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# Scattering of elastic waves by an anisotropic sphere


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# Abstract

Scattering of a plane wave by a single spherical obstacle is the archetype of many scattering problems in physics and geophysics. Spherical objects can provide a good approximation for many real objects, and the analytic formulation for a single sphere can be used to investigate wave propagation in more complicated structures like particle composites or grainy materials, which may have application in non-destructive testing, material characterization, medical ultrasound, etc. The main direction of this thesis is to investigate an analytical solution for scattering of elastic waves by an anisotropic sphere in the special case with transverse isotropy. Throughout the thesis a systematic series expansion approach is used to derive displacement and traction fields outside and inside the sphere. For the surrounding isotropic medium such an expansion is made conveniently in terms of the traditional vector spherical wave functions. However, describing the fields inside the anisotropic sphere is more complicated since the classical methods are not applicable anymore. The first step is to describe the anisotropy in spherical coordinates, then the expansion inside the sphere is made in the vector spherical harmonics in the angular directions and power series in the radial direction. The governing equations inside the sphere provide recurrence relations among the unknown expansion coefficients. The remaining expansion coefficients outside and inside the sphere can be found using the boundary conditions on the sphere. Thus, this gives the scattered wave coefficients from which the transition (**T**) matrix can be found. This is convenient as the **T** matrix fully describes the sphere and is independent of the incident wave. The expressions of the general **T** matrix elements are complicated, but in the low frequency limit it is possible to obtain explicit expressions.

The **T** matrices may be used to solve more complicated problems like the wave propagation in polycrystalline materials. The attenuation and wave velocity in a polycrystalline material with randomly oriented transversely isotropic grains is thus investigated. These quantities are calculated analytically using the simple theory of Foldy and show a very good correspondence for low frequencies with previously published results and numerical computations with FEM.

Keywords: Scattering, Anisotropy, Sphere, **T** matrix, Distribution of inclusions, Effective wave number, Attenuation, Phase velocity.

*to my wife, Asieh.*

# Preface

The work presented in this thesis was accomplished at the Division of Dynamics at the Department of Mechanics and Maritime Sciences, at Chalmers University of Technology between December 2018 and May 2021. This research has been funded by the Swedish Research Council and this is gratefully acknowledged. The author took part in planning the project, performed analytical derivation and wrote most of the papers. The author also wishes to greatly thank Anders Boström and Peter Folkow, whose insightful supervision and knowledge the author has benefited very much from.

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# Thesis

This thesis consists of an extended summary and the following appended papers:

| | |
|---|---|
| **Paper A** | A. Jafarzadeh, P. D. Folkow, and A. Boström. "Scattering of elastic SH waves by transversely isotropic sphere". *Proceedings of the International Conference on Structural Dynamic , EURODYN*. vol. 2. 2020, pp. 2782–2797 |
| **Paper B** | A. Jafarzadeh, P. D. Folkow, and A. Boström. Scattering of elastic waves by a transversely isotropic sphere and ultrasonic attenuation in hexagonal polycrystalline materials. *To be submitted* |

# Contents

# Part I

# Extended Summary

# 1 Introduction

Wave propagation in elastic solids with an inhomogeneity is an interesting and very broad research field. The results of this subject find many applications in diverse fields of engineering such as material characterization, nondestructive materials testing by ultrasonics, in-situ safety and reliability control of complex structural components by acoustic emission, dynamic fracture mechanics, seismology and ground vibrations. Among all these subjects, the focus of the current study is on specific areas mostly related to material characterization and nondestructive testing.

## 1.1 Background and motivation

Engineering materials often contain various types of inhomogeneities and anomalies, herafter called inclusions, which may be in the nature of micro grains in metals or may be induced by materials processing, manufacturing and in-service conditions like fiber and particle composites or cracks and cavities in materials. The detection and characterization of inclusions in materials have great importance in engineering applications since the materials integrity, stiffness and strength are significantly affected by the presence of such inclusions. This purpose is often achieved by means of nondestructive testing. Nondestructive testing techniques are based on the emission of arbitrary waves into a medium and then studying the propagation of the wave in them. This technique is very useful for characterization of the materials, since there is a direct connection between material characteristics like inclusion distribution, density, location, size and orientation with the properties of the wave propagating in the medium, like effective wave speed and intensity. Unlike the wave propagation in an ideally homogeneous elastic solid, waves propagating in an elastic solid with inclusions are generally experiencing diffraction and scattering. Diffraction refers to the wave deviating from its original path and scattering refers to the wave radiation from inclusions. Inclusions can be considered as secondary sources of radiation which are due to the excitation by the incident waves. The diffraction and scattering of the incident wave lead to some interesting phenomena in the medium. An incident wave propagating inside a medium carries energy, and when there are inclusions inside the medium this energy is converted into the scattered wave energy. Such conversion of energy results in intensity reduction and shape distortion of the incident wave, in other words the incident wave attenuate and disperse. Consequently, an elastic solid with elastic inclusions is seen by an incident wave as an attenuative and dispersive medium [3].

Analysis of scattering in an elastic medium with inclusions normally starts with a single scattering model. Scattering of waves by a single scatterer is a classical problem in various fields of mathematical physics like acoustics, electromagnetics and elasticity. The basic nature of the problem is the same for all fields. Namely, a propagating wave encounters a discontinuity in the form of an inclusion. The methods used to study such problems are generally based on techniques shown to be successful in acoustics and electromagnetic fields such as separation-of-variables, the **T** matrix methods, integral equation methods, and finite element methods (FEM). However, the differences of elastodynamic fields compared with the other fields are the coupled wave equations and the existence of two wave speeds. Such properties increase the complexity of the elastodynamic equations. Therefore, most of the studies in elastic fields are limited to isotropic materials, which results in relative ease in mathematical treatment and still has important applications since many materials are approximately isotropic or can be homogenised as an isotropic material. A comprehensive overview of scattering of acoustic, electromagnetic, and elastic waves in isotropic media is covered in the literature [4, 5]. However, recent investigations of anisotropic materials, like composites, biological materials and grainy materials (typically metals), have demonstrated the importance of studying wave propagation in these materials.

Wave propagation in a medium with anisotropic inclusions is more complicated to study since many of the classical methods are not applicable any longer. Scattering in anisotropic materials has mostly been studied for electromagnetic waves [6, 7, 8]. For mechanical waves, most of the studies are done for spherically and cylindrically anisotropic inclusions [9, 10, 11, 12, 13]. Scattering of elastic waves by an anisotropic obstacle when the anisotropy is in Cartesian coordinates is investigated in 2D by Boström [14, 15]. The method presented there is pursued by this research to extend the possible solutions to 3D problems.

The analysis of multiple scattering has also been extensively studied. Theoretically, a sequence of equations moving from single scattering to the inclusion of two-obstacle interactions, then three-obstacle interactions and upwards can be set up using, for instance, the T-matrix method. However, the complexity of the equations increases in each step and thus it is normally pursued only for two inclusions [16]. On the other hand, there are plenty of materials which contain or are composed of a distribution of inclusions. Detailed investigation of structures of this kind requires a large number of parameters and leads to extremely complex equations while, in practical applications, such detailed investigation is not necessary. More efficiently, such materials may be described by their statistics, like number density of inclusions and the mean size of the inclusions. The usual basis for the analysis of scattering of these structures has been volume integral equation methods combined with some perturbation method, often the Born approximation. Such methods have been used frequently to study the scattering problem and calculate attenuation and wave speed in grainy materials like simple or complex polycrystals with various types of anisotropy of the grains [17, 18, 19, 20]. However, these studies all seem to have restrictions to more or less weak anisotropy. A different approach to estimate the attenuation and the effective wave speed in polycrystalline materials is to use the **T** matrix to calculate the scattering cross section which is related to total energy carried by the scattered waves. The scattering cross section together with statistical information of the structure like number density of inclusions and their mean size is used in some approximate methods like the theory of Foldy [21] to calculate the attenuation and wave speed. Such an approach is used to estimate the attenuation of 2D polycrystalline materials with grains of cubic materials [22]. These approximate methods does not have any limitation on the degree of anisotropy, however, since they normally neglect multiple scattering effects, they are restricted to low frequencies for polycrystalline materials. Recently FEM have also been used to study polycrystalline materials and investigate the attenuation and phase velocity in them [23, 24, 25, 26, 27, 28].

The main aim of the present project is to extend the possible type of analytical solutions by solving the canonical 3D problem of scattering of elastic waves by an anisotropic inclusion. The inclusion has a spherical shape an its material properties is assumed to be transversely isotropic or orthotropic. Meanwhile, the surrounding material is considered to be isotropic. A general solution may be presented by calculating the linear relationship between the expansion coefficient of the incident wave with those of the scattered wave in the spherical basis. Such a relation defines the transition **T** matrix of the sphere . This is done in the appended papers for a transversely isotropic sphere. First, in Paper A, the problems is considered for a situation when only a torsional wave is considered and is incident along the symmetry axis of the anisotropic sphere. Therefore the problem is simplified to an axisymmetric situation. Then, using the same methodology in Paper B, the scattering of elastic waves by a transversely isotropic sphere with an arbitrary incident wave is studied. In both papers the **T** matrix is calculated and presented explicitly for low frequencies. The **T** matrix can then be used to calculate the scattering for any incident wave, a plane wave, a wave from an ultrasonic probe, etc. The **T** matrix can also be used as a tool when considering multiple scattering problems, like the scattering by two or more spheres, or the scattering by a sphere close to a planar interface.

Another important purpose of the project is to use the **T** matrix to study particle composites and grainy materials. Specifically, the **T** matrix may be used to calculate the attenuation and effective wave speed in these materials as long as the scattering by each particle or grain is so small that multiple scattering may be neglected. Such an assumption may be reasonable for the cases with low concentrations of the particles or very small scattering by each grain. This purpose is pursued in Paper B for polycrystalline materials with transversely isotropic grains.

## 1.2 Outline of the thesis

The extended summary of this thesis is structured as follows:

In Chapter 2, the elastodynamic relations describing a scattering problem are introduced for general anisotropic materials. Then some special cases including isotropic, orthotropic, cubic and transversely isotropic materials are introduced and their effect on the elastodynamic wave equations is discussed. Finally, the transformation of the wave equations into different system of coordinates is explained. The derivation of the general solution of the wave equations for isotropic materials in the polar and spherical system of coordinates is explained and the solutions are presented in terms of the vector wave functions.

Chapter 3 focuses on different scattering problems, specifically when the inclusion is not isotropic. An analytical approach to derive the solution in terms of the **T** matrix is discussed. First the solution of the 2D scattering problem by an anisotropic circle is outlined. Then a general approach to solve a group of 3D scattering problems by an anisotropic sphere is explained. In the last section an approximate method to derive the attenuation and the wave velocity in materials with a distribution of inclusions is briefly explained and the implementation to analyse polycrystalline materials is discussed.

A summary of the appended papers is presented in Chapter 4 and is followed by the last chapter with some concluding remarks and sharing some ideas for future works.

# 2 Elastodynamics

In this section, an outline is given of the principles of elasticity that are relevant to elastic wave propagation and scattering. This includes stress and strain definitions, constitutive relations, and governing equations. Isotropic media are in particular treated and different anisotropic media are introduced. Also, expressing an elastodynamic problem in different system of coordinates and possible solutions of them are discussed. There is a wide body of literature that covers several aspects of continuum mechanics and elastic wave propagation ([29, 30]). One can refer to them for comprehensive explanations of the concepts discussed in this section.

## 2.1 Basic equations

To start developing governing equations in an elastic medium, consider an infinitesimal surface with normal vector $\hat{\boldsymbol{n}}$ and surface area $dS$. The traction on this surface is defined by the force acting per unit surface area and is given by

$$\boldsymbol{t}^n = \boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}, \tag{2.1}$$

where $\boldsymbol{\sigma}$ is the stress tensor and is a crucial quantity to describe the governing equation in an elastic medium since having the stress tensor provides the force acting on any surface using eq. (2.1). Conservation of angular momentum leads to the symmetry property of the stress tensor, thus

$$\sigma_{ij} = \sigma_{ji}. \tag{2.2}$$

In general the stress in a medium depends on the deformation of the medium. The deformation can be defined using the displacement field $\boldsymbol{u}(\boldsymbol{x}, t)$, which varies with position

$\boldsymbol{x}$ and time $t$. The constant displacement field with respect to time is the static situation and the constant displacement field with respect to position shows rigid body motion and does not generate internal deformations. In the Cartesian tensor the displacement is written

$$\boldsymbol{u} = u_i \boldsymbol{e}_{x_i}, \tag{2.3}$$

where $(x_1, x_2, x_3)$ are Cartesian coordinates, $\boldsymbol{e}_{x_i}$ is the unit vector in $x_i$ direction and Einstein's summation convention is used throughout the section so that a repeated index is summed over $i = 1, 2, 3$. The quantity to describe deformation is strain and for small displacements and deformations, the linear strain tensor is defined by

$$\epsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j). \tag{2.4}$$

It is obvious from this definition that the strain tensor is symmetric.

The relation between strain and resulting stress is the constitutive relation which can be linearised for materials experiencing small deformations. Such a linearisation is expressed by Hooke's law and the constitutive relation can be written as

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}. \tag{2.5}$$

The fourth rank tensor $C_{ijkl}$ is the stiffness tensor which in three dimensions has 81 elements. However, due to symmetry of stress and strain together with energy considerations the stiffness tensor has the following properties

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}. \tag{2.6}$$

This reduces the number of independent elements of the fourth rank stiffness tensor to the number of independent elements of a symmetric $6 \times 6$ matrix which is 21. Such an analogy facilitates in expressing the stiffness tensor, thus an appropriate matrix representation of the stiffness tensor is used wherever needed. Most natural materials have fewer than 21 independent stiffness components. For an isotropic elastic materials the stiffness tensor depends only on the Lamé parameters $\lambda$ and $\mu$

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{2.7}$$

where $\delta_{ij}$ is the Kronecker delta. In this section a general elasticity tensor is considered and later on the isotropic case and some examples of other constitutive relations are discussed.

Considering the definition of traction, the force acting on a volume $V$ due to stresses can be calculated by integrating the traction over the surface of the volume $S$ ($\oint \boldsymbol{t}\, dS$). Using eq. (2.1) and Gauss theorem it can be shown that the force due to the stress acting on a volume is given by integrating the divergence of the stress tensor over the volume ($\int \partial_i \sigma_{ij} dV$). In addition to this force which is resulting from the deformation of the volume, forces like gravity or other external excitations may act on the volume. These forces are denoted the body forces. Now considering a unit volume Newton's second law can be written as

$$\partial_j \sigma_{ij} + b_i = \rho \ddot{u}_i, \tag{2.8}$$

where $\boldsymbol{b}$ is the body force per unit volume, $\rho$ is the density (mass per unit volume) and $\partial_j \sigma_{ij}$ shows the force acting on a unit volume due to the internal deformation. This is called the wave equation and is the governing equation inside an elastic medium. By considering eqs. (2.4) and (2.5) the governing equation can be written in terms of the displacement field $\boldsymbol{u}$

$$\partial_j (C_{ijkl} \partial_k u_l) + b_i = \rho \ddot{u}_i. \tag{2.9}$$

A common and facilitating approach to study the wave equation is to transform the governing equation into the frequency domain. The frequency domain formulation and the time harmonic domain formulation are related by the Fourier transform. For the temporal Fourier transform between time $t$ and angular frequency $\omega$ the following convention is used:

$$\tilde{\boldsymbol{u}}(\omega) = \int_{-\infty}^{\infty} \boldsymbol{u}(t) e^{i\omega t} dt, \tag{2.10}$$

with the inverse

$$\boldsymbol{u}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\boldsymbol{u}}(\omega) e^{-i\omega t} d\omega. \tag{2.11}$$

A frequency domain formulation removes all time derivatives and greatly facilitates the solution of the wave equation. In this research all the derivations are given in the frequency domain and the body forces are neglected. Considering these assumptions the wave equation simplifies to

$$\partial_j (C_{ijkl} \partial_k u_l) + \rho \omega^2 u_i = 0. \tag{2.12}$$

Such a wave equation is a second order differential equation for the displacement field $\boldsymbol{u}$. This differential equation needs to be supplemented by some boundary conditions on the boundary of the medium to complete a boundary value problem. In scattering problems the domain of study consists of at least two different parts where the boundary conditions apply to the interface of these domains. This interface may be a closed surface in the case of a bounded domain. By such definition, a wide range of different boundary value problems like wave propagation in a half space or wave propagation in an infinite medium consisting of a distribution of inclusions, may be considered as a scattering problem. From all these various types of scattering problems, the main interest of this research is type of the problems where an infinite domain named matrix bound at least one finite domain named obstacle or scatterer.

One of the main parameters of these scattering problems is the shape of the obstacle. The most common shapes, especially for analytical approaches, are circular shapes like spheres and cylinders in 3D and circles in 2D. The simplicity of these shapes in comparison with more complex ones is crucial for analytical analysis. They are useful to model grainy materials or fiber composites or on a bigger scale buried pipelines, and other practical problems. These types of geometry make it necessary to perform the calculation in curvilinear system of coordinates. Some of these system of coordinates are explained in detail in section 2.3.

The other main parameter is the material properties of the matrix and obstacle. The simplest cases are those with all the material properties being isotropic. However, plenty of synthetic and natural materials are not isotropic and this makes it necessary to study cases with anisotropic materials. Some more common types of anisotropy, which are particularly relevant of this study, are explained in section 2.2.

Another important parameter is the number of obstacles. The simplest case is when there is a single obstacle in the matrix. Such problems are briefly explained for 2D and 3D problems in sections 3.1 and 3.2, respectively, and for 3D cases a detailed discussion is presented in Paper A and Paper B. For the cases with two or more obstacles, multiple scattering (which is the influence of different obstacles on each other) makes the analysis more complicated. Different methods are developed to model multiple scattering with many obstacles, a simple one of these is discussed in section 3.3 and used in Paper B.

## 2.2 Anisotropy of solids

As it shown in eq. (2.6) the stiffness tensor which describes the material properties may have 21 independent elements. In most cases the material behaves similarly in some directions and consequently this number of elements is reduced. The simplest case is when the material behaviour is similar in all directions. These are called isotropic materials and the number of independent elements are reduced to two. The independent elements are the Lamé parameters $\lambda$ and $\mu$. In this case the equation of motion can be simplified to

$$(\lambda + 2\mu)\nabla(\nabla \cdot \boldsymbol{u}) - \mu\nabla \times (\nabla \times \boldsymbol{u}) = -\rho\omega^2\boldsymbol{u}. \tag{2.13}$$

For such a partial differential equation, Helmholtz decomposition is useful for the analysis. The Helmholtz decomposition for an arbitrary vector field like $\boldsymbol{u}$ is

$$\boldsymbol{u} = \nabla\Phi + \nabla \times \boldsymbol{\Psi}, \tag{2.14}$$

where $\Phi$ and $\boldsymbol{\Psi}$ are scalar and vector potentials, respectively. Substituting eq. (2.14) into the wave equation eq. (2.13) shows that the potentials should satisfy a Helmholtz equation

$$\begin{aligned} \nabla^2\Phi + k_p^2\Phi = 0, \\ \nabla\boldsymbol{\Psi} + k_s^2\boldsymbol{\Psi} = 0, \end{aligned} \tag{2.15}$$

where $k_p = \omega\sqrt{\rho/(\lambda + 2\mu)}$ and $k_s = \omega\sqrt{\rho/\mu}$. This shows that there are two types of waves propagating in the medium. One is the wave corresponding to the scalar potential $\boldsymbol{u} = \nabla\Phi$ which is a compressional wave and propagates with the wave number $k_p$. It can be observed that this wave is irrotational ($\nabla \times \boldsymbol{u} = 0$) and the displacement vector and propagation direction are aligned for a plane wave. Therefore, it is called a longitudinal wave and is often denoted a P wave. The other type of wave is corresponding to the vector potential $\boldsymbol{u} = \nabla \times \Psi$ which is a shear wave and propagates with the wave number $k_s$. It can be observed that this wave is equivoluminal ($\nabla \cdot \boldsymbol{u} = 0$) and the displacement vector and propagation direction are perpendicular to each other for a plane wave. This wave is called a transverse wave and is often denoted a S wave.

Using eq. (2.14) three quantities $u_i$ are related to four new dependent variables $\Phi$ and $\Psi_i$. Therefore, obviously one degree of arbitrariness is left unspecified for $\Phi$ and $\boldsymbol{\Psi}$ potentials. A simple and useful additional restriction on the potentials is to take the vector potential $\boldsymbol{\Psi}$ as divergence free i.e. $\nabla \cdot \boldsymbol{\Psi} = 0$. However, other types of restrictions are considered in the literature, especially to facilitate the solution of the vector potential by uncoupling some components of the fields in curvilinear coordinates. Such restrictions and the solution of scalar and vector potentials in polar and spherical coordinates is discussed in chapter 3.

Besides the isotropic case two special cases of an orthotropic material, which have plenty of applications, is mentioned here. An orthotropic material is characterized by three mutually orthogonal symmetry planes. Thus the number of stiffness constants for an orthotropic material are reduced to nine. The constitutive relations for such a material in Voigt notation are

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{yz} \\ \epsilon_{zx} \\ \epsilon_{xy} \end{Bmatrix}. \tag{2.16}$$

This is the stress-strain relation of a general orthotropic material. A special case is a cubic material where the material stiffness in all the three coordinate directions are equivalent so the material has three independent stiffness constants and the extra relations among the stiffness components of a general orthotropic material are

$$\begin{aligned} C_{11} &= C_{22} = C_{33}, \\ C_{12} &= C_{13} = C_{23}, \\ C_{44} &= C_{55} = C_{66}. \end{aligned} \tag{2.17}$$

Another special case is a transversely isotropic material. The stiffness for such materials are equal in all directions in a plane which is called the isotropic plane. Consequently the number of independent constants in a transversely isotropic material is five. Taking the $xy$ plane as the isotropic plane the extra relations among the stiffness constants of a general orthotropic material are

$$\begin{aligned} C_{11} &= C_{22}, \\ C_{13} &= C_{23}, \\ C_{44} &= C_{55}, \\ 2C_{66} &= C_{11} - C_{12}. \end{aligned} \tag{2.18}$$

In section 3.1 an example of the scattering by a circle with orthotropic material properties is explained briefly. In Paper A and Paper B the material properties of the obstacle (or obstacles) are assumed to be transversely isotropic.

## 2.3 System of coordinates

As mentioned it is more convenient to use curvilinear system of coordinates in many scattering problems. Here polar and spherical coordinates for 2D and 3D problems are explained and general solution of the wave equation (wave functions) in these system of coordinates are expressed.

For a 2D situation all fields are independent of $z$, the problem can be divided into out of plane and in plane waves and the displacement field can be decomposed as

$$\boldsymbol{u}(x,y) = \nabla\Phi + \nabla \times \boldsymbol{e}_z\Psi + \boldsymbol{e}_z u_z. \tag{2.19}$$

Here, three displacement components are related to three scalar quantities $\Phi$, $\Psi$ and $u_z$, all of which must satisfy Helmholtz equation

$$\begin{aligned} \nabla^2 u_z + k_s^2 u_z &= 0, \\ \nabla^2\Psi + k_s^2\Psi &= 0, \\ \nabla^2\Phi + k_p^2\Phi &= 0. \end{aligned} \tag{2.20}$$

Equations (2.19) and (2.20) decompose the displacement field into three waves, one P wave and two S waves. To distinguish the two different S waves, the third term in eq. (2.19), which is decoupled from the other two, is called the SH wave and the second term, which couples to P waves, is called the SV waves.

To shorten the discussion only in plane waves (P-SV) are considered for the 2D case. Therefore the third term in eq. (2.19) may be neglected and any convenient coordinates in the $xy$ plane may be used. Here the polar system of coordinates $(r,\varphi)$, which is commonly used in most scattering problems, is considered. This system of coordinates is defined by the following relations with respect to the Cartesian coordinates $(x,y)$

$$x = r\cos\varphi, \quad y = r\sin\varphi. \tag{2.21}$$

Consequently the expressions for the strains given in eq. (2.4) change to the following relations for polar coordinates

$$\begin{aligned} \epsilon_{rr} &= \partial_r u_r, \\ \epsilon_{\varphi\varphi} &= \frac{1}{r}(\partial_\varphi u_\varphi + u_r), \\ \epsilon_{r\varphi} &= \frac{1}{2r}(\partial_\varphi u_r - u_\varphi + r\partial_r u_\varphi) \end{aligned} \tag{2.22}$$

The equations of motion with respect to stresses in polar coordinates are

$$
\begin{aligned}
\partial_r \sigma_{rr} + \frac{1}{r}\partial_\varphi \sigma_{r\varphi} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} &= -\rho\omega^2 u_r, \\
\partial_r \sigma_{r\varphi} + \frac{1}{r}\partial_\varphi \sigma_{\varphi\varphi} + \frac{2\sigma_{r\varphi}}{r} &= -\rho\omega^2 u_\varphi.
\end{aligned} \tag{2.23}
$$

To get the equation of motion (eq. (2.23)) in terms of the displacement, constitutive relations must be used. The constitutive relations may depend on the system of coordinates. However, for an isotropic material the constitutive relations are independent of the system of coordinates and constitutive relations in 2D for polar coordinates may be written as

$$
\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\varphi\varphi} \\ \sigma_{r\varphi} \end{Bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 2\mu \end{bmatrix} \begin{Bmatrix} \epsilon_{rr} \\ \epsilon_{\varphi\varphi} \\ \epsilon_{r\varphi} \end{Bmatrix}. \tag{2.24}
$$

Substituting eqs. (2.22) and (2.24) into eq. (2.23) gives the equations of motion in terms of the displacement field in an isotropic medium. These equations of motion may be represented as in Equations (2.19) and (2.20). Solution of this system of equations is done by solving the Helmholtz equations in eq. (2.20) for the $\Phi$ and $\Psi$ potentials. Considering the definition of the $\nabla$ operator in polar coordinates

$$
\nabla = \frac{\partial}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial}{\partial \varphi}\hat{\boldsymbol{\varphi}}, \tag{2.25}
$$

and using separation of variables, the potentials $\Phi$ and $\Psi$ may be written as

$$
\begin{aligned}
\Phi^0 &= J_m(k_p r)\begin{pmatrix} \cos m\varphi \\ \sin m\varphi \end{pmatrix} \\
\Psi^0 &= J_m(k_s r)\begin{pmatrix} \cos m\varphi \\ \sin m\varphi \end{pmatrix}
\end{aligned} \tag{2.26}
$$

The upper index 0 on the potentials denotes that they are regular waves, containing Bessel functions $J_m$. The corresponding outgoing waves contain Hankel functions $H_m^{(1)}$ to satisfy the Sommerfeld radiation condition in the far field and are denoted by an upper index $+$.

To express the general solution of the equations of motion, it is convenient to introduce the following polar vector wave functions

$$
\begin{aligned}
\boldsymbol{\chi}^0_{1\sigma m} &= \sqrt{\epsilon_m}\frac{1}{k_s}\left(\nabla \times \hat{\boldsymbol{z}}\Psi^0\right) \\
&= \sqrt{\epsilon_m}\left[\boldsymbol{e}_r \frac{m}{k_s r} J_m(k_s r)\begin{pmatrix} \sin m\varphi \\ \cos m\varphi \end{pmatrix} + \boldsymbol{e}_\varphi J'_m(k_s r)\begin{pmatrix} -\cos m\varphi \\ \sin m\varphi \end{pmatrix}\right] \\
\boldsymbol{\chi}^0_{2\sigma m} &= \sqrt{\epsilon_m}\frac{1}{k_s}\left(\nabla\Phi^0\right) \\
&= \sqrt{\epsilon_m}\frac{k_p}{k_s}\left[\boldsymbol{e}_r J'_m(k_p r)\begin{pmatrix} \cos m\varphi \\ \sin m\varphi \end{pmatrix} + \boldsymbol{e}_\varphi \frac{m}{k_p r} J_m(k_p r)\begin{pmatrix} \sin m\varphi \\ -\cos m\varphi \end{pmatrix}\right]
\end{aligned} \tag{2.27}
$$

These are vector wave functions constructed from the scalar wave functions (eq. (2.26)) and Helmholtz decomposition of the displacement field (eq. (2.19)). Here the first index 1 or 2 on the wave functions denotes transverse or longitudinal waves, respectively. The second index $\sigma = e$ (even) or $o$ (odd) corresponds to the upper or lower row of the trigonometric functions, respectively. The even and odd vector wave functions are symmetric or antisymmetric with respect to $\varphi$, respectively. The Neumann factor is defined as $\epsilon_0 = 1$ and $\epsilon_m = 2$ for $m = 1, 2, \ldots$.

Now it is convenient to express the displacement field as a sum of the incident wave $\boldsymbol{u}^{in}$ (corresponds to the regular wave) and the scattered wave $\boldsymbol{u}^{sc}$ (corresponds to the outgoing wave)

$$\boldsymbol{u} = \boldsymbol{u}^{in} + \boldsymbol{u}^{sc} = \sum_{\tau,\sigma,m} (a_{\tau\sigma m}\boldsymbol{\chi}^{0}_{\tau\sigma m} + b_{\tau\sigma m}\boldsymbol{\chi}^{+}_{\tau\sigma m}), \tag{2.28}$$

where the coefficients $a_{\tau\sigma m}$ and $b_{\tau\sigma m}$ are incident and scattered wave coefficients, respectively. In the scattering problems considered in this thesis, the incident wave coefficients are in principle known and the scattered wave coefficients need to be determined. Here, the scattered wave is expanded in terms of the outgoing vector wave functions since the scattered wave as the additional part must satisfy radiation conditions.

In the spherical coordinates $(r, \theta, \varphi)$ the relations with Cartesian coordinates $(x, y, z)$ are

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta. \tag{2.29}$$

Also the strain displacement relations are

$$\begin{aligned}
\epsilon_{rr} &= \frac{\partial u_r}{\partial r}, \quad \epsilon_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + \frac{\cot\theta}{r}u_\theta + \frac{u_r}{r}, \\
\epsilon_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}, \quad \epsilon_{\theta\varphi} = \frac{1}{2r}\left(\frac{\partial u_\varphi}{\partial\theta} - \cot\theta u_\varphi + \frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial\varphi}\right), \\
\epsilon_{\varphi r} &= \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right), \quad \epsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right),
\end{aligned} \tag{2.30}$$

and the equations of motion in terms of the stresses in this system of coordinates are

$$\begin{aligned}
&\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{r\varphi}}{\partial\varphi} + \frac{1}{r}\left(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{r\varphi} + \cot\theta\sigma_{r\theta}\right) - \rho\frac{\partial^2 u_r}{\partial t^2} = 0, \\
&\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\theta\varphi}}{\partial\varphi} + \frac{1}{r}\left(\cot\theta\left(\sigma_{\theta\theta} - \sigma_{\varphi\varphi}\right) + 3\sigma_{r\theta}\right) - \rho\frac{\partial^2 u_\theta}{\partial t^2} = 0, \\
&\frac{\partial\sigma_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\varphi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\varphi\varphi}}{\partial\varphi} + \frac{1}{r}\left(3\sigma_{r\varphi} + 2\cot\theta\sigma_{\theta\varphi}\right) - \rho\frac{\partial^2 u_\varphi}{\partial t^2} = 0.
\end{aligned} \tag{2.31}$$

The constitutive relations may be transformed from the Cartesian to the spherical coordinates using the appropriate transformation matrix which is discussed in section 3.2. However, for an isotropic medium the constitutive relations are similar in the spherical and the Cartesian system of coordinates

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{\varphi\varphi} \\ \sigma_{\theta\varphi} \\ \sigma_{\varphi r} \\ \sigma_{r\theta} \end{Bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{Bmatrix} \epsilon_{rr} \\ \epsilon_{\theta\theta} \\ \epsilon_{\varphi\varphi} \\ \epsilon_{\theta\varphi} \\ \epsilon_{\varphi r} \\ \epsilon_{r\theta} \end{Bmatrix}. \tag{2.32}$$

Substituting eq. (2.32) into eq. (2.31) gives the equations of motion in terms of the displacements for an isotropic medium. This system of equations may be decomposed into three scalar potentials as

$$\boldsymbol{u}(\boldsymbol{r}) = \nabla\Phi + \nabla \times (\boldsymbol{r}\Psi_1) + \nabla \times \nabla \times (\boldsymbol{r}\Psi_2), \tag{2.33}$$

where $\Phi$, $\Psi_1$ and $\Psi_2$ are potentials associated to P, SH and SV waves, respectively. These potentials satisfy Helmholtz equations with wavenumbers $k_p$ for $\Phi$ and $k_s$ for $\Psi_1$ and $\Psi_2$. The $\nabla$ operator in spherical coordinates is defined as

$$\nabla = \frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\boldsymbol{e}_\varphi. \tag{2.34}$$

Using separation of variables to solve the Helmholtz equations in spherical coordinates leads to trigonometric functions $\cos m\varphi$ or $\sin m\varphi$, where $m = 0, 1, 2, ...$ for the azimuthal factor ($\varphi$), associated Legendre functions with $\cos\theta$ argument as $P_l^m(\cos\theta)$ with $l = m, m+1, m+2, ...$ for the polar factor ($\theta$) and spherical Bessel $j_m$ or Hankel $h_m^{(1)}$ functions for the radial factor ($r$), and the potentials may be written as

$$\begin{aligned} \Phi^0 &= j_m(k_p r)Y_{\sigma ml}(\theta,\varphi), \\ \Psi_1^0 &= j_m(k_s r)Y_{\sigma ml}(\theta,\varphi), \\ \Psi_2^0 &= j_m(k_s r)Y_{\sigma ml}(\theta,\varphi). \end{aligned} \tag{2.35}$$

Here again the upper index 0 denotes that they are regular waves, containing spherical Bessel functions $j_m$, and the corresponding outgoing waves denoted by + contain spherical Hankel functions $h_m^{(1)}$. The $Y_{\sigma ml}(\theta,\varphi)$ are called spherical harmonics with the following definition

$$Y_{\sigma ml}(\theta,\varphi) = \sqrt{\frac{\epsilon_m(2l+1)(l-m)!}{4\pi(l+m)!}}P_l^m(\cos\theta)\begin{Bmatrix} \cos m\varphi \\ \sin m\varphi \end{Bmatrix}, \tag{2.36}$$

where $\sigma = e$ is for the upper row which is even with respect to $\varphi$ and $\sigma = o$ is for the lower row which is odd with respect to $\varphi$. Indices $l$ and $m$ run through $m = 0, 1, 2, ...$ and $l = m, m+1, m+2, ....$

Here the general solution of the displacement field may be written in terms of the spherical vector wave functions introduced as

$$
\begin{aligned}
\boldsymbol{\psi}^0_{1\sigma ml}(r,\theta,\varphi) &= \frac{1}{\sqrt{l(l+1)}}\nabla\times\left(\Psi^0_1\right) = j_l(k_s r)\boldsymbol{A}_{1\sigma ml}(\theta,\varphi),\\
\boldsymbol{\psi}^0_{2\sigma ml}(r,\theta,\varphi) &= \frac{1}{\sqrt{l(l+1)}}\frac{1}{k_s}\nabla\times\nabla\times\left(\Psi^0_2\right)\\
&= \left(j_l'(k_s r)+\frac{j_l(k_s r)}{k_s r}\right)\boldsymbol{A}_{2\sigma ml}(\theta,\varphi)+\sqrt{l(l+1)}\frac{j_l(k_s r)}{k_s r}\boldsymbol{A}_{3\sigma ml}(\theta,\varphi),\\
\boldsymbol{\psi}^0_{3\sigma ml}(r,\theta,\varphi) &= \left(\frac{k_p}{k_s}\right)^{3/2}\frac{1}{k_p}\nabla\left(\Phi^0\right)\\
&= \left(\frac{k_p}{k_s}\right)^{3/2}\left((j_l'(k_p r)\boldsymbol{A}_{3\sigma ml}(\theta,\varphi)+\sqrt{l(l+1)}\frac{j_l(k_p r)}{k_p r}\boldsymbol{A}_{2\sigma ml}(\theta,\varphi)\right),
\end{aligned}
\tag{2.37}
$$

where the first index is denoted $\tau = 1, 2, 3$ for SH, SV and P wavefunctions, respectively. $\mathbf{A}_{\tau\sigma ml}$ are the vector spherical harmonics which are defined as

$$
\begin{aligned}
\boldsymbol{A}_{1\sigma ml}(\theta,\varphi) &= \frac{1}{\sqrt{l(l+1)}}\nabla\times(\boldsymbol{r}Y_{\sigma ml}(\theta,\varphi))\\
&= \frac{1}{\sqrt{l(l+1)}}\left(\boldsymbol{e}_\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}Y_{\sigma ml}(\theta,\varphi)-\boldsymbol{e}_\varphi\frac{\partial}{\partial\theta}Y_{\sigma ml}(\theta,\varphi)\right),\\
\boldsymbol{A}_{2\sigma ml}(\theta,\varphi) &= \frac{1}{\sqrt{l(l+1)}}r\nabla Y_{\sigma ml}(\theta,\varphi)\\
&= \frac{1}{\sqrt{l(l+1)}}\left(\boldsymbol{e}_\theta\frac{\partial}{\partial\theta}Y_{\sigma ml}(\theta,\varphi)+\boldsymbol{e}_\varphi\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}Y_{\sigma ml}(\theta,\varphi)\right),\\
\boldsymbol{A}_{3\sigma ml}(\theta,\varphi) &= \boldsymbol{e}_r Y_{\sigma ml}(\theta,\varphi).
\end{aligned}
\tag{2.38}
$$

Similarly as in 2D the general solution of the displacement field may be written as the sum of an incident and a scattered part

$$
\boldsymbol{u}(\boldsymbol{r}) = \boldsymbol{u}^{in}+\boldsymbol{u}^{sc} = \sum_{\tau\sigma ml}(a_{\tau\sigma ml}\boldsymbol{\psi}^0_{\tau\sigma ml}(\boldsymbol{r})+b_{\tau\sigma ml}\boldsymbol{\psi}^+_{\tau\sigma ml}(\boldsymbol{r})). \tag{2.39}
$$

In the following section, solving scattering problems is discussed with the means of concepts and quantities described here.

# 3 Scattering problems

A general description of a scattering problem is discussed in Chapter 2. The main purpose of this study is to extend the possible type of analytical solutions by solving the canonical problem of scattering of elastic waves by an anisotropic (transversely isotropic and orthotropic) sphere. To do so, first, the methodology of the current approach is briefly explained within an example of scalar 2D scattering by a circle with orthotropic material, then the expansion of such an approach to 3D scattering by a sphere is discussed. Later on the simple theory due to Foldy is explained to study scattering of elastic waves in a medium with a distribution of inclusions (obstacles).

## 3.1 2D scattering by a circle

2D scattering by a circle with orthotropic material properties is studied by Boström ([14, 15]) and here a brief description of this approach is presented. Consider a circle of radius $a$ with orthotropic material properties, residing in an infinite homogeneous isotropic medium. The material properties of the surrounding medium are density $\rho$ and Lamé parameters $\lambda$ and $\mu$. The material properties of the circle are density $\rho_1$ and stiffness constants $C_1, C_2, C_3$ and $C_4$, in which the constitutive relations in Cartesian coordinates $(x, y)$ are

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{yx} \end{Bmatrix} = \begin{bmatrix} C_1 & C_2 & 0 \\ C_2 & C_3 & 0 \\ 0 & 0 & 2C_4 \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{yx} \end{Bmatrix} \tag{3.1}$$

Due to the geometry of the problem, it is of course natural to use polar coordinates $(r, \varphi)$. It is then convenient to use the polar wave functions introduced in eq. (2.27) to describe field quantities outside the circle.

The displacement field outside the circle can be defined as in eq. (2.28), where the coefficients $a_{\tau\sigma m}$ of the incident wave are in principle known and the coefficients $b_{\tau\sigma m}$ of the scattered wave need to be determined. A suitable approach is to find the relation between the scattered wave coefficients and the incident wave coefficients. This relation defines the transition $\mathbf{T}$ matrix as

$$b_{\tau\sigma m} = \sum_{\tau'\sigma'm'} T_{\tau\sigma m,\tau'\sigma'm'} a_{\tau'\sigma'm'}. \tag{3.2}$$

With this approach the field description is available for an arbitrary incident wave.

To find the $\mathbf{T}$ matrix elements boundary conditions on the boundary of the circle are required. The boundary conditions are continuity of the displacement and traction on $r = a$. The displacement field of the surrounding medium is described as in eq. (2.28) and the traction in radial direction may be derived by the following traction operator

$$\boldsymbol{t}^{(r)}(\boldsymbol{u}) = \lambda \nabla \cdot \boldsymbol{u} \boldsymbol{e}_r + 2\mu \frac{\partial \boldsymbol{u}}{\partial r} + \mu \boldsymbol{e}_r \times (\nabla \times \boldsymbol{u}). \tag{3.3}$$

To apply the boundary conditions, the displacement and traction field inside the circle also need to be determined. The first step to do so is to express the governing equations inside the circle in polar coordinates. Therefore, the constitutive relations need to be transformed to polar coordinates as

$$\begin{aligned}
\sigma_{rr} =& (\alpha_1 + 2\alpha_2)\,\epsilon_{rr} + \alpha_1\epsilon_{\varphi\varphi} + 2\beta_1\,(\epsilon_{rr}\cos 2\varphi - \epsilon_{r\varphi}\sin 2\varphi) \\
&+ \beta_2\,((\epsilon_{rr} - \epsilon_{\varphi\varphi})\cos 4\varphi - 2\epsilon_{r\varphi}\sin 4\varphi)\,, \\
\sigma_{\varphi\varphi} =& (\alpha_1 + 2\alpha_2)\,\epsilon_{\varphi\varphi} + \alpha_1\epsilon_{rr} - 2\beta_1\,(\epsilon_{\varphi\varphi}\cos 2\varphi + \epsilon_{r\varphi}\sin 2\varphi) \\
&+ \beta_2\,((\epsilon_{\varphi\varphi} - \epsilon_{rr})\cos 4\varphi - 2\epsilon_{r\varphi}\sin 4\varphi)\,, \\
\sigma_{r\varphi} =& 2\alpha_2\epsilon_{r\varphi} + \beta_1\,((\epsilon_{\varphi\varphi} + \epsilon_{rr})\sin 2\varphi) + \beta_2\,((\epsilon_{\varphi\varphi} - \epsilon_{rr})\sin 4\varphi - 2\epsilon_{r\varphi}\cos 4\varphi)\,,
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
\alpha_1 &= \frac{1}{8}(C_1 + 6C_3 + C_2 - 4C_4), \quad \alpha_2 = \frac{1}{8}(C_1 + C_2 - 2C_3 + 4C_4), \\
\beta_1 &= \frac{1}{4}(C_1 - C_2), \quad \beta_2 = \frac{1}{8}(C_1 + C_2 - 2C_3 - 4C_4).
\end{aligned} \tag{3.5}$$

It is observed that $\beta_1$ and $\beta_2$ are measures of the anisotropy degree and vanish for an isotropic medium ($\beta_1 = \beta_2 = 0$). Also, $\alpha_1$ and $\alpha_2$ can be considered as some mean stiffnesses and for an isotropic medium with Lamé constants $\lambda_1$ and $\mu_1$, they simplify to $\alpha_1 = \lambda_1$ and $\alpha_2 = \mu_1$.

With the constitutive relations and strain definition in polar coordinates the equations of motion inside the circle are transformed to polar coordinates. As the stress strain relation (eq. (3.4)), the equations of motion also depend explicitly on trigonometric functions $\cos 2\varphi$, $\cos 4\varphi$, $\sin 2\varphi$ and $\sin 4\varphi$ in the azimuthal angle. Due to symmetry the solution inside the circle can be divided into four independent parts that are symmetric or antisymmetric with respect to the $x$ and $y$ axes.

The procedure for solving such a system of equations is to make an expansion of the displacement field in terms of appropriate orthogonal functions. Looking at the polar wave functions where the displacement field is expressed in trigonometric functions of the azimuthal coordinate, and considering that inside the circle the stress relations and equations of motion depend on trigonometric functions, it is useful to expand $\boldsymbol{u}$ inside the circle in a series of $\cos m\varphi$ and $\sin m\varphi$ where the expansion coefficients are functions of $r$. Consequently, the governing equations change to a coupled system of ordinary differential equations for the expansion coefficients, which may be solved by a power series expansion in $r$. For instance, the displacement field expansion for the part that is doubly symmetric can be written as

$$\begin{aligned}
u_r(r,\varphi) &= f_{01}r + f_{03}r^3 + \ldots + \cos 2\varphi(f_{21}r + f_{23}r^3 + \ldots) + \ldots, \\
u_\varphi(r,\varphi) &= \sin 2\varphi(g_{21}r + g_{23}r^3 + \ldots) + \ldots.
\end{aligned} \tag{3.6}$$

Here $f_{ij}$ and $g_{ij}$ are the unknown expansion coefficients which are labelled so that the first index refers to the azimuthal order and the second one refers to the power of the radial coordinate. The power series expansion follows from the regularity condition at $r = 0$ and the trigonometric expansion follows from the symmetry requirements. The corresponding stresses can be calculated by substituting these expansions in eq. (3.4). Insertion of the resulting stress relations into the equations of motion leads to some recursion relations among the unknown coefficients. For the doubly symmetric case and considering only the terms expressed explicitly in eq. (3.6) these relations are

$$\begin{aligned} &g_{21} = -f_{21}, \qquad 8(\alpha_1 + 2\alpha_2) f_{03} + 12\beta_1 f_{23} + 24\beta_2 f_{43} + \rho_1\omega^2 f_{01} = 0, \\ &12\beta_1(f_{03} + f_{43}) + (8\alpha_1 + 12\alpha_2 + 4\beta_2) f_{23} + 4(\alpha_1 - \beta_2) g_{23} + \rho_1\omega^2 f_{21} = 0. \end{aligned} \tag{3.7}$$

With the solution inside the circle complete, the scattering problem may be solved by finding the unknown expansion coefficients using the boundary conditions. The elements of the transition (**T**) matrix of the circle can then be calculated explicitly. However, a general expression of the **T** matrix elements are too complicated and cumbersome to calculate (but it can be done numerically). Therefore, only the dominant low frequency **T** matrix elements are calculated to lowest order and given explicitly. To obtain the dominant **T** matrix elements for the low frequency, it is sufficient to consider the equations for $m = 0$, $m = 1$ and $m = 2$. Considering the example of doubly symmetric case for the even values of $m$ ($m = 0$ and $m = 2$), the boundary conditions for the displacement and traction fields provide four equations for the radial components and two equations for the azimuthal components. The unknowns are the scattered field coefficients $b_{2e0}, b_{2e2}$ and $b_{1e2}$ and the expansion coefficients inside the circle which are those expressed explicitly in eq. (3.6) $(f_{01}, f_{03}, f_{21}, f_{23}, g_{21}, g_{23})$. Therefore, there exist nine unknowns that can be found with the system of equations consisting of six equations from boundary conditions and three recursion relations among the unknown coefficients inside the circle (eq. (3.7)). Finally, the scattered wave coefficients can be derived in terms of the incident wave coefficients and expressed in form of the **T** matrix. The **T** matrix elements of a circle with orthotropic material and a detailed explanation of such a scattering problems are given by Boström [15].

## 3.2 3D scattering by a sphere

For the 3D scattering by an anisotropic sphere a similar approach is taken to solve the problem. Consider the scattering of an elastic wave by a transversely isotropic spherical obstacle contained in a three-dimensional, homogeneous, isotropic, and infinite elastic medium. The material properties of the surrounding medium are density $\rho$ and stiffnesses $\lambda$ and $\mu$, and the transversely isotropic medium is defined by density $\rho_1$ and stiffness constants $C_{11}, C_{12}, C_{13}, C_{33}$ and $C_{44}$. The stiffness constants obey the constitutive relation as in eqs. (2.16) and (2.18), which means that the $z$ axis is considered as the axis of anisotropy for the sphere.

Here the spherical system of coordinates is an appropriate choice based on the geometry of the problem. Thus, the displacement field outside the sphere may be written in terms of the spherical vector wave functions as in eq. (2.39) and the radial traction which is necessary for applying boundary conditions may be derived as

$$\boldsymbol{t}^{(r)} = \boldsymbol{e}_r \lambda \nabla \cdot \boldsymbol{u} + \mu \left( 2 \frac{\partial \boldsymbol{u}}{\partial r} + \boldsymbol{e}_r \times (\nabla \times \boldsymbol{u}) \right). \tag{3.8}$$

The starting point for the fields inside the sphere is to state the anisotropic elastodynamic equations with respect to displacement in spherical coordinates. This is done by first transforming the constitutive relations expressed in eqs. (2.16) and (2.18) into spherical coordinates. To do so, the following relation for the transformation of a second rank tensor like stress and strain from the Cartesian to the spherical coordinates is used

$$\mathbf{S}_s = \mathbf{R}^T \mathbf{S}_c \mathbf{R}, \tag{3.9}$$

where $\mathbf{S}_s$ and $\mathbf{S}_c$ are second rank tensors in the spherical and the Cartesian coordinates, respectively, and $\mathbf{R}$ is the rotation matrix with the following appearance

$$\mathbf{R} = \begin{bmatrix} \cos\varphi \sin\theta & \cos\varphi \cos\theta & -\sin\varphi \\ \sin\varphi \sin\theta & \sin\varphi \cos\theta & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}. \tag{3.10}$$

Substituting the transformed relations into the equations of motion in terms of stresses (eq. (2.31)) leads to a system of partial differential equations that contains trigonometric functions with argument $2\theta$ and $4\theta$ similarly as in the 2D example. As the 2D case the idea is to expand the displacement field in terms of orthogonal bases. It can be observed from the spherical vector wave functions that the displacement field outside the sphere is in terms of vector spherical harmonics in the angular directions. The vector spherical harmonics constitute an orthonormal base and thus are appropriate candidates for expansion of the displacement field inside the sphere. Such expansions facilitate the application of the boundary conditions as well. Such an approach is first taken in Paper A where the problem is axisymmetric (meaning that all fields are $\varphi$ independent) and the vector spherical harmonics consist of only the associated Legendre functions in the $\theta$ coordinate. Then a general expansion is explained in Paper B. In both cases these expansions change the equations of motion inside the sphere to a system of ordinary differential equations with respect to $r$, which can be solved using power series expansions in $r$.

Finally, using the boundary conditions on the surface of the sphere results in a system of equations for all the unknown expansion coefficients of the fields outside and inside the sphere. This makes it possible to determine the $\mathbf{T}$ matrix elements for the scattering by a single sphere. Details of such calculations are discussed in Paper A and Paper B.

## 3.3 Wave numbers in polycrystals

Polycrystalline materials are solids that consist of many small crystals which are usually called grains. The grains are normally anisotropic, have random crystallographic orientations and are separated by grain boundaries. Therefore, propagation of a wave through polycrystalline materials can be considered a special case of a distribution of inclusions in a matrix. Propagation of a wave in an elastic solid with a distribution of elastic inclusions induce scattered waves and consequently some of the incident wave energy is transferred into the scattered waves. Due to the multiple scattering the wave experiences attenuation and dispersion, meaning that the wave decays and its effective phase velocity changes in a way so that both the attenuation and the phase velocity are dependent on frequency. In other words, an elastic solid with a distribution of inclusions acts like an attenuative and dispersive solid, even though the matrix and inclusions are perfectly elastic and do not dissipate energy [3]. Note also that there are other factors rather than scattering that is affecting wave attenuation and dispersion. However, here only the scattering effect is considered.

The interest of the current study is mainly to investigate the overall average response of the polycrystalline material, rather than the local effects of individual grains. Therefore, the basic idea is to model the original, inhomogeneous polycrystal as a statistically homogeneous medium, where both mediums should approximately have the same overall average response. Various approximate methods are developed to obtain the attenuation and effective phase velocity of the wave in polycrystals. Most of these estimations are using volume integral equation methods combined with some perturbation method, often the Born approximation (see for instance [17, 18, 19, 20]). However, these studies are restricted to polycrystalline media with weakly anisotropic grains. In this research another approach based on the theory of Foldy ([21]) is used to calculate attenuation and effective phase velocity of the wave. This theory was developed for waves propagating in an acoustic medium with a distribution of point scatterers, but has been generalized to the elastic case [31]. However, it is still reasonable to apply such an approach to polycrystalline materials.

In the theory of Foldy, the solution procedure consists of three steps. First the average forward scattering amplitude of a single grain is determined, then the equation of Foldy is used to calculate the complex effective wave number based on the average forward scattering of a single grain and the density of grains. The effective wave velocity and attenuation coefficient are then obtained by taking the real and imaginary parts of the complex effective wave number.

To calculate the average forward scattering, first the far field amplitude of the scattered wave needs to be calculated. This can be done by using the asymptotic behaviour of the scattered wave when $r \to \infty$. The far field scattered wave in spherical coordinates may be written as

$$\boldsymbol{u}^{sc}(r,\theta,\varphi) = \sum_{\tau\sigma ml} b_{\tau\sigma ml}\boldsymbol{\psi}^{+}_{\tau\sigma ml}(\boldsymbol{r}) = \frac{e^{ik_s r}}{k_s r}\boldsymbol{f}_s(\theta,\varphi) + \frac{e^{ik_p r}}{k_p r}\boldsymbol{f}_p(\theta,\varphi), \tag{3.11}$$

where the $e^{ikr}/kr$ terms are the asymptotic form of the spherical Hankel functions in the far field. The vector functions $\boldsymbol{f}_p$ and $\boldsymbol{f}_s$ are the far field amplitudes for P and S waves respectively, and are defined as

$$
\begin{aligned}
\boldsymbol{f}_s(\theta,\varphi) &= \sum_{\sigma ml} i^{-l}(-ib_{1\sigma ml}\boldsymbol{A}_{1\sigma ml}(\theta,\varphi) + b_{2\sigma ml}\boldsymbol{A}_{2\sigma ml}(\theta,\varphi)),\\
\boldsymbol{f}_p(\theta,\varphi) &= \left(\frac{k_p}{k_s}\right)^{3/2} \sum_{\sigma ml} i^{-l} b_{3\sigma ml}\boldsymbol{A}_{3\sigma ml}(\theta,\varphi).
\end{aligned} \tag{3.12}
$$

The avarage forward scattering amplitude can be calculated by taking the mean of the far field amplitudes over all directions of the incident wave, which is the same as the mean over all orientations of the inclusions. Using the definition of the **T** matrix the average forward scattering amplitude may be written in terms of the **T** matrix elements as [32]

$$
\begin{aligned}
\bar{f}_p &= -\frac{i}{k_p}\sum_{\sigma ml} T_{3\sigma ml,3\sigma ml},\\
\bar{f}_s &= -\frac{i}{2k_s}\sum_{\substack{\tau\sigma ml\\ \tau=1,2}} T_{\tau\sigma ml,\tau\sigma ml}.
\end{aligned} \tag{3.13}
$$

In the next step, the complex effective wavenumbers need to be calculated using the equation of Foldy which may be stated as [31]

$$
K_i^2 = k_i^2 + 4\pi N\bar{f}_i, \tag{3.14}
$$

where the index $i$ can be P or S for longitudinal and transverse waves, $N$ is the number density of scatterers and $k_i$ is the wave number in the absence of scatterers. The essential assumption made in this relation is the neglect of interaction among the scatterers or multiple scattering effects. Therefore it is valid for a dilute distribution of scatterers (small values of $N$). However, under the condition that each grain scatters extremely little, neglect of the multiple scattering should be valid even for a high concentration of inclusions, and this is the case for the low frequency wave scattering in polycrystalline materials.

Finally, in analogy to viscoelastic wave propagation in a homogeneous material, the average dynamic response of the medium including attenuation and phase velocity can be described by the effective wave number as

$$
K(\omega) = \frac{\omega}{C(\omega)} + i\alpha(\omega), \tag{3.15}
$$

where $C$ is the effective phase velocity of the wave and $\alpha$ is the attenuation coefficient. Therefore having the effective wave number as in eq. (3.14) immediately leads to the determination of phase velocity and attenuation coefficient as

$$
\begin{aligned}
\frac{\alpha_i}{k_i} &= \operatorname{Im}\frac{K_i}{k_i},\\
\frac{C_i}{c_i} &= \operatorname{Re}\frac{k_i}{K_i},
\end{aligned} \tag{3.16}
$$

Such an approach is taken in paper B and explicit expressions are presented for calculating attenuation and phase velocity for low frequencies for polycrystalline materials with transversely isotropic grains.

# 4 Summary of Appended Papers

## 4.1 Paper A

In this paper, a single anisotropic spherical obstacle contained in a three-dimensional, homogeneous, isotropic and infinite elastic medium is considered. For this canonical problem a transversely isotropic sphere is considered in which the axis of material symmetry is perpendicular to the $xy$ plane. A spherical coordinate system ($r$, $\theta$, $\varphi$) is set at the center of the sphere and an incident plane wave propagates in the $z$ direction. Such restriction on the incident wave makes the problem rotationally symmetric and consequently the displacement and stress fields are independent of the azimuthal angle $\varphi$. Based on the polarization of the incident wave the problem can be decomposed into two independent problems, one relating to motion in the $\varphi$ direction (torsional waves, also known as SH waves) and the other relating to motion in the $r\theta$ plane (P-SV waves). Only the SH waves which is a scalar case is considered in Paper A.

To solve the problem, the displacement field in the isotropic medium outside the sphere is constructed as a superposition of incident and scattered waves, which are expanded in spherical vector wave functions. In the anisotropic sphere first the elastodynamic equations are transformed into spherical coordinates and then the displacement field is expanded in associated Legendre functions in $\theta$ and powers in $r$. Substituting the expansion into the equation of motion inside the sphere leads to recursion relations for the expansion coefficients that couple different polar orders (in contrast to an isotropic sphere where there is no such coupling). Using the boundary conditions on the surface of the sphere results in a system of equations for all the expansion coefficients for the fields outside and inside the sphere. As a result, the transition ($\mathbf{T}$) matrix elements are calculated and given explicitly for low frequencies. Some numerical examples of scattered far fields are also presented.

## 4.2 Paper B

Paper B is in continuation of Paper A. Here the same problem as in Paper A is considered, however, this time no restriction is applied on the incident wave, meaning that it may be of any type, for example a plane wave of arbitrary type and in any direction. Therefore, the axisymmetric assumption made in Paper A is not valid anymore and all fields depend on all the spherical coordinates. The general approach taken in this paper is similar to the one in Paper A. Here again the elastodynamic equations inside the sphere are transformed to spherical coordinates and the displacement field is expanded in the vector spherical harmonics in the angular coordinates and powers in the radial coordinate. Then recurrence relations among the expansion coefficients can be found using the governing equations inside the sphere. In the surrounding medium the displacement and traction fields are expanded in terms of the spherical vector wave functions. All the remaining expansion coefficients for the fields outside and inside the sphere are found using the boundary conditions on the surface of the sphere. As a result, the transition (**T**) matrix elements are calculated and given explicitly for low frequencies.

In the B, this **T** matrix is used to study wave propagation in polycrystalline materials with transversely isotropic crystals. Here the theory of Foldy is used to give an explicit expression for the effective complex wavenumber of such materials for low frequencies. The attenuation coefficients and phase velocities of the wave in the material are then directly derived from the effective wave number. These quantities are numerically compared with previously published results and with recent FEM results. The comparison with FEM for low frequencies show a very good agreement regardless of the degree of anisotropy while the validity of other published methods is restricted to weakly anisotropic materials.

# 5 Concluding Remarks and Future Work

This thesis demonstrates an analytical approach to study scattering by an anisotropic sphere. A valuable advantage of analytical approaches is the better insights of the various aspects of the problem. Such understanding in this research starts with the expression of the stresses for a transversely isotropic sphere transformed to spherical coordinates. It is worth to highlight the direct dependence of the stresses on trigonometric functions of the polar coordinate. It is shown in the thesis how such a dependence leads to coupling between different orders of the incident wave, which do not happen in the isotropic cases. The complexity of the transformed wave equations for an anisotropic material is then overcome by introducing a systematic series expansion in terms of appropriate orthogonal functions in the angular directions and power series in the radial direction. The governing equations inside the sphere lead to recurrence relations, which relate the expansion coefficients inside the sphere to each other.

Another insight comes from the explicit expressions of the **T** matrix elements for low frequencies. It can clearly be observed that for low frequencies it is the P-SV waves that play an dominant role and the SH waves are of limited interest. To be more specific, the leading order **T** matrix elements of the SH waves behave as $(ka)^5$ while for the P-SV waves they behave as $(ka)^3$. Furthermore, such explicit expressions clearly demonstrate how different **T** matrix elements depend on the stiffness constants and density of the sphere and the surrounding medium. It is worth mentioning that there is no particular problem in numerically computing the **T** matrix elements for higher frequencies.

One important application of these **T** matrices is to derive explicit expressions for attenuation coefficients and effective phase velocities in a polycrystalline materials with transversely isotropic grains. Although these expressions are limited to low frequencies, they have no restriction on the polycrystalline degree of anisotropy in contrast to other published analytical approaches.

The present methods can be extended in various directions. It is, of course, of interest to consider more general anisotropy, in particular cubic and orthotropic materials. A cubic material has only three stiffness constants, while a transversely isotropic material has five, still this is a more difficult problem because there is no rotational symmetry so there is coupling both in the polar and azimuthal directions. For a cubic material there is published results both with analytical methods valid for low degree of anisotropy and with FEM.

Another extension of the present work is to consider somewhat higher frequencies. The **T** matrix must then be computed numerically, but the restriction of the Foldy approach used so far may then be important. There exist refinements of Foldy's approach that take some care of multiple scattering that can be tried. Another way to extend the present results is to consider a distribution of grain sizes and this should be straightforward. A further extension is to consider a material with different types of grains, for example duplex materials.

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