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On the orthogonality of generalized eigenspaces for the Ornstein–Uhlenbeck operator

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Abstract. We study the orthogonality of the generalized eigenspaces of an Ornstein–Uhlenbeck operator \mathcal{L} in \mathbb{R}^N , with drift given by a real matrix B whose eigenvalues have negative real parts. If B has only one eigenvalue, we prove that any two distinct generalized eigenspaces of \mathcal{L} are orthogonal with respect to the invariant Gaussian measure. Then we show by means of two examples that if B admits distinct eigenvalues, the generalized eigenspaces of \mathcal{L} may or may not be orthogonal.

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1. Introduction. In this note, we discuss the orthogonality of the generalized eigenspaces associated to a general Ornstein–Uhlenbeck operator \mathcal{L} in \mathbb{R}^N .

Recently, the authors started studying some harmonic analysis issues in a nonsymmetric Gaussian context [1–3]. In particular, the Ornstein–Uhlenbeck semigroup $(\mathcal{H}_t)_{t>0}$ generated by \mathcal{L} is not assumed to be self-adjoint in $L^2(\gamma_\infty)$; here γ_∞ denotes the unique invariant probability measure under the action of the semigroup, and will be specified later.

In this general framework, the Ornstein–Uhlenbeck operator \mathcal{L} admits a complete system of generalized eigenfunctions; see [8]. But without self-adjointness, the orthogonality of distinct eigenspaces of \mathcal{L} is not guaranteed. In fact, while the kernel of \mathcal{L} is always orthogonal to the other generalized eigenspaces of \mathcal{L} in $L^2(\gamma_\infty)$, the question of orthogonality between generalized

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eigenspaces associated to nonzero eigenvalues is more delicate. As expected, the spectral properties of B play a prominent role here. Indeed, we prove in Section 3 that if B has a unique eigenvalue, then any two generalized eigenfunctions of \mathcal{L} corresponding to different eigenvalues are orthogonal in $L^2(\gamma_\infty)$.

Then in Sections 4 and 5, we exhibit two examples showing, respectively, that if B admits two distinct eigenvalues, the generalized eigenspaces associated to \mathcal{L} may or may not be orthogonal. The last section also contains a result which relates the orthogonality of the eigenspaces of \mathcal{L} to that of the eigenspaces of the drift matrix, under some restrictions.

In the following, the symbol I_k will denote the identity matrix of size k , and we omit the subscript when the size is obvious. We will write $\langle \cdot, \cdot \rangle$ for scalar products both in \mathbb{R}^N and in $L^2(\gamma_\infty)$. By \mathbb{N} we mean $\{0, 1, \dots\}$.

2. The Ornstein–Uhlenbeck operator. In this section, we specify the definition of the Ornstein–Uhlenbeck operator \mathcal{L} and recall some known facts concerning its spectrum.

We consider the Ornstein–Uhlenbeck semigroup $(\mathcal{H}_t^{Q,B})_{t>0}$, given for all bounded continuous functions f in \mathbb{R}^N , $N \geq 1$, and all $t > 0$ by the Kolmogorov formula

$$\mathcal{H}_t^{Q,B} f(x) = \int f(e^{tB}x - y) d\gamma_t(y), \quad x \in \mathbb{R}^N,$$

(see [6] and [7, Theorem 9.1.1]). Here B is a real $N \times N$ matrix whose eigenvalues have negative real parts, and Q is a real, symmetric, and positive-definite $N \times N$ matrix. Then we introduce the covariance matrices

$$Q_t = \int_0^t e^{sB} Q e^{sB^*} ds, \quad t \in (0, +\infty],$$

all symmetric and positive definite. Finally, the normalized Gaussian measures γ_t are defined for $t \in (0, +\infty]$ by

$$d\gamma_t(x) = (2\pi)^{-\frac{N}{2}} (\det Q_t)^{-\frac{1}{2}} e^{-\frac{1}{2} \langle Q_t^{-1} x, x \rangle} dx.$$

As mentioned above, γ_∞ is the unique invariant probability measure of the Ornstein–Uhlenbeck semigroup.

The Ornstein–Uhlenbeck operator is the infinitesimal generator of the semigroup $(\mathcal{H}_t^{Q,B})_{t>0}$, and it is explicitly given by

$$\mathcal{L}^{Q,B} f(x) = \frac{1}{2} \operatorname{tr} (Q \nabla^2 f)(x) + \langle Bx, \nabla f(x) \rangle, \quad f \in \mathcal{S}(\mathbb{R}^N),$$

where ∇ is the gradient and ∇^2 the Hessian.

By convention, we abbreviate $\mathcal{H}_t^{Q,B}$ and $\mathcal{L}^{Q,B}$ to \mathcal{H}_t and \mathcal{L} , respectively. We can thus write $\mathcal{H}_t = e^{t\mathcal{L}}$.

In [8, Theorem 3.1], it is verified that the spectrum of \mathcal{L} is the set

$$\left\{ \sum_{j=1}^r n_j \lambda_j : n_j \in \mathbb{N} \right\}, \quad (1)$$

where $\lambda_1, \dots, \lambda_r$ are the eigenvalues of the drift matrix B . In particular, 0 is an eigenvalue of \mathcal{L} , and the corresponding eigenspace $\ker \mathcal{L}$ is one-dimensional and consists of all constant functions, as proved in [8, Section 3].

We also recall that, given a linear operator T on some L^2 space, a number $\lambda \in \mathbb{C}$ is a generalized eigenvalue of T if there exists a nonzero $u \in L^2$ such that $(T - \lambda I)^k u = 0$ for some positive integer k . Then u is called a generalized eigenfunction, and those u span the generalized eigenspace corresponding to λ . As already recalled, it is known from [8, Section 3] that the Ornstein–Uhlenbeck operator \mathcal{L} admits a complete system of generalized eigenfunctions, that is, the linear span of the generalized eigenfunctions is dense in $L^2(\gamma_\infty)$. It is also known that all generalized eigenfunctions of \mathcal{L} are polynomials, see [7, Theorem 9.3.20].

2.1. Use of Hermite polynomials. As proved in [9], a suitable linear change of coordinates in \mathbb{R}^N makes $Q = I$ and Q_∞ diagonal. When applying this, we adhere to the notation introduced in [4, Lemma 1], where also the following facts can be found. Let \mathbf{H}_n denote the space of Hermite polynomials of degree n in these coordinates, adapted by means of a dilation to γ_∞ in the sense that the \mathbf{H}_n are mutually orthogonal in $L^2(\gamma_\infty)$ (they are called $H_{\lambda,k}$ in [4]). The classical Hermite expansion (called the Itô–Wiener decomposition in [4]) says that $L^2(\gamma_\infty)$ is the closure of the direct sum of the \mathbf{H}_n ; we refer to [10, p. 64] for a proof in dimension one and note that the extension to higher dimension is trivial. In other words, we can decompose any function $u \in L^2(\gamma_\infty)$ as

$$u = \sum_j u_j \quad (2)$$

with $u_j \in \mathbf{H}_j$ and convergence in $L^2(\gamma_\infty)$. Further, each \mathbf{H}_n is invariant under \mathcal{L} ; see [4, Proposition 1].

The Hermite decomposition implies, in particular, that each generalized eigenfunction of \mathcal{L} with a nonzero eigenvalue is orthogonal to the space of constant functions, that is, to the kernel of \mathcal{L} . Anyway, we provide here a proof of this fact which is independent of Hermite polynomials.

Lemma 2.1. *Let $\lambda \neq 0$. If $u \in L^2(\gamma_\infty)$ and $(\mathcal{L} - \lambda)^k u = 0$ for some $k \in \{1, 2, \dots\}$, then $\int u d\gamma_\infty = 0$.*

Proof. The implication is trivial if we set $k = 0$, so assume it holds for some $k \geq 0$ and that $(\mathcal{L} - \lambda)^{k+1} u = 0$.

Then

$$\mathcal{L}(\mathcal{L} - \lambda)^k u = \lambda(\mathcal{L} - \lambda)^k u,$$

and thus for any $t > 0$,

$$e^{t\mathcal{L}}(\mathcal{L} - \lambda)^k u = e^{t\lambda}(\mathcal{L} - \lambda)^k u.$$

These operators commute, so

$$(\mathcal{L} - \lambda)^k e^{t\mathcal{L}} u = (\mathcal{L} - \lambda)^k e^{t\lambda} u,$$

that is,

$$(\mathcal{L} - \lambda)^k (e^{t\mathcal{L}} u - e^{t\lambda} u) = 0.$$

The induction assumption now implies that

$$\int (e^{t\mathcal{L}} u - e^{t\lambda} u) d\gamma_\infty = 0.$$

Since γ_∞ is invariant under the semigroup, this means that

$$\int u d\gamma_\infty = e^{t\lambda} \int u d\gamma_\infty$$

for all $t > 0$. Thus the integral vanishes. \square

3. The case when B has only one eigenvalue.

Proposition 3.1. *If the drift matrix B has only one eigenvalue, then any two generalized eigenfunctions of \mathcal{L} with different eigenvalues are orthogonal with respect to γ_∞ .*

Let λ be the unique eigenvalue of B , which is necessarily real and negative. We first state a lemma and use it to prove the proposition. Recall that any generalized eigenfunction of \mathcal{L} is a polynomial.

Lemma 3.2. *Let u be a generalized eigenfunction of \mathcal{L} which is a polynomial of degree $n \geq 0$. Then the corresponding eigenvalue is $n\lambda$.*

Proof of Proposition 3.1. Let u be a generalized eigenfunction of \mathcal{L} , thus satisfying $(\mathcal{L} - \mu)^k u = 0$ for some $\mu \in \mathbb{C}$ and $k \in \mathbb{N}$. Applying the coordinates from Subsection 2.1, we can decompose u as in (2), where the sum is now finite. Since then

$$\sum_j (\mathcal{L} - \mu)^k u_j = 0$$

and each term here is in the corresponding \mathbf{H}_j , all the terms are 0. But this is compatible with Lemma 3.2 only if there is only one nonzero term in the decomposition of u . Thus $u \in \mathbf{H}_n$, where n is the polynomial degree of u .

Lemma 3.2 then implies that two generalized eigenfunctions with different eigenvalues are of different degrees and thus belong to different \mathbf{H}_n . The desired orthogonality now follows from that of the \mathbf{H}_n . \square

Proof of Lemma 3.2. Let u be a generalized eigenfunction of \mathcal{L} of polynomial degree n . We denote the corresponding eigenvalue by μ . Decomposing u as in (2), we see that this sum is for $j \leq n$ and that the term u_n is nonzero and a generalized eigenfunction of \mathcal{L} with eigenvalue μ . For some m , the function $(\mathcal{L} - \mu)^m u_n$ will then be an eigenfunction with the same eigenvalue. This function is in \mathbf{H}_n and thus a polynomial of degree n . As a result, we can assume that u is actually an eigenfunction of \mathcal{L} when proving the lemma.

We now choose coordinates in \mathbb{R}^N that give a Jordan decomposition of B . This means that $B = \lambda I + R$, where $R = (R_{i,j})$ is a matrix with nonzero entries only in the first subdiagonal. More precisely, $R_{i,i-1} = 1$ for $i \in P$, where P is a subset of $\{2, \dots, N\}$, and all other entries of R vanish.

We write $\mathcal{L} = \mathcal{S} + \mathcal{B}$, where

$$\mathcal{B}f(x) = \langle Bx, \nabla f(x) \rangle,$$

and \mathcal{S} is the remaining, second-degree part of \mathcal{L} . Notice that, when applied to polynomials, \mathcal{B} preserves the degree whereas \mathcal{S} decreases it by 2. So if v is the n th-degree part of u , we must have $\mathcal{B}v = \mu v$.

We let \mathcal{B} act on a monomial x^α , where $\alpha \in \mathbb{N}^N$ is a multiindex of length $|\alpha| = n$, getting

$$\begin{aligned}\mathcal{B}x^\alpha &= \sum_j \lambda x_j \frac{\partial x^\alpha}{\partial x_j} + \sum_{i \in P} x_{i-1} \frac{\partial x^\alpha}{\partial x_i} \\ &= \lambda \sum_j \alpha_j x^\alpha + \sum_{i \in P} \alpha_i \frac{x_{i-1}}{x_i} x^\alpha = \lambda n x^\alpha + \sum_{i \in P} \alpha_i x^{\alpha^{(i)}},\end{aligned}$$

where $\alpha^{(i)} = \alpha + e_{i-1} - e_i$ for $i \in P$. Here $\{e_j\}_{j=1}^n$ denotes the standard basis in \mathbb{R}^N . Thus the restriction of \mathcal{B} to the space of homogeneous polynomials of degree n is given as $\lambda n I + \mathcal{R}$, where \mathcal{R} is the linear operator that maps x^α to $\sum_{i \in P} \alpha_i x^{\alpha^{(i)}}$.

We claim that the only eigenvalue of \mathcal{R} is 0. If so, the only eigenvalue of the restriction of \mathcal{B} mentioned above is λn , which would prove the lemma since $\mathcal{B}v = \mu v$.

In order to prove this claim, we define for any $\alpha \in \mathbb{N}^N$ with $|\alpha| = n$,

$$V(\alpha) = \sum_1^N j \alpha_j.$$

Clearly $V(\alpha^{(i)}) = V(\alpha) - 1$. We select a basis in the linear space of all homogeneous polynomials of degree n consisting of all monomials x^α with $|\alpha| = n$, enumerated in such a way that V is nondecreasing. The definition of \mathcal{R} now shows that its matrix with respect to this basis is upper triangular with zeros on the diagonal. The claim follows, and so does the lemma. \square

4. B has two distinct eigenvalues: a first example. The following example shows that the generalized eigenspaces of the Ornstein–Uhlenbeck operator may be orthogonal even in the case when B has more than one eigenvalue. We show that \mathcal{L} , while not being self-adjoint, is normal; then the orthogonality of its eigenspaces follows from the spectral theorem.

In two dimensions, we let

$$Q = I_2 \quad \text{and} \quad B = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \quad (3)$$

whose eigenvalues are $-1 \pm i$.

One finds that

$$e^{sB} = e^{-s} \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}$$

and

$$e^{sB} e^{sB^*} = e^{-2s} I_2,$$

so that

$$Q_\infty = \frac{1}{2} I_2, \quad Q_\infty^{-1} = 2 I_2.$$

We write

$$B = -I_2 + R, \quad \text{where} \quad R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since $R = -R^*$, [8, Proposition 2.1] implies that \mathcal{L} is normal (observe that $I_2 = \frac{1}{2} D_{1/\lambda}$ in the notation of [8]).

However, we give below a brief, direct proof of this fact, independent of the change of variables adopted in [8, 9]. In the following, we write

$$\mathcal{L} = \mathcal{L}^0 + \mathcal{R},$$

where

$$\mathcal{L}^0 = \frac{1}{2} \Delta - \langle x, \nabla \rangle$$

is the standard Ornstein–Uhlenbeck operator and so self-adjoint in $L^2(\gamma_\infty)$. Further,

$$\mathcal{R} = \langle Rx, \nabla \rangle$$

is seen to be an antisymmetric operator in $L^2(\gamma_\infty)$. This leads to

$$[\mathcal{L}, \mathcal{L}^*] = [\mathcal{L}^0 + \mathcal{R}, \mathcal{L}^0 - \mathcal{R}] = -\mathcal{L}^0 \mathcal{R} + \mathcal{R} \mathcal{L}^0 - \mathcal{L}^0 \mathcal{R} + \mathcal{R} \mathcal{L}^0 = 2[\mathcal{R}, \mathcal{L}^0].$$

If we write ∂_i for ∂_{x_i} , $i = 1, 2$, this amounts to

$$2 \left[x_2 \partial_1 - x_1 \partial_2, \frac{1}{2} \Delta - x_1 \partial_1 - x_2 \partial_2 \right].$$

A straightforward computation shows that this vanishes, and so \mathcal{L} is normal. The spectral theorem for normal operators now implies the following result.

Proposition 4.1. *With $N = 2$, let Q and B be as in (3). Then each generalized eigenfunction of \mathcal{L} is an eigenfunction. Moreover, any two eigenfunctions of \mathcal{L} with different eigenvalues are orthogonal with respect to γ_∞ .*

5. B has two distinct eigenvalues: a second example. In this section, we exhibit a class of drift matrices B with two different eigenvalues (which, in contrast to those in the example in Section 4, are real), but such that the generalized eigenspaces associated to the corresponding Ornstein–Uhlenbeck operator \mathcal{L} are not orthogonal.

In \mathbb{R}^2 , we consider $Q = I_2$ and

$$B = \begin{pmatrix} -a + d & 0 \\ c & -a - d \end{pmatrix}, \quad (4)$$

with $a > d > 0$ and $c \neq 0$. To compute the exponential of sB , we write $B = -aI + M$, where

$$M = \begin{pmatrix} d & 0 \\ c & -d \end{pmatrix}.$$

Since $MM = d^2I$, we get for $s > 0$,

$$\exp(sB) = e^{-as} (\cosh(sd) I + d^{-1} \sinh(sd) M).$$

This leads to

$$\exp(sB) \exp(sB^*) = e^{-2as} \begin{pmatrix} e^{2sd} & \frac{c}{d} e^{sd} \sinh(sd) \\ \frac{c}{d} e^{sd} \sinh(sd) & \frac{c^2}{d^2} \sinh^2(sd) + e^{-2sd} \end{pmatrix}.$$

Integrating this matrix over $0 < s < \infty$, we obtain

$$Q_\infty = \begin{pmatrix} \frac{1}{2(a-d)} & \frac{\frac{c}{4a(a-d)}}{\frac{c^2}{4a(a-d)(a+d)} + \frac{1}{2(a+d)}} \\ \frac{\frac{c}{4a(a-d)}}{\frac{c^2}{4a(a-d)(a+d)} + \frac{1}{2(a+d)}} & \frac{1}{2(a+d)} \end{pmatrix},$$

and so

$$\frac{1}{2} Q_\infty^{-1} = \frac{1}{c^2 + 4a^2} \begin{pmatrix} 2a[c^2 + 2a(a-d)] - 2ac(a+d) & \\ & -2ac(a+d) \quad 4a^2(a+d) \end{pmatrix}.$$

The invariant measure γ_∞ is thus proportional to

$$\begin{aligned} & \exp \left(-\frac{2a[c^2 + 2a(a-d)]}{c^2 + 4a^2} x_1^2 + \frac{4ac(a+d)}{c^2 + 4a^2} x_1 x_2 - \frac{4a^2(a+d)}{c^2 + 4a^2} x_2^2 \right) dx \\ &= \exp \left(-(a-d) x_1^2 \right) \exp \left(-\frac{a+d}{c^2 + 4a^2} (cx_1 - 2ax_2)^2 \right) dx. \end{aligned}$$

Writing $z_1 = \sqrt{a-d} x_1$ and $z_2 = \sqrt{\frac{a+d}{c^2+4a^2}} (2ax_2 - cx_1)$ and recalling that γ_∞ is a probability measure, we see that

$$d\gamma_\infty = \pi^{-1} \exp \left(-z_1^2 - z_2^2 \right) dz.$$

To find some eigenfunctions of \mathcal{L} , we consider polynomials in x_1, x_2 of degree 2. One finds that

$$\begin{aligned} v_1 &= x_1^2 - \frac{1}{2(a-d)}, \\ v_2 &= x_1^2 - \frac{2d}{c} x_1 x_2 - \frac{1}{2a}, \\ v_3 &= x_1^2 - \frac{4d}{c} x_1 x_2 + \frac{4d^2}{c^2} x_2^2 - \frac{c^2 + 4d^2}{2c^2(a+d)} \end{aligned}$$

are eigenfunctions, with eigenvalues $-2(a-d)$, $-2a$, and $-2(a+d)$, respectively.

Any two of these polynomials turn out not to be orthogonal with respect to the invariant measure, as follows by straightforward computations. We sketch one example.

One simply multiplies v_1 and v_3 and rewrites the product in terms of z_1 and z_2 . Doing so, one can neglect all terms of odd order in z_1 or z_3 , when integrating with respect to γ_∞ . Writing "odd" for such terms, we find that the product is

$$\begin{aligned} & \frac{1}{a^2} z_1^4 + \frac{d^2(c^2 + 4a^2)}{a^2 c^2 (a^2 - d^2)} z_1^2 z_2^2 - \left[\frac{c^2 + 4d^2}{2c^2(a^2 - d^2)} + \frac{1}{2a^2} \right] z_1^2 \\ & - \frac{d^2(c^2 + 4a^2)}{2a^2 c^2 (a^2 - d^2)} z_2^2 + \frac{c^2 + 4d^2}{4c^2(a^2 - d^2)} + \text{odd}. \end{aligned}$$

Integrating and simplifying, we get

$$\int v_1 v_3 d\gamma_\infty = \frac{1}{2a^2} > 0,$$

so v_1 and v_3 are not orthogonal.

Remark 5.1. Let now $d = a/2$ in this example. Then the fourth-degree polynomial

$$v_4 = x_1^4 - \frac{6}{a} x_1^2 + \frac{3}{a^2}$$

is an eigenfunction of \mathcal{L} with eigenvalue $-2a$, like v_2 . Thus eigenfunctions of different polynomial degrees can have the same eigenvalue. This shows that for an eigenfunction u , the sum in (2) may consist of more than one term, and a (generalized) eigenspace need not be contained in one \mathbf{H}_n .

The eigenvalues of the matrix B defined in (4) are $-a \pm d$, and it is easily seen that the corresponding eigenspaces are not orthogonal in \mathbb{R}^2 . This turns out to be related to the non-orthogonality of the eigenspaces of \mathcal{L} , at least in two dimensions, in the following way.

Proposition 5.2. *Let $N = 2$ and $Q = I$, and assume that B has two different, real eigenvalues. Then the generalized eigenspaces of \mathcal{L} are orthogonal in $L^2(\gamma_\infty)$ if and only if the two eigenspaces of B are orthogonal in \mathbb{R}^2 .*

Proof. To begin with, we consider a coordinate change $\tilde{x} = Hx$, where H is an orthogonal matrix. Simple computations show that the operator $\mathcal{L}^{Q,B}$ is transformed to $\mathcal{L}^{\tilde{Q},\tilde{B}}$ in the new coordinates, with $\tilde{Q} = HQH^*$ and $\tilde{B} = H\tilde{B}H^*$; cf. [9, p. 474]. In our case, $\tilde{Q} = Q = I$. The eigenvalues of B and the angle between its eigenvectors will not change.

To prove the proposition, assume first that the (real) eigenvectors of B are orthogonal in \mathbb{R}^2 . Then B is symmetric since it can be diagonalized by means of an orthogonal change of coordinates as just described. This implies that \mathcal{L} is symmetric ([7, Proposition 9.3.10]), so that the orthogonality of its eigenspaces is trivial.

Next, we assume that the eigenvectors of B are not orthogonal in \mathbb{R}^2 . By Schur's decomposition theorem (see [5, Theorem 2.3.1]), there exists an orthogonal change of coordinates which makes B lower triangular, though not diagonal. We are thus in the situation described in (4). As we have seen, some eigenspaces of \mathcal{L} are then not orthogonal with respect to the invariant measure. \square

We finally remark that the “if” part of this proposition easily extends to arbitrary dimension N . Then it is assumed that B has N different, real eigenvalues with mutually orthogonal eigenspaces.

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