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SL(5) Supersymmetry

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We consider supersymmetry in five dimensions, where the fermionic parameters are a 2-form under $SL(5)$. Supermultiplets are investigated using the pure spinor superfield formalism, and are found to be closely related to infinite-dimensional extensions of the supersymmetry algebra: the Borcherds superalgebra $\mathscr{B}(E_4)$, the tensor hierarchy algebra $S(E_4)$ and the exceptional superalgebra $E(5,10)$. A theorem relating $\mathscr{B}(E_4)$ and $E(5,10)$ to all levels is given.

1. Introduction and Overview

Supersymmetry provides an extension of bosonic space-time symmetries with fermionic generators. These are generically spinors under space-time rotations (and may also transform under R-symmetry). In certain situations, supersymmetry generators in non-spinorial modules may be considered. The main example is provided by "twisting", where one considers a fermionic generator which is a singlet under some subalgebra.

More generically, one may a priori consider an assignment where supersymmetry generators come in a module S of a space-time "structure group" G, which we think of as corresponding to the double cover of the Lorentz group together with R-symmetry. A supersymmetry algebra will take the form[1] $[Q_a, Q_b] = c_{ab}{}^m P_m$, with some invariant tensor c, and the rest of the brackets vanishing. The only condition is that the symmetric product $\vee^2 S$ contains the vector representation V.

Presently, we will consider one specific such assignment, namely when the structure group is $G = SL(5)$, $V = \overline{\mathbf{5}}$ and $S = \mathbf{10}$. The supersymmetry algebra then is[2]

$$[Q^{mn}, Q^{pq}] = 2\epsilon^{mnpqr}\partial_r, \qquad (1.1)$$

There are several observations that make this choice special, and exploring them is the purpose of the paper.

A general method for formulating supersymmetric field theories on superspace is provided by "pure spinor superfield theory",[1–14] where the superspace coordinates $x \in V$ and $\theta \in S$ are complemented by a "bosonic spinor" $\lambda \in S$, which is subject to the constraint $c_{ab}{}^m \lambda^a \lambda^b = 0$. The content of supermultiplets can be deduced already from the partition function (Hilbert series) of λ.

In the simple case of $D = 10$ super-Yang–Mills theory, the constraint on λ turns it into a Cartan pure spinor. This (more generally, the fact that λ belongs to a minimal S-orbit under G) enables a Koszul duality[15] between the associative algebra generated by λ and the positive levels of a Borcherds superalgebra. The structure of the Borcherds superalgebra, in the $D = 10$ super-Yang–Mills case $\mathscr{B}(E_5)$, is thus closely connected to the supersymmetry multiplet.[16–18]

In Section 3, we will establish such a relation for the $SL(5)$ supersymmetry algebra (1.1), the Borcherds superalgebra $\mathscr{B}(E_4)$ and a certain supermultiplet[3]. This is achieved by applying, in Section 2.1, the principles of pure spinor superfield theory to the case at hand. This supermultiplet found turns out, as a vector space, to be the adjoint module of the (infinite-dimensional) exceptional superalgebra $E(5,10)$,[19–21] of which the global supersymmetry algebra (1.1) is a subalgebra. This observation then leads to a surprising relation between $E(5,10)$ and the Borcherds superalgebra $\mathscr{B}(E_4)$ or the tensor hierarchy algebra[22] $S(E_4)$. More precisely, the half of $S(E_4)$ at levels ≥ 3 turns out to be freely generated by the coadjoint module of $E(5,10)$. We also describe $E(5,10)$ as a "restricted tensor hierarchy algebra" in terms of the generalisation of Chevalley generators introduced in refs. [23, 24].

It may be noted that the supersymmetry algebra (1.1) is a subalgebra of the $D = 10$, $N = 1$ supersymmetry algebra. Under the subgroup $SL(5) \subset Spin(10)$, the vector and spinor branch as $\mathbf{10} \to \mathbf{5} \oplus \overline{\mathbf{5}}$, $\mathbf{16} \to \mathbf{1} \oplus \mathbf{10} \oplus \overline{\mathbf{5}}$, and the module $\mathbf{10}$ parametrises the infinitesimal (projective) deformations of a pure spinor $\mathbf{1}$. The (on-shell) $D = 10$ super-Yang–Mills supermultiplet, effectively encoded in a pure spinor of $Spin(10)$, must be possible to describe in terms of $SL(5)$ supermultiplets. We shall comment on this in Section 2.2.

Infinite-dimensional superalgebras, in particular Borcherds superalgebras or tensor hierarchy algebras, play an important rôle in the context of extended geometry.[25–28] The superalgebra underlying the extended geometry for $D = 11$ supergravity

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[1] We use a notation with $[\cdot,\cdot]$ for the graded Lie brackets or graded commutators; the present bracket is of course symmetric.

[2] A factor i may be included in the right hand side, depending on conventions.

[3] The notation $E_4 \simeq A_4$ is used since it is part of the E series, and the super-extension is associated to the leftmost node in the Dynkin diagram, see **Figure 1**.

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reduced to d dimensions is $S(E_{12-d})$. In the series, we find $S(E_4)$, which (most likely) coincides with $\mathscr{B}(E_4)$ at positive levels, governing the gauge structure. $S(E_5)$ (or $\mathscr{B}(E_5)$) lies behind extended geometry for maximal supergravity in $d = 7$, but is also central to $D = 10$ super-Yang–Mills theory. It would be interesting to understand whether there is some deeper reason that the exact same algebraic structures appear in two seemingly different contexts.

The Koszul duality described in Section 3 holds for any object in a minimal orbit and the corresponding Borcherds superalgebra. Our impression is that this is not enough for the meaningful description of a supersymmetry multiplet, which also seems to rely on the superalgebra having some freely generated part. This happens also in cases, such as $D = 11$ supergravity, where the orbit is not minimal, and the dual superalgebra is not even a Lie superalgebra.[18,29] One may for example ask if $\mathscr{B}(E_6)$ or $S(E_6)$ (which differ slightly even at positive levels[30]) is relevant for some kind of supersymmetry based on E_6.

We expect relations similar to the one between $E(5, 10)$ and $S(E_4)$ to hold also for some other of the exceptional superalgebras,[19–21] such as $E(3, 6)$ or $E(3, 8)$, that may have a relation to $S(E_3)$, but this issue has yet to be investigated.

2. $SL(5)$ Supersymmetry

The supersymmetry algebra (1.1) is realised by

$$Q^{mn} = \frac{\partial}{\partial\theta_{mn}} + \epsilon^{mnpqr}\theta_{pq}\partial_r, \tag{2.1}$$

with $\partial_m = \frac{\partial}{\partial x^m}$. Then, $[D^{mn}, Q^{pq}] = 0$, where the covariant fermionic derivative is

$$D^{mn} = \frac{\partial}{\partial\theta_{mn}} - \epsilon^{mnpqr}\theta_{pq}\partial_r. \tag{2.2}$$

Now,

$$[D^{mn}, D^{pq}] = -2\epsilon^{mnpqr}\partial_r, \tag{2.3}$$

and the superspace torsion is $T^{mn,pq,r} = 2\epsilon^{mnpqr}$.

The BRST operator of pure spinor field theory is, as usual, constructed as

$$Q = \lambda_{mn}D^{mn}. \tag{2.4}$$

Its nilpotency is guaranteed by the bilinear constraint

$$\lambda_{[mn}\lambda_{pq]} = 0. \tag{2.5}$$

Denoting representations and representation modules by the Dynkin label of the highest weight, and letting $\lambda \in S = (0100) = R(\Lambda_2)$, the general symmetric product is

$$\vee^2 S = \vee^2(0100) = (0200) \oplus (0001).$$

The constraint implies that only the module with highest weight $2\Lambda_2$ survives, so λ belongs to the (unique) minimal S-orbit under $SL(5)$, which is a cône over the 6-dimensional Grassmannian $Gr(2, 5)$.

2.1. The Cohomology of a Scalar Superfield

We will be quite brief about the technicalities of the calculations of "pure spinor field theory" leading to the supermultiplets in this and the following subsection. They go along the general principles explained e.g. in ref. [14]. The calculation of the zero-mode cohomologies is a matter of comparing components in different superfields, a pure algebraic problem well suited for a computer. Here, we choose to put stronger focus on the way in which a supermultiplet appears in the partition function of the constrained object λ, and in the following Section on the Koszul duality to a superalgebra.

"Physical states" may be defined as cohomology of the BRST operator Q of eq. (2.4). They are also directly encoded in the partition function of λ. This partition function encodes the modules S_p of monomials of degree of homogeneity n in λ as the coefficient of t^n in a formal power series with coefficients in the representation ring as $Z_\lambda(t) = \oplus_{p=0}^{\infty} S_p t^p$. We choose the conventions that $S_p = R(p\Lambda_3)$ are the modules of the *components* in the expansion[4], which are conjugate to the ones of the basis elements $\lambda_{m_1n_1} \dots \lambda_{m_pn_p}$. Thus,

$$Z_\lambda(t) = \bigoplus_{p=0}^{\infty}(00p0)t^p. \tag{2.6}$$

Factoring out the partition function of an unconstrained object in S, which we denote $(1-t)^{-(0010)} \equiv \oplus_{p=0}^{\infty} \vee^p (0010)t^p$ (and which is compensated by θ),

$$Z_\lambda(t) = (1-t)^{-(0010)} \otimes \big((0000) \ominus (1000)t^2 \oplus (0001)t^3 \ominus (0000)t^5\big). \tag{2.7}$$

The interpretation of the numerator is as the zero-mode cohomology, i.e., the cohomology of $\lambda_{mn}\frac{\partial}{\partial\theta_{mn}}$ on a scalar field $\Psi(\theta, \lambda)$. Assigning ghost number 1 to Ψ (and of course 0 to θ and 1 to λ), the interpretation of the zero-mode cohomology is:

- A ghost c;
- A 1-form α, appearing in Ψ as $\epsilon^{mnpqr}\lambda_{mn}\theta_{pq}\alpha_r$;
- A vector ξ, appearing as $\epsilon^{mnpqr}\lambda_{mn}\theta_{pq}\theta_{rs}\xi^s$;
- An "antifield" γ, appearing as $\epsilon^{mnrst}\epsilon^{pquvw}\lambda_{mn}\lambda_{pq}\theta_{rs}\theta_{tu}\theta_{vw}\gamma$.

Further extracting a factor $(1-t^2)^{(1000)} \equiv \oplus_{i=0}^{5} \wedge^i (1000)t^{2i}$ (which is compensated by the x-dependence),

$$Z_\lambda(t) = (1-t)^{-(0010)} \otimes (1-t^2)^{(1000)} \otimes \Big((0000) \oplus \bigoplus_{i=0}^{\infty}(i001)t^{3+2i} \ominus \bigoplus_{i=0}^{\infty}(i100)t^{4+2i}\Big). \tag{2.8}$$

The terms in the rightmost factor are interpreted as the ghost zero-mode together with the derivative expansions of ξ and $\chi = d\alpha$.

4 The duality formulated in ref. [15] and in Section 3 employs a relation to the *coalgebra*. We account for this by this definition of the partition function; an alternative would be to employ negative instead of positive levels.

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Table 1. The zero-mode cohomology in Ψ. The superfields at different ghost numbers are shifted so that fields in the same row has the same dimension. Black dots denote the absence of cohomology.

λ^0	λ^1	λ^2	λ^3
(0000)			
•	•		
•	(1000)	•	
•	(0001)	•	•
•	•	•	•
•	•	(0000)	•
•	•	•	•

Table 2. The zero-mode cohomology in a vector field Φ^m.

λ^0	λ^1	λ^2
(0001)		
(0100)	•	
•	•	•
•	(0010)	•
•	(1000)	•
•	•	•

In the full cohomology, closedness implies the "equation of motion" $\partial_m\xi^m = 0$, and quotienting out exact functions gives the gauge invariance $\alpha \sim \alpha + d\beta$. This cohomology is reflected by the partition function (2.8). This off-shell supermultiplet has 4 bosonic and 4 fermionic local degrees of freedom. As we will see later, it corresponds to the exceptional Lie superalgebra $E(5, 10)$. The same cohomology arises from twisting of $D = 11$ supergravity.[31]

2.2. Cohomology of Other Superfields

This Section lies outside the main line of the paper, and may be skipped in a linear reading.

One may attempt to assign some non-trivial module to a field. Then, some so-called shift symmetry[7] must also be introduced, otherwise the cohomology would just become the tensor product of the module of the field with the cohomology found in a scalar field. In the language of ref. [32] this amounts to considering not functions on the minimal orbit, but sections of some geometric sheaf (such as products of the (co-)tangent sheaf) over the minimal orbit.

Taking a field $\Phi^m(x, \theta, \lambda)$ in the vector module, and imposing equivalence under the shift symmetry $\Phi^m \sim \Phi^m + \lambda_{np}\varrho^{mnp}$ for completely antisymmetric $\varrho^{mnp}(x, \theta, \lambda)$ leads to the zero-mode cohomology of **Table 2**.

In the full cohomology, the vector $v^m(x)$ will obey $\partial_m\partial_n v^n = 0$ and the 2-form ω is closed. In this sense, the same local degrees of freedom are reproduced as the ones in a scalar field (there is only a single extra linear singlet mode in the derivative expansion of v). On the other hand, if Ψ is taken to be fermionic and of ghost number 1, and Φ^m bosonic of ghost number 0, they can be combined into a description of the $D = 10$ super-Yang–Mills multiplet with $SL(5)$ (or $SU(5)$) covariance.[33] Then the 1-form α and the vector v are the components of the $D = 10$ connection, and the vector ξ and the 2-form ω are part of the spinor.

Yet another possibility is a 1-form field $\Xi_m(x, \theta, \lambda)$, with shift symmetry $\Xi_m \sim \Xi_m + \lambda_{mn}\varrho^n$. The zero-mode cohomology is given in **Table 3**.

Table 3. The zero-mode cohomology in a 1-form field Ξ_m.

λ^0	λ^1	λ^2
(1000)		
(0001)	•	
•	(2000)	•
•	(0000) $\oplus$ (1001)	•
•	(0010)	•
•	•	•

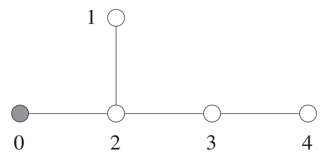

Figure 1. The Dynkin diagram of $\mathscr{B}(E_4)$, $S(E_4)$ and $E(5, 10)$.

The supermultiplet at λ^1 is interesting. There is a symmetric tensor h_{mn} with a gauge transformation $\delta_u h_{mn} = \partial_{(m}u_{n)}$, which looks like the linearised transformation of a gravity field. The fermions $\psi_m{}^n$ have a gauge symmetry $\delta_\epsilon\psi_m{}^n = \partial_m\epsilon^n$, and there is a bosonic 3-form (field strength). There are 20 bosonic and 20 fermionic local degrees of freedom. The multiplet will certainly be a part of a decomposition of $D = 10$, $N = 1$ supergravity, but may also have some significance of its own.

There may be more interesting supermultiplets that we have not found.

3. Minimal Orbit Partition Function and Koszul Duality

It is known that there in general is a Koszul duality between the associative algebra generated by an object $\lambda \in S$ in a minimal orbit under G and the positive part $\mathscr{B}_+$ of a Borcherds superalgebra.[15] The Lie superalgebra $\mathscr{B}$ is defined by a Dynkin diagram obtained by extending the Dynkin diagram of $\mathfrak{g} = \mathrm{Lie}(G)$ by a "fermionic" null root connected according to the Dynkin label of S. The diagram is depicted in Figure 1.

The concrete relation is

$$Z_\lambda(t) \otimes Z_{\mathscr{B}_+}(t) = 1, \tag{3.1}$$

where $Z_{\mathscr{B}_+}$ is the partition function of (the universal enveloping algebra of) $\mathscr{B}_+$, defined as

$$Z_{\mathscr{B}_+}(t) = \bigotimes_{p=1}^{\infty}(1 - t^p)^{-(-1)^p R_p}, \tag{3.2}$$

R_p being the module of the level p generators in $\mathscr{B}$. The duality (3.1) can be understood as a factorised version of the summation form $Z_\lambda(t) = \bigoplus_{p=0}^{\infty} \overline{R(p\Lambda)}\, t^p$, with Λ the highest weight of S, and thus as a denominator formula for $\mathscr{B}$. A concrete way of understanding the duality is in terms of the BRST operator for the bilinear constraint on λ. The modules R_p are the modules of the corresponding infinite tower of ghosts.[15,34–36] This BRST operator can be identified as the coalgebra differential of the co-superalgebra $\mathscr{B}_+^\star$. The identification relies on the absence of Lie superalgebra cohomology other than polynomials of level 1, which holds for Borcherds superalgebras.

In the present case, we thus have

$$\begin{aligned} Z_{\mathscr{B}_+}(t) &= (1-t)^{(0010)} \otimes (1-t^2)^{-(1000)} \\ &\otimes \left(1 \oplus \bigoplus_{i=0}^{\infty} (i001) t^{3+2i} \ominus \bigoplus_{i=0}^{\infty} (i100) t^{4+2i}\right)^{-1} \\ &= (1-t)^{(0010)} \otimes (1-t^2)^{-(1000)} \otimes (1-t^3)^{(0001)} \otimes (1-t^4)^{-(0100)} \\ &\otimes (1-t^5)^{(1001)} \otimes (1-t^6)^{-(0002)-(1100)} \otimes \ldots \end{aligned} \tag{3.3}$$

The factor $(1 \oplus \bigoplus_{i=0}^{\infty} (i001) t^{3+2i} \ominus \bigoplus_{i=0}^{\infty} (i100) t^{4+2i})^{-1}$ is now read as the partition function for the superalgebra freely generated[5] by the supermultiplet with partition

$$P(t) = \ominus \bigoplus_{i=0}^{\infty} (i001) t^{3+2i} \oplus \bigoplus_{i=0}^{\infty} (i100) t^{4+2i}. \tag{3.4}$$

Note that this partition function is consistently graded in the sense that when the products $\otimes^i P(t)$ are evaluated, all terms at odd level are negative and at even level positive, so no cancellations may arise. One can then safely identify the algebra at levels ≥ 3 as freely generated by the multiplet. The picture is in complete analogy with how the Borcherds superalgebra $\mathscr{B}(E_5)$ is freely generated by the $D=10$ Yang–Mills on-shell supermultiplet from level 3.[16–18]

We believe that the Koszul duality in general can be extended to fields in non-trivial modules with shift symmetry. Then, just as the Koszul duality described above can be seen as providing a denominator formula for the Borcherds superalgebra, it may be conjectured that a corresponding relation involving a non-scalar field will provide a character formula for a representation of the superalgebra.[18]

4. Superalgebras Based on *SL*(5) Supersymmetry

There is a number of infinite-dimensional superalgebras that all contain generators in (0010) at level 1 and (1000) at level 2. They can all be described by the same Dynkin diagram, Figure 1. They have standard Chevalley generators e_a, $a=0,\ldots,4$ and f_i, h_i, $i=1,\ldots,4$, but differ in terms of the level -1 generators. The Borcherds superalgebra $\mathscr{B}(E_4)$ is contragredient, having a Chevalley generators f_0 and h_0 with the usual relations. The generator e_0, with the Serre relations $[e_0,e_0]=0$ following from the Cartan matrix, is the lowest weight state in the 2-form module at level 1, and the Serre relation implies that level 2 only contains a vector.

[5] The partition function for an algebra freely generated by generators in $P(t)$ is $(1-P(t))^{-1}$. Even if the freely generated algebra itself is complicated to describe level by level, its universal enveloping algebra has the simple partition function $\bigoplus_{i=0}^{\infty} \otimes^i P(t) = (1-P(t))^{-1}$.

4.1. $S(E_4)$

While the superalgebra $\mathscr{B}(E_4)$ is contragredient, so the module at level $-p$ is conjugate to the one at level p, this does not apply to the tensor hierarchy algebras[22] $S(E_4)$. In $S(E_4)$, h_0 is removed, and the generator f_0 of the Borcherds superalgebra is replaced by f_{0j}, $i=1,3,4$ (the nodes not connected to node 0). New brackets are $[h_i,f_{0j}]=-A_{0i}f_{0j}$, $[e_0,f_{0j}]=h_j$.[23,24,37] The presence of f_{01} implies the presence of (2000) at level -1, while f_{03}, f_{04} give (0011). The reducibility of level -1 comes from the disconnectedness of the Dynkin diagram of A_4 with node 2 removed.

Although it has not been proven, we strongly believe that the positive levels of $\mathscr{B}(E_4)$ and $S(E_4)$ coincide, $S_+(E_4) \simeq \mathscr{B}_+(E_4)$, and that consequently the Koszul duality involving the supersymmetry multiplet, as described in Section 3, applies equally well to $S(E_4)$. It seems likely that a proof of this may be based on an argument that the freely generated property of the part of $\mathscr{B}(E_4)$ at levels ≥ 3 does not allow for any ideal annihilated by the negative level generators in $S(E_4)$. This will be postponed to future examination[18]

Instead of standard contragredience, $S(E_4)$ allows for an invariant bilinear form pairing level p with level $5-p$, so that $R_{5-p} = \overline{R_p}$.[38]

4.2. $E(5,10)$

The superalgebra $E(5,10)$[19–21] is one of the “exceptional” simple superalgebras which are linearly compact (which for our purposes holds if the elements arise as a power series of the coordinates of some finite-dimensional space) and of finite depth (meaning that there is a maximal level[6]), and has attracted some interest in the mathematics literature.[39–41] It is defined as a super-extension of volume-preserving diffeomorphisms in 5 dimensions by fermionic generators, the parameters of which are closed 2-forms. Letting ξ, η be divergence-free vector fields and χ, ψ closed fermionic 2-forms, the brackets are

$$\begin{aligned} [P_\xi, P_\eta] &= P_{L_\xi \eta}, \\ [P_\xi, Q_\chi] &= Q_{L_\xi \chi}, \\ [Q_\chi, Q_\psi] &= P_{\star(\chi\wedge\psi)}. \end{aligned} \tag{4.1}$$

The Jacobi identity with three Q’s relies on the identity $L_{\star(\gamma\wedge\gamma)}\gamma = 0$ for a bosonic closed 2-form γ, which when written out in components leads to antisymmetrisation in 6 indices.

From a derivative expansion of the parameters, we get the level decomposition. At level $2-2i$, there are generators $P^{a_1\ldots a_i}{}_b$, symmetric in $(a_1\ldots a_i)$ with $P^{a_1\ldots a_{i-1}b}{}_b = 0$, i.e., in $(100i)$. At level $1-2i$, there is $Q^{a_1\ldots a_i,bc}$, symmetric in $(a_1\ldots a_i)$ and antisymmetric in $[bc]$, with $Q^{a_1\ldots a_{i-1}[a,bc]} = 0$, i.e., in $(001i)$.

We observe that the supermultiplet of Section 2.1, related to $\mathscr{B}(E_4)$ (or $S(E_4)$) in Section 3, is in fact the adjoint module of $E(5,10)$. The theorem, stating a relation between $E(5,10)$ and $\mathscr{B}(E_4)$, then immediately follows:

[6] In most of the mathematical literature, level is defined with a minus sign compared to our conventions, so this would read as minimal level.

Theorem. *The part of $\mathscr{B}(E_4)$ at levels ≥ 3 is freely generated by the coadjoint module of $E(5,10)$, with the lowest states assigned to level 3.*

If it holds that $S_+(E_4) \simeq \mathscr{B}_+(E_4)$ (see the discussion in Section 4.1), the theorem applies to the part of $S(E_4)$ at levels ≥ 3. This is precisely half of $S(E_4)$, and the theorem then gives complete information concerning the $SL(5)$ modules in $S(E_4)$ at all levels, since the invariant quadratic form relates the remaining half as $R_{2-i} = \overline{R_{3+i}}$.

One may also understand $E(5,10)$ in terms of Chevalley-like generators associated to the Dynkin diagram of Figure 1 and the corresponding Cartan matrix. The disconnectedness of the Dynkin diagram of $A_1 \oplus A_2$ obtained by deleting node 2 presents the option not to include all three f_{0j} from the definition of $S(E_4)$[7]. We call such a superalgebra a restricted tensor hierarchy algebra. By including $f_{0j'}$, $j'=3,4$, but not f_{01}, one obtains only (0011) at level -1. Then, levels ≥ 3, as constructed in $S(E_4)$, become an ideal, which follows from the observation that level 3 is annihilated by level -1, since $(0011)\otimes(0001) \not\supset (1000)$. The result is $E(5,10)$.

The difference between the coadjoint module and and the superalgebra it generates freely appears first at level 6, where the symmetric product of level 3, i.e., (0002), enters. Its dual module in $S(E_4)$ is (2000) at level -1, which accounts for the difference between $S(E_4)$ and $E(5,10)$ at level -1.

Yet another superalgebra can be defined as a restricted tensor hierarchy algebra by making the complementary choice to the one leading to $E(5,10)$: keeping f_{01} and omitting f_{03}, f_{04}. Then, only (0002) enters at level -1. Now, level -1 annihilates level 2, since $(2000)\otimes(1000) \not\supset (0010)$, so levels ≥ 2 form an ideal. In addition, level -2 is empty (the generators in (0002) have vanishing brackets among themselves in $S(E_4)$). The resulting superalgebra is the "strange" (finite-dimensional) superalgebra $P(4)$.[42] This observation is due to Jakob Palmkvist.[30]

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Conflict of Interest

The author has declared no conflict of interest.

Keywords

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7 The rôle of the generators f_{0j} in a tensor hierarchy algebra $S(\mathfrak{g})$ is as the Cartan part of the adjoint of $\mathfrak{g}^-$, the Lie algebra obtained by deleting the node(s) connected to the fermionic one. The list is typically redundant. Here, by including either f_{03} or f_{04}, the other one can be obtained by the action of $e_{3,4}$ and $f_{3,4}$.

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