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Tutorial on Tom and Jerry: the two smoothings of the anticanonical cone over $\mathbb{P}(1, 2, 3)$

Gavin Brown, Miles Reid, and Jan Stevens

Abstract. This is a first introduction to unprojection methods, and more specifically to Tom and Jerry unprojections. These two harmless tricks deserve to be better known, since they answer many practical questions about constructing codimension 4 Gorenstein subschemes. In particular, we discuss here the two smoothing components of the anticanonical cone over $\mathbb{P}(1, 2, 3)$.

Constructing Gorenstein rings in codimension 4 remains a difficult problem. Kustin–Miller [7] provide one approach: take as input a codimension 3 Gorenstein scheme X containing a divisor $D \subset X$ that is also Gorenstein (here X is local, affine, or graded in positive degree). From this they construct a codimension 4 Gorenstein scheme Y . Papadakis and Reid [10] state and prove the Kustin–Miller unprojection theorem via adjunction for the Serre–Grothendieck dualising sheaf: the Poincaré residue homomorphism $\mathcal{H}om(\mathcal{I}_D, \omega_X) = \omega_X(D) \twoheadrightarrow \omega_D$ leads to a rational function s on X with pole on D . Then \mathcal{O}_Y is given by adjoining the element s , to give $\mathcal{O}_Y = \mathcal{O}_X[s]$ where s satisfies only linear relations. See [10] for the full statement.

To apply unprojection, the problem is how to construct the input data $D \subset X$. The cases we consider here have X given by the 4×4 Pfaffians of a 5×5 skew matrix, and D a codimension 4 complete intersection contained in X . Tom and Jerry [4] are two explicit ways of constructing pairs $D \subset X$.

Consider a codimension 3 variety $X \subset \mathbb{A}^5$ defined by the 4×4 Pfaffians of a skew 5×5 matrix

$$M = \begin{pmatrix} m_{12} & m_{13} & m_{14} & m_{15} \\ & m_{23} & m_{24} & m_{25} \\ & & m_{34} & m_{35} \\ & & & m_{45} \end{pmatrix}$$

and a codimension 4 complete intersection ideal $I_D \subset \mathcal{O}_{\mathbb{A}^5}$. (We omit the zeros on the diagonal and the $m_{ji} = -m_{ij}$ with $i < j$.) The Tom and Jerry conditions on M are two methods of ensuring that the Pfaffians of M belong to I_D . You should think of them as related to maximal linear subspaces of $\text{Grass}(2, 5)$ in its Plücker embedding. Tom_i requires that the 6 entries m_{jk} with $j, k \neq i$ are in I_D ; we view this as 2 conditions on the m_{jk} .

Jerry $_{jk}$ requires that the 7 entries m_{ij} and m_{ik} in the j th and k th row and column of M are in I_D ; we view this as 3 conditions.

This paper gives a substantial illustration of Tom and Jerry to show their flexibility in practice, their ancestral relations to $\mathbb{P}^2 \times \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and other formats, and their application to the construction and especially the deformation theory of varieties.

Section 1 works out the 9 equations of the cone on $\mathbb{P}(1, 2, 3)$. Section 2 treats the “ 6×6 extrasymmetric format”, that describes the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ and some of its degenerations. One can view this as just algebraic manipulations, or as a typical case of Tom unprojection. In a similar vein, Section 3 treats the “Double Jerry construction”, that describes the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and some of its degenerations. In Section 4 we put these two unprojection constructions together as a versal deformation of the anticanonical cone over $\mathbb{P}(1, 2, 3)$ over a reducible base, with the obstructions also controlled by the matrix format. We conclude with some general remarks, mnemonics, slogans, and FAQ. We do not pretend any generality, or any theoretical treatment of Gorenstein codimension 4 (compare [12]).

1. The anticanonical cone over $\mathbb{P}(1, 2, 3)$

Let $X \subset \mathbb{A}^7$ be the anticanonical cone over $\mathbb{P}(1, 2, 3)_{(u,v,w)}$; this is also the quotient by the group action $\frac{1}{6}(1, 2, 3)$ on $\mathbb{A}_{(u,v,w)}^3$. We set out its 7 coordinate monomials as the Newton polygon

$$\begin{array}{cccc} u^6 & u^4v & u^2v^2 & v^3 & a & b & c & x \\ u^3w & uvw & & & d & e & & \\ w^2 & & & & f & & & \end{array} \quad (1)$$

The somewhat idiosyncratic choice of coordinates on \mathbb{A}^7 relates to the extrasymmetric format of Section 2.

One finds the equations defining X without difficulty. The semigroup ideal of internal monomials of the Newton polygon is generated by the single monomial $e = uvw$. There are tag relations between any three consecutive boundary monomials, that involve e if we turn a corner:

$$ac - b^2, \quad xb - c^2, \quad cf - e^2, \quad xdf - e^3, \quad af - d^2, \quad bd - ae.$$

Note in particular the equations $cf = e^2$ (that is, the tag at x is 0) and $xd = f^{-1}e^3$ or $xdf = e^3$ (the tag at f is -1).

These equations define the toric variety X in the complement of the coordinate hyperplanes, where e is invertible. The remaining generators of I_X come by colonizing out e : for example, $cf - e^2$ and $xdf - e^3$ give

$$(c(xdf - e^3) - xd(cf - e^2))e^{-2} = xd - ce,$$

where e is invertible. The ideal is generated by the 9 binomials:

$$\begin{aligned} ac - b^2, \quad xb - c^2, \quad cf - e^2, \quad af - d^2, \quad bd - ae \\ xd - ce, \quad bf - de, \quad dc - be, \quad xa - bc. \end{aligned} \tag{2}$$

Another way to view the equations is that they describe a singular del Pezzo surface S of degree 6. The monomials $U = u^3, V = uv, W = w$ base $H^0(\mathbb{P}(1, 2, 3), \mathcal{O}(3))$. We view them as coordinates on \mathbb{P}^2 . Then multiplying (1) by u^3 gives the 7 monomials

$$\begin{aligned} U^3 \quad U^2V \quad UV^2 \quad V^3 \\ U^2W \quad UVW \\ UW^2 \end{aligned} \tag{3}$$

that base the linear system of cubics in $\mathbb{P}_{U,V,W}^2$ with flex line $U = 0$ at $(0 : 0 : 1)$. It is an amusing exercise to recover from this the A_1 singularity $cf = e^2$ at P_x and the A_2 singularity $xdf = e^3$ at P_f .

2. Extrasymmetric format

2.1. Extrasymmetric format

Tom unprojections frequently lead to equations in extrasymmetric format. Consider for example the 6×6 skew matrix

$$N = \begin{pmatrix} z & y & a & b & d \\ & x & b & c & e \\ & & d & e & f \\ & & & \lambda z & \lambda y \\ & & & & \lambda x \end{pmatrix} = \begin{pmatrix} B & A \\ -A & \lambda B \end{pmatrix}. \tag{4}$$

A matrix of this shape is *extrasymmetric* (the term also covers slightly more general cases, see [4, 9.1]). It is made up of 3×3 blocks, where the top right block A is symmetric, the top left block B is skew, and the bottom right block λB repeats the information contained in the top left block, in this case with a nonzero scalar factor λ .

The 4×4 Pfaffians of N generate the ideal of the Segre embedding

$$\text{Segre}(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}_{\langle a,b,c,d,e,f,x,y,z \rangle}^8.$$

More precisely, the extrasymmetry means that the 15 upper-triangular entries of N consist of 9 independent entries and 6 repeats. The same is true of the 4×4 Pfaffians of N ,

which give 9 relations and 6 repeats. The resulting 9 equations define a variety in $\mathbb{A}_{\langle x,y,z,a,b,c,d,e,f \rangle}^9$ that, for $\lambda \neq 0$, is a linear transformation away from the affine cone over $\text{Segre}(\mathbb{P}^2 \times \mathbb{P}^2)$. The linear transformation involves taking $\mu = \sqrt{-\lambda}$. If we write

$$\lambda x^2 - cf + e^2 = (e - \mu x)(e + \mu x) - cf = e'x' - cf$$

and similarly for d', e' and y', z' , leaving a, c, f fixed, the Pfaffian equations become linear combinations of the 2×2 minors of

$$\begin{pmatrix} a & b' & d' \\ z' & c & e' \\ y' & x' & f \end{pmatrix} = A + \sqrt{-\lambda}B.$$

This agrees with the more banal way of defining $\text{Segre}(\mathbb{P}^2 \times \mathbb{P}^2)$ as $\bigwedge^2 M = 0$ for a generic 3×3 matrix M . Any such matrix M may be written $M = A + \mu B$ with A symmetric and B skew, and the ideal of 2×2 minors of M equals the ideal of 4×4 Pfaffians of the extrasymmetric matrix $N = \begin{pmatrix} B & A \\ -A & \lambda B \end{pmatrix}$. Swapping the signs of the square root $\mu = \sqrt{-\lambda}$ transposes the matrix M and so interchanges the two copies of \mathbb{P}^2 in $\mathbb{P}^2 \times \mathbb{P}^2$.

In more geometrical terms, this format displays $\mathbb{P}^2 \times \mathbb{P}^2$ as a nongeneric linear section of $\text{Grass}(2, 6)$.

2.2. Specialise to $v_6(\mathbb{P}(1, 2, 3))$

Now we consider λ as a variable and specialise the matrix (4) by setting $\lambda = 0$, $z = 0$ and $y = c$; the Pfaffian equations specialise to (2). That is, the anticanonical cone X over $\mathbb{P}(1, 2, 3)$ is the particular section $\lambda = 0$, $z = 0$ and $y = c$ of a degeneration of the cone over $\mathbb{P}^2 \times \mathbb{P}^2$. Wiggling the section gives one of the smoothing components of the deformations of X .

2.3. The same viewed as a Tom unprojection

As we said, the extrasymmetric matrix N in (4) has 6 repeated entries. The entries that are not repeated are the three diagonal entries a, c, f of the top right 3×3 block A . They correspond to the three coordinate points of $\mathbb{P}^2 \times \mathbb{P}^2$ such as $P_a = (1 : 0 : 0; 1 : 0 : 0)$, etc. Here again λ is a nonzero scalar, and P_a is unmoved by the linear coordinate changes or choices of $\sqrt{-\lambda}$ above.

Now project from P_a , and view the original equations as the result of undoing this projection. A practical point of view on unprojection is that it groups the 9 equations according to how they involve a . Because of the format of (4), a only appears linearly in 4 equations

$$\begin{aligned} ac &= b^2 + \lambda z^2, & ae &= bd + \lambda yz, \\ af &= d^2 + \lambda y^2, & ax &= yb - zd, \end{aligned} \tag{5}$$

and the remaining 5 equations not involving a are the Pfaffians of

$$N_{\hat{4}} = \begin{pmatrix} z & y & b & d \\ & x & c & e \\ & & e & f \\ & & & \lambda x \end{pmatrix} \quad (6)$$

(delete row and column 4 from N of (4)). What makes $N_{\hat{4}}$ a Tom_1 matrix is that the 6 entries not in row and column 1 are in the codimension 4 complete intersection ideal (x, c, e, f) . The coincidences $m_{25} = m_{34} = e$ and $m_{45} = \lambda x = \lambda m_{23}$ that bring this about are remnants of the extrasymmetry of N . From this point of view a is an unprojection variable: the equations (5) describe a map from the ideal (x, c, e, f) to the coordinate ring of the affine cone over $\text{Segre}(\mathbb{P}^2 \times \mathbb{P}^2)$ that is not a multiple of the natural inclusion (cf. §4.1 below). If we did not already have them, the main theorem of [10] would allow us to recover the variable a up to a unit and the equations (5) from $N_{\hat{4}}$ and (x, c, e, f) ; see [10, Remark 1.3].

Geometrically, the Pfaffians of (6) define the projection of $\mathbb{P}^2 \times \mathbb{P}^2$ from P_a . It is a 4-fold section of $\text{Grass}(2, 5)$ containing the 3-plane $\mathbb{P}^3_{(b,d,z,y)}$ defined by the ideal (x, c, e, f) .

2.4. Finding the Tom format from $v_6(\mathbb{P}(1, 2, 3))$

We can start from the other end, dividing the 9 equations (2) of $v_6(\mathbb{P}(1, 2, 3))$ into 4 that are linear in a and 5 not involving a . One gets $af = d^2$, $ae = bd$, $ac = b^2$ and $ax = bc$ together with the five Pfaffians of

$$\begin{pmatrix} 0 & c & b & d \\ & x & c & e \\ & & e & f \\ & & & 0 \end{pmatrix}. \quad (7)$$

If we hope to describe the set of all 9 equations as Pfaffians of a special 6×6 skew matrix, we must put a where it multiplies x, c, e, f and *not* b, d , so put it at the end of the first row as m_{16} ; row and column operations can take it to m_{14} as in the matrix (4).

3. Double Jerry format

3.1. Double Jerry

A neat starting point [4, 9.2] is to view Double Jerry as a theorem saying that a codimension 2 complete intersection $m_1 = m_2 = 0$ that contains two different codimension 3 complete intersections (x_1, x_2, x_3) and (y_1, y_2, y_3) is defined by two bilinear forms

$$m_1(x_1, x_2, x_3; y_1, y_2, y_3) \quad \text{and} \quad m_2(x_1, x_2, x_3; y_1, y_2, y_3).$$

We can then introduce two parallel sets of unprojection equations

$$s \cdot (x_1, x_2, x_3) = \cdots \quad \text{and} \quad t \cdot (y_1, y_2, y_3) = \cdots,$$

each taking us to codimension 3, together with a *long equation* $st = \dots$. Each unprojection separately is given by Cramer’s rule, leading to a 5×5 Pfaffian Jerry matrix, but the long equation is an intriguing and in general surprisingly complicated function of m_1, m_2, x_i, y_i . A particular case is worked out in Brown and Georgiadis [3].

3.2. Our particular case

Rather than rework the general material of [4, 9.2], consider only the case of the Newton polygon (1). As before, a only appears linearly in 4 equations, so can be eliminated or “projected out”, expressing the variety as an unprojection. The 5 equations not involving a are again the Pfaffians of (7). However, we now view it as a Jerry₂₃ matrix: in fact, the 7 entries $\{0, x, c, e\} \cup \{c, x, e, f\}$ of its 2nd and 3rd rows and columns consist of the regular sequence x, c, e, f with repeats:

$$\begin{array}{cccc}
 0 & & & \\
 x & c & e & \\
 \text{row 2} & \subset & \begin{pmatrix} 0 & c & b & d \\ \boxed{x} & c & e & \\ & e & f & \\ & & & 0 \end{pmatrix} & \supset \text{row 3} \\
 & & & c \\
 & & & x \\
 & & & e & f.
 \end{array}$$

What makes it a *double* Jerry is that the pivot $m_{23} = x$ is one of the variables on the nose, rather than a linear combination.

The matrix

$$\begin{pmatrix} \mu f & c + \nu f & b & d \\ & x & c & e \\ & & e & f \\ & & & -g \end{pmatrix}, \tag{8}$$

is a deformation respecting the Jerry₂₃ requirements just described. Here μ and ν scalars, and g is a new indeterminate of degree 1, not constrained by the Jerry format to be in the ideal (x, c, e, f) , that arises naturally as an additional degree of freedom. Putting back a as unprojection variable defines a family of del Pezzo 3-folds

$$W_{\mu, \nu} \subset \mathbb{P}_{\langle a, b, c, d, e, f, g, x \rangle}^7.$$

We recover $v_6(\mathbb{P}(1, 2, 3))$ on setting $\mu = \nu = 0$ and taking the hyperplane section $g = 0$.

3.3. Interpretation as double Jerry

Two of the Pfaffians of (8) do not involve x :

$$be - cd + \mu fg \quad \text{and} \quad bf + cg - de + \nu fg. \tag{9}$$

The codimension 2 complete intersection $U_{\mu, \nu}$ defined by these contains as divisors two different codimension 3 complete intersection $V(b, d, g)$ and $V(c, e, f)$. Unprojecting these lead to x and a respectively: schematically we have

$$\begin{array}{ccccc}
 \mathbb{P}^8 & \xrightarrow{\text{project out } a} & \mathbb{P}^7 & \xrightarrow{\text{project out } x} & \mathbb{P}^6 \\
 \cup^{\text{codim } 4} & & \cup^{\text{codim } 3} & & \cup^{\text{codim } 2} \\
 W_{\mu, \nu} & \dashrightarrow & V_{\mu, \nu} & \dashrightarrow & U_{\mu, \nu}
 \end{array}$$

and unprojection allows us to work backwards from $U_{\mu,v}$ and its two distinguished complete intersection divisors to construct $W_{\mu,v}$.

In more detail, first write (9) as

$$(b \quad d \quad g) \begin{pmatrix} e & f \\ -c & -e \\ \mu f & c + \nu f \end{pmatrix} = 0.$$

By Cramer's rule, (b, d, g) is proportional to the minors of the 3×2 matrix. This predicts the remaining 3 minors of (8):

$$\begin{aligned} xb &= c^2 - \mu ef + \nu cf, \\ xd &= ce + \nu ef - \mu f^2, \\ xg &= -cf + e^2. \end{aligned} \tag{10}$$

For c, e, f , working in the same way, (9) gives

$$\begin{pmatrix} -d & b & \mu g \\ g & -d & b + \nu g \end{pmatrix} \begin{pmatrix} c \\ e \\ f \end{pmatrix} = 0.$$

Adjoining a as the unprojection variable gives the other half of the double Jerry:

$$\begin{aligned} ac &= b^2 + \nu bg + \mu dg, \\ ae &= bd + \nu dg + \mu g^2, \\ af &= -bg + d^2, \end{aligned} \quad \text{and} \quad \begin{pmatrix} -a & b & -d & \mu g \\ & -d & g & b + \nu g \\ & & f & -c \\ & & & e \end{pmatrix}.$$

We get the long equation for ax by cancelling b, c, d, e, f or g from a linear combination of the other equations. There are many such derivations: for example, start from $xg = e^2 - fc$, multiply by a and rewrite the right-hand side until it is divisible by g . The result is

$$ax = (b + \nu g)(c + \nu f) - \mu(df - eg).$$

The symmetry between the two unprojections is underlined by the fact that the 9 equations are simply interchanged¹ by the involution

$$\mu \longleftrightarrow -\mu, \quad a \longleftrightarrow x, \quad b \longleftrightarrow c, \quad d \longleftrightarrow e, \quad f \longleftrightarrow g.$$

¹They are also invariant under $(d, e, \mu) \leftrightarrow (-d, -e, -\mu)$. In these calculations there may be several correct choices of signs (and many incorrect ones). Getting the signs right can be a major headache, with no perfect solutions.

3.4. S_3 symmetry

For general μ, ν , the 3-fold $W_{\mu,\nu}$ is projectively equivalent to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Carrying this out requires an S_3 Galois field extension.

The little exercise in A_2 symmetry is fun and not quite obvious: the three equations involving x in (8) are (10). From them we deduce that in the deformation given by (8), the tag equation $xdf = e^3$ of (1) deforms to

$$x(df + eg) = e^3 + \nu ef^2 - \mu f^3 = \Phi(e, f).$$

The projective equivalence of $W_{\mu,\nu}$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ holds when the discriminant of Φ does not vanish, and involves the roots of Φ . It thus takes place over its splitting field. The Galois group action permutes the 3 copies of \mathbb{P}^1 . This reflects the Weyl group $W(A_2) = S_3$ symmetry behind the deformation theory of the A_2 singularity.

Write s, t, u for the roots of Φ , so that $s + t + u = 0$,

$$\begin{aligned} \nu &= st + ut + su = -(s^2 + st + t^2), & \mu &= ust = -st(s + t), \\ \text{and } \Phi(e, f) &= (e - sf)(e - tf)(e - uf). \end{aligned}$$

Now set y_0, y_1, y_2 and z_0, z_1, z_2 to be the following linear combinations of (b, d, g) and (c, e, f) :

$$\begin{aligned} y_0 &= c + se + tuf, & z_0 &= b - sd + tug, \\ y_1 &= c + te + suf, & \text{and } z_1 &= b - td + sug, \\ y_2 &= c + ue + stf, & z_2 &= b - ud + stg. \end{aligned}$$

After a calculation, we find

$$\begin{aligned} xz_i &= y_j y_k, \\ ay_i &= z_j z_k \quad \text{for } \{i, j, k\} = \{0, 1, 2\}, \\ xa &= y_i z_i. \end{aligned}$$

These are the standard equations of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ as the 2×2 minors of the 3-cube.

4. Unprojection and deformations

4.1. Unprojection

The general theory of unprojection was initiated by Kustin and Miller [7] and developed in the present form by Papadakis and Reid, see [9–11].

Let X be a Gorenstein scheme (local, affine or projective) containing a Gorenstein codimension 1 subscheme $D \subset X$. Consider the adjunction sequence

$$0 \rightarrow \omega_X \rightarrow \text{Hom}(\mathcal{I}_D, \omega_X) \rightarrow \omega_D \rightarrow 0.$$

By [10, Lemma 1.1], the \mathcal{O}_X -module $\text{Hom}(\mathcal{I}_D, \omega_X)$ is generated by two elements; we can take one of these as an injective map $s: \mathcal{I}_D \hookrightarrow \omega_X \cong \mathcal{O}_X$ that projects to a basis element of $\omega_D \cong \mathcal{O}_D$. The *unprojection* Y of D in X is the spectrum of the \mathcal{O}_X -algebra $\mathcal{O}_X[S]/(Sf_i - s(f_i))$, where the f_i generate the ideal $\mathcal{I}_D \subset \mathcal{O}_X$. The scheme Y is again Gorenstein.

As X is Gorenstein,

$$\text{Hom}(\mathcal{I}_D, \omega_X) \cong \text{Hom}(\mathcal{I}_D, \mathcal{O}_X).$$

We calculate generators of $\text{Hom}(\mathcal{I}_D, \mathcal{O}_X)$ in concrete cases by computer algebra, cf. [2]. This construction also applies in a relative situation, over a base space T . The most general T is the base of a versal deformation of the inclusion map $i: D \hookrightarrow X$.

4.2. Combining the two deformation families

In our case, the first order infinitesimal deformations of $i: D \hookrightarrow X$ are described as the Pfaffian perturbations of the equations contained in the ideal (x, c, e, f) . The trivial deformations are given by vector fields $\text{Der}(-\log D)$ preserving D . For deformations of weight -1 , this means that we make the matrix as general as possible, with no coordinate transformations of x, c, e and f allowed. The result is

$$\begin{pmatrix} z & c + y & b & d \\ & x & c & e \\ & & e & f \\ & & & -g \end{pmatrix}.$$

The minus sign conforms with the deformation (8).

For deformations of weight ≥ 0 a short computation² in SINGULAR [6] shows that the above deformations generate the module of deformations: we can replace y and z with polynomials in f , and g with a polynomial in x having deformation variables as coefficients. Since our singularity is nonisolated some care is needed with the meaning of infinite dimensional versal deformation. We restrict ourselves here to deformations of nonpositive weight, that globalise to deformations of the projective cone. Then the first order infinitesimal deformations are given by

$$\begin{pmatrix} z + \mu f & c + y + \nu f & b & d \\ & x & c & e \\ & & e & f \\ & & & -g + \lambda x \end{pmatrix}. \tag{11}$$

For higher order deformations, the equations are the Pfaffians of the matrix (11), as the deformation is in particular a deformation of X . The obstruction is that they must lie in

²available from <http://www.math.chalmers.se/~stevens/singular.html>

the ideal (x, c, e, f) . Hence setting these variables to zero in (11) we find $gy = gz = 0$ as the equations of the base space.

We compute $\text{Hom}(\mathcal{I}_D, \mathcal{O}_X)$ using SINGULAR [6] to determine the unprojection, obtaining the equations

$$\begin{aligned} ac - b(b + vg) - \lambda(z + \mu f)^2 - \mu dg + \lambda v c(c + y + vf), \\ ae - (b + vg)d - \lambda(c + y)(z + \mu f) - \mu g^2 + \lambda v x d + \lambda \mu x g, \\ af - d^2 + bg - \lambda(c + y)^2 - \lambda v(c + y)f, \\ ax - (b + vg + \lambda v x)(c + y + vf) + d(z + \mu f) - \mu e g. \end{aligned}$$

We find two components, with total spaces that are isomorphic up to a smooth factor with the Tom and Jerry formats of Sections 2 and 3.3. We replace y by $y + c$ in the Tom equations, to obtain the cone as section $\lambda = 0$, $z = 0$ and $y = 0$. The coordinate transformations needed are $a \mapsto a - \lambda v(c + y)$, $y \mapsto y - vf$ and $z \mapsto z - \mu f$ for the Tom component and $a \mapsto a + \lambda \mu e$, $g \mapsto g + \lambda x$ for Jerry. Note that these coordinate transformations mix the deformation and the space variables.

4.3. The versal deformation of the cone over $v_6(\mathbb{P}(1, 2, 3))$

Altmann [1, Table 5.1] records the result of our computation of the infinite dimensional versal deformation. What we have actually computed is the part in nonpositive weight, giving the (embedded) versal deformation of the projective cone. After a simple coordinate transformation and translation to our present coordinates, the formulas there give exactly the same ideal as computed above in terms of unprojection.

4.4. The cone over an elliptic curve of degree 6

An elliptic curve of degree 6 is a hyperplane section of a smooth dP_6 but also of our $v_6(\mathbb{P}(1, 2, 3))$. We describe the versal deformation of the cone over the curve with unprojection methods. We relate this to Tom and Jerry, and come back to discuss it further at the end.

The versal deformation of the cone over an elliptic normal curve of degree 6 is described without equations by Mérindol [8]. The base space is the product of the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ with the germ of an appropriate modular curve.

Deformations of negative weight can be described by Pinkham's construction of "sweeping out the cone". More precisely, the total space over a line in the base space is the cone over the anticanonical model of an almost del Pezzo surface of degree 6, with the given elliptic curve E as hyperplane section. Such a surface is obtained by blowing up three points on the curve, embedded in the plane by a linear system of degree 3. Mérindol's construction starts with a family of such surfaces over an Abelian variety A , which is the hypersurface in $\text{Pic}^3 \times E^3$ given by $3H - (P_1 + P_2 + P_3) = 6O$. The Weyl

group $W = A_1 \times A_2$ acts on this: A_2 permutes the three points, and A_1 acts by

$$(H; P_1, P_2, P_3) \\ \mapsto (2H - P_1 - P_2 - P_3; H - P_2 - P_3; H - P_1 - P_3, H - P_1 - P_2).$$

Thus, the base space of the versal deformation in negative weight is the cone over $A/W \cong \mathbb{P}^1 \times \mathbb{P}^2$.

We find the elliptic curve as hyperplane section of the singular del Pezzo surface $v_6(\mathbb{P}(1, 2, 3))$. In affine coordinates of \mathbb{P}^2 related to (3) we take the curve

$$w^2 = v^3 + \gamma v^2 + v,$$

realising the cone as the hyperplane section

$$f - x - \gamma c - b = 0.$$

Thus, the variable a does not appear in the equation.

For the deformations of negative weight, we perturb the matrix (7) (with $b = f - x - \gamma c$) with independent variables, subject to the resulting equations lying in the ideal (x, c, e, f) . This means that the entries multiplied by $m_{1,5} = d$ are not perturbed, and moreover, no perturbation of x, c, e , or f is absorbed by coordinate transformations. We take

$$\begin{pmatrix} z & c + y & b + u & d \\ & x & c & e + q \\ & & e & f + p \\ & & & s \end{pmatrix}. \quad (12)$$

The Pfaffians of this matrix with x, c, e and f (and therefore also b) equated to zero give the equations of the base space: the minors of

$$\begin{pmatrix} z & y & u \\ q & p & s \end{pmatrix}.$$

As for the space of deformations of weight zero, a computation with SINGULAR shows that it has dimension two. One deformation is given by the modulus γ , but there is another, corresponding to the choice of point from which to project the curve.

The matrix (12) is neither a Tom nor a Jerry matrix. But it can be written in these forms after a small resolution of the base space. We do this here for the Tom format. The cone over $\mathbb{P}^1 \times \mathbb{P}^2$ is resolved by $\mathbb{P}^1 \times \mathbb{A}^3$. We introduce an inhomogeneous coordinate λ on \mathbb{P}^1 and set $q = \lambda z$, $s = \lambda y$ and $s = \lambda u$. Then we can make the matrix into a Tom₁ by row and column operations. After the coordinate transformation

$$(c, d, e, f, x, u, y, z) \mapsto (c - \lambda x, d - \gamma \lambda z + \lambda^2 z, e, \\ f + \lambda c + \gamma \lambda x - 2\lambda^2 x, x, u + b, y - c + \lambda x, z),$$

(so that $b = f - x - \gamma c + \lambda c - 2\lambda^2 x$), the matrix takes the form

$$\begin{pmatrix} z & y & u & & d \\ & x & c & & e \\ & & e & & f \\ & & & & (\lambda + \gamma\lambda^2 + \lambda^3)x \end{pmatrix}.$$

5. General remarks and FAQ

5.1. Which is Tom, and which is Jerry?

We offer three answers as useful mnemonics. We do not assume any prior familiarity with the Hanna–Barbera characters.

- (i) Tom is fatter. The ancestral Tom is the projective 4-fold $\mathbb{P}^2 \times \mathbb{P}^2$, whereas for Jerry it is the 3-fold $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.
- (ii) The Tom_i condition (that the 6 entries m_{jk} with $j, k \neq i$ are in I_D) means in simple cases two coincidences on the m_{jk} . On the other hand, the Jerry_{jk} condition (that the 7 elements in the j th and k th rows and columns are in I_D) means 3 conditions.
- (iii) Weight-for-weight, Jerry is more singular. In fact, any point $P \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ lies on 3 lines, and the linear projection from P contracts these to nodes. In contrast, if we take the 3-fold hyperplane section V of $\mathbb{P}^2 \times \mathbb{P}^2$ to get the flag variety of \mathbb{P}^2 , the linear projection of V from P only has two nodes.

Trying to fit a Jerry unprojection into a 6×6 skew matrix format is invariably a waste of time.

5.2. What's it all about?

A hypersurface or complete intersection is determined by the coefficients of its defining equations, so its deformations are unobstructed. The subtlety of the deformation theory in these cases is nothing to do with obstructions, but how to pass to the quotient by the appropriate equivalence relation, which involves dividing by the groupoid of local diffeomorphisms.

The Buchsbaum–Eisenbud theorem [5] puts codimension 3 Gorenstein ideals in the same framework: the variety is given by a skew $(2k + 1) \times (2k + 1)$ matrix (most commonly 5×5), that encodes both the defining equations and the syzygies, so that the entries of the matrix can be freely deformed. In other words, the skew matrix is a given mould, into which one can simply pour functions on the ambient space in a liquid manner.

In contrast, one usually expects codimension 4 constructions to be obstructed. A typical case is the cone over $d\mathbb{P}_6$, whose deformation theory has the 2 components we have mentioned many times.

The point of Tom and Jerry is that, in most commonly occurring cases, our variety admits a Gorenstein projection to codimension 3, with the projected variety given

by the Pfaffians of a 5×5 skew matrix; that is, the projected variety is a regular pull-back from $\text{Grass}(2, 5)$ in its Plücker embedding, marked with an unprojection divisor that corresponds to a linear subspace of $\text{Grass}(2, 5)$. Every geometer must have done the easy exercise of seeing that any linear subspace of $\text{Grass}(2, 4)$ (the Klein quadric) either consists of lines of \mathbb{P}^3 passing through a point P , or dually, of lines contained in a plane $\mathbb{P}^2 \subset \mathbb{P}^3$. The Tom and Jerry formats discussed here answer the same question for $\text{Grass}(2, 5)$; see [4, 2.1].

5.3. Do they do everything?

Unfortunately, no. Tom and Jerry provide two smooth components of the deformation theory, and for deformation problems entirely contained within one component or the other, they can be relied on to do everything. However, we know other cases in codimension 4 that appear not to have any useable structure of Kustin–Miller unprojection.

A general structure theorem for Gorenstein codimension 4 ideals is described in [12]. It should account for the singular total spaces of versal deformations, but it does not lead to tractable calculations. The discussion of the cone over an elliptic curve of degree 6 in §4.4 above illustrates this point. Once we know the answer, we may verify its place in the general structure theorem, but that structure theorem does not predict the answer or help to find it. Deformations of the hyperplane sections, that is, the cone over 6 points in \mathbb{P}^4 , are more complicated still.

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Gavin Brown

Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK;
g.brown@warwick.ac.uk

Miles Reid

Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK;
miles.reid@warwick.ac.uk

Jan Stevens

Department of Mathematical Sciences, Chalmers University of Technology
and University of Gothenburg, SE 412 96 Gothenburg, Sweden; stevens@chalmers.se