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Descriptive Set Theory and Applications

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Abstract

The systematic study of Polish spaces within the scope of Descriptive Set Theory furnishes the working mathematician with powerful techniques and illuminating insights. In this thesis, we start with a concise recapitulation of some classical aspects of Descriptive Set Theory which is followed by a succinct review of topological groups, measures and some of their associated algebras.
The main application of these techniques contained in this thesis is the study of two families of closed subsets of a locally compact Polish group G, namely $\mathcal{U}(G)$ - closed sets of uniqueness - and $\mathcal{U}_0(G)$ - closed sets of extended uniqueness. We locate the descriptive set theoretic complexity of these families, proving in particular that $\mathcal{U}(G)$ is Π_1^1-complete whenever $G/\overline{[G,G]}$ is non-discrete, thereby extending the existing literature regarding the abelian case. En route, we establish some preservation results concerning sets of (extended) uniqueness and their operator theoretic counterparts. These results constitute a pivotal part in the arguments used and entail alternative proofs regarding the computation of the complexity of $\mathcal{U}(G)$ and $\mathcal{U}_0(G)$ in some classes of the abelian case.
We study $\mathcal{U}(G)$ as a calibrated Π_1^1 σ-ideal of $\mathcal{F}(G)$ - for G amenable - and prove some criteria concerning necessary conditions for the inexistence of a Borel basis for $\mathcal{U}(G)$. These criteria allow us to retrieve information about G after examination of its subgroups or quotients. Furthermore, a sufficient condition for the inexistence of a Borel basis for $\mathcal{U}(G)$ is proven for the case when G is a product of compact (abelian or not) Polish groups satisfying certain conditions.
Finally, we study objects associated with the point spectrum of linear bounded operators $T \in \mathcal{L}(X)$ acting on a separable Banach space X. We provide a characterization of reflexivity for Banach spaces with an unconditional basis : indeed such space X is reflexive if and only if for all closed subspaces $Y \subseteq X, Z \subseteq X^*$ and $T \in \mathcal{L}(Y), S \in \mathcal{L}(Z)$ it holds that the point spectra $\sigma_p(T), \sigma_p(S)$ are Borel. We study the complexity of sets prescribed by eigenvalues and prove a stability criterion for Jamison sequences.

Keywords : Descriptive Set Theory, Harmonic Analysis, Thin Sets, Sets of Uniqueness, Operator U-sets, Operator U_0-sets, Fourier Algebra, Point Spectrum, Reflexivity, Jamison Sequences

Contents

1 Preamble

1.1 A background for Descriptive Set Theory

The mathematical landscape of the late 1800's and early 1900's was inexorably shaped by the dawn of revolutions which brought metamathematical fertility to the soil. From the ideas of Cantor - that not only introduced *one* but *many* completed infinities - to the logicism[1] project of Frege - sponsored by Russell and Hilbert, eventually driven to pessimism by Gödel - the foundations of mathematics were at stake. From the intuitionism[2] and constructivism[3] of Brouwer and Kleene to the innumerous ideas emerging from tangential areas concerning ontological and epistemological aspects of mathematics, philosophy of language and philosophy of science in general, this was a time for skepticism and attempts towards a robust axiomatization of mathematical truth. Within this state of affairs, Lebesgue, Borel and Baire were studying regularity properties of subsets of real numbers, aiming to provide a reasonable framework to study the abstract idea of a *function*, introduced by Riemann and Dirichlet. In particular, Lebesgue was concerned with definability issues (cf. [51]), i.e. how to decide when such objects are permissible to *exist*, based on which expressions define them. It was indeed a mistake in [51], detected by Suslin, that sparked the birth of what is nowadays called (classical) Descriptive Set Theory. From a modern point of view, Descriptive Set Theory can be defined as the systematic study of Polish spaces (i.e. completely metrizable and separable topological spaces) and their properties. In the early days, several *regularity* properties were proven to hold for the set of Borel subsets of a Polish space X (for instance, in ZFC the Continuum Hypothesis holds automatically for those sets) - that is stratified as follows :

(i) $\mathbf{\Sigma^0_1} = \{\text{open sets of } X\}$ and $\mathbf{\Sigma^0_\alpha} = \{\bigcup_n A_n : A_n \in \mathbf{\Pi^0_{\beta(n)}},\ \beta(n) < \alpha\}$, for all $1 \leq \alpha < \omega_1$

(ii) $\mathbf{\Pi^0_\alpha} = \neg\mathbf{\Sigma^0_\alpha}$, for all $1 \leq \alpha < \omega_1$

(iii) $\mathbf{\Delta^0_\alpha} = \mathbf{\Sigma^0_\alpha} \cap \mathbf{\Pi^0_\alpha}$, for all $1 \leq \alpha < \omega_1$

We start with *elementary* sets, that ought to exist - these are the open sets, which are essential for the working mathematician and perhaps not too hard

[1] *"The logicist project consists in attempting to reduce mathematics to logic..."*, in Stanford Encyclopedia of Philosophy

[2] *"Intuitionism is based on the idea that mathematics is a creation of the mind. The truth of a mathematical statement can only be conceived via a mental construction that proves it to be true (...).This view on mathematics has far reaching implications for the daily practice of mathematics, one of its consequences being that the principle of the excluded middle,* $(A \vee \neg A)$*, is not longer valid."*, in Stanford Encyclopedia of Philosophy

[3] *"Constructive mathematics is distinguished from its traditional counterpart, classical mathematics, by the strict interpretation of the phrase 'there exists' as 'we can construct'"*., in Stanford Encyclopedia of Philosophy

to be tamed by our intuition - at least in separable metric spaces. We accept countable unions and taking complements as admissible operations available to us in order to generate new sets. It is clear that through this process, we generate all Borel sets. Surely these sets can be quite complicated but at least, from an heuristic perspective, their right to exist and their constructivist status quo is hardly challenged by most of mathematicians. Considering projections as an additional available tool to construct objects, we get the analytic sets. Lebesgue erroneously claimed in [51] that a projection of a Borel subset of the plane onto the real line is again a Borel set. Suslin detected the mistake, establishing the non redundancy of projections. The properties of analytic sets were then extensively studied in Moscow and Warsaw, led respectively by Suslin, Lusin and Sierpinski. The projective hierarchy on X is defined as follows :

(i) $\mathbf{\Sigma^1_1} = \{A \subseteq X : A = \pi(B), B \subseteq X^2, B \text{ Borel}\}$ and $\mathbf{\Pi^1_1} = \neg\mathbf{\Sigma^1_1}$

(ii) $\mathbf{\Sigma^1_{n+1}} = \{A \subseteq X : A = \pi(B), B \subseteq X^2, B \in \mathbf{\Pi^1_n}\}$ and $\mathbf{\Pi^1_{n+1}} = \neg\mathbf{\Sigma^1_{n+1}}$

The study of these hierarchies - where one keeps generating new sets which are, in some vague sense, more *complex* than the pre-existing ones - is a quintessential topic of research in Descriptive Set Theory. This meritorious interest is certainly justified at the level of the set theorist. In fact, the study of regularity properties of these *new* projective sets revealed to be a much harder task than with the case of Borel sets. And there was a very fundamental reason for this : indeed much of those properties for projective sets - provable in ZFC for Borel sets - strongly rely on extra set theoretic assumptions, being independent statements within the standard axioms accepted by most of the mathematical community. As it turned out, several combinatorial principles and regularity properties no longer hold automatically for some of these sets, as it was the case with the Borel hierarchy.[4] As a consequence, the study of these hierarchies was inevitably of interest for those who study choice principles and weaker/alternative axioms (to those of ZFC). Perhaps, it is worth to highlight how much permeable every other branch of mathematics is to the ramifications that come from starting with *even just slightly* different axioms. Unsurprisingly, this catalyses further interest for any working mathematician. If truth is painted by words through social practices, then the paintbrush was embodied by large cardinal axioms and the study of determinacy of infinite games. Throughout the years, Descriptive Set Theory established itself as an active and important part of Logic and Set Theory with a remarkably strong affinity for interdisciplinary efforts ranging from Harmonic Analysis to Dynamical Systems, from Group Theory to Computer Science.

We notify the reader that in the remaining sections of this thesis, we will drop the ***bold face*** *notation for the Borel (and projective) hierarchy - we use it here in order to distinguish those hierarchies from the (hyper)arithmetical setting, denoted in the remaining of this section with the light face notation.*

[4] In this thesis, we merely mention some examples such as the Perfect Set Property, Property of Baire (Section 3.3) or Borel determinacy (Section 3.4).

Conceivably, the more scrutinous reader could raise some objections against the argument that open sets ought to exist as elementary particles just because they are ubiquitous in the life of the analyst. In order to provide plausibility to the idea, we try to unify the Borel and the projective hierarchies within a bigger picture. This leads us to a small detour in logic and recursion theory. The latter was heavily influenced by Kleene and Mostowski who developed a deep body of work remarkably reminiscent of many aspects of the classical descriptive set theory framework. In fact, the analogies are so strong that when Addison formalized them in a rigorous way, the marriage between recursion theory and descriptive set theory was successfully consummated in what is called *effective* descriptive set theory. Through an admittedly shallow and rather informal overview of these matters, we attempt nevertheless to advocate for the *naturality* of the classical hierarchies. The reader is referred to [62] for an in depth treatment of the topic, a reference that we shall follow closely.

Without any pretension to provoke any radical constructivist or ultrafinitist, it is perhaps not too controversial to accept the idea that functions which can be computed by a Turing machine should be, intuitively, legit mathematical objects. Let's consider the case of functions with tuples of positive integers as input arguments and with a positive integer as output. Surely, few people would contest the legitimacy of *simple* functions such as the successor function $S(n) = n + 1$, the constant function $c(n_1, ..., n_k) = c$ or projections like $p_i(x_1, ..., x_k) = x_i$. Moreover, it seems likely that almost all skeptics would accept that the composition of two *simple* functions should yield a *simple* function. If we have $g(x)$ and $h(u, n, x)$ as allowed entities, we may think about defining f by primitive recursion :

$$\begin{cases} f(0, x) = g(x) \\ f(n + 1, x) = h(f(n, x), n, x) \end{cases}$$

Finally, given $g(n, x)$ one considers the minimization procedure to define $f(x)$ as the least natural number such that $g(n, x) = 0$. To the elements of the smallest class which contains all simple functions S, c, p_i and is closed under composition, primitive recursion and minimization, we call recursive (or computable) functions. It is well known that the class of recursive functions coincides with the class of functions which are possible to be computed via a Turing machine. In turn, this provides some favourable evidence towards the Church-Turing thesis which claims that *every* function which is not obnoxious to the spirit of the most conservative one is in fact Turing computable. A set is said to be recursive (or decidable) if its characteristic function is recursive. In other words, if there is an effective algorithm to decide the membership of the set. A famous example of an undecidable membership problem, essentially by virtue of a diagonal argument, is the so called Turing's Halting Problem. Now let's analyse the choice of open sets as *simple* sets under this lens. Consider the following :

$$\mathcal{U} = \bigcup_n (a_n, b_n), \text{ with } a_n, b_n \in \mathbb{Q}$$

An idea for a method to determine whether or not $x \in \mathcal{U}$ is to choose a sequence of rationals $q_i \to x$ with $|x - q_i| < 2^{-i}$ and search for n and i such that $a_n + 2^{-i} < x < b_n - 2^{-i}$. If $x \in \mathcal{U}$ and if we can search for the intervals in a *computable* way, then the process stops. However, if $x \notin \mathcal{U}$, the process does not stop. In general, whenever this happens we say that a set is semi-recursive (or semi-decidable). Motivated by this observation, let $X = X_1 \times ... \times X_n$ be a product space, with each X_i being a Polish space endowed with a recursive presentation, i.e. a dense countable set $\{r_i\}$ for which the relations $d(r_i, r_j) \leq \frac{m}{k+1}$ and $d(r_i, r_j) < \frac{m}{k+1}$ are recursive. One can then endow the product space X with a recursive presentation and appropriately codify the basic open neighborhoods (cf. [62], Section 3B). A set $G \subseteq X$ is said to be semi-recursive if :

$$G = \bigcup_n N(X, \epsilon(n)), \text{ where } n \mapsto \epsilon(n) \text{ is a recursive enumeration}$$

In other words, a set is called semi-recursive if it is a *recursive* union of basic neighborhoods. It turns out that a set $P \subseteq \omega^k$ is recursive if and only if both P and $\omega^k \setminus P$ are semi-recursive (cf. [62], Theorem 3C.2). We use the notation $P \in \exists^\omega \Gamma$ as usual to indicate that $P(x) \Leftrightarrow \exists n : Q(x, n)$ with $Q \in \Gamma$ and similarly with $\exists^{\omega^\omega}$. We can then define the effective versions of the classical hierarchies :

(i) $\Sigma^0_1 = \{\text{semi-recursive sets}\}$ and $\Sigma^0_{n+1} = \exists^\omega \neg \Sigma^0_n$

(ii) $\Pi^0_n = \neg \Sigma^0_n$

(iii) $\Delta^0_n = \Sigma^0_n \cap \Pi^0_n$, so that if $X = \omega^k$ the class Δ^0_1 coincides with the class of recursive sets. Sometimes, in this context the class Δ^0_0 is defined to be the class of primitive recursive sets, i.e. sets which membership can be decided without minimization.

Moreover, we similarly define higher hierarchies :

(i) $\Sigma^1_1 = \exists^{\omega^\omega} \Pi^0_1$ and $\Sigma^1_{n+1} = \exists^{\omega^\omega} \neg \Sigma^1_n$

(ii) $\Pi^1_n = \neg \Sigma^1_n$

(iii) $\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$

All these constructions share analogous closure properties with the classical hierarchies. In order to state the big analogy in a rigorous way, we introduce the notation for relativization : with each pointclass Γ and each $z \in Z$ we associate $\Gamma(z)$ by declaring that $P \subseteq X$ is in $\Gamma(z)$ if there is some $Q \subseteq Z \times X$ in Γ such that $P(x) \Leftrightarrow Q(z, x)$. We can now state the correspondence between the effective and the classical hierarchies ([62], Theorem 3E.4) :

Let $\boldsymbol{\Gamma}$ denote the classes $\boldsymbol{\Sigma^0_n}$, $\boldsymbol{\Sigma^1_n}$, $\boldsymbol{\Pi^0_n}$ or $\boldsymbol{\Pi^1_n}$ and Γ denote the correspondent effective classes. Then, $P \subseteq X$ is in $\boldsymbol{\Gamma}$ if and only if $P \in \Gamma(\epsilon)$ for some $\epsilon \in \omega^\omega$.

Another related idea - arguably hinting towards a more formalist or logicist flavour - for a criterion that classifies objects according to their simplicity would be to look at the complexity of the formulas used to define them. A structure $\mathfrak{U}$ is defined by $\mathfrak{U} = (\{A_i\}_{i\in I}, \{f_j\}_{j\in J}, \{R_k\}_{k\in K}, \{c_l\}_{l\in L})$ where each A_i is a non-empty set, f_j are fuctions from some cartesian product $A_{i_1} \times ... \times A_{i_p}$ to some A_n , R_k are relations on some cartesian product of elements in $\{A_i\}_{i\in I}$ and each c_l is an element of some A_i. For instance, the structure of *second order arithmetic* :

$$\mathcal{A}^2 = (\omega, \omega^\omega, +, \times, \mathrm{ap}, 0, 1)$$

with ap: $\omega^\omega \times \omega \to \omega$ given by $\mathrm{ap}(\alpha, n) = \alpha(n)$. We then associate a formal language $\mathcal{L}^{\mathfrak{U}}$ to a structure $\mathfrak{U}$, providing an alphabet of symbols and a grammar, which consists on a list of rules which determines the well-formed formulae in $\mathcal{L}^{\mathfrak{U}}$. We can then talk about a formula being satisfied in a language. In a sense, it can be argued that this furnishes our syntax with the *vitality* of *meaning.* The rules of satisfaction are meant to be self-evident and can be understood as to translate statements from $\mathcal{L}^{\mathfrak{U}}$ to natural language. We will use the symbol $\models$ to indicate satisfaction. For a concise introduction, the reader is refererred to [62] (Section 8A). In this context, we say that a set $A \subseteq \omega^k \times (\omega^\omega)^n$ is definable in $\mathcal{A}^2$ if there is a formula $\varphi(m_1, ..., m_k, f_1, ..., f_n)$ such that :

$$A(m_1, ..., m_k, f_1, ..., f_n) \Leftrightarrow \mathcal{A}^2 \models \varphi[m_1, ..., m_k, f_1, ...f_n]$$

For $A \subseteq \omega^k \times (\omega^\omega)^n$, with $k, n \in \omega$, we define the arithmetical hierarchy (if A is definable in $\mathcal{A}^2$ without quantifying over elements of ω^ω) and the analytical hierarchy (for A definable in $\mathcal{A}^2$) :

(i) $A \in \Delta^0_0$ if A is definable in $\mathcal{A}^2$ by a formula with only bounded quantifiers[5]

(ii) $A \in \Sigma^0_n$ if A is definable in $\mathcal{A}^2$ by $\exists m_1 \forall m_2 ... \mathcal{Q} m_n \varphi$, where φ has only bounded quantifiers and $\mathcal{Q} = \exists$ if n is odd and $\mathcal{Q} = \forall$, otherwise

(iii) $A \in \Pi^0_n$ if A is definable in $\mathcal{A}^2$ by $\forall m_1 \exists m_2 ... \mathcal{Q} m_n \varphi$, where φ has only bounded quantifiers and $\mathcal{Q} = \exists$ if n is even and $\mathcal{Q} = \forall$, otherwise

(iv) More generally, $\Delta^0_n = \Sigma^0_n \cap \Pi^0_n$

If we quantify over elements in ω^ω, we get the analytical hierarchy :

(i) $\Sigma^1_0 = \Sigma^0_1$ and $\Pi^1_0 = \Pi^0_1$

(ii) $A \in \Sigma^1_n$ if A is definable in $\mathcal{A}^2$ by $\exists f_1 \forall f_2 ... \mathcal{Q} f_n \varphi$, where φ has only quantifiers over ω and $\mathcal{Q} = \exists$ if n is odd and $\mathcal{Q} = \forall$, otherwise

(iii) $A \in \Pi^1_n$ if A is definable in $\mathcal{A}^2$ by $\forall f_1 \exists f_2 ... \mathcal{Q} f_n \varphi$, where φ has only quantifiers over ω and $\mathcal{Q} = \exists$ if n is even and $\mathcal{Q} = \forall$, otherwise

(iv) $\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$

[5] In other words, when we restrict the range of the quantified variable under $\forall$ or $\exists$

The correspondence between these and the classical hierarchies is then given by the following fact (cf. [28]) :

Let $A \subseteq (\omega^\omega)^k$. Then, $A \in \mathbf{\Sigma^0_n}$ if and only if $A \in \Sigma^0_n(\alpha)$, for some $\alpha \in \omega^\omega$. Similarly for $\mathbf{\Pi^0_n}$. Regarding projective and analytical sets, $A \in \mathbf{\Sigma^1_n}$ if and only if $A \in \Sigma^1_n(\alpha)$, for some $\alpha \in \omega^\omega$. Similarly for $\mathbf{\Pi^1_n}$.

In this thesis, in Section 3, we present a synthesis of some fundamental results and examples from *classical* Descriptive Set Theory. In Section 5, we investigate some applications of this methods to the study of closed sets of uniqueness in locally compact groups. In Section 6, under this lens, we study the point spectra of operators.

1.2 A brief overview on sets of uniqueness

Thin sets have been a central object of study for Harmonic Analysis. Among those, sets of uniqueness constitute a particularly important topic of interest with a long and illustrious history, honoured by fruitful interdisciplinary collaborations and witnessed by the development and usage of techniques whose range of applicability goes way beyond the study of Analysis related objects. In his *Habilitationsschrift*, Riemann suggested the problem of determining whether or not a representation of a function by a trigonometric series, whenever it exists, is unique. In modern language this amounts to ask whether or not the empty set is a set of uniqueness. Working on this problem, Cantor was led to study ordinal numbers and consequently to *inaugurate* the field of Set Theory, igniting a profound change of paradigm within the mathematical community. A set $E \subset \mathbb{T}$ is said to be a set of uniqueness if :

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} = 0 \text{ off } E, \text{ implies that } c_n = 0 \text{ for all } n \in \mathbb{Z}$$

Other exceptional sets arise naturally in relation with sets of uniqueness, for instance sets of extended uniqueness. Indeed, a set $E \subseteq \mathbb{T}$ is said to be a set of extended uniqueness if :

$$\sum_{n=-\infty}^{\infty} \hat{\mu}(n) e^{inx} = 0 \text{ off } E, \text{ implies that } \hat{\mu}(n) = 0 \text{ for all } n \in \mathbb{Z}$$

where μ is a Borel measure on $\mathbb{T}$ and $\hat{\mu}(n)$ its Fourier-Stieltjes coefficients. Clearly, all sets of uniqueness are sets of extended uniqueness.

Lebesgue, Bernstein and Young strengthened Cantor's results and eventually proved that every countable set of $\mathbb{T}$ is a set of uniqueness. The structure of sets of uniqueness was extensively studied during the first three decades of the 20^{th} century, most notably by the Russian and Polish schools. Some of the early landmark successes of the subject include Bari's theorem - which guarantees that a countable union of closed sets of uniqueness is still a set of uniqueness

- and Salem-Zygmund theorem - establishing a bridge between number-theoretic properties and sets of uniqueness, fully characterizing a certain type of the latter - as well as rather counter-intuitive examples, such as Menshov's example of a null set which is not a set of uniqueness or Bari's and Rajchman's constructions of uncountable perfect sets of uniqueness. Despite of the early successes, a characterization of the set of sets of uniqueness in terms of a *definable* subset or even in terms of a measure theoretic or topological *qualitative* description, was consistently elusive. There were plenty of difficulties which seemed technically unsurpassable. However, in the 1950's the subject regained a vigorous interest by virtue of a new functional analytic flavoured framework initiated by Piatetski-Shapiro which reformulated the language of the area, naturally refocusing efforts to study the family of closed sets of uniqueness. One important accomplishment of his works was establishing the existence of (closed) sets of extended uniqueness which are not (closed) sets of uniquess.

In the early 60's, the notion of (closed) set of uniqueness is extended to abelian locally compact groups by Herz and in the late 70's, to general locally compact groups by Bożejko within the framework of Fourier algebras, previously introduced by Eymard. Curiously enough, and despite the fact that one can trace back the origins of Set Theory to the early investigations on sets of uniqueness, both subjects remained relatively isolated from each other until the 80's. At this point, the incorporation of descriptive set-theoretic tools in the study of sets of uniqueness not only sharpened the understanding of the subject, providing new insights to classical results from a more abstract perspective, but also led to the solution of old open problems and to the development of powerful methods. The successful merging of set theoretic techniques with a modern functional analytic framework to approach (and generalize) classical problems arising from Analysis related contexts was led notably by Debs, Saint Raymond, Kechris, Louveau and Woodin. For a comprehensive introduction to this angle on sets of uniqueness (for $\mathbb{T}$), the reader is referred to [47].

In this thesis, in Section 4 we recall the theory involving groups, measures and algebras on groups which is required to introduce the Fourier algebra of a locally compact group G. In Section 5, we carefully define sets of (extended) uniqueness in the general setting of such groups as well as their operator-theoretic counterpart as defined in [74]. We prove new results on preservation properties of closed sets of uniqueness of G, extend previous descriptive set-theoretic complexity classifications and establish criteria for the non existence of Borel bases for the set of closed sets of uniqueness under certain assumptions on G.

1.3 Papers included in this thesis

In what follows, we provide a brief summary of each paper whose content is contained in this thesis. Any reference of a result by a number (*eg.* Th. X.YY) is with respect to this thesis.

• **Paulos J.,** ***Descriptive set theoretic aspects of closed sets of uniqueness in the non abelian setting*** - submitted

Henceforth, G is a locally compact group. We prove some results concerning the preservation of (operator) M_0-sets under certain inverse images and finite products. Moreover, we locate the descriptive set theoretic complexity of $\mathcal{U}(G)$ and $\mathcal{U}_0(G)$ under further assumptions on the group G.
In [74], the notion of closed set of (extended) uniqueness is related with its operator theoretic counterpart. Using operator theoretic flavoured techniques, several *functorial* results were established. In particular, concerning finite products one can extract the following :

(**Cor. 5.17**) Let G_i be second countable and $E_i \subseteq G_i$ closed subsets, for $i \in \{1,2\}$. Then, if either E_1 or E_2 is a U-set it follows that so is the product $E_1 \times E_2$. On the other hand, if both E_1 and E_2 are M-sets, so is their product.

We prove that the analogous result still holds for U_0-sets :

(**Th. 5.18**) Let (X_i,μ_i) and (Y_i,ν_i) be standard measure spaces and $\kappa_i \subseteq X_i \times Y_i$ be ω-closed sets for $i \in \{1,2\}$. The set $\rho(\kappa_1 \times \kappa_2)$ is an operator M_0-set if and only if both κ_1 and κ_2 are operator M_0-sets. Here, the map ρ is defined as follows :

$$\rho : (X_1 \times Y_1) \times (X_2 \times Y_2) \to (X_1 \times X_2) \times (Y_1 \times Y_2)$$
$$((x_1,y_1),(x_2,y_2)) \mapsto ((x_1,x_2),(y_1,y_2))$$

As a consequence, we can conclude that :

(**Cor. 5.19**) Let G_i be second countable groups and $E_i \subseteq G_i$ closed subsets.

(a) If both E_1 and E_2 are M_0-sets, so is $E_1 \times E_2$

(b) If either E_1 or E_2 is a U_0-set, so is $E_1 \times E_2$

Concerning inverse images, the work developed in [74] allows to extract the following :

(**Cor. 5.21**) : Let G be second countable, $H \subseteq G$ be a closed normal subgroup and $E \subseteq G/H$ a closed subset. Then, E is a U-set if and only if $q^{-1}(E)$ - where q denotes the quotient map - is a U-set.

In [80], the analogous result for U_0-sets is proven for abelian groups. We prove partial generalizations of the result for U_0-sets for some non-necessarily abelian groups. In particular, we prove that :

(**Th. 5.24**) : Let G be amenable, $H \subseteq G$ a closed normal subgroup and $E \subseteq G/H$ a closed subset. If $q^{-1}(E)$ is a M_0-set, then so is E.

A closed normal subgroup $H \subseteq G$ is said to have *property* $|l|^2$ if for every proper compact $K \subseteq G/H$ there exists some $\theta \in C_c(G) \cap A(G)$ such that $\phi(|\theta|) = c_1$ and $\phi(|\theta|^2) = c_2$ for some positive real constants on a neighbourhood of K. Here, $\phi(f)([x])$ is defined to be the integral $\int_H f(xh)dh$. Under the assumption of this property (easily seen to hold, for instance, for $\mathbb{Z} \subseteq \mathbb{R}$) we have the following :

(**Th. 5.28**) : Let $H \subseteq G$ be a closed normal subgroup with *property* $|l|^2$ and such that G/H is second countable. Let $E \subseteq G/H$ be a compact subset. If E is a M_0-set, then so is $q^{-1}(E)$.

Solovay and Kaufman proved that $\mathcal{U}(\mathbb{T})$ and $\mathcal{U}_0(\mathbb{T})$ are Π^1_1-complete. The reader is referred to [47] for a proof (alternatively, cf. Theorem 5.41 and Theorem 5.43). In [80], it was proven that $\mathcal{U}(G)$ and $\mathcal{U}_0(G)$ are Π^1_1-complete whenever G is abelian, second countable and non-discrete. Using an entirely different approach, after noting that :

(**Th. 5.38 and Th. 5.39**) : $\mathcal{U}(G)$ and $\mathcal{U}_0(G)$ are always coanalytic if G is second countable (not necessarily abelian).

We rely on functorial properties and provide a *direct* proof of the following descriptive complexities :

(i) (**Th. 5.44**) : $\mathcal{U}(G)$ is Π^1_1-complete if G is a connected Lie group (not necessarily abelian).

(ii) (**Th. 5.45**) : $\mathcal{U}_0(G)$ is Π^1_1-complete if G is a connected and abelian Lie group.

Finally, using the results of [80], we easily establish more generally that :

(**Th. 5.46**) : Let G be a locally compact Polish group such that $G/\overline{[G,G]}$ is non-discrete. Then, $\mathcal{U}(G)$ is Π^1_1-complete.

For the sake of completeness, we also include the following observation (which does not appear in any other draft) concerning a generalization of a famous result due to Bari :

(**Th. 5.36**) : Under the negation of the CH and assumption of MA(κ), suppose that $\{E_\alpha\}_{\alpha<\kappa}$ is a family of closed sets of uniqueness of $\mathbb{T}$. Then, $\bigcup_{\alpha<\kappa} E_\alpha$ is a set of uniqueness.

In [17] it was proven that $\mathcal{U}(\mathbb{T})$ does not have a Borel basis and in [56] this result was extended for general second countable non-discrete locally compact abelian groups G. We provide some criteria for the inexistence of a Borel basis for $\mathcal{U}(G)$ when G is not necessarily abelian. Henceforth, G is assumed to be locally compact and we prove the following results :

(**Th. 5.56**) : Let G be amenable and second countable and $H \subseteq G$ a countable closed subgroup such that :

(a) The quotient map $q : G \to G/H$ is a closed map

(b) There is no Borel basis for $\mathcal{U}(G/H)$

Then, $\mathcal{U}(G)$ does not have a Σ_1^1 pre-basis. In particular, if G is compact it follows that $\mathcal{U}(G)$ is Π_1^1-complete.

(**Th. 5.60**) : Let G be amenable and $H \subseteq G$ be an open subgroup such that $\mathcal{U}(H)$ does not have Borel basis. Then, $\mathcal{U}(G)$ does not have a Borel basis and in particular, if G is compact then $\mathcal{U}(G)$ is Π_1^1-complete.

We prove that if G is Polish and amenable, then $\mathcal{U}(G)$ is calibrated. In particular, this holds for whenever G is compact - a fact we use in order to establish a sufficient condition for the inexistence of Borel basis for $\mathcal{U}(G)$ for certain product groups G :

(**Th. 5.62**) : Let G be a Polish amenable group. Then, $\mathcal{U}(G)$ is a calibrated coanalytic σ-ideal of $\mathcal{F}(G)$.

(**Th. 5.67**) : Suppose G is a product of the form $G_1 \times ... \times G_n$ with each G_i compact, second countable and such that :

(a) For every $i \in [n]$ there is a closed set $E_i \subseteq G_i$ such that $E_i \notin \mathcal{U}(G_i)^{\text{loc}}$ and a G_δ-set $F_i \subseteq E_i$ which is dense in E_i and such that $F_i \in \mathcal{U}(G_i)^{\text{int}}$.

(b) There is some $N \in [n]$ such that for every M-set $E \subseteq G_N$, $\mathcal{F}(E) \cap \mathcal{U}(G_N)$ is not Borel

Then, $\mathcal{U}(G)$ is Π_1^1-complete and does not have a Borel basis.

Open question : *By [56], we know that $\mathcal{U}(G)$ does not have a Borel basis whenever G is second countable, abelian and non-discrete. Can we generalize this fact for a larger class of locally compact groups ? A natural candidate is the class of (infinite) compact groups as they contain an infinite abelian subgroup by a result of Zelmanov. However, if G is connected this subgroup is necessarily non open and it is not clear for the author how to deal with this case. Another natural case to consider is the class of Lie groups or totally disconnected Lie groups, as structural descriptions such as the Gleason-Yamabe theorem appear to be particularly relevant. However, the author was unsuccessful in his attempts to extend these inexistence results to these larger classes.*

• Paulos J., ***On reflexivity and point spectrum*** **- to appear in Real Analysis Exchange**

We prove a characterization of reflexivity for Banach spaces with unconditional basis (for instance, this includes all Hilbert spaces) and we locate the descriptive complexity of certain sets associated with the point spectrum of operators acting on separable Banach spaces. In particular, we prove that :

(**Th. 6.10**) : Let X be a Banach space with an unconditional basis. Then, X is reflexive if and only if for all closed subspaces $Y \subseteq X$, $Z \subseteq X^*$ and operators $T \in \mathcal{L}(Y), T' \in \mathcal{L}(Z)$ it holds that $\sigma_p(T)$ and $\sigma_p(T')$ are Borel.

Open question : *Can we improve the previous result by virtue of considering only X instead of X and its continuous dual X^* ? Can we prove some similar characterization of reflexivity for spaces without unconditional basis ?*

Let X be a separable Banach space and fix some $T \in \mathcal{L}(X)$. We study the map $\Gamma_T : \mathbb{C} \to \mathcal{F}(X)$ which sends λ to $\ker(T - \lambda 1)$ and provide some upper bounds for the descriptive complexity of sets arising *naturally* within this context. In particular, we prove that :

(**Th. 6.15**) : Let X be a reflexive and separable Banach space and fix some $T \in \mathcal{L}(X)$. Then, the set $K_T = \{\ker(T - \lambda 1) : \lambda \in \mathbb{C}\}$ is Borel in the Effros-Borel space of $\mathcal{F}(X)$.

• **Paulos J., *Stability of Jamison sequences under certain perturbations*, NWEJM vol. 5, 89-99 (2019)**

An increasing sequence (n_k) of positive integers is said to be a Jamison sequence if whenever $T \in \mathcal{L}(X)$ - with X a separable Banach space - is such that $\sup_k ||T^{n_k}|| < \infty$, then the unimodular point spectrum $\sigma_p(T) \cap \mathbb{T}$ is countable. We prove the following stability criterion :

(**Th. 6.28**) : For increasing sequences of positive integers (n_k) and (t_k) define :

$$(r_k)_{\frac{t_k}{n_k}} := (|t_k - [\frac{t_k}{n_k}]n_k|)$$

where [.] denotes the closest integer function. Suppose that (t_k) is (non) Jamison. If one of the following conditions hold, (n_k) is also (non) Jamison :

(i) $\sup_k(\frac{t_k}{n_k}) < \infty$ and $\sup_k(r_k)_{\frac{t_k}{n_k}} < \infty$

(ii) $\sup_k(\frac{n_k}{t_k}) < \infty$ and $\sup_k(r_k)_{\frac{n_k}{t_k}} < \infty$

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2 Terminology

The terminology and notation adopted throughout this thesis, bearing in mind what is the literature's jargon, is mostly standard. Nevertheless, and hoping to clarify any potential ambiguities, we include this short section which contains certain definitions which may sporadically appear without previous context. Moreover, we declare that unless otherwise stated, we work within ZFC.

We start with a combinatorial tool : let A be any non-empty set and A^n be the set of finite sequences of elements in A with length $n \in \mathbb{N}$, i.e. the set of functions from the ordinal n to the set A. Naturally, if $n = 0$ then A^0 has a single element : the empty sequence $\emptyset$. In other words, the empty set is the initial object in the category of sets and functions. We denote by $A^{<\omega}$ the set of all finite sequences of elements in A, i.e. :

$$A^{<\omega} := \bigcup_{n \in \mathbb{N}} A^n$$

Whenever referring to the set of natural numbers we will use - in different contexts and in an univocal way - both ω and $\mathbb{N}$. Whenever $(x_n) \in A^{\mathbb{N}}$ and $m \in \mathbb{N}$, by $x|_m$ we refer to the element $(x_0, ..., x_{m-1}) \in A^m$. Given $t \in A^k$ and $s \in A^m$, with $k \leq m$, we say that s extends t - denoted by $t \subseteq s$ - if $s|_k = t$. We will denote the usual concatenation of two sequences s and t by $s \frown t$.

Definition 2.1. Let A be any non-empty set. A tree T on A is a subset $T \subseteq A^{<\omega}$ such that if $t \in T$ and $s \subseteq t$, then $s \in T$. An infinite branch of T is a sequence $x \in A^{\mathbb{N}}$ such that $x|_n \in T$ for every $n \in \mathbb{N}$ and the set of all infinite branches of T is called the body T, denoted by $[T]$. A tree T is said to be pruned if for every $s \in T$ there is a proper extension $t \in T$ of s.

Remark 2.2. Let A be any non-empty set and R be an entire binary relation on it, i.e. such that for every $x \in A$, there is some $y \in A$ with xRy. Assume for a moment that every non-empty pruned tree has an infinite branch. Define a tree T on A by declaring that $(a_0, ..., a_m) \in T$ if and only if $a_k R a_{k+1}$ for every $0 \leq k < m$. Then, since R is entire it follows that T is pruned and by assumption, it has an infinite branch. This means that there is a sequence $(a_n) \in A^{\mathbb{N}}$ such that $a_n R a_{n+1}$ for every $n \geq 0$. The latter statement - i.e. given any non-empty set A and entire binary relation R on it, there is a sequence (a_n) such that $a_n R a_{n+1}$ - is known as the Axiom of Dependent Choice (DC). It is a weak form of the Axiom of Choice, easily seen to be equivalent (within ZF) to the assertion that every non-empty pruned tree has an infinite branch. It turns out that the DC is equivalent (within ZF) to the Baire category theorem for complete metric spaces ([7]) and strictly stronger than the Axiom of Countable Choice ([39], Theorem 8.12).

If T is a tree and $p \in T$, let $T_p := \{s : p \frown s \in T\}$ which is also a tree. If T is a tree on a product of two non-empty sets $A_1 \times A_2$ and $x \in A_1^{\mathbb{N}}$, we define the section tree on A_2 as follows :

$$T(x) := \{s \in A_2^{<\omega} : s \in A_2^k \text{ and } (x|_k, s) \in T\}$$

It is furthermore worth to point out that there is a bijection between pruned trees on A and closed subsets of $A^{\mathbb{N}}$ given by $T \mapsto [T]$, whose inverse is given by $F \mapsto \{x|_n : x \in F, n \in \mathbb{N}\}$ (cf. [45], Proposition 2.4).

We finish this short digression on trees by mentioning how we will denote open basic sets in the product topology of discrete spaces. For $(x_n) \in A^{\mathbb{N}}$, let :

$$\Sigma(x|n) := \{x_0\} \times ... \times \{x_{n-1}\} \times A \times A \times ...$$

We define analogously any open basic set induced by an element $s \in A^{<\omega}$.

Again, for the sake of clarity, we proceed by recalling that an algebra $\mathcal{A}$ over $\mathbb{C}$ (or $\mathbb{R}$) endowed with a submultiplicative norm with respect to which it becomes a complete normed space is called a Banach algebra. If, furthermore, $\mathcal{A}$ has an isometric involution $* : a \mapsto a^*$ then it is said to be a $*$-Banach algebra and finally, if the C*-equation holds - i.e. $||a^*a|| = ||a||^2$ for all $a \in \mathcal{A}$ - $\mathcal{A}$ is said to be a C*-algebra.

As usual, given a topological space X we denote the set of complex-valued continuous functions on X by $C(X)$. We often consider $C(X)$ equipped with pointwise multiplication and complex conjugation as involution. The subset of bounded continuous functions is denoted by $C_b(X)$ and when endowed with the supremum norm $||f||_\infty := \sup_{x \in X}\{|f(x)|\}$ it becomes a C*-algebra. The subset of continuous functions with compact support is denoted by $C_c(X)$ and the subset of continuous functions that vanish at infinity by $C_0(X)$. We recall that $C_c(X)$ is a dense subalgebra of $C_0(X)$.

Given a Banach space X, the space of continuous linear functionals - itself a Banach space - will be denoted by X^*. The weak topology on X is the coarsest topology on X for which all functionals in X^* are continuous and the weak*-topology (or w*-topology) on X^* is the coarsest topology for which all evaluation functionals (i.e. such that $\varphi \mapsto \varphi(x)$ for some $x \in X$) are continuous. Arguably, one of the most useful well-known facts about the latter topology is the Banach-Alaoglu theorem which states that the closed unit ball $B_1(X^*)$ is weakly*-compact.

Given a bounded linear operator $T : X \to Y$ between Banach spaces X and Y, the adjoint operator $T^* : Y^* \to X^*$ is defined by $T^*(\varphi) := \varphi \circ T$.

For a commutative Banach algebra $\mathcal{A}$, the set of algebra homomorphisms from $\mathcal{A}$ onto $\mathbb{C}$ is denoted by $\sigma(\mathcal{A})$ and the Gelfand transform $\Gamma : \mathcal{A} \to C(\sigma(\mathcal{A}))$, where $\sigma(\mathcal{A})$ is endowed with the weak*-topology, is defined by :

$$\Gamma : a \mapsto \hat{a}, \text{ such that } \hat{a}(\varphi) = \varphi(a) \text{ for } a \in \mathcal{A}, \varphi \in \sigma(\mathcal{A})$$

The relative weak*-topology on $\sigma(\mathcal{A})$ is also referred to as the Gelfand topology. It is a well-known fact that $\sigma(\mathcal{A})$ is then a locally compact Hausdorff space, compact if $\mathcal{A}$ is unital (i.e., if $\mathcal{A}$ has an identity). Another standard fact about Banach algebras is the acclaimed Gelfand duality : if $\mathcal{A}$ is a commutative C*-algebra then Γ is an isometric isomorphism onto $C_0(\sigma(\mathcal{A}))$.

Remark 2.3. Let CHaus be the category of compact Hausdorff spaces and continuous functions and $\mathrm{C}^*_{\mathrm{com}}$ be the category of unital commutative C*-algebras and $*$-homomorphisms. Then, the functors sending X to $C(X)$ and $\mathcal{A}$ to $\sigma(\mathcal{A})$ induce a duality between CHaus and $\mathrm{C}^*_{\mathrm{com}}$. If instead we consider the subcategory $\mathrm{C}^*_{\mathrm{com,nu}}$ of non-unital commutative C*-algebras and the category $CHaus^*$ of pointed compact Hausdorff spaces, similarly one has a duality between $\mathrm{C}^*_{\mathrm{com,nu}}$ and $CHaus^*$, providing clear justification for the terminology of the latter theorem.

3 Elements of Descriptive Set Theory

The aim of this section is to introduce elementary notions in Descriptive Set Theory while providing a not so narrow, yet not so comprehensive overview of the field - particularly its classical aspects.

3.1 Borel and Projective hierarchies

Throughout this section, and unless otherwise stated, X is a Polish space - i.e. a completely metrizable second countable topological space. We recall that a topological space is said to be zero-dimensional if it has a basis consisting of clopen sets.[6] As usual, the Borel σ-algebra of any topological space X is the σ-algebra generated by the open subsets of X and it will be denoted by $\mathcal{B}(X)$. A measurable space (X, Σ) is said to be a standard Borel space if there is a Polish topology $\mathcal{T}$ on X such that Σ coincides with the σ-algebra generated by $\mathcal{T}$.

Remark 3.1. Every separable metrizable space is homeomorphic to a subspace of the Hilbert cube $I^{\mathbb{N}}$. Indeed, consider such space (X, d) with $d \leq 1$ and $\{x_n\} \subseteq X$ a countable dense set. It is then the case that the assignment $x \mapsto (d(x, x_n))_{n\in\mathbb{N}}$ is a homeomorphism onto its image. It is a well-known fact that a subspace of a Polish space is Polish if and only if it is G_δ (cf. [45],

[6] It is well known that if X is separable and metrizable, then X is zero-dimensional if and only if its Lebesgue covering dimension is zero.

Theorem 3.11). Hence, up to homeomorphism Polish spaces are precisely the G_δ subspaces of $I^{\mathbb{N}}$. Furthermore, all Polish groups (i.e. topological groups whose topology is Polish) are isomorphic to a closed subgroup of $H(I^{\mathbb{N}})$, the group of homeomorphisms of $I^{\mathbb{N}}$ (cf. [45], Theorem 9.18).

Definition 3.2. We define, via transfinite recursion for all $1 \leq \alpha < \omega_1$, the Borel hierarchy of sets of a topological space X :

$$\Sigma_1^0 = \{\mathcal{U} : \mathcal{U} \subseteq X \text{ is open}\} \text{ and } \Pi_1^0 = \{F : F \subseteq X \text{ is closed}\}$$

$$\Sigma_\alpha^0 = \{\bigcup_n A_n : A_n \in \Pi_{\beta(n)}^0, \beta(n) < \alpha\} \text{ and } \Pi_\alpha^0 = \{X \setminus A : A \in \Sigma_\alpha^0\}$$

Remark 3.3. Note that the Borel hierarchy does indeed provide an ordered stratification of $\mathcal{B}(X)$ for a metric space X. Indeed, it is immediate that :

$$\mathcal{B}(X) = \bigcup_{\alpha<\omega_1} \Sigma_\alpha^0 = \bigcup_{\alpha<\omega_1} \Pi_\alpha^0$$

and that if $\beta < \alpha$, then $\Sigma_\beta^0 \subseteq \Sigma_\alpha^0$ and $\Sigma_\beta^0 \subseteq \Pi_\alpha^0$ (and similarly for Π_β^0). Since there are $\mathfrak{c}$ many open sets in any separable infinite metric space X, it is then clear that in that case $|\mathcal{B}(X)| = \mathfrak{c}$. Consequently, one can provide a simple proof of the existence of Lebesgue measurable sets of $\mathbb{R}$ which are not Borel. Indeed, since the Lebesgue measure is complete it follows that every subset of the Cantor set is Lebesgue measurable. However, there are $2^{\mathfrak{c}}$ such subsets.

Moreover, if X is uncountable we need all ω_1 steps to exhaust all Borel sets :

Theorem 3.4. *Let X be an uncountable Polish space. Then, for every $\alpha < \omega_1$ one has a strict inclusion $\Sigma_\alpha^0 \subsetneq \Sigma_{\alpha+1}^0$.*

Proof. The reader can find a proof in [77] (Corollary 3.6.8). ■

Remark 3.5. Let X be any topological space and define $\mathrm{Ord}(X)$ to be the least ordinal α such that $\Sigma_\alpha^0 = \mathcal{B}(X)$. Note that if X is a discrete space, then $\mathrm{Ord}(X) = 1$ and that $\mathrm{Ord}(\mathbb{Q}) = 2$. If one assumes the Continuum Hypothesis, then for every $\alpha \leq \omega_1$ there is a separable metric space X such that $\mathrm{Ord}(X) = \alpha$ (cf. [60], Corollary 10.4). This shows how important the assumptions in Theorem 3.4 are.

Definition 3.6. Let X be a Polish space. We define the Projective hierarchy as follows :

$$\Sigma_1^1 = \{\pi(B) : B \in \mathcal{B}(X \times X)\} \text{ and } \Pi_1^1 = \{X \setminus A : A \in \Sigma_1^1\}$$

$$\Sigma_{n+1}^1 = \{\pi(B) : B \subseteq X \times X,\ B \in \Pi_n^1\} \text{ and } \Pi_{n+1}^1 = \{X \setminus A : A \in \Sigma_{n+1}^1\}$$

Proposition 3.7. Let X be a Polish space and $A \subseteq X$. Then, the following are equivalent :

(i) $A \in \Sigma_1^1$

(ii) There is a Polish space Y and some $B \in \mathcal{B}(X \times Y)$ such that $\pi_X(B) = A$

(iii) There is a continuous function $f : \omega^\omega \to X$ such that $f(\omega^\omega) = A$

(iv) There is a closed subset $C \subseteq X \times \omega^\omega$ such that $\pi_X(C) = A$

(v) For every uncountable Polish space Y there is a G_δ set $G \subseteq X \times Y$ such that $\pi_X(G) = A$

If A satisfies any of the above conditions, then A is said to be analytic. If $X \setminus A$ is analytic, then A is said to be coanalytic.

Proof. The reader can find a proof in [77] (Proposition 4.1.1). ■

Remark 3.8. Let X be a Polish space and $A \subseteq X$. Then, $A \in \Sigma_1^1$ if and only if A is the continuous image of a Polish space. On one hand, if A is analytic then A is the continuous image of ω^ω which is a Polish space. On the other hand, suppose that there is some Polish space Y and a continuous function $f : Y \to X$ such that $f(Y) = A$. By Theorem 7.9 in [45], there is a continuous surjective function $g : \omega^\omega \to Y$ and thus, $h = f \circ g : \omega^\omega \to X$ is a continuous function such that $h(\omega^\omega) = A$. This is a quite commonly used definition of analytic set.

Historically speaking, the idea of analytic sets emerged from a famous mistake detected by Lusin. Indeed, in [51] Lebesgue erroneously claimed that the projection of a Borel set of $\mathbb{R}^2$ onto $\mathbb{R}$ is still a Borel set. In truth, this is not always the case. We end this section with two classical results and a central definition which are related with this issue. The following result establishes the non-redundancy of the Projective hierarchy :

Theorem 3.9. *Let X be an uncountable Polish space. Then, $\mathcal{B}(X) \subsetneq \Sigma_1^1$ and $\Sigma_n^1 \subsetneq \Sigma_{n+1}^1$ for all n.*

Proof. The reader can find a proof in [45] (Theorems 14.2 and 37.7). ■

The class of Borel sets of a Polish space coincides with the class of sets which are simultaneously analytic and coanalytic :

Theorem 3.10. *Let X be a Polish space. Then, $\mathcal{B}(X) = \Sigma_1^1 \cap \Pi_1^1$.*

Proof. It is enough to prove that if $A, B \subseteq X$ are two disjoint analytic sets, then they can be separated by a Borel set - i.e. there is a Borel set C such that $A \subseteq C$ and $C \cap B = \emptyset$. Let $f, g : \omega^\omega \to X$ be continuous functions such that $f(\omega^\omega) = A$ and $g(\omega^\omega) = B$ and assume, towards a contradiction, that A and B can't be separated by a Borel set. Note that if $A = \bigcup_n A_n$ and $B = \bigcup_m B_m$ with A_n and B_m separated for each n, m by a Borel set $R_{n,m}$, then A and B are separated by $\bigcup_n \bigcap_m R_{n,m}$. Hence, we can define recursively $x(n), y(n) \in \mathbb{N}$ such that $f(\Sigma(x|n))$ and $g(\Sigma(y|n))$ are not separated by a Borel set. However, since A and B are disjoint, there are disjoint open neighbourhoods $\mathcal{U}$ and $\mathcal{V}$ of $f(x)$ and $g(y)$ respectively which separate $f(\Sigma(x|N))$ and $g(\Sigma(y|N))$ for large enough N, which is a contradiction. ■

A commonly used technique to identify sets which are not Borel relies on the concept of pointclass completeness. Due to the scope of this thesis, we will restrict our focus to the pointclass of (co)analytic sets :

Definition 3.11. Let X be a Polish space and $A \subseteq X$. Then, A is said to be Σ_1^1-hard if for every zero dimensional Polish space Z and $B \subseteq Z$ which is analytic, there is a continuous function $f : Z \to X$ such that $B = f^{-1}(A)$. If furthermore $A \in \Sigma_1^1$, then A is said to be Σ_1^1-complete. The definitions of Π_1^1-hardness and Π_1^1-completeness are entirely analogous.

We note that if $A \subseteq X$ is Σ_1^1-hard, then A it is not Borel. Indeed, by Theorem 3.9 there is some $B \subseteq \omega^\omega$ which is analytic and not Borel. Since A is Σ_1^1-hard then there is a continuous function $f : \omega^\omega \to X$ such that $f^{-1}(A) = B$. However, since Borel sets are preserved under the pre-image of continuous functions we can conclude that A is not Borel. Similarly, if A is Π_1^1-hard then A is not Borel. Note as well that if X and Y are Polish spaces, $A \subseteq X$ is Σ_1^1-hard and $B \subseteq Y$ is such that there is a continuous function $f : X \to Y$ with $A = f^{-1}(B)$, then B is also Σ_1^1-hard. Similarly with Π_1^1-hardness.

3.2 A few examples

In this section we provide a few examples of Polish spaces and (co)analytic sets. We focus mainly on two standard examples : the set of well-founded trees WF which constitutes an elementary, though important, example of a Π_1^1-complete set and the space $\mathcal{F}(X)$ of closed subsets of a space X which is central in further sections of this text. On our way, we comment very briefly on some applications to the study of Banach spaces.

3.2.1 Two standard examples

Our first example is the archetype of a Π_1^1-complete set. Recall that a tree T is said to be well-founded if and only if $[T] = \emptyset$. We consider the set $\mathrm{Tr} \subseteq 2^{\omega^{<\omega}}$ of trees on ω, after identifying each tree with its characteristic map. It is easily verified that Tr is a G_δ subset of $2^{\omega^{<\omega}}$ and thus, a Polish space. Then :

$$WF := \{T \in \mathrm{Tr} : [T] = \emptyset\} \text{ and } IL := \mathrm{Tr} \setminus WF$$

Now consider the following set $E \subseteq 2^{\omega^{<\omega}} \times \omega^\omega$:

$$E = \{(T, \beta) : T \in \mathrm{Tr} \text{ and there is some } n \text{ such that } T(\beta|_n) = 0\}$$

It is easily shown that E is Borel and thus, noting that :

$$WF = \{T \in \mathrm{Tr} : (T, \beta) \in E, \text{ for all } \beta \in \omega^\omega\} := \forall^{\omega^\omega} E$$

it follows from Theorem 3.26 that $WF \in \Pi_1^1$.

It is a well-known fact that every zero dimensional Polish space is homeomorphic to a closed subspace of ω^ω (cf. [45], Theorem 7.8). Hence, if X is a zero dimensional Polish space there is a homeomorphism $\varphi : X \to [T]$, where $T \in \mathrm{Tr}$ is some pruned tree. Let $A \subseteq X$, $\iota : [T] \hookrightarrow \omega^\omega$ be the inclusion map and $f : \omega^\omega \to [T]$ be a continuous function whose restriction to $[T]$ is the identity - such map is guaranteed to exist (cf. [45], Proposition 2.8). Then, setting $B = f^{-1}(\varphi(A))$ it is clear that $\psi := \varphi^{-1} \circ f : \omega^\omega \to X$ and $\phi := \iota \circ \varphi : X \to \omega^\omega$ are continuous functions such that $\psi^{-1}(A) = B$ and $\phi^{-1}(B) = A$. Hence, in order to prove that WF is Π^1_1-complete it suffices to show that for any coanalytic subset $C \subseteq \omega^\omega$ we can find a continuous function $f : \omega^\omega \to \mathrm{Tr}$ such that $f^{-1}(WF) = C$. When it comes to (co)analytic subsets of ω^ω one has the following useful characterization :

Proposition 3.12. Let $C \subseteq \omega^\omega$. Then, C is coanalytic if and only if there is a tree on $\omega \times \omega$ such that $\alpha \in C$ if and only if $T(\alpha)$ is well-founded.

Proof. The reader can find a proof in [45] (Proposition 25.2). ■

Remark 3.13. In [42] it was proven that every bounded analytic set $C \subseteq \mathbb{C}$ can be realized as the point spectrum $C = \sigma_p(T)$ of some bounded linear operator T acting on a separable Banach space, which may depend on C. In [44], this characterization was refined and it was proven that such T can be chosen to act on c_0, regardless of our choice of C. In other words (and since the point spectrum of an operator acting on a separable Banach space is always analytic - cf. Proposition 6.1), bounded analytic subsets of the complex plane coincide with the point spectrum sets of linear bounded operators acting on c_0.

Consider now any coanalytic subset $C \subseteq \omega^\omega$ and let T be as in Proposition 3.12. Define $f : \omega^\omega \to \mathrm{Tr}$ by $f(\alpha) = T(\alpha)$. It is clear that f is a continuous function with $f^{-1}(WF) = C$ and thus, we can conclude that :

Theorem 3.14. *The set $WF \subseteq Tr$ is Π^1_1-complete.*

For our second example, let X be a topological space. Let $\mathcal{F}(X)$ and $\mathcal{K}(X)$ denote respectively the set of closed and compact subsets of X. On $\mathcal{F}(X)$ we consider the σ-algebra generated by the family of sets of the form :

$$\{F \in \mathcal{F}(X) : F \cap \mathcal{U} \neq \emptyset\}, \text{ for open subsets } \mathcal{U} \subseteq X$$

The set $\mathcal{F}(X)$ endowed with this σ-algebra is often called the Effros space of $\mathcal{F}(X)$. On $\mathcal{K}(X)$, consider the topology generated by sets of the form :

$$\{K \in \mathcal{K}(X) : K \subseteq \mathcal{U}\} \text{ and } \{K \in \mathcal{K}(X) : K \cap \mathcal{U} \neq \emptyset\}, \text{ for open subsets } \mathcal{U} \subseteq X$$

This topology on $\mathcal{K}(X)$ is often referred to as the Vietoris topology.
Recall that if (X, d) is a metric space (with $d \leq 1$) one defines the Hausdorff metric d_H on $\mathcal{K}(X)$ as follows : if $K, L \in \mathcal{K}(X)$ are both non-empty, then :

$$d_H(K, L) = \max\{\max_{x \in K} d(x, L), \max_{x \in L} d(x, K)\}$$

Moreover, if K and L are both empty, then $d_H(K, L) = 0$ and if exactly one of K and L is empty, then $d_H(K, L) = 1$. It is well known that the Hausdorff metric is compatible with the Vietoris topology (cf. [77], Proposition 2.4.14). If X is separable with a countable dense set D, then so is $\mathcal{K}(X)$ by considering the countable dense subset $\{K \in \mathcal{K}(X) : K \subseteq D,\ K \text{ is finite}\}$. Moreover, if X is Polish we have the following :

Theorem 3.15. *If X is completely metrizable, so is $\mathcal{K}(X)$ with the Vietoris topology. In particular, if X is Polish so is $\mathcal{K}(X)$.*

Proof. The reader can find a proof in [45] (Theorem 4.25). ∎

Similarly, if X is Polish we are still in a *safe* setting regarding $\mathcal{F}(X)$:

Theorem 3.16. *Let X be a Polish space. Then, the Effros space on $\mathcal{F}(X)$ is a standard Borel space.*

Proof. The reader can find a proof in [45] (Theorem 12.6). ∎

Remark 3.17. For a separable metric space X, $\mathcal{F}(X)$ is a standard Borel space if and only if X is the union of a Polish space and a K_σ (cf. [72]).

Let X be a locally compact Polish space and define the Fell topology on $\mathcal{F}(X)$ by considering the following sets as basic open sets :

$$\{F \in \mathcal{F}(X) : F \cap K = \emptyset, F \cap \mathcal{U}_1 \neq \emptyset, ..., F \cap \mathcal{U}_n \neq \emptyset\}$$

where $K \in \mathcal{K}(X)$ and $\mathcal{U}_i \subseteq X$ are open sets. It follows that the Borel σ-algebra induced by the Fell topology on $\mathcal{F}(X)$ coincides with the Effros space. If furthermore X is compact and thus $\mathcal{K}(X) = \mathcal{F}(X)$ as sets, we have that our previous definitions are put together as follows :

Theorem 3.18. *Let X be a compact Polish space. Then, the Vietoris topology on $\mathcal{K}(X)$ induces the Effros space.*

Proof. The reader can find a proof in [77] (pp. 96-7). ∎

Another important topology one can consider on $\mathcal{F}(X)\setminus\{\emptyset\}$, for any metric space (X, d), is the so called Wijsman topology. This is the weak topology generated by $\{\varphi_x : x \in X\}$, where $\varphi_x : \mathcal{F}(X) \setminus \{\emptyset\} \to \mathbb{R}$ is given by $\varphi_x(F) := d(x, F)$.

Theorem 3.19. *Let X be a Polish space with a compatible complete metric d. Then, $\mathcal{F}(X) \setminus \{\emptyset\}$ endowed with the Wijsman topology with respect to d is a Polish space. Moreover, it induces the Effros space on $\mathcal{F}(X) \setminus \{\emptyset\}$.*

Proof. The reader can find a proof in [5] (Theorem 4.3). ∎

Remark 3.20. One can consider the extended Wijsman topology on $\mathcal{F}(X)$ by considering a local base at $\emptyset$ to be constituted by sets of the form (for some fixed $x_0 \in X$) :

$$\{\emptyset\} \cup \{F \in \mathcal{F}(X) : d(x_0, F) > n, n \in \mathbb{N}_{>0}\}$$

The extended Wijsman topology on $\mathcal{F}(X)$ is the subspace topology it inherits from the one-point compactification of $\overline{\mathcal{F}(X) \setminus \{\emptyset\}}$ - after identifying the point at infinity with $\{\emptyset\}$. We conclude that if X is Polish, then $\mathcal{F}(X)$ with the extended Wijsman topology is also Polish (cf. [5], Theorem 4.4).

3.2.2 Other examples and some applications

In this section we aim to provide, through rather general lines, different settings where descriptive set theoretic methods have found their use.

Let X be a separable (real) Banach space and consider :

$$\mathrm{Subs}(X) = \{Y \subseteq X : Y \text{ is a closed linear subspace}\} \subseteq \mathcal{F}(X)$$

The Kuratowski-Ryll-Nardzewski theorem states that if X is Polish then there is a sequence of Borel functions $d_n : \mathcal{F}(X) \to X$ such that for every non-empty closed set F, then $\{d_n(F)\}$ is dense in F (cf. [45], Theorem 12.13). Let then (d_n) be such a sequence and note that :

$$F \in \mathrm{Subs}(X) \Leftrightarrow 0 \in F \text{ and } \forall m, n \in \mathbb{N} \forall p, q \in \mathbb{Q} : (pd_n(F) + qd_m(F) \in F)$$

Therefore, $\mathrm{Subs}(X)$ is a Borel set in $\mathcal{F}(X)$. Note that for complex Banach spaces, one simply considers $\mathbb{Q} + i\mathbb{Q}$ instead of $\mathbb{Q}$.

It is a standard result that every (non-empty) compact metrizable space is a quotient of the Cantor space $\mathcal{C}$. Hence, and since $B_1(X^*)$ is weak*-compact, there is a surjective continuous function $f : \mathcal{C} \to B_1(X^*)$. Now for each $x \in X$, let $f_x \in C(\mathcal{C})$ be defined by $f_x(y) := f(y)(x)$. It follows that the assignment $x \mapsto f_x$ is an isometric isomorphism and thus, every separable Banach space is isometrically isomorphic to a closed subspace of $C(\mathcal{C})$. As such, we can study separable Banach spaces as elements of $\mathrm{Subs}(C(\mathcal{C}))$ - a standard Borel space.
If X is a separable Banach space, we follow [11] and let $\langle X \rangle$ denote the equivalence class $\{Y \in \mathrm{Subs}(C(\mathcal{C})) : Y \approx X\}$ induced by isomorphism $\approx$ of Banach spaces. We can thus encode information about separable Banach spaces in a canonical way via $c : \mathrm{Subs}(C(\mathcal{C})) \to \mathrm{Subs}(C(\mathcal{C}))/\approx$, such that $X \mapsto \langle X \rangle$. It turns out that within this setting, $\approx$ and other natural relations are not Borel :

Theorem 3.21. *The isomorphism relation $\approx$ and the relation $\{(X, Z) : \exists Y \in$ Subs$(C(\mathcal{C})) : Z \approx X \oplus Y\}$ are analytic and non-Borel in Subs$(C(\mathcal{C}))^2$.*

Proof. The reader can find a proof in [11] (Theorem 2.3). ∎

Recall that a Banach space is said to be universal if it contains an isomorphic copy of every separable Banach space, like $C(\mathcal{C})$. In [11], the following powerful criterion is proved :

Theorem 3.22. *Let $\mathcal{A}$ be an analytic family of separable Banach spaces which is stable under isomorphism, contains all reflexive separable spaces. Then, $\mathcal{A}$ contains a space which is universal for all separable Banach spaces.*

Proof. The reader can find a proof in [11] (Theorem 3.2). ∎

Let X be a separable Banach space and $\mathcal{A}$ be the family of separable Banach spaces which have an isomorphic copy in X. By Theorem 3.21 this is an analytic family and thus, by Theorem 3.22, it follows that X is universal if and only if it contains a copy of every reflexive separable space, i.e. universal for reflexive (separable) spaces. This outstanding result was firstly proved in [12]. In fact, more can be said by choosing different analytic families :

Theorem 3.23. *Let X be a separable Banach space. Then :*

(i) Every reflexive separable space has an isomorphic copy in X if and only if X is universal.

(ii) Every reflexive separable space is isomorphic to a subspace of a quotient of X if and only if X contains an isomorphic copy of ℓ_1.

(iii) Every reflexive separable space is isomorphic to a quotient of X if and only if X an isomorphic complemented copy of ℓ_1.

Proof. The reader can find a proof in [11] (Corollary 3.4). ■

It follows, once again by Theorem 3.22 that the family Ref $\subseteq$ Subs$(C(\mathcal{C}))$ of reflexive separable spaces is not analytic and thus, not Borel. Matter of fact, it can be proven that Ref is actually Π_1^1-complete (cf. [11], Lemma 2.4).

We finish this section with a few more examples, this time concerning continuous functions $f \in C([0,1])$. For more considerations on reflexive (separable) Banach spaces and other examples of complete sets, the reader is referred to the appended papers. In these papers, examples and applications concerning closed sets of uniqueness, point spectra and reflexivity are discussed.

Theorem 3.24. *For any $f \in C([0,1])$, let $ND(f)$ be the set of points in $[0,1]$ where f is not differentiable. Then, the following sets are Π_1^1-complete :*

(i) $\{f \in C([0,1]) : ND(f) \in \mathcal{F}\}$, where $\mathcal{F}$ is a family of countable subsets of $[0,1]$ containing $\emptyset$.

(ii) NDIFF $= \{f \in C([0,1]) : f$ is nowhere differentiable$\}$

Proof. The reader can find a proof of (i) in [30] and a proof of (ii) in [58]. ■

Note that it follows from Theorem 3.24 that the set of continuous functions on the unit interval which are differentiable everywhere is coanalytic complete, a result originally proven in [59]. From the next example, one can deduce the classical result on the existence of continuous functions on $\mathbb{T}$ with divergent Fourier series :

Theorem 3.25. *The set of continuous functions on $[0,1]$ with everywhere convergent Fourier series is Π_1^1-complete in $C([0,1])$.*

Proof. The reader can find a proof in [45] (Theorem 33.13). ■

3.3 A few properties

In this section, for the sake of completeness, we present a short compilation of well known properties of (co)analytic. For any subset $B \subseteq X \times Y$ of an arbitrary product of sets we define :

$$\exists^Y B = \{x \in X : (x,y) \in B, \text{ for some } y \in Y\}$$

$$\forall^Y B = \{x \in X : (x,y) \in B, \text{ for all } y \in Y\}$$

Proposition 3.26. For a Polish space X, the following statements are true :

(i) For every $1 \leq \alpha < \omega_1$, the classes Σ^0_α and Π^0_α are closed under finite unions and finite intersections. Moreover, the classes Σ^0_α are closed under countable unions and the classes Π^0_α are closed under countable intersections. All the classes Σ^0_α and Π^0_α are closed under pre-images of continuous functions.

(ii) For every n, the classes Σ^1_n and Π^1_n are closed under countable unions, countable intersections and pre-images of Borel functions. Moreover, the classes Σ^1_n are closed under $\exists^Y$ and the classes Π^1_n are closed under $\forall^Y$, for any Polish space Y.

Proof. The statement (i) is trivial. For a proof of the statement (ii), the reader can check [77] (Proposition 4.1.7). ■

As noted previously, it is not the case that arbitrary continuous images of Borel sets remain Borel. However, one has the following :

Theorem 3.27. *Let X, Y be Polish spaces and $f : X \to Y$ be a Borel function which is countable-to-one. Then, if $B \subseteq X$ is Borel so is $f(B) \subseteq Y$.*

Proof. The reader can find a proof in [77] (Theorem 4.12.4). ■

Given two topological spaces X, Y, we say that $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$ are Borel isomorphic if there is a bijective function $f : X \to Y$ such that f and f^{-1} are Borel functions. It turns out, perhaps surprisingly, that the cardinality of a Polish space is an invariant :

Theorem 3.28. *Let X and Y be standard Borel spaces. Then, X and Y are Borel isomorphic if and only if $|X| = |Y|$. In particular, any two uncountable standard Borel spaces are isomorphic.*

Proof. The reader can find a proof in [45] (Theorem 15.6). ■

Next, we recall three important *regularity* properties :

Definition 3.29. Let X be a Polish space and $A \subseteq X$. Then :

(i) A has the Perfect Set Property (PSP) if A is either countable or contains a non-empty perfect set.

(ii) A has the Baire Property (BP) if there is an open set $\mathcal{U} \subseteq X$ such that the symmetric difference $A\Delta\mathcal{U}$ is meager.

(iii) A is universally measurable if it is μ-measurable with respect to any σ-finite Borel measure μ on X.

Remark 3.30. Recall that every complete metrizable space without isolated points contains a homeomorphic copy of the Cantor set and thus, it is uncountable (cf. [77], Proposition 2.6.1). Therefore, the Continuum Hypothesis holds for the class of subsets of a Polish space with the PSP. It is also a well known fact that any separable metric space X can be decomposed as $X = Y \sqcup Z$, with Y countable and Z closed and without isolated points (cf. [77], Proposition 2.6.2). Thus, every G_δ subset of a Polish space has the PSP - in particular any closed subset of a Polish space which is not countable, has the cardinality $\mathfrak{c}$. This result can be generalized as follows (cf. [78], Theorem 3.6) : Let X be a metric space of cardinality α and weigth $\beta \geq \aleph_0$. Then, the number of closed subsets of X of cardinality γ is :

(i) α^γ, if $\gamma \leq \beta$

(ii) 2^β, if $\gamma = \alpha$

(iii) 0, if X is complete and $\beta < \gamma < \alpha$

This contrasts with what happens with non metrizable spaces. For instance, the closed subsets of $\beta\mathbb{N}$ are either finite or of size $2^{\mathfrak{c}}$.

Theorem 3.31. *Let X be a Polish space. The following statements are true :*

(i) All analytic subsets of X have the PSP.

(ii) All analytic subsets of X have the BP.

(iii) All analytic subsets of X are universally measurable. In particular, all analytic subsets of $\mathbb{R}$ are Lebesgue measurable.

Proof. The reader can find a proof of statements (i) and (ii) in [77] (respectively, Theorems 4.3.5 and 4.3.2) and of statement (iii) in [45] (Theorem 21.10). ■

Recall that $B \subseteq \mathbb{R}$ is said to be a Bernstein set if it intersects each uncountable closed set but contains none of them. It can be shown via a transfinite recursion argument that $\mathbb{R}$ is indeed an union of $\mathfrak{c}$ many pairwise disjoint Bernstein sets. It follows from the definition that if B is a Bernstein set, then B is not Lebesgue measurable and that B does not have the BP.

Remark 3.32. It follows immediately from Theorem 3.31 that all coanalytic sets also have the BP and are universally measurable. However, the issue with the PSP is more delicate. Indeed, the question of whether or not all coanalytic sets have the PSP is independent from ZFC : assuming $V = L$, there is an uncountable coanalytic set which does not contain any perfect set (cf. [33])

and, on the other hand, assuming the Σ^1_1-determinacy all coanalytic sets have the PSP (see section 3.4). Note as well that Theorem 3.31 establishes that the CH holds for analytic subsets of a Polish space. Within ZFC one can prove that uncountable coanalytic sets either have cardinality $\aleph_1$ or $\mathfrak{c}$ (cf. [77], Theorem 4.3.18). Interestingly enough, if E is an analytic equivalence relation on a Polish space with uncountably many equivalence classes, then this number is either $\aleph_1$ or $\mathfrak{c}$. This is known as Burgess theorem (cf. [77], Theorem 5.13.4). On the other hand, if E is a coanalytic equivalence relation, then the number of equivalence classes is either countable or $\mathfrak{c}$. This is known as Silver's theorem (cf. [77], Theorem 5.13.11).

Remark 3.33. Let $A \subseteq X$ be a subset with the BP. Then A is, up to a meager set, just *like* an open set. The Baire property appears naturally in many contexts. In order to provide a perhaps less obvious context for this property, consider the following : for a set X, let $[X]^{\aleph_0}$ be the set of subsets of X with cardinality $\aleph_0$. We consider $[\mathbb{N}]^{\aleph_0}$ endowed with the Ellentuck topology. This is the topology which has the following sets as basic open sets, for $a \in \mathcal{P}_{\text{fin}}(\mathbb{N})$ and $A \in \mathcal{P}(\mathbb{N}) \setminus \mathcal{P}_{\text{fin}}(\mathbb{N})$:

$$[a, A] = \{S \in [\mathbb{N}]^{\aleph_0} : a \subseteq S \subseteq a \cup A\}, \text{ with } \max(a) < \min(A)$$

Then, $X \subseteq [\mathbb{N}]^{\aleph_0}$ is said to be completely Ramsey if for every such a, A there is some $B \subseteq A$ with either $[a, B] \subseteq X$ or $[a, B] \subseteq [\mathbb{N}]^{\aleph_0} \setminus X$. It turns out that X is completely Ramsey if and only if it has the Baire property in the Ellentuck topology (cf. [45], Theorem 19.14). As a consequence, an infinitary analog of Ramsey's theorem[7] - usually referred to as Galvin-Prikry theorem - can be easily proven (cf. [45], Theorem 19.11) : if $[\mathbb{N}]^{\aleph_0} = \bigcup_{i=1}^n P_i$ is a partition with each P_i Borel, then there exists an infinite subset H of $\mathbb{N}$ such that $[H]^{\aleph_0} \subseteq P_k$ for some $k \leq n$. A different context where the Baire property is relevant will be mentioned in section 3.5.

We finish this section with some brief comments on ranks. In turn, this provides yet another method to identify sets which are not Borel.
A rank φ on a set S is simply a function from S into ω_1. We can associate with φ a relation $\leq_\varphi$ on S as follows :

$$x \leq_\varphi y \text{ if and only if } \varphi(x) \leq \varphi(y)$$

If S is a strict subset of another set X, we simply declare $\varphi(x) = \omega_1$ whenever $x \notin S$. This extends $\leq_\varphi$ to a relation $\leq^*_\varphi$ on X as follows :

$$x \leq^*_\varphi y \Leftrightarrow x \in S \text{ and } \varphi(x) \leq \varphi(y)$$

Naturally we can consider $<^*_\varphi$ on X as follows :

$$x <^*_\varphi y \Leftrightarrow x \in S \text{ and } \varphi(x) < \varphi(y)$$

[7]For any set X, let $[X]^n = \{A \subseteq X : |A| = n\}$. Then, if $[\mathbb{N}]^n = \bigcup_{i=1}^n P_i$ is a partition, there is an infinite $H \subseteq \mathbb{N}$ such that $[H]^n \subseteq P_k$ for some $k \leq n$.

Definition 3.34. Let X be a Polish space and $P \subseteq X$ be coanalytic. A coanalytic rank (or Π^1_1-rank) on P is a rank φ such that the relations $x \leq^*_\varphi y$ and $x <^*_\varphi y$ are both coanalytic subsets of X^2.

As a central example, consider $X = \mathrm{Tr}$ and $P = WF$. For a tree $T \in WF$ and $s \in \omega^{<\omega}$ we define recursively the height of s in T as follows :

$$h(s,T) = 0, \text{ if } s \notin T$$

$$h(s,T) = \sup\{h(t,T) + 1 : s \subsetneq t\}, \text{ if } s \in T$$

The height of T is defined as $h(T) := h(\emptyset, T)$. Note that since $T \in WF$, then $h(T) < \omega_1$. It follows that h is a coanalytic rank on WF (cf. [47], Lemma V.I.2).

Let X be a Polish space and $\prec$ be a binary relation on X. We say that $\prec$ is well-founded if there is no sequence $\{x_n\} \subseteq X$ such that for all n one has that $x_{n+1} \prec x_n$. If $\prec$ is well-founded we can define recursively - similarly to our definition of height of a well-founded tree - the length lh of $\prec$ as follows :

$$lh(x, \prec) = 1, \text{ if there is no } y \text{ such that } y \prec x$$

$$lh(x, \prec) = \sup\{lh(y, \prec) + 1 : y \prec x\}, \text{ otherwise}$$

The length of $\prec$ is then defined as $lh(\prec) = \sup\{lh(x, \prec) + 1 : x \in X\}$. An important property of lh is that if X is a Polish space and $\prec$ is a well-founded analytic relation, then $lh(\prec) < \omega_1$ (cf. [47], Theorem V.I.6).

Remark 3.35. The previous statement about the length of well-founded relations can be generalized as follows (cf. [45], Theorem 31.5) : let X be a Polish space, $A \subseteq X$ and Y any discrete space. Then, A is said to be Y-Suslin if there is a closed set $F \subseteq X \times Y^{\mathbb{N}}$ such that $A = \pi_X(F)$. In particular, $\mathbb{N}$-Suslin sets are just analytic sets. If κ is an infinite ordinal and $\prec$ is a well-founded κ-Suslin relation on X, then $lh(\prec) < \kappa^+$.

Theorem 3.36. *Every coanalytic set admits a coanalytic rank. Moreover, let φ be a coanalytic rank on some coanalytic subset P of a Polish space X and suppose that $Q \subseteq P$. Then, if Q is analytic one has that* $\sup\{\varphi(x) : x \in Q\} < \omega_1$.

Proof. The fact that every Π^1_1 set admits a Π^1_1-rank follows easily from the fact that WF is Π^1_1-complete and that the height h is a Π^1_1-rank on WF. In order to prove the second part of the theorem, consider the relation :

$$x \prec y \Leftrightarrow x \in Q \text{ and } y \in Q \text{ and } \varphi(x) < \varphi(y)$$

This is clearly a well-founded relation on X and, since φ is a Π^1_1-rank and Q is analytic, it follows from the definition that $\prec$ is analytic. Consequently, $\prec$ has a countable length and we are done. ∎

Theorem 3.36 is usually referred to as the Boundedness Theorem. It provides a method to identify non Borel sets by means of finding a Π^1_1-rank which is unbounded on a given set.

3.4 Games

In this section we briefly consider infinite games and some of their immediate relations with (co)analytic sets. The framework is the following : we consider an arbitrary discrete space A, a non-empty pruned tree $T \subseteq A^{<\omega}$ (which defines the legal positions of the game) and a subset $X \subseteq [T]$ (payoff set). The game $G(T, X)$ is played as follows : there are two players, P1 and P2 (we usually assume that P1 starts the game), which take turns playing $a_i \in A$ such that $(a_0, a_1, ..., a_n) \in T$ for every n. In the *end*, we say that P1 wins $G(T, X)$ if and only if $(a_n) \in X$.
A strategy for P1 is then a non-empty pruned subtree $\sigma \subseteq T$ such that :

(i) If $(a_0, ..., a_{2j}) \in \sigma$, then for all $a_{2j+1} \in A$ such that $(a_0, ..., a_{2j}, a_{2j+1}) \in T$, we have that $(a_0, ..., a_{2j}, a_{2j+1}) \in \sigma$

(ii) If $(a_0, ..., a_{2j-1}) \in \sigma$, there is an unique $a_{2j} \in A$ such that $(a_0, ..., a_{2j-1}, a_{2j}) \in \sigma$

In other words, a strategy for P1 is a rule which tells at each point what P1 should play in response to P2. Appropriately, we say that σ is a winning strategy if and only if $[\sigma] \subseteq X$ - i.e. no matter what P2 plays, if P1 follows σ then it is guaranteed to win the game. We define (winning) strategy for P2 in an entirely analogous way. We say that a game $G(T, X)$ is determined if one of the players has a winning strategy.

Remark 3.37. Consider $T = \omega^{<\omega}$. It is clear that there are $2^{\aleph_0}$ strategies for each player. Let $\{\sigma_\alpha\}_{\alpha \in I}$ and $\{\tau_\alpha\}_{\alpha \in I}$ be the set of strategies, respectively for P1 and P2, indexed by some well-ordered set I with cardinality $2^{\aleph_0}$. Pick $p_0 \in [\tau_0]$ and $q_0 \in [\sigma_0]$ such that $p_0 \neq q_0$. For $\alpha \in I$ suppose we picked for every $\beta < \alpha$ elements $p_\beta \in [\tau_\beta]$ and $q_\beta \in [\sigma_\beta]$ such that $p_\beta \neq q_\beta$. Then, pick some $p_\alpha \notin \{q_\beta\}_{\beta<\alpha}$ and $q_\alpha \notin \{p_\beta\}_{\beta<\alpha} \cup \{p_\alpha\}$ such that $p_\alpha \in [\tau_\alpha]$ and $q_\alpha \in [\sigma_\alpha]$. Set $P = \{p_\alpha\}$ and $Q = \{q_\alpha\}$. Then, $G(\omega^{<\omega}, P)$ is not determined : indeed, suppose towards a contradiction that P1 has a winning strategy, say σ_γ. Since $q_\gamma \in Q$ and by definition $Q \cap P = \emptyset$, it follows that $[\sigma_\gamma] \subsetneq P$. Similarly, P2 does not have any winning strategy either.

It follows from Remark 3.37 that at least within ZFC, there are undetermined games. It is consistent with ZF that all games are determined and we shall laconically comment on this issue in the end of this section. In contrast, open and closed games (i.e. infinite games whose payoff set is open or closed) are always determined :

Theorem 3.38. *Suppose that T is a non-empty pruned tree on an arbitrary set A and that $X \subseteq [T]$ is either closed or open. Then, $G(T, X)$ is determined.*

Proof. Suppose that X is closed and that P2 does not have a winning strategy. A position $p = (a_0, ..., a_{2n-1}) \in T$ with P1 to play next is said to be a non-losing position for P1 if P2 does not have a winning strategy for the game onwards, i.e. for the game $G(T_p, X_p)$. Clearly, $\emptyset$ is a non-losing position for P1 and for every non-losing position p, there is some $a_{2n} \in T_p$ such that for any a_{2n+1} with $(a_{2n}, a_{2n+1}) \in T_p$, then $p \frown (a_{2n}, a_{2n+1})$ is also a non-losing position for P1. This produces a strategy for P1 which we claim to be a winning strategy. Indeed, let (a_n) be a run of this game where P1 followed this strategy and suppose, towards a contradiction, that $(a_n) \notin X$. Since X is closed, there is some k such that $\Sigma((a_0, ..., a_{2k-1})) \cap [T] \subseteq [T] \setminus X$. In turn, this implies that $(a_0, ..., a_{2k-1})$ is losing for P1 which is a contradiction. If, on the other hand X is open, the argument is entirely similar (switching the roles of the players). ■

Remark 3.39. We note that, due to the single-valuedness condition in the definition of strategy, the proof of Theorem 3.38 relies on the Axiom of Choice. Using the notion of quasistrategy instead, the argument used in Theorem 3.38 provides a proof for the analogous result - without appealing to the Axiom of Choice (cf. [45], p. 139). On the other hand, given any set X such that $\emptyset \notin X$, consider the tree T on $X \cup (\bigcup X)$ given by $(a_0) \in T$ if and only if $a_0 \neq \emptyset$, $(a_0, a_1) \in T$ if and only if $a_1 \in a_0$ and $(a_0, a_1, ..., a_n) \in T$ if and only if $a_1 = a_n$, for all $n \geq 1$. Then, if Theorem 3.38 holds it follows that $G(T, \emptyset)$ is determined and thus, there is a choice function. Hence, in ZF the statement that closed games are determined (as in Theorem 3.38) is actually equivalent to the Axiom of Choice.

After determinacy was established by Gale-Stewart for closed and open payoff sets in [27], Wolfe proved in [82] that games with a Σ^0_2 payoff set are determined and Davis proved in [16] that Σ^0_3 payoff sets are also determined. Finally, Martin proved in [55] (and later heavily simplified in [54]) that Borel sets are determined. This determinacy result also has metamathematical interest for at least a couple of reasons : in [26], Friedman showed that there is a model of ZF minus the Axiom of Replacement (and where the Axiom of Choice holds) where Borel determinacy fails. Hence, in any proof of Borel determinacy the Axiom of Replacement is needed. In any proof of Borel determinacy, it is needed to use sets of very high type : for instance, in order to prove that Borel sets are determined in $\{0, 1\}$ one needs to use $\mathcal{P}_\alpha(\mathbb{N})$ - the α^{th}-iterated power set of $\mathbb{N}$ - for all $\alpha < \omega_1$. Furthermore, Borel determinacy is in a sense an optimal result : indeed, the statement that all analytic sets are determined is independent from ZFC and constitutes an alternative axiom usually referred to as Σ^1_1-determinacy.

In order to sketch Martin's inductive argument for Borel determinacy, we shall need the concept of a covering. Let T be a non-empty pruned tree on an arbitrary set A. A covering of T is a triple $(\tilde{T}, \pi, \varphi)$ such that :

(i) $\tilde{T}$ is a non-empty pruned tree on some set $\tilde{A}$.

(ii) $\pi : \tilde{T} \to T$ is a monotone and length preserving function which extends to a continuous function $\pi : [\tilde{T}] \to [T]$.

(iii) φ maps strategies for P1 (P2) in $\tilde{T}$ into strategies for P1 (P2) in T in such way that $\varphi(\tilde{\sigma})|_n$ depends only on $\tilde{\sigma}|_n$.

(iv) If $\tilde{\sigma}$ is a strategy in $\tilde{T}$ and $x \in [\varphi(\tilde{\sigma})]$, then there is some $\tilde{x} \in [\tilde{\sigma}]$ such that $\pi(\tilde{x}) = x$.

(v) Moreover, for each $k \in \mathbb{N}$ we say that $(\tilde{T}, \pi, \varphi)$ is a k-covering if $T|_{2k} = \tilde{T}|_{2k}$ and $\pi|_{\tilde{T}_{2k}}$ is the identity.

Note that it follows from condition (iv) that if $(\tilde{T}, \pi, \varphi)$ is a covering of T and $X \subseteq [T]$, then a winning strategy $\tilde{\sigma}$ for P1 in $G(T, X)$ is mapped to a winning strategy $\varphi(\tilde{\sigma})$ for P1 in $G(\tilde{T}, \pi^{-1}(X))$. Similarly for P2. We say that $(\tilde{T}, \pi, \varphi)$ unravels $X \subseteq [T]$ if $\pi^{-1}(X)$ is clopen in $[\tilde{T}]$. Hence, it follows by Theorem 3.38 that if $(\tilde{T}, \pi, \varphi)$ unravels $G(T, X)$ then $G(T, X)$ is determined.
Endowed with this terminology, in order to prove Martin's result on the determinacy of Borel sets it is thus sufficient to prove that for every game $G(T, X)$ with X Borel, there is a covering of T which unravels X. In order to enable an inductive argument, something stronger is proved for the base case :

Theorem 3.40. *Let T be a non-empty pruned tree and $X \subseteq [T]$ be closed. For each $k \in \mathbb{N}$ there is a k-covering of T that unravels X.*

Proof. The reader can find a proof in [45] (Lemma 20.7). ■

A final piece of machinery is needed in the shape of the following result which asserts the existence of inverse limits :

Theorem 3.41. *Let $k \in \mathbb{N}$ and $(T_{i+1}, \pi_{i+1}, \varphi_{i+1})$ be a $k+1$-covering of T_i, for $i \geq 0$. Then, there is a pruned tree T_∞ and maps $\pi_{\infty,i}, \varphi_{\infty,i}$ such that $(T_{\infty,i}, \pi_{\infty,i}, \varphi_{\infty,i})$ is a $(k+i)$-covering of T_i and moreover, $\pi_{i+1} \circ \pi_{\infty,i+1} = \pi_{\infty,i}$ and $\varphi_{i+1} \circ \varphi_{\infty,i+1} = \varphi_{\infty,i}$.*

Proof. The reader can find a proof in [45] (Lemma 20.8). ■

Assuming Theorems 3.40 and 3.41, we can establish the following :

Theorem 3.42. *Let T be a non-empty pruned tree on an arbitrary set A. If $X \subseteq [T]$ is Borel, then for each $k \in \mathbb{T}$ there is a k-covering which unravels X. In particular, $G(T, X)$ is determined.*

Proof. Since X is Borel, then $X \in \Sigma^0_\alpha$ for some $1 \leq \alpha < \omega_1$. We prove the result by induction : the base case of $\alpha = 1$ follows from Theorem 3.40 and the observation that if a k-covering unravels X then it also unravels $[T] \setminus X$. We fix some k and suppose that there is always some k-covering which unravels X if $X \in \Pi^0_\beta$, for all $\beta < \alpha$. We represent our payoff set as $X = \bigcup_{n \in \mathbb{N}} X_n$, with $X_n \in \Pi^0_{\beta_n}$ for $\beta_n < \alpha$. Let (T_1, π_1, φ_1) be a k-covering of $T_0 = T$ unravelling X_0 and define $(T_{n+1}, \pi_{n+1}, \varphi_{n+1})$ by recursion to be a $(k+n)$-covering of T_n which unravels $\pi_n^{-1} \circ ... \circ \pi_1^{-1}(X_n)$. Let $(T_\infty, \pi_{\infty,n}, \varphi_{\infty,n})$ to be as in Theorem 3.41. Since $\pi^{-1}_{\infty,0}(X)$ is open in $[T_\infty]$, let $(\tilde{T}, \pi, \phi)$ be a k-covering of T_∞ that unravels $\pi^{-1}_{\infty,0}(X)$. It follows that $(\tilde{T}, \pi_{\infty,0} \circ \pi, \varphi_{\infty,0} \circ \varphi)$ is a k-covering of T that unravels X as we wanted. ■

Remark 3.43. Let Γ be a class of sets of ω^ω. A statement of the following type is usually referred to as Γ-determinacy :

For every $A \in \Gamma$, the game $G(\omega^{<\omega}, A)$ is determined

If Σ^1_1-determinacy holds, then coanalytic sets have the PSP (see Remark 3.32) :

(i) First note that if Γ is a pointclass of subsets of ω^ω closed under continuous preimages, then all sets $A \in \Gamma$ are determined if and only if all sets $\omega^\omega \setminus A$ with $A \in \Gamma$ are determined. In particular, under the Σ^1_1-determinacy all coanalytic subsets of ω^ω are also determined : suppose that $A \in \Pi^1_1$ and define $f : \omega^\omega \to \omega^\omega$ by $f(x)(n) := x(n+1)$. Since f is continuous, the game $G(\omega^{<\omega}, \omega^\omega \setminus f^{-1}(A))$ is determined. Suppose then that τ is a winning strategy for P2. It is then easy to verify that σ defined by $\sigma(s) = \tau(0 \frown s)$ for all $s \in \omega^{<\omega}$ is a winning strategy for P1 on $G(\omega^{<\omega}, A)$. The case is analogous if we switch the players.

(ii) Given a non-empty Polish space X with a compatible complete metric d, fix a countable basis $\{\mathcal{V}_n\}$ and a payoff set $A \subseteq X$. We then define the following game : P1 starts by choosing two basic open sets with disjoint closure and diameter less than 1 and P2 picks one of them. In the next turn, P1 picks again two basic open sets with smaller diameter and still disjoint closure which are contained in the one picked by P2 and the process iterates. In other words, at the n^{th}-turn P1 picks a pair $(\mathcal{V}^n_0, \mathcal{V}^n_1)$ with $\text{diam}(\mathcal{V}^n_i) < \frac{1}{2^n}$ and $\overline{\mathcal{V}^n_0} \cap \overline{\mathcal{V}^n_1} = \emptyset$ for $i \in \{0,1\}$ and P2 responds with $i_n \in \{0,1\}$. In the next turn, P1 picks another pair with the additional restriction that $\overline{\mathcal{V}^{n+1}_0 \cup \mathcal{V}^{n+1}_1} \subseteq \mathcal{V}^n_{i_n}$ and so on. This uniquely defines an element $x \in \bigcap_n \overline{\mathcal{V}^n_{i_n}}$ and we say that P1 wins the game if and only if $x \in A$. Naturally, we can translate this game into the setting of games in $\omega^{<\omega}$ by means of simply using enumerations since $\{\mathcal{V}_n\}$ is countable. Let's denote this game by $G^*(A)$. It turns out that P1 has a winning strategy for $G^*(A)$ if and only if A contains a Cantor set and P2 has a winning strategy for $G^*(A)$ if and only if A is countable (cf. [45], Theorem 21.1).

(iii) Now assume that the Σ^1_1-determinacy holds and consider any (non-empty) Polish space X and an uncountable coanalytic subset $A \subseteq X$. By Theorem 3.28, let $\varphi : X \to \omega^\omega$ be a Borel isomorphism and thus, it follows from (i) that under the Σ^1_1-determinacy the game $G^*(\varphi(A))$ is determined. It follows then from (ii) that $\varphi(A)$ contains a Cantor set and thus we can conclude that so does A as we wanted.

Remark 3.44. A more radical statement about determinacy is the Axiom of Determinacy (AD) :

For every subset A, the game $G(\omega^{<\omega}, A)$ is determined

This statement can be seen as the natural extension of a standard rule of propositional logic to infinitary logic since it can be restated as :

$$\neg\exists a_0 \forall a_1 ... (a_n) \in A \Leftrightarrow \forall a_0 \exists a_1 ... \neg(a_n) \in A$$

Moreover, note that by Remark 3.37, the AD is incompatible with the Axiom of Choice. For the sake of completeness the author feels compelled to include two somewhat lengthy observations concerning the AD. In [63], Mycielski and Steinhaus - who had introduced the AD - proved that under ZF+AD, all subsets of $\mathbb{R}$ are Lebesgue measurable. Without going too deep into the details, we sketch the idea. Fix some $\epsilon > 0$ and for each $n \in \mathbb{N}$ consider the collection $\{G_k^n\}_{k\in\mathbb{N}}$ of all finite unions G_k^n of intervals with rational endpoints such that $\mu^*(G_k^n) < \frac{\epsilon}{2^{2n+1}}$, where μ^* denotes the Lebesgue outer measure. We assume, without loss of generality, that we fix some arbitrary subset $S \subseteq [0,1]$ and for each $x := (x_n) \in \omega^\omega$ let $f(x) := \sum_n \frac{x_n}{2^{n+1}} \in \mathbb{R}$. We play the following game : at each turn, P1 picks some $a_i \in \mathbb{N}$ and afterwards P2 responds by picking another $b_i \in \mathbb{N}$ so that at the end of the game we have a play $(a_0, b_0, a_1, b_1, ...)$. We declare that P1 wins the game if and only if each $a_i \in \{0,1\}$, $f((a_i)) \in S$ and $f((a_i)) \notin \bigcup_n G_{b_n}^n$. Under the AD this game is determined and as a consequence, the following statement holds :

if $S \subseteq \mathbb{R}$ is such that every measurable $N \subseteq S$ is null, then S is null

We note that this is false in ZFC as a Bernstein set provides a counter-example. However, in ZF+AD this implies that every subset of $\mathbb{R}$ is Lebesgue measurable. As a final note, we point out that an universe where the existence of sets is declared by ZF and every subset of $\mathbb{R}$ is Lebesgue measurable (LM) turns out to be a quite bizarre place :

$$\text{(ZF+LM) implies that } |\mathbb{R}| < |\mathbb{R}/\mathbb{Q}|$$

In other words, there is an equivalence relation on $\mathbb{R}$ with more than $\mathfrak{c}$ equivalence classes. The reader can read more about this type of paradoxical divisions in [81].

We end this section with an useful result, usually referred to as Wadge's Lemma. It is a consequence of Borel determinacy, with self-contained interest, that we shall use in Section 5. In order to prove Wadge's Lemma it is convenient to introduce the setting for a Wadge game. We start with two fixed subsets $A, B \subseteq \omega^\omega$ which define the game $GW(A, B)$ under the following rules : there are two players, P1 and P2 respectively picking natural numbers $x(i) \in \omega$ and $y(i) \in \omega$ at each turn. P2 wins $GW(A, B)$ if and only if $(x(i)) \in A$ and $(y(i)) \in B$. Within this context, we will use the standard notation $A \leq_W B$ in order to indicate that there is a continuous function $f : \omega^\omega \to \omega^\omega$ such that $A = f^{-1}(B)$. It is worth to note that in the literature, this type of game is also referred to as a Lipschitz game - denoted $GL(A, B)$. Whenever that is the case, $GW(A, B)$ is usually defined under slightly different rules, with P2 having the ability of *passing a move* and not picking a positive integer on its turn.

Theorem 3.45. *Let $A, B \subseteq \omega^\omega$ be Borel subsets. Then, either $A \leq_W B$ or $B \leq_W \omega^\omega \setminus A$.*

Proof. By Theorem 3.42, the game $GW(A, B)$ is determined and thus, either P2 or P1 has a winning strategy. Suppose that P2 has a winning strategy and note that we can view it as a length preserving monotone map $\varphi : \omega^{<\omega} \to \omega^{<\omega}$. In turn, this induces a continuous map $f : \omega^\omega \to \omega^\omega$ such that $x \in A$ if and only if $f(x) \in B$, i.e. $A \leq_W B$. Analogously, if P1 has a winning strategy we conclude that $B \leq_W \omega^\omega \setminus A$. ■

Remark 3.46. The notation $\leq_W$ is rather suggestive. Indeed, for Borel sets $A, B \subseteq \omega^\omega$ we define the following equivalence relation :

$$A \equiv_W B \Leftrightarrow A \leq_W B \text{ and } B \leq_W A$$

Its classes, called Wadge degrees, are denoted by $[A]_W$. The so called coarse Wadge classes are then defined as $A^* := [A]_W \cup [\omega^\omega \setminus A]_W$ and we shall denote the set of these by WADGE^*_B. Finally, if we define $A^* \leq^* B^*$ to mean that either $A \leq_W B$ or $A \leq_W \omega^\omega \setminus B$ holds, it follows from Theorem 3.45 that $\leq^*$ is a linear ordering in WADGE^*_B. Furthermore, one can prove that $(\text{WADGE}^*_B, \leq^*)$ is in fact a well-ordering (cf. [45], Theorem 21.15).

3.5 Ideals and bases

This section contains a very brief overview of some techniques concerning the study of ideals of closed and compact subsets of Polish spaces. These techniques can be extremely powerful and general, finding their usefulness in interdisciplinary efforts where descriptive set theoretic approaches reveal to be fruitful. As an example, the application of these techniques in the study of closed sets of uniqueness turned out to enable a deeper insight on the topic. Throughout this section, X will be a non-empty Polish space.

Definition 3.47. A collection $I \subseteq \mathcal{F}(X)$ is called a σ-ideal if :

(i) $K \in I$, $L \subseteq K$ and $L \in \mathcal{F}(X)$ imply that $L \in I$ (I is hereditary)

(ii) $\{K_n\} \subseteq I$ and $K := \bigcup_n K_n \in \mathcal{F}(X)$ imply that $K \in I$

We often consider σ-ideals in $\mathcal{K}(X)$ whose definition is entirely analogous.

Let $A \subseteq X$. Clearly, $\mathcal{F}(A) := \{F \in \mathcal{F}(X) : F \subseteq A\}$ is a σ-ideal and likewise, $I_{\text{meag}} = \{F \in \mathcal{F}(X) : F \text{ is meager}\}$ is also easily verified to be a σ-ideal. A slightly more sophisticated example of a σ-ideal is as follows : consider a subset $M \subseteq P(X)$ of Borel probability measures and define :

$$I_M = \{F \in \mathcal{F}(X) : \forall \mu \in M : \mu(F) = 0\}$$

In particular, letting M be the set of Rajchman measures on an abelian locally compact group G this means that the set $\mathcal{U}_0(G)$ of closed sets of extended uniqueness of G (cf. section 5) is a σ-ideal. It turns out that the set $\mathcal{U}(G)$ of closed sets of uniqueness of an amenable locally compact group G is also a σ-ideal, though this is not a trivial result (cf. section 5). When $G = \mathbb{T}$, this is a consequence of Bari's Theorem (cf. [4]) which asserts that the countable union of closed sets of uniqueness of $\mathbb{T}$ is still a set of uniqueness of $\mathbb{T}$.

One has the following, perhaps surprising, dichotomy result :

Theorem 3.48. *Suppose that I is Π_1^1 σ-ideal of $\mathcal{K}(X)$. Then, either I is G_δ or I is Π_1^1-complete.*

Proof. The reader can find a proof in [45] (Theorem 33.3). ■

In order to illustrate Theorem 3.48, suppose that $A \subseteq X$ is Π_1^1. It is not hard to verify that $\mathcal{K}(A) = \{K \in \mathcal{K}(X) : K \subseteq A\}$ and $K_\omega(A) = \{K \in \mathcal{K}(X) : K \subseteq A \text{ and } K \text{ is countable}\}$ are both Π_1^1 σ-ideals. It follows immediately by Theorem 3.48 that if A is not G_δ, then $\mathcal{K}(A)$ is Π_1^1-complete. Furthermore, an application of Baire's Category Theorem and Theorem 3.48 shows that if A contains a Cantor set, then $\mathcal{K}_\omega(A)$ is also Π_1^1-complete.

Definition 3.49. Let $\mathcal{C} \subseteq \mathcal{F}(X)$ be an arbitrary collection. Then :

(i) $\mathcal{C}^{\text{ext}} = \{A \subseteq X : \ A \text{ is covered by countably many elements in } \mathcal{C}\}$

(ii) If $\mathcal{C}$ is hereditary, then :

$$\begin{aligned}\mathcal{C}^{\text{loc}} &= \{C \in \mathcal{F}(X) : \exists \mathcal{V} \subseteq X \text{ open} : \mathcal{V} \cap C \neq \emptyset \text{ and } \overline{\mathcal{V} \cap C} \in \mathcal{C}\}\\ &\{C \in \mathcal{F}(X) : \exists \mathcal{V} \subseteq X \text{ open} : \mathcal{V} \cap C \neq \emptyset \text{ and } \mathcal{V} \cap C \in \mathcal{C}^{\text{ext}}\}\end{aligned}$$

(iii) $\mathcal{C}^{\text{perf}} = \mathcal{F} \setminus \mathcal{C}^{\text{loc}}$ and $\mathcal{C}_\sigma = \mathcal{F}(X) \cap \mathcal{C}^{\text{ext}}$

(iv) If $\mathcal{C}$ is a σ-ideal, we define $\mathcal{C}^{\text{int}} = \{A \subseteq X : \mathcal{F}(A) \subseteq \mathcal{C}\}$

We can now formulate the useful definition of (pre)basis :

Definition 3.50. Let $I \subseteq \mathcal{F}(X)$ be a σ-ideal. A subset $B \subseteq I$ is said to be a basis if B is hereditary and $B_\sigma = I$. If B is not necessarily hereditary, we say that B is a pre-basis. A basis B for a σ-ideal I on $\mathcal{F}(X)$ is said to be non-trivial in $E \subseteq X$ if there is some $F \in \mathcal{F}(E)$ such that $F \in I \setminus B$.

Note that it follows immediately from the definition that if $A \subseteq X$ is not Borel, then $\mathcal{F}(A)$ does not have a Borel basis. On the other hand, $\mathcal{F}_\omega(X)$ has $\{\emptyset\} \cup \{\{x\} : x \in X\}$ as a Borel basis.

Definition 3.51. Let I be a σ-ideal on $\mathcal{F}(X)$. Then :

(i) $\mathcal{I}$ is said to have the covering property if $\mathcal{I}^{\text{int}} \cap \Sigma_1^1 \subseteq \mathcal{I}^{\text{ext}}$

(ii) $\mathcal{I}$ is said to be calibrated if whenever G is a G_δ in $\mathcal{I}^{\text{int}}$, $\{F_n\} \subseteq \mathcal{I}$ and $F := \bigcup_n F_n \cup G \in \mathcal{F}(X)$ then $F \in \mathcal{I}$

One can motivate the definition of calibrated σ-ideal as asserting a certain inner approximation property by Π_2^0-sets :

Proposition 3.52. Let $\mathcal{I}$ be a σ-ideal of $\mathcal{K}(X)$. Then, $\mathcal{I}$ is calibrated if and only if $\mathcal{I}^{\text{int}} \cap \Pi_2^0$ is a σ-ideal of Π_2^0 sets.

Proof. The reader can find a proof in [48] (Proposition 3.2.1). ■

Note that while I_M is calibrated, if X is perfect then I_{meag} is not.

For the sake of completeness we end this section with important results concerning the covering property. These results will be of crucial importance when applied to the context of sets of uniqueness.

Theorem 3.53. *Let $\mathcal{I} \subseteq \mathcal{F}(X)$ be an arbitrary collection. If $A \subseteq X$ is Σ_1^1 and $A \notin \mathcal{I}^{ext}$, then A contains a G_δ set which is not in I^{ext}.*

Proof. The reader can find a proof in [76] (Theorem 1). ■

Theorem 3.54. *Let $\mathcal{I}$ and $\mathcal{J}$ be σ-ideals in $\mathcal{F}(X)$. If $\mathcal{I}$ is calibrated with a basis B which is non-trivial for every closed set $E \notin J$, then every G_δ set in $\mathcal{I}^{int}$ is in $\mathcal{J}^{ext}$.*

Proof. The reader can find a proof in [57] (Theorem 3.8). ■

As an important corollary of Theorems 3.53 and 3.54 we immediately obtain the following result (cf. [57] and [17]) :

Theorem 3.55. *Let $\mathcal{I}$ be a σ-ideal in $\mathcal{F}(X)$ such that :*

(i) $\mathcal{I}$ is calibrated

(ii) I has a basis which is non-trivial in each closed set $E \notin \mathcal{I}$

Then, $\mathcal{I}$ has the covering property.

4 Groups and algebras

The aim of this section is to define several Banach algebras which are associated with a locally compact group and state some of their properties. *En route*, we briefly visit classic results about groups such as the Pontryagin duality, the Riesz-Markov-Kakutani representation theorem and the existence of the Haar measure for locally compact groups.

4.1 Groups and measures

4.1.1 Groups

Recall that a topological group G is a set endowed with a group structure and a topology which makes the group product and group inverse continuous operations on $G \times G$ and G respectively. We assume, by default, that topological groups are Hausdorff. Indeed, a T_0 topological group is automatically regular and in particular, Hausdorff (cf. [19], Theorem 3.1).

For $f : G \to \mathbb{C}$ and $g \in G$ we define :

$$L_g(f)(h) := f(g^{-1}h), \text{ for all } h \in G$$
$$\tilde{f}(h) := \overline{f(h^{-1})} \text{ and } \check{f}(h) := f(h^{-1}), \text{ for all } h \in G$$

For a locally compact abelian group G let $\hat{G}$ be the set of characters, i.e. the set of continuous homomorphisms $\chi : G \to \mathbb{T}$. Then, $\hat{G}$ endowed with pointwise multiplication and with the topology of uniform convergence on compact sets, is again a locally compact abelian group - usually called the (Pontryagin) dual group of G. In fact, we have the following result usually referred to as Pontryagin duality :

Theorem 4.1. *Let G be a locally compact abelian group. Then, the assignment $x \mapsto \alpha_x$ where $\alpha_x(\chi) := \chi(x)$ for all $\chi \in \hat{G}$, is an isomorphism of topological groups between G and the dual group of $\hat{G}$.*

Proof. The reader can find a proof in [61] (Theorem 23). ■

Remark 4.2. Note that Theorem 4.1 establishes indeed a duality. In other words, the category of locally compact abelian groups is equivalent to its opposite category via the functor which sends G to $\hat{G}$ and $f : G_1 \to G_2$ to $f^\sharp : G_2^* \to G_1^*$ such that $f^\sharp(\varphi) := \varphi \circ f$.

A bounded linear operator T acting on a Hilbert space $\mathcal{H}$ is said to be unitary if $T^*T = TT^* = 1$ or, equivalently, if T is surjective and preserves inner products. The set of unitary operators acting on $\mathcal{H}$ is denoted by $\mathcal{U}(\mathcal{H})$.

Let G be a locally compact group. A continuous unitary representation of G is a pair $(\pi, \mathcal{H}(\pi))$ with $\mathcal{H}(\pi)$ a Hilbert space and a homomorphism $\pi : G \to \mathcal{U}(\mathcal{H}(\pi))$ such that :

$$\forall \xi, \eta \in \mathcal{H}(\pi) : \varphi_{\xi,\eta} : x \mapsto \langle \pi(x)\xi, \eta \rangle \quad \text{is continuous}$$

In other words, $\pi : G \to \mathcal{U}(\mathcal{H}(\pi))$ is continuous with respect to the WOT. Since the WOT and SOT coincide on $\mathcal{U}(\mathcal{H}(\pi))$, this is equivalent to require that :

$$\forall \xi \in \mathcal{H}(\pi) : x \mapsto \pi(x)\xi \quad \text{is continuous}$$

We say that two representations π, σ of G are unitarily equivalent if there is an unitary map $U : \mathcal{H}(\pi) \to \mathcal{H}(\sigma)$ such that $U\pi(x) = \sigma(x)U$ for all $x \in G$.

Definition 4.3. The left regular representation $\lambda_G : G \to \mathcal{U}(L^2(G))$ of G is defined as follows :

$$x \mapsto (\lambda_G(x)(f)(y) := f(x^{-1}y)),\ f \in L^2(G), y \in G$$

4.1.2 Measures

Now we *forget* about the group structure of G and will instead consider an arbitrary locally compact Hausdorff space X. Recall that a positive Borel measure μ on X is said to be regular if for all $E \in \mathcal{B}(X)$ the following holds :

$$\begin{aligned}\mu(E) &= \inf\{\mu(\mathcal{U}) : E \subseteq \mathcal{U},\ \mathcal{U} \text{ open}\} \\ &= \sup\{\mu(K) : K \subseteq E,\ K \text{ compact}\}\end{aligned}$$

A complex Borel measure μ is said to be regular if $|\mu|$ is regular.
A Radon measure μ on X is a positive Borel measure on X such that :

(i) μ is inner regular, i.e. for all open sets $\mathcal{U} \subseteq X$ we have that $\mu(\mathcal{U}) = \sup\{\mu(K) : K \subseteq \mathcal{U},\ K \text{ compact}\}$

(ii) μ is outer regular, i.e. for all Borel sets $E \subseteq X$ we have that $\mu(E) = \inf\{\mu(\mathcal{V}) : E \subseteq \mathcal{V},\ \mathcal{V} \text{ open}\}$

(iii) $\mu(K) < \infty$, for every compact $K \subseteq X$

As usual, we consider the space of all regular complex Borel measures on X - henceforth denoted by $M(X)$ - endowed with the total variation norm, i.e. $||\mu|| = |\mu|(X)$ for $\mu \in M(X)$.

Recall that we have a relation of duality between $M(X)$ and $C_0(X)$ which is captured by Riesz-Markov-Kakutani theorem. It is worth to note that this result of cornerstone importance allows us to look at measures through a more functional analytic lens.

Theorem 4.4. *Let X be a locally compact Hausdorff space. Then, every bounded linear functional $\Psi \in C_0(X)^*$ is of the form*

$$\Psi(f) = \Psi_\mu(f) := \int_X f(x)d\mu(x), \text{ for all } f \in C_0(X)$$

for an unique regular complex Borel measure μ. Moreover, the assignment $\mu \mapsto \Psi_\mu$ is a surjective isometry and in particular, a bijection between positive measures and positive linear functionals on $C_c(X)$.

Proof. The reader can find a proof in [70] (Theorems 2.14 and 6.19). ∎

Remark 4.5. If X is compact (and Hausdorff), then Theorem 4.4 admits a rather elegant proof (cf. [34]). Indeed, for compact Hausdorff spaces we can consider the natural transformation (ι_X) where $\iota_X : M(X) \to C(X)^*$ is such that $\iota_X : \mu \mapsto \Psi_\mu$ and the following diagram commutes :

$$\begin{tikzcd} M(X) \arrow[r, "\alpha^*"] \arrow[d, "\iota_X"'] & M(Y) \arrow[d, "\iota_Y"] \\ C(X)^* \arrow[r, "\alpha^{\sharp\sharp}"] & C(Y)^* \end{tikzcd}$$

where if $\alpha : X \to Y$ is a map, then $\alpha^*(\mu) := \mu \circ \alpha^{-1}$, $\alpha^\sharp(g) := g \circ \alpha$ and $\alpha^{\sharp\sharp}(\Psi) := \Psi \circ \alpha^\sharp$.

Note that every compact space X is a continuous image of an extremally disconnected space : indeed, the Stone-Čech compactification of a discrete is an extremally disconnected space and one can consider the extension of the map $1 : X_d \to X$, where X_d is simply X equipped with the discrete topology. For compact extremally disconnected spaces Z, it is easy to verify that the assignment $M(Z) \ni \mu \mapsto \Psi_\mu \in C(Z)^*$ is surjective (cf. Lemma 1 in [34]) and hence, choosing $\alpha := 1 : \beta(X_d) \to X$ in the above diagram we conclude that ι_X is surjective as required.

4.1.3 $L^p(G)$ and $M(G)$

Let G be a locally compact group. A Borel measure μ on G is said to be left-invariant if $\mu(xE) = \mu(E)$ for every $x \in G$ and $E \subseteq G$, Borel set.

Theorem 4.6. *Let G be a locally compact group. Then, there exists a non-zero left-invariant Radon measure μ on G. If μ and ν are two such measures, then there is positive real constant c such that $\mu = c\nu$.*

Proof. We merely sketch the idea of the argument given in [19] for the existence of such measure. For the uniqueness (up to a constant) of such measure, the reader can find a proof in [19] (Theorem 7.32).
Let $C_c^+(G)$ denote the set of non-zero functions $f : G \to \mathbb{R}$ which take non-negative values and fix some $\omega \in C_c^+(G)$. For any $f, \varphi \in C_c^+(G)$, define :

$$(f : \varphi) := \inf\{\sum_{i=1}^{n} c_i : f \leq \sum_{i=1}^{n} c_i L_{g_i}(\varphi), \text{ some } c_i \geq 0, g_i \in G\}$$

Furthermore, for any $f, \varphi \in C_c^+(G)$ we also define :

$$\mu_\varphi(f) := \frac{(f : \varphi)}{\omega : \varphi}$$

Then, by Lemma 7.22 in [19] one has that μ_φ is subadditive and positively homogeneous, monotone non-decreasing, $\mu_\varphi(f) > 0$, $\mu_\varphi(L_g(f)) = \mu_\varphi(f)$ for all $g \in G$ and $\frac{1}{(\omega:f)} \leq \mu_\varphi(f) \leq (f : \omega)$.
Moreover, by Lemma 7.23 in [19], for every $\epsilon > 0$ and $f_1, f_2 \in C_c^+(G)$ there exists an open neighborhood $\mathcal{V}$ of the identity such that whenever $\varphi \in C_c^+(G)$ with $\text{supp}(\varphi) \subseteq \mathcal{V}$, then $\mu_\varphi(f_1) + \mu_\varphi(f_2) \leq \mu_\varphi(f_1 + f_2) + \epsilon$.
We now consider intervals of the form $I(f) := [\frac{1}{(\omega:f)}, (f : \omega)]$ so that by Tychonoff theorem, the following is compact :

$$I = \prod_{f \in C_c^+(G)} I(f)$$

For any open neighborhood $\mathcal{V}$ of the identity, let $\mathcal{F}_\mathcal{V}$ be the set of all points $(\mu_\varphi(f))_{f \in C_c^+(G)} \in I$ such that $\text{supp}(\varphi) \subseteq \mathcal{V}$. By compactness, there is a point :

$$(\mu(f))_{f \in C_c^+(G)} \in \bigcap_{\mathcal{V}} \overline{\mathcal{F}_\mathcal{V}}$$

It follows (from Lemma 7.22 and Lemma 7.23 in [19]) that $\mu(L_g(f)) = \mu(f)$ for all $g \in G$, $\mu(f_1 + f_2) = \mu(f_1) + \mu(f_2)$, $\mu(\lambda f) = \lambda\mu(f)$ and $\mu(f) > 0$, for $\lambda > 0$ and $f_1, f_2 \in C_c^+(G)$. We set $\mu(0) := 0$ and appeal to Theorem 4.4 to conclude the argument. ∎

Definition 4.7. The non-zero left-invariant Radon measure μ on G given by Theorem 4.6 is called the Haar measure on G.

Remark 4.8. The existence of Haar measures for compact groups (cf. [19]) and for locally compact abelian groups (cf. [35]) can be established through the existence of certain fixed points. More generally, a simple proof of the existence of Haar measures for amenable groups is given in [36].
For the case of compact groups, we can use a version of the Ryll-Nardzewski fixed point theorem. A statement of Ryll-Nardzewski theorem, as given in [19] (Theorem 5.23) is as follows : Suppose that $\mathcal{S}$ is a semigroup of continuous linear operators on a Hausdorff locally convex topological linear space E and that $K \subseteq E$ is a non-empty compact convex subset. Assume that $S(K) \subseteq K$ for every $S \in \mathcal{S}$ and that whenever $k_1 \neq k_2$, then $0 \notin \{S(k_1 - k_2) : S \in \mathcal{S}\}$. Then, there exists some $k \in K$ such that $S(k) = k$ for every $S \in \mathcal{S}$. One can then verify that the conditions for the previous statement hold in the case $E = M(G)$ and $K = P(G)$ - the set of probability measures on G - with respect to the weak*-topology and $\mathcal{S} = \{T_g^* : g \in G\}$, where T_g^* is the adjoint of $T_g : C(G) \to C(G)$ defined by $T_g(f)(h) = f(gh)$ (Corollary 5.25 in [19]).
For the case of locally compact abelian groups, we can use the following version of the Markov-Kakutani fixed point theorem (as given in [35]) : Let K be a non-empty compact convex subset of a Hausdorff topological space and let $\mathcal{F}$ be a commuting family of continuous affine mappings of K into itself. Then, there is a point $k \in K$ such that $T(k) = k$ for all $T \in \mathcal{F}$. In this case, K is specified to be a certain subset of positive linear functionals on $C_c(G)$ and similarly as in the compact case, we let $\mathcal{F} = \{T_g : C_c(G)^* \to C_c(G)^* : g \in G\}$ such that $T_g(\Psi)(f) = \Psi(f_g)$, where $f_g(x) := f(a + x)$.

Definition 4.9. For $1 \leq p < \infty$, the Lebesgue space $L^p(G)$ is defined to be space of (equivalence classes) Borel measurable functions f on G such that $||f||_p := (\int_G |f(x)|^p d\mu)^{\frac{1}{p}}$ is finite, where μ is the Haar measure. If $p = \infty$, then $L^\infty(G)$ is the space of μ-essentially bounded functions endowed with the supremum norm.

Recall that Lebesgue spaces $L^p(G)$ are Banach spaces and that $C_c(G)$ is dense in each $L^p(G)$ with $1 \leq p < \infty$. In case $p = 2$, then $L^2(G)$ is furthermore a Hilbert space with inner product given by :

$$\langle f, g \rangle = \int_G f(x)\overline{g(x)} d\mu(x)$$

In case $p = 1$, the convolution $f * g$ of functions defined as :

$$(f * g)(x) := \int_G f(y) g(y^{-1}x) dx$$

makes $L^1(G)$ a Banach $*$-algebra under the involution :

$$f^*(x) := \Delta_G(x^{-1})\overline{f(x^{-1})}$$

where $y \mapsto \Delta_G(y)$ is the unimodular function, i.e. for every Borel set $E \subset G$ and $y \in G$, $\mu(Ey) = \Delta_G(y)\mu(E)$, where μ is the Haar measure on G. Note that the existence of such function follows from the uniqueness (up to a constant) of the Haar measure.

If $\mu, \nu \in M(G)$ one defines the convolution $\mu * \nu \in M(G)$ as follows :

$$\int_G \varphi(x) d(\mu * \nu)(x) = \int_G \int_G \varphi(xy) d\mu(x) d\nu(y), \text{ for all } \varphi \in C_c(G)$$

Furthermore, one defines an involution on $M(G)$ in the following way :

$$\mu \in M(G), \text{ then } \mu^*(E) := \overline{\mu(E^{-1})}$$

Definition 4.10. The set $M(G)$ of regular complex Borel measures equipped with the total variation norm, convolution of measures and involution μ^* becomes a Banach $*$-algebra usually called the measure algebra of G.

Note that $L^1(G)$ embeds as a closed 2-sided ideal in $M(G)$:

$$f \mapsto \mu_f, \text{ with } \mu_f(E) := \int_E f(x) d\mu(x)$$

We finish this section with a few remarks on $L^1(G)$ whenever G is a locally compact *abelian* group. Henceforth, unless otherwise stated, in the context of integration with locally compact groups, dx will refer to the Haar measure.

Definition 4.11. Let G be a locally compact abelian group and $f \in L^1(G)$. The Fourier transform $\hat{f}$ of f is the map $\hat{f} : \hat{G} \to \mathbb{C}$ given by :

$$\hat{f}(\chi) = \int_G f(x) \chi(x) dx$$

The Fourier transform extends to $M(G)$ as follows :

$$\hat{\mu} : \hat{G} \to \mathbb{C}, \text{ given by } \hat{\mu}(\chi) = \int_G \chi(x) d\mu(x)$$

It is worth to point out that $f \mapsto \hat{f}$ is an injective $*$-homomorphism of $L^1(G)$ into $C_0(\hat{G})$ and that the image of $L^1(G)$ under the Fourier transform is dense in $C_0(\hat{G})$ (cf. [41], Theorem 1.5.14).

Remark 4.12. The Fourier transform $L^1(G) \to C_0(\hat{G})$ coincides with the Gelfand transform $L^1(G) \to C_0(\sigma(L^1(G)))$. Indeed, on one hand there is a bijection between $\hat{G}$ and $\sigma(L^1(G))$ given by : (cf. [40], Theorem 2.7.2)

$$\hat{G} \ni \alpha \mapsto (\varphi_\alpha : f \mapsto \int_G f(x) \overline{\alpha(x)} dx), \text{ for } f \in L^1(G)$$

On the other hand, the Gelfand topology and the compact open topology coincide (cf. [40], Theorem 2.7.5).

4.2 Group algebras

4.2.1 $C^*(G)$, $C_r^*(G)$ and $VN(G)$

Let $\mathcal{A}$ be a normed $*$-algebra. A $*$-representation of $\mathcal{A}$ is a pair $(\pi, \mathcal{H}(\pi))$, with $\mathcal{H}(\pi)$ a Hilbert space and $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H}(\pi))$ a homomorphism such that $\pi(a^*) = \pi(a)^*$ for all $a \in \mathcal{A}$. The representation is said to be non-degenerate if the subspace $\{\xi \in \mathcal{H}(\pi) : \pi(a)\xi = 0, \text{ for all } a \in \mathcal{A}\}$ is trivial.

Let G be a locally compact group. Then, every unitary representation π of G determines a non-degenerate $*$-representation $\tilde{\pi}$ of $L^1(G)$ on $\mathcal{H}(\pi)$ as :

$$\langle \tilde{\pi}(f)\xi, \eta \rangle = \int_G f(x)\langle \pi(x)\xi, \eta \rangle dx, \text{ for all } f \in L^1(G), \xi, \eta \in \mathcal{H}(\pi)$$

In particular, note that if $\pi = \lambda_G$, then $\langle \tilde{\lambda}_G(f)\xi, \eta \rangle = \langle f * \xi, \eta \rangle$.[8]

In general, for a locally compact group G and an unitary representation π on $\mathcal{H}(\pi)$, the assignment $x \mapsto \varphi_{\chi,\eta}(x) = \langle \pi(x)\xi, \eta \rangle$ is called a coefficient function.

A representation π is said to be irreducible if $\{0\}$ and $\mathcal{H}(\pi)$ are the only closed π-invariant subspaces of $\mathcal{H}(\pi)$ (i.e. the only closed subspaces E such that for all $x \in G$, $\pi(x)\xi \in E$ whenever $\xi \in E$). Let $\mathcal{R}$ be the set of unitary (equivalence classes) of irreducible representations of G. Then, for $f \in L^1(G)$, it turns out that the following is a C*-norm on $L^1(G)$:

$$||f||_* := \sup\{||\tilde{\pi}(f)|| : \pi \in \mathcal{R}\}$$

Definition 4.13. Let G be a locally compact group. The group C*-algebra of G is defined to be the completion of $L^1(G)$ under the $||.||_*$ norm and denoted by $C^*(G)$.

Another norm on the Banach $*$-algebra $L^1(G)$ which satisfies the C*-condition is given by :

$$||f||_r := ||\lambda_G(f)||$$

Definition 4.14. Let G be a locally compact group. The reduced C*-algebra of G is defined to be the completion of $L^1(G)$ under the $||.||_r$ norm and denoted by $C_r^*(G)$.

Remark 4.15. Note that if G is a locally compact abelian group, then the Fourier transform extends to an isomorphism between $C_r^*(G)$ and $C_0(\hat{G})$.

The way the norm $||.||_*$ was defined on $L^1(G)$ is such that every representation π of $L^1(G)$ extends uniquely to a representation of $C^*(G)$. Moreover, note that since λ_G extends to a surjective $*$-homomorphism of $C^*(G)$ onto $C_r^*(G)$, then $C_r^*(G)$ is a quotient of $C^*(G)$. When G is amenable, $C_r^*(G)$ and $C^*(G)$ coincide.

[8]We often write, by slight abuse of notation, λ_G (or just λ) instead of $\tilde{\lambda}_G$. Whenever the context is clear, we also often just write π instead of $\tilde{\pi}$.

Definition 4.16. Let $E \subseteq L^\infty(G)$ be a linear subspace containing the constant functions. A mean on E is an element $m \in E^*$ satisfying $||m|| = 1$. If E is left translation invariant, then a mean on E is called left invariant if for every $x \in G$ and $f \in E$, then $m(L_x(f)) = m(f)$. A locally compact group is said to be amenable if there is a left invariant mean on $L^\infty(G)$. [9]

In the theorem that follows, we list some well-known preservation properties on amenable groups :

Theorem 4.17. *The following statements hold true :*

(i) Let G be a locally compact group and $N \subseteq G$ be a closed normal subgroup. If N and G/N are amenable then so is G.

(ii) Let G be a locally compact amenable group. Then, every closed subgroup of G is amenable.

(iii) Let G and H be locally compact groups and $\varphi : G \to H$ be a continuous homomorphism with dense range. Then, if G is amenable so is H. In particular, quotients of amenable groups are amenable.

Example 4.18. We provide a short list of classes of (non)amenable groups :

(a) Every finite group is amenable : $f \mapsto |G|^{-1} \sum_{g \in G} f(g)$ defines an invarian mean on $L^\infty(G)$.

(b) Every compact group is amenable : Let μ be the normalized Haar measure on G. Then, $\langle m, f \rangle := \int_G f(x) d\mu(x)$, for $f \in L^\infty(G)$ defines an invariant mean.

(c) Every abelian group is amenable : We appeal to Day's fixed point theorem, as stated in [41] (Theorem 1.8.7) : G is amenable if and only if whenever G acts affinely on a non-empty convex compact set K of a separated locally convex vector space E and the action $G \times E \ni (g, x) \mapsto g.x \in E$ is separately continuous, then there is some $x \in E$ such that $g.x = x$ for all $g \in G$. It follows from Markov-Kakutani fixed point theorem (cf. Remark 4.8) that if G is abelian then the assumptions of Day's fixed point theorem hold.

(d) $\mathbb{F}_2$ is not amenable : Let a and b be the free generators of $\mathbb{F}_2$ and let the set of reduced words starting with a, a^{-1}, b and b^{-1} be respectively denoted by $S_a, S_{a^{-1}}, S_b$ and $S_{b^{-1}}$. Let $C = \{1, b, b^2, ...\}$ and note that :

$$\begin{aligned}\mathbb{F}_2 =& S_a \sqcup S_{a^{-1}} \sqcup (S_b \setminus C) \sqcup (S_{b^{-1}} \cup C) \\ =& S_a \sqcup a S_{a^{-1}} \\ =& b^{-1}(S_b \setminus C) \sqcup (S_{b^{-1}} \cup C)\end{aligned}$$

[9] *'Amenable groups admit approximately* $10^{10^{10}}$ *different characterizations'*, in [15].

Assume, towards a contradiction that $\mathbb{F}_2$ is amenable and thus, by invariance of m we have that :

$$\begin{aligned} 1 = m(1) = & m(\chi_{S_a}) + m(\chi_{S_{a^{-1}}}) + m(\chi_{S_b \setminus C}) + m(\chi_{S_{b^{-1}} \cup C}) \\ = & m(\chi_{S_a}) + m(a\chi_{S_{a^{-1}}}) + m(b^{-1}\chi_{S_b \setminus C}) + m(\chi_{S_{b^1} \cup C}) \\ = & 2m(1) = 2 \end{aligned}$$

which is a contradiction.

e) It follows from (d) and from Theorem 4.17 that every group which contains $\mathbb{F}_2$ as a closed subgroup is not amenable. Examples include $SL(n, \mathbb{C})$ and $GL(n, \mathbb{C})$.

Let S and T be two sets of representations of a C*-algebra. We say that S is weakly contained in T, denoted by $S \prec T$, if $\bigcap_{\tau T} \ker(\tau) \subseteq \bigcap_{\sigma \in S} \ker(\sigma)$. Let G be a locally compact group, $\mathcal{R}$ be the set of unitary equivalence classes of irreducible representations of G and $\mathcal{R}_r$ be the set of all $\pi \in \mathcal{R}$ such that $\pi \prec \lambda_G$. Note that if $1 \prec \lambda_G$ then $\pi \prec \lambda_G$ for every $\pi \in \mathcal{R}$ and we have that $\mathcal{R} = \mathcal{R}_r$. A well-known characterization of amenability is as follows (cf. [41], Theorem 1.8.18) : G is amenable if and only if the trivial representation of G is weakly contained in λ_G. In particular, this characterization entails that :

Theorem 4.19. *Let G be an amenable group. Then, $C^*(G)$ and $C_r^*(G)$ coincide.*

Remark 4.20. Let G be an étale locally compact groupoid. By Corollary 5.6.17 in [15], if G is amenable then $C^*(G) = C_r^*(G)$, where $C^*(G)$ and $C_r^*(G)$ are respectively the full groupoid C*-algebra and the reduced groupoid C*-algebra.

Definition 4.21. Let G be a locally compact group. The von Neumann group algebra of G, denoted by $VN(G)$, is defined to be $VN(G) = \overline{C_r^*(G)}^{w^*}$.

Remark 4.22. If G is a locally compact abelian group, $VN(G) = L^\infty(\hat{G})$. This can be seen as a consequence of what is developed in the next subsection, as in this setting the Fourier algebra $A(G)$ of G is isomorphic to $L^1(\hat{G})$ (cf. Remark 4.29) and is the predual of $VN(G)$ (cf. Theorem 4.27).

4.2.2 $B(G)$ and $A(G)$

Throughout this section, G is a locally compact group unless otherwise stated. The aim of this section is to introduce the Fourier-Stieltjes algebra of G - $B(G)$ - and the Fourier algebra of G - $A(G)$. We will mainly follow [41] which in turn follows [24] closely, where these algebras were introduced.

Let $B(G)$ be the collection of all coefficient functions $G \ni x \mapsto \langle \pi(x)\xi, \eta \rangle \in \mathbb{C}$, where π is a continuous unitary representation of G and $\xi, \eta \in \mathcal{H}(\pi)$.

By Proposition 2.1 and 1.19 in [24], $B(G)$ is identified with the Banach dual of $C^*(G)$ via the pairing :

$$\langle f, u\rangle = \int_G f(x)u(x)dx, \text{ for } f \in L^1(G), u \in B(G)$$

Consequently, if $u(.) = \langle \pi(.)\xi, \eta\rangle$ then $\langle g, u\rangle = \langle \pi(g)\xi, \eta\rangle$, for all $g \in C^*(G)$. The norm on $B(G)$ is given by :

$$||u|| = \sup\{|\int_G f(x)u(x)dx| : \ f \in L^1(G), \ ||f||_* \leq 1\}$$

It turns out that $B(G)$ equipped with this norm and pointwise multiplication, is an unital commutative Banach algebra (cf. [41], Theorem 2.1.11).

Definition 4.23. $B(G)$ is called the Fourier-Stieltjes algebra of G.

Remark 4.24. If G is a locally compact abelian group, then $B(G)$ is isometrically isomorphic to $M(\hat{G})$. Indeed, by Bochner's theorem (cf. [41], Theorem 1.5.19) and Lemma 2.1.4 in [41], for any $u \in B(\hat{G})$ there is an element $\mu \in M(G)$ such that $u(\chi) = \hat{\mu}(\chi)$ for every $\chi \in \hat{G}$. Furthermore, one can use the Fourier inversion formula to prove that $||u|| = ||\mu||$ (cf. [41], Remark 2.1.15). It follows that the Fourier transform provides an isometric isomorphism between $M(G)$ and $B(\hat{G})$ and the initial statement follows from Pontryagin duality (Theorem 4.1).

Consider the closure of the linear span of $B(G) \cap C_c(G)$. This ideal of $B(G)$ admits various characterizations (cf. [41], Proposition 2.3.3) and will be denoted by $A(G)$.

Definition 4.25. $A(G)$ as a subalgebra of $B(G)$ is called the Fourier algebra of G.

Theorem 4.26. *Let G be a locally compact group. For each $x \in G$ let :*

$$\varphi_x : A(G) \to \mathbb{C} \textit{ be such that } \varphi_x(u) = u(x)$$

Then, the assignment $x \mapsto \varphi_x$ is a homeomorphism between G and $\sigma(A(G))$. Morever, $A(G)$ is a regular algebra of functions on G.

Proof. The reader can find a proof in [41] (Theorem 2.3.8). ∎

A fundamental property of $A(G)$ is that it is the pre-dual of $VN(G)$. In order to state this rigorously, we recall that λ_G (or sometimes - when no notational confusion arise - simply λ) also denotes the representation of $M(G)$ in $\mathcal{L}(L^2(G))$ prescribed by :

$$\mu \mapsto \mu(g) := \mu * g, \text{ i.e. } \lambda(\mu)(g)(t) := \int_G g(s^{-1}t)d\mu(s)$$

Theorem 4.27. *Let G be a locally compact group. For any $\varphi \in A(G)^*$ there is an unique operator $T_\varphi \in VN(G)$ such that the assignment $\varphi \mapsto T_\varphi$ is a surjective linear isometry and a homeomorphism with respect to the w^*-topology on $A(G)^*$ and the ultraweak topology on $VN(G)$. Moreover, if $\mu \in M(G)$ and $\varphi_\mu \in A(G)^*$ is such that $\langle \varphi_\mu, u \rangle = \int_G u(x) d\mu(x)$ for all $u \in A(G)$, then $T_{\varphi_\mu} = \lambda_G(\mu)$.*

Proof. The reader can find a proof in [41] (Theorem 2.3.9). ■

Another, quite concrete, characterization of $A(G)$ as a set is as follows :

Theorem 4.28. *$A(G)$ is precisely the set of all functions of the form $f * \tilde{g}$, with $f, g \in L^2(G)$.*

Proof. The reader can find a proof in [41] (Theorem 2.4.3). ■

It follows from Theorem 4.28 that $A(G)$ coincides with the coefficient functions of λ_G. Indeed, note that :

$$(f * \tilde{g})(x) = \int_G f(xy)\overline{g(y)} dy = \langle \lambda_G(x^{-1}) f, g \rangle$$

Remark 4.29. If G is a locally compact abelian group, then $A(G)$ can be identified with $L^1(\hat{G})$ via the Fourier transform. Indeed, let $u = \xi * \check{\eta} \in A(\hat{G})$ with $\xi, \eta \in L^2(\hat{G})$. By Plancherel theorem (cf. [41], Theorem 1.5.15) there are $f, g \in L^2(G)$ such that $\hat{f} = \xi$ and $\hat{g} = \eta$. Thus, $\widehat{f\check{g}} = u$ and we conclude that $A(\hat{G}) \subseteq \widehat{L^1(G)}$. On the other, since the image of the Fourier transform is dense one has that $\widehat{L^1(G)} \subseteq A(\hat{G})$. Finally, we note that (cf. [41], Remark 2.1.15) for $\mu \in M(G)$, then $||\hat{\mu}||_{B(G)} = ||\mu||$ and consequently, $||\hat{f}||_{A(G)} = ||f||_1$.

Let $u \in B(G)$ and $T \in VN(G)$. We define an operator $u.T \in VN(G)$ via the following relation :

$$\langle u.T, v \rangle = \langle T, uv \rangle, \text{ for all } v \in A(G)$$

Note that if $T = \lambda_G(\mu)$, for some $\mu \in M(G)$, then for any $u \in B(G)$ one has that $u.\lambda_G(\mu)$ coincides with $\lambda_G(u\mu)$ (cf. [41], Remark 2.5.2).

Proposition 4.30. Let $T \in VN(G)$ and $a \in G$. Then, the following conditions are equivalent :

(i) The operator $\lambda_G(a)$ is the w^*-limit in $VN(G)$ of operators of the form $v.T$, where $v \in A(G)$.

(ii) For every neighbourhood $\mathcal{V}$ of a in G, there exists some $v \in A(G)$ such that $\mathrm{supp}(v) \subseteq \mathcal{V}$ and $\langle T, v \rangle \neq 0$.

(iii) If $u \in A(G)$ is such that $u.T = 0$, then $u(a) = 0$.

Proof. The reader can find a proof in [41] (Proposition 2.5.3). ■

Definition 4.31. Let $T \in VN(G)$. Then, the set of elements in G which satisfy any of the equivalent conditions of Proposition 4.30 is called the support of T and denoted by $\mathrm{supp}(T)$.

Remark 4.32. If $\mu \in M(G)$, then $\mathrm{supp}(\lambda_G(\mu)) = \mathrm{supp}(\mu)$ (cf. [24], Remarque 4.7).

Lemma 4.33. Let $T \in VN(G)$. Then, $T \neq 0$ if and only if $\mathrm{supp}(T) \neq \emptyset$.

Proof. If $T = 0$, it is clear that $\mathrm{supp}(T) = \emptyset$. For the converse, the reader can find a proof in [41] (Lemma 2.5.5). ■

We finish the section with a powerful result relating $A(G)$ with $A(H)$, whenever H is a closed subgroup of G. This is sometimes referred to as Herz restriction theorem :

Theorem 4.34. *Let G be a locally compact group and H a closed subgroup of G. Then, for every $u \in A(H)$ there is some $v \in A(G)$ such that $v|_H = u$ and $||v||_{A(G)} = ||u||_{A(H)}$.*

Proof. The reader can find a proof in [41] (Theorem 2.6.4). ■

5 Sets of uniqueness

This section is devoted to the study of sets of uniqueness. We start with a somewhat crude overview of the topic, firstly focusing on the classical case (concerning the group $G = \mathbb{T}$) and thereafter considering the general case of locally compact groups G. For a comprehensive account of the matter in the classical setting, the reader is referred to [47]. The main goal of this section is to apply descriptive set-theoretic methods to the study of the set of closed sets of uniqueness of a locally compact group G and in this context, new results are presented. On the way, we present some properties of this family of closed sets of G and briefly explore operator-theoretic connections, mainly following [74].

5.1 Definitions

5.1.1 Classical case : $\mathbb{T}$

Consider a trigonometric series, i.e. a formal expression of the form :

$$\sum_{n \in \mathbb{Z}} c_n e^{inx}, \ c_n \in \mathbb{C}, \ x \in \mathbb{T}$$

A rather natural question to ask is whether or not it is the case that whenever two trigonometric series $\sum_n c_n e^{inx}$ and $\sum_n d_n e^{inx}$ converge everywhere to the same value, then $c_n = d_n$ for all $n \in \mathbb{Z}$, i.e. they are the same series. This motivates the following :

Definition 5.1. A set $E \subseteq \mathbb{T}$ is said to be a set of uniqueness if whenever a trigonometric series $\sum_n c_n e^{inx}$ converges to zero outside E, then all $c_n = 0$. A set which is not a set of uniqueness is said to be a set of multiplicity.

If instead one considers trigonometric series whose coefficients arise from Borel measures on $\mathbb{T}$, one has the following :

Definition 5.2. Let μ be a Borel measure on $\mathbb{T}$ and $\hat{\mu}(n)$ its Fourier-Stieltjes coefficients. A set $E \subseteq \mathbb{T}$ is said to be a set of extended uniqueness if for every trigonometric series of the form $\sum_n \hat{\mu}(n) e^{inx}$ which converges to zero outside E, then $\hat{\mu}(n) = 0$ for all $n \in \mathbb{Z}$. A set which is not a set of extended uniqueness is said to be a set of restricted multiplicity.

Remark 5.3. Note that every set of uniqueness is itself a set of extended uniqueness. However, the converse is false. This is due to Piatetski-Shapiro (cf. [66]) and the reader is referred to [47] (Corollary 16 - VII.3).

Example 5.4. We provide a short list of examples of sets of uniqueness and multiplicity :

(a) The empty set is a set of uniqueness (cf. [47], Theorem 7 - I.3). In fact, every set $E \subseteq \mathbb{T}$ which does not contain a perfect set is a set of uniqueness (cf. [47], Theorem 6 - I.5) - in particular, every countable set is a set of uniqueness. However, as sketched in (c), there are uncountable perfect sets which are sets of uniqueness.

(b) Every set of uniqueness has measure zero and thus, in particular, every subset of $\mathbb{T}$ with positive measure is a set of multiplicity. Indeed, if $E \subseteq \mathbb{T}$ has positive Lebesgue measure then there is a closed subset $F \subseteq E$ with $\lambda(F) > 0$. Consider the trigonometric series whose coefficients are prescribed by the Fourier-Stieltjes coefficients of the characteristic function χ_F. On one hand, $\hat{\chi_F}(0) = \lambda(F) > 0$. On the other hand, this series converges to zero outside F. However, there are sets of Lebesgue measure zero which are not sets of uniqueness. A proof of this result, originally due to Menshov, can be found in [47] (Theorem 5 - III.4). For an example of a set of restricted multiplicity (and thus, multiplicity) of measure zero, see Example 5.77.

(c) As gently hinted in (a) and (b), it is extremely difficult to provide a characterization of sets of uniqueness. This statement will be made more precise further in this section. Exceptionally, there are certain families of sets whose characterization in terms of sets of uniqueness is known. In order to exemplify this, start with an arbitrary set of real parameters η_i such that :

$$0 = \eta_0 < \eta_1 < ... < \eta_k < \eta_{k+1} = 1, \text{ with } 1 - \eta_k := \xi < \eta_{i+1} - \eta_i$$

Given a closed interval $[a, b]$, with $l = b - a$, we consider the following pairwise disjoint closed intervals :

$$[a + l\eta_i, a + l\eta_i + l\xi], \text{ for } 0 \leq i \leq k$$

One says that their union was obtained from $[a, b]$ by a dissection of type $(\xi, \eta_1, ..., \eta_k)$. In our case of interest, we start with $E_0 = [0, 2\pi]$ and successively obtain by dissection a decreasing chain of closed intervals $E_0 \supseteq ... \supseteq E_n \supseteq ...$ and finally consider their intersection :

$$E(\xi, \eta_1, ..., \eta_k) := \bigcap_n E_n$$

This process mimics the usual construction of the Cantor set which in this terminology appears as $E(\frac{1}{3}, \frac{2}{3})$.
The Salem-Zygmund theorem (cf. [47], Theorem 1 - III.4) reveals a rather surprising connection between number theory and sets of uniqueness - it states that $E(\xi, \eta_1, ..., \eta_k)$ is a set of uniqueness if and only if $\theta := \frac{1}{\xi}$ is a Pisot number and each $\eta_i \in \mathbb{Q}(\theta)$.

(d) The question of whether or not every set of uniqueness must be topologically negligible, i.e. meagre, was firstly raised in [4] and answered affirmatively for analytic sets by Debs-Saint Raymond in [17] (Theoreme 13) - see Theorem 5.76.

The modern theory of sets of uniqueness, driven by a functional analytic flavoured reformulation due to Piatetski-Shapiro's work, frames our focus within the family of *closed* sets of (extended) uniqueness - $\mathcal{U}(\mathbb{T})$ ($\mathcal{U}_0(\mathbb{T})$).

Recall that $A(\mathbb{T}) \approx \ell^1$ via the Fourier transform and that $VN(\mathbb{T}) \approx \ell^\infty$, with :

$$\langle f, S\rangle = \sum_n \hat{f}(-n)S(n), \text{ for } f = \sum_n c_n e^{inx} \in A(\mathbb{T}) \text{ and } S \in \ell^\infty$$

Since each $\mu \in M(\mathbb{T})$ yields an element $S = (\hat{\mu}(n)) \in \ell^\infty$, one usually refers to ℓ^∞ as the space of pseudomeasures - $PM(\mathbb{T})$. Furthermore, since each $f \in L^1(\mathbb{T})$ is associated with an element $(\hat{f}(n)) \in c_0$ it is usual to refer to c_0 as the space of pseudofunctions - $PF(\mathbb{T})$. For a closed subset E, the subset of pseudomeasures supported in E is defined to be the subset of pseudomeasures S such that for every open interval I disjoint from E and every infinitely differentiable $f \in C(\mathbb{T})$ which is supported by I, it follows that $\sum_n \hat{f}(n)S(-n) = 0$. The latter set is denoted by $PM(E)$. With this terminology, one can reformulate the definition of *closed* sets of (extended) uniqueness in the following crucially useful functional analytic turn : Let $E \subseteq \mathbb{T}$ be a closed subset, then

$$E \text{ is a set of uniqueness iff } PM(E) \cap PF(\mathbb{T}) = \{0\}$$

$$E \text{ is a set of extended uniqueness iff } M(E) \cap PF(\mathbb{T}) = \{0\}$$

The reader can find a proof of the equivalence of definitions in [47], respectively in Theorem 1 - II.4 and Proposition 6 - II.5.

More generally, for a closed subset E of a locally compact group G we define :

$$J(E) = \overline{\{u \in A(G) : \ u \text{ has compact support disjoint from } E\}}$$

It is known that (cf. [24]) :

$$J(E)^\perp = \{T \in VN(G) : \text{supp}(T) \subseteq E\}$$

Furthermore, it is the case that $PM(E) = J(E)^\perp$ (cf. [32]). Another general fact about abelian (or compact) groups G is that the class of measures $\mu \in M(G)$ which satisfy $\lambda(\mu) \in C_r^*(G)$ coincides with the class of Rajchman measures, i.e. measures whose Fourier-Stieltjes coefficients vanish at infinity (cf. [8]). Consequently, we conclude that if $E \subseteq \mathbb{T}$ is closed, then :

$$E \text{ is a set of uniqueness iff } J(E)^\perp \cap C_r^*(\mathbb{T}) = \{0\}$$

$$E \text{ is a set of extended uniqueness iff } \lambda(M(E)) \cap C_r^*(\mathbb{T}) = \{0\}$$

In turn, as it will be introduced in the next subsection, this motivates the definition of sets of uniqueness for general locally compact groups.

Example 5.5. A compact subset $E \subseteq \mathbb{T}$ is said to be a Helson set if every continuous function on E can be extended to an element of $A(\mathbb{T})$. More generally, if G is a locally compact group, then a closed subset $E \subseteq G$ is said to be a Helson set if the restriction $A(G) \to C_0(E)$ is surjective. For an element $\mu \in M(\mathbb{T})$, by slight abuse of notation we set $||\mu||_{PM} = ||\hat{\mu}||_{PM}$. It follows that if $E \subseteq \mathbb{T}$ is a Helson set then the norms arising from $M(\mathbb{T})$ and from $PM(\mathbb{T})$ are equivalent in $M(E) \cap PF(\mathbb{T})$ (cf. [47], Corollary 3 - VII.3). We include a sketch of a proof that Helson sets are sets of extended uniqueness (cf. [32], Theorem 4.5.2 for full details) : suppose, towards a contradiction, that E is a Helson set and there is a non-zero $\nu \in M(E) \cap PF(\mathbb{T})$. Since ν is continuous, the complement of its support is a countable union of disjoint open intervals. We pick some $x \in \text{supp}(\nu)$ which is not an endpoint of any of the intervals so that for every $\epsilon > 0$ the following holds :

$$|\nu|(x, x+\epsilon) \neq 0 \text{ and } |\nu|(x-\epsilon, x) \neq 0$$

Furthermore, we define the following function :

$$f(t) = \begin{cases} 1 & \text{if } t \in (x, x+1) \\ 0 & \text{if } t \in (x-1, x] \end{cases}$$

and extend it to a continuous function on $\mathbb{T} \setminus \{x\}$. The assignment $\mu \mapsto \int f d\mu$ is a bounded linear functional on $M(E) \cap PF(\mathbb{T})$ on which, since E is a Helson set, the norms from $M(\mathbb{T})$ and $PM(\mathbb{T})$ agree. Hence there is some element $g \in A(\mathbb{T}) = (PF(\mathbb{T}))^*$ such that :

$$\int f d\mu = \int g d\mu, \text{ for } \mu \in M(E) \cap PF(\mathbb{T})$$

However, it follows from our choice of x that g is not continuous at that point which is a contradiction.

5.1.2 General case : locally compact groups

Throughout this section, unless otherwise mentioned, G is a non-discrete locally compact group.

Definition 5.6. Let G be a locally compact group and $E \subseteq G$ be a closed subset. Then, E is said to be a set of uniqueness (U-set, for short) if $J(E)^{\perp} \cap C_r^*(G) = \{0\}$. Otherwise, E is said to be a set of multiplicity (M-set, for short). The set of closed sets of uniqueness of G is denoted by $\mathcal{U}(G)$ and $\mathcal{M}(G) := \mathcal{F}(G) \setminus \mathcal{U}(G)$.

Definition 5.7. Let G be a locally compact group and $E \subseteq G$ be a closed subset. Then, E is said to be a set of extended uniqueness (U_0-set, for short) if $\lambda(M(E)) \cap C_r^*(G) = \{0\}$, where λ is the extension of the left regular representation to $M(G)$. Otherwise, E is said to be a set of restricted multiplicity (M_0-set, for short). The set of closed sets of extended uniqueness is denoted by $\mathcal{U}_0(G)$ and we set $\mathcal{M}_0(G) := \mathcal{F}(G) \setminus \mathcal{U}_0(G)$.

Remark 5.8. Note that since $\lambda(M(E)) \subseteq J(E)^{\perp}$, it follows that $\mathcal{U}(G) \subseteq \mathcal{U}_0(G)$. Again, as in the case of $G = \mathbb{T}$, this inclusion is strict for general non-discrete locally compact abelian groups. Indeed, it is the case that every M-set contains a Helson set (and as such, a U_0-set) which is M-set (cf. [50], [43] for $G = \mathbb{T}$ and [71] for the general case). This fact will be presented in greater detail in subsection 5.3 as it plays an important role in establishing a certain descriptive set-theoretic description of $\mathcal{U}(G)$.

Remark 5.9. Note that by a duality argument, if G is amenable then $E \subseteq G$ is a U-set if and only if $\overline{J(E)}^{w^*} = B(G)$.

Example 5.10. Similarly as in the previous subsection, we present a short list of examples of U-sets (M-sets) and U_0-sets (M_0-sets) :

(a) Every compact subset of a non-discrete locally compact group which does not contain a non-empty perfect set, is a set of uniqueness ([13], Theorem 1). In particular, every countable compact set is a U-set. If furthermore G is second countable, then any countable closed set is a U-set ([74], Corollary 5.3).

(b) Any closed subset E of a second countable locally compact group G with positive Haar measure is a M_0-set. Indeed, let $K \subseteq E$ be a compact such that $\mu(K) > 0$ and consider the measure τ prescribed by $d\tau(x) := \chi_K(x)d\mu(x)$. Clearly, $\mathrm{supp}(\tau) \subseteq E$ and $0 \neq \lambda(\tau) \in C_r^*(G)$.

(c) Similarly to the case $G = \mathbb{T}$, Helson sets are U_0-sets in non-discrete locally compact abelian groups (see Example 5.5).

5.1.3 Sets of operator multiplicity

In this section we present a short digression on an operator-theoretic point of view towards sets of uniqueness. We will follow mainly [74]. The goal of this section is restricted to define sets of operator multiplicity and state the relation between M and M_0-sets and their operator-theoretic counterparts. This context is then revealed to be fruitful in the next section, where we prove some properties about sets of (restricted) multiplicity. In order to achieve our goal, we start the subsection by introducing the needed terminology.

Recall that a measure space (X, μ) is called a standard measure space if μ is a Radon measure with respect to some completely metrizable separable and locally compact topology on X. For standard measure spaces (X, μ) and (Y, ν), a subset $R \subseteq X \times Y$ is said to be a rectangle if it is of the form $R = A \times B$ for A and B measurable. We consider the product measure in $X \times Y$:

(i) $E \subseteq X \times Y$ is said to be *marginally null* if $E \subseteq (X_0 \times Y) \cup (X \times Y_0)$ for $\mu(X_0) = 0$ and $\nu(Y_0) = 0$.

(ii) $E, F \subseteq X \times Y$ are said to be *marginally equivalent* if $E \Delta F$ is marginally null. We denote this by $E \sim F$.

(iii) A subset $E \subseteq X \times Y$ is said to be ω-open if it is marginally equivalent to a countable union of rectangles. The complement of a ω-open set will be called ω-closed.

Given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we denote as usual the space of compact operators in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, by $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$. Henceforth, and throughout this subsection, we set $\mathcal{H}_1 = L^2(X, \mu)$ and $\mathcal{H}_2 = L^2(Y, \nu)$. The space $\mathcal{C}_1(\mathcal{H}_2, \mathcal{H}_1)$ of nuclear operators is identified with the Banach space dual of $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ via $\langle T, S \rangle = \operatorname{tr}(TS)$. Moreover, one can identify $\mathcal{C}_1(\mathcal{H}_2, \mathcal{H}_1)$ with the space $\Gamma(X, Y)$ of all (marginal equivalence classes of) fuctions $h : X \times Y \to \mathbb{C}$ which admit a representation :

$$h(x, y) = \sum_{i=1}^{\infty} f_i(x) g_i(y)$$

with $f_i \in \mathcal{H}_1$ and $g_i \in \mathcal{H}_2$ such that $\sum_{i=1}^{\infty} ||f_i||_2^2 < \infty$ and $\sum_{i=1}^{\infty} ||g_i||_2^2 < \infty$. The duality between $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $\Gamma(X, Y)$ is given by :

$$\langle T, f \otimes g \rangle = (Tf, \overline{g}), \text{ for } T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \text{ and } f \in L^2(X, \mu), g \in L^2(Y, \nu)$$

If $f \in L^\infty(X, \mu)$, let $M_f \in \mathcal{B}(\mathcal{H}_1)$ be the operator of multiplication by f. The collection $\{M_f\}_{f \in L^\infty(X,\mu)}$ is a maximal abelian selfadjoint algebra (masa). If $A \subseteq X$ is measurable, we write $P(A) = M_{\chi_A}$ where χ_A is the characteristic map of A. A subspace $W \subseteq \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is then called a masa-bimodule if $M_\Psi T M_\varphi \in W$ for all $\Psi \in L^\infty(Y, \nu)$, $T \in W$ and $\varphi \in L^\infty(X, \mu)$.

Now let $\kappa \subseteq X \times Y$ be a ω-closed set and $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. We say that κ supports T (or that T is supported on κ) if $P(B)MP(A) = 0$ whenever $A \times B \cap \kappa \sim \emptyset$. For a subset $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, there exists a smallest (up to marginal equivalence) ω-closed set which supports every operator $T \in \mathcal{M}$, that we denote by *supp*$(\mathcal{M})$ (cf. [23]). On the other hand, it is known that for every ω-closed set κ there exists the smallest and the largest weak* closed masa-bimodule - respectively $\mathfrak{M}_{min}(\kappa)$ and $\mathfrak{M}_{max}(\kappa)$ - with support κ (cf. [1], [75]).

Definition 5.11. Let (X,μ) and (Y,ν) be standard measure spaces and $\kappa \subseteq X \times Y$ be a ω-closed set. Then, κ is an operator M-set if :

$$\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2) \cap \mathfrak{M}_{max}(\kappa) \neq \{0\}$$

Otherwise, κ is said to be an operator U-set.

In order to define the operator-theoretic counterpart to M_0-sets, we need to introduce additional terminology and notation concerning Arveson measures. We mainly follow the seminal work [1].

Let σ be a complex measure of finite total variation defined on the product σ-algebra of $X \times Y$ and let $|\sigma|$ denote the variation of σ and $|\sigma|_X, |\sigma|_Y$ be the marginal measures of $|\sigma|$. Such measure σ is said to be an Arveson measure if there is a constant $c > 0$ such that the following holds :

$$|\sigma|_X \leq c\mu \ \text{ and } \ |\sigma|_Y \leq c\nu$$

The set of all Arveson measures on $X \times Y$ is denoted by $\mathbb{A}(X,Y)$ and for some $\sigma \in \mathbb{A}(X,Y)$, we denote the smallest constant satisfying its defining inequalities by $||\sigma||_{\mathbb{A}}$. For a σ-closed subset $\kappa \subseteq X \times Y$, we denote by $\mathbb{A}(\kappa)$ the set of all Arveson measures σ in $X \times Y$ such that $\mathrm{supp}(\sigma) \subseteq \kappa$.

An Arveson measure $\sigma \in \mathbb{A}(Y,X)$ defines an operator $T_\sigma : \mathcal{H}_1 \to \mathcal{H}_2$ which will be called pseudointegral. These operators were introduced in [1]. Indeed, for $\sigma \in \mathbb{A}(Y,X)$ one can consider the sesquilinear form $\phi : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C}$ given by :

$$\phi(f,g) = \int_{Y \times X} f(x)\overline{g(y)} d\sigma(y,x)$$

By the Riesz Representation Theorem, it follows that there is an unique operator $T_\sigma : \mathcal{H}_1 \to \mathcal{H}_2$ such that $(T_\sigma f, g) = \phi(f,g)$.

For a given ω-closed subset $\kappa \subseteq X \times Y$ we let $\hat{\kappa} = \{(y,x) : (x,y) \in \kappa\}$. We then have the following :

Theorem 5.12. *Let $\sigma \in \mathbb{A}(Y,X)$. There exists an unique $T_\sigma : \mathcal{H}_1 \to \mathcal{H}_2$ such that :*

$$(T_\sigma f, g) = \int_{Y \times X} f(x)\overline{g(y)} d\sigma(y,x), \ \textit{for } f \in \mathcal{H}_1, g \in \mathcal{H}_2$$

Moreover, $||T_\sigma|| \leq ||\sigma||_{\mathbb{A}}$ and for a given ω-closed subset $\kappa \subseteq X \times Y$ the operator T_σ is supported on κ if and only if $supp(\sigma) \subseteq \hat{\kappa}$.

Proof. The reader can find a proof in [74] (Theorem 3.2). ■

We finally define the operator-theoretic counterpart of M_0-sets :

Definition 5.13. Let (X, μ) and (Y, ν) be standard measure spaces and $\kappa \subseteq X \times Y$ be a ω-closed set. Then, κ is an operator M_0-set if :

$$\text{There is a non-zero measure } \sigma \in \mathbb{A}(\hat{\kappa}) \text{ such that } T_\sigma \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$$

Otherwise, κ is said to be an operator U_0-set.

In the remaining of this subsection, we bridge the gap between the notions of multiplicity and operator multiplicity. As mentioned in the beginning, this is a fruitful effort. Let G be a group and $E \subseteq G$. We define :

$$E^* = \{(s,t) \in G \times G : ts^{-1} \in E\}$$

We note that if G is second countable and E is closed, then E^* is ω-closed. One has the following central results :

Theorem 5.14. *Let G be a locally compact second countable group and $E \subseteq G$ be a closed subset. Then, the following are equivalent :*

(i) E is a M-set.

(ii) E^* is an operator M-set.

Proof. The reader can find a proof in [74] (Theorem 4.9). ■

Theorem 5.15. *Let G be a locally compact second countable group and $E \subseteq G$ be a closed subset. Then, the following are equivalent :*

(i) E is a M_0-set.

(ii) E^* is an operator M_0-set.

Proof. The reader can find a proof in [74] (Theorem 4.12). ■

5.2 More properties

In this subsection we state and prove some preservation properties that hold for closed sets of (extended) uniqueness. We omit proofs for most of those results one can refer to literature, providing nevertheless the proof whenever some novelty is asserted. For the sake of organization we divide the section according to the nature of the properties discussed.

5.2.1 Products

Let (X_i, μ_i) and (Y_i, ν_i) be standard measure spaces. We define the following map :

$$\rho : (X_1 \times Y_1) \times (X_2 \times Y_2) \to (X_1 \times X_2) \times (Y_1 \times Y_2)$$
$$((x_1, y_1), (x_2, y_2)) \mapsto ((x_1, x_2), (y_1, y_2))$$

We note that the following useful identity holds :

$$\rho(E_1^* \times E_2^*) = (E_1 \times E_2)^*$$

for $E_i \subseteq G_i$ where G_1 and G_2 are groups. Since we already know how to relate operator M-sets with M-sets, the key element to prove that preservation of M-sets under finite direct products holds is the following result :

Theorem 5.16. *Let (X_i, μ_i) and (Y_i, ν_i) be standard measure spaces and $E_i \subseteq X_i \times Y_i$ be ω-closed sets. The set $\rho(E_1 \times E_2)$ is an operator M-set if and only if both E_1 and E_2 are operator M-sets.*

Proof. The reader can find a proof in [74] (Theorem 5.11). ■

Corollary 5.17. Let G_1 and G_2 be locally compact second countable groups and let $E_1 \subseteq G_1$ and $E_2 \subseteq G_2$ be closed sets. Then, the following holds :

(i) If E_1 or E_2 are sets of uniqueness, then $E_1 \times E_2$ is a set of uniqueness

(ii) If E_1 and E_2 are sets of multiplicity, then $E_1 \times E_2$ is a set of multiplicity

Proof. We only prove (i), since the argument for (ii) is entirely analogous (cf. [74], Corollary 5.12). Suppose that E_1 is a set of uniqueness. By Theorem 5.14, E_1^* is an operator U-set. Since $\rho(E_1^* \times E_2^*) = (E_1 \times E_2)^*$, it follows from Theorem 5.16 that $(E_1 \times E_2)^*$ is an operator U-set and thus, once again by Theorem 5.14, we conclude that $E_1 \times E_2$ is a set of uniqueness. ■

In the remaining of this subsection, we establish the analogous preservation property for M$_0$-sets. It will be useful to consider the left and right slice maps. Given $\omega \in (\mathcal{K}(H_2, K_2))^* = \mathcal{C}_1(K_2, H_2)$, the left slice map L_ω is the function $L_\omega : \mathcal{K}(H_1 \otimes H_2, K_1 \otimes K_2) \to \mathcal{K}(H_1, K_1)$ defined on elementary tensors by :

$$L_\omega(A \otimes B) = \omega(B)A$$

A useful property of L_ω is that if $T \in \mathcal{K}(H_1 \otimes H_2, K_1 \otimes K_2)$ is supported on $\rho(\kappa_1 \times \kappa_2)$ then, $\operatorname{supp}(L_\omega(T)) \subseteq \kappa_1$ (cf. [74]). The right slice map R_ω : $\mathcal{K}(H_1 \otimes H_2, K_1 \otimes K_2) \to \mathcal{K}(H_2, K_2)$ is defined analogously.

Theorem 5.18. *Let $(X_i, \nu_i), (Y_i, \mu_i)$ be standard measure spaces and $\kappa_i \subseteq X_i \times Y_i$ be ω-closed sets, for $i = 1, 2$. The set $\rho(\kappa_1 \times \kappa_2)$ is an operator M_0-set if and only if both κ_1 and κ_2 are operator M_0-sets.*

Proof. Let κ_1 and κ_2 be operator M_0-sets, so that there are non zero measures $\sigma_1 \in \mathbb{A}(\hat{\kappa_1})$ and $\sigma_2 \in \mathbb{A}(\hat{\kappa_2})$ such that $T_{\sigma_1} \in \mathcal{K}(H_1, K_1)$ and $T_{\sigma_2} \in \mathcal{K}(H_2, K_2)$. Following the proof of Theorem 3.8 in [74], we conclude that $T_\sigma := T_{\sigma_1} \otimes T_{\sigma_2} \in \mathcal{K}(L^2(X_1 \times X_2), L^2(Y_1 \times Y_2))$ for a non zero Arveson measure σ supported in $\rho(\widehat{\kappa_1 \times \kappa_2})$. Hence, $\rho(\kappa_1 \times \kappa_2)$ is an operator M_0-set.
Conversely, suppose that $\rho(\kappa_1 \times \kappa_2)$ is an operator M_0-set so that there is a non zero measure $\sigma \in \mathbb{A}(\rho(\widehat{\kappa_1 \times \kappa_2}))$ and $T_\sigma \in \mathcal{K}(H_1 \otimes H_2, K_1 \otimes K_2)$. Note that

$L_\omega(T_\sigma) \in \mathcal{K}(H_1, K_1)$ and $\operatorname{supp}(L_\omega(T_\sigma)) \subseteq \kappa_1$. Thus, it suffices to prove that for some ω, $L_\omega(T_\sigma)$ is a pseudo-integral operator - say T_γ - for a non-zero Arveson measure γ. It will follow from Theorem 5.12 that $\gamma \in \mathbb{A}(\hat{\kappa_1})$ and thus, κ_1 is an operator M_0-set. The case for κ_2 is entirely analogous, using the right slice operator R_ω instead.

We let $\Omega = Y_1 \times Y_2 \times X_1 \times X_2$, π, π_{X_i}, π_{Y_i} be respectively the projections of Ω onto $Y_1 \times X_1$, X_i and Y_i and $f_i \in L^2(X_i, \nu_i)$, $g_i \in L^2(Y_i, \mu_i)$, for $i = 1, 2$ with $\omega = f_2 \otimes g_2$.

Note that if σ is Arveson (in particular with finite total variation), then so is $|\sigma|$. Consequently, we can assume that $\sigma \in \mathbb{A}(\rho(\widehat{\kappa_1 \times \kappa_2}))$ is finite, non-zero and non-negative and as Ω is completely metrizable and separable, it follows that σ is Radon.[10] Since $\sigma \neq 0$ and is Radon, there is a compact $K \subseteq \Omega$ with $\sigma(K) > 0$. Note that $\pi_{X_2}(K) := C \subseteq X_2$ and $\pi_{Y_2}(K) := C' \subseteq Y_2$ are compact sets and let $\mathcal{U}, \mathcal{V}$ be open sets such that $C \subseteq \mathcal{U}$ and $C' \subseteq \mathcal{V}$. Since X_2 and Y_2 are normal, there are continuous functions $f_2 : X_2 \to [0, 1]$ and $g_2 : Y_2 \to [0, 1]$ such that $\chi_C \leq f_2 \leq \chi_\mathcal{U}$ and $\chi_{C'} \leq g_2 \leq \chi_\mathcal{V}$. Since ν_2 is Radon, then :

$$\inf\{\nu_2(\mathcal{W}) : C \subseteq \mathcal{W} \text{ and } \mathcal{W} \text{ is open}\} = \nu_2(C) < \infty$$

and we can choose $\mathcal{V}$ such that $\nu_2(\mathcal{V}) < \infty$. Similarly, we can choose $\mathcal{W}$ such that $\mu_2(\mathcal{W}) < \infty$. Consequently :

$$\int_{X_2} f_2^2(x_2) d\nu_2 \leq \int_{X_2} \chi_\mathcal{U} d\nu_2 = \nu_2(\mathcal{U}) < \infty$$

and we conclude that $f_2 \in L^2(X_2, \nu_2)$. Similarly, we conclude that $g_2 \in L^2(Y_2, \mu_2)$. This is our choice of f_2 and g_2, defining ω. Now note that :

$$(L_\omega(T_\sigma) f_1, \overline{g_1}) = \langle L_\omega(T_\sigma), f_1 \otimes g_1 \rangle = \langle T_\sigma, (f_1 \otimes g_1) \otimes \omega \rangle = (T_\sigma(f_1 \otimes f_2), \overline{g_1 \otimes g_2})$$

Since σ is an Arveson measure on Ω we have :

$$(L_\omega(T_\sigma) f_1, g_1) = \int_\Omega f_1(x_1) f_2(x_2) \overline{g_1(y_1) g_2(y_2)} d\sigma((y_1, y_2), (x_1, x_2))$$

Letting $y = (y_1, y_2)$ and $x = (x_1, x_2)$, we define a measure γ on $Y_1 \times X_1$ as follows :

$$\gamma(E) := \int_\Omega \chi_{\pi^{-1}(E)}((y, x)) f_2(x_2) \overline{g_2(y_2)} d\sigma(y, x)$$

Therefore, we may conclude that :

$$(L_\omega(T_\sigma) f_1, g_1) = \int_{Y_1 \times X_1} f_1(x_1) \overline{g_1(y_1)} d\gamma(y_1, x_1)$$

[10] Let X be a completely metrizable and separable space and μ a (positive) Borel measure on X. Then, X is Radon (cf. [9], Theorem 7.1.7). In fact, if $w(X)$ is the weight of X, then there is a non-tight finite (positive) Borel measure on X if and only if $w(X)$ is a measurable cardinal (cf. [25], Theorem 438H). Existence of measurable cardinals is not provable within ZFC.

If one proves that γ an Arveson measure, we may conclude that $L_\omega(T_\sigma)$ is pseudo-integral. It will then be enough to check that γ is non-zero and we are done. On one hand :

$$\gamma(Y_1 \times X_1) = \int_\Omega \chi_\Omega(y,x) f_2(x_2) g_2(y_2) d\sigma(y,x) \leq \int_\Omega \chi_\Omega(y,x) d\sigma < \infty$$

and thus, γ has finite total variation. On the other hand, let $E = \alpha \times X_1$, with a measurable set $\alpha \subseteq Y_1$. Then, we have that :

$$\begin{aligned} |\gamma|(\alpha \times X_1) &\leq \int_\Omega \chi_{\alpha \times Y_2 \times X_1 \times X_2}((y,x)) f_2(y_2) g_2(x_2) d|\sigma|((y,x)) \\ &= \int_\Omega \chi_\alpha(y_1) \chi_{Y_2 \times X_1 \times X_2}(y_2, x_1, x_2) f_2(y_2) g_2(x_2) d|\sigma|((y,x)) \\ &\leq (\int_\Omega f_2^2 g_2^2 d|\sigma|)^{\frac{1}{2}} (\int_\Omega \chi_\alpha \chi_{Y_2 \times X_1 \times X_2} d|\sigma|)^{\frac{1}{2}} := A^{\frac{1}{2}} B^{\frac{1}{2}} \\ &\leq k\nu_1(\alpha), \text{for some finite positive } k. \text{ In fact :} \end{aligned}$$

Since σ is Arveson :

$$A^{\frac{1}{2}} \leq c(\int_{X_2} f_2^2(x_2) d\nu_2)^{\frac{1}{2}} (\int_{Y_2} g_2^2(y_2) d\mu_2)^{\frac{1}{2}} = c||f_2|| ||g_2|| < \infty$$

Furthermore, note that given measurable spaces (X_i, ν_i) and (Y_i, μ_i), a measure σ on $(X_1 \times X_2) \times (Y_1 \times Y_2)$ can be identified with a measure $\tilde{\sigma}$ on $X_1 \times (X_2 \times Y_1 \times Y_2)$ as the product σ-algebras can be canonically identified. In this way, the marginal measures $|\tilde{\sigma}|(\alpha)$ and $|\sigma|(\alpha \times X_2)$ - for $\alpha \subseteq X_1$ measurable - coincide. If σ is Arveson, it then follows that there is some positive finite constant d such that $|\tilde{\sigma}(\alpha)| \leq d\nu_1^2(\alpha)$. Thus, $B^{\frac{1}{2}} \leq (d\nu_1^2(\alpha))^{\frac{1}{2}}$.
It remains to check that γ is non-zero : let $E = \pi_{Y_1}(K) \times \pi_{X_1}(K)$ which is compact, hence closed and thus Borel subset of $Y_1 \times X_1$. Then :

$$\gamma(E) = \int_\Omega \chi_{\pi^{-1}(E)}(y,x) f_2(x_2) g_2(y_2) d\sigma(y,x) \geq \sigma(K) > 0$$

since if $(y_1, y_2, x_1, x_2) \in K$, then $\chi_{\pi^{-1}(E)} = 1$ and $f_2(x_2) = g_2(y_2) = 1$, by construction. ■

We can thus establish the corresponding fact for M_0-sets :

Corollary 5.19. Let G_1 and G_2 be locally compact second countable groups and $E_1 \subseteq G_1$ and $E_2 \subseteq G_2$ be closed sets. Then :

(i) If E_1 and E_2 are sets of restricted multiplicity, then $E_1 \times E_2$ is a set of restricted multiplicity.

(ii) If E_1 or E_2 are sets of extended uniqueness, then $E_1 \times E_2$ is a set of extended uniqueness.

Proof. Let E_1 and E_2 be M_0-sets. By Theorem 5.15, E_1^* and E_2^* are operator M_0-sets. Since $\rho(E_1^* \times E_2^*) = (E_1 \times E_2)^*$ it follows by Theorem 5.18 and Theorem 5.15 that $E_1 \times E_2$ is a M_0-set. The proof is entirely analogous for the case of sets of extended uniqueness. ■

5.2.2 Inverse images

We begin with a quite general result concerning the preservation of U-sets under inverse images. In [80] (Theoreme 7) it is proven that if G is a locally compact abelian second countable group, $H \subseteq G$ a normal closed subgroup, $E \subseteq G/H$ a closed subset and $q : G \to G/H$ the quotient map, then :

$$E \in \mathcal{U}(G/H) \text{ if and only if } q^{-1}(E) \in \mathcal{U}(G)$$

We extend the latter result for locally compact second countable groups, not necessarily abelian. Indeed, this generalization is a direct consequence of the following fulcral result :

Theorem 5.20. *Let $(X, \mu), (Y, \nu), (X_1, \mu_1), (Y_1, \nu_1)$ be standard Borel spaces and suppose that $\varphi : X \to X_1$ and $\Psi : Y \to Y_1$ are measurable maps. Let $E \subseteq X_1 \times Y_1$ and $F = \{(x, y) \in X \times Y : (\varphi(x), \Psi(y)) \in E\}$. If $\varphi_*\mu$ and $\Psi_*\nu$ are equivalent, respectively, to μ_1 and ν_1, then E is an operator M-set if and only if F is an operator M-set.*

Proof. The result follows from Theorem 5.5 in [74]. As noted in Remark 5.6 (cf. [74]), the injectivity assumption in Theorem 5.5 can be dropped (cf. [22]). ■

Corollary 5.21. Let G be a locally compact second countable group, $H \subseteq G$ a normal closed subgroup and $E \subseteq G/H$ a closed subset. Then :

$$E \in \mathcal{U}(G/H) \text{ if and only if } q^{-1}(E) \in \mathcal{U}(G)$$

Proof. Note that the pushforward measure of the Haar measure m_G of G by q is equivalent to the Haar measure $m_{G/H}$ on G/H. Indeed, $q_*(m_G)$ is Borel, non-trivial and left-invariant since for $S \subseteq G/H$ one has that :

$$q_* m_G(gHS) = m_G(q^{-1}(gHS)) = m_G(SH) = m_G(q^{-1}(S)) = q_* m_G(S)$$

Since G/H is Polish it follows that $q_* m_G$ is Radon and thus, by uniqueness of Haar measure, equivalent to $m_{G/H}$. The result now follows immediately from Theorem 5.20 and Theorem 5.14 noting that since q is a homomorphism, then $q^{-1}(E)^* = (q^{-1} \times q^{-1})(E^*)$. ■

Corollary 5.22. Let G and H be locally compact second countable groups with Haar measures m_G and m_H respectively and $E \subseteq H$ be a closed set. If $\varphi : G \to H$ is a continuous isomorphism, then :

$$E \in \mathcal{U}(H) \text{ if and only if } \varphi^{-1}(E) \in \mathcal{U}(G)$$

Proof. Since φ is a homomorphism, $\varphi^{-1}(E)^* = (\varphi^{-1} \times \varphi^{-1})(E^*)$ and since it is an isomorphism, $\varphi_* m_G$ and m_H are equivalent. Thus, the result follows from Theorem 5.14 and Theorem 5.20. ■

The following characterization of closed subgroups was proved in [74] (Corollary 5.10), answering a questions posed in [14]. As noted previously, the problem of characterizing sets of uniqueness is extremely difficult in general (and, as one could argue in the next section, *an effort doomed to fail*), hence the following result is of particular interest and we include its proof :

Corollary 5.23. Let G be a locally compact second countable group and $H \subseteq G$ a closed subgroup. Then, H is a M-set if and only if H is open. In particular, if G is connected then the only closed subgroup which is a M-set is G itself.

Proof. Let m denote the Haar measure of G. Then, by Steinhaus' theorem H is open if and only if $m(H) > 0$ and since U-sets have necessarily zero measure, it follows by Theorem 5.14 that it is enough to prove that if $m(H) = 0$ then H^* is an operator U-set. Since G/H is Polish, $(G/H, q_*m)$ is standard. Consider $D = \{(z, z) : z \in G/H\}$. Since every operator on $L^2(G/H, q_*m)$ supported on D is a multiplication operator and q_*m is non atomic, the only compact operator supported on D must be the zero operator and consequently, D is an operator U-set. By Theorem 5.20, it follows that $H^* = (q^{-1} \times q^{-1})(D)$ is an operator U-set, as we wanted. ∎

As with U-sets, in [80] it was proven that U_0-sets are also preserved under the inverse image of quotients of abelian groups (cf. [80], Theoreme 8). More concretely, if G is a locally compact abelian second countable group, H is a closed normal subgroup and E is a closed subset of G/H, then :

$$E \in \mathcal{U}_0(G/H) \text{ if and only if } q^{-1}(E) \in \mathcal{U}_0(G)$$

In the remaining of the subsection we partly extend the result for groups which are not necessarily abelian. We recall that if G is either compact or abelian, the class of measures $\mu \in M(G)$ satisfying $\lambda(\mu) \in C_r^*(G)$ coincides with the class of Rajchman measures.

Consider the case $q : \mathbb{R} \to \mathbb{T}$ and suppose that $E \subseteq \mathbb{T}$ is a closed set such that $q^{-1}(E)$ is a M_0-set. Thus, there is a Rajchman measure $\nu \neq 0$ supported on $q^{-1}(E)$. Take the pushforward measure $\mu := q_*\nu$ on $\mathbb{T}$, which is a non-zero measure on $\mathbb{T}$ supported on E. Furthermore, μ is Rajchman. Indeed, this follows from the observation that if f is a function on $\mathbb{T}$ and g is its 2π-periodic extension as a function on $\mathbb{R}$, then $\int_{\mathbb{R}} g(x)d\nu(x) = \int_{\mathbb{T}} f(t)d\mu(t)$. In particular, $\hat{\nu}(n) = \hat{\mu}(n)$. Hence, since $\hat{\nu} \in C_0(\mathbb{R})$ it follows that $\lim_{|n|\to\infty} \hat{\mu}(n) = 0$ and μ is Rajchman. We conclude that E is a M_0-set. In order to use this idea within a more general setting, we establish first some notation.

Let G be locally compact and $H \subseteq G$ be a closed normal subgroup. Given $f \in C_c(G)$, we define :

$$\phi(f)([x]) := \int_H f(xh)dh$$

One can prove that $\phi(f) \in C_c(G/H)$ and that in fact, $\phi : C_c(G) \to C_c(G/H)$ is surjective (cf. Proposition 1.3.7, in [41]). The Haar measures on G and on H

can be normalized in such way that the Weil's formula holds for all $f \in C_c(G)$:

$$\int_G f(x)dx = \int_{G/H} (\int_H f(xh)dh)d\mu[x]$$

where μ is a G-invariant measure on G/H (since H is closed normal, in this case μ is a Haar measure on G/H). One can verify that given $f \in L^1(G)$, Weil's formula still holds and that $||\phi(f)||_1 \leq ||f||_1$ and thus, ϕ is actually a *-homomorphism from $L^1(G)$ onto $L^1(G/H)$.

Recall (see section 3) that every unitary representation of G determines a non degenerate *-representation $\tilde{\pi}$ of $L^1(G)$ as follows :

$$\langle \tilde{\pi}(f)\xi, \eta \rangle = \int_G f(x)\langle \pi(x)\xi, \eta \rangle dx, \text{ for } f \in L^1(G) \text{ and } \xi, \eta \in \mathcal{H}(\pi)$$

For any representation π of G/H and $\xi, \eta \in \mathcal{H}(\pi)$, one shows that :

$$\langle \pi \circ q(f)\xi, \eta \rangle = \langle \pi(\phi(f))\xi, \eta \rangle, \text{ for } f \in L^1(G)$$

where $q : G \to G/H$ is the quotient map. This implies that $||\phi(f)||_{C^*(G/H)} \leq ||f||_{C^*(G)}$ and thus, ϕ can be extended to a *-homomorphism from $C^*(G)$ onto $C^*(G/H)$. For more details, the interested reader is again referred to [41].

Theorem 5.24. *Let G be an amenable locally compact group. Let $H \subseteq G$ be a normal closed subgroup and $E \subseteq G/H$ a closed set. Then, if $q^{-1}(E)$ is a M_0-set, so is E.*

Proof. Since $q^{-1}(E)$ is a M_0-set, there is a measure $\mu \in M(q^{-1}(E))$ such that $\lambda_G(\mu) \neq 0$ and $\lambda_G(\mu) \in C^*(G)$. Consider the pushforward measure $q_*\mu \in M(E)$ which is non-zero and thus, $\lambda_{G/H}(q_*\mu) \neq 0$. It then suffices to prove that $\lambda_{G/H}(q_*\mu) \in C^*(G/H)$ and we can conclude that E is a M_0-set. Indeed :

$$\begin{aligned}\phi(\lambda_G(\mu))\lambda_{G/H}(\phi(f)) &= \phi(\lambda_G(\mu)\lambda_G(f)) = \phi(\lambda_G(\mu * f)) \\ &= \lambda_{G/H}(\phi(\mu * f)) = (\lambda_{G/H} \circ q)(\mu * f) \\ &= (\lambda_{G/H} \circ q)(\mu)(\lambda_{G/H} \circ q)(f)\end{aligned}$$

Since $\pi \circ q(\mu) = \pi(q_*\mu)$ for any representation π we conclude that :

$$\phi(\lambda_G(\mu)) = \lambda_{G/H} \circ q(\mu) = \lambda_{G/H}(q_*\mu) \in C^*(G/H)$$

and we are done. ■

In order to prove a converse to Theorem 5.24 we rely on a certain operator that allows going from $C_r^*(G/H)$ to $C_r^*(G)$ while respecting the support in a convenient way.

Let G be a locally compact group, $\theta \in A(G) \cap C_c(G)$ and $T \in VN(G/H)$, where H is a closed normal subgroup of G. Then, one can show that the following functional is bounded (Theorem 3.7 in [73]) :

$$A(G) \ni u \mapsto \langle T, \phi(\theta u) \rangle$$

Thus, there is an operator $\Phi_\theta(T) \in VN(G)$ such that :

$$\langle \Phi_\theta(T), u \rangle = \langle T, \phi(\theta u) \rangle \quad , u \in A(G)$$

Lemma 5.25. Let $\theta \in A(G) \cap C_c(G)$. Then, the following holds :

(i) Φ_θ maps $C_r^*(G/H)$ into $C_r^*(G)$.

(ii) If $T \in C_r^*(G/H) \cap J(E)^\perp$, then $\Phi_\theta(T) \in C_r^*(G) \cap J(q^{-1}(E))^\perp$, where $q : G \to G/H$ is the quotient map.

Proof. The reader can find a proof in [73] (Theorem 3.7). ■

Motivated by the definition of property *(l)* in [73], we introduce the following property :

Definition 5.26. Let G be a locally compact group and $H \subseteq G$ be a closed normal subgroup. For any $\theta \in C_c(G)$, let $|\theta|(x) := |\theta(x)|$. We say that H has the property $(|l|^2)$ if for every proper compact $K \subseteq G/H$ there exists $\theta \in A(G) \cap C_c(G)$ such that $\phi(|\theta|) = c_1$ and $\phi(\theta^2) = c_2$, for positive real constants c_1, c_2 on a neighborhood of K.

Remark 5.27. As an important example, $\mathbb{Z} \subseteq \mathbb{R}$ has the property $(|l|^2)$. Indeed, after identifying $\mathbb{T}$ with $[0, 1)$ let's fix a proper compact $K \subseteq \mathbb{T}$. We may assume that $0 \notin K$ - otherwise replacing K with a suitable translation. Using the regularity of $A(\mathbb{R})$, let $\theta \in A(\mathbb{R})$ such that $\theta(x) = 1$ on a neighborhood $\mathcal{U}$ of K and $\theta(x) = 0$ whenever $x \notin (0, 1)$. If $x \in \mathcal{U}$, note that :

$$\phi(|\theta|)([x]) = \sum_{h\in\mathbb{Z}} |\theta(x + h)| = \sum_{h\in\mathbb{Z}} \theta(x + h)\theta(x + h) = \phi(\theta^2)([x]) = 1$$

An entirely analogous argument shows that $\mathbb{Z}^n \subseteq \mathbb{R}^n$ and $\mathbb{Z}^n \times \{1\}^m \subseteq \mathbb{R}^n \times \mathbb{T}^m$ also have the property $(|l|^2)$.

Theorem 5.28. *Let G be a locally compact group, $H \subseteq G$ a closed normal subgroup with property $(|l|^2)$ and $E \subseteq G/H$ a compact set. Furthermore, assume that G/H is second countable. Let $q : G \to G/H$ denote the quotient map and suppose that E is a M_0-set. Then, $q^{-1}(E)$ is also a M_0-set.*

Proof. Since $E \subseteq G/H$ is a M_0-set, there is some $\mu \in M(E)$ such that $\lambda(\mu) \neq 0$ and $\lambda(\mu) \in C_r^*(G/H)$. We note that since G/H is second countable and $\mu \neq 0$, it follows that $\mu(E) > 0$.
For any $\theta \in A(G) \cap C_c(G)$, $\Phi_\theta(\lambda(\mu)) \in C_r^*(G)$ by Lemma 5.25. Thus, if one shows that $\Phi_\theta(\lambda(\mu)) = \lambda(\nu)$ for some $\nu \in M(q^{-1}(E))$ such that $\lambda(\nu) \neq 0$, it follows that $q^{-1}(E)$ is a M_0-set.

In order to obtain such measure ν, appealing to Riesz-Markov-Kakutani Representation Theorem it is enough to choose θ such that the following linear functional Ψ is bounded :

$$C_0(G) \ni u \mapsto \int_{G/H} (\int_H \theta(xh)u(xh)dh)d\mu(x)$$

Indeed, if Ψ is bounded then there is some measure $\nu \in M(G)$ such that :

$$\Psi(u) = \int_G u(s)d\nu(s)$$

By construction, for $u \in A(G)$ we have that :

$$\langle \Phi_\theta(\lambda(\mu)), u\rangle = \langle \lambda(\mu), \phi(\theta u)\rangle = \Psi(u) = \langle \lambda(\nu), u\rangle$$

and we can conclude that $\Phi_\theta(\lambda(\mu)) = \lambda(\nu)$. Thus, it remains to choose an appropriate θ. Since E is compact and $H \subseteq G$ has the property $(|l|^2)$, let $\theta \in A(G) \cap C_c(G)$ be such that $\phi(|\theta|) = c_1 > 0$ and $\phi(\theta^2) = c_2 > 0$ on a neighborhood of E. It follows that :

$$\begin{aligned} |\int_{G/H} (\int_H \theta(xh)u(xh)dh)d\mu(x)| &\leq \int_{G/H} |\int_H \theta(xh)u(xh)dh| d\mu(x) \\ &\leq ||u||_\infty \int_{G/H} |\phi(\theta)(x)| d\mu(x) \\ &\leq ||u||_\infty c_1 \mu(E) \end{aligned}$$

Hence, Ψ is bounded. Moreover, it follows from Lemma 5.25 that $\nu \in M(q^{-1}(E))$. It remains to check that indeed $\lambda(\nu) \neq 0$. For this purpose, it suffices to choose $u \in A(G)$ such that $\Psi(u) \neq 0$. We let $u = \theta$ and note that :

$$\int_{G/H} (\int_H \theta(xh)\theta(xh)dh)d\mu(x) = \int_{G/H} \phi(\theta^2)(x)d\mu(x) = c_2\mu(E) > 0$$

■

Corollary 5.29. Let $q : \mathbb{R}^n \times \mathbb{T}^m \to \mathbb{T}^{n+m}$ be the quotient map for integers $n, m \geq 0$ and $E \subseteq \mathbb{T}^{n+m}$ be a closed subset. Then, E is a M_0-set if and only if $q^{-1}(E)$ is a M_0-set.

Proof. The result follows from Theorem 5.24 and Theorem 5.28. ■

5.2.3 Unions

In this subsection we approach the problem of closure properties of U-sets under unions. An early accomplishment in this direction is the result due to Bari (cf. [4]) stating that a countable union of closed sets of uniqueness (in $\mathbb{T}$) is still a set of uniqueness. It should be stressed that these matters are far from being well understood : for instance, insofar the author is aware, it is still an open problem to determine whether or not the union of even *two* G_δ sets of uniqueness of $\mathbb{T}$ is still a set of uniqueness (cf. [47], pp. 46). Our goal for the subsection is more modest : we follow [83] with the purpose of establishing that $\mathcal{U}(G)$ is a σ-ideal whenever G is a locally compact amenable group. We finish the subsection with a brief digression on Bari's classical result within certain set-theoretic assumptions.

Let G be a locally compact group. Following [83], we say that a closed set $E \subseteq G$ is a J-set whenever the following holds :

$$E = \overline{\bigcup_{T \in J(E)^{\perp} \cap C_r^*(G)} \operatorname{supp}(T)}$$

Proposition 5.30. Let G be a locally compact group and $E \subseteq G$ a closed subset. Then :

(i) $E \in \mathcal{U}(G)$ if and only if E does not contain any non-empty J-set

(ii) If G is amenable, $E \in \mathcal{U}(G)$ if and only if $\overline{F \setminus E} = F$ for every J-set F

Proof. The proof of (i) is immediate from the definitions and properties of $\operatorname{supp}(T)$. For a proof of (ii), the reader is referred to [83] (Corollary 4.6). ■

Theorem 5.31. *Let G be a locally compact amenable group, $\{E_n\} \subseteq \mathcal{U}(G)$ and set $E = \bigcup_n E_n$. Then, if E is closed it follows that $E \in \mathcal{U}(G)$.*

Proof. By Proposition 5.30-(i) it is enough to show that E does not contain any non-empty J-set. Suppose, towards a contradiction, that E contains a J-set F and let $O_n := F \setminus E_n$. By Proposition 5.30-(ii) it follows that O_n is dense in F and by the Baire Category theorem, so is $\bigcap_n O_n$. However, since $F \subseteq \bigcup_n E_n$ this is impossible unless $F = \emptyset$. ■

Corollary 5.32. Let G be a locally compact amenable group. Then, $\mathcal{U}(G)$ is a σ-ideal of $\mathcal{F}(G)$.

Proof. The fact that $\mathcal{U}(G)$ is hereditary is straightforward and the requirement concerning countable unions is satisfied from Theorem 5.31. ■

For the purpose of this thesis, Corollary 5.32 is all that is needed. However, and for the sake of completeness, we end the subsection with the observation that the proof of Bari's result as given in [47] (Theorem 5 - I.5) still holds under more general set-theoretic assumptions. With this in mind, we momentarily go back to trigonometric series $S \sim \sum c_n e^{inx}$, with $x \in \mathbb{T}$. For such formal series with bounded coefficients $\{c_n\}$ and for a continuous function φ on $\mathbb{T}$ with absolutely convergent Fourier coefficients, one defines the following formal series :

$$S(\varphi).S \sim \sum C_n e^{inx}, \text{ with } C_n = \sum_k c_k \hat{\varphi}(n-k)$$

If φ has Fourier coefficients converging sufficiently rapidly to 0 (indeed, it is enough that $\hat{\varphi}(n) = O(|n|^{-3})$) we can relate the formal product $S(\varphi).S$ with $\varphi(x).S$ in an useful way. In particular :

Proposition 5.33. If $\varphi \in C^{\infty}(\mathbb{T})$ and $S \sim \sum c_n e^{inx}$ is such that $c_n \to 0$, the following holds :

$$\sum_{n=-\infty}^{\infty} (C_n - \varphi(x)c_n)e^{inx} = 0, \text{ uniformly on } x$$

Proof. The reader can find a proof of a more general result in [47] (Lemma 2 - I.4). ■

Another result which is needed in our digression is the following :

Proposition 5.34. Suppose that $S \sim \sum c_n e^{inx}$ is such that $c_n \to 0$, with partial sums $S_n(x)$ bounded at each point x and $\sum c_n e^{inx} = 0$ almost everywhere. Then, $c_n = 0$ for all $n \in \mathbb{Z}$.

Proof. The reader can find a proof in [47] (Theorem 3 - I.5). ■

Meager sets and null sets are trivially closed under countable unions and under certain set-theoretic assumptions, for $\aleph_0 \leq \kappa < 2^{\aleph_0}$, the same still holds true for κ-unions. The reader which is unfamiliar with the terminology is referred to the Appendix.

Theorem 5.35. *Under $ZFC + MA(\kappa)$, the following assertions are true :*

(i) Let $\{M_\alpha\}_{\alpha<\kappa}$ be a collection of meager sets of $\mathbb{R}$. Then, $\bigcup_\alpha M_\alpha$ is meager

(ii) Let $\{N_\alpha\}_{\alpha<\kappa}$ be a collection of null sets of $\mathbb{R}$. Then, $\bigcup_\alpha N_\alpha$ is null

Proof. The reader can find a proof in [49] (respectively, Theorem 2.20 and Theorem 2.21). ■

We can finally conclude our digression :

Theorem 5.36. *Assume $ZFC + MA(\kappa)$ and let $\{E_\alpha\}_{\alpha<\kappa}$ be a family of closed sets of uniqueness of $\mathbb{T}$. Then, $\bigcup_{\alpha<\kappa} E_\alpha$ is a set of uniqueness.*

Proof. Let $E = \bigcup_\alpha E_\alpha$ and suppose that $S \sim \sum c_n e^{inx} = 0$ for all $x \in \mathbb{T} \setminus E$. Since each $E_\alpha \in \mathcal{U}(\mathbb{T})$, it follows by Theorem 5.35-(ii) that $\mu(E) = 0$ and thus, $c_n \to 0$.[11] Suppose, towards a contradiction, that not all coefficients c_n are zero and consider the following set :

$$G = \{x \in \mathbb{T} : \{S_N(x)\} \text{ is unbounded}\}, \text{ with } S_N = \sum_{n=-N}^{N} c_n e^{inx}$$

[11]By Cantor-Lebesgue theorem, if $\sum c_n e^{inx} = 0$ on a set of positive measure then $c_n \to 0$ (cf. [47], Lemma 6 - I.3).

It follows by Proposition 5.34 that $G \neq \emptyset$ and as such, a non-empty Polish space. Let $G_\alpha = G \cap E_\alpha$ and note that since $G = \bigcup_\alpha G_\alpha$, it follows from Proposition 5.35-(i) that there is some β and interval I_β such that $G \cap I_\beta = G_\beta \cap I_\beta \neq \emptyset$. We prove that $\sum c_n e^{inx} = 0$ on I_β and thus, $I_\beta \cap G = \emptyset$, yielding a contradiction. Pick any $f \in C^\infty(\mathbb{T})$ which vanishes on $\mathbb{T} \setminus I_\beta$ and is positive on I_β and consider the formal product $T := S(f).S \sim C_n e^{inx}$. Since $E_\beta \in \mathcal{U}(\mathbb{T})$, it follows by Proposition 5.33 that it is enough to prove that $\sum C_n e^{inx}$ vanishes off E_β. Let $x \notin E_\beta$ and note that it is sufficient to consider the case when $x \in I_\beta \cap E$. By regularity, let J be an interval such that $x \in J$ and $\overline{J} \cap E_\beta = \emptyset$. Pick any $g \in C^\infty(\mathbb{T})$ such that $g(x) = 1$ with $\mathrm{supp}(g) \subseteq \overline{J}$ and consider the formal product $R := S(g).T \sim \sum D_n e^{inx}$. Since $\sum c_n e^{inx} = 0$ a.e., the same is true for T and consequently, for R as well. On the other hand, $\sum D_n e^{inx}$ has bounded partial sums outside $\overline{J} \cap G = \overline{J} \cap G_\beta$ as S and T have that property as well. It follows from Proposition 5.34 that $D_n = 0$ for all n and thus, by Proposition 5.33, that $\sum C_n e^{inx} = 0$. ■

Remark 5.37. Since $MA(\aleph_0)$ holds in ZFC, we recover Bari's result. Moreover, since singletons are sets of uniqueness of $\mathbb{T}$, we conclude that under $ZFC + MA(\kappa)$ every subset $E \subseteq \mathbb{T}$ with $|E| = \kappa$ is a set of uniqueness.

5.3 Descriptive set theory and sets of uniqueness

Some applications of descriptive set-theoretic tools to the study of closed sets of uniqueness are presented. In turn, these provide insight on the problem of characterization of such sets from which other properties can be entailed. We prove some generalizations of previous results, aiming to provide the appropriate context for the sake of completeness.

5.3.1 Complexity

In this subsection we locate the descriptive set-theoretic complexity of $\mathcal{U}(G)$ and $\mathcal{U}_0(G)$ under certain assumptions on G. In [80], it was proven that if G is a non-discrete locally compact *abelian* second countable group then $\mathcal{U}(G)$ and $\mathcal{U}_0(G)$ are Π^1_1-complete subsets of $\mathcal{F}(G)$. We observe that $\mathcal{U}(G)$ and $\mathcal{U}_0(G)$ are always coanalytic whenever G is a locally compact second countable group, abelian or not. Relying on the functorial properties of section 5.2 and on the fact that $\mathcal{U}(\mathbb{T})$ is Π^1_1-complete, we provide a *direct* proof that $\mathcal{U}(G)$ is Π^1_1-complete for connected locally compact Lie groups G (abelian or not) and that $\mathcal{U}_0(G)$ is Π^1_1-complete for connected abelian Lie groups G. Using the aforementioned result from [80], we easily extend the result on the descriptive complexity of $\mathcal{U}(G)$ for general locally compact Polish groups (non necessarily abelian) such that the quotient $G/\overline{[G,G]}$ is non-discrete. We start by identifying the Polish spaces on which we will work.

Recall that for a locally compact space X, we consider the Fell topology on

$\mathcal{F}(X)$ which is compact metrizable, induces the Effros Borel space and coincides with the Vietoris topology whenever X is compact. Moreover, recall that if X is a locally compact, Hausdorff and second countable space then $C_0(X)$ is separable[12] and since $M(X) = C_0(X)^*$, it follows that $B_1(M(G))$ is w^*-metrizable whenever G is a locally compact and second countable (Hausdorff) group G. Thus, by Banach-Alaoglu Theorem, it follows that :

$$G \text{ locally compact second countable group, then } (B_1(M(G)), w^*) \text{ is Polish}$$

Let G be a locally compact and second countable group and recall that any element of $u \in A(G)$ is of the form $u = f * \check{g}$ for $f, g \in L^2(G)$, with norm $||u|| = \inf\{||f||_2||g||_2\}$ with infimum taken over all representations of the form $u = f*\check{g}$. Since G is second countable, then $L^2(G)$ is separable and consequently, $A(G)$ is also separable. Thus, and since $VN(G) = A(G)^*$, $B_1(VN(G))$ is w^*-metrizable. Similarly as before, we conclude that :

$$G \text{ locally compact second countable group, then } (B_1(VN(G)), w^*) \text{ is Polish}$$

Moreover, if G is a locally compact and second countable group, it follows that $L^1(G)$ is separable. Since the norm in $C^*(G)$ is dominated by the $||.||_1$-norm, we conclude that $C^*(G)$ is separable. Since it is defined as a completion, it follows that $C^*(G)$ is Polish. We consider the representation λ of $M(G)$ restricted to the unit ball and with the following topologies :

$$\lambda : (B_1(M(G)), w^*) \to (VN(G), w^*)$$

Note that λ is continuous. Indeed, since $B_1(M(G))$ is w^*-metrizable, it is sufficient to verify sequential continuity : if $\mu_n \to \mu$, it follows by definition that $\varphi_{\mu_n} \to \varphi_\mu$ (cf. Theorem 4.27) and thus, $\lambda(\mu_n) \to \lambda(\mu)$. In fact, the definition of the pairing $A(G)^* = VN(G)$ entails the continuity of λ even if one considers the WOT topology in the codomain.

Theorem 5.38. *Let G be a locally compact second countable group. Then, $\mathcal{U}(G) \subseteq \mathcal{F}(G)$ is coanalytic.*

Proof. For the sake of readability, we divide the proof in 3 steps :

Step 1 : Let $||.||_r$ and $||.||_*$ be the norms on $C_r^*(G)$ and $C^*(G)$ respectively and recall that since $||.||_r \leq ||.||_*$, this entails a continuous surjective map : $\Upsilon : C^*(G) \to C_r^*(G) \subseteq \mathcal{L}(L^2(G))$, taking the operator norm in the codomain. Note that $C := B_1(VN(G)) \cap C_r^*(G)$ is closed in the operator norm and thus, since Υ is continuous, $P := \Upsilon^{-1}(C)$ is also closed. As a closed subspace of a Polish space, it follows that P is also Polish. Let's denote the restriction of Υ to P still as Υ. Then, $\Upsilon : P \to B_1(VN(G))$ is a function whose image is $B_1(VN(G)) \cap C_r^*(G)$, as Υ is surjective. Since Υ was continuous with respect

[12]Since X is second countable, so it is its one-point compactification $\tilde{X}$. Thus, $C(\tilde{X})$ is separable as $\tilde{X}$ is compact, Hausdorff and second countable. Since subspaces of separable metric spaces are separable, we are done.

to the operator norm, it will remain continuous with respect to the w^*-topology in the range. Thus :

$$\Gamma_1 := C_r^*(G) \cap B_1(VN(G)) \subseteq (B_1(VN(G)), w^*) \text{ is analytic}$$

Step 2 : Considering $\mathcal{F}(G)$ with the Fell topology and $(B_1(VN(G)), w^*)$, the following set is closed, thus Borel :

$$\Gamma_2 = \{(S, E) \in B_1(VN(G)) \times \mathcal{F}(G) : \operatorname{supp}(S) \subseteq E\}$$

Let $\{(S_n, E_n)\} \subseteq \Gamma_2$ such that $(S_n, E_n) \to (S, E)$. It is enough to prove that $\operatorname{supp}(S) \subseteq E$. Let $u \in A(G) \cap C_c(G)$ such that $\operatorname{supp}(u) \cap E = \emptyset$ and $\mathcal{W} = \{F \in \mathcal{F}(G) : F \cap \operatorname{supp}(u) = \emptyset\}$ which is an open neighborhood of E (in the Fell topology). Since $E_n \to E$, then $E_k \subseteq \mathcal{W}$ for all sufficiently large k. Since $\operatorname{supp}(S_k) \subseteq E_k$, then for all sufficiently large k one has that $\langle u, S_k \rangle = 0$. On the other hand, since $S_n \xrightarrow{w^*} S$, it follows that $\langle u, S \rangle = 0$ and thus $\operatorname{supp}(S) \subseteq E$ as we wanted.

Step 3 : Define $P \subseteq B_1(VN(G)) \times \mathcal{F}(G)$ to be the following subset :

$$P = \{(S, E) : (S, E) \in \Gamma_2, S \neq 0\} \cap \pi_1^{-1}(\Gamma_1)$$

where π_i $(i = 1, 2)$ denotes the projection onto the factors of the product $B_1(VN(G)) \times \mathcal{F}(G)$. By Step 1, Γ_1 is analytic and thus so is $\pi_1^{-1}(\Gamma_1)$ since π_1 is Borel. By Step 2, and since the condition $S \neq 0$ is Borel, P is analytic. Again, since π_2 is Borel, it follows that $\pi_2(P)$ is analytic. Note that :

$$\pi_2(P) = \mathcal{M}(G)$$

and we can thus conclude that $\mathcal{U}(G)$ is coanalytic. ∎

Theorem 5.39. *Let G be a locally compact second countable group. Then, $\mathcal{U}_0(G) \subseteq \mathcal{F}(G)$ is coanalytic.*

Proof. For the sake of readability, we divide the proof in 3 steps :

Step 1 : Since the ball $B_1(VN(G))$ is w^*-closed by Banach-Alaoglu Theorem, it follows that $P := \lambda^{-1}(B_1(VN(G))) \subseteq (B_1(M(G)), w^*)$ is closed and thus, Polish. Consequently, $\lambda(P) \subseteq (B_1(VN(G)), w^*)$ is analytic. By the proof of Theorem 5.38, $C_r^*(G) \cap B_1(VN(G))$ is also analytic. Since the intersection of two analytic sets is analytic, the following set is also analytic in $(B_1(VN(G)), w^*)$:

$$A := \lambda(P) \cap C_r^*(G) \cap B_1(VN(G)) \setminus \{0\}$$

Consequently, the following set is analytic in $(B_1(M(G)), w^*)$:

$$\Lambda_1 := \lambda^{-1}(A) = \{\mu \in B_1(M(G)) : \lambda(\mu) \in C_r^*(G) \cap B_1(VN(G)) \setminus \{0\}\}$$

Step 2 : Consider $(B_1(M(G)), w^*)$ and $\mathcal{F}(G)$ with the Fell topology and let :

$$\Lambda_2 = \{(\mu, E) : \mathit{supp}^*(\lambda(\mu)) \subseteq E\}$$

where π_i (for $i = 1, 2$) is the projection onto the factors of $B_1(M(G)) \times \mathcal{F}(G)$. The proof that Λ_2 is closed, hence analytic, goes exactly as in the proof that Γ_2 is closed in Theorem 5.38. Indeed, as pointed out earlier, $\mu_n \xrightarrow{w^*} \mu$ implies that $\lambda(\mu_n) \xrightarrow{w^*} \lambda(\mu)$ and we can reproduce the argument used in the proof of Theorem 5.38.

Step 3 : Define $P \subseteq B_1(M(G)) \times \mathcal{F}(G)$ as the following subset :

$$P = \{(\mu, E) : \mu \in M(E),\ \lambda(\mu) \in C_r^*(G) \cap B_1(VN(G)),\ \lambda(\mu) \neq 0\}$$

Since $P = \pi_1^{-1}(\Lambda_1) \cap \Lambda_2$ it follows by Step 1 and Step 2 that P is analytic. Thus, to prove that $\mathcal{U}_0(G)$ is coanalytic it is enough to show that $\mathcal{M}_0(G) = \pi_2(P)$:
On one hand, if $E \in \pi_2(P)$, then there is a measure $\mu \in B_1(M(G))$ such that $(\mu, E) \in P$. By definition of P, this means that $\lambda(M(E)) \cap C_r^*(G) \neq \{0\}$ and thus, E is a M_0-set.
Conversely, suppose that $E \subseteq G$ is a M_0-set and thus there is some $\mu \in M(E)$ such that $\lambda(\mu) \in C_r^*(G) \setminus \{0\}$. In particular, $\mu \neq 0$. Then, $\tilde{\mu} = \frac{\mu}{||\mu||} \in B_1(M(G)) \cap M(E)$ and $T := \lambda(\tilde{\mu}) \in C_r^*(G) \setminus \{0\}$. If $||T|| \leq 1$, then $\lambda(\tilde{\mu}) \in C_r^*(G) \cap B_1(VN(G))$. Otherwise, we consider $\mu' = \frac{\tilde{\mu}}{||T||}$ and $\lambda(\mu')$. In any case, there is a measure $\mu \in B_1(M(G)) \cap M(E)$ such that $\lambda(\mu) \in C_r^*(G) \cap B_1(VN(G))$ and $\lambda(\mu) \neq 0$. Thus, $E \in \pi_2(P)$ ■

For the sake of completeness we provide a proof that $\mathcal{U}(\mathbb{T}) \subseteq \mathcal{K}(\mathbb{T})$ is Π_1^1-complete - fact on which we will further rely on - as given in [47]. It should be stressed out that a different proof of this fact (in fact of a more general statement), using the techniques presented in the next subsection, was given in [17]. We use the following result, where $\mathbb{Q}'$ denotes $\mathbb{Q} \cap [0, 1]$:

Proposition 5.40. The set $\mathcal{K}(\mathbb{Q}') = \{K \in \mathcal{K}([0, 1]) : K \subseteq \mathbb{Q}'\}$ is Π_1^1-complete.

Proof. For the sake of readability we divide the proof in steps :

Step 1 : $\mathcal{K}(\mathbb{Q}')$ is coanalytic. Indeed, let $N = [0, 1] \setminus \mathbb{Q}'$ and define :

$$G = \{(K, x) \in \mathcal{K}([0, 1]) \times [0, 1] : x \in K \cap N\}$$

Let π_1 and π_2 be, respectively, the projections of $\mathcal{K}([0, 1]) \times [0, 1]$ onto $\mathcal{K}([0, 1])$ and $[0, 1]$ and $C = \{(K, x) \in \mathcal{K}([0, 1]) \times [0, 1] : x \in K\}$. Since C is closed[13] and N is G_δ it follows that $G = \pi_2^{-1} \cap C$ is Polish. Consequently, $\mathcal{K}([0, 1]) \setminus \mathcal{K}(\mathbb{Q}') = \pi_1(G)$ is analytic. Hence, $\mathcal{K}(\mathbb{Q}')$ is coanalytic.

Step 2 : Suppose that $F \subseteq 2^\omega$ is a F_σ set and that $E \subseteq 2^\omega$ is the set of eventually periodic sequences. By Theorem 3.45, either $F \leq_W E$ or $E \leq_W 2^\omega \setminus F$.[14]

[13]It is a well known result that if X is metrizable, then $\{(x, K) \in X \times \mathcal{K}(X) : x \in K\}$ is closed (cf. [45], Exercise 4.29).

[14]Recall that if X is a zero dimensional Polish space and $A \subseteq X$, then there is some $B \subseteq \omega^\omega$ such that $A \equiv_W B$ (cf. section 3.2.1).

However, if there is a continuous map f with $E = f^{-1}(2^\omega \setminus E)$ it follows that E is a dense G_δ set. On the other, and since E is countable, its complement is also a dense G_δ set which contradicts Baire Category theorem. Thus, we conclude that $F \leq_W E$ necessarily. Let $\Theta = \mathcal{K}(E)$ and note that, similarly as in Step 1, this is a coanalytic subset of 2^ω. After seeing WF as a Π^1_1-complete set of 2^ω, it follows that there is some F_σ set $F \subseteq 2^\omega \times 2^\omega$ such that :

$$x \in WF \Leftrightarrow \forall y \in 2^\omega : (x, y) \in F$$

Consider a continuous function f such that $F = f^{-1}(E)$ and define $\Psi : 2^\omega \to \mathcal{K}(2^\omega)$ as $\Psi(x) := f(\{x\} \times 2^\omega)$. As WF is Π^1_1-complete, this proves that so is Θ.

Step 3 : Define $f : 2^\omega \to [0,1]$ by $x \mapsto \sum_{n=0} \frac{x(n)}{2^{n+1}}$. Note that $x \in E$ if and only if $f(x) \in \mathbb{Q}'$. Let $F : \mathcal{K}(2^\omega) \to \mathcal{K}([0,1])$ be prescribed by $K \mapsto f(K)$. Note that F is continuous [15] and that $K \in \mathcal{K}(E)$ if and only $F(K) \in \mathcal{K}(\mathbb{Q}')$. By Step 2, we conclude that $\mathcal{K}(\mathbb{Q}')$ is Π^1_1-complete. ■

We can now prove the complexity results. First for $\mathbb{T}$ and thereafter for a more general case.

Theorem 5.41. *The set* $\mathcal{U}(\mathbb{T}) \subseteq \mathcal{K}(\mathbb{T})$ *is* Π^1_1*-complete.*

Proof. By Theorem 5.40 it is enough to define a continuous map $F : \mathcal{K}([0,1]) \to \mathcal{K}(\mathbb{T})$ such that $\mathcal{K}(\mathbb{Q}') = F^{-1}(\mathcal{U}(\mathbb{T}))$. We start by defining a continuous map $f : [0,1] \to \mathcal{K}(\mathbb{T})$ prescribed as follows :

$$x \mapsto E(\frac{1}{4}, \frac{3}{8} + \frac{x}{9}, \frac{3}{4})$$

It follows by Salem-Zygmund theorem (cf. Example 5.4) that $\mathbb{Q} = f^{-1}(\mathcal{U}(\mathcal{K}(\mathbb{T})))$. Now, we define a map $F : \mathcal{K}([0,1]) \to \mathcal{K}(\mathbb{T})$ as follows :

$$K \mapsto \bigcup_{x \in K} f(x)$$

By the properties of the Vietoris topology, F is continuous. Moreover, by Bari's theorem (cf. Theorem 5.36), $F^{-1}(\mathcal{U}(\mathbb{T})) = \mathcal{K}(\mathbb{Q}')$ as we wanted. ■

Remark 5.42. The proof of Salem-Zygmund theorem given in [47] (Theorem 1 - III.4) actually shows a stronger statement : if $\frac{1}{\xi}$ is not a Pisot number or some $\eta_i \notin \mathbb{Q}(\frac{1}{\xi})$, then $E(\xi, \eta_1, ..., \eta_k)$ is a set of restricted multiplicity (cf. [47], Theorem 4 - III.4). Since sets of uniqueness are also sets of extended uniqueness, the proof of Theorem 5.41 also shows the following :

Theorem 5.43. *The set* $\mathcal{U}_0(\mathbb{T}) \subseteq \mathcal{K}(\mathbb{T})$ *is* Π^1_1*-complete.*

Relying on Theorems 5.41 and 5.43 and on some of the previously established preservation properties of U-sets and U_0-sets, one proves the following :

[15] It is a well known result that if X, Y are metrizable and $f : X \to Y$ is continuous, then the function $g : \mathcal{K}(X) \to \mathcal{K}(Y)$ defined by $C \mapsto f(C)$ is continuous (cf. [45], Exercise 4.29).

Theorem 5.44. *Let G be a connected locally compact Lie group. Then, $\mathcal{U}(G) \subseteq \mathcal{F}(G)$ is Π_1^1-complete.*

Proof. By Theorem 5.38, it is enough to prove that $\mathcal{U}(G)$ is Π_1^1-hard. Therefore, we divide the analysis in two cases :

Case 1 : Suppose that G is abelian and thus, $G \approx \mathbb{R}^n \times \mathbb{T}^m$. By Corollary 5.22 it suffices to show that $\mathcal{U}(\mathbb{R}^n \times \mathbb{T}^m)$ is Π_1^1-hard. Firstly, note that for any positive integer k, the function $g : \mathcal{F}(\mathbb{T}) \to \mathcal{F}(\mathbb{T}^k)$ such that $E \mapsto E^k$ is measurable and, by Theorem 5.17, is such that $\mathcal{U}(\mathbb{T}) = g^{-1}(\mathcal{U}(\mathbb{T}^k))$. Thus, $\mathcal{U}(\mathbb{T}^k)$ is Π_1^1-hard. Now, let $q : \mathbb{R}^n \times \mathbb{T}^m \to \mathbb{T}^{n+m}$ be the quotient map and $f : \mathcal{F}(\mathbb{T}^{n+m}) \to \mathcal{F}(\mathbb{R}^n \times \mathbb{T}^m)$ be given by $E \mapsto q^{-1}(E)$.
Note that this map is measurable. Indeed, if $A = \{F \in \mathcal{F}(\mathbb{R}^n \times \mathbb{T}^m) : F \cap \mathcal{U} \neq \emptyset\}$, then $f^{-1}(A) = \{F \in \mathcal{F}(\mathbb{T}^{n+m}) : F \cap q(\mathcal{U}) \neq \emptyset\}$. Since q is the quotient map by an action of a subgroup, $q(\mathcal{U})$ is open for each open set $\mathcal{U}$ and we may conclude that f is measurable. Furthermore, by Corollary 5.21 one has that $E \in \mathcal{U}(\mathbb{T}^{n+m})$ if and only if $q^{-1}(E) \in \mathcal{U}(\mathbb{R}^n \times \mathbb{T}^m)$ and we are done.

Case 2 : To consider the non-abelian case, note that $G/\overline{[G,G]}$ is a connected abelian Lie group and thus, by Step 1, $\mathcal{U}(G/\overline{[G,G]})$ is Π_1^1-hard and we use the same argument as before with the quotient map $q : G \to G/\overline{[G,G]}$. ■

Theorem 5.45. *Let G be a connected abelian Lie group. Then, $\mathcal{U}_0(G) \subseteq \mathcal{F}(G)$ is Π_1^1-complete.*

Proof. By Theorem 5.39 it is enough to prove that $\mathcal{U}_0(G)$ is Π_1^1-hard. For the sake of readability we divide the proof in 3 steps :

Step 1 : By Corollary 5.19 and considering the map $h : \mathcal{F}(\mathbb{T}) \mapsto \mathcal{F}(\mathbb{T}^k)$ given by $E \mapsto E^k$ we conclude that $\mathcal{U}_0(\mathbb{T}^k)$ is Π_1^1-hard.

Step 2 : Let G_1 and G_2 be locally compact abelian groups, $\varphi : G_1 \to G_2$ be an isomorphism (of topological groups) and $\mu \in M(G_1)$. This induces a pushforward measure $\varphi_*\mu$ on G_2 and an isomorphism $\varphi_* : \hat{G}_1 \to \hat{G}_2$ given by $\chi \mapsto \chi \circ \varphi^{-1}$ such that :

$$\hat{\mu}(\chi) = \int_{G_1} \overline{\chi(x)} d\mu(x) = \int_{G_2} \overline{\varphi_*(\chi)(y)} d\varphi_*\mu(y) = \widehat{\varphi_*\mu}(\varphi_*(\chi))$$

Hence, if $\hat{\mu} \in C_0(\hat{G_1})$ it follows that $\widehat{\varphi_*\mu} \in C_0(\hat{G}_2)$. If $E \subseteq G_1$ is a M_0-set there is some $\mu \in M(E)$ such that $\mu \neq 0$ is Rajchman and thus, $\varphi_*(\mu) \in M(\varphi(E))$ and is a non zero Rajchman measure implying that $\varphi(E)$ is also a M_0-set. Analogously, we may conclude that $E \subseteq G_1$ is a M_0-set if and only if $\varphi(E)$ is a M_0-set.

Step 3 : Any such G is isomorphic to $\mathbb{R}^n \times \mathbb{T}^m$ for some integers $n, m \geq 0$ and consequently, by Step 2, it suffices to prove that $\mathcal{U}_0(\mathbb{R}^n \times \mathbb{T}^m)$ is Π_1^1-hard. In order to do so, we appeal to Corollary 5.29 and consider the Borel reduction

$f : \mathcal{F}(\mathbb{T}^{n+m}) \to \mathcal{F}(\mathbb{R}^n \times \mathbb{T}^m)$ given by $E \mapsto q^{-1}(E)$, where $q : \mathbb{R}^n \times \mathbb{T}^m \to \mathbb{T}^{n+m}$ is the quotient map. ■

We conclude the section with a generalization of Theorem 5.44. In order to do so, we recall once again that in [80] it was proven that $\mathcal{U}(G)$ is Π_1^1-complete whenever G is a non-discrete second countable locally compact abelian group. Thus, repeating the argument given in the proof of Theorem 5.44 - applied to the quotient map $q : G \to G/\overline{[G,G]}$ we easily conclude that :

Theorem 5.46. *Let G be a locally compact Polish group such that $G/\overline{[G,G]}$ is non-discrete. Then, $\mathcal{U}(G)$ is Π_1^1-complete.*

5.3.2 Borel bases : U-sets

Debs and Saint-Raymond proved in [17] that $\mathcal{U}(\mathbb{T})$ does not have a Borel basis, a result which is generalized by Matheron in [56] : if G is a locally compact abelian second countable group, then $\mathcal{U}(G)$ does not have a Borel basis. In particular, $\mathcal{U}(G)$ is not Borel and, if G is compact, we recover the complexity result proven in [80] for the abelian case. Furthermore, this provides a deep insight on the question regarding the characterization of the set of (closed) sets of uniqueness. Indeed, there is no *reasonable* (Borel, hereditary) subset $\mathcal{B}$ of $\mathcal{U}(G)$ for which one can express every closed set of uniqueness as a countable union of elements in $\mathcal{B}$.
In this subsection we include a sketch of the proof of Matheron's result (as given in [56]) and prove three sufficient conditions for the non-existence of a Borel basis for $\mathcal{U}(G)$ - for G not necessarily abelian. The reader who is unfamiliar with the terminology is referred to section 3.5.

We start by recalling that $\mathcal{U}(G)$ is a σ-ideal of $\mathcal{F}(G)$ whenever G is a locally compact amenable group (cf. Corollary 5.32).

Proposition 5.47. Let G be a locally compact abelian second countable non-discrete group. Then, the σ-ideal $\mathcal{U}(G)$ is calibrated.

Proof. It is enough to prove that if $\{E_n\} \subseteq \mathcal{U}(G)$ and $E \in \mathcal{F}(G)$ is such that $E \setminus \bigcup_n E_n \in \mathcal{U}(G)^{\text{int}}$ then $E \in \mathcal{U}(G)$. Suppose, by contradiction, that $E \notin \mathcal{U}(G)$ and let $S \in PF(G)$ be such that $S \neq 0$ and is supported by E. By Proposition 2.1 in [56] there is some $T \neq 0$ such that $\text{supp}(T) \subseteq E \setminus \bigcup_n E_n$ and thus, $\text{supp}(T)$ is a M-set which is a contradiction. ■

Proposition 5.48. Let G be a locally compact abelian second countable non-discrete group and $E \in \mathcal{M}(G)$. Then, $\mathcal{U}(G) \cap \mathcal{F}(E)$ is Π_1^1-complete.

Proof. The reader can find a proof in [56] (Corollaire 2.3). ■

Remark 5.49. A proof of Proposition 5.48 using different arguments can be found in [47] (Theorem 2-VII.2) for the case $G = \mathbb{T}$: if $E \in \mathcal{M}(\mathbb{T})$, there is an unbounded rank on $\mathcal{U}(\mathbb{T}) \cap \mathcal{K}(\mathbb{T})$ and thus, it follows by Theorem 3.36 that $\mathcal{U}(\mathbb{T}) \cap \mathcal{K}(\mathbb{T})$ is Π_1^1-complete.

Proposition 5.50. Let G be a locally compact abelian second countable non-discrete group. There is a G_δ-set of G such that all its closed subsets are in $\mathcal{U}(G)$ but it can't be covered by countably many elements in $\mathcal{U}(G)$.

Proof. By a result of Saeki (cf. [71]) every M-set contains a Helson set of multiplicity and by Lemme 7 in [56] if $E \subseteq G$ is a Helson set, then there is a G_δ-set $F \subseteq E$ dense in E and such that all its closed subsets are U-sets. We can thus choose E to be a Helson set of multiplicity and assume that $E \in \mathcal{U}(G)^{\text{perf}}$. Finally, suppose by contradiction that $F \subseteq \bigcup_n E_n$ with $E_n \in \mathcal{U}(G)$. Since $E \in \mathcal{U}(G)^{\text{perf}}$ it follows that each E_n is meager in E and consequently so is F which contradicts the Baire Category theorem. ∎

We can finally conclude the sketch of the proof of inexistence of Borel bases for $\mathcal{U}(G)$ as in [56] :

Theorem 5.51. *Let G be a non-discrete second countable locally compact abelian group. Then, $\mathcal{U}(G)$ does not have a Borel basis.*

Proof. The result is immediate from Theorem 3.55, Proposition 5.47, Proposition 5.48 and Proposition 5.50. ∎

Remark 5.52. Note that if G is also compact, it follows from Theorem 5.51 and Theorem 3.48 that $\mathcal{U}(G)$ is Π^1_1-complete.

In the remaining of this section we identify sufficient conditions on which the set of closed sets of uniqueness of a not necessarily abelian group does not admit a Borel basis, based on insofar its quotients or open subgroups admit such basis or not. Thereafter, we prove a sufficient condition for the inexistence of Borel basis for groups which are a product of a certain form. An useful result on bases and pre-bases is the following :

Proposition 5.53. Let $I \subseteq \mathcal{F}(X)$ be a σ-ideal. Then :

(i) I has a Σ^1_1 basis if and only if I has a Σ^1_1 pre-basis.

(ii) If I is Π^1_1 and has a Σ^1_1 basis, then I has a Borel basis.

Proof. The reader can find a proof in [57] (Proposition 1.11). ∎

We will need the following well-known properties of the support of $T \in VN(G)$:

Proposition 5.54. Let G be a locally compact group, $T, T_1, T_2 \in VN(G)$ and $u \in B(G)$. Then :

(i) If $T = \lambda(\mu)$ for some $\mu \in M(G)$, then $\text{supp}(T) = \text{supp}(\mu)$.

(ii) If T_1 or T_2 have compact support then $\text{supp}(T_1T_2) \subseteq \text{supp}(T_1)\text{supp}(T_2)$.

(iii) $\text{supp}(u.T) \subseteq \text{supp}(u) \cap \text{supp}(T)$.

Proof. The reader can find a proof in [24] (Remarque 4.7 and Proposition 4.8, respectively). ∎

Proposition 5.55. Let G be a locally compact group, $g \in G$ and $E \subseteq G$ a closed subset. Then, if E is a U-set so is $gE := \{gx : x \in E\}$.

Proof. Suppose that gE is not a U-set and let $T \in J(gE)^{\perp} \cap C_r^*(G)$ such that $T \neq 0$. Consider $\tilde{T} := \lambda(\delta_{g^{-1}})T \in C_r^*(G)$ and note that by Prop. 5.54 one has that $\mathrm{supp}(\tilde{T}) \subseteq g^{-1}(gE) = E$. Moreover, $\tilde{T} \neq 0$, since $T \neq 0$ and $\lambda(\delta_{g^{-1}})$ is unitary. Thus, $\tilde{T} \in J(E)^{\perp} \cap C_r^*(G) \setminus \{0\}$ and it follows that E is not a U-set. ■

Theorem 5.56. *Let G be a locally compact second countable amenable group and H be a countable closed subgroup such that :*

(i) The quotient map $q : G \to G/H$ is a closed map.

(ii) There is no Borel basis for $\mathcal{U}(G/H)$.

Then, $\mathcal{U}(G)$ does not have a Σ_1^1 pre-basis. In particular, $\mathcal{U}(G)$ does not have a Borel basis and consequently, if G is also compact then $\mathcal{U}(G)$ is Π_1^1-complete.

Proof. Suppose that $\mathcal{U}(G)$ has a Σ_1^1 pre-basis $\mathcal{B}$. We prove that in this case, $\tilde{\mathcal{B}} := \{q(B) : B \in \mathcal{B}\}$ is a Σ_1^1 pre-basis for $\mathcal{U}(G/H)$. It follows by Proposition 5.53 that $\mathcal{U}(G/H)$ has a Borel basis which is a contradiction. Indeed, consider the map :

$$f : \mathcal{F}(G) \to \mathcal{F}(G/H),\ E \mapsto q(E)$$

Note that since q was assumed to be a closed map, f is a well-defined Borel map and moreover, since $\mathcal{B}$ was assumed to be Σ_1^1, it follows that $\tilde{\mathcal{B}}$ is Σ_1^1.[16] It remains to show that $\tilde{\mathcal{B}}$ is a pre-basis for $\mathcal{U}(G/H)$:
(i) Each $q(B) \in \mathcal{U}(G/H)$: by Corollary 5.21, $q(B) \in \mathcal{U}(G/H)$ if and only if $q^{-1}(q(B)) \in \mathcal{U}(G)$. Note that $q^{-1}(q(B)) = \bigcup_{h \in H} hB$ and that since each $B \in \mathcal{U}(G)$ it follows from Proposition 5.55 that $hB \in \mathcal{U}(G)$ for each $h \in H$. Since q was assumed to be closed, $q^{-1}(q(B))$ is closed and thus, since we assume that H is countable, it follows from Theorem 5.31 that $q^{-1}(q(B))$ is a U-set, as required.
(ii) We now check that $\mathcal{U}(G/H) = \tilde{\mathcal{B}}_\sigma$: let $E \in \mathcal{U}(G/H)$ so that by Corollary 5.21, $q^{-1}(E) \in \mathcal{U}(G)$. Since $\mathcal{B}$ is a pre-basis it follows that $q^{-1}(E) \subseteq \bigcup_n B_n$ for some $\{B_n\} \subseteq \mathcal{B}$. Since q is surjective, $q(q^{-1}(E)) = E \subseteq \bigcup_n q(B_n)$ and we are done. ■

Example 5.57. Theorem 5.56 and Theorem 5.51 imply that, for instance, $\mathcal{U}(G \times D)$ - where G is any compact non-discrete second countable abelian group and D any non-abelian compact countable group - does not have a Borel basis and thus, is Π_1^1-complete. Indeed, recall that for general topological groups, G/H is discrete if and only if $H \subseteq G$ is open and thus, one considers the commutator $[G \times D, G \times D]$ as the subgroup H.

[16] If X, Y are Polish and $f : X \to Y$ is continuous then $F \mapsto \overline{f(F)}$ is a Borel map from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ (cf. Exercise 12.11 in [70]).

Now let G be a locally compact group and consider $H \subseteq G$ to be an open subgroup, so that one can consider the (non-zero) Haar measure on H induced by the Haar measure on G. We denote by f° the trivial extension from H to G of a function defined on H. It follows that given $T \in VN(G)$ and $f \in L^2(H)$, if we define $T|_H(f) := T(f^\circ)|_H$ then $T|_H : f \mapsto T|_H(f)$ is an element of $VN(H)$. Furthermore, we define the restriction map $r : u \mapsto u|_H$, for $u \in A(G)$.

Proposition 5.58. Let H be an open subgroup of a locally compact group G. Then :

(i) The map $\phi : u \mapsto u^\circ$ is an isometric isomorphism of $A(H)$ into $A(G)$ and the map $T \mapsto T|_H$ is its the adjoint map ϕ^*. Moreover :

$$\phi^*(VN(G)) = VN(H) \text{ and } \phi^*(C_r^*(G)) = C_r^*(H)$$

(ii) The restriction map r maps $A(G)$ onto $A(H)$ and its adjoint map r^* is an isomorphism onto its image such that $r^*(C_r^*(H)) \subseteq C_r^*(G)$

Proof. The reader can find a proof in [41] (Proposition 2.4.1). ■

Theorem 5.59. *Let G be a locally compact group and $H \subseteq G$ be an open subgroup.*[17] *Then :*

(i) If $E \in \mathcal{U}(G)$, then $E \cap H \in \mathcal{U}(H)$.

(ii) If $E \in \mathcal{U}(H)$, then $E \in \mathcal{U}(G)$.

Proof. (i) Suppose that $E \cap H$ is not a U-set of H and thus, there is some $T \in C_r^*(H) \cap J(E \cap H)^\perp$ such that $T \neq 0$. By Proposition 5.58, $r^*(T) \neq 0$ is an element of $C_r^*(G)$. Moreover, $r^*(T) \in J(E)^\perp$. Indeed, let $u \in J(E)$ so that $u|_H \in J(E \cap H)$ and $\langle T, u|_H \rangle = \langle r^*(T), u \rangle = 0$. Thus, E is not a U-set of G.
(ii) Suppose that E is not a U-set of G and let $T \in C_r^*(G) \cap J(E)^\perp$ such that $T \neq 0$. By Proposition 5.58, $T|_H \in C_r^*(H)$. We note that if $h \in \text{supp}(T) \subseteq E \subseteq H$, then $h \in \text{supp}(T|_H)$ and since $T \neq 0$, it follows that $T|_H \neq 0$. Indeed, suppose towards a contradiction that there is some $u \in A(H)$ such that $u(h) \neq 0$ and that $u.T|_H = 0$. Then, $\langle T|_H, uv \rangle = 0$ for all $v \in A(H)$ and thus, $\langle T, u^\circ w \rangle = 0$ for all $w \in A(G)$. Consequently, $u^\circ.T = 0$ which is impossible since $u^\circ(h) \neq 0$. ■

Theorem 5.60. *Let G be a locally compact amenable group and H be an open subgroup such that $\mathcal{U}(H)$ does not have a Borel basis. Then, $\mathcal{U}(G)$ does not have a Borel basis and if G is also compact, then $\mathcal{U}(G)$ is Π_1^1-complete.*

Proof. Suppose that $\mathcal{B}$ is a Borel basis for $\mathcal{U}(G)$. By Proposition 5.53 it is enough to show that $\tilde{\mathcal{B}} = \{B \cap H : B \in \mathcal{B}\}$ is a Σ_1^1 pre-basis for $\mathcal{U}(H)$. Note that $\tilde{\mathcal{B}}$ is Σ_1^1 since H is a fixed closed subset of G. Furthermore, and since H is open, $\tilde{\mathcal{B}}$ is indeed a pre-basis :
(i) By Theorem 5.59, $\tilde{\mathcal{B}} \subseteq \mathcal{U}(H)$.
(ii) Let E be a U-set of H. By Theorem 5.59, E is also a U-set of G and thus, $E \subseteq \bigcup_n B_n$ for $\{B_n\} \subseteq \mathcal{B}$. It is then enough to note that $E = E \cap H \subseteq \bigcup_n B_n \cap H$ and we conclude that $\mathcal{U}(H) = \tilde{\mathcal{B}}_\sigma$. ■

[17] Recall that an open subgroup is also closed.

Example 5.61. Recall that if G is a topological group, then the identity component G^0 is open if and only if G is locally connected. In particular, this holds for any Lie group G. Furthermore, if G is a compact, nilpotent Lie group then G^0 is an abelian normal subgroup. Hence, it follows from Theorem 5.60 that $\mathcal{U}(G)$ does not have a Borel basis (and in particular is Π^1_1-complete) if G is a compact, nilpotent Lie group such that G^0 is not discrete.

Finally, we prove a sufficient condition for the non existence of a Borel basis for $\mathcal{U}(G)$ whenever G is a product of a certain form. We follow closely the proof that $\mathcal{U}(\mathbb{T})$ does not have a Borel basis as given in [47], [17] and [46].

Henceforth, until the end of this subsection, we consider $G = G_1 \times ... \times G_n$ such that each factor G_i is a compact and second countable group verifying the following properties :

(1) For every $i \in \{1, ..., n\}$ there is a closed set $E_i \subseteq G_i$ such that $E_i \notin \mathcal{U}(G_i)^{\mathrm{loc}}$ and a G_δ-set $F_i \subseteq E_i$, dense in E_i and such that $F_i \in \mathcal{U}(G_i)^{\mathrm{int}}$.

(2) There is some $N \in \{1, ..., n\}$ such that for and every M-set $E \subseteq G_N$, then $\mathcal{U}(G_N) \cap \mathcal{F}(E)$ is not Borel.

We note that G is compact and consequently, $\mathcal{U}(G)$ is calibrated :

Proposition 5.62. Let G be a Polish amenable group, not necessarily abelian. Then, the coanalytic σ-ideal $\mathcal{U}(G)$ is calibrated.

Proof. For the sake of readability we divide the proof in three steps :

Step 1 : Let $Z \subseteq B(G)$ be a convex subset and for $S \in C^*_r(G)$ define $Z \cdot S = \{u \cdot S : u \in Z\}$. Then, $\overline{Z}^{w^*} \cdot S \subseteq \overline{Z \cdot S}$. Indeed, suppose that $Z \cdot S \subseteq C^*_r(G)$ so that $\overline{Z \cdot S}$ is a closed convex subset of $C^*_r(G)$, the predual of $B(G)$. Let $T \in C^*_r(G) \setminus \overline{Z \cdot S}$. By a version of Hahn-Banach Theorem[18] there is some $v \in B(G)$ such that :

$$\Re(\langle T, v\rangle) > s := \sup\{\Re(\langle u \cdot S, T\rangle) : u \in Z\}$$

Now $\{u \in B(G) : \Re(\langle v, u{\cdot}S\rangle) \leq s\}$ is w^*-closed subset of $B(G)$ containing Z and thus, $\overline{Z}^{w^*}$. Hence, $T \notin \overline{Z}^{w^*} \cdot S$ as long as we check that $Z \cdot S \subseteq C^*_r(G)$. Indeed, for any $u \in B(G)$ and $T = \lambda_G(\mu)$ with $\mu \in M(G)$ one has that $u \cdot T = \lambda_G(u\mu)$ (cf. [41], Rem. 2.5.2). Consequently, $\Phi_u : VN(G) \to VN(G)$ given by $T \mapsto u{\cdot}T$ is a bounded map such that $\Phi_u(\lambda_G(C_c(G))) \subseteq C^*_r(G)$ from where it follows that $\Phi_u(C^*_r(G)) \subseteq C^*_r(G)$ as we wanted.

Step 2 : Let $S \in C^*_r(G), E \in \mathcal{U}(G)$ and $\epsilon > 0$. Then, there is some $T \in C^*_r(G)$ such that $||T - S|| < \epsilon$ and $\mathrm{supp}(T) \subseteq \mathrm{supp}(S) \setminus E$. Indeed, if E is a U-set, then

[18] Let C be a non-empty closed convex subset of a Banach space X and let $x_0 \in X \setminus C$. Then, there is some $x^* \in X^*$ such that $\sup_{x \in C} \Re(\langle x, x^*\rangle) < \Re(\langle x_0, x^*\rangle)$.

$\overline{J(E)}^{w^*} = B(G)$ and consequently, applying Step 1 to $Z = J(E)$, we conclude that $S \in \overline{J(E) \cdot S}$. Hence, there is some $u \in J(E)$ such that by Proposition 5.54 $T := u \cdot S$ has the required properties.

Step 3 : Let $\{E_n\} \subseteq \mathcal{U}(G)$ and $E \in \mathcal{F}(G)$ such that $\mathcal{F}(E \setminus \bigcup_n E_n) \subseteq \mathcal{U}(G)$. We want to prove that $E \in \mathcal{U}(G)$. Suppose not, so that there is some $S \in J(E)^{\perp} \cap C_r^*(G)$ such that $S \neq 0$. Without loss of generality, assume $||S|| = 1$. Appealing to Step 2, one defines by induction (starting with $S_0 = S$) a sequence $\{S_n\} \subseteq C_r^*(G)$ such that $||S_{n+1} - S_n|| \leq 2^{-(n+2)}$ and $\operatorname{supp}(S_{n+1}) \subseteq \operatorname{supp}(S_n) \setminus E_n$. We let $T := \lim_n S_n$ and note that by construction, $T \neq 0$ and $\operatorname{supp}(T) \subseteq E \setminus \bigcup_n E_n$. However, this contradicts the assumption that $E \setminus \bigcup_n E_n \in \mathcal{U}^{\text{int}}$ and we conclude that $\mathcal{U}(G)$ is indeed calibrated. ■

In particular $G = \mathbb{T}^n$ is of this form. Indeed Propositions 5.48 and 5.50 imply that the required conditions hold. We note that the content which follows, provides an alternative proof of the inexistence of Borel basis for $\mathcal{U}(\mathbb{T}^n)$ (and consequently for any $G \approx \mathbb{T}^n$) which in particular only relies on the existence of Helson sets of multiplicity for $\mathbb{T}$ (cf. [17] and [43]) - in order to verify property (1) for the torus. For a proof of property (2) for the case $G_i = \mathbb{T}$, the reader can find a rank-theoretic argument in [47] (Theorem VII.2.2) (cf. Remark 5.49).

Proposition 5.63. There is a closed set $E \subseteq G$ such that $E \notin \mathcal{U}(G)^{\text{loc}}$ and a G_δ-set $F \subseteq E$, dense in E and such that $F \in \mathcal{U}(G)^{\text{int}}$.

Proof. For each i, let $F_i \subseteq E_i \subseteq G_i$ be as in property (1) and consider :

$$E := \prod_{i=1}^{n} E_i \subseteq G \text{ and } F := \prod_{i=1}^{n} F_i \subseteq E$$

Note that E is a closed set, F is a G_δ-set and that since $\overline{F} = \overline{F_1} \times ... \overline{F_n} = E$, then F is dense in E. In order to check that $E \notin \mathcal{U}(G)^{\text{loc}}$ consider an open set $\mathcal{W}$ of G such that $\mathcal{W} \cap E \neq \emptyset$. Assume, without loss of generality, that $\mathcal{W} = \mathcal{W}_1 \times ... \times \mathcal{W}_n$, with each $\mathcal{W}_i$ and open set of G_i. By choice of E_i, it follows that $\overline{\mathcal{W}_i \cap E_i} \notin \mathcal{U}(G_i)$ for each i and by Corollary 5.17, $L := \prod_{i=1}^{n} \overline{\mathcal{W}_i \cap E_i} \notin \mathcal{U}(G)$. It suffices to note that $L \subseteq \overline{\mathcal{W} \cap E}$ and we can conclude that $\overline{\mathcal{W} \cap E} \notin \mathcal{U}(G)$. In order to check that $F \in \mathcal{U}(G)^{\text{int}}$ note that by property (1) if $C \subseteq F$ is any closed set, then $C_i := \pi_i(C) \in \mathcal{F}(G_i)$. By choice of F_i, each $C_i \in \mathcal{U}(G_i)$ and by Corollary 5.17, $\prod_{i=1}^{n} C_i \in \mathcal{U}(G)$. Since $C \subseteq \prod_{i=1}^{n} C_i$, we conclude that C is a U-set and we're done. ■

Theorem 5.64. *Let $E \in \mathcal{M}(G)$ be of the form $E = \prod_{i=1}^{n} E_i$. Then, $\mathcal{U}(G) \cap \mathcal{F}(E)$ is not Borel.*

Proof. Let N be as in property (2) and without loss of generality, let $N = 1$. Consider the following function after fixing $E = E_1 \times ... \times E_n$:

$$\varphi : \mathcal{F}(G_1) \to \mathcal{F}(G) \;\text{ such that }\; F \mapsto F \times E_2 \times ... \times E_n$$

Since this is a well-defined Borel map it suffices to verify that $\varphi^{-1}(\mathcal{U}(G) \cap \mathcal{F}(E)) = \mathcal{U}(G_1) \cap \mathcal{F}(E_1)$, as it follows by Corollary 5.17 that each E_i is a M-set. Let $F \in \mathcal{U}(G_1) \cap \mathcal{F}(E_1)$ so that $\varphi(F) \in \mathcal{F}(E)$. It follows from Corollary 5.17 that $\varphi(F) \in \mathcal{U}(G)$. Conversely, either F is a M-set of G_1 in which case it also follows from Corollary 5.17 that $\varphi(F) \notin \mathcal{U}(G)$ or $F \subsetneq E_1$ in which case $\varphi(F) \notin \mathcal{F}(E)$. ■

We will need a modified version of Theorem 3.55 - which we shall prove in Theorem 5.66. Its proof relies on the following important result :

Theorem 5.65. *Let X be a compact metric space and I be a σ-ideal of $\mathcal{F}(X)$. Assume that I is calibrated and has a basis $\mathcal{B}$ such that for every non-empty open set $\mathcal{V} \subseteq X$ there is some $K \subseteq \mathcal{V}$ such that $K \in I \setminus \mathcal{B}$. Then, if $A \subseteq X$ has the Baire property and $\mathcal{K}(A) \subseteq I$, then A is meager.*

Proof. The reader can find a proof in [46] (Theorem 23.2). ■

Theorem 5.66. *Let X be a compact metric space of the form $X = X_1 \times X_2$ and I be a σ-ideal of $\mathcal{K}(X)$. Assume that :*

(i) I is calibrated

(ii) I admits a basis B which is non-trivial on each closed set $L \notin I$ of the form $L = C_1 \times C_2$

Then, I has the covering property.

Proof. We prove that if there is some analytic set $A \subseteq X$ which cannot be covered by countably many elements of I, then there is some closed $K \subseteq A$ such that $K \notin I$. Note that since A is Σ^1_1, there is a G_δ-set $G \subseteq X \times \mathcal{C}$ such that $A = \pi_X(G)$. Consider the following set :

$$\tilde{G} = G \setminus \bigcup \{\mathcal{V} \subseteq X \times \mathcal{C} : \mathcal{V} \text{ is open and } \pi_X(\mathcal{V} \cap G) \text{ can be countably covered in } I\}$$

Since we assumed that such A exists, it is clear that $\tilde{G} \neq \emptyset$ and thus $F = \overline{\tilde{G}}$ is a non-empty compact. We now define the following σ-ideal J on $\mathcal{K}(F)$:

$$K \in J \Leftrightarrow \pi_X(K) \in I$$

Suppose that J verifies the conditions of Theorem 5.65 and notice that $\tilde{G}$ is not meager, by the Baire Category Theorem. Hence, it will follow that there is some closed set $L \subseteq \tilde{G}$ such that $L \notin J$, i.e. $K := \pi_X(L) \in \mathcal{K}(A) \setminus I$ and we are done. Thus, it suffices to show that the hypothesis of Theorem 5.65 hold :

(i) J is calibrated : Let $K \in \mathcal{K}(F)$, $\{K_n\} \subseteq J$ and $\mathcal{K}(K \setminus \bigcup_n K_n) \subseteq J$. Suppose, towards a contradiction, that $K \notin J$ and thus $\pi_X(K) \notin I$. Note that since each $K_n \in J$, then $\pi_X(K_n) \in I$. Hence, since I is calibrated, it follows that there is a compact $L \subseteq \pi_X(K) \setminus \bigcup_n \pi_X(K_n)$ such that $L \notin I$. But this implies that $T := K \cap (L \times \mathcal{C}) \subseteq K \setminus \bigcup_n K_n$ is such that $T \notin J$, which is a contradiction.

(ii) Consider $D = \{K \in \mathcal{K}(F) : \pi_X(K) \in B\}$ and note that D is a basis for J. Now consider any open set $\mathcal{V}$ which intersects F. We shall prove that $J \cap \mathcal{K}(\overline{\mathcal{V} \cap F}) \neq D \cap \mathcal{K}(\overline{\mathcal{V} \cap F})$ and thus conclude that $J \cap \mathcal{K}(\mathcal{U}) \neq D \cap \mathcal{K}(\mathcal{U})$ for all non-empty open sets $\mathcal{U}$ in F establishing that D is a non-trivial basis. So let's fix an open set $\mathcal{V}$ which intersects F. Since X and $\mathcal{C}$ are metric, one can find another open set $\mathcal{W}$ intersecting F and such that : $\overline{\pi_X(\mathcal{W} \cap F)} \subseteq \overline{\pi_{X_1}(\mathcal{W} \cap F)} \times \overline{\pi_{X_2}(\mathcal{W} \cap F)} \subseteq \overline{\pi_X(\mathcal{V} \cap F)}$. By definition of $\tilde{G}$, $\pi_X(\mathcal{W} \cap \tilde{G})$ can't be countably covered by elements in I, otherwise $\pi_X(\mathcal{W} \cap G)$ can be so covered which implies that $\mathcal{W} \cap \tilde{G} = \emptyset$, contradicting the density of $\tilde{G}$ in F. Hence, $\overline{\pi_X(\mathcal{W} \cap F)} \notin I$. Since I is an ideal, $L := \overline{\pi_{X_1}(\mathcal{W} \cap F)} \times \overline{\pi_{X_2}(\mathcal{W} \cap F)} \notin I$. By condition (ii) in our hypothesis, there is some $K \in \mathcal{K}(L)$ such that $K \in I \setminus B$. By compactness, $\overline{\pi_X(\mathcal{V} \cap F)} = \pi_X(\overline{\mathcal{V} \cap F})$ and thus, considering $C := (K \times \mathcal{C}) \cap \overline{\mathcal{V} \cap F}$ one has that $\pi_X(C) = K$ and thus, $C \notin D$ and $C \in J$. Furthermore, since $K \subseteq L \subseteq \pi_X(\overline{\mathcal{V} \cap F})$, we have that $C \in \mathcal{K}(\overline{\mathcal{V} \cap F})$. ■

Theorem 5.67. *The σ-ideal $\mathcal{U}(G) \subseteq \mathcal{F}(G)$ has no Borel basis.*

Proof. Suppose, by contradiction, that there is a Borel basis for $\mathcal{U}(G)$. Since $\mathcal{U}(G)$ is calibrated it follows from Theorem 5.64 and Theorem 5.66 that $\mathcal{U}(G)$ has the covering property. But this is impossible by Theorem 5.63, in an entirely analogous way to the argument given in the proof of Proposition 5.50. ■

5.3.3 Borel bases : U_0-sets

A substantial structural difference between $\mathcal{U}(\mathbb{T})$ and $\mathcal{U}_0(\mathbb{T})$ was established in [17] : while the former does not have a Borel basis, the latter does. In turn, this result synthesises in itself important cornerstone moments of the classic theory of sets of uniqueness. As such, and for the sake of completeness, we include in this final subsection an extended comment - omitting most of the proofs - on the issue. We mainly follow [56], where the topic is treated with greater generality. For a comprehensive source regarding the existence of Borel bases for $\mathcal{U}_0(\mathbb{T})$, the reader is referred to [47].

Let X be a locally compact and second countable space and A be a regular Banach algebra of functions in $C_b(X)$. For an element of $S \in A^*$, its support $\mathrm{supp}(S)$ is defined to be the smallest closed $F \subseteq X$ with the property that for each $f \in A$ with compact support disjoint from F, then $\langle S, f\rangle = 0$. This is reminiscent of the definition of $\mathrm{supp}(T)$ whenever $T \in VN(G) = A(G)^*$.
For elements $\varphi \in A$ and $S \in A^*$ we define $S.\varphi \in A^*$ via the relation :

$$\langle S.\varphi, g\rangle = \langle S, \varphi.g\rangle, \text{ for every } g \in A$$

Henceforth we fix some $P \subseteq A$ witnessing the regularity of A and we say that a subset $\mathcal{B} \subseteq A^*$ is stable under P-multiplication if for every $S \in \mathcal{B}$ and $\varphi \in P$ we have that $S.\varphi \in \mathcal{B}$. Morever, if $E \subseteq X$ is closed :

$$\mathcal{B}(E) := \{S \in \mathcal{B} : \mathrm{supp}(S) \subseteq E\} \text{ and } \mathcal{I}_{\mathcal{B}} := \{E \in \mathcal{F}(X) : \mathcal{B}(E) = \{0\}\}$$

Proposition 5.68. If $\mathcal{B} \subseteq A^*$ is a (norm) closed convex cone stable under P-multiplication, then $\mathcal{I}_\mathcal{B}$ is a σ-ideal of $\mathcal{F}(X)$.

Proof. The reader can find a proof in [56] (Proposition 1.1). ■

Let G be a locally compact abelian second countable and non-discrete group. We extend the terminology previously introduced when $G = \mathbb{T}$ and define $PM(G)$ - the space of pseudomeasures of G - to be the dual of $A(G)$. Similarly, $PF(G)$ - the space of pseudofunctions of G - is defined to be the set of those pseudomeasures S such that $\hat{S} \in C_0(\hat{G})$, where $\hat{G}$ is the dual group of G.

Remark 5.69. Let G be a locally compact abelian group. It follows by Proposition 5.68, setting $A = P = A(G)$ and $\mathcal{B} = PF$ that $\mathcal{U}(G)$ is a σ-ideal. If instead we fix $\mathcal{B} = M(G) \cap PF$, we conclude that $\mathcal{U}_0(G)$ is a σ-ideal.

Following the notation in [56], henceforth E is a non-empty compact metrizable space and $M_+(E) \subseteq M(E)$ is the subset of positive measures.

Definition 5.70. A $\mathcal{B} \subseteq M_+(E)$ is called a band if it is a closed convex cone such that whenever $\mu \leq \nu$ and $\nu \in \mathcal{B}$, then $\mu \in \mathcal{B}$. Furthermore, we say that $\mathcal{B}$ is a band of type I if $\mathcal{B}$ is a band and there is a locally compact second countable space Y and a continuous function $\theta : Y \to C(E)$ such that $\theta(Y)$ is bounded and $\mathcal{B} = \{\mu \in M_+(E) : \mu \circ \theta \in C_0(Y)\}$, with $\circ$ referring to duality.

Example 5.71. Let G be a locally compact abelian second countable group and $E \subseteq G$ be a compact subset. Then, it is well known that the set $\mathcal{B}$ of positive Rajchman measures supported by E is a band of type I (cf. [29]).

For a band $\mathcal{B}$ of type I and $\mu \in M_+(E)$, let $\tilde{\mu} := \mu \circ \theta$ and $R(\mu) = \overline{\lim}_{y \to \infty} |\tilde{\mu}(y)|$. Furthermore, we define the following set :

$$\mathcal{I}'_\mathcal{B} = \{F \in \mathcal{K}(E) : \exists c > 0 \forall \mu \in P(F) : R(\mu) \geq c\}$$

where $P(F)$ denotes the set of probability measures supported by F.

Remark 5.72. If $\mathcal{B}$ is the set of positive Rajchman measures supported by E, $\mathcal{I}'_\mathcal{B}$ coincides with the set $\mathcal{U}'_0$ as defined in [47].

Theorem 5.73. *If $\mathcal{B}$ is a band of type I, then $\mathcal{I}'_\mathcal{B}$ is a basis for $\mathcal{I}_\mathcal{B}$.*

Proof. The reader can find a proof in [56] (Theoreme 3.1). ■

Corollary 5.74. Let G be a compact abelian second countable group. Then, $\mathcal{U}'_0(G)$ is a Borel basis for $\mathcal{U}_0(G)$.

Proof. Clearly, $\mathcal{U}'_0(G)$ is Borel and thus, the conclusion follows immediately from Theorem 5.73 and Example 5.71. ■

Theorem 5.75. *Let G be a compact abelian second countable group which is non discrete. Then, $\mathcal{U}_0(G)$ has the covering property.*

Proof. Firstly, note that $\mathcal{U}_0(G)$ is calibrated. It is known that given any $E \in \mathcal{M}_0(G)$, then $\mathcal{U}_0(G) \cap \mathcal{F}(E)$ is Π^1_1-complete (cf. [56], Corollaire 3.4). Since by Corollary 5.74 $\mathcal{U}_0(G)$ has a Borel basis, the conclusion follows from Theorem 3.55. ■

Finally, we can provide a rigorous justification of something claimed in the beginning of the section for sets of uniqueness (not necessarily closed) of $\mathbb{T}$:

Theorem 5.76. *Let $A \subseteq \mathbb{T}$ be a set of extended uniqueness with the BP. Then, A is meager. In particular, all analytic sets of uniqueness of $\mathbb{T}$ are meager.*

Proof. For the sake of readability we divide the proof in 2 steps :

Step 1 : First, note that it is enough to prove that every analytic set of extended uniqueness $A \subseteq \mathbb{T}$ is meager. Indeed, if this is the case, let $\tilde{A} \subseteq \mathbb{T}$ be any set of extended uniqueness with the BP. Either $\tilde{A}$ is meager or comeager in some open set $\mathcal{V}$ and thus, contains a G_δ set G which is dense in $\mathcal{V}$. But this is impossible since G is analytic and a meager set of extended uniqueness. Accordingly, we fix an analytic set of extended uniqueness $A \subseteq \mathbb{T}$. By Theorem 5.75, $\mathcal{U}_0(\mathbb{T})$ has the covering property and thus, it suffices to prove that all closed subsets of A are sets of extended uniqueness. In order to do so, we prove that any Rajchman measure μ is such that $\mu(A) = 0$. Suppose, towards a contradiction, that this is not the case. By Theorem 3.31, A is measurable and consequently there is a closed set $F \subseteq A$ with $\mu(F) > 0$. We define $\nu(E) := \mu(E \cap F)$ for all μ-measurable subsets E and prove in Step 2 that $\hat{\nu}(n) \to 0$. It follows from Theorem 5-II.3 in [47] that $\sum \hat{\nu}(n)e^{inx} = 0$ for all $x \notin F$ and thus, for all $x \notin A$. However, since $\mu(F) > 0$ this contradicts the assumption that A is a set of extended uniqueness.

Step 2 : For every $\epsilon > 0$ there is a trigonometric polynomial $P(x) = \sum_{k=-N}^{N} c_k e^{ikx}$ such that :

$$\int |\chi_F - P| d\mu < \epsilon$$

Now let $d_n := \int P(t)e^{-int} d\mu$ and note that $|\hat{\nu}(n) - d_n| = |\int (\chi_F(t) - P(t))e^{-int} d\mu|$. Hence, in order to prove that $\hat{\nu}(n) \to 0$, it is enough to prove that $d_n \to 0$. Indeed, this is the case since :

$$d_n = \int (\sum_{k=-N}^{N} c_k e^{ikx}) e^{-int} d\mu = \sum_{k=-N}^{N} c_k \hat{\mu}(n-k)$$

and by assumption, μ is a Rajchman measure. ■

Example 5.77. As an application of Theorem 5.76 we provide an interesting example of a set of restricted multiplicity which has measure zero (cf. Example 5.4). We follow the proof given in [47] of a result which had been proven by Lyons in [53], providing a negative answer to an open question asked by Kahane and Salem. A sequence $(x_k) \subseteq [0, 2\pi]$ is said to be uniformly distributed if for every interval $I \subseteq [0, 2\pi]$ of length $|I|$, the following holds :

$$\lim_{N\to\infty} \frac{1}{N} |\{k \leq N : x_k \in I\}| = \frac{|I|}{2\pi}$$

A set $P \subseteq [0, 2\pi]$ is called a W^*-set if there is some increasing sequence of positive integers (n_k) such that for every $x \in P$ the sequence $(n_k x)$ - considered mod 2π - is not uniformly distributed. What Kahane and Salem asked was whether or not it is the case that every W^*-set is a set of uniqueness. Lyons proved that the set of non-normal numbers (of base $q \geq 2$) :

$$W^*_{q^k} = \{x \in [0, 2\pi] : \ (q^k x \bmod 2\pi) \text{ is not uniformly distributed}\}$$

is a set of restricted multiplicity. Recall that a number is said to be normal if it is normal for every base $q \geq 2$. It is a classic result due to Borel (cf. [10]) that almost every real number is normal. Thus, $W^*_{q^k}$ provides an example of a set of (restricted) multiplicity with measure zero. In order to establish this, we consider the following set :

$$P = \{x \in [0, 2\pi] : \ \overline{\lim}_N (|\frac{1}{N} \sum_{k=1}^{n} e^{iq^k x}|) = 1\}$$

The fact that $P \subseteq W^*_{q^k}$ follows from Weyl's criterion.[19] It is then enough to check that P is comeager since by Theorem 5.76 this implies that P is a set of restricted multiplicity. Indeed, the complement of P is contained in the union of the following closed sets :

$$F_{k,r} = \{x \in [0, 2\pi] : \ \forall N \geq k : \ |\frac{1}{N} \sum_{k=1}^{N} e^{iq^k x}| \leq r\}, \text{ for } k \in \mathbb{N}, r \in \mathbb{Q} \cap [0, 1)$$

Moreover, each $F_{k,r}$ has empty interior. Otherwise, suppose that there is some open interval $\mathcal{V}$ contained in some $F_{k,r}$. It follows that $F_{k,r}$ would contain a point $x = \frac{2m\pi}{k_0}$ for some k_0 and thus, $e^{iq^k x} = 1$ for $k \geq k_0$ which is a contradiction.

6 Point spectrum

This section contains a few results concerning the point spectrum of linear bounded operators acting on separable Banach spaces. Despite its independence from previous sections, we decide to include it due to the nature of this work. We recall that, given a Banach space X and a linear bounded operator $T \in \mathcal{L}(X)$, its point spectrum is defined as $\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(T - \lambda 1) \neq \{0\}\}$.

[19] A sequence (x_k) is said uniformly distributed if and only if for all non-zero $m \in \mathbb{Z}$, then $\lim_N (\frac{1}{N} \sum_{k=1}^{N} e^{imx_k}) = 0$.

6.1 A characterization of reflexivity

In this subsection we provide a characterization of Banach spaces with unconditional basis in terms of the complexity of operators acting on their closed subspaces. We start with the following observation (cf. [64]) :

Proposition 6.1. Let X be a separable Banach space and $T \in \mathcal{L}(X)$. Then, the set $\sigma_p(T) \subseteq \mathbb{C}$ is analytic.

Proof. Let $W = \{x \in X : ||x|| = 1 \text{ and } \exists \lambda(x) \in \mathbb{C} : T(x) = \lambda(x)x\}$ and consider the function $\lambda : W \to \mathbb{C}$ prescribed by $x \mapsto \lambda(x)$. We note that λ is well-defined and moreover, $\lambda(W) = \sigma_p(T)$. Hence, it suffices to prove that W is closed and λ is continuous. Suppose that $(x_n) \subseteq W$ is such that $x_n \to x$. Since T is bounded, note that:

$$|\lambda(x_n)| = \frac{||T(x_n)||}{||x_n||} \to \frac{||T(x)||}{||x||} < \infty$$

and thus, there is a convergent subsequence $\lambda(x_{n_k}) \to w$ from where it follows that $T(x_{n_k}) \to T(x) = wx$ and thus, W is closed. Moreover, λ is continuous. ∎

Recall that a Banach space X is said to be reflexive if the following map is surjective :

$$J : X \to (X^*)^*, \text{ such that } x \mapsto J_x(f) := f(x)$$

Whenever X is reflexive, the statement of Proposition 6.1 can be sharpened :

Theorem 6.2. *Let X be a reflexive separable Banach space and $T \in \mathcal{L}(X)$. Then, $\sigma_p(T)$ is Borel.*

Proof. The reader can find a proof in [64]. ∎

Recall that a (Schauder) basis (e_i) for a Banach space X is a sequence such that for every element $x \in X$ there is an unique sequence of scalars (α_i) for which $\sum \alpha_i e_i$ converges to x in norm. Evidently, every Banach space with a basis is separable.[20] To each element e_i we associate a functional e_i^* such that for every $x \in X$ one has that $x = \sum e_i^*(x)e_i$.

Definition 6.3. Let X be a Banach space with a basis (e_i). Then :

(a) (e_i) is shrinking if (e_i^*) is a basis for X^*.

(b) (e_i) is boundedly complete if whenever (α_i) is a sequence of scalars such that $\sup_n ||\sum_{i=1}^n \alpha_i e_i|| < \infty$, then $\sum \alpha_i e_i$ converges in norm.

(c) (e_i) is an unconditional basis if the basis expansion for each $x \in X$ converges unconditionally. Otherwise, (e_i) is said to be a conditional basis.

Theorem 6.4. *Let (e_i) be a basis for X. Then, X is reflexive iff (e_i) is both shrinking and boundedly complete.*

[20] The converse is false, as proven by P. Enflo providing a negative answer to *Problem 153* in the *Scottish Book*.

Proof. The reader can find a proof in [37]. ■

The next result concerns Banach spaces with an unconditional basis :

Theorem 6.5. *Let (e_i) be an unconditional basis for X. Then, the following hold :*

(i) (e_i) is boundedly complete iff X has no complemented subspace isomorphic to c_0.

(ii) (e_i) is shrinking iff X has no complemented subspace isomorphic to ℓ^1.

Proof. The reader can find a proof in [52]. ■

Remark 6.6. It is known that every separable infinite dimensional Banach space with a basis has a conditional basis (cf. [65]) and every infinite dimensional Banach space contains an infinite dimensional subspace which either has an unconditional basis or is hereditarily indecomposable (cf. [31]). In the case when X is reflexive, it is known that every reflexive Banach space with unconditional basis is isomorphic to a complemented subspace of a reflexive Banach space with symmetric basis (cf. [79]).

Theorem 6.7. *Let $S \subseteq \mathbb{C}$ be a bounded analytic set. Then, there is some $T \in \mathcal{L}(c_0)$ such that $\sigma_p(T) = S$.*

Proof. The reader can find a proof in [44]. ■

An immediate consequence of Theorem 6.7 is the following :

Corollary 6.8. Let X be a Banach space which contains a closed subspace Y isomorphic to c_0. Then, there is some $T \in \mathcal{L}(Y)$ with non Borel $\sigma_p(T)$.

Proof. By Theorem 6.7, one can choose an operator $T \in \mathcal{L}(c_0)$ such that its point spectrum $\sigma_p(T)$ is not Borel. Let $S : Y \to c_0$ be an isomorphism and consider $R = S^{-1}TS \in \mathcal{L}(Y)$. Then, note that $\sigma_p(R) = \sigma_p(T)$: indeed, suppose that $\lambda \in \sigma_p(T)$ so that there is some $z \in c_0 \setminus \{0\}$ such that $Tz = \lambda z$. As S is bijective, there is an unique $y \in Y \setminus \{0\}$ such that $S(y) = z$. Since :

$$R(y) = S^{-1}(T(S(y))) = S^{-1}(Tz) = S^{-1}(\lambda z) = \lambda S^{-1}(z) = \lambda y$$

we conclude that $\lambda \in \sigma_p(R)$. Conversely, let $\lambda \in \sigma_p(R)$ so that there is some $y \in Y \setminus \{0\}$ such that $Ry = \lambda y$. Consider $z = S(y) \in c_0 \setminus \{0\}$ and note that :

$$R(y) = S^{-1}(T(S(y))) = S^{-1}(Tz) = \lambda y \Rightarrow S(S^{-1}(Tz)) = S(\lambda y) \Rightarrow Tz = \lambda z$$

from where we conclude that $\lambda \in \sigma_p(T)$. It follows that $\sigma_p(T) = \sigma_p(R)$. ■

In order to establish a characterization of reflexivity for Banach spaces with unconditional basis, we need a final observation. For the sake of completeness, we state the result we need with its converse as well :

Theorem 6.9. *A Banach space X contains a complemented isomorphic copy of ℓ^1 iff X^* contains an isomorphic copy of c_0.*

Proof. The reader can find a proof in [6]. ■

We conclude the subsection with the following characterization of reflexivity :

Theorem 6.10. *Let X be a Banach space with unconditional basis. Then, X is reflexive if and only if for every closed subspaces $Y \subseteq X$ and $Z \subseteq X^*$ and operators $T \in \mathcal{L}(Y)$ and $T' \in \mathcal{L}(Z)$ it holds that $\sigma_p(T)$ and $\sigma_p(T')$ are Borel sets.*

Proof. Suppose that X is reflexive and note that since it has a basis, then X is separable. It follows that X^* is also reflexive and separable and thus, each closed subspaces Y and Z are also reflexive (and separable). By Theorem 6.2, we are done.
Now suppose that X is not reflexive and let (e_i) be an unconditional basis. By Theorem 6.4, either (e_i) is not shrinking or (e_i) is not boundedly complete. If (e_i) is not shrinking, by Theorem 6.5 and Theorem 6.9, X^* contains an isomorphic copy of c_0. If, on the other hand, (e_i) is not boundedly complete, then by Theorem 6.5 X contains an isomorphic copy of c_0. Hence, we conclude by Corollary 6.8 that there is either a closed subspace $Y \subseteq X$ or $Z \subseteq X^*$ with operators $T \in \mathcal{L}(Y)$ with non Borel point spectrum or $T' \in \mathcal{L}(Z)$ with non Borel point spectrum. ■

6.2 Subsets of Subs(X) associated with T

In this subsection, following some ideas in [64], we prove that if X is a reflexive separable Banach space and $T \in \mathcal{L}(X)$, then the following is a Borel subset of Subs(X) :

$$K_T = \{\ker(T - \lambda 1) : \lambda \in \mathbb{C}\}$$

For each $T \in \mathcal{L}(X)$, define a function $\Gamma_T : \mathbb{C} \to \mathcal{F}(X) \setminus \{\emptyset\}$ prescribed by :

$$\lambda \mapsto \ker(T - \lambda 1)$$

For fixed $x \in X$ and $p \geq 0$, define the following sets :

$$A_x^p := \{\lambda \in \mathbb{C} : d(x, \ker(T - \lambda 1)) \leq p\} \text{ and } \hat{A}_x^p := A_x^p \cap \sigma_p(T)$$

Proposition 6.11. Let X be a Banach space and $T \in \mathcal{L}(X)$ such that $\sigma_p(T)$ is Borel. Then, every A_x^p is analytic.

Proof. First, we introduce some notation. Let d be the metric induced by the norm on X and let $r, p > 0$ and $x \in X$. Define :

$$C_{r,p}^x = \{y \in X : r \leq ||y||, d(x, y) \leq p \text{ and } \exists \lambda(y) \in \mathbb{C} : T(y) = \lambda(y) y\}$$

As $C_{r,p}^x$ consists of eigenvectors, the map $\varphi_{r,p}^x : C_{r,p}^x \to \mathbb{C}$ given by $y \mapsto \lambda(y)$ is well-defined. For the sake of readability, we divide the proof in three steps :

Step 1: First, we prove that $\varphi^x_{r,p}(C^x_{r,p}) \subseteq \mathbb{C}$ is analytic. It is enough to prove that $C^x_{r,p} \subseteq X$ is closed and that $\varphi^x_{r,p}$ is continuous. Let $\{y_n\} \subseteq C^x_{r,p}$ such that $y_n \to y$. Clearly, $||y|| \geq r$. Moreover, and since T is bounded :

$$|\lambda(y_n)| = \tfrac{||T(y_n)||}{||y_n||} \to \tfrac{||T(y)||}{||y||} < \infty$$

Thus, $\{\lambda(y_n)\}$ is bounded and we can consider a convergent subsequence $\lambda(y_{n_k}) \to \lambda$. It follows that $T(y_{n_k}) = \lambda(y_{n_k})y_{n_k} \to \lambda y$. Hence, $T(y) = \lambda y$. Furthermore, it is clear that $d(x, y) \leq p$ and thus, $C^x_{r,p}$ is closed. Finally, we note that in order to prove that $\varphi^x_{r,p}$ is continuous, it is enough to prove that $(\lambda(y_n))$ is Cauchy :

$$|\lambda(y_n)-\lambda(y_{n+1})|||y|| \leq |\lambda(y_n)|||y-y_n||+||T(y_n)-T(y_{n+1})||+|\lambda(y_{n+1})|||y_{n+1}-y||$$

Since T is continuous, $y_n \to y$ and $\{\lambda(y_n)\}$ is bounded, it is clear that $(\lambda(y_n))$ is Cauchy and we are done.

Step 2: We prove that for each $p > 0$, the sets $\hat{A}^p_x$ are analytic. We divide the analysis in two cases :
(i) Suppose that $d(x, 0) \leq p$. Then, $\hat{A}^p_x = \sigma_p(T)$ which is Borel by assumption.

(ii) Suppose that $d(x, 0) > p$. By Step 1, it is enough to prove that for each $r_m = \frac{1}{m}$, the following holds :

$$\hat{A}^p_x = \bigcup_m \bigcap_n \varphi^x_{r_m,p+\frac{1}{n}}(C^x_{r_m,p+\frac{1}{n}})$$

Indeed, let $\lambda \in \hat{A}^p_x$. Since $d(x, 0) > p$, there is some m such that for every $n \geq m$ one can choose some $y_n \neq 0$ such that $y_n \in \ker(T - \lambda 1)$ and $d(x, y_n) \leq p + \frac{1}{n}$. One can assume without loss of generality that there is some $\delta > 0$ such that $||y_n|| \geq \delta$, for all n. Indeed, towards a contradiction, suppose that for every $\delta > 0$ there is some N such that for all $k \geq N$ then $d(y_k, 0) < \delta$. Then, $y_n \to 0$ which implies that $d(x, 0) \leq p$. Hence, there is some $\delta > 0$ such that for all n, there is some $k \geq n$ such that $d(y_k, 0) \geq \delta$. Thus, we can assume without loss of generality that $||y_n|| \geq \delta$. If $\delta \geq 1$, then certainly $\lambda \in \varphi^x_{r_1,p+\frac{1}{n}}(C^x_{r_1,p+\frac{1}{n}})$ for all n. Otherwise, let m be such that $\frac{1}{m} \leq \delta$. In this case, it is clear that $\lambda \in \varphi^x_{r_m,p+\frac{1}{n}}(C^x_{r_m,p+\frac{1}{n}})$ for all n.
Conversely, let $\lambda \in \bigcap_n \varphi^x_{r,p+\frac{1}{n}}(C^x_{r,p+\frac{1}{n}})$ for some $r > 0$. Then, for all n there is some $y_n \in \ker(T - \lambda 1)$ such that $||y_n|| \geq r > 0$ and $d(x, y_n) \leq p + \frac{1}{n}$. Thus, $\lambda \in \sigma_p(T)$ and $d(x, \ker(T - \lambda 1)) \leq p$ so that $\lambda \in \hat{A}^p_x$.

Step 3: (i) Let $p > 0$. By Step 2, $\hat{A}^p_x$ is analytic and thus, it is enough to prove that $A^p_x \cap \neg\sigma_p(T)$ is analytic. Note that $A^p_x \cap \neg\sigma_p(T) \subseteq \neg\sigma_p(T)$ and that if $\lambda \in \neg\sigma_p(T)$, then $\ker(T - \lambda 1) = \{0\}$. Either $d(x, 0) \leq p$ in which case $\lambda \in A^p_x$ and thus $A^p_x \cap \neg\sigma_p(T) = \neg\sigma_p(T)$ or $d(x, 0) > p$ in which case $A^p_x \cap \neg\sigma_p(T) = \emptyset$. Since $\sigma_p(T)$ is assumed to be Borel, in both cases we have that $A^p_x \cap \neg\sigma_p(T)$ is

analytic.
(ii) If $p = 0$, then either $A_0^0 = \mathbb{C}$ or A_x^0 has at most one element, for $x \neq 0$. In either case, A_x^0 is analytic. ■

For $T \in \mathcal{L}(X)$, $K \subseteq X$ and $M \geq 0$ we define the following set :

$$\Lambda_T(K, M) = \{\lambda \in \mathbb{C} : \ker(T - \lambda 1) \cap K \neq \emptyset \text{ and } |\lambda| \leq M\}$$

The importance of the sets $\Lambda_T(K, M)$ is due to the following result :

Proposition 6.12. Let $T \in \mathcal{L}(X)$, for X separable and Banach and let $K \subseteq X$ be weakly compact. Then, for any $M \geq 0$, the set $\Lambda_T(K, M)$ is compact.

Proof. The reader can find a proof in [64]. ■

Proposition 6.13. Let X be a separable and reflexive Banach space, with $T \in \mathcal{L}(X)$. Then, A_x^p is Borel.

Proof. For any $x \in X$, let $K_x^r = \overline{B(x, r)}$. Since X is reflexive, it follows by Proposition 6.12 that each $\Lambda_T(K_x^r, M)$ is compact and thus, it is enough to prove that :

$$A_x^p = \bigcup_M \bigcap_n \Lambda_T(K_x^{p+\frac{1}{n}}, M)$$

Let $\lambda \in A_x^p$ such that $|\lambda| \leq M$. Then, $d(x, \ker(T - \lambda 1)) \leq p$ and thus, for any n, there is some $y_n \in \ker(T - \lambda 1) \cap K_x^{p+\frac{1}{n}}$. Hence, $\lambda \in \Lambda_T(K_x^{p+\frac{1}{n}}, M)$.
Conversely, suppose that for some M, $\lambda \in \Lambda_T(K_x^{p+\frac{1}{n}}, M)$ for every n. If $d(x, \ker(T - \lambda 1)) > p$, then there is some m such that for every $y \in \ker(T - \lambda 1)$ one has that $d(x, y) \geq p + \frac{1}{m}$. It follows that $\lambda \notin \Lambda_T(K_x^{p+\frac{1}{m+1}}, M)$, which contradicts our assumption. ■

We now consider the measurability of Γ_T (for a fixed $T \in \mathcal{L}(X)$) with respect to the σ-algebra generated by the Wijsman topology $\mathcal{W}$ on $\mathcal{F}(X) \setminus \{\emptyset\}$. The reader who is unfamiliar with the terminology is referred to section 3.2.

Proposition 6.14. Let X be a separable and reflexive Banach space. Then, Γ_T is a measurable map.

Proof. Since $\mathcal{W}$ is in particular second countable, it is enough to verify that the pre-image of each subbasis element is a measurable subset of $\mathbb{C}$. Thus, it suffices to show that $\Gamma_T^{-1}(\varphi_x^{-1}((p, q)))$ is Borel for any $x \in X$ and $p, q \in \mathbb{Q}_0^+$. Consider :

$$B_x^p = \{\lambda \in \mathbb{C} : d(x, \ker(T - \lambda 1)) > p\} \text{ and } C_x^q = \{\lambda \in \mathbb{C} : d(x, \ker(T - \lambda 1)) < q\}$$

Note that $B_x^p = \neg A_p^x$ and thus, by Proposition 6.13, is Borel. Moreover, note that $C_x^q = \bigcup_n A_x^{q-\frac{1}{n}}$ and thus, is Borel. ■

Theorem 6.15. *Let X be a reflexive separable Banach space and fix some $T \in \mathcal{L}(X)$. Let $K_T = \{\ker(T - \lambda 1) : \lambda \in \mathbb{C}\}$. Then, K_T is a Borel set of Subs(X). In particular, $\mathcal{K}_T$ is an element of the Effros-Borel space of $\mathcal{F}(X)$.*

Proof. By Theorem 6.2, $\sigma_p(T)$ is Borel and by Proposition 6.14 the map Γ_T is measurable. Since $\Gamma_T|_{\sigma_p(T)}$ is injective, Theorem 3.27 implies that $K_T = \Gamma_T(\sigma_p(T)) \cup \{0\}$ is a Borel subset of Subs(X) and $\mathcal{F}(X)$. ■

6.3 Jamison sequences

We finish this section with a brief overview on Jamison sequences, leading to the proof of a stability result concerning certain perturbations on the set of such sequences. Henceforth, X is a separable Banach space and whenever we consider increasing sequences (n_k) of positive integers we assume, without loss of generality, that $n_1 = 1$.

Definition 6.16. Let (n_k) be an increasing sequence of positive integers and $T \in \mathcal{L}(X)$. We say that T is partially power bounded with respect to (n_k) if :

$$\sup_k ||T^{n_k}|| < \infty$$

An increasing sequence (n_k) of positive integers is said to be a Jamison sequence if whenever $T \in \mathcal{L}(X)$ is partially power bounded with respect to it, then the unimodular point spectrum $\sigma_p(T) \cap \mathbb{T}$ of T is countable. The set of Jamison sequences will be denoted by $\mathcal{J}$.

Remark 6.17. Let X be a separable infinite dimensional Banach space. We say that X is an universal Jamison space if whenever an increasing sequence of positive integers (n_k) is not a Jamison sequence, then there is some $T \in \mathcal{L}(X)$ partially power bounded with respect to (n_k) and with uncountable unimodular point spectrum. It was proven in [21] that $\ell^2(\mathbb{N})$ is an universal Jamison space and more generally in [18] that separable Banach spaces which admit an unconditional Schauder decomposition, are universal Jamison spaces.[21]

Example 6.18. It turns out that for a given increasing sequence (n_k) of positive integers, the growth of $\{||T^k||\}$ is related with how *small* $\sigma_p(T) \cap \mathbb{T}$ is :

(a) The sequence $(n_k) = (k)$ is a Jamison sequence (cf. [38]). In order to prove this, define the following equivalence relation in $\mathbb{T}$:

$$z \sim w \text{ if and only if } \exists i, j \in \mathbb{N} \setminus \{0\} \text{ such that } z^i w^j = 1$$

It is a well known fact that if z and w are not equivalent, then the subset $\{(z^n, w^n) : n \in \mathbb{N}\}$ is dense in $\mathbb{T} \times \mathbb{T}$ (cf. [67]). Moreover, suppose that $T \in \mathcal{L}(X)$ is such that $\sup_k ||T^k|| = M < \infty$ and that $\lambda_1, \lambda_2 \in \sigma_p(T) \cap \mathbb{T}$ are not equivalent, with norm one eigenvectors x_1 and x_2, respectively. Then :

$$||x_1 - x_2|| \geq \frac{2}{M+1}$$

[21] One says that X admits an unconditional Schauder decomposition if there is a sequence of closed subspaces (X_n) of X such that $X_n \neq \{0\}$ and that any element $x \in X$ can be uniquely written as an unconditionally convergent series $x = \sum x_n$, with $x_n \in X_n$.

Indeed, since λ_1 and λ_2 are not equivalent, we can pick a sequence (n_k) such that $(\lambda_1^{n_k}, \lambda_2^{n_k}) \to (-1, 1)$. By triangular inequality, one has that :

$$||x_1 - \lambda_1^{n_k} x_1|| \leq ||x_1 - x_2|| + ||x_2 - \lambda_2^{n_k} x_2|| + ||\lambda_2^{n_k} x_2 - \lambda_1^{n_k} x_1||$$

The left hand side converges to $||2x_1|| = 2$ and, on the other hand one has that $||x_2 - \lambda_2^{n_k} x_2|| \to 0$ and that $||\lambda_2^{n_k} x_2 - \lambda_1^{n_k} x_1|| \leq M||x_1 - x_2||$. Now, suppose towards a contradiction that $\sup_k ||T^k|| < \infty$ and $\sigma_p(T) \cap S^1$ is uncountable. It follows that there is an uncountable and mutually disjoint collection of open balls $\{B(x, \frac{1}{M+1})\}$, which contradicts the separability of X. Hence, (k) is a Jamison sequence.

(b) It follows from (a) that if $\sigma_p(T) \cap \mathbb{T}$ is uncountable, then $\sup_k ||T^k|| = \infty$. It is thus natural to ask whether or not if $\sigma_p(T) \cap \mathbb{T}$ uncountable implies $\lim_k ||T^k|| = \infty$. A negative answer to the former question was provided in [69]. However, in [68] it is proven that under certain additional assumptions the answer is positive :

(i) If $\sigma_p(T) \cap \mathbb{T}$ has positive Lebesgue measure or is of second category, then $\lim_k ||T^k|| = \infty$

(ii) If $\sigma_p(T) \cap \mathbb{T}$ is uncountable, then there is a subset $Z \subseteq \mathbb{N}$ of density zero for which $\lim_{k \notin Z} ||T^k|| = \infty$

In order to provide an useful characterization of Jamison sequences we introduce a metric $d_{(n_k)}$ on $\mathbb{T}$ associated with each increasing sequence (n_k) of positive integers :

$$d_{(n_k)}(\lambda, \mu) := \sup_k |\lambda^{n_k} - \mu^{n_k}|, \text{ for } \lambda, \mu \in \mathbb{T}$$

Theorem 6.19. *Let (n_k) be an increasing sequence of positive integers. The following are equivalent :*

(i) The sequence (n_k) is a Jamison sequence

(ii) For every uncountable subset $K \subseteq \mathbb{T}$, the metric space $(K, d_{(n_k)})$ is non separable

(iii) For every uncountable subset $K \subseteq \mathbb{T}$ there exists a positive ϵ such that K contains an uncountable ϵ-separated family for the distance $d_{(n_k)}$.

(iv) There exists a positive ϵ such that every uncountable subset $K \subseteq \mathbb{T}$ contains an uncountable ϵ-separated family for the distance $d_{(n_k)}$.

(v) There exists an $\epsilon > 0$ such that any two distinct points $\lambda, \mu \in \mathbb{T}$ are ϵ-separated for the distance $d_{(n_k)}$.

Proof. The reader can find a proof in [3]. ∎

Remark 6.20. Let G be a Lie group and (n_k) an increasing sequence of positive integers. In [2] it is proven that (n_k) is Jamison if and only if there exists an open neighborhood $\mathcal{U}$ of the identity 1_G such that if $g^{n_k} \in \mathcal{U}$ for every k, then $g = 1_G$ (cf. Lemma 6.22, for the case when $G = \mathbb{T}$).

For a fixed sequence (n_k) and $\epsilon > 0$, define the following subset of $\mathbb{T}$:

$$\Lambda_\epsilon^{(n_k)} := \{\lambda \in \mathbb{T} : \sup_k |\lambda^{n_k} - 1| < \epsilon\}$$

Lemma 6.21. Let $T \in \mathcal{L}(X)$ and suppose that (n_k) is an increasing sequence of positive integers such that $\sup_k ||T^{n_k}|| < \infty$. Then, given $\epsilon > 0$, there is a countable subset $\{\mu_n\} \subset \mathbb{T}$ such that :

$$\sigma_p(T) \cap \mathbb{T} \subset \bigcup_l \mu_l E$$

where

$$E = \bigcap_k \{\lambda \in \mathbb{T} : |\lambda^{n_k} - 1| \leq \epsilon\}$$

Proof. The reader can find a proof in [69]. ■

Lemma 6.22. Let (n_k) be an increasing sequence of positive integers and $\epsilon > 0$. Then, $(n_k) \in \mathcal{J}$ if and only if $\Lambda_\epsilon^{(n_k)} = \{1\}$.

Proof. For the sake of readability we divide the proof in two steps :

Step 1 : We first prove that $(n_k) \in \mathcal{J}$ if and only if there is some $\epsilon > 0$ for which $\Lambda_\epsilon^{(n_k)}$ is countable. Suppose that $(n_k) \in \mathcal{J}$ and note that by Theorem 6.19(v) there is some ϵ such that $\Lambda_\epsilon^{(n_k)} = \{1\}$. Conversely, suppose that there is some $\epsilon > 0$ such that $\Lambda_\epsilon^{(n_k)}$ is countable and let $T \in \mathcal{L}(X)$ be such that $\sup_k ||T^{n_k}|| < \infty$. Then, the set E in Lemma 6.21 (for instance for $\tilde{\epsilon} = \frac{\epsilon}{2}$) is countable and thus, $\sigma_p(T) \cap \mathbb{T}$ is countable, from where it follows that (n_k) is a Jamison sequence.

Step 2 : If there is some $\epsilon > 0$ such that $\Lambda_\epsilon^{(n_k)} = \{1\}$, then it follows immediately from Step 1 that $(n_k) \in \mathcal{J}$. Conversely, suppose that each $\Lambda_\epsilon^{(n_k)}$ has at least two elements. If there is any $\delta > 0$ such that $\Lambda_\delta^{(n_k)}$ is countable, by Step 1 we have that (n_k) is a Jamison sequence. But this is impossible, since by Theorem 6.19(v) there is some $\epsilon > 0$ such that $\Lambda_\epsilon^{(n_k)}$ has only one element. ■

Example 6.23. We provide a short list of examples of (non) Jamison sequences (cf. [3]). In what follows, (n_k) is an increasing sequence of positive integers :

(a) If $\sup_k\{\frac{n_{k+1}}{n_k}\} < \infty$, then (n_k) is Jamison. If furthermore $n_k | n_{k+1}$, then (n_k) is Jamison if and only if $\sup_k\{\frac{n_{k+1}}{n_k}\} < \infty$.

(b) If $\lim_k \frac{n_{k+1}}{n_k} = \infty$, then (n_k) is non Jamison.

(c) If (n_k) is a set of positive upper density, then (n_k) is Jamison. If (n_k) has density zero, it is not possible to conclude anything : (k^2) is Jamison but $(k!)$ is not.

We finish the section with a few results on the stability of $\mathcal{J}$ under certain perturbations. These criteria allow us to easily generate Jamison sequences.

Lemma 6.24. Suppose that (n_k) is an increasing sequence of positive integers and $\epsilon > 0$. If $(n_k) \notin \mathcal{J}$, then $1 \in \overline{\Lambda_\epsilon^{(n_k)} \setminus \{1\}}$.

Proof. Since (n_k) is non Jamison, it follows by Lemma 6.22 that each $\Lambda_\epsilon^{(n_k)} \setminus \{1\}$ is non empty. Moreover, since $\Lambda_\epsilon^{(n_k)} \subseteq \Lambda_\delta^{(n_k)}$ for $\epsilon < \delta$, one has that $\{\overline{\Lambda_\epsilon^{(n_k)} \setminus \{1\}}\}_{\epsilon>0}$ is a family of closed non empty subsets of S^1 with the finite intersection property. It follows by compactness of S^1 that there is some $z \in \bigcap_{\epsilon>0} \overline{\Lambda_\epsilon^{(n_k)} \setminus \{1\}}$. Thus, one has that for each $\epsilon > 0$, $|z-1| \leq \epsilon$. Hence, $z = 1 \in \bigcap_{\epsilon>0} \overline{\Lambda_\epsilon^{(n_k)} \setminus \{1\}}$. ∎

Theorem 6.25. *Let (n_k) and (t_k) be increasing sequences of positive integers such that $\sup_k |t_k - n_k| < \infty$. Then, $(n_k) \in \mathcal{J}$ if and only if $(t_k) \in \mathcal{J}$.*

Proof. Let $\sup_k |t_k - n_k| = M < \infty$, fix some small $\epsilon > 0$ and assume that $(n_k) \notin \mathcal{J}$. By Lemma 6.22 it is enough to prove that $\Lambda_\epsilon^{(t_k)} \neq \{1\}$. Since (n_k) is non Jamison, using Lemma 6.24 we can pick some $z = e^{i\theta}$ such that $z \neq 1$, $z \in \Lambda_{\frac{\epsilon}{3}}^{(n_k)}$ and with θ small enough so that $\frac{M\theta}{2\pi} < \frac{\epsilon}{3}$. Then, for any k :

$$|z^{t_k} - 1| \leq |z^{t_k} - z^{n_k}| + |z^{n_k} - 1| < \frac{2\epsilon}{3} < \epsilon$$

Hence, $\Lambda_\epsilon^{(t_k)} \neq \{1\}$ and thus, (t_k) is non Jamison. ∎

Lemma 6.26. Let c be any positive integer and (n_k) be an increasing sequence of positive integers. Then, $(n_k) \in \mathcal{J}$ if and only if $(cn_k) \in \mathcal{J}$.

Proof. Suppose that (cn_k) is non Jamison and fix some $\epsilon > 0$. We prove that $\Lambda_\epsilon^{(n_k)} \neq \{1\}$, from where it follows that (n_k) is non Jamison by Lemma 6.22. By Lemma 6.24, there is some sequence $(\lambda_n) \subseteq \Lambda_\epsilon^{(cn_k)} \setminus \{1\}$ such that $\lambda_n \to 1$. We pick any element from this sequence, say $\lambda_k := \lambda$. Then, $\lambda \neq 1$ and $\sup_k |\lambda^{cn_k} - 1| < \epsilon$. If $\lambda^c \neq 1$, then $\lambda^c \in \Lambda_\epsilon^{(n_k)} \setminus \{1\}$ and we are done. Otherwise, suppose that $\lambda^c = 1$ and note that any other cth-root of unity μ is such that $|\lambda - \mu| \geq \frac{2}{c}$. Since (λ_n) is a Cauchy sequence, let $\mu = \lambda_m$ be such that $|\mu - \lambda| < \frac{2}{c}$. It is clear that $\mu^c \neq 1$ and since $\mu \in \Lambda_\epsilon^{(cn_k)}$, it follows that $\mu^c \in \Lambda_\epsilon^{(n_k)} \setminus \{1\}$. Conversely, let (cn_k) be a Jamison sequence and suppose that for some operator $T \in \mathcal{L}(X)$ one has that $\sup_k ||T^{n_k}|| = M < \infty$. For any k :

$$||T^{cn_k}|| \leq c||T^{n_k}|| \leq cM < \infty$$

Hence, $\sup_k ||T^{cn_k}|| < \infty$. By assumption, $(cn_k) \in \mathcal{J}$ and thus $\sigma_p(T) \cap \mathbb{T}$ is countable. $\blacksquare$

In fact, one can further generalize Proposition 6.26 as follows :

Proposition 6.27. Let (t_k) and (n_k) be increasing sequences of positive integers such that $(t_k) = (a_k n_k)$, for some sequence (a_k) of positive integers with $\sup_k(a_k) = A < \infty$. Then, $(t_k) \in \mathcal{J}$ if and only if $(n_k) \in \mathcal{J}$.

Proof. Suppose that (t_k) is non Jamison. Since $\sup_k(a_k) = A$ one has that $a_k \in \{1, ..., A\}$. Without loss of generality, we can assume that for each $j \leq A$ the subset $I_j := \{k \in \mathbb{N} : a_k = j\}$ is either infinite or empty. Moreover, we will see that it is enough to consider the case when each $I_j \neq \emptyset$. Fix any $\epsilon > 0$ and let $P := \prod_{i=1}^{A} i < \infty$. Since (t_k) is non Jamison, there is some $\lambda \in \Lambda_{\frac{\epsilon}{P}}^{(t_k)}$ such that $\lambda \neq 1$. Note that for each $j \leq A$ one has that $\sup_{k \in I_j} |\lambda^{jn_k} - 1| < \frac{\epsilon}{P}$. Our strategy is to use Lemma 6.22 to prove that (Pn_k) is non Jamison and then, by Lemma 6.26 it follows that (n_k) is also non Jamison. We can assume that for any k there is some $j \leq A$ such that $k \in I_j$ and thus :

$$|\lambda^{Pn_k} - 1| \leq (\prod_{i \neq j}^{A} i)|\lambda^{jn_k} - 1| < (\prod_{i \neq j}^{A} i)\frac{\epsilon}{P} = \frac{\epsilon}{j} < \epsilon$$

Conversely, suppose that (t_k) is Jamison. Suppose that $T \in \mathcal{L}(X)$ is such that $\sup_k ||T^{n_k}|| = M < \infty$. Then :

$$||T^{t_k}|| = ||T^{a_k n_k}|| \leq ||T^{n_k}||^{a_k} \leq M^A < \infty$$

By assumption, (t_k) is Jamison and thus, $\sigma_p(T) \cap \mathbb{T}$ is countable. $\blacksquare$

It is convenient to introduce some notation : Given two increasing sequences of positive integers (t_k) and (n_k), we define another sequence $(r_k)_{\frac{t_k}{n_k}}$ prescribed by (with $[.]$ denoting the closest integer function) :

$$r_k := |t_k - [\frac{t_k}{n_k}]n_k|$$

Theorem 6.28. *Let (t_k) and (n_k) be increasing sequences of positive integers and suppose that (t_k) is (non) Jamison. Then, if one of the following conditions hold, (n_k) is also (non) Jamison :*

(i) $\sup_k(\frac{t_k}{n_k}) < \infty$ *and* $\sup_k(r_k)_{\frac{t_k}{n_k}} < \infty$

(ii) $\sup_k(\frac{n_k}{t_k}) < \infty$ *and* $\sup_k(r_k)_{\frac{n_k}{t_k}} < \infty$

Proof. Suppose that (i) holds. Since $\sup_k(r_k) < \infty$, it follows by Theorem 6.25 that $[\frac{t_k}{n_k}]n_k$ is non Jamison. Furthermore, since $\sup_k(\frac{t_k}{n_k}) < \infty$ it follows by Proposition 6.27 that (n_k) is non Jamison. Alternatively, suppose that (ii) holds. Note that $n_k = [\frac{n_k}{t_k}]t_k \pm r_k$ for all k. Since $\sup_k(\frac{n_k}{t_k}) < \infty$, it follows by Proposition 6.27 that $([\frac{n_k}{t_k}]t_k)$ is non Jamison and since $\sup_k(r_k) < \infty$, it follows by Theorem 6.25 that (n_k) is non Jamison. $\blacksquare$

7 Appendix

Definition 7.1. Let $(P, \leq)$ be a partially ordered set and $\mathcal{F} \subseteq P$. The subset $\mathcal{F}$ is said to be a filter if :

(i) $\mathcal{F} \neq \emptyset$

(ii) For every $x, y \in \mathcal{F}$ there is some $z \in \mathcal{F}$ such that $z \leq x$ and $z \leq y$

(iii) Whenever $x \in \mathcal{F}$ and $y \in P$ is such that $x \leq y$, then $y \in \mathcal{F}$

Definition 7.2. Let $(P, \leq)$ be a partially ordered set. A subset $D \subseteq P$ is said to be dense if for every element $x \in P$ there is some $y \in D$ such that $y \leq x$. Two elements $x, y \in P$ are said to be incompatible if there is no element $z \in P$ such that $z \leq x$ and $z \leq y$ otherwise, they are said to be compatible. A subset $A \subseteq P$ is said to be an antichain if every distinct $x, y \in A$ are incompatible. A partially ordered set $(P, \leq)$ is said to be c.c.c. (or to have the countable chain condition) if every antichain in P is countable.

Example 7.3. Let X be a topological space and consider :

$$P = (\{\mathcal{V} \subseteq X : \mathcal{V} \text{ is non-empty open subset}\}, \leq)$$

with $\mathcal{U} \leq \mathcal{V}$ if and only if $\mathcal{U} \subseteq \mathcal{V}$. Clearly, P is c.c.c. if and only if there is no uncountable family of pairwise disjoint non-empty open sets of X. Thus, if X is separable it follows that P is c.c.c. It is worth to note that the converse is false : if $\kappa > 2^{\aleph_0}$, then 2^κ is c.c.c. (cf. [49], Proposition 1.9) but not separable.

Definition 7.4. Let $\kappa < 2^{\aleph_0}$ be an infinite cardinal. Then, $MA(\kappa)$ is the following statement : if $(P, \leq)$ is a non-empty c.c.c. partial order and $\mathcal{D}$ is a family of $\leq \kappa$ dense subsets of P, then there is a filter $\mathcal{F} \subseteq \mathcal{P}(P)$ such that for every $D \in \mathcal{D}$ we have that $D \cap \mathcal{F} \neq \emptyset$.

The assumption $\kappa < 2^{\aleph_0}$ is justified by the following proposition :

Proposition 7.5. The statement $MA(\aleph_0)$ holds in ZFC, while the statement $MA(2^{\aleph_0})$ does not.

Proof. Suppose that $\mathcal{D} = \{D_n\}$ is a family of dense subsets of $(P, \leq)$. We pick $d_1 \in D_1$ and since D_2 is dense, there is some $d_2 \in D_2$ such that $d_2 \leq d_1$. We define by induction a set $S = \{d_n\}$ such that $d_n \in D_n$ and $d_{n+1} \leq d_n$ for all n. In order to prove that $MA(\aleph_0)$ holds, it suffices to consider the filter $\mathcal{F} = \{x \in P : \exists n(d_n \leq x)\}$.
In order to verify that $MA(2^{\aleph_0})$ is inconsistent with ZFC, consider :

$$P = (\{\text{finite partial functions } f : \omega \to 2\}, \leq)$$

with $f \leq g$ if and only if f extends g. We note that f, g are compatible if they both agree on the intersection of their domains and that since P is countable, it is immediate that P is c.c.c. For each $n \in \omega$ and $F \in 2^\omega$, define :

$$A_n = \{f \in P : n \in \text{dom}(f)\} \text{ and } B_F = \{f \in P : \exists n \in \text{dom}(f)(f(n) \neq F(n))\}$$

Set $\mathcal{D} = \{A_n\} \cup \{B_F\}$ and note that $\mathcal{D}$ is a collection of dense subsets of P such that $|\mathcal{D}| = 2^{\aleph_0}$. Suppose that $\mathcal{F}$ is a filter on P and assume, towards a contradiction, that $\mathcal{F} \cap D \neq \emptyset$ for every $D \in \mathcal{D}$. Since $\mathcal{F}$ is a filter, for every subset $S \subset \mathcal{F}$ there is an element f_S which extends all elements in S. Choose $S = \{\mathcal{F} \cap D\}_{D \in \mathcal{D}}$ and consider the element f_S. Then, $\mathrm{dom}(f_S) = \omega$ but f_S does not coincide with any element in 2^ω, which is a contradiction. ■

Remark 7.6. For evident reasons, one often assumes the negation of the Continuum Hypothesis while working under $MA(\kappa)$. What may not be so evident is the reason why we assume that $(P, \leq)$ is c.c.c., since this is not needed in order to prove that $MA(\aleph_0)$ holds in ZFC (cf. Theorem 7.5). Indeed, if $(P, \leq)$ is not c.c.c. then $MA(\aleph_1)$ may not hold : consider the following partial order :

$$P = (\{\text{finite partial functions } f : \omega \to \omega_1\}, \leq)$$

where $f \leq g$ if and only if f extends g. Clearly, P is not c.c.c. as $\{\langle 0, \alpha \rangle\}_{\alpha \in \omega_1}$ is an uncountable family of incompatible elements in P. For each $\alpha \in \omega_1$ we define $D_\alpha = \{f \in P : \alpha \in \mathrm{ran}(f)\}$ and note that $\mathcal{D} = \{D_\alpha\}_{\alpha \in \omega_1}$ is a family of dense subsets of P. Suppose, towards a contradiction, that $\mathcal{F}$ is a filter on P such that $\mathcal{F} \cap D_\alpha \neq \emptyset$ for every $\alpha \in \omega_1$. Then, consider the element g_S which extends all elements in $S = \{\mathcal{F} \cap D_\alpha\}$ and thus, $\mathrm{ran}(g_S) = \omega_1$ which is impossible.

References

[1] Arveson W., *Operator algebras and invariant subspaces*, Ann. of Math. (2) 100, 433-532 (1974)

[2] Badea C., Devinck V., Grivaux S., *Escaping a neighborhood along a prescribed sequence in Lie groups and Banach algebras*, Canadian Math. Bull. 63(3), 1-25 (2019)

[3] Badea C., Grivaux S., *Size of the peripheral point spectrum under power or resolvent growth conditions*, J. Funct. Anal. 246, 302-339 (2007)

[4] Bari N., *Sur l'unicité du développement trigonometrique*, Fund. Math. 9, 62-115 (1927)

[5] Beer G., *A Polish topology for the closed subsets of a Polish space*, Proc. Amer. Math. Soc. vol. 113 (4), 1123-1133 (1991)

[6] Bessaga C., Pełczyński A., *On bases and unconditional convergence of series in Banach spaces*, Studia Math. 17, 151-164 (1958)

[7] Blair C., *The Baire category theorem implies the principle of dependent choices*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 25(10), 933-934 (1977)

[8] Blümlinger M., *Characterization of measures in the group C^*-algebra of a locally compact group*, Math. Ann. 289 (3), 393-402 (1991)

[9] Bogachev V., *Measure theory, vol. 2*, Springer-Berlin (2007)

[10] Borel É., *Les probabilités dénombrables et leurs applications arithmétiques*, Supplemento di rend. circ. Mat. Palermo 27, 247-271 (1909)

[11] Bossard B., *A coding of separable Banach spaces. Analytic and coanalytic families of Banach spaces*, Fund. Math. 172(2), 117-152 (2002)

[12] Bourgain J., *On separable Banach spaces, universal for all separable reflexive spaces*, Proc. Amer. Math. Soc. 79, 241-246 (1980)

[13] Bożejko M., *Sets of uniqueness on non commutative locally compact groups*, Proc. Amer. Math. Soc. 64 (1), 93-96 (1977)

[14] Bożejko M., *Sets of uniqueness on non commutative locally compact groups II*, Colloq. Math. 42, 39-41 (1979)

[15] Brown N., Ozawa N., *C^*-algebras and finite dimensional approximations*, Graduate Studies in Mathematics vol. 88, Amer. Math. Soc. (2008)

[16] Davis M., *Infinite games of perfect information*, Ann. Math. Studies 52, 85- 101 (1964)

[17] Debs G., Saint-Raymond J., *Ensembles boréliens d'unicité et unicité au sens large*, Ann. Inst. Fourier 37 (3), 217-239 (1987)

[18] Devinck V., *Universal Jamison spaces and Jamison sequences for C_0-semigroups*, Studia Math. 214(1) (2013)

[19] Diestel J., Spalsbury A., *The Joys of Haar Measure*, Graduate Studies in Mathematics, Amer. Math. Soc. 150 (2014)

[20] Dunkl C., Ramirez D., *Translation in measure algebras and the correspondence to Fourier transforms vanishing at infinity*, Mich. Math. J. 17, 311-320 (1970)

[21] Eisner T., Grivaux S., *Hilbertian Jamison sequences and rigid dynamical systems*, J. Funct. Anal. 261(7), 2013-2052 (2011)

[22] Eleftherakis G., *Bilattices and Morita equivalence of masa bimodules*, Proc. Edinb. Math. Soc. 59(3), 605-621 (2016)

[23] Erdos J., Katavolos A., Shulman V., *Rank one subspaces of bimodules over maximal abelian selfadjoint algebras*, J. Funct. Anal. 157(2), 554-587 (1998)

[24] Eymard P., *L'algèbre de Fourier d'un groupe localement compact*, Bulletin de la S. M. F., tome 92 (1964), 181-236

[25] Fremlin D., *Measure theory, vol. 4* (2004)

[26] Friedman H., *Higher set theory and mathematical practice*, Ann. Math. Logic 2, 325-357 (1971)

[27] Gale D., Stewart F., *Infinite games with perfect information*, Ann. Math. Studies 28 (1953), 245-266

[28] Galicki A., *Aspects of classical Descriptive Set Theory*

[29] Ghandehari M., *Derivations on the algebra of Rajchman measures*, Complex Analysis and its Synergies 5 (2019)

[30] Glab S., *On complexity of continuous functions differentiable on cocountable sets*, Real Anal. Exchange 34, 521-530 (2008)

[31] Gowers T., *A new dichotomy for Banach spaces*, GAFA 6 (1996)

[32] Graham C., McGehee C., *Essays in Commutative Harmonic Analysis*, Springer-Verlag, Berlin, New York (1979)

[33] Gödel K., *The consistency of the axiom of choice and the generalized continuum hypothesis*, Proc. Nat. Acad. Sci. USA 24 (1938), 556 – 557

[34] Hartig, D. G., *The Riesz representation revisited*, Amer. Math. Monthly 90 (1983), 277–280

[35] Izzo A.J., *A Functional Analysis proof of the existence of Haar measure on locally compact abelian groups*, Proc. Amer. Math. Soc. 115 no.2 (1992), 581–583

[36] Izzo A.J., *A simple proof of the existence of Haar measure on amenable groups*, Math. Scand. 120 no. 2 (2017), 317–319

[37] James R., *Basis and reflexivity of Banach spaces*, Ann. of Math. 52, 518-527 (1950)

[38] Jamison B., *Eigenvalues of modulus 1*, Proc. Amer. Math. Soc. 16, 375-377 (1965)

[39] Jech T., *The Axiom of Choice*, North-Holland Pub. Co. (1973)

[40] Kaniuth E., *A course in commutative Banach algebras*, Springer Verlag GTM 246 (2009)

[41] Kaniuth E., Lau A., *Fourier and Fourier-Stieltjes Algebras on Locally Compact Groups*, Mathematical Surveys and Monographs 231, Amer. Math. Soc. (2018)

[42] Kaufman R., *Lipschitz spaces and Suslin sets*, Jour. Funct. Anal. 42, 271-273 (1981)

[43] Kaufman R., *M-sets and distributions*, Astérisque 5, 225-230 (1973)

[44] Kaufman R., *On some operators in* c_0, Israel Jour. Math. 50, 353-356 (1985)

[45] Kechris A., *Classical Descriptive Set Theory*, Springer-Verlag GTM 156 (1995)

[46] Kechris A., *Set theory and uniqueness for trigonometric series*

[47] Kechris A., Louveau A., *Descriptive Set Theory and the Structure of Sets of Uniqueness*, London Mathematical Society Lecture Notes Series, Camb. Univ. Press (1987)

[48] Kechris A., Louveau A., Woodin W., *The structure of σ-ideals of compact sets*, Trans. Amer. Math. Soc. 310(2), 263-288 (1987)

[49] Kunen K., *Set Theory. An introduction to independence proofs*, North-Holland (1983)

[50] Körner T., *A pseudofunction on a Helson set*, Astérisque 5, 3-224 and 231-239 (1973)

[51] Lebesgue H., Sur les fonctions représentables analytiquement, Journal de Mathématiques Pures et Appliquées, vol. 1 (1905), 139-216

[52] Lindberg K., *On subspaces of Orlicz sequence spaces*, Studia Math. 45, 119-146 (1973)

[53] Lyons R., *A characterization of measures whose Fourier-Stieltjes transforms vanish at infinity*, PhD Thesis, Univ. of Michigan (1983)

[54] Martin D., *A purely inductive proof of Borel determinacy*, Recursion Theory, Proc. Symp. Pure Math. 42, 303-308 (1985)

[55] Martin D., *Borel determinacy*, Annals of Mathematics Second Series, Vol. 102, No. 2, 363-371 (1975)

[56] Matheron É., *Sigma-idéaux polaires et ensembles d'unicité dans les groupes abéliens localement compacts*, Ann. de l'institut Fourier 46 no 2, 493-533 (1996)

[57] Matheron É., Zeleny M. , *Descriptive Set Theory of Families of Small Sets*, Bull. Symb. Logic 13(4), 482-537 (2007)

[58] Mauldin R., *The set of continuous nowhere differentiable functions*, Pac. Journ. Math. 83, 199-205 (1979)

[59] Mazurkiewicz S., *Uber die menge der differenzierbaren functionen*, Fund. Math. 27, 244-249 (1936)

[60] Miller A., *Descriptive Set Theory and Forcing: How to prove theorems about Borel sets in a hard way*, Lecture notes in Logic 4, Springer-Verlag, New York (1995)

[61] Morris S., *Pontryagin duality and the structure of locally compact abelian groups*, Cambridge University Press (1977)

[62] Moschovakis Y., *Descriptive Set Theory, 2nd edition*, Mathematical Surveys and Monographs (2009)

[63] Mycielski J., Steinhaus H., *On the Lebesgue measurability and the axiom of determinateness*, Fund. Math. 54, 67-71 (1964)

[64] Nimiec P., *Borel parts of the spectrum of an operator and of the operator algebra of a separable Hilbert space*, Studia Math. 208, 77-85 (2010)

[65] Pełczyński A., Singer I., *On non-equivalent bases and conditional bases in Banach spaces*, Studia Math. 25, 5-25 (1964)

[66] Piatetski-Shapiro I., *On the problem of uniqueness of expansion of a function in trigonometric series*, Moscov. Gos. Univ. Uc. Zap. 165 Math. 7, 79-97 (1954)

[67] Pontrjagin L., *Topological groups*, Princeton Univ. Press (1939)

[68] Ransford T., *Eigenvalues and power growth*, Israel J. Math. 146, 93-110 (2005)

[69] Ransford T., Roginskaya M., *Point spectra of partially power-bounded operators*, J. Funct. Anal. 230, 432-445 (2006)

[70] Rudin W., *Real and Complex Analysis (3^{rd} edition)*, McGraw-Hill, Inc. (1987)

[71] Saeki S., *Helson sets which disobey spectral synthesis*, Proc. Amer. Math. Soc. 47(2), 371-377 (1975)

[72] Saint-Raymond J., *La structure borélienne d'Effros est-elle standard?*, Fund. Math. 100, 201-210 (1978)

[73] Shulman V., Todorov I., Turowska L., *Reduced spectral synthesis and compact operator synthesis*, Adv. Math. 367 (2020)

[74] Shulman V., Todorov I., Turowska L., *Sets of multiplicity and closable multipliers on group algebras*, J. Funct. Anal. 268, 1454-1508 (2015)

[75] Shulman V., Turowska L., *Operator synthesis I, bilattices and tensor algebras*, J. Funct. Anal. 209, 293-331(2004)

[76] Solecki S., *Covering analytic sets by families of closed sets*, Journ. Symb. Logic 59 (3), 1022-1031 (1994)

[77] Srivastava S., *A course on Borel sets*, Springer-Verlag GTM 180 (1998)

[78] Stone A.H., *Cardinals of closed sets*, Mathematika vol. 6, Issue 2 (1959), 99 - 107

[79] Szankowski A., *Embedding Banach spaces with unconditional bases into spaces with symmetric bases*, Israel J. Math. 15, 53-59 (1973)

[80] Tardivel V., *Fermés d'unicité dans les groupes abéliens localement compacts*, Studia Math. 91, 1-15 (1988)

[81] Taylor A., Wagon S., *A Paradox Arising from the Elimination of a Paradox*, Amer. Math. Monthly 126(4), 306-318 (2009)

[82] Wolfe P., *The strict determinateness of certain infinite games*, Pac. Jour. Math. 5, 841-847 (1955)

[83] Ülger A., *Relatively weak*-closed ideals of $A(G)$, sets of synthesis and sets of uniqueness*, Coll. Math. 136(2), 271-296 (2014)