ and $G_k^{(m)}(x)$ of degree k, $k \in \mathbb{Z}_{\geq 0}$, such that

$$c_{m,2k} E_k^{(m)}(x^2) = H_{2k}^{(m)}(x), \quad c_{m,2k+2m+1} x^{2m+1} G_k^{(m)}(x^2) = H_{2k+2m+1}^{(m)}(x).$$

Let us recall that the generalised Laguerre polynomials $L_n^{(\alpha)}$ are polynomials of degree n with the highest order coefficient $(-1)^n/n!$, which satisfy the differential equation

$$z\frac{d^2 L_n^{(\alpha)}(z)}{dz^2} + (\alpha + 1 - z)\frac{dL_n^{(\alpha)}(z)}{dz} + nL_n^{(\alpha)}(z) = 0. \tag{20}$$

We have the following relation of m-Hermite polynomials with the generalised Laguerre polynomials for special values of the parameter α (cf. [BDS20] where general Wronskians of Hermite polynomials were expressed via Wronskians of Laguerre polynomials).

Proposition 2. *We have*

$$E_n^{(m)}(x) = (-1)^n n! L_n^{(\alpha)}(\frac{1}{2}x^2), \quad \alpha = -m - \frac{1}{2},$$

and

$$G_n^{(m)}(x) = (-1)^n n! L_n^{(\alpha)}(\frac{1}{2}x^2), \quad \alpha = m + \frac{1}{2},$$

for any $n \in \mathbb{Z}_{\geq 0}$.

Proof. Let us do the change of variables $z = \frac{1}{2}x^2$ in equation (20). We get the following equation:

$$\frac{d^2 L_n^{(\alpha)}(x^2)}{dx^2} + \frac{2\alpha + 1}{x}\frac{dL_n^{(\alpha)}(x^2)}{dx} - x\frac{dL_n^{(\alpha)}(x^2)}{dx} + 2nL_n^{(\alpha)}(x^2) = 0. \tag{21}$$

which coincides with the equation in property 3 of Proposition 1 for $H_{2n}(x)$ when $\alpha = -m - \frac{1}{2}$.

Let us conjugate the equation (21) by $x^{2\alpha}$. It follows that $f = x^{2\alpha} L_n^{(\alpha)}(x^2)$ satisfies the equation

$$\frac{d^2 f}{dx^2} + \frac{1 - 2\alpha}{x}\frac{df}{dx} - x\frac{df}{dx} + (2\alpha + 2n)f = 0,$$

which coincides with the equation in property 3 of Proposition 1 for $H_{2n+2m+1}(x)$ when $\alpha = m + \frac{1}{2}$. □

Recall that generalised Laguerre polynomials are orthogonal with respect to the bilinear form

$$(f, g) = \int_0^\infty x^\alpha f(x)g(x)e^{-x}dx.$$

Note that the integral in general diverges for $\alpha \leq -1$ but it can be regularised using complex contour integrals. A related problem for the corresponding m-Hermite polynomials was discussed in a wider context of exceptional Hermite polynomials in [HHV16].

4. $\mathcal{A}$-Hermite Polynomials: Definition and Properties

Let us assume now that the configuration $\mathcal{A}$ is such that the rational BA function $\phi(x, \lambda)$ exists and satisfies the condition $\phi(0, 0) \neq 0$. This function is both symmetric and homogeneous in the sense that

$$\phi(x, \lambda) = \phi(\lambda, x) \tag{22}$$

and

$$\phi(tx, \lambda) = \phi(x, t\lambda), \quad t \in \mathbb{C}, \tag{23}$$

respectively. Symmetry follows from Theorem 2.3 in [CFV99], whereas homogeneity is a simple consequence of the defining conditions and uniqueness.

It also satisfies the following integral identity

$$\int_{i\xi+\mathbb{R}^N} \frac{\phi(-iz, \lambda)\phi(iz, \mu)}{A_m(z)^2} d\gamma(z) = e^{-(\lambda^2+\mu^2)/2}\phi(\lambda, \mu), \tag{24}$$

where $d\gamma(z) = (2\pi)^{-N/2}e^{-z^2/2}dz$, and $\xi \in \mathbb{R}^N$ is such that $A_m(\xi) \neq 0$, which was established in [FHV13].

For each homogeneous $q \in \mathcal{Q}_\mathcal{A}$, there exists a unique homogeneous differential operator

$$L_q = q(\partial_1, \ldots, \partial_N) + \text{l.o.t.}$$

of homogeneity degree $-\deg q$ such that $[L_q, L] = 0$. Such operators form a commutative ring isomorphic to $\mathcal{Q}_\mathcal{A}$, with L corresponding to x^2 and the ring isomorphism given by

$$L_q\phi(x, \lambda) = q(\lambda)\phi(x, \lambda). \tag{25}$$

Moreover, as it follows from [Ber98,Cha98,CFV99],

$$L_q\mathcal{Q}_\mathcal{A} \subset \mathcal{Q}_\mathcal{A}, \quad \forall q \in \mathcal{Q}_\mathcal{A}. \tag{26}$$

Attached to $\mathcal{Q}_\mathcal{A}$ is a natural bilinear form, given by

$$(p, q)_m = (L_p q)(0), \quad p, q \in \mathcal{Q}_\mathcal{A}, \tag{27}$$

which can be seen to be symmetric [EG02A,FV03]. Then Theorem 3.1 in [FHV13] yields the integral representation

$$(p, q)_m = \phi(0, 0) \int_{i\xi+\mathbb{R}^N} \frac{(e^{L/2}p)(-ix)(e^{L/2}q)(ix)}{A_m(x)^2} d\gamma(x).$$

To proceed further, we find it convenient to renormalise the bilinear form $(\cdot,\cdot)_m$: we set

$$\langle p, q\rangle_m = \phi(0,0)^{-1}(p,q)_m, \quad p, q \in \mathcal{Q}_{\mathcal{A}}. \tag{28}$$

Choosing a homogeneous basis q_i, $i \in \mathbb{Z}_{\geq 0}$, in $\mathcal{Q}_{\mathcal{A}}$, the defining condition (II) for ϕ entails that

$$\phi(x,\lambda) = \sum_{i=0}^{\infty} q_i(x) q^i(\lambda) \tag{29}$$

for some polynomials q^i. By symmetry (22) and homogeneity (23), we have $q^i \in \mathcal{Q}_{\mathcal{A}}$ and $\deg q^i = \deg q_i$. Etingof and Ginzburg [EG02A] showed that the quasi-invariants q^i, $i \in \mathbb{Z}_{\geq 0}$, constitute a dual basis in $\mathcal{Q}_{\mathcal{A}}$:

$$\langle q_i, q^j\rangle_m = \delta_{ij}. \tag{30}$$

(Strictly speaking, they considered only the Coxeter case but their arguments work in our more general case as well, see also [FV03].)

Let us introduce now the analogues of the Hermite polynomials in the algebra of quasi-invariants $\mathcal{Q}_{\mathcal{A}}$. Let

$$F(x,\lambda) = \phi(x,\lambda)\exp(-\lambda^2/2). \tag{31}$$

We define a linear "Hermitisation" map $\chi_H : \mathcal{Q}_{\mathcal{A}} \to \mathcal{Q}_{\mathcal{A}}, q \mapsto H_q$, by

$$H_q(x) = \langle F(x,\cdot), q(\cdot)\rangle_m, \quad q \in \mathcal{Q}_{\mathcal{A}}. \tag{32}$$

Polynomials $H_q(x)$ for homogeneous $q \in \mathcal{Q}_{\mathcal{A}}$ are called *$\mathcal{A}$-Hermite polynomials*. For a quasi-invariant polynomial q of degree d the polynomial $H_q(x)$ has the form

$$H_q(x) = q(x) + \sum_{n=1}^{\lfloor d/2\rfloor} H_q^{(d-2n)}(x), \tag{33}$$

with $H_q^{(d')} \in \mathcal{Q}_{\mathcal{A}}[d']$, where $\mathcal{Q}_{\mathcal{A}}[d']$ denotes the space of homogeneous quasi-invariants of degree d'.

For the basis q_i of quasi-invariants $\mathcal{Q}_{\mathcal{A}}$ we can define the corresponding $\mathcal{A}$-Hermite polynomials by the generating function expansion

$$F(x,\lambda) = \sum_{i=0}^{\infty} H_{q_i}(x) q^i(\lambda). \tag{34}$$

Proposition 3. *The $\mathcal{A}$-Hermite polynomials $H_{q_i}(x)$ form a basis in the space of quasi-invariants $\mathcal{Q}_{\mathcal{A}}$.*

Proof. Quasi-invariance of the polynomials $H_{q_i}(x)$ follows from formulae (31), (34) and property (II) of the BA function. It is also clear from formula (33) that these polynomials are linearly independent and span the space $\mathcal{Q}_{\mathcal{A}}$. □

Now we establish some properties of $\mathcal{A}$-Hermite polynomials. Let us introduce the Euler operator

$$E = E_x = \sum_{i=1}^{N} x_i \partial_i$$

and the following "harmonic" version of the operator (7):

$$\mathcal{L}^{\mathcal{H}} = L - E = \Delta - \sum_{\alpha \in \mathcal{A}} \frac{2m_\alpha}{(\alpha, x)} \partial_\alpha - \sum_{i=1}^{N} x_i \partial_i. \tag{35}$$

Proposition 4. *For $q \in \mathcal{Q}_{\mathcal{A}}[d]$, $H_q(x)$ satisfies the differential equation*

$$\mathcal{L}^{\mathcal{H}} H_q = -d H_q. \tag{36}$$

Proof. Taking $t \to 0$ in the identity $(\phi(tx, \lambda) - \phi(x, t\lambda))/t = 0$ (see (23)), we obtain

$$E_x \phi(x, \lambda) - E_\lambda \phi(x, \lambda) = 0.$$

When combining this identity with equation (6), we readily find that $F(x, \lambda)$ satisfies the differential equation

$$\mathcal{L}^{\mathcal{H}}_x F = -E_\lambda F.$$

Note that the operator E is self-adjoint with respect to $\langle \cdot, \cdot \rangle_m$ since homogeneous quasi-invariants of different degrees are orthogonal. Hence, for any $q \in \mathcal{Q}_{\mathcal{A}}$, we get

$$\big(\mathcal{L}^{\mathcal{H}} \chi_H\big)(q) = \big\langle \mathcal{L}^{\mathcal{H}}_x F(x, \cdot), q(\cdot) \big\rangle_m = -\langle F(x, \cdot), Eq(\cdot) \rangle_m = -(\chi_H E)(q),$$

which implies the statement. □

Proposition 5. *For $q \in \mathcal{Q}_{\mathcal{A}}[d]$, $H_q(x)$ is the unique quasi-invariant polynomial of the form (33) that satisfies the differential equation (36).*

Proof. Let us treat (33) as an ansatz. From [Cha98, CFV99] we derive that

$$L(\mathcal{Q}_{\mathcal{A}}[d]) \subset \mathcal{Q}_{\mathcal{A}}[d-2], \quad d \geq 2.$$

Substituting the right-hand side of (33) for H_q in (36), and using

$$E H_q^{(d-2n)} = (d - 2n) H_q^{(d-2n)},$$

we arrive at the recurrence relation

$$2n H_q^{(d-2n)} = -L H_q^{(d-2n+2)}, \quad n = 1, \ldots \lfloor d/2 \rfloor,$$

which implies uniqueness of the solution $H_q(x)$ such that $H_q^{(d)} = q$. □

The next proposition details an integral representation for H_q, obtained as a straightforward consequence of formula (31) and the integral identity (24).

Proposition 6. *Let $q \in \mathcal{Q}_{\mathcal{A}}[d]$. Then we have*

$$H_q(x) = \exp(x^2/2) \int_{i\xi+\mathbb{R}^N} \frac{q(-iz)\phi(iz,x)}{A_m(z)^2} d\gamma(z), \tag{37}$$

where $d\gamma(z) = (2\pi)^{-N/2} e^{-z^2/2} dz$ and $\xi \in \mathbb{R}^N$ satisfies $A_m(\xi) \neq 0$ as before.

Proof. By linearity in q, it suffices to prove (37) for $q = q_i$, $i \in \mathbb{Z}_{\geq 0}$. Multiplying (24) by $\exp(\mu^2/2)$ and taking $\mu \to x$, we obtain

$$F(x,\lambda) = \exp(x^2/2) \int_{i\xi+\mathbb{R}^N} \frac{\phi(-iz,\lambda)\phi(iz,x)}{A_m(z)^2} d\gamma(z). \tag{38}$$

Due to formula (29) we have

$$\phi(-iz,\lambda) = \sum_{j=0}^{\infty} q_j(-iz) q^j(\lambda).$$

By substituting this expansion into formula (38) and comparing it with the expansion (34) for $F(x,\lambda)$ we get the claim. □

Proposition 7. *For $q \in \mathcal{Q}_{\mathcal{A}}[d]$, we have*

$$H_q(x) = (-1)^d \exp(x^2/2) L_q \exp(-x^2/2).$$

Proof. By using formula (25) and the fact that $q \in \mathcal{Q}_{\mathcal{A}}[d]$, we rewrite (37) as

$$H_q(x) = (-1)^d \exp(x^2/2) L_q \int_{i\xi+\mathbb{R}^N} \frac{\phi(iz,x)}{A_m(z)^2} d\gamma(z).$$

From (29) it is evident that $\phi(x,0) = \phi(0,\lambda) = \phi(0,0)$. Setting $\lambda = 0$ in the identity (24), we thus find that

$$\int_{i\xi+\mathbb{R}^N} \frac{\phi(iz,x)}{A_m(z)^2} d\gamma(z) = \exp(-x^2/2),$$

and the claim follows. □

The polynomials H_q can also be written in the following form (cf. Theorem 3.4 in Rösler [Ros98]).

Proposition 8. *For $q \in \mathcal{Q}_{\mathcal{A}}[d]$, we have*

$$H_q(x) = e^{-L/2} q.$$

Proof. Note that $[E, L] = -2L$, which implies $[E, e^{-L/2}] = L e^{-L/2}$. Therefore

$$\mathcal{L}^{\mathcal{H}} e^{-L/2} q = (L - E) e^{-L/2} q = -e^{-L/2} E q = -d(e^{-L/2} q),$$

and the statement follows by Proposition 5. □

As the following theorem demonstrates, the Hermitisation map χ_H becomes a unitary map when the domain has the bilinear form $\langle \cdot, \cdot \rangle_m$ and the codomain is equipped with the bilinear form

$$\{p, q\}_m \equiv \int_{i\xi+\mathbb{R}^N} \frac{p(z)q(z)}{A_m(z)^2} d\gamma(z), \quad p, q \in \mathcal{Q}_{\mathcal{A}}.$$

Theorem 2. *For $p, q \in \mathcal{Q}_{\mathcal{A}}$, we have*

$$\{H_p, H_q\}_m = \langle p, q \rangle_m.$$

Proof. By linearity, we may and shall assume $p \in \mathcal{Q}_{\mathcal{A}}[d]$ and $q \in \mathcal{Q}_{\mathcal{A}}[d']$ for some $d, d' \in \mathbb{Z}_{\geq 0}$. From the integral identity (24), and using property (23) with $t = \pm i$, we infer

$$\int_{i\xi+\mathbb{R}^N} \frac{F(z, -i\lambda)F(z, i\mu)}{A_m(z)^2} d\gamma(z) = \phi(\lambda, \mu). \tag{39}$$

Note that

$$\langle p(v), F(z, \pm i v) \rangle_m = (\pm i)^d \langle p(v), F(z, v) \rangle_m. \tag{40}$$

Therefore by applying $\langle p, \cdot \rangle_m$ in λ to the equality (39), it follows from fornulas (32) and (5) that

$$(-i)^d \int_{i\xi+\mathbb{R}^N} \frac{H_p(z)F(z, i\mu)}{A_m(z)^2} d\gamma(z) = p(\mu).$$

The claim now follows by applying $\langle q, \cdot \rangle_m$ in μ to this equality and using again formulae (32) and (40). □

5. The Lassalle–Nekrasov Correspondence and Multivariable m-Hermite Polynomials

In this section, we consider the case of the Coxeter configuration $\mathcal{A}$ of type A_{N-1} in which $\mathcal{A}$ is the set of positive roots $A_{N-1+} = \{e_i - e_j : 1 \leq i < j \leq N\} \subset \mathbb{R}^N$ with all the roots taken with the same multiplicity $m \in \mathbb{Z}_{\geq 0}$. We will call the corresponding $\mathcal{A}$-Hermite polynomials *multivariable m-Hermite polynomials*.

The one-dimensional m-Hermite polynomials from Sect. 3 correspond to a different embedding of the root system $A_1 \subset \mathbb{R}$. Moreover, as explained in further detail at the end of this section, Lassalle's multivariable Hermite polynomials [Las91] with parameter $\alpha = -1/m$ can be viewed as special instances of multivariable m-Hermite polynomials.

Throughout this section, we use the standard notation

$$\Lambda_N = \mathbb{C}[x_1, \ldots, x_N]^{S_N}$$

for the algebra of symmetric polynomials in N variables $x_1, \ldots, x_N$.

In the present case, the operator (35) takes the form

$$\mathcal{L}^{\mathcal{H}} = \Delta - \sum_{i<j}^{N} \frac{2m}{(x_i - x_j)}(\partial_i - \partial_j) - \sum_{i=1}^{N} x_i \partial_i \tag{41}$$

and is conjugated to the N-particle CM operator

$$L^{\mathcal{H}} = \Delta - \sum_{i<j}^{N} \frac{2m(m+1)}{(x_i - x_j)^2} - \frac{1}{4}\sum_{i=1}^{N} x_i^2,$$
$$\mathcal{L}^{\mathcal{H}} = \widehat{\Psi}_0^{-1}(L^{\mathcal{H}} - \frac{1}{2}mN(N-1) + \frac{N}{2})\widehat{\Psi}_0,$$

where $\widehat{\Psi}_0$ is the operator of multiplication by $\Psi_0 = \prod_{i<j}^{N}(x_i - x_j)^{-m}\exp(-x^2/4)$.

It can be included in a commutative ring of differential operators $\mathcal{L}_p^{\mathcal{H}}$, $p \in \Lambda_N$, with highest order terms $p(\partial_1^2, \dots, \partial_N^2)$ (see [OP83,Poly92]), given explicitly by formula (48) below.

Lassalle [Las91] (in the quantum case) and Nekrasov [Nek97] (both in the classical and quantum cases) discovered a remarkable correspondence between this system and Sutherland's version of the CM system with the Hamiltonian

$$H_N = -\Delta + \sum_{i<j}^{N} \frac{2m(m+1)}{\sin^2(z_i - z_j)}.$$

It has an eigenfunction $\Phi_0 = \prod_{i<j}^{N}\sin^{-m}(z_i - z_j)$ with eigenvalue $\lambda_0 = m^2N(N^2-1)/3$. Its gauged version $\frac{1}{4}\widehat{\Phi}_0^{-1}(H_N - \lambda_0)\widehat{\Phi}_0$, written in the exponential coordinates $x_j = e^{2iz_j}$, takes the form

$$\mathcal{H}_N = \sum_{i=1}^{N}(x_i\partial_i)^2 - m\sum_{i<j}^{N}\frac{x_i + x_j}{x_i - x_j}(x_i\partial_i - x_j\partial_j). \tag{42}$$

To construct the corresponding quantum integrals, Heckman [Hec91B] introduced the following version of the Dunkl operators:

$$\mathcal{D}_i = x_i\partial_i - \frac{m}{2}\sum_{j\neq i}^{N}\frac{x_i + x_j}{x_i - x_j}(1 - s_{ij}), \quad i = 1, \dots, N,$$

where s_{ij} is the transposition acting on functions by permuting the coordinates x_i and x_j. They are related to the original Dunkl operators [Dun89]

$$D_i = \partial_i + m\sum_{j\neq i}\frac{1}{x_i - x_j}(s_{ij} - 1)$$

by

$$\mathcal{D}_i = x_i D_i - \frac{m}{2}\sum_{j\neq i}(s_{ij} - 1).$$

Heckman's operators $\mathcal{D}_i$ do not commute, but the differential operators

$$\mathbb{L}_k = \mathrm{Res}\,(\mathcal{D}_1^k + \dots + \mathcal{D}_N^k), \tag{43}$$

where Res means the restriction to the space of symmetric polynomials Λ_N, do commute with each other. Since $\mathbb{L}_2 = \mathcal{H}_N$ they are integrals of the quantum trigonometric CM system (42).

For our purposes, it is convenient to replace Heckman's operators $\mathcal{D}_i$ by Polychronakos' operators [Poly92]

$$\pi_i = x_i D_i, \quad i = 1, \dots, N.$$

The corresponding differential operators

$$\mathcal{L}_k = \mathrm{Res}\,(\pi_1^k + \cdots + \pi_N^k) \tag{44}$$

pair-wise commute and $\mathcal{L}_2 = \mathcal{H}_N$, (see [SV15, Proposition 5.2] for the precise relation between the two sets of quantum integrals (43) and (44)). Note that

$$\mathcal{L}_1 = \mathbb{L}_1 = x_1\partial_1 + \cdots + x_N\partial_N$$

is simply the total momentum of the system (in the exponential coordinates).

More generally, for every $p \in \Lambda_N$, there exists a unique S_N-invariant differential operator $\mathcal{L}_p$ such that

$$\mathcal{L}_p q = p(\pi_1, \dots, \pi_N)q, \quad \forall q \in \Lambda_N,$$

which commutes with all other such operators $\mathcal{L}_r$, $r \in \Lambda_N$. The operator (44) corresponds to the Newton power sum $p_k(\xi) = \xi_1^k + \cdots + \xi_N^k$.

Consider now the algebra of quasi-invariants $\mathcal{Q}_{\mathcal{A}}$, which we in this case will denote by $\mathcal{Q}_m$. It consists of the polynomials $p(\xi_1, \dots, \xi_N)$ which are invariant under permutations s_{ij} of ξ_i and ξ_j up to order $2m$:

$$p(\xi) - p(s_{ij}\xi) \equiv 0 \mod (\xi_i - \xi_j)^{2m+1}.$$

The symmetric algebra Λ_N is a subalgebra of the algebra $\mathcal{Q}_m$.

First we show that the algebra $\mathcal{Q}_m$ is preserved by the operators $\mathcal{L}_p$.

Proposition 9. *For each $p \in \Lambda_N$, we have*

$$\mathcal{L}_p \mathcal{Q}_m \subset \mathcal{Q}_m.$$

Proof. Let H_m be the rational Cherednik algebra corresponding to the symmetric group S_N and parameters $t = 1, c = m$ [EG02B]. It is generated by operators D_i, x_i, and s_{ij}.

Consider the spherical subalgebra $eH_me \subset H_m$, where $e = \frac{1}{N!}\sum_{w\in S_N} w$. One can identify operators $p(\pi_1, \dots, \pi_N)$, $p \in \Lambda_N$, with elements

$$m_e(p) := p(\pi_1, \dots, \pi_N)e \in eH_me.$$

Since, up to analogous identifications, $\mathbb{C}[x_1, \dots, x_N]^{S_N}$ and $\mathbb{C}[D_1, \dots, D_N]^{S_N}$ generate eH_me [BEG02], we can write $p(\pi_1, \dots, \pi_N)$ as a polynomial in (non-commuting) operators of multiplication by $q(x_1, \dots, x_N)$ and $q(D_1, \dots, D_N)$ with $q \in \Lambda_N$. Therefore the differential operator $\mathcal{L}_p$ can be expressed as a polynomial in $q(x)$ and L_q. The statement follows from formula (26). □

To proceed further, we need the following lemma (cf. [DJO94] where a related property was studied).

Lemma 1. *The restriction of* $(\cdot,\cdot)_m$ *to* Λ_N *is non-degenerate.*

Proof. Let

$$\mathrm{Ker}(m) = \{p \in \Lambda_N : (p,q)_m = 0,\ \forall q \in \Lambda_N\}.$$

For $p \in \mathrm{Ker}(m)$ and $q, q' \in \Lambda_N$, we have

$$(L_q p, q')_m = (p, qq')_m = 0.$$

Hence $\mathrm{Ker}(m)$ is a graded ideal in Λ_N which is invariant under the operators L_q. Assuming $\mathrm{Ker}(m) \neq \{0\}$, we choose a non-zero homogenous $p \in \mathrm{Ker}(m)$ of minimal degree. By the invariance of $\mathrm{Ker}(m)$ and minimality of p,

$$L_q p = 0, \quad \forall q \in \Lambda_N,$$

that is p is m-harmonic in the terminology of [FV02]. Since the subspace of S_N-invariant m-harmonic polynomials coincides with $\mathbb{C}$ we must have $p \in \mathbb{C}$. Then $(p,p)_m = p^2$, which implies $p = 0$, and the statement follows. □

This enables us to establish self-adjointness of the operators $\mathcal{L}_p$ with respect to the bilinear form $(\cdot,\cdot)_m$.

Proposition 10. *For each* $p \in \Lambda_N$, *we have*

$$(\mathcal{L}_p q, q')_m = (q, \mathcal{L}_p q')_m, \quad q, q' \in \mathcal{Q}_m. \tag{45}$$

Proof. We recall that any differential operator in variables $x_1, \dots, x_N$ that has rational coefficients with poles located only along the hyperplanes $x_i = x_j$, $1 \le i < j \le N$, is completely determined by its action on Λ_N, see e.g. Lemma 2.8 in [Opd95]. Both $\mathcal{L}_p$ and its dual $\mathcal{L}_p^*$ are such differential operators. This is immediate for $\mathcal{L}_p$, and it follows for $\mathcal{L}_p^*$ from the proof of Proposition 9 and the fact that $L_q^* = q, q^* = L_q$ for all $q \in \mathcal{Q}_m$. By Lemma 1, it thus suffices to prove (45) for $q, q' \in \Lambda_N$.

To this end, we recall the bilinear form [DJO94]

$$[p,q]_m := \big(p(D_1, \dots, D_N)q\big)(0), \quad p, q \in \mathbb{C}[x_1, \dots, x_N],$$

and its properties

$$[x_j p, q]_m = [p, D_j q]_m, \quad [D_j p, q]_m = [p, x_j q]_m, \tag{46}$$

for all $j = 1, \dots, N$. It follows from relations (46) that

$$[\pi_j p, q]_m = [p, \pi_j q]_m. \tag{47}$$

Now let $q, q' \in \Lambda_N$. We deduce by the property (47) that

$$(\mathcal{L}_p q, q')_m = \big[p(\pi_1, \dots, \pi_N)q, q'\big]_m = \big[q, p(\pi_1, \dots, \pi_N)q'\big]_m = (q, \mathcal{L}_p q')_m,$$

and the claim follows. □

As a simple consequence, we can now prove that the BA function $\phi(x, \lambda)$ satisfies the following remarkable symmetry property.

Theorem 3. *For any $p \in \Lambda_N$, we have*

$$\mathcal{L}_{p,x}\phi(x,\lambda) = \mathcal{L}_{p,\lambda}\phi(x,\lambda),$$

where notations $\mathcal{L}_{p,x}$ and $\mathcal{L}_{p,\lambda}$ stand for the operator $\mathcal{L}_p$ acting in the variables $x = (x_1,\dots,x_N)$ and $\lambda = (\lambda_1,\dots,\lambda_N)$, respectively.

Proof. Let q_i and q^i be dual bases of quasi-invariants $\mathcal{Q}_m$ so that formulas (29)–(30) hold. Defining

$$a_i^j := \langle \mathcal{L}_p q_i, q^j\rangle_m, \quad b_i^j := \langle q_i, \mathcal{L}_p q^j\rangle_m,$$

we have by Proposition 10 and formula (28) that $a_i^j = b_i^j$ for all $i, j \in \mathbb{Z}_{\geq 0}$. It follows that

$$\mathcal{L}_{p,x}\phi(x,\lambda) = \sum_i q^i(\lambda)\sum_j a_i^j q_j(x) = \sum_j q_j(x)\sum_i b_i^j q^i(\lambda) = \mathcal{L}_{p,\lambda}\phi(x,\lambda).$$

□

For $p \in \Lambda_N$, let $\mathcal{L}_p = \mathcal{L}_{p,x}$, and let L be given by (7), which in this case is

$$L = \Delta - \sum_{i<j}\frac{2m}{x_i - x_j}(\partial_i - \partial_j).$$

Following Baker and Forrester [BF97], we consider the differential operators

$$\mathcal{L}_p^{\mathcal{H}} = \mathcal{L}_p + \frac{1}{2}[\mathcal{L}_p, L] + \frac{1}{2^2 2!}[[\mathcal{L}_p, L], L] + \dots + \frac{1}{2^d d!}[\dots[\mathcal{L}_p, L], \dots, L], \quad (48)$$

where the last term contains $d = \deg p$ commutators.

The operators $\mathcal{L}_p^{\mathcal{H}}$ have order $2\deg p$ and the constant coefficient highest order terms $(-1)^d p(\partial_1^2,\dots,\partial_N^2)$. In particular, when $p(\xi) = \xi_1 + \dots + \xi_N$ the corresponding operator

$$\mathcal{L}_p^{\mathcal{H}} = \sum_{i=1}^N x_i\partial_i - L = -\mathcal{L}^{\mathcal{H}}$$

coincides up to a sign with the CM operator (41). As we will see below, these operators commute and thus are quantum integrals of the CM system with harmonic term. This statement can also be extracted from the work of Baker and Forrester [BF97].

Let $\chi_H : \mathcal{Q}_m \to \mathcal{Q}_m$ be the Hermitisation map defined by (32), and $\mathcal{L}_p, \mathcal{L}_p^{\mathcal{H}}$ the operators defined above. Note that both of them preserve the algebra of quasi-invariants due to Proposition 9 and the fact that L preserves $\mathcal{Q}_m$.

We are now ready to state and prove our generalisation of the Lassalle-Nekrasov correspondence to the algebra of quasi-invariants $\mathcal{Q}_m$.

Theorem 4. *The Hermitisation map χ_H intertwines the trigonometric CM operator and the rational CM operator in an harmonic field and their quantum integrals. More precisely, the diagram*

$$\begin{array}{ccc} \mathcal{Q}_m & \xrightarrow{\chi_H} & \mathcal{Q}_m \\ {\scriptstyle \mathcal{L}_p}\downarrow & & \downarrow{\scriptstyle \mathcal{L}_p^{\mathcal{H}}} \\ \mathcal{Q}_m & \xrightarrow{\chi_H} & \mathcal{Q}_m \end{array}$$

is commutative for all $p \in \Lambda_N$, that is

$$\mathcal{L}_p^{\mathcal{H}} \circ \chi_H = \chi_H \circ \mathcal{L}_p.$$

Proof. As before we will use subscript x or λ to indicate the variables the operators act in. Let us start with the following conjugation relation

$$e^{\lambda^2/2}\mathcal{L}_{p,\lambda}e^{-\lambda^2/2} = \mathcal{L}_{p,\lambda} + \frac{1}{2}[\lambda^2, \mathcal{L}_{p,\lambda}] + \frac{1}{2^2 2!}[\lambda^2, [\lambda^2, \mathcal{L}_{p,\lambda}]] + \cdots + \frac{1}{2^d d!}[\lambda^2, \ldots, [\lambda^2, \mathcal{L}_{p,\lambda}]\cdots], \tag{49}$$

where the last commutator is applied $d = \deg p$ times. Here we used the standard formula $Ad_{e^X} = e^{ad_X}$ with X being the operator of multiplication by $\lambda^2/2$, and the fact that the operator ad_X is lowering the order of a differential operator at least by one, so that after $d = \deg p$ times the commutator becomes zero.

Now using formula (49), Theorem 3, and the eigenfunction property $L_x\phi = \lambda^2\phi$ we get

$$\begin{aligned}\mathcal{L}_{p,\lambda}\, F(x,\lambda) &= e^{-\lambda^2/2}(e^{\lambda^2/2}\mathcal{L}_{p,\lambda}e^{-\lambda^2/2})\phi \\ &= \Big(\mathcal{L}_{p,x} + \frac{1}{2}[\mathcal{L}_{p,x}, L_x] + \frac{1}{2^2 2!}[[\mathcal{L}_{p,x}, L_x], L_x] \\ &\quad + \cdots + \frac{1}{2^d d!}[\cdots[\mathcal{L}_{p,x}, L_x], \ldots, L_x]\Big)\phi\, e^{-\lambda^2/2} = \mathcal{L}_{p,x}^{\mathcal{H}} F(x,\lambda).\end{aligned}$$

Therefore by Proposition 10 we get

$$\big(\mathcal{L}_p^{\mathcal{H}}\chi_H\big)(q) = \big\langle \mathcal{L}_{p,x}^{\mathcal{H}} F(x,\cdot), q(\cdot)\big\rangle_m = \langle F(x,\cdot), \mathcal{L}_p q(\cdot)\rangle_m = (\chi_H \mathcal{L}_p)(q),$$

as required. □

As a corollary, the commutativity of the operators $\mathcal{L}_p^{\mathcal{H}}$ immediately follows.

Corollary 1. *The operators* (48) *commute:*

$$[\mathcal{L}_p^{\mathcal{H}}, \mathcal{L}_q^{\mathcal{H}}] = 0 \tag{50}$$

for all $p, q \in \Lambda_N$ and thus are quantum integrals of the CM system in an harmonic field (41).

Note that the coefficients of the operators $\mathcal{L}_p^{\mathcal{H}}$ depend polynomially on m, and therefore commutativity (50) holds for any value of the parameter m, not necessarily integer.

In the quantum case, the *Lassalle–Nekrasov correspondence* [Las91,Nek97] is given by the linear map $\chi_{LN}\colon \Lambda_N \to \Lambda_N$, $\chi_{LN}(p) = e^{-L/2}p$, $p \in \Lambda_N$, which is such that the diagram

$$\begin{array}{ccc} \Lambda_N & \xrightarrow{\chi_{LN}} & \Lambda_N \\ {\scriptstyle \mathcal{L}_p}\downarrow & & \downarrow{\scriptstyle \mathcal{L}_p^{\mathcal{H}}} \\ \Lambda_N & \xrightarrow{\chi_{LN}} & \Lambda_N \end{array} \tag{51}$$

is commutative for all $p \in \Lambda_N$. (See the introduction for an outline of the correspondence in the classical case.) Strictly speaking, the case corresponding to $p(\xi) = \xi_1 + \cdots + \xi_N$

appeared in [Las91], whereas the general case can be found in [BF97]. Although this formulation of the correspondence is not present in these works, it is readily extracted from the latter paper. In our case with $m \in \mathbb{Z}_{\geq 0}$ we have by Proposition 8 the relation $\chi_H|_{\Lambda_N} = \chi_{LN}$. Moreover, our definition in (32) of the Hermitisation map χ_H is readily modified to yield χ_{LN} for non-integer m: the BA function should be replaced by a particular generalised hypergeometric series (cf. (3.2b) in [BF97]) and $\langle \cdot, \cdot \rangle_m$ by a suitable L^2-inner product with respect to which the Jack polynomials are orthogonal.

This establishes the equivalence between the rational CM system in an harmonic field and Sutherland's version of this system, which sounds surprising, especially at the classical level. Indeed, these two systems have very different dynamics: the orbits of CM system with harmonic field are known to be periodic with the same period 2π, while for the Sutherland case they are much more complicated. In fact, there is no contradiction here since the Hamiltonian $\mathcal{L}_1^{\mathcal{H}}$ corresponds to the momentum $\mathcal{L}_1$, but not to the Hamiltonian $\mathcal{L}_2$.

Let us demonstrate this in the simplest case $m = 0$. In this case the quasi-invariant algebra $\mathcal{Q}_0 = \mathbb{C}[x_1, \dots, x_N]$ coincides with the set of all polynomials. For any polynomial p the corresponding operators are

$$\mathcal{L}_p = p(x_1\partial_1, \dots, x_N\partial_N), \quad \mathcal{L}_p^{\mathcal{H}} = p(x_1\partial_1 - \partial_1^2, \dots, x_N\partial_N - \partial_N^2).$$

The Hermitisation χ_H sends the monomial $x^\mu = x_1^{\mu_1} \dots x_N^{\mu_N}$ to the product

$$H_\mu(x) = H_{\mu_1}(x_1) \dots H_{\mu_N}(x_N)$$

of the usual Hermite polynomials, and extends to the general polynomial by linearity.

Note that the monomials x^μ are the joint eigenvectors of all operators $\mathcal{L}_p$:

$$\mathcal{L}_p x^\mu = p(\mu) x^\mu,$$

while $H_\mu(x)$ are the joint eigenvectors of $\mathcal{L}_p^{\mathcal{H}}$:

$$\mathcal{L}_p^{\mathcal{H}} H_\mu(x) = p(\mu) H_\mu(x)$$

with the same eigenvalue $p(\mu)$.

On the other hand, allowing any $m \in \mathbb{N}$ while restricting attention to $\Lambda_N \subset \mathcal{Q}_m$, we can recover Lassalle's multivariable Hermite polynomials. More specifically, for partitions $\lambda = (\lambda_1, \dots, \lambda_N)$ consider the (monic) Jack polynomials $P_\lambda^{(\alpha)}(x)$ in N variables $x = (x_1, \dots, x_N)$, see e.g. [Mac95]. For us, the relevant parameter values are $\alpha = -1/m$. Note that not all Jack polynomials can be specialized to such a parameter value, e.g.

$$P_{(3,1)}^{(-1/m)}(x) = m_{(3,1)}(x) + \frac{2m}{m-1} m_{(2,2)}(x) + \frac{m(5m-3)}{(m-1)^2} m_{(2,1,1)}(x) + \frac{12m^2}{(m-1)^2} m_{(1,1,1,1)}(x)$$

is well-defined if and only if $m \neq 1$. (Here, the monomial symmetric polynomials $m_\lambda(x) = \sum_{a \in S_N(\lambda)} x_1^{a_1} \cdots x_N^{a_N}$.) For m and λ such that no α-poles are encountered, we consider the symmetric m-Hermite polynomials

$$H_\lambda^{(m)} := \chi_H \left(P_\lambda^{(-1/m)} \right).$$

From Proposition 5, we infer that $H^{(m)}_\lambda$ is the unique symmetric polynomial of the form

$$H^{(m)}_\lambda(x) = P^{(-1/m)}_\lambda(x) + \text{lower degree terms},$$

with lower degrees of the form $|\lambda| - 2k, k = 1, \ldots, \lfloor \frac{1}{2}|\lambda| \rfloor$, that satisfies the differential equation

$$\mathcal{L}^H H^{(m)}_\lambda = -|\lambda| H^{(m)}_\lambda.$$

Since Lassalle's multivariable Hermite polynomials also have these two properties (see Théorème 1 and Corollaire to Théorème 3 in [Las91]), they must coincide with the symmetric multivariable m-Hermite polynomials (up to a choice of normalisation and after setting the parameter $\alpha = -1/m$).

6. Lassalle–Nekrasov Correspondence and Automorphisms of Cherednik Algebras

Recall the rational Cherednik algebra H_m associated with the symmetric group S_N and parameter m [EG02B] (we assume the parameter $t = 1$). This algebra is generated by two polynomial rings $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_N]$, $\mathbb{C}[y] = \mathbb{C}[y_1, \ldots, y_N]$ and the group algebra $\mathbb{C}S_N$ with defining relations

$$wp(x) = p(w^{-1}x)w, \quad wq(y) = q(w^{-1}y)w, \quad y_i x_j - x_j y_i = S_{ij},$$

where $S_{ij} = m s_{ij}$ if $i \neq j$, $S_{ii} = 1 - m \sum_{k \neq i} s_{ik}$ if $i = j$, and $p(x) \in \mathbb{C}[x]$, $q(y) \in \mathbb{C}[y]$, $w \in S_N$.

The algebra H_m has a faithful polynomial representation $\rho\colon H_m \to \mathrm{End}(\mathbb{C}[x])$ where the elements $x_i \in H_m$ act on $\mathbb{C}[x]$ by multiplication, the elements $y_i \in H_m$ act by Dunkl operators D_i, and the elements s_{ij} act as transpositions of x_i and x_j.

Consider the following automorphisms $a_{\tau,l}, b_{\tau,l}$ of the rational Cherednik algebra H_m [EG02B]:

$$\begin{aligned} a_{\tau,l}(x_i) &= x_i + \tau y_i^l, \quad a_{\tau,l}(y_i) = y_i, \quad a_{\tau,l}(w) = w, \\ b_{\tau,l}(x_i) &= x_i, \quad b_{\tau,l}(y_i) = y_i + \tau x_i^l, \quad b_{\tau,l}(w) = w, \end{aligned} \tag{52}$$

where $\tau \in \mathbb{C}, l \in \mathbb{Z}_{\geq 0}, w \in S_N$, and $1 \leq i \leq N$.

Let $\mathbb{G}$ be the group of automorphisms of the rational Cherednik algebra H_m generated by these elements $a_{\tau,l}, b_{\tau,l}$. Note that elements of $\mathbb{G}$ preserve the spherical subalgebra $eH_m e \subset H_m$, where $e = \frac{1}{N!} \sum_{w \in S_N} w$.

Consider the subalgebra of H_m generated by the elements $\sigma_k = e \sum_{i=1}^N \pi_i^k$, where $\pi_i = x_i y_i$, $k \in \mathbb{Z}_{\geq 0}$ become Polychronakos operators [Poly92] in the polynomial representation. The elements π_i satisfy the commutation relation

$$[\pi_i, \pi_j] = m(\pi_i - \pi_j) s_{ij}$$

for any i, j, which implies that the elements σ_k pairwise commute, cf. Polychronakos [Poly92].

Recall that operators $\mathcal{L}_k = \mathrm{Res}\, \sigma_k$ generate an algebra of (symmetric) quantum integrals of the trigonometric CM system. In particular $\mathcal{L}_1 = E = \sum_{i=1}^N x_i \partial_i$ is the

Euler operator, and $\mathcal{L}_2 = \mathcal{H}_N$, where $\mathcal{H}_N$ is the Hamiltonian of the trigonometric CM system (42).

Let us describe the action of the generating automorphisms $a_{\tau,l}, b_{\tau,l} \in \mathbb{G}$ in terms of the adjoint action of the elements

$$y^l := \sum_{i=1}^{N} y_i^l, \quad x^l := \sum_{i=1}^{N} x_i^l.$$

Namely, the following statement takes place (cf. [EG02B]).

Proposition 11. *The action of the automorphisms $a_{\tau,l}$ and $b_{\tau,l}$ can be given as*

$$a_{\tau,l}(h) = e^{ad_{\frac{\tau}{l+1}y^{l+1}}} h, \quad b_{\tau,l}(h) = e^{-ad_{\frac{\tau}{l+1}x^{l+1}}} h, \quad (h \in H_m), \tag{53}$$

where $ad_A(X) = AX - XA$.

Proof. It is well known that the exponential of any nilpotent derivation is a homomorphism, so it is sufficient to prove formulas (53) for $h = x_i, y_i$, $(i = 1, \dots, N)$. The following commutation relations can be checked directly:

$$[x_i, y^{l+1}] = -(l+1)y_i^l, \quad [y_i, x^{l+1}] = (l+1)x_i^l. \tag{54}$$

The right-hand side of the formula (53) for $a_{\tau,l}(x_i)$ has two non-zero terms only by formula (54), which are $a_{\tau,l}(x_i) = x_i + \tau y_i^l$ as required. It is also clear that the first formula (53) states $a_{\tau,l}(y_i) = y_i$ as required. The second formula (53) can be checked similarly. □

Recall the standard formula from linear algebra

$$Ad_{e^A} = e^{ad_A}, \tag{55}$$

where $Ad_C X = CXC^{-1}$.

In the polynomial representation we have a well defined action of the operators of the form $e^{p(D)}$, $p \in \mathbb{C}[x]$ since $p(D)$ are locally nilpotent operators. We also have the natural action of the operators $p(D)$ on the functions of the form $e^{-q(x)}r(x)$, where $r \in \mathbb{C}[x]$, $q \in \mathbb{C}[x]^{S_N}$, which allows us to consider the operator

$$Ad_{e^{q(x)}} p(D) := e^{q(x)} p(D) e^{-q(x)}$$

acting on the space of polynomials $\mathbb{C}[x]$.

In view of formula (55) we can now restate Proposition 11 as follows.

Proposition 12. *In the polynomial representation the action of elements $a_{\tau,l}(h)$ and $b_{\tau,l}(h)$ for any $h \in H_m$ can be given as*

$$a_{\tau,l}(h) = e^{\frac{\tau}{l+1}D^{l+1}} h e^{-\frac{\tau}{l+1}D^{l+1}}, \quad b_{\tau,l}(h) = e^{-\frac{\tau}{l+1}x^{l+1}} h e^{\frac{\tau}{l+1}x^{l+1}},$$

where $D^{l+1} := \sum_{i=1}^{N} D_i^{l+1}$.

This leads us to another interpretation of the Lassalle–Nekrasov correspondence φ_{LN} between quantum integrals of the trigonometric CM system and the rational CM system in an harmonic field, sending $\mathcal{L}_p$ to $\mathcal{L}_p^{\mathcal{H}}$, $p \in \Lambda_N$, see diagram (51).

Let $\mathcal{T}$ be the commutative subalgebra in H_m generated by the elements

$$\pi^k := \sum_{i=1}^{N} \pi_i^k, \quad k \in \mathbb{Z}_{\geq 0}. \tag{56}$$

Define $\mathcal{R} = \gamma(\mathcal{T})$ to be its image under the automorphism $\gamma = a_{-1,1} \in \mathbb{G}$. More explicitly, the algebra $\mathcal{R}$ is generated by the elements

$$\gamma(\pi^k) = \sum_{i=1}^{N} (x_i y_i - y_i^2)^k.$$

Let $\mathcal{D}_{CM}^T$ and $\mathcal{D}_{CM}^R$ be the algebras of quantum integrals of the trigonometric CM system and the rational CM system with an harmonic term generated by the differential operators $\mathcal{L}_p$ and $\mathcal{L}_p^{\mathcal{H}}$, $p \in \Lambda_N$, respectively.

Proposition 13. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\gamma} & \mathcal{R} \\ {\scriptstyle \mathrm{Res}}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}} \\ \mathcal{D}_{CM}^T & \xrightarrow{\varphi_{LN}} & \mathcal{D}_{CM}^R, \end{array}$$

where φ_{LN} is the Lassalle–Nekrasov correspondence between the quantum integrals:

$$\varphi_{LN}(\mathcal{L}_p) = \mathcal{L}_p^{\mathcal{H}},$$

with the operator $\mathcal{L}_p^{\mathcal{H}}$ defined by (48). Furthermore, this operator can also be given as

$$\mathcal{L}_p^{\mathcal{H}} = e^{-L/2} \mathcal{L}_p e^{L/2}, \quad L = \Delta - \sum_{i<j}^{N} \frac{2m}{x_i - x_j} (\partial_i - \partial_j). \tag{57}$$

Proof. All the maps in the diagram are homomorphisms so it is sufficient to compare the maps $\varphi_{LN} \circ \mathrm{Res}$ and $\mathrm{Res} \circ \gamma$ applied to $\pi^k \in \mathcal{T}$.

Recall that L given in (57) is the Hamiltonian of the rational CM system (without harmonic term) which can be expressed as

$$L = \mathrm{Res} \sum_{i=1}^{N} y_i^2,$$

see [Dun89, Hec91B]. Therefore by Proposition 12 we have $\mathrm{Res}\,(\gamma(\pi^k)) = e^{-L/2} \mathcal{L}_k e^{L/2}$, where $\mathcal{L}_k = \mathrm{Res}\,\pi^k$ is the quantum integral of the trigonometric CM system.

On the other hand by Proposition 8 and Theorem 4 we have $e^{-L/2} \mathcal{L}_k e^{L/2} = \mathcal{L}_k^{\mathcal{H}}$ as required. □

7. Higher Analogues of Multivariable m-Hermite Polynomials and Bispectrality

Consider now the triangular automorphism γ of the rational Cherednik algebra H_m which is a composition of automorphisms (52):

$$\gamma(x_i) = x_i + \mathcal{P}_\gamma(y_i), \quad \gamma(y_i) = y_i, \quad \gamma(w) = w, \tag{58}$$

where $\mathcal{P}_\gamma(z) = \sum_{j=0}^{l} \tau_j z^j$ is a polynomial of degree $l \in \mathbb{Z}_{\geq 0}$, $\tau_j \in \mathbb{C}$. The same arguments as above in the case $l = 1$ lead to the next proposition (see also [Hor16, Hor18, VZh09] who used automorphisms of the Weyl algebra in a related context).

Recall that $L_j = \mathrm{Res} \sum_{i=1}^{N} D_i^j$ defines a higher order quantum integral of the rational CM system, see Heckman [Hec91A]. Define L_γ as

$$L_\gamma = \sum_{j=0}^{l} \tau_j L_{j+1}.$$

Proposition 14. *The algebra $\mathcal{D}_{CM}^{R,\gamma} = \mathrm{Res}\, \mathcal{R}_\gamma$, where $\mathcal{R}_\gamma = \gamma(\mathcal{T})$, is a commutative algebra of differenial operators with rational coefficients which is isomorphic to the algebra of polynomials in N variables. The lowest order differential operator in the algebra $\mathcal{D}_{CM}^{R,\gamma}$ has the form*

$$K_\gamma = \mathrm{Res}\, \gamma(\sum_{i=1}^{N} x_i y_i) = \mathrm{Res} \sum_{i=1}^{N} (x_i y_i + y_i \mathcal{P}_\gamma(y_i)) = E + L_\gamma, \tag{59}$$

where $E = \sum x_i \partial_i$ is the Euler operator.

Note that this proposition holds for any value of m, not necessarily integer. For integer m the operator K_γ admits an operator intertwining it with a corresponding constant coefficient differential operator. Let $T = T(x, \partial)$ be a differential operator such that the BA function is given as

$$\phi(x, \lambda) = T e^{(x,\lambda)}. \tag{60}$$

We have the following result, which for $l = 1$ is contained in [BC20].

Proposition 15. *For any $m \in \mathbb{Z}_{\geq 0}$ the operator K_γ satisfies the intertwining relation*

$$K_\gamma T = T(\sum_{j=0}^{l} \tau_j \partial^{j+1} + E),$$

where $\partial^j = \sum_{i=1}^{N} \partial_i^j$.

Proof. Relation (60) and the property $L_j \phi(x, \lambda) = \lambda^j \phi(x, \lambda)$, where $\lambda^j = \sum_{i=1}^{N} \lambda_i^j$, imply the intertwining relation

$$L_j T = T \partial^j.$$

The operator T is homogeneous of degree 0 by the homogeneity relation (23) of the BA function $\phi(x, \lambda)$. Hence $[E, T] = 0$ and the statement follows. □

Recall that the operator L_γ preserves the space of quasi-invariant polynomials $\mathcal{Q}_m$ and it acts on this space as a locally nilpotent operator. Similarly to Propositions 4 and 8, we have the following result.

Proposition 16. *For any homogeneous $q \in \mathcal{Q}_m$ the polynomial*

$$P_q^\gamma = e^{\sum_{j=0}^{l} \frac{\tau_j}{j+1} L_{j+1}} q \tag{61}$$

is an eigenfunction of the operator K_γ:

$$K_\gamma P_q^\gamma = \deg q \; P_q^\gamma .$$

More generally, we have the operator $e^{\sum_{j=0}^{l} \frac{\tau_j}{j+1} L_{j+1}}$ intertwining the action of the algebras $\mathcal{D}_{CM}^{R,\gamma}$ and $\mathcal{D}_{CM}^{T}$ on quasi-invariants $\mathcal{Q}_m$.

The spectrum of the operator K_γ on $\mathcal{Q}_m$ thus coincides with the spectrum of the Euler operator E.

In the simplest one-dimensional case with $m = 0$ and $\tau_j = 0$ for $j < l$, $\tau_l = \tau$ we have the operator

$$K_\gamma = x\frac{d}{dx} + \tau \frac{d^{l+1}}{dx^{l+1}}.$$

The corresponding polynomials

$$P_n(x;\tau) = e^{\frac{\tau}{l+1}\frac{d^{l+1}}{dx^{l+1}}} x^n$$

are known as Gould–Hopper polynomials [GH62]. These polynomials appear in relation with various problems from probability, quantum mechanics and integrable systems [Cha11, DLMT96, VL13].

Note that for $l \geq 2$ these polynomials do not satisfy any three-term recurrence relation and hence by Favard's theorem do not form an orthogonal system (but satisfy a more general vector orthogonality property, see Horozov [Hor16]).

The corresponding eigenvalue equation with $\tau = -l - 1$

$$K_\gamma \Psi = x\frac{d\Psi}{dx} - (l+1)\frac{d^{l+1}\Psi}{dx^{l+1}} = \lambda \Psi \tag{62}$$

has interesting non-polynomial solutions

$$\Psi = m_k = \int_0^{+\infty} z^k e^{-z^{2(l+1)}+xz^2} dz, \quad k \geq 0,$$

which are the moments of the generalised Shohat–Freud weight [Fre76] (see e.g. [CJ20] where moments are expressed through generalised hypergeometric functions for $l = 2, 3, 4$). Indeed, it is easy to see that

$$K_\gamma e^{-z^{2(l+1)}+xz^2} = \frac{1}{2} z \frac{d}{dz} e^{-z^{2(l+1)}+xz^2},$$

which after integration by parts gives the relation (62) with $\lambda = -\frac{k}{2}$.

Going back to the general case consider the function

$$F_\gamma(x,\lambda) = \phi(x,\lambda) e^{\sum_{j=0}^{l} \frac{\tau_j}{j+1}\lambda^{j+1}},$$

where $\lambda^{j+1} = \sum_{i=1}^{N} \lambda_i^{j+1}$. For a homogeneous basis q^i, $i \in \mathbb{Z}_{\geq 0}$ in the space of quasi-invariants $\mathcal{Q}_m$ consider the polynomials $P_i^\gamma(x) := P_{q_i}^\gamma(x)$ given by formula (61), where q_i is the dual basis in quasi-invariants $\mathcal{Q}_m$.

Proposition 17. *We have the following expansion*

$$F_\gamma(x,\lambda) = \sum_i P_i^\gamma(x) q^i(\lambda).$$

Proof. The function $F_\gamma(x,\lambda)$ can be expanded as

$$F_\gamma(x,\lambda) = \sum_i \widetilde{P}_i^\gamma(x) q^i(\lambda)$$

for some quasi-invariant polynomials $\widetilde{P}_i^\gamma(x)$ with the highest degree term $q_i(x)$. Similarly to the proof of Proposition 4 we have the relation

$$K_\gamma F_\gamma(x,\lambda) = E_\lambda F_\gamma(x,\lambda),$$

where the operator K_γ is given by formula (59). Therefore

$$K_\gamma \widetilde{P}_i^\gamma(x) = \deg q^i \, \widetilde{P}_i^\gamma(x). \tag{63}$$

Since the polynomial solution of differential equation (63) with the highest degree term q_i is unique, it follows by Proposition 16 that $\widetilde{P}_i^\gamma(x) = P_i^\gamma(x)$. □

This function satisfies the following bispectrality property in the sense of Duistermaat and Grünbaum [DG86].

Proposition 18. *For any quasi-invariant polynomial $q \in \mathcal{Q}_m$ the following differential equations hold:*

$$L_{q,x} F_\gamma(x,\lambda) = q(\lambda) F_\gamma(x,\lambda), \quad \widehat{L}_{q,\lambda} F_\gamma(x,\lambda) = q(x) F_\gamma(x,\lambda),$$

where $\widehat{L}_{q,\lambda} = e^{\sum_{j=0}^{l} \frac{\tau_j}{j+1}\lambda^{j+1}} L_{q,\lambda} e^{-\sum_{j=0}^{l} \frac{\tau_j}{j+1}\lambda^{j+1}}$.

The proof follows directly from property (5) of the BA function and its symmetry (22). This leads to the following result.

Theorem 5. *For any homogeneous quasi-invariant $q(x) \in \mathcal{Q}_m[d]$ the polynomials $P_i^\gamma(x)$ satisfy the recurrence relation*

$$q(x) P_i^\gamma(x) = \sum_{j : d_i - ld \leq \deg q_j \leq d_i + d} C_i^j P_j^\gamma(x), \quad C_i^j = \langle \widehat{L}_q q^j, q_i \rangle_m, \tag{64}$$

where $d_i = \deg P_i^\gamma = \deg q_i$, which together with the differential equation

$$K_\gamma P_i^\gamma(x) = d_i \, P_i^\gamma(x)$$

gives the differential–difference bispectral property for the polynomials $P_i^\gamma(x)$.

Proof. Let us write the operator $\widehat{L}_{q,\lambda}$ in the basis $q^i(\lambda)$:

$$\widehat{L}_{q,\lambda} q^i(\lambda) = \sum_j C^i_j q^j(\lambda), \tag{65}$$

where the coefficients $C^i_j = \langle \widehat{L}_q q^i, q_j \rangle_m$ since $\langle q^j, q_k \rangle_m = \delta^j_k$. The operator $\widehat{L}_{q,\lambda}$ can be given by replacing partial derivatives ∂_i in $L_{q,\lambda}$ with $\partial_i - \mathcal{P}_\gamma(\lambda_i)$. Since the operator $L_{q,\lambda}$ is lowering the degree of quasi-invariants by d and the multiplication by $\mathcal{P}_\gamma(\lambda_i)$ is increasing the degree by l the inner product $C^i_j = \langle \widehat{L}_q q^i, q_j \rangle_m = 0$ if $\deg q_j > d_i + ld$ or $\deg q_j < d_i - d$.

By applying the operator $\widehat{L}_{q,\lambda}$ to the function $F_\gamma(x,\lambda)$ and using formula (65) and Propositions 17 and 18 we get the required relation (64). □

Let us illustrate Theorem 5 by a one-dimensional example with $\mathcal{P}_\gamma(z) = \tau z^l$. Let us define polynomials p^l_k, $k \in \mathbb{Z}_{\geq 0}$ by the expansion

$$\phi_m(x,\lambda) e^{\frac{\tau}{l+1}\lambda^{l+1}} = \sum_{k=0}^{\infty} p^l_k(x) \lambda^k,$$

where the BA function ϕ_m is given by (16). Note that $p^l_{2s-1}(x) = 0$ for $s = 1, \dots, m$. The relation of polynomials p^l_k at $l = 1$, $\tau = -1$ with m-Hermite polynomials is given by the formula

$$H^{(m)}_k(x) = k! p^1_k(x).$$

In this case the operator $\widehat{L}_{q,\lambda}$ for $q = x^2$ takes the form

$$\widehat{L}_{x^2,\lambda} = \partial^2_\lambda - 2(\tau\lambda^l + m\lambda^{-1})\partial_\lambda + \tau^2\lambda^{2l} + (2m-l)\tau\lambda^{l-1},$$

and hence

$$x^2 p^l_k = (k+2)(k-2m+1) p^l_{k+2} + \tau(l - 2(k-m+1)) p^l_{k-l+1} + \tau^2 p^l_{k-2l}. \tag{66}$$

Let now $m = 1$. Then $x^3 \in \mathcal{Q}_m$ and the corresponding differential operators $L_{x^3,\lambda}$ and $\widehat{L}_{x^3,\lambda}$ have the form

$$\begin{aligned} L_{x^3,\lambda} &= \partial^3_\lambda - 3\lambda^{-1}\partial^2_\lambda + 3\lambda^{-2}\partial_\lambda, \\ \widehat{L}_{x^3,\lambda} &= L_{x^3,\lambda} - 3\tau\lambda^l\partial^2_\lambda + 3(\tau(2-l)\lambda^{l-1} + \tau^2\lambda^{2l})\partial_\lambda \\ &\quad -\tau(l-1)(l-3)\lambda^{l-2} + 3(l-1)\tau^2\lambda^{2l-1} - \tau^3\lambda^{3l}. \end{aligned}$$

The corresponding recurrence relation has the form

$$\begin{aligned} x^3 p^l_k &= (k+3)(k+1)(k-1) p^l_{k+3} \\ &\quad -\tau(3k^2 - 3kl + l^2 + 3k - l - 3) p^l_{k-l+2} + 3\tau^2(k-l) p^l_{k-2l+1} \\ &\quad -\tau^3 p^l_{k-3l}. \end{aligned} \tag{67}$$

We note that when $l = 1$ recurrence relations of this type were considered by Gómez-Ullate et al. [GKMM16] in the context of exceptional Hermite polynomials [GGM14].

In this case formulas (66), (67) take the form of the following recurrence relations for m-Hermite polynomials:

$$x^2 H_k^{(1)} = \frac{k-1}{k+1} H_{k+2}^{(1)} + (2k-1) H_k^{(1)} + k(k-1) H_{k-2}^{(1)},$$
$$x^3 H_k^{(1)} = \frac{k-1}{k+2} H_{k+3}^{(1)} + 3(k-1) H_{k+1}^{(1)} + 3k(k-1) H_{k-1}^{(1)} + k(k-1)(k-2) H_{k-3}^{(1)}.$$

8. Deformed Calogero–Moser Systems

Let us recall the following generalised CM Hamiltonian which describes interaction of two types of particles [CFV98,SV04]:

$$L_{N_1,N_2} = k^{-1} \sum_{i=1}^{N_1} \partial_{z_i}^2 + \sum_{j=1}^{N_2} \partial_{w_j}^2 - \sum_{i_1<i_2}^{N_1} \frac{2}{z_{i_1} - z_{i_2}} (\partial_{z_{i_1}} - \partial_{z_{i_2}})$$
$$- \sum_{j_1<j_2}^{N_2} \frac{2k^{-1}}{w_{j_1} - w_{j_2}} (\partial_{w_{j_1}} - \partial_{w_{j_2}}) - \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \frac{2}{z_i - w_j} (k^{-1} \partial_{z_i} - \partial_{w_j}), \quad (68)$$

where $N_1, N_2 \in \mathbb{Z}_{\geq 0}$. This Hamiltonian is known to be integrable for any non-zero $k \in \mathbb{C}$.

Consider the rational Cherednik algebra $H_{1/k}$ associated with the symmetric group S_N, where $k \in \mathbb{N}$, $k \leq N$. In this case the polynomial representation $\mathbb{C}[x]$ has a submodule I consisisting of polynomials $p(x)$ vanishing on the S_N orbit of the subspace Π given by

$$x_1 = \ldots = x_k =: z_1,$$
$$x_{k+1} = \ldots = x_{2k} =: z_2,$$
$$\vdots$$
$$x_{(N_1-1)k+1} = \ldots = x_{N_1 k} =: z_{N_1},$$

where $N_1 \in \mathbb{N}$ is such that $N_2 := N - kN_1 \in \mathbb{N}$ (see [ESG09] and the arXiv version of [Fei12], where this statement first appeared).

Let us consider the action of an S_N–invariant element $h \in H_{1/k}$ in the quotient module $\mathbb{C}[x]/I$. It gives rise to a differential operator $\mathrm{Res}_\Pi h$ acting on functions in variables z, w, where $z = (z_1, \ldots, z_{N_1})$, and $w = (w_1, \ldots, w_{N_2})$ is given by $w_i = x_{N_1 k+i}$, $i = 1, \ldots, N_2$. Let $\Sigma \subset \Pi$ be the union of hyperplanes in Π where two coordinates from the set z_i, w_j become equal. Then this operator may be defined by the formula

$$\mathrm{Res}_\Pi h(f) = h(\widetilde{f})|_\Pi,$$

where f is a regular function on a small open set $U \subset \Pi \setminus \Sigma$, and $\widetilde{f}$ is its S_N-invariant regular extension to an S_N-invariant open set $V \subset \mathbb{C}^N$, $U \subset V$, such that $\widetilde{f}|_U = f$. It was established in [Fei12] that

$$\mathrm{Res}_\Pi \sum_{i=1}^{N} y_i^2 = L_{N_1,N_2}. \quad (69)$$

Let us also consider the trigonometric version of the Hamiltonian (68) which (in the exponential coordinates) has the form [CFV98, SV04]

$$\mathcal{L}_{N_1,N_2} = k^{-1}\sum_{i=1}^{N_1}(z_i\partial_{z_i})^2 + \sum_{j=1}^{N_2}(w_j\partial_{w_j})^2 - \sum_{i_1<i_2}^{N_1}\frac{z_{i_1}+z_{i_2}}{z_{i_1}-z_{i_2}}(z_{i_1}\partial_{z_{i_1}} - z_{i_2}\partial_{z_{i_2}})$$
$$-k^{-1}\sum_{j_1<j_2}^{N_2}\frac{w_{j_1}+w_{j_2}}{w_{j_1}-w_{j_2}}(w_{j_1}\partial_{w_{j_1}} - w_{j_2}\partial_{w_{j_2}})$$
$$-\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\frac{z_i+w_j}{z_i-w_j}(k^{-1}z_i\partial_{z_i} - w_j\partial_{w_j}).$$

Lemma 2. *The following relations hold:*

$$\mathrm{Res}_{\Pi}\sum_{i=1}^{N}(x_iy_i)^2 = \mathcal{L}_{N_1,N_2}, \tag{70}$$

and

$$\mathrm{Res}_{\Pi}\sum_{i=1}^{N}x_iy_i = \sum_{i=1}^{N_1}z_i\partial_{z_i} + \sum_{j=1}^{N_2}w_j\partial_{w_j} =: E_{N_1,N_2}. \tag{71}$$

Proof. Recall that

$$\mathcal{H}_N = \mathrm{Res}\sum_{i=1}^{N}\mathcal{D}_i^2 = \mathrm{Res}\sum_{i=1}^{N}(x_iy_i)^2$$
$$= \sum_{i=1}^{N}(x_i\partial_{x_i})^2 - k^{-1}\sum_{i<j}^{N}\frac{x_i+x_j}{x_i-x_j}(x_i\partial_{x_i} - x_j\partial_{x_j}).$$

Let us consider orthonormal coordinates $(\tilde{z}, w)$ on the space Π, where $\tilde{z} = k^{1/2}z$. In order to calculate the operator $\mathrm{Res}_{\Pi}\sum_{i=1}^{N}(x_iy_i)^2$ one has to replace in the operator $\mathcal{H}_N$ coordinates x_{jk+s} with $k^{-1/2}\tilde{z}_j$ $(0 \le j \le N_1-1,\ 1 \le s \le k)$, and derivatives $\partial_{x_{jk+s}}$ with $k^{-1/2}\partial_{\tilde{z}_j}$, coordinates x_{N_1k+i} with w_i, $i = 1,\dots,N_2$ and derivatives $\partial_{x_{N_1k+i}}$ with ∂_{w_i} and to replace with zero all the derivatives normal to the space Π (cf. [Fei12]). This produces the operator

$$\mathrm{Res}_{\Pi}\sum_{i=1}^{N}(x_iy_i)^2 = k^{-1}\sum_{i=1}^{N_1}(\tilde{z}_i\partial_{\tilde{z}_i})^2 + \sum_{j=1}^{N_2}(w_j\partial_{w_j})^2$$
$$-\sum_{i_1<i_2}^{N_1}\frac{\tilde{z}_{i_1}+\tilde{z}_{i_2}}{\tilde{z}_{i_1}-\tilde{z}_{i_2}}(\tilde{z}_{i_1}\partial_{\tilde{z}_{i_1}} - \tilde{z}_{i_2}\partial_{\tilde{z}_{i_2}}) - k^{-1}$$
$$\sum_{j_1<j_2}^{N_2}\frac{w_{j_1}+w_{j_2}}{w_{j_1}-w_{j_2}}(w_{j_1}\partial_{w_{j_1}} - w_{j_2}\partial_{w_{j_2}})$$
$$-\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\frac{k^{-1/2}\tilde{z}_i+w_j}{k^{-1/2}\tilde{z}_i-w_j}(k^{-1}\tilde{z}_i\partial_{\tilde{z}_i} - w_j\partial_{w_j}).$$

Formula (70) follows by changing variables $\tilde{z} = k^{1/2}z$. Formula (71) is easy to see in a similar way. □

Define the algebra $\Lambda_{N_1,N_2} \subset \mathbb{C}[z,w]$ as the algebra generated by the deformed Newton sums (cf. [SV04])

$$p_r = k\sum_{i=1}^{N_1} z_i^r + \sum_{i=1}^{N_2} w_i^r, \quad r \geq 0.$$

Let I^{S_N} be the S_N-invariant part of the ideal I, it consists of S_N-invariant polynomials $p(x)$ which vanish on Π.

Lemma 3. *We have an isomorphism of algebras $\Lambda_{N_1,N_2} \cong \Lambda_N/I^{S_N}$.*

Proof. Consider the map $\varphi : \Lambda_N \to \Lambda_{N_1,N_2}$ given by the natural restriction $\varphi(p) = p|_\Pi$, $p \in \Lambda_N$. It is clear that $\operatorname{Im}\varphi = \Lambda_{N_1,N_2}$ and $\operatorname{Ker}\varphi = I^{S_N}$. □

We have the following version of the Lassalle–Nekrasov correspondence for the deformed CM operators, (which also can be extracted from [DH12]).

Theorem 6. *The following diagram is commutative for any $p \in \Lambda_N$:*

$$\begin{array}{ccc} \Lambda_{N_1,N_2} & \xrightarrow{\chi_{N_1,N_2}} & \Lambda_{N_1,N_2} \\ {\scriptstyle \mathcal{L}_{p,N_1,N_2}}\downarrow & & \downarrow{\scriptstyle \mathcal{L}^{\mathcal{H}}_{p,N_1,N_2}} \\ \Lambda_{N_1,N_2} & \xrightarrow{\chi_{N_1,N_2}} & \Lambda_{N_1,N_2}, \end{array}$$

with $\chi_{N_1,N_2}(f) = e^{-\frac{1}{2}L_{N_1,N_2}} f$, $f \in \Lambda_{N_1,N_2}$, and $\mathcal{L}_{p,N_1,N_2} = \operatorname{Res}_\Pi p(\pi_1,\dots,\pi_N)$, $\mathcal{L}^{\mathcal{H}}_{p,N_1,N_2} = Ad_{e^{-\frac{1}{2}L_{N_1,N_2}}}\mathcal{L}_{p,N_1,N_2}$.

Furthermore, the following equalities hold:

$$\mathcal{L}_{\sum x_i y_i,N_1,N_2} = E_{N_1,N_2}, \quad \mathcal{L}_{\sum (x_i y_i)^2,N_1,N_2} = \mathcal{L}_{N_1,N_2}, \tag{72}$$

and

$$\mathcal{L}^{\mathcal{H}}_{\sum x_i y_i,N_1,N_2} = -L_{N_1,N_2} + E_{N_1,N_2}. \tag{73}$$

Proof. Let us recall commutative diagram (51). This diagram is equivalent to the following commutative diagram where Cherednik algebra elements act via Dunkl operators:

$$\begin{array}{ccc} \Lambda_N & \xrightarrow{\chi} & \Lambda_N \\ {\scriptstyle p(\pi)}\downarrow & & \downarrow{\scriptstyle p^{\mathcal{H}}(\pi)} \\ \Lambda_N & \xrightarrow{\chi} & \Lambda_N \end{array}$$

with $\chi(f) = e^{-\frac{1}{2}\sum y_i^2}(f)$, $f \in \Lambda_N$, and the operator $p^{\mathcal{H}}(\pi) = Ad_{e^{-\frac{1}{2}\sum y_i^2}} p(\pi)$. By considering actions in the quotient space Λ_{N_1,N_2} (see Lemma 3) and applying the operation Res_Π and formula (69) the first statement follows.

Formulas (72) follow by Lemma 2. In order to establish (73) let us observe that $p^{\mathcal{H}}(\pi) = \operatorname{Res}_\Pi(x_i y_i - \sum y_i^2)$. Then formula (73) follows from (69). □

We also have a quasi-invariant version of Theorem 6. Let us introduce the following algebra $\Lambda_{N_1,N_2}(k) \subset \mathbb{C}[z_1,\dots,z_{N_1},w_1,\dots,w_{N_2}]$ of generalised quasi-invariants following [ERF16]. A polynomial $q \in \Lambda_{N_1,N_2}(k)$ if the following three conditions are satisfied:

1. $\sigma(q) = q$ for any $\sigma \in S_{N_2}$ which acts by permuting w-variables,
2. for any fixed w, the polynomial q is k-quasi-invariant as a polynomial in variables $z_1,\dots,z_{N_1}$,
3. $\partial_{z_i} q = k\partial_{w_j} q$ if $z_i = w_j$ for any $1 \le i \le N_1$, $1 \le j \le N_2$.

Note that we have an inclusion of algebras $\Lambda_{N_1,N_2} \subset \Lambda_{N_1,N_2}(k)$. We also have the property that the action of the differential operator L_{N_1,N_2} preserves the space $\Lambda_{N_1,N_2}(k)$, and, furthermore, $\Lambda_{N_1,N_2}(k)$ is a module over the spherical subalgebra $eH_{1/k}e$ [ERF16]. In this module the elements $p(\pi_1,\dots,\pi_N)$, $p \in \Lambda_N$ act as differential operators $\mathcal{L}_{p,N_1,N_2}$, in particular, these operators preserve the space $\Lambda_{N_1,N_2}(k)$. This leads us to the following extension of Theorem 6.

Theorem 7. *The following diagram is commutative for any $p \in \Lambda_N$:*

$$\begin{array}{ccc} \Lambda_{N_1,N_2}(k) & \xrightarrow{\chi_{N_1,N_2}} & \Lambda_{N_1,N_2}(k) \\ {\scriptstyle \mathcal{L}_{p,N_1,N_2}}\downarrow & & \downarrow{\scriptstyle \mathcal{L}^{\mathcal{H}}_{p,N_1,N_2}} \\ \Lambda_{N_1,N_2}(k) & \xrightarrow{\chi_{N_1,N_2}} & \Lambda_{N_1,N_2}(k). \end{array}$$

Indeed, as we explained all the maps from the space $\Lambda_{N_1,N_2}(k)$ to itself are well-defined. Since a differential operator acting on functions in variables z, w is fully determined by its action on Λ_{N_1,N_2} the statement follows from Theorem 6.

Operators $\mathcal{L}_{p,N_1,N_2}$ in Theorems 6, 7 were obtained as restrictions Res_Π of the elements $Ad_{e^{-\frac{1}{2}\sum y_i^2}} p(\pi)$ of the rational Cherednik algebra. In terms of automorphisms of the rational Cherednik algebras considered in Sect. 6 we have $\mathcal{L}^{\mathcal{H}}_{p,N_1,N_2} = \mathrm{Res}_\Pi a_{-1,1}(p(\pi))$, where $a_{-1,1}$ is now an automorphism of the algebra $H_{1/k}$ defined in (52).

Let us now consider a more general automorphism γ of the algebra $H_{1/k}$ given by formula (58). Define the algebra of differential operators $\mathcal{D}^{R,\gamma}_{N_1,N_2} = \mathrm{Res}_\Pi \mathcal{R}_\gamma$, where $\mathcal{R}_\gamma = \gamma(\mathcal{T})$, and $\mathcal{T}$ is the commutative subalgebra in the algebra $H_{1/k}$ generated by elements π^r, $r \in \mathbb{Z}_{\ge 0}$, given by formula (56).

We have the following generalisation of Proposition 14 to the deformed case.

Theorem 8. *The algebra $\mathcal{D}^{R,\gamma}_{N_1,N_2}$ is a commutative algebra of differenial operators with rational coefficients which contains $N_1 + N_2$ algebraically independent elements. The lowest order differential operator in the algebra $\mathcal{D}^{R,\gamma}_{N_1,N_2}$ has the form*

$$\begin{aligned} K_{N_1,N_2,\gamma} &= \mathrm{Res}_\Pi\, \gamma(\sum_{i=1}^N x_i y_i) = \mathrm{Res}_\Pi \sum_{i=1}^N (x_i y_i + y_i \mathcal{P}_\gamma(y_i)) \\ &= E_{N_1,N_2} + L_{N_1,N_2,\gamma}, \end{aligned} \tag{74}$$

where

$$L_{N_1,N_2,\gamma} = \mathrm{Res}_\Pi \sum_{i=1}^N y_i \mathcal{P}_\gamma(y_i)$$

is the order $l+1$ quantum integral of the rational generalised CM Hamiltonian (68).

Indeed, algebraic independence of (the highest order terms of) the operators

$$\mathrm{Res}_{\Pi}\gamma(\pi^s) = \mathrm{Res}_{\Pi}\sum_{i=1}^{N}(x_i y_i + y_i \mathcal{P}_{\gamma}(y_i))^s$$

with $s = 1, \ldots, N_1 + N_2$ follows from [SV04, Proposition 4]. The rest of the proof is immediate from Lemma 2 and other previous considerations.

In the case $P_{\gamma}(z) = -z$ the operator (74) is equal to the operator $-L_{N_1,N_2} + E_{N_1,N_2} = \mathcal{L}^{\mathcal{H}}_{\sum x_i y_i, N_1, N_2}$ considered above. Its integrability was established independently in [Fei12] for integer k and in [DH12] for general $k \in \mathbb{C}$. Furthermore, in the latter paper so-called super Hermite polynomials were introduced and shown to be joint eigenfunctions of $\mathcal{L}^{\mathcal{H}}_{\sum x_i y_i, N_1, N_2}$ and its higher order integrals.

9. Concluding Remarks

We have defined a new class of $\mathcal{A}$-Hermite polynomials related to special configurations $\mathcal{A}$ of hyperplanes with multiplicities, and we showed that these polynomials are the eigenvectors of the corresponding generalised CM operator (35). Of particular interest is the case of multivariable m-Hermite polynomials. A natural question is whether these polynomials are joint eigenvectors of the corresponding commuting quantum integrals. The answer is in general negative already for the symmetric m-Hermite polynomials.

The reason is that if the multiplicity m is non-zero then the bilinear forms on the space $\mathcal{Q}_m$ become indefinite and the invariant subspaces of the algebra of integrals in general are not one-dimensional and contain multivariable m-Hermite polynomials as generalised eigenvectors.

One can see this already in the simplest two-particle case, when the second integral has Jordan blocks. Indeed, let us consider the trigonometric side of the Lassalle–Nekrasov correspondence instead. Then this integral corresponds to the second order Hamiltonian (42). Let us fix $m = 1$. Then for any $l \in \mathbb{N}$ we have a 2×2 Jordan block given by

$$\mathcal{H}_2(x_1^l x_2^l) = 2l^2 x_1^l x_2^l, \quad \mathcal{H}_2(x_1^{l-1} x_2^{l-1}(x_1^2 + x_2^2)) = 2l^2 x_1^{l-1} x_2^{l-1}(x_1^2 + x_2^2) - 4x_1^l x_2^l.$$

The corresponding spectral decomposition of the space of quasi-invariants $\mathcal{Q}_m$ seems to be very interesting.

Another interesting direction is to extend the considerations to the difference versions of the Calogero–Moser systems, see [Rui87] and [vDie95]. We hope to come back to some of these questions elsewhere.

Acknowledgements. We are very grateful to Nikita Nekrasov for attracting our attention to his work [Nek97], and to Margit Rösler for useful discussions.

Funding Open access funding provided by Chalmers University of Technology.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory

regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[Adl77] Adler, M.: Some finite dimensional integrable systems and their scattering behaviour. Commun. Math. Phys. **55**, 195–230 (1977)

[BF97] Baker, T.H., Forrester, P.J.: The Calogero–Sutherland model and generalized classical polynomials. Commun. Math. Phys. **188**, 175–216 (1997)

[Ber98] Berest, Y.: Huygens' principle and the bispectral problem. CRM Proc. Lect. Notes **14**, 11–30 (1998)

[BC20] Berest, Y., Chalykh, O.: Deformed Calogero–Moser operators and ideals of rational Cherednik algebras. arXiv:2002.08691

[BEG02] Berest, Y., Etingof, P., Ginzburg, V.: Cherednik algebras and differential operators on quasi-invariants. Duke Math. J. **118**, 279–337 (2002)

[BDS20] Bonneux, N., Dunning, C., Stevens, M.: Coefficients of Wronskian Hermite polynomials. Stud. Appl. Math. **144**, 245–288 (2020)

[BHSS18] Bonneux, N., Hamaker, Z., Stembridge, J., Stevens, M.: Wronskian Appell polynomials and symmetric functions. Adv. Appl. Math. **111**, 101932 (2019)

[Cal71] Calogero, F.: Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials. J. Math. Phys. **12**, 419–436 (1971)

[Cha98] Chalykh, O.A.: Darboux transformations for multidimensional Schrödinger operators. Russ. Math. Surv. **53**, 377–379 (1998)

[CFV98] Chalykh, O.A., Feigin, M.V., Veselov, A.P.: New integrable generalizations of Calogero–Moser quantum problem. J. Math. Phys. **39**, 695–703 (1998)

[CFV99] Chalykh, O.A., Feigin, M.V., Veselov, A.P.: Multidimensional Baker–Akhiezer functions and Huygens' principle. Commun. Math. Phys. **206**, 533–566 (1999)

[CV90] Chalykh, O.A., Veselov, A.P.: Commutative rings of partial differential operators and Lie algebras. Commun. Math. Phys. **126**, 597–611 (1990)

[Cha11] Chang, J.-H.: The Gould–Hopper polynomials in the Novikov–Veselov equation. J. Math. Phys. **52**, 092703 (2011)

[CJ20] Clarkson, P.A., Jordaan, K.: A generalised sextic Freud weight. arXiv:2004.00260

[DLMT96] Dattoli, G., Lorenzutta, S., Maino, G., Torre, A., Cesarano, C.: Generalized Hermite polynomials and supergaussian forms. J. Math. Anal. Appl. **203**, 597–609 (1996)

[DH12] Desrosiers, P., Hallnäs, M.: Hermite and Laguerre symmetric functions associated with operators of Calogero–Moser–Sutherland type. SIGMA Symmetry Integr. Geom. Methods Appl. **8**, 049 (2012)

[DG86] Duistermaat, J.J., Grünbaum, F.A.: Differential equations in the spectral parameter. Commun. Math. Phys. **103**, 177–240 (1986)

[Dun89] Dunkl, C.F.: Differential-difference operators associated to reflection groups. Trans. Am. Math. Soc. **311**, 167–183 (1989)

[DJO94] Dunkl, C.F., de Jeu, M.F.E., Opdam, E.M.: Singular polynomials for finite reflection groups. Trans. Am. Math. Soc. **346**, 237–256 (1994)

[EG02A] Etingof, P., Ginzburg, V.: On m-quasi-invariants of a Coxeter group. Mosc. Math. J. **2**, 555–566 (2002)

[EG02B] Etingof, P., Ginzburg, V.: Symplectic reflection algebras, Calogero–Moser space, and deformed Harish–Chandra homomorphism. Invent. Math. **147**, 243–348 (2002)

[ERF16] Etingof, P., Rains, E.: On Cohen–Macaulayness of algebras generated by generalized power sums. With an appendix by Misha Feigin. Commun. Math. Phys. **347**(1), 163–182 (2016)

[ESG09] Etingof, P., Stoica, E.: Unitary representations of rational Cherednik algebras. With an appendix by Stephen Griffeth. Represent. Theory **13**, 349–370 (2009)

[Fei12] Feigin, M.: Generalized Calogero–Moser systems from rational Cherednik algebras. Sel. Math. (N.S.) **18**(1), 253–281 (2012), arXiv:0809.3487

[FJ14] Feigin, M., Johnston, D.: A class of Baker–Akhiezer arrangements. Commun. Math. Phys. **328**, 1117–1157 (2014)

[FHV13] Feigin, M.V., Hallnäs, M.A., Veselov, A.P.: Baker–Akhiezer functions and generalised Macdonald-Mehta integrals. J. Math. Phys. **54**, 052106 (2013)

[FV02] Feigin, M.V., Veselov, A.P.: Quasi-invariants of Coxeter groups and m-harmonic polynomials. Int. Math. Res. Not. IMRN **2002**, 521–545 (2002)

[FV03] Feigin, M.V., Veselov, A.P.: Quasi-invariants and quantum integrals of the deformed Calogero–Moser systems. Int. Math. Res. Not. IMRN **2003**, 2487–2511 (2003)

[Fre76] Freud, G.: On the coefficients in the recursion formulae of orthogonal polynomials. Proc. R. Ir. Acad. A **76**, 1–6 (1976)

[GGM14] Gómez-Ullate, D., Grandati, Y., Milson, R.: Rational extensions of the quantum harmonic oscillator and exceptional Hermite polynomials. J. Phys. A **47**, 015203 (2014)

[GKMM16] Gómez-Ullate, D., Kasman, A., Kuijlaars, A.B.J., Milson, R.: Recurrence relations for exceptional Hermite polynomials. J. Approx. Theory **204**, 1–16 (2016)

[GH62] Gould, H.W., Hopper, A.T.: Operational formulas connected with two generalizations of Hermite polynomials. Duke Math. J. **29**, 51–63 (1962)

[HHV16] Haese-Hill, W.A., Hallnäs, M.A., Veselov, A.P.: Complex exceptional orthogonal polynomials and quasi-invariance. Lett. Math. Phys. **106**, 583–606 (2016)

[Hec91A] Heckman, G.J.: A remark on the Dunkl differential–difference operators In: Analysis on Reductive Groups. Progress in Mathematics **101**, pp. 181–191 (1991)

[Hec91B] Heckman, G.J.: An elementary approach to the hypergeometric shift operators of Opdam. Invent. Math. **103**, 341–350 (1991)

[Hor16] Horozov, E.: Automorphisms of algebras and Bochner's property for vector orthogonal polynomials. SIGMA Symmetry Integr. Geom. Methods Appl. **12**, 050 (2016)

[Hor18] Horozov, E.: d-orthogonal analogs of classical orthogonal polynomials. SIGMA Symmetry Integr. Geom. Methods Appl. **14**, 063 (2018)

[KKS78] Kazhdan, D., Kostant, B., Sternberg, S.: Hamiltonian group actions and dynamical systems of Calogero type. Commun. Pure Appl. Math. **31**, 481–507 (1978)

[Las91] Lassalle, M.: Polynômes de Hermite généralisés. C. R. Acad. Sci. Paris Sér. I Math. **313**, 579–582 (1991)

[Mac95] Macdonald, I.G.: Symmetric Functions and Hall Polynomials, 2nd edn. Oxford University Press, New York (1995)

[Mos75] Moser, J.: Three integrable Hamiltonian systems connected with isospectral deformations. Adv. Math. **16**, 197–220 (1975)

[Nek97] Nekrasov, N.: On a duality in Calogero–Moser–Sutherland systems. arXiv:hep-th/9707111

[OP76A] Olshanetsky, M.A., Perelomov, A.M.: Completely integrable Hamiltonian systems connected with semisimple Lie algebras. Invent. Math. **37**, 93–108 (1976)

[OP76B] Olshanetsky, M.A., Perelomov, A.M.: Geodesic flows on symmetric spaces of zero curvature, and explicit solutions of the generalized Calogero model for the classical case. Funkt. Anal. Appl. **10**, 86–87 (1976)

[OP83] Olshanetsky, M.A., Perelomov, A.M.: Quantum integrable systems related to Lie algebras. Phys. Rep. **94**, 313–404 (1983)

[Opd95] Opdam, E.M.: Harmonic analysis for certain representations of graded Hecke algebras. Acta Math. **175**, 75–121 (1995)

[Poly92] Polychronakos, A.P.: Exchange operator formalism for integrable systems of particles. Phys. Rev. Lett. **69**(5), 703–705 (1992)

[Ros98] Rösler, M.: Generalized Hermite polynomials and the heat equation for Dunkl operators. Commun. Math. Phys. **192**, 519–542 (1998)

[Rui87] Ruijsenaars, S.N.M.: Complete integrability of relativistic Calogero–Moser systems and elliptic function identities. Commun. Math. Phys. **110**, 191–213 (1987)

[Sch29] Schur, I.: Einige Sätze über Primzahlen mit Anwendungen auf Irreduzibilitäts Fragen, II. Sitzungsber. Preuss. Akad. Wiss. Berl. Phys. Math. Kl. **14**, 370–391 (1929)

[Sch31] Schur, I.: Affektlose Gleichungen in der Theorie der Laguerreschen und Hermitschen Polynome. J. Reine Angew. Math. **165**, 52–58 (1931)

[SV04] Sergeev, A.N., Veselov, A.P.: Deformed quantum Calogero–Moser problems and Lie superalgebras. Commun. Math. Phys. **245**, 249–278 (2004)

[SV15] Sergeev, A.N., Veselov, A.P.: Jack–Laurent symmetric functions. Proc. Lond. Math. Soc. **111**, 63–92 (2015)

[Sut71] Sutherland, B.: Exact results for a quantum many-body problem in one dimension. Phys. Rev. A **4**, 2019–2021 (1971)

[Sze39] Szegö, G.: Orthogonal Polynomials. American Mathematical Society, New York (1939)

[vDie95] van Diejen, J.F.: Multivariable continuous Hahn and Wilson polynomials related to integrable difference systems. J. Phys. A Math. Gen. **28**, L369 (1995)

[vDie97] van Diejen, J.F.: Confluent hypergeometric orthogonal polynomials related to the rational quantum Calogero system with harmonic confinement. Commun. Math. Phys. **188**, 467–497 (1997)

[VSC93] Veselov, A.P., Styrkas, K.L., Chalykh, O.A.: Algebraic integrability for the Schrödinger equation, and groups generated by reflections. Theor. Math. Phys. **94**, 182–197 (1993)

[VL13] Vignat, C., Lèvêque, O.: Proof of a conjecture by Gazeau et al. using the Gould–Hopper polynomials. J. Math. Phys. **54**, 073513 (2013)

[VZh09] Vinet, L., Zhedanov, A.: Automorphisms of the Heisenberg–Weyl algebra and d-orthogonal polynomials. J. Math. Phys. **50**, 033511 (2009)

Communicated by K. Johansson