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IASS 2020/21 SURREY 7
International Conference on Spatial Structures 2020/21

Proceedings of the IASS Annual Symposium 2020/21 and the 7th International Conference on Spatial Structures
Inspiring the Next Generation
23 – 27 August 2021, Guildford, UK
S.A. Behnejad, G.A.R. Parke and O.A. Samavati (eds.)

The construction of new masonry bridges inspired by Paul Séjourné

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Abstract

Masonry arch bridges for rail transport in Europe are still in use 200 years after they were built. Traditional masonry bridges support granular fill which in turn supports the roadway or railway tracks spreading the load on the arch. The stiffness requirements for high speed rail bridges mean that they have to be heavy, in which case masonry bridges supporting fill may again become attractive in comparison with steel or concrete. In this paper we describe the optimization of the bridge geometry to carry the approximately hydrostatic loading from the fill.

Keywords: masonry bridge, membrane theory, differential geometry, hydrostatic shell, form finding.

1 Introduction

The young civil engineer has a quite different relationship to masonry than to steel, concrete and timber. The engineer might live in a house of load-bearing masonry or travel across masonry bridges. These structures were made by previous generations of engineers, they are now a fairy-tale, and get little attention in education or practice. However, masonry arch bridges for railways are still being used 200 years after they were built and carry much higher loads than they were designed for. Masonry requires less maintenance than steel and concrete and many civil engineering structures from the 1960's are reaching the end of their lives [15], and some have collapsed, such as the Ponte Morandi in Genoa.

Masonry arch bridges are shell structures acting as a container and support for the granular fill, which in turn carries the railway and spreads the loads more uniformly on the arches. This can also be seen in the beautiful diagrams by Willis [24] of the fan vaults Peterborough Cathedral, where the fill is necessary for the stability [11].

The Maidenhead viaduct which still carries the railway westwards from London, Figures 1 and 2, by Isambard Kingdom Brunel has very shallow elliptical arches necessitated by the span and the desire to not raise the level of the track, which would have meant long inclines either side. The increased curvature of the arch near the supports balances the increased load from the fill. On the other hand Paul Séjourné used secondary arches over the supports to avoid the large fill load for the pont des Catalans, figure 3. Séjourné's *Grandes voûtes* [21] and *Progrès depuis cinquante ans dans l'art de projeter et d'exécuter grandes voûtes en maçonnerie* (Fifty years of progress in the art of designing and building large masonry arches) [22] provide the last word on masonry arches. *Progrès depuis cinquante ans* is available on the internet at the link given in the references, as is Tome III of *Grandes voûtes* [20] from which we have taken figure 4 which shows part of Séjourné's discussion of arch geometry. Figure 5 shows an elliptical arch by Séjourné.

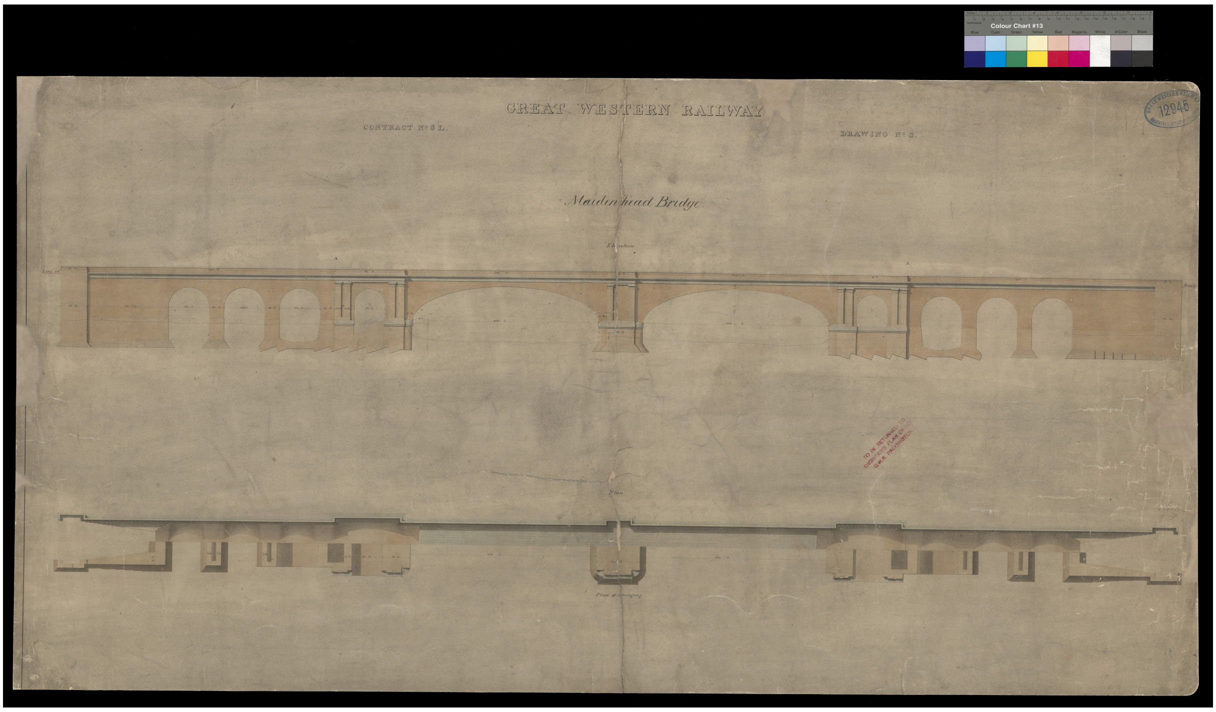


Figure 1: Elevation and plan of Maidenhead Railway Bridge, image from Network Rail Corporate Archive

It is generally understood that masonry arches stopped being built because they were too expensive and took too long to build, although it could have just as much been that they were unfashionable. But now the issue of sustainability means that the long life of masonry structures may make them again competitive using new methods of construction.

Figure 6 shows the construction of a typical masonry arch. The fill transmits the load from the road or railway to the arch. The fill also produces a lateral thrust on the spandrel walls. The spandrel walls in turn help stabilise the arch turning it from an arch to a cylindrical shell. We thus have a complex composite structure which would be difficult to design and analyse, even with the most sophisticated computer software. Physical model testing could be part of the design process, and we will discuss this briefly.

2 Background

There has been much recent work on the assessment of existing masonry arch bridges, see for example Hughes [12], Page [16] and Sarhosis et al. [18], and Bill Harvey and Hamish Harvey [8]. Page estimates that there are 40,000 masonry arch bridges in use in the UK. However, the techniques used for assessment and design are very different, even though the structural analysis techniques must be similar. The designer can use intelligence to design out uncertainty and avoid problems by removing them rather than solving them.

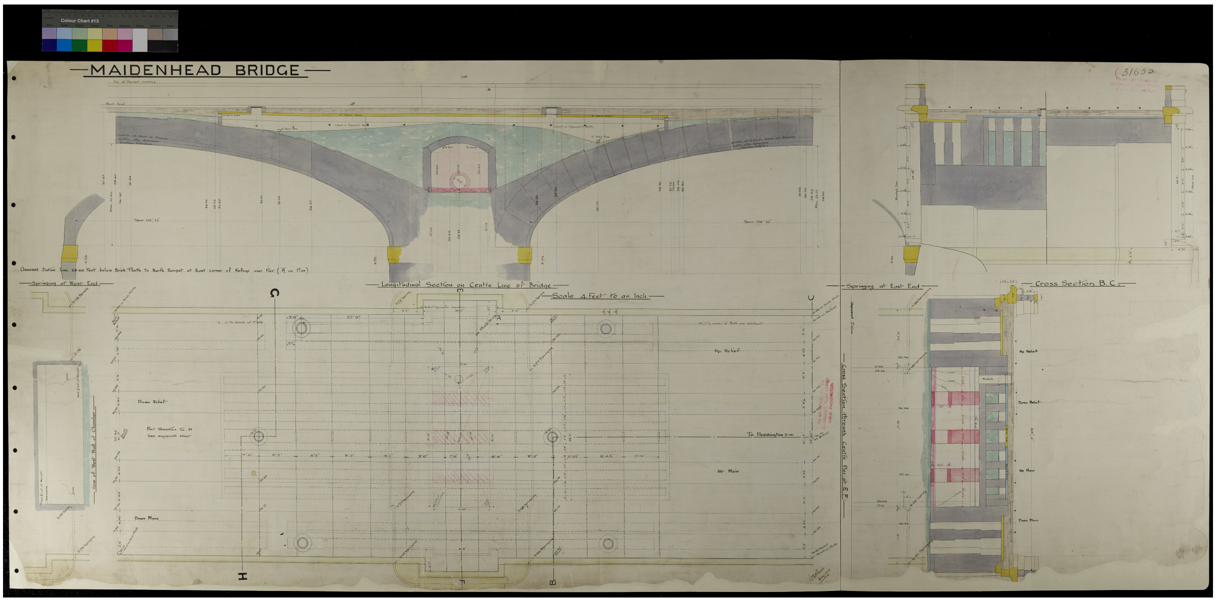


Figure 2: Cross section through chamber of Maidenhead Railway Bridge, image from Network Rail Corporate Archive

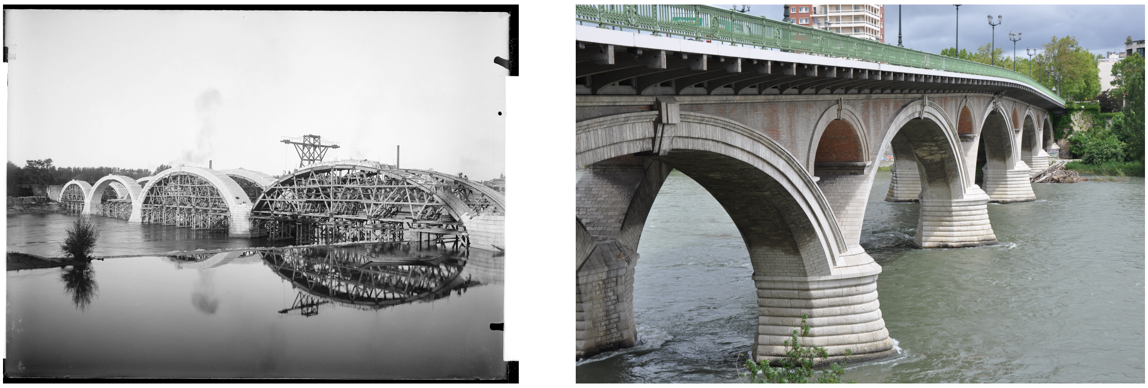
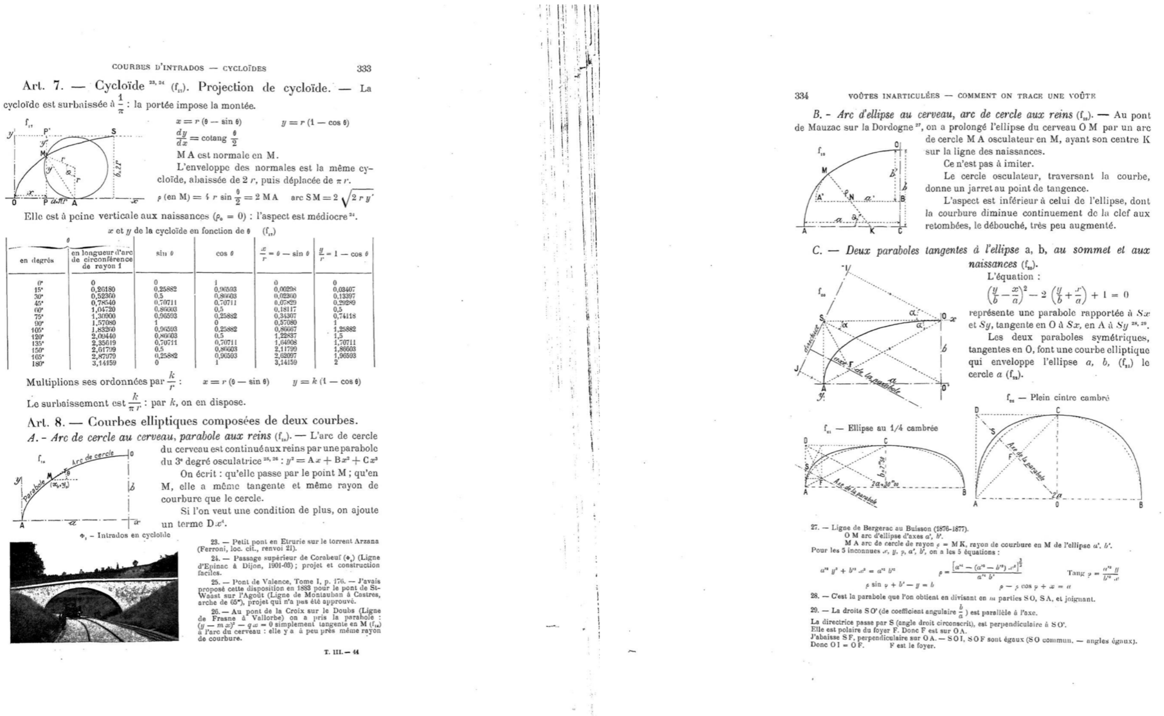


Figure 3: Le pont des Catalans, Paul Séjourné, 1908. Left image, licensed under public domain from Fonds Eugène Trutat, preserved by the muséum de Toulouse. Right photo by M.Strīķis licensed under CC BY-SA 3.0 https://commons.wikimedia.org/w/index.php?curid=56702265

3 Some historical notes on theory of arches and design of bridges

In 1717 Gautier stated five problems needed to be addressed [10], which are relevant for the design of bridges,

1. The thickness of abutment piers for all kinds of bridges.
2. The dimension of internal piers as a proportion of the span of the arches.
3. The thickness of the voussior between extrados and intrados in the neighbourhood of the keystone.
4. The shape of arches.
5. The dimensions of retaining walls to hold back soil.



COURBES D'INTRADOS — CYCLOÏDES 333

Art. 7. — Cycloïde [23, 24] (f_{17}). **Projection de cycloïde.** — La cycloïde est surbaissée à $\frac{1}{\pi}$: la portée impose la montée.

$x = r(\theta - \sin\theta)$ $\quad y = r(1 - \cos\theta)$

$\frac{dy}{dx} = \text{cotang}\ \frac{\theta}{2}$

M A est normale en M.

L'enveloppe des normales est la même cycloïde, abaissée de $2r$, puis déplacée de πr.

$\rho\ (\text{en M}) = 4r \sin\frac{\theta}{2} = 2\,\text{MA}$ $\quad$ arc $\text{SM} = 2\sqrt{2ry'}$

Elle est à peine verticale aux naissances ($\rho_0 = 0$) : l'aspect est médiocre [24].

x et y de la cycloïde en fonction de θ (f_{17})

θ en degrés	θ en longueur d'arc de circonférence de rayon 1	$\sin\theta$	$\cos\theta$	$\frac{x}{r} = \theta - \sin\theta$	$\frac{y}{r} = 1 - \cos\theta$
0°	0	0	1	0	0
15°	0,26180	0,25882	0,96593	0,00298	0,03407
30°	0,52360	0,5	0,86603	0,02360	0,13397
45°	0,78540	0,70711	0,70711	0,07829	0,29289
60°	1,04720	0,86603	0,5	0,18117	0,5
75°	1,30900	0,96593	0,25882	0,34307	0,74118
90°	1,57080	1	0	0,57080	1
105°	1,83260	0,96593	0,25882	0,86667	1,25882
120°	2,09440	0,86603	0,5	1,22837	1,5
135°	2,35619	0,70711	0,70711	1,64908	1,70711
150°	2,61799	0,5	0,86603	2,11799	1,86603
165°	2,87979	0,25882	0,96593	2,62097	1,96593
180°	3,14159	0	1	3,14159	2

Multiplions ses ordonnées par $\frac{k}{r}$: $\quad x = r(\theta - \sin\theta)$ $\quad y = k(1 - \cos\theta)$

Le surbaissement est $\frac{k}{\pi r}$: par k, on en dispose.

Art. 8. — Courbes elliptiques composées de deux courbes.

A. - Arc de cercle au cerveau, parabole aux reins (f_{18}). — L'arc de cercle du cerveau est continué aux reins par une parabole du 3e degré osculatrice [25, 26] : $y^2 = Ax + Bx^2 + Cx^3$

On écrit : qu'elle passe par le point M ; qu'en M, elle a même tangente et même rayon de courbure que le cercle.

Si l'on veut une condition de plus, on ajoute un terme Dx^4.

Φ_1 – Intrados en cycloïde

23. — Petit pont en Etrurie sur le torrent Arzana (Ferroni, loc. cit., renvoi 21).

24. — Passage supérieur de Corabeuf (Φ_1) (Ligne d'Epinac à Dijon, 1901-03) ; projet et construction faciles.

25. — Pont de Valence, Tome I, p. 176. — J'avais proposé cette disposition en 1883 pour le pont de St-Waast sur l'Agoût (Ligne de Montauban à Castres, arche de 65m), projet qui n'a pas été approuvé.

26. — Au pont de la Croix sur le Doubs (Ligne de Frasne à Vallorbe) on a pris la parabole : $(y - mx)^2 - qx = 0$ simplement tangente en M (f_{18}) à l'arc du cerveau : elle y a à peu près même rayon de courbure.

T. III. — 44

334 VOÛTES INARTICULÉES — COMMENT ON TRACE UNE VOÛTE

B. - Arc d'ellipse au cerveau, arc de cercle aux reins (f_{19}). — Au pont de Mauzac sur la Dordogne [27], on a prolongé l'ellipse du cerveau O M par un arc de cercle M A osculateur en M, ayant son centre K sur la ligne des naissances.

Ce n'est pas à imiter.

Le cercle osculateur, traversant la courbe, donne un jarret au point de tangence.

L'aspect est inférieur à celui de l'ellipse, dont la courbure diminue continuement de la clef aux retombées, le débouché, très peu augmenté.

C. — Deux paraboles tangentes à l'ellipse a, b, *au sommet et aux naissances* (f_{20}).

L'équation :

$$\left(\frac{y}{b} - \frac{x}{a}\right)^2 - 2\left(\frac{y}{b} + \frac{x}{a}\right) + 1 = 0$$

représente une parabole rapportée à Sx et Sy, tangente en O à Sx, en A à Sy [28, 29].

Les deux paraboles symétriques, tangentes en O, font une courbe elliptique qui enveloppe l'ellipse a, b, (f_{21}) le cercle a (f_{22}).

f_{21} — Ellipse au 1/4 cambrée

f_{22} — Plein cintre cambré

27. — Ligne de Bergerac au Buisson (1876-1877).
O M arc d'ellipse d'axes a', b'.
M A arc de cercle de rayon ρ = M K, rayon de courbure en M de l'ellipse a', b'.
Pour les 5 inconnues x, y, φ, a', b', on a les 5 équations :

$$a'^2 y^2 + b'^2 x^2 = a'^2 b'^2 \qquad \rho = \frac{\left[a'^4 - (a'^2 - b'^2)\,x^2\right]^{\frac{3}{2}}}{a'^4 b'} \qquad \text{Tang}\ \varphi = \frac{a'^2 y}{b'^2 x}$$

$$\rho \sin\varphi + b' - y = b \qquad \rho - \rho\cos\varphi + x = a$$

28. — C'est la parabole que l'on obtient en divisant en m parties SO, SA, et joignant.

29. — La droite SO' (de coefficient angulaire $\frac{b}{a}$) est parallèle à l'axe.
La directrice passe par S (angle droit circonscrit), est perpendiculaire à SO'.
Elle est polaire du foyer F. Donc F est sur OA.
J'abaisse SF, perpendiculaire sur OA. — SOI, SOF sont égaux (SO commun, — angles égaux).
Donc OI = OF. F est le foyer.

Figure 4: Arch geometry from Tome III of Paul Séjourné's *Grandes voûtes* (Large arches) [20]

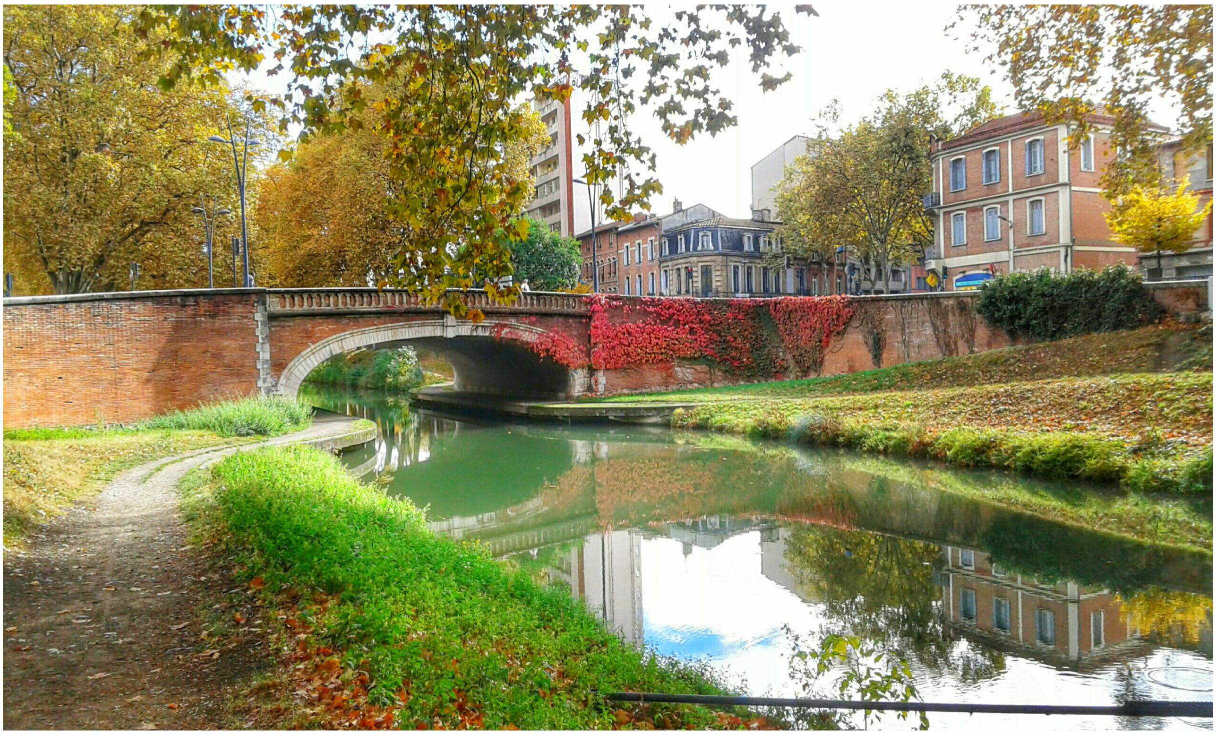


Figure 5: Pont de l'Avenue by Paul Séjourné over the Canal de Brienne in Toulouse. Photo by Cesar Sabas licensed under CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=44584947

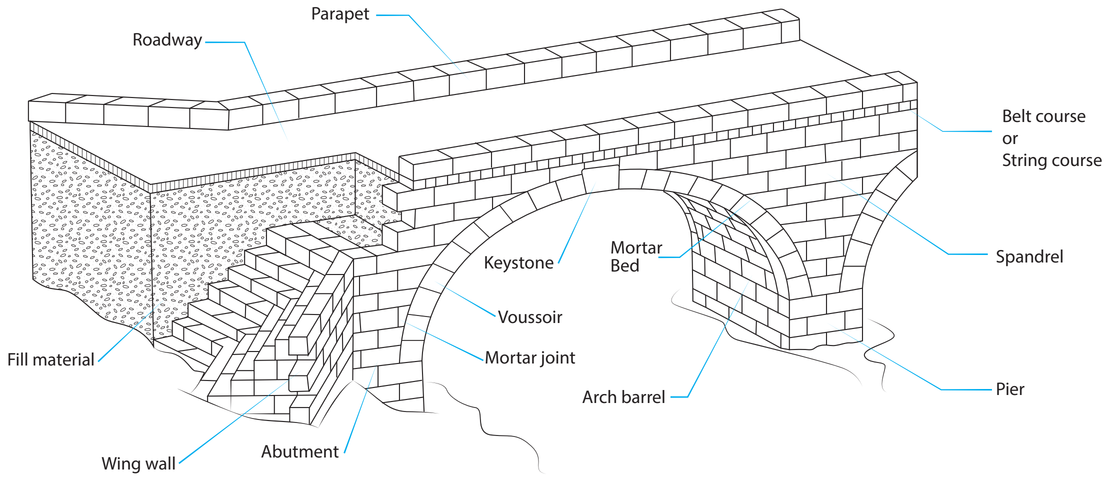


Figure 6: Typical stone arch bridge. Redrawn from Commonwealth of Pennsylvania

La Hire addressed the first problem. La Hire had returned his attention to masonry arches in 1712, and at this stage, abandoned his frictionless theory. The significance in La Hire's work at this stage was that he turned the attention to finding the value of the thrust such that abutments could be dimensioned to withstand the collapse of the vault. If the abutments were weak, La Hire stated that the arch would collapse somewhere between the springing and the crown. Choosing a plane of rupture, La Hire could set up a static procedure solving the internal force at the hinge check the necessary weight of the abutments to prevent an overturning. Bélidor specified this angle of 45° in his making this method applicable to architects and engineers. In 1730 Couplet made further progress in the understanding of masonry arches. Couplet did understand the fundamental concepts of masonry structures and stated that the friction in practice prevents the voussoirs from sliding, the compressive stresses are usually so low that no crushing can occur and that there exists no tensile strength to keep voussoirs from separating. However, empirically derived rules already existed during the 17th century, such as Blondel's rule, found in the book by Derand [4].

The fourth question was investigated by Hooke, Gregory, La Hire [14] and Jakob Bernoulli in the 17th century and early 18th century [3]. The importance was apparent in the bridge at Pontypridd by William Edwards, of which the third bridge, according to Heyman [10] collapsed due to its shape. The memoir by Coulomb from 1776, found translated in Heyman [9], is of much interest. Coloumb addresses many key issues relating to the problems stated by Gautier. Coloumb takes on the subject of fill pressure on retaining walls as well as the finite thickness of an arch, expanding on the work by La Hire and Couplet, resulting in an early formulation of the maximum and minimum thrust of the arch. Thus, describing the domain in which the line of thrust must lie. Later illustrated in the model by Barlow [2], and the line of thrust theory described by Moseley in England and Méry in France [13]. The line of thrust method was used in the assessment of bridges, and Castagliano proposed a method in 1879 for ensuring the engineer with a single answer [7]. During the first half of the 20th century, heavier vehicles increased the loads on the masonry bridges, leading to an interest in new assessment methods. The method developed by Pippard and Chitty in collaboration Davey became influential. It led to developing the quicker method, Military Engineering Experimental Establishment (MEXE) method, when the assessment was needed on the spot.

Alongside developments in the theory of arches, there is the question of the fill, that is soil mechanics after Coulomb [23] including Critical State Soil Mechanics [19].

4 Design consideration for arch bridges and viaducts interacting with fill

In designing a masonry bridge one will need to have compression only solution, although there is the possibility of external prestressing by post tensioning.

We have said that the spandrel walls turn an arch into a shell. The Alloz Aqueduct by Eduardo Torroja (Figure 7) is a cylindrical prestressed concrete shell supporting the hydrostatic load from the water. 'Cylindrical' means that it has the same cross-section all the way along, not that it is circular. It is an exceptionally interesting structure and clearly it is much simpler than a masonry arch supporting fill. If the Alloz aqueduct is treated as a shell the boundary condition at the free edge requires a shear stress which Torroja provided by prestressing. If it is treated as a beam the prestressing is still required to control the position of the neutral axis. Study of the Alloz aqueduct shows that lateral pressure from the water, or the fill in a bridge, is just as important as the vertical load.

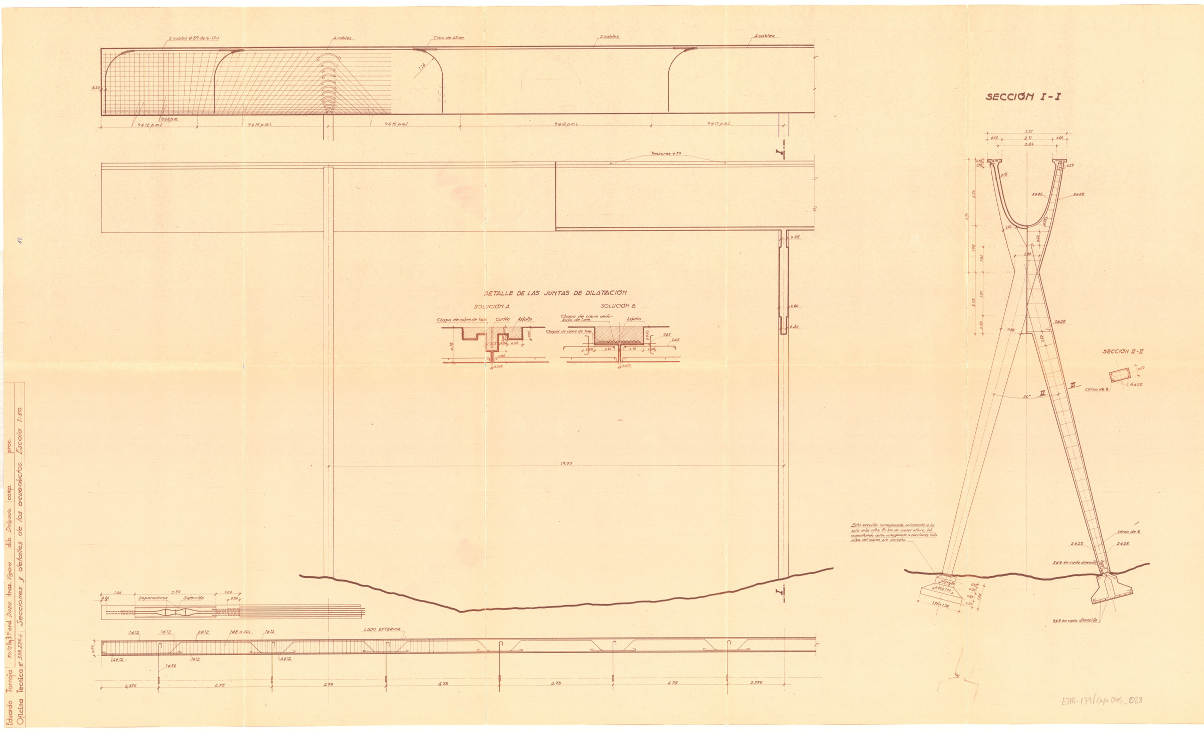


Figure 7: Section of Alloz aqueduct by Eduardo Torroja, 1939, source: Archivo Torroja, CEHOPU-CEDEX

Clearly the load from the fill is greatest near the supports where the depth of the fill is greatest. If we assume that the load from the fill on the arch is a hydrostatic pressure, and we ignore the weight of the arch in comparison to the fill, then the curvature of the arch must be proportional to the vertical distance from the top of the fill. The shape of the arch should therefore be that of the Euler's elastica, we need to take Euler's Figure 11 from our Figure 8 and rotate it clockwise by 90º to get the form of the arch. This is illustrated in Figure 90 in Rankine [17].

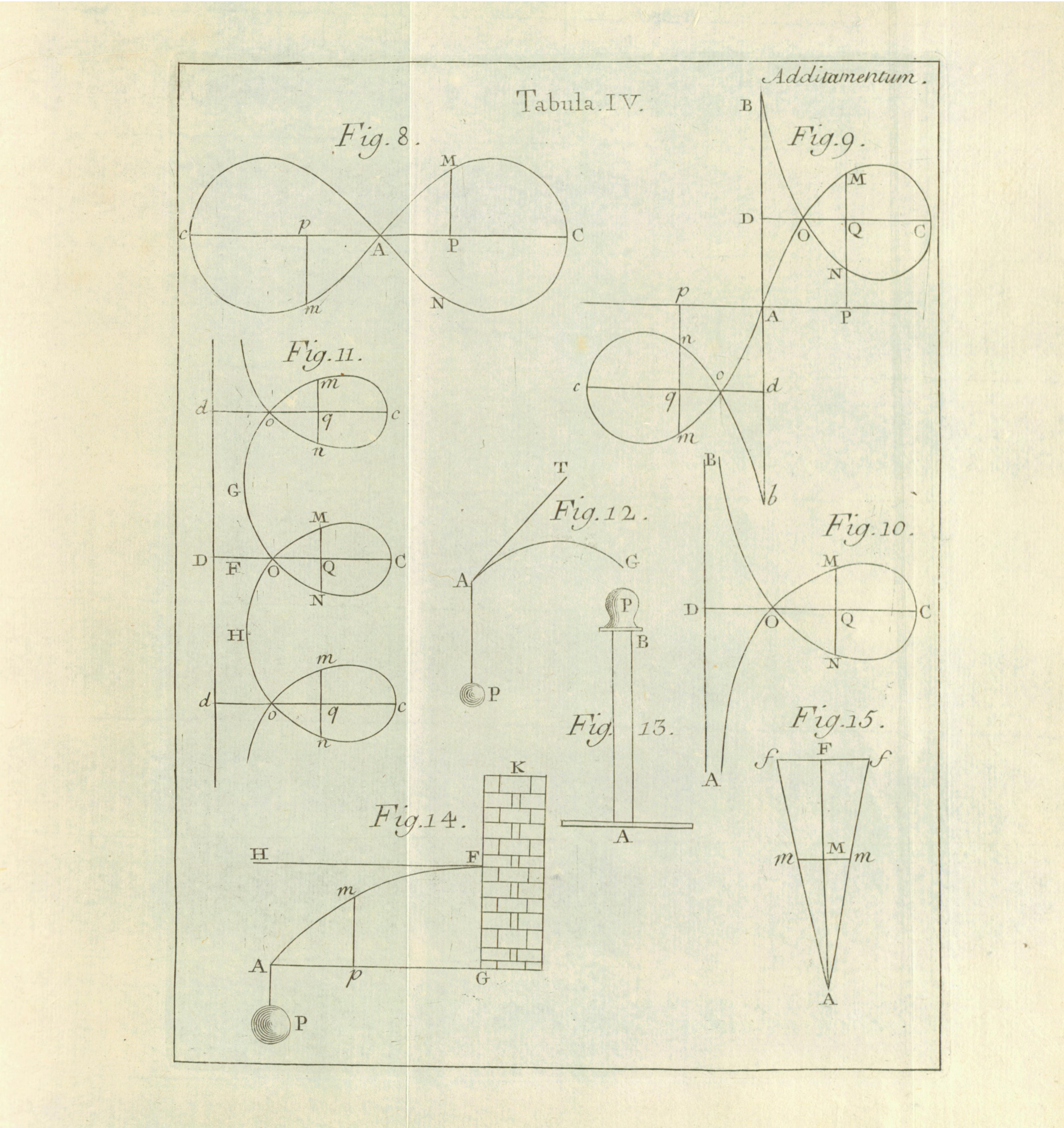


Figure 8: The elastica from Euler [5]. Fig. 11 in this figure should be rotated clockwise by 90º to get the form of a hydrostatic arch.

The fill spreads the load from the road on the arch and the weight of the fill stabilises the arch. So from this perspective the more fill the better. On the other hand from the point of view of foundations, the less fill the better. So it will always be a compromise.

5 Geometry and form finding of hydrostatic shells

It is difficult to generate the form of a hydrostatic shell. This is particularly due to the boundary conditions at the top of the of the wall, as we have discussed briefly for Torroja's Alloz aqueduct. However, we have to generate a form in order to start analysis, and in order to do this we have to make a number of assumptions, as indeed we have to do for any form finding.

The equilibrium equations for a general membrane shell element, as found in chapter 11 of Green & Zerna [6] are:

$$\frac{1}{\sqrt{a}}\left(\sigma^{\alpha\beta}\sqrt{a}\mathbf{a}_\beta\right)_{,\alpha} + p^\beta\mathbf{a}_\beta + p\mathbf{n} = 0 \tag{1}$$

$$\nabla_\alpha\sigma^{\alpha\beta} + p^\beta = 0 \tag{2}$$

$$\sigma^{\alpha\beta}b_{\alpha\beta} + p = 0 \tag{3}$$

$$p^\beta\mathbf{a}_\beta + p\mathbf{n} = \text{load per unit area} \tag{4}$$

$$\sigma^{\alpha\beta} = \text{components of the membrane stress tensor} \tag{5}$$

Green & Zerna use $n^{\alpha\beta}$ instead of $\sigma^{\alpha\beta}$ for the membrane stress tensor and $\mathbf{a}_3$ instead of $\mathbf{n}$ for the normal. We use ∇ for the covariant derivative.

The lateral pressure from the fill may well be less than hydrostatic and the active pressure [23] is a lower bound to this pressure. However, if we assume hydrostatic loading from the fill,

$$\rho g = \text{weigth per unit volume} \tag{6}$$

$$p = -\rho g z \tag{7}$$

$$p^\beta = 0 \tag{8}$$

We can always align the coordinate curves with the principal stresses to give

$$\sigma^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta = \frac{Q}{a_{11}}\mathbf{a}_1 \otimes \mathbf{a}_1 + T a^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta \tag{9}$$

$$a_{\alpha\beta}\sigma^{\alpha\beta} = Q + 2T \tag{10}$$

$$\sigma^{\alpha\beta}\sqrt{a}\mathbf{a}_\beta = \frac{Q\sqrt{a}}{a_{11}}\delta_1^\alpha\mathbf{a}_1 + T\sqrt{a}\mathbf{a}^\alpha \tag{11}$$

$$\left(\frac{Q\sqrt{a}}{a_{11}}\mathbf{a}_1\right)_{,1} + \left(T\sqrt{a}\mathbf{a}^\alpha\right)_{,\alpha} + \sqrt{a}p\mathbf{n} = 0 \tag{12}$$

$$\frac{Qb_{11}}{a_{11}} + Tb_\alpha^\alpha + p = 0. \tag{13}$$

We will now impose the condition that one set of coordinate curves are geodesics, thus the coordinate net becomes a geodesic coordinate net, as in Adiels et al. [1],

$$\mathbf{a}^2 \cdot \mathbf{a}_{1,1} = \Gamma_{11}^2 = 0 \tag{14}$$

producing

$$0 = \mathbf{a}_1 \cdot \left(\left(\frac{Q\sqrt{a}}{a_{11}}\mathbf{a}_1\right)_{,1} + \left(T\sqrt{a}\mathbf{a}^\alpha\right)_{,\alpha}\right) = \left(\frac{Q\sqrt{a}}{a_{11}}\right)_{,1} a_{11} + \frac{Q\sqrt{a}}{2a_{11}}a_{11,1} + \left(T\sqrt{a}\right)_{,1} - T\sqrt{a}\Gamma_{\alpha 1}^\alpha \tag{15}$$

and

$$0 = \mathbf{a}^2 \cdot \left(\left(\frac{Q\sqrt{a}}{a_{11}} \mathbf{a}_1 \right)_{,1} + \left(T\sqrt{a}\mathbf{a}^\alpha \right)_{,\alpha} \right) = \left(T\sqrt{a} \right)_{,\alpha} a^{2\alpha} - T\sqrt{a} a^{2\beta} \Gamma^\alpha_{\alpha\beta}. \tag{16}$$

Thus we finally arrive at the differential equations

$$\left(\frac{Q\sqrt{a}}{\sqrt{a_{11}}} \right)_{,1} \frac{\sqrt{a_{11}}}{\sqrt{a}} + T_{,1} = 0 \tag{17}$$

$$T_{,\alpha} a^{2\alpha} = 0 \tag{18}$$

$$\frac{Qb_{11}}{a_{11}} + Tb^\alpha_\alpha - \rho g z = 0 \tag{19}$$

which we have solved numerically to produce figures 9, 10 and 11.

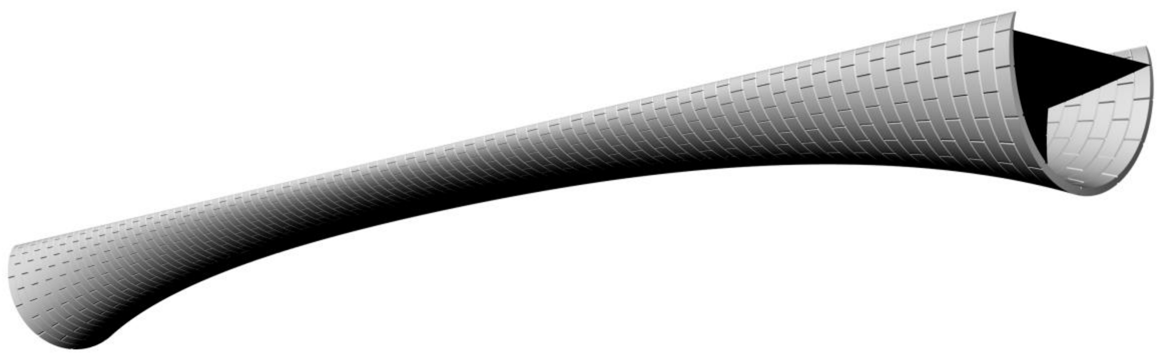


Figure 9: Design for single span bridge

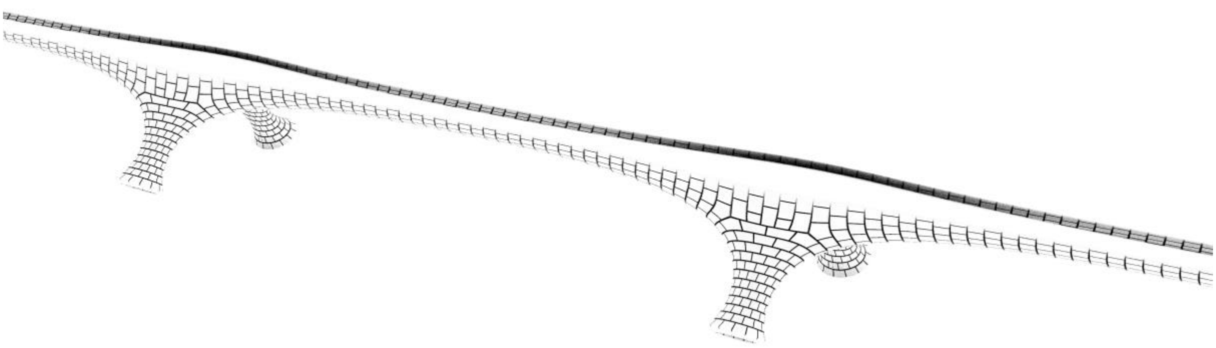


Figure 10: Design for multi span bridge

Figure 11: Design for multi span bridge

6 Simulation and model testing

Since a masonry arch is a composite shell involving the arch, the fill and the spandrel walls, it is a very complex problem in analysis in which the latest techniques in discrete element analysis and particle methods would be struggling.

In this situation physical models still have a place, if only to validate the numerical models. If the masonry can only carry tension and the fill is purely frictional, then scaling of the results is not a problem and there would be no need for equipment such as the large centrifuge at Cambridge University to model the own weight of the fill.

We still have the option of lower bound plasticity type analysis in which we postulate possible equilibrium states. But of course the frictional nature of the fill means that we break the normality condition stating that the yield strain has to be normal to the yield surface and so our analysis may no longer be a lower bound.

7 Discussion and conclusions

In this study, a simplification was made approximating hydrostatic loading from the fill. However, the form finding is only the starting point in the design process followed by further analysis considering more load cases and actions on the structure. Nevertheless, Rankine [17] examines geostatic arches, assuming a more accurate load from the earth, considering friction between the shell and the soil. Thus, future work is to investigate form finding, design and analysis of geostatic shells. Further studies will ideally also include physical model testing as a complement to numerical simulations.

McKibbins et al. [15] state that masonry bridges are some of the most durable and sustainable structures. Many have been in service for hundreds of years without significant repair or strengthening works — exceeding the design life requirements of modern structures. By contrast, many of the steel and concrete bridges built in the last century have required considerable expenditure on maintenance and repair or even replacement within the first 30 - 40 years of service. Civil engineers of today and the future have the challenge of working with local resources to produce durable solutions. Hence, designing efficient structures utilising soil or fill is one of the viable alternatives to contemporary practice for sustainable civil engineering for the future.

Perhaps we should now revisit Paul Séjourné's work to see if we can start building masonry bridges again.

8 Acknowledgements

We would like to thank Bill Baker of SOM who made us aware of the studies by Rankine [17] on hydrostatic arches.

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