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BMS algebra from residual gauge invariance in light-cone gravity

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Abstract: We analyze the residual gauge freedom in gravity, in four dimensions, in the light-cone gauge, in a formulation where unphysical fields are integrated out. By checking the invariance of the light-cone Hamiltonian, we obtain a set of residual gauge transformations, which satisfy the BMS algebra realized on the two physical fields in the theory. Hence, the BMS algebra appears as a consequence of residual gauge invariance in the bulk and not just at the asymptotic boundary. We highlight the key features of the light-cone BMS algebra and discuss its connection with the quadratic form structure of the Hamiltonian.

Keywords: Classical Theories of Gravity, Gauge Symmetry, Space-Time Symmetries

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1 Introduction

The Bondi-van der Burg-Metzner-Sachs (BMS) group is an infinite-dimensional enhancement of the Poincaré group, that arises as the asymptotic symmetry group at null infinity for asymptotically flat spacetimes [1–3]. In [4], the BMS group was extended to include superrotations. These asymptotic symmetries have been related to soft theorems for gauge theories in the recent years [5], following which there has been a renewed interest in the study of these symmetries.

The study of asymptotic symmetries is very sensitive to the boundary conditions and gauge choices imposed on the fields. At spatial infinity, for instance, only the Poincaré algebra (and not the BMS algebra) can be canonically realized under the standard boundary conditions in the Hamiltonian formulation of general relativity [6]. Recent work however showed that, by relaxing those boundary conditions, the BMS group can be recovered at spatial infinity, thereby, resolving the puzzle between the asymptotic structure at spatial and null infinity [7, 8]. This raises an important question: how much of the gauge freedom can be fixed in the theory without losing the residual reparameterizations that are associated with the BMS symmetry?

In the light-cone formulation of gravity in four dimensions, one can gauge away the unphysical degrees of freedom and describe the dynamics of the theory in terms of the two physical states of the graviton. In the usual Bondi gauge, the BMS group appears as the asymptotic symmetry of gravity at infinity. In light-cone gravity, we will show instead that we obtain the BMS algebra from the invariance of the Hamiltonian under residual gauge transformations in the bulk.

In a previous paper [9] we have studied the problem by constructing non-linear representations of the Poincaré algebra on two field degrees of freedom with helicity $+2$ and -2 respectively. We performed the study in the light-cone frame where one of the light-cone coordinates is the evolution parameter "time" and the conjugate momentum, the Hamiltonian. The representation is unique up to possible counterterms [10]. When we ask if the Hamiltonian is invariant under an extended symmetry we find indeed that it is invariant under the BMS symmetry, which shows that this symmetry is also present in the bulk. It should be mentioned though that our formulation is a perturbative one in κ, and we only demonstrate this to the lowest order, but experience from earlier studies suggests that the symmetries will hold at higher orders. Furthermore since it is an infinite series with an ever-increasing number of graviton fields we have to restrict our studies to weak fields.

The question here is then to ask if the remaining residual gauge invariance corresponds to just the BMS algebra on the two helicity fields h and $\bar{h}$ or if it can be further extended. This formulation is often thought to be one where the gauge symmetry is completely fixed apart from some freedom in the definition of the inverse ∂_- (the light-cone momentum that is taken to be a space derivative). But, in this paper, we show that there is still some residual gauge freedom in the theory, which leads to the light-cone realization of the BMS algebra and only that.

The paper is organized as follows. We start with a brief review of the light-cone gauge-fixing of the Einstein-Hilbert action followed by the perturbative expansion in terms of the helicity fields h and $\bar{h}$. In section 2.2, we discuss the remaining reparameterization freedom in the theory. In section 3, we focus on a special class of reparameterizations, dubbed "helicity-preserving", and derive how the fields transform under them by demanding invariance of the light-cone Hamiltonian. We then present the symmetry algebra underlying these reparameterizations, which yields the BMS algebra in light-cone gravity. In section 3.3, we define a canonical generator for the supertranslations, from which we obtain the quadratic form structure for the Hamiltonian previously found in [10, 11]. We conclude with some remarks on the structure of the BMS algebra in the light-cone gauge and possible extensions to Yang-Mills theory, higher-spin and supersymmetric theories.

2 Gravity in the light-cone gauge

With the metric $(-,+,+,+)$, the light-cone coordinates are defined as

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^3)\,, \tag{2.1}$$

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with the corresponding derivatives $\partial_\pm$. The transverse coordinates and derivatives are

$$x = \frac{1}{\sqrt{2}}\,(\,x_1 + i\,x_2\,)\,; \qquad \bar{\partial} = \frac{1}{\sqrt{2}}\,(\,\partial_1 - i\,\partial_2\,)\,,$$
$$\bar{x} = \frac{1}{\sqrt{2}}\,(\,x_1 - i\,x_2\,)\,; \qquad \partial = \frac{1}{\sqrt{2}}\,(\,\partial_1 + i\,\partial_2\,)\,. \tag{2.2}$$

The Einstein-Hilbert action on a Minkowski background reads

$$S_{\rm EH} = \int d^4x\,\mathcal{L} = \frac{1}{2\,\kappa^2}\int d^4x\,\sqrt{-g}\,\mathcal{R}\,, \tag{2.3}$$

where $g = det\,(\,g_{\mu\nu}\,)$ is the determinant of the metric. $\mathcal{R}$ is the curvature scalar and $\kappa^2 = 8\pi G$ is the coupling constant derived from the gravitational constant. The corresponding field equations are

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\mathcal{R} = 0\,. \tag{2.4}$$

We impose the following *three* gauge choices [12, 13] on the dynamical variable $g_{\mu\nu}$

$$g_{--} = g_{-i} = 0\,, \quad i = 1, 2\,. \tag{2.5}$$

These choices are motivated by the fact that in Minkowski space, we have $\eta_{--} = \eta_{-i} = 0$. We also choose

$$\begin{aligned} g_{+-} &= -e^{\phi}\,, \\ g_{i\,j} &= e^{\psi}\,\gamma_{ij}\,. \end{aligned} \tag{2.6}$$

where ϕ, ψ are real parameters and γ^{ij} is a real, symmetric matrix with unit determinant. Field equations that do not involve time derivatives (∂_+) are constraint relations as opposed to *true* equations of motion, which have explicit time derivatives. The $\mu = \nu = -$ constraint from (2.4) yields

$$2\,\partial_-\phi\,\partial_-\psi - 2\,\partial_-^2\psi - (\partial_-\psi)^2 + \frac{1}{2}\partial_-\gamma^{ij}\,\partial_-\gamma_{ij} = 0\,, \tag{2.7}$$

which may be solved by making the *fourth and final* gauge choice

$$\phi = \frac{\psi}{2}\,. \tag{2.8}$$

Note that this gauge choice relates g_{+-} and g_{ij}. This choice implies, from (2.7), that

$$\psi = \frac{1}{4}\,\frac{1}{\partial_-^2}\,(\partial_-\gamma^{ij}\,\partial_-\gamma_{ij})\,. \tag{2.9}$$

Other constraint relations eliminate g_{++} and g_{+i} resulting in the following action

$$\begin{aligned} S = \frac{1}{2\kappa^2}\int d^4x\; e^{\psi}\Big(2\,\partial_+\partial_-\phi + \partial_+\partial_-\psi - \frac{1}{2}\,\partial_+\gamma^{ij}\partial_-\gamma_{ij}\Big) \\ - e^{\phi}\gamma^{ij}\Big(\partial_i\partial_j\phi + \frac{1}{2}\partial_i\phi\partial_j\phi - \partial_i\phi\partial_j\psi - \frac{1}{4}\partial_i\gamma^{kl}\partial_j\gamma_{kl} + \frac{1}{2}\partial_i\gamma^{kl}\partial_k\gamma_{jl}\Big) \\ - \frac{1}{2}e^{\phi-2\psi}\gamma^{ij}\frac{1}{\partial_-}R_i\frac{1}{\partial_-}R_j\,, \end{aligned} \tag{2.10}$$

where

$$R_i \equiv e^{\psi}\left(\frac{1}{2}\partial_-\gamma^{jk}\partial_i\gamma_{jk} - \partial_-\partial_i\phi - \partial_-\partial_i\psi + \partial_i\phi\partial_-\psi\right) + \partial_k(e^{\psi}\,\gamma^{jk}\partial_-\gamma_{ij})\,.$$

This is the closed form expression for the light-cone gravity action [12, 13] — purely in terms of the physical degrees of freedom in the theory.

2.1 Perturbative expansion

We now examine the perturbative expansion of the closed form expression obtained above. The order κ^2 result was first presented in [13, 14] while the κ^3 result was derived in [15]. We parameterize the matrix γ_{ij} as

$$\gamma_{ij} = (e^{\kappa H})_{ij}\,, \tag{2.11}$$

where H is a traceless matrix since $det\,(\,\gamma_{ij}) = 1$. We choose

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & -h_{11} \end{pmatrix}\,; \quad h = \frac{(h_{11} + i\,h_{12})}{\sqrt{2}}\,, \qquad \bar{h} = \frac{(h_{11} - i\,h_{12})}{\sqrt{2}}\,. \tag{2.12}$$

The Lagrangian (density) in terms of h and $\bar{h}$ to order κ now reads

$$\mathcal{L} = \frac{1}{2}\,\bar{h}\,\Box\,h + 2\,\kappa\,\bar{h}\,\partial_-^2\left[-h\,\frac{\bar{\partial}^2}{\partial_-^2}h + \frac{\bar{\partial}}{\partial_-}h\,\frac{\bar{\partial}}{\partial_-}h\right] + \text{c.c.}\,, \tag{2.13}$$

with the d'Alembertian $\Box = 2\,(\,\partial\,\bar{\partial} - \partial_+\,\partial_-\,)$. At the next order, time derivatives need to be removed using field redefinitions and the resulting quartic Lagrangian was computed in [13, 14].

The corresponding Hamiltonian density for gravity reads

$$\mathcal{H} = \partial\bar{h}\,\bar{\partial}h + 2\,\kappa\,\partial_-^2\,\bar{h}\,\left(h\,\frac{\bar{\partial}^2}{\partial_-^2}h - \frac{\bar{\partial}}{\partial_-}h\,\frac{\bar{\partial}}{\partial_-}h\right) + c.c. + \mathcal{O}(\kappa^2)\,. \tag{2.14}$$

The light-cone action for gravity is invariant under Poincaré transformations in four dimensions [16]. As x^+ is treated as the evolution parameter, the conjugate momentum P^- is the Hamiltonian operator. The light-cone Poincaré generators split into two kinds — the kinematical ones ($\mathbb{K}$) which do not involve time derivatives ∂_+ and the dynamical ones ($\mathbb{D}$) that do and hence receive non-linear contributions in the interacting theory. The dynamical generators or "Hamiltonians" in Dirac's language [17] take the field forward in light-cone time.

$$\begin{aligned} \mathbb{K}:&\quad \{P, \bar{P}, P^+, J, J^+, \bar{J}^+, J^{+-}\} \\ \mathbb{D}:&\quad \{P^- \equiv H, J^-, \bar{J}^-\} \end{aligned} \tag{2.15}$$

All the relevant commutators of the Poincaré algebra in four dimensions are listed in appendix A. The commutators fall broadly into three varieties

$$[\,\mathbb{K},\,\mathbb{K}\,] = \mathbb{K}\,, [\,\mathbb{K},\,\mathbb{D}\,] = \mathbb{D}\,, [\,\mathbb{D},\,\mathbb{D}\,] = 0\,. \tag{2.16}$$

The fields h and $\bar{h}$ transform with helicity, $\lambda = 2$ and $\lambda = -2$, respectively under the little group in four dimensions and thus, represent the two physical states of graviton.

The Hamiltonian. The Hamiltonian $H \equiv P^-$ in (2.14) follows from the action in (2.13) — which was obtained by gauge-fixing the covariant Einstein-Hilbert Lagrangian. There is an alternate approach that also leads to the same result — this is to recognize that the Hamiltonian is also an element of the Poincaré algebra and hence can be determined entirely simply closing all the commutators in the symmetry algebra [16]. This is a necessary step because Lorentz invariance is not manifest on the light-cone and must be explicitly checked. It is interesting to note, however, that the Hamiltonian is explicitly helicity-covariant when expressed in terms of h and $\bar{h}$.

Having reached this point, we note now that there are still some reparameterizations allowed — more specifically, transformations that (i) leave the Hamiltonian invariant and (ii) preserve all gauge choices made so far. These residual reparameterizations will be the focus of the next subsection.

2.2 Residual gauge transformations

Having obtained a perturbative expansion of the gauge-fixed action, we now turn to examining the issue of residual reparameterization invariace. To this end, we note that under

$$x^\mu \to x^\mu + \xi^\mu \,,$$

the first gauge choice $g_{--} = 0$ in (2.5) holds as long as

$$\xi^+ = f(x^+, x^j) \quad \text{so} \quad \partial_- \xi^+ = 0 \,. \tag{2.17}$$

The second gauge condition $g_{-i} = 0$ in (2.5) then requires that

$$\partial_- \xi^j \, g_{ij} + \partial_i \xi^+ \, g_{+-} = 0 \,. \tag{2.18}$$

This relates ξ^j to $\xi^+ = f(x^+, x^j)$

$$\xi^k = -\partial_i f \, \frac{1}{\partial_-} (g_{+-} g^{ik}) + Y^k \,, \tag{2.19}$$

where Y^k does not depend on x^-. The fourth gauge condition (2.8) further restricts the form of different components of ξ^μ.

Confining ourselves to residual gauge transformations on the fields h, $\bar{h}$ we look for transformations that leave the light-cone Hamiltonian invariant. The light-cone gauge conditions and the subsequent perturbative expansion of γ_{ij} constrain the form of allowed residual reparameterizations, the details of which can be found in appendix B. This choice of residual gauge transformations, when expressed in the $(x, \bar{x})$ coordinates, corresponds to the following "helicity-preserving" reparameterizations

$$x \to x + Y(x) \,, \tag{2.20}$$

$$\bar{x} \to \bar{x} + \overline{Y}(\bar{x}) \,, \tag{2.21}$$

$$x^+ \to x^+ + f(x, \bar{x}, x^+) \,, \tag{2.22}$$

$$x^- \to x^- + \xi^- \,, \tag{2.23}$$

such that the parameters satisfy

$$\partial Y = \partial_- Y = 0\,, \quad \bar{\partial}\overline{Y} = \partial_- \overline{Y} = 0\,, \quad \partial_- f = 0\,, \tag{2.24}$$

and the parameter ξ^- is completely determined in terms of $f, Y, \overline{Y}$ as in (B.13).

3 BMS symmetry in light-cone gravity

3.1 Invariance of the light-cone Hamiltonian

Before we compute the symmetry algebra underlying these reparameterizations, we must ensure invariance of the light-cone Hamiltonian expressed in terms of the helicity fields (2.14). We must do this since the elimination of the unphysical degrees of freedom might further reduce the residual symmetry. Thus, invariance of the Hamiltonian under the new reparameterizations guarantees that Lorentz covariance is not violated. This requirement will indeed put some additional constraints on the parameters Y and $\overline{Y}$ as we show below.

This will, in turn, ensure that we can define a canonical generator, G_ξ for these residual gauge transformations in the reduced phase space of $(h, \bar{h})$ such that it commutes with the Hamiltonian

$$\delta_\xi H = 0 \quad \Rightarrow \quad [G_\xi, H] = 0. \tag{3.1}$$

We can then study the algebra of these residual gauge generators with the canonical Poincaré generators defined in the $(h, \bar{h})$ phase space, which are presented in appendix A.

One can check the invariance of the light-cone Hamiltonian under a given transformation order by order in κ as follows

$$\delta_\xi H = \delta_\xi^{(0)}\, H^{(0)} + \delta_\xi^{(\kappa)}\, H^{(0)} + \delta_\xi^{(0)}\, H^{(\kappa)} + \mathcal{O}(\kappa^2) = 0\,. \tag{3.2}$$

Under the reparameterizations (2.20) and (2.21) with the parameters satisfying (2.24), the fields h and $\bar{h}$ transform as

$$\begin{aligned}\delta_{Y,\overline{Y}}\, h &= Y(x)\,\bar{\partial} h + \overline{Y}(\bar{x})\,\partial h + (\partial\overline{Y} - \bar{\partial}Y)\,h\,,\\ \delta_{Y,\overline{Y}}\, \bar{h} &= Y(x)\,\bar{\partial}\bar{h} + \overline{Y}(\bar{x})\,\partial\bar{h} - (\partial\overline{Y} - \bar{\partial}Y)\,\bar{h}\,.\end{aligned} \tag{3.3}$$

These transformations leave the Hamiltonian invariant if one assumes

$$\bar{\partial}^2 Y = \partial^2 \overline{Y} = 0\,, \tag{3.4}$$

which restricts the parameters Y and $\overline{Y}$ to be at most linear in x and $\bar{x}$ respectively. Thus, the above condition reduces the $Y, \overline{Y}$ reparameterizations to a part of the Poincaré transformations. With these assumptions, it is easy to show that the light-cone Hamiltonian is invariant under (3.3) up to the cubic order. More importantly, no new corrections to $\delta_{Y,\overline{Y}} h$ of order κ or higher are required, thus, classifying these under the kinematical part of the light-cone Poincaré transformations (2.15).

We now examine the time reparameterizations labeled by the parameter f, which transforms the fields as

$$\delta_f h = f\partial_+ h = f\frac{\partial\bar{\partial}}{\partial_-}h\,. \tag{3.5}$$

This transformation leaves the free light-cone Hamiltonian invariant. However, in order to establish the invariance at order κ

$$\delta_f^{(\kappa)}\, H^{(0)} + \delta_f^{(0)}\, H^{(\kappa)} = 0\,, \tag{3.6}$$

we must add corrections to $\delta_f h$ at $\mathcal{O}(\kappa)$. Thus, the field h transforms non-linearly under f as follows

$$\begin{aligned}\delta_f h = f(x,\bar{x},x^+)\bigg\{ &\frac{\partial\bar{\partial}}{\partial_-}h + 2\,\kappa\,\partial_-\left(h\,\frac{\bar{\partial}^2}{\partial_-^2}h - \frac{\bar{\partial}}{\partial_-}h\,\frac{\bar{\partial}}{\partial_-}h\right) \\ &+ 2\,\kappa\,\frac{1}{\partial_-^3}\left(\frac{\partial^2}{\partial_-^2}\bar{h}\,\partial_-^2 h - 2\,\frac{\partial}{\partial_-}\bar{h}\,\partial_-\partial h + \bar{h}\,\partial_-^2\partial^2 h\right) + \mathcal{O}(\kappa^2)\bigg\}\,, \end{aligned} \tag{3.7}$$

and $\delta_f \bar{h}$ is simply the complex conjugate of the above expression. Therefore, the above time reparameterizations are a symmetry of the light-cone Hamiltonian, provided one adds corrections to $\delta_f h$ at every order in κ, making these transformations dynamical. Interestingly, in order to prove the invariance of the Hamiltonian, the explicit form of f is not required.

It is important to note though that *the invariance of the Hamiltonian under these transformations is strictly proven only to first order in the coupling constant, κ.* But, extensions to higher orders should follow without any formal difficulties, although the explicit calculations could prove cumbersome.

A key difference from the previous analysis in [9] is that the condition $\partial_- f = 0$ here follows from (2.17), which is a consequence of the light-cone gauge fixing of the Einstein-Hilbert action. In [9], the same condition is obtained by demanding the invariance of the light-cone Hamiltonian under local extensions of the Poincaré transformations. This reflects how the focus of this paper is primarily on the residual gauge symmetry of the graviton fields h and $\bar{h}$, in contrary to [9], where the goal was to obtain possible extensions of the light-cone Poincaré algebra with local parameters.

3.2 Light-cone realization of the BMS algebra

The fourth gauge condition (2.8) precisely fixes the x^+ dependence of f as in (B.10)

$$f(x^+,x,\bar{x}) = T(x,\bar{x}) + \frac{1}{2}\,x^+\,(\partial\overline{Y} + \bar{\partial}Y)\,, \tag{3.8}$$

which reproduces the light-cone BMS symmetry in [9]. Here, the parameters Y and $\overline{Y}$ obey (2.24) and (3.4), indicating that the only independent parameter in f is $T(x,\bar{x})$. Besides, this choice of f also coincides with the conformal Carroll case as we have discussed in appendix C. As opposed to the geometric aspects of the conformal Carroll group in [18], our focus lies on the field-theoretic aspects of gravity and the Lie algebra representation associated with the symmetries of the theory.

On the initial surface $x^+ = 0$ such that $f = T$, the BMS transformations in light-cone gravity to order κ read

$$\begin{aligned}\delta_{Y,\overline{Y},T}\, h = {} & Y(x)\,\bar\partial h + \overline{Y}(\bar x)\,\partial h + (\partial\overline{Y} - \bar\partial Y)\,h + T\,\frac{\partial\bar\partial}{\partial_-}\,h \\ & - 2\,\kappa\,T\,\partial_-\left(h\,\frac{\bar\partial^2}{\partial_-^2}h - \frac{\bar\partial}{\partial_-}h\,\frac{\bar\partial}{\partial_-}h\right) - 2\,\kappa\,T\,\frac{1}{\partial_-}\left(\frac{\partial^2}{\partial_-^2}\bar h\,\partial_-^2 h\right) \\ & - 2\,\kappa\,T\,\frac{\partial^2}{\partial_-^3}\,(\bar h\,\partial_-^2 h) + 4\,\kappa\,T\,\frac{\partial}{\partial_-^2}\left(\frac{\partial}{\partial_-}\bar h\,\partial_-^2 h\right)\,, \end{aligned} \tag{3.9}$$

where the parameters Y, $\overline{Y}$ and T satisfy

$$\partial Y = \partial_- Y = 0\,, \quad \bar\partial\overline{Y} = \partial_-\overline{Y} = 0\,, \quad \partial_- T = 0\,, \tag{3.10}$$

and

$$\bar\partial^2 Y = \partial^2\overline{Y} = 0\,. \tag{3.11}$$

Two such transformations close on another reparameterization

$$\Big[\,\delta(Y_1,\overline{Y}_1,T_1)\,,\ \delta(Y_2,\overline{Y}_2,T_2)\Big]\ h = \delta(Y_{12},\overline{Y}_{12},T_{12})\,h\,, \tag{3.12}$$

with the new parameters defined as

$$Y_{12} \equiv Y_2\,\bar\partial\,Y_1 - Y_1\,\bar\partial\,Y_2\,, \tag{3.13}$$

$$\overline{Y}_{12} \equiv \overline{Y}_2\,\partial\,\overline{Y}_1 - \overline{Y}_1\,\partial\,\overline{Y}_2\,, \tag{3.14}$$

$$T_{12} \equiv \left[Y_2\,\bar\partial\,T_1 + \overline{Y_2}\,\partial\,T_1 + \frac{1}{2}\,T_2(\bar\partial Y_1 + \partial\overline{Y}_1)\right] - (1 \leftrightarrow 2)\,. \tag{3.15}$$

This is the light-cone realization of the BMS algebra in four dimensions [9]. The "super-translations" labeled by the function $T(x,\bar x)$ enhance the dynamical part of the Poincaré algebra into an infinite-dimensional set, while the kinematical part of the algebra is the same as in (2.15)

$$\begin{aligned} &\mathbb{K} \to \mathbb{K}\,, \\ &\mathbb{D} \to \mathbb{D}(T)\,, \end{aligned} \tag{3.16}$$

such that the light-cone BMS algebra in (3.12) takes the form

$$[\,\mathbb{K},\,\mathbb{K}\,] = \mathbb{K}\,,\,[\,\mathbb{K},\,\mathbb{D}(T)\,] = \mathbb{D}(T)\,,\,[\,\mathbb{D}(T),\,\mathbb{D}(T)\,] = 0\,. \tag{3.17}$$

Thus, the key feature of the light-cone BMS algebra is that the dynamical part of the Poincaré algebra is enlarged to accommodate the supertranslations. This is different from the BMS algebra in covariant formulations, where the four spacetime translations of the Poincaré algebra are enhanced by the supertranslations. However, Lorentz invariance of the theory dictates that the enhancement involves *only one single local parameter* $T(x,\bar x)$, which is in keeping with the original BMS group found in [1–3].

The Poincaré subgroup of the BMS algebra can be obtained by imposing the condition $\partial^2 T = \bar\partial^2 T = 0$. This restricts T to be at most linear in x or $\bar x$, thereby, reducing the dynamical part to $\mathbb{D}$ given in (2.15). In appendix B, we discuss in more details the Poincaré subgroup of the light-cone BMS algebra (B.18).

3.3 Supertranslations and the quadratic form Hamiltonian

As we have established that $\delta_T h$ are canonical transformations in the $(h,\bar{h})$ phase space, we can, now, define a canonical generator G_T for supertranslations through (3.1)

$$G_T = \int d^3x\, \partial_-\bar{h}\,(\delta_T h) = \int d^3x\, \partial_-\bar{h}\, T\,\frac{\partial\bar{\partial}}{\partial_-} h + \mathcal{O}(\kappa)\,, \tag{3.18}$$

where we have used the equation of motion for h obtained from (2.13). One can derive the transformation law for the fields h and $\bar{h}$ order by order in κ from the above generator using the brackets[1]

$$\delta_T h = [G_T, h]\,, \delta_T \bar{h} = [G_T, \bar{h}]\,. \tag{3.19}$$

At the lowest order, the generator can be brought to the form

$$\begin{aligned} G_T^{(0)} &= \int d^3x\, \left(T\,\partial\bar{h}\,\bar{\partial}h + \partial T\,\bar{h}\,\bar{\partial}h\right) \\ &= \int d^3x\, \left(T\,\partial\bar{h}\,\bar{\partial}h + \frac{1}{2}\partial T\,\bar{h}\,\bar{\partial}h + \frac{1}{2}\,\bar{\partial}T\,h\,\partial\bar{h}\right)\,. \end{aligned} \tag{3.20}$$

At the cubic order, G_T reads [16]

$$\begin{aligned} G_T^{(\kappa)} = \kappa \int d^3x\, T\,\partial_-\bar{h}\Bigg\{ & 2\,\partial_-\left(h\,\frac{\bar{\partial}^2}{\partial_-^2}h - \frac{\bar{\partial}}{\partial_-}h\,\frac{\bar{\partial}}{\partial_-}h\right) \\ & + \frac{1}{\partial_-^3}\left(\frac{\partial^2}{\partial_-^2}\bar{h}\,\partial_-^2 h - 2\,\frac{\partial}{\partial_-}\bar{h}\,\partial_-\partial h + \bar{h}\,\partial_-^2\partial^2 h\right)\Bigg\}\,. \end{aligned} \tag{3.21}$$

After some partial integrations, the details of which can be found in appendix D, the supertranslation generator up to order κ reads

$$\begin{aligned} G_T = \int d^3x\, T\,\mathcal{D}\bar{h}\,\overline{\mathcal{D}}h + \int d^3x \Bigg\{ & \frac{1}{2}\,\partial T\,\bar{h}\,\bar{\partial}h - \kappa\,\partial T\,\bar{h}\frac{1}{\partial_-^2}\left(\frac{\partial}{\partial_-}\bar{h}\,\partial_-^3 h - \bar{h}\partial_-^2\partial h\right) \\ & - 2\,\kappa\,\partial T\,\frac{1}{\partial_-^2}(\bar{h}\partial_-^2 h)\,\partial\bar{h} + \frac{1}{2}\,\kappa\,\partial^2 T\,\bar{h}\,\frac{1}{\partial_-^2}(\bar{h}\partial_-^2 h)\Bigg\} + \text{c.c.}\,, \end{aligned} \tag{3.22}$$

where the "covariant" derivative $\overline{\mathcal{D}}h$ reads [10, 11]

$$\overline{\mathcal{D}}h = \bar{\partial}h + 2\,\kappa\,\frac{1}{\partial_-^2}\left(\frac{\partial}{\partial_-}\bar{h}\,\partial_-^3 h - \bar{h}\partial_-^2\partial h\right)\,. \tag{3.23}$$

For constant time translations, $T = a$, the corresponding generator becomes the Hamiltonian of the theory given by

$$G_{T=a} = H = \int d^3x\, a\,\mathcal{D}\bar{h}\,\overline{\mathcal{D}}h\,. \tag{3.24}$$

[1] The fields h and $\bar{h}$ satisfy

$$[h(x), \bar{h}(y)] = \frac{1}{\partial_-}\delta^{(3)}(x-y)\,, \quad [\,h(x), h(y)\,] = [\,\bar{h}(x), \bar{h}(y)\,] = 0\,.$$

Thus, the light-cone Hamiltonian for gravity in four dimensions can be recast into a *positive semi-definite structure*, which we call *quadratic form*, indicating that the energy of the system is manifestly positive. In an earlier work [11], we had derived the order κ^2 terms in $\bar{\mathcal{D}}h$ proving that this feature extends to the second order in coupling constant as well. However, in this analysis, we learn that the quadratic form structure of the Hamiltonian is actually fixed by the BMS symmetry. The terms involving derivatives of T in (3.22) are not arbitrary as these are related to the *spin corrections* to the boost generators J^- and $\bar{J}^-$ required for the closure of the light-cone Poincaré algebra. Any further partial integrations in ∂ and $\bar{\partial}$ will spoil this structure and hence, lead to ill-defined boost generators.[2] In fact, we can read off corrections to the boost generators order by order in κ from the general expression for supertranslation generator to higher orders.

The quadratic form expression (3.24) resembles the generator of supertranslations at null infinity proposed by Bondi, van der Burg, Metzner, and Sachs [1–3], which shows that the flux of the gravitational energy at null infinity is positive. Proof of the positive energy theorem in gravity [19–21] typically involves some spinor-like variables. In the light-cone formulation, however, one can derive the quadratic form Hamiltonian without resorting to any spinor variables. It is important to note though that we are concerned with the energy density in the bulk, not the flux of gravitational energy at null infinity.

Interestingly, the covariant derivative $\overline{\mathcal{D}}h$ in the quadratic form Hamiltonian plays a role similar to the "News tensor" in the Bondi frame, which is related to vacuum configurations. Therefore, we can define the vacuum configurations of gravity in the light-cone formulation as the states which satisfy the condition

$$\overline{\mathcal{D}}h = \bar{\partial}h + 2\,\kappa\,\frac{1}{\partial_-^2}\left(\frac{\partial}{\partial_-}\bar{h}\,\partial_-^3 h - \bar{h}\partial_-^2\partial h\right) + \mathcal{O}(\kappa^2) = 0\,. \tag{3.25}$$

It would be instructive to use this equation to classify different vacuum configurations and explore its possible implications for the quantum theory, as the fields h and $\bar{h}$ indeed correspond to the two physical states of the graviton.

4 Concluding remarks

In this paper, we discussed the BMS symmetry in the light-cone gauge from the perspective of residual gauge freedom in the theory. The invariance of the Hamiltonian under these reparameterizations leads to the correct transformation laws for the fields, which then realize the BMS algebra in four dimensions. The most notable characteristics of the light-cone BMS algebra is the enhancement of the dynamical part of the Poincaré algebra. In the light-cone formalism, the dynamical transformations, which are non-linearly realized on the fields, provide us with a powerful framework to construct interacting actions from the closure of the symmetry algebra [16]. This is particularly interesting for higher-spin theories as constructing interacting actions with higher-spin fields is a very complex issue. It would,

[2] This can be compared to the asymptotic symmetry analysis at spatial infinity, where the boost generators require special treatment in order to have well-defined Poincaré generators in the Hamiltonian formulation [7, 8].

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therefore, be worthwhile to explore the implications of these non-linear transformations for the interacting theory.

Although ours is a perturbative analysis, this light-cone approach offers a unique insight into the structure of the BMS symmetry, that also shares some interesting features with the BMS literature for the full Einstein theory. One important similarity with the spatial infinity analysis is that the invariance of the Hamiltonian reduces the parameters, Y and $\overline{Y}$, to Lorentz rotations, which eliminates superrotations from the theory. In light-cone gravity, one can study the BMS symmetry in a physical gauge, which might help us better understand its connection with on-shell amplitudes [22, 23] and soft theorems [24, 25].

The fact that the light-cone Hamiltonian for gravity in four dimensions can be expressed as a positive semi-definite quadratic form puts this theory in a special class of theories that admit such Hamiltonians. The occurrence of such simple structures in field theories is often indicative of hidden symmetries [26, 27]. In this paper, we have attributed the quadratic form structure in light-cone gravity to the invariance of the Hamiltonian under BMS supertranslations. Similar analyses for residual gauge symmetry in Yang-Mills and higher-spin theories, which also admit such quadratic form Hamiltonians [28, 29], could bring forth some deeper links between these theories and gravity. Similarly, extensions to supersymmetric theories [30] could be instrumental in explaining the quadratic form Hamiltonians found in maximally supersymmetric Yang-Mills and supergravity in four dimensions.

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A Light-cone Poincaré algebra in $d = 4$

In this section, we present the light-cone Poincaré algebra in four dimensions. A general Poincaré trasnformation in four dimensions

$$\delta x^\mu = \omega^\mu_\nu x^\nu + a^\mu \,, \tag{A.1}$$

in light-cone coordinates, takes the unusual form

$$\delta \begin{pmatrix} x^+ \\ x^- \\ x \\ \bar{x} \end{pmatrix} = \begin{pmatrix} \omega_{+-} & 0 & -\bar{\omega}_- & -\omega_- \\ 0 & -\omega_{+-} & \bar{\omega}_+ & \omega_+ \\ \omega_+ & \bar{\omega_-} & -i\omega_{12} & 0 \\ \bar{\omega_+} & \omega_- & 0 & i\omega_{12} \end{pmatrix} \begin{pmatrix} x^+ \\ x^- \\ x \\ \bar{x} \end{pmatrix} + \begin{pmatrix} a^+ \\ a^- \\ a \\ \bar{a} \end{pmatrix} \tag{A.2}$$

where the two real $(\omega_{+-}, \omega_{12})$ and two complex (ω_+, ω_-) parameters label the Lorentz group. We define

$$J^+ = \frac{J^{+1} + iJ^{+2}}{\sqrt{2}} \,, \quad \bar{J}^+ = \frac{J^{+1} - iJ^{+2}}{\sqrt{2}} \,, \quad J = J^{12} \,, \quad H = P^- \,. \tag{A.3}$$

All the *non-vanishing* commutators of the Poincaré algebra are listed below

$$\begin{aligned}
&[H,J^{+-}]=-iH\,, && [H,J^{+}]=-iP\,, && [H,\bar{J}^{+}]=-i\bar{P}\\
&[P^{+},J^{+-}]=iP^{+}\,, && [P^{+},J^{-}]=-iP\,, && [P^{+},\bar{J}^{-}]=-i\bar{P}\\
&[P,\bar{J}^{-}]=-iH\,, && [P,\bar{J}^{+}]=-iP^{+}\,, && [P,J]=P\\
&[\bar{P},J^{-}]=-iH\,, && [\bar{P},J^{+}]=-iP^{+}\,, && [\bar{P},J]=-\bar{P}\\
&[J^{-},J^{+-}]=-iJ^{-}\,, && [J^{-},\bar{J}^{+}]=iJ^{+-}+J\,, && [J^{-},J]=J^{-}\\
&[\bar{J}^{-},J^{+-}]=-i\bar{J}^{-}\,, && [\bar{J}^{-},J^{+}]=iJ^{+-}-J\,, && [\bar{J}^{-},J]=-\bar{J}^{-}\\
&[J^{+-},J^{+}]=-iJ^{+}\,, && [J^{+-},\bar{J}^{+}]=-i\bar{J}^{+}\,, &&\\
&[J^{+},J]=J^{+}\,, && [\bar{J}^{+},J]=-\bar{J}^{+}\,. &&
\end{aligned} \tag{A.4}$$

In case of light-cone gravity, the Poincaré generators are canonically realised on the phase space of h and $\bar{h}$ as follows

$$\begin{aligned}
&P^{-}=\int d^3x\mathcal{H}\,,\quad P=\int d^3x\partial_-\bar{h}\,\partial h\,,\qquad \overline{P}=\int d^3x\partial_-\bar{h}\,\bar{\partial}h\,,\quad P^{+}=\int d^3x\partial_-\bar{h}\,\partial_-h\,,\\
&J=i\int d^3x\,\partial_-\bar{h}\,(x\bar{\partial}-\bar{x}\partial-\lambda)\,h\,,\qquad J^{+-}=\int d^3x\,(\partial_-\bar{h}x^{-}\partial_-h-x^{+}\mathcal{H})\,,\\
&J^{+}=\int d^3x\,\partial_-\bar{h}\,(x^{+}\partial+x\partial_-)\,h\,,\qquad J^{-}=\int d^3x\,\left\{x\mathcal{H}+\partial_-\bar{h}\left(x^{-}\partial-\lambda\frac{\partial}{\partial_-}\right)h+\mathcal{S}^{-}\right\},\\
&\bar{J}^{+}=\int d^3x\,\partial_-\bar{h}\,(x^{+}\bar{\partial}+\bar{x}\partial_-)\,h\,,\qquad \bar{J}^{-}=\int d^3x\,\left\{\bar{x}\mathcal{H}+\partial_-\bar{h}\left(x^{-}\bar{\partial}-\lambda\frac{\bar{\partial}}{\partial_-}\right)h+\bar{\mathcal{S}}^{-}\right\},
\end{aligned} \tag{A.5}$$

where the spin corrections read

$$\begin{aligned}
&\delta_{s^-}h=-2\frac{\partial}{\partial_-}h-4\kappa\,\partial_-\left(h\frac{\bar{\partial}}{\partial_-^2}h-\frac{1}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h\right)+\mathcal{O}(\kappa^2)\,,\\
&\delta_{\bar{s}^-}h=2\frac{\bar{\partial}}{\partial_-}h-4\kappa\frac{1}{\partial_-^3}\left(\frac{\partial}{\partial_-^2}\bar{h}\partial_-^4h-\frac{1}{\partial_-}\bar{h}\partial_-^3\partial h+3\frac{\partial}{\partial_-}\bar{h}\partial_-^3h-3\bar{h}\partial_-^2\partial h\right)+\mathcal{O}(\kappa^2)\,.
\end{aligned} \tag{A.6}$$

B Allowed residual gauge transformations

Any residual gauge transformation must leave the light-cone gauge choices invariant. These put some constraints on the form of the allowed gauge parameters ξ^μ.

The first gauge condition $g_{--}=0$ leads to the constraint

$$\delta g_{--}=0\Rightarrow\partial_-\xi^+\,g_{+-}=0 \tag{B.1}$$

This condition is easily satisfied if we chose the parameter ξ^+ as

$$\xi^+=f(x^+,x^j)\quad\text{such that}\quad\partial_-f=0$$

The condition $g_{-i}=0$ leads to

$$\delta g_{-i}=0\Rightarrow\partial_-\xi^j\,g_{ij}+\partial_i\xi^+\,g_{+-}=0 \tag{B.2}$$

For the second condition to hold, we can solve for ξ^j in terms of $\xi^+ = f$

$$\xi^k = -\partial_i f \, \frac{1}{\partial_-}(g_{+-}g^{ik}) + Y^k \,, \tag{B.3}$$

where the integration constant Y^k does not depend on x^-.

The gauge condition (2.8) relates the g_{-+} component to the determinant of the metric g_{ij}. Thus, we first consider δg_{-+} to obtain

$$\delta\phi = f\,\partial_+\phi + \xi^-\partial_-\phi + \xi^k\,\partial_k\phi + \partial_+ f + \partial_-\xi^- - \partial_k f\, g_{+i}\, g^{ik} \tag{B.4}$$

As evident from the expression of ϕ in (2.9), the above equation is complicated involving the non-local $\frac{1}{\partial_-}$ operators. But in the perturbative expansion, one can simplify the equation in orders of κ and obtain non-trivial relations between the gauge parameters. At the lowest order, one finds

$$\partial_+ f + \partial_-\xi^- = 0 \tag{B.5}$$

We can, alternatively, obtain $\delta\phi$ from the variation of the determinant of g_{ij}

$$\delta g = g\, g^{ij}\delta g_{ij} \,, \tag{B.6}$$

with g given by

$$g = det(g_{ij}) = 2\psi = \phi \tag{B.7}$$

where the last equality follows from the fourth gauge choice (2.8). When compared with $\delta\phi$ in (B.4) to the lowest order, we get

$$\partial_+ f = \frac{1}{2}\partial_i Y^i \,. \tag{B.8}$$

This constraint fixes the time dependence of the parameter f

$$f(x^+, x^i) = \frac{1}{2}\partial_i Y^i\, x^+ + T(x^i) \,, \tag{B.9}$$

which in the $(x, \bar{x})$ coordinate reads

$$f(x^+, x, \bar{x}) = \frac{1}{2}x^+(\partial\overline{Y} + \bar{\partial}Y) + T(x, \bar{x}) \,. \tag{B.10}$$

Further consistency checks may be performed on the remaining components of the metric. Since both g_{++} and g_{+i} are at least of order κ, we have the following conditions at the zeroth order

$$\delta g_{++} = 0 \;\Rightarrow\; \partial_+\xi^- = 0 \,, \tag{B.11}$$
$$\delta g_{+i} = 0 \;\Rightarrow\; \partial_i\xi^- = \partial_+\xi_i \,, \tag{B.12}$$

which along with (B.5) completely determines the form of ξ^- in terms of f and ξ^i

$$\xi^- = -(\partial_+ f)\, x^- + (\partial_+\xi_i)\, x^i \,. \tag{B.13}$$

Poincaré subgroup within the BMS. From the above conditions, one can write a general BMS transformation on the light-cone coordinates as

$$\delta x = \alpha + \beta x + \sigma x^+ \,, \tag{B.14}$$
$$\delta \bar{x} = \bar{\alpha} + \bar{\beta}\bar{x} + \bar{\sigma} x^+ \,, \tag{B.15}$$
$$\begin{aligned} \delta x^+ &= Re(\beta) x^+ + T(x,\bar{x}) \\ &= Re(\beta) x^+ + [\, t_0 + t_1 x + \bar{t_1}\bar{x} + \ldots] \,, \end{aligned} \tag{B.16}$$
$$\delta x^- = \gamma - Re(\beta) x^- + \sigma x + \bar{\sigma}\bar{x} \,, \tag{B.17}$$

where the parameters t_0, γ are real and $\alpha, \beta, \sigma, t_1$ are all complex. We can then read off the Poincaré subgroup inside the BMS group by identifying the parameters in the above set of equations to those in (A.2) as follows

$$\begin{aligned} &(a^+, a^-, a, \bar{a}) = (t_0, \gamma, \alpha, \bar{\alpha}) \\ &\omega_{+-} = Re(\beta) \,, \quad \omega_{12} = Im(\beta) \,, \quad (\omega_+, \omega_-) = (\sigma, t_1) \,. \end{aligned} \tag{B.18}$$

The extension of the Poincaré group to the BMS group is, thus, parameterized *only* by the higher order expansions of $T(x,\bar{x})$ in (B.16).

C Connections to the Carroll group

For null hypersurfaces like the light-cone surfaces $x^\pm = \mathit{constant}$, the BMS algebra in four dimensions is isomorphic to the conformal Carroll group [18]. The conformal Carroll group consists of two-dimensional conformal algebra on the spatial coordinates augmented by supertranslations in the time direction, u.

$$x \to \phi(x) \,, \qquad u \to \Omega(x)^{\frac{1}{2}} \, [u + \alpha(x)] \,, \tag{C.1}$$

where $\Omega(x)$ is the scaling factor associated with the conformal transformations on x.

We, therefore, consider conformal Carroll transformations in the light-cone coordinates on a constant x^- surface

$$x \to \omega(x) \,, \quad \bar{x} \to \overline{\omega}(\bar{x}) \,, \tag{C.2}$$
$$x^+ \to \Omega(x,\bar{x})^{\frac{1}{2}} \, [\, x^+ + \alpha(x,\bar{x}) \,] \,. \tag{C.3}$$

An infinitesimal conformal transformation on $x, \bar{x}$ is given by

$$x \to \omega(x) = x + Y(x) \,, \tag{C.4}$$
$$[0.2cm] \bar{x} \to \overline{\omega}(\bar{x}) = \bar{x} + \overline{Y}(\bar{x}) \,. \tag{C.5}$$

Thus, from (C.3), we find that the time-coordinate x^+ transforms infinitesimally as

$$\begin{aligned} &x^+ \to \Omega(x,\bar{x})^{\frac{1}{2}} \left[x^+ + \alpha(x,\bar{x}) \right] = [1 + (\bar{\partial} Y + \partial \overline{Y})]^{\frac{1}{2}} \left(x^+ + \alpha(x,\bar{x}) \right) \\ &[0.3cm] \sim \left\{ 1 + \frac{1}{2}(\bar{\partial} Y + \partial \overline{Y}) \right\} \left(x^+ + \alpha(x,\bar{x}) \right) \sim x^+ + f(x^+, x, \bar{x}) \,, \end{aligned} \tag{C.6}$$

where

$$f(x^+, x, \bar{x}) = T(x, \bar{x}) + \frac{1}{2}\, x^+ \,(\bar{\partial} Y + \partial \overline{Y})\,. \tag{C.7}$$

Here we have ignored the terms of higher orders in Y and $\overline{Y}$. Thus, the residual reparameterizations considered in (3.8) can indeed be interpreted as infinitesimal conformal Carroll transformation on the constant x^- null hypersurface.

D Supertranslation generator at order κ

We consider the supertranslation generator at order κ

$$\begin{aligned} G_T^{(\kappa)} &= \kappa \int d^3x\, T\, \partial_- \bar{h} \Bigg\{ 2\, \partial_- \left(h\, \frac{\bar{\partial}^2}{\partial_-^2} h - \frac{\bar{\partial}}{\partial_-} h\, \frac{\bar{\partial}}{\partial_-} h \right) \\ &\qquad\qquad + \frac{1}{\partial_-^3} \left(\frac{\partial^2}{\partial_-^2} \bar{h}\, \partial_-^2 h - 2\, \frac{\partial}{\partial_-} \bar{h}\, \partial_- \partial h + \bar{h}\, \partial_-^2 \partial^2 h \right) \Bigg\} \\ &= \mathcal{X} + \mathcal{Y}\,, \end{aligned} \tag{D.1}$$

where $\mathcal{X}$ contains the terms involving $\bar{h}hh$

$$\mathcal{X} = 2\,\kappa \int d^3x\, T\, \partial_- \bar{h}\, \partial_- \left(h\, \frac{\bar{\partial}^2}{\partial_-^2} h - \frac{\bar{\partial}}{\partial_-} h\, \frac{\bar{\partial}}{\partial_-} h \right)\,, \tag{D.2}$$

and $\mathcal{Y}$ contains terms involving $\bar{h}\bar{h}h$

$$\mathcal{Y} = \kappa \int d^3x\, T\, \partial_- \bar{h} \frac{1}{\partial_-^3} \left(\frac{\partial^2}{\partial_-^2} \bar{h}\, \partial_-^2 h - 2\, \frac{\partial}{\partial_-} \bar{h}\, \partial_- \partial h + \bar{h}\, \partial_-^2 \partial^2 h \right)\,. \tag{D.3}$$

Let us focus on the $\mathcal{X}$ terms, which upon partial integrations, can be expressed as

$$\mathcal{X} = 2\kappa \int d^3x\, T \partial_- \bar{h}\, \partial_- \left(h\, \frac{\bar{\partial}^2}{\partial_-^2} h \right) - 2\kappa \int d^3x\, T \partial_- \bar{h}\, \partial_- \left(\frac{\bar{\partial}}{\partial_-} h\, \frac{\bar{\partial}}{\partial_-} h \right) \tag{D.4}$$

$$= -2\kappa \int d^3x\, \frac{\bar{\partial}}{\partial_-^2} (T\, \partial_-^2 \bar{h} h)\, \bar{\partial} h + 2\kappa \int d^3x\, T\, \frac{1}{\partial_-} \left(\frac{\bar{\partial}}{\partial_-} h\, \partial_-^2 \bar{h} \right) \tag{D.5}$$

$$\begin{aligned} &= -2\kappa \int d^3x\, \bar{\partial} T\, \frac{1}{\partial_-^2} (\partial_-^2 \bar{h} h)\, \bar{\partial} h - 2\kappa \int d^3x\, T\, \frac{\bar{\partial}}{\partial_-^2} (\partial_-^2 \bar{h} h)\, \bar{\partial} h \\ &\quad + 2\kappa \int d^3x\, T\, \frac{1}{\partial_-} \left(\bar{\partial} h\, \partial_-^2 \bar{h} + \frac{\bar{\partial}}{\partial_-} h\, \partial_-^3 \bar{h} \right) \end{aligned} \tag{D.6}$$

$$= -2\,\kappa \int d^3x\, \bar{\partial} T\, \frac{1}{\partial_-^2} (h \partial_-^2 \bar{h})\, \bar{\partial} h + 2\,\kappa \int d^3x\, T\, \frac{1}{\partial_-^2} \left(\frac{\bar{\partial}}{\partial_-} h\, \partial_-^3 \bar{h} - h\, \partial_-^2 \bar{\partial} \bar{h} \right) \bar{\partial} h\,. \tag{D.7}$$

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Similarly, the $\mathcal{Y}$ terms can be simplified as follows

$$\mathcal{Y} = \kappa \int d^3x\, T\, \partial_- \bar{h} \frac{1}{\partial_-^3} \left(\frac{\partial^2}{\partial_-^2} \bar{h} \partial_-^4 h - 2 \frac{\partial}{\partial_-} \bar{h} \partial_-^3 \partial h + \bar{h} \partial_-^2 \partial^2 h \right) \tag{D.8}$$

$$= -\kappa \int d^3x\, T\, \partial_-^2 \left(\frac{1}{\partial_-^2} \bar{h} \frac{\partial^2}{\partial_-^2} \bar{h} \right) \partial_-^2 h + 2\kappa \int d^3x\, \partial_- \partial \left(T \frac{1}{\partial_-^2} \bar{h} \frac{\partial}{\partial_-} \bar{h} \right) \partial_-^2 h$$
$$- \kappa \int d^3x\, \partial^2 \left(T\, \bar{h} \frac{1}{\partial_-^2} \bar{h} \right) \partial_-^2 h \tag{D.9}$$

$$= -2\kappa \int d^3x\, \partial T\, \bar{h} \frac{1}{\partial_-^2} \left(\frac{\partial}{\partial_-} \bar{h}\, \partial_-^3 h - \bar{h} \partial_-^2 \partial h \right) + \kappa \int d^3x\, \partial^2 T\, \bar{h}\, \frac{1}{\partial_-^2} (\bar{h} \partial_-^2 h)$$
$$- 2\kappa \int d^3x\, T\, \partial_-^2 h \left(\bar{h} \frac{\partial^2}{\partial_-^2} \bar{h} - \frac{\partial}{\partial_-} \bar{h}\, \frac{\partial}{\partial_-} \bar{h} \right). \tag{D.10}$$

Note that the second line in the above equation is the complex conjugate of $\mathcal{X}$ in (D.2). Thus, we obtain

$$\mathcal{Y} = -2\kappa \int d^3x\, \partial T\, \bar{h} \frac{1}{\partial_-^2} \left(\frac{\partial}{\partial_-} \bar{h}\, \partial_-^3 h - \bar{h} \partial_-^2 \partial h \right) + \kappa \int d^3x\, \partial^2 T\, \bar{h}\, \frac{1}{\partial_-^2} (\bar{h} \partial_-^2 h)$$
$$- 2\,\kappa \int d^3x\, \partial T\, \frac{1}{\partial_-^2} (\bar{h} \partial_-^2 h)\, \partial \bar{h} + 2\,\kappa \int d^3x\, T\, \frac{1}{\partial_-^2} \left(\frac{\partial}{\partial_-} \bar{h}\, \partial_-^3 h - \bar{h}\, \partial_-^2 \partial h \right) \partial \bar{h}\,. \tag{D.11}$$

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