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Amendment to: populations in environments with a soft carrying capacity are eventually extinct

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Abstract

This sharpens the result in the paper Jagers and Zuyev (J Math Biol 81:845–851, 2020): consider a population changing at discrete (but arbitrary and possibly random) time points, the conditional expected change, given the complete past population history being negative, whenever population size exceeds a carrying capacity. Further assume that there is an $\epsilon > 0$ such that the conditional probability of a population decrease at the next step, given the past, always exceeds ϵ if the population is not extinct but smaller than the carrying capacity. Then the population must die out.

Keywords Population dynamics · Extinction · Martingales · Stochastic stability

Mathematics Subject Classification 92D25 · 60G42 · 60K40

1 Three assumptions and one result

Denote population sizes, starting at time $\tau_0 = 0$, by Z_0, changing into $Z_1, Z_2, \ldots \in \mathbb{N}$ at subsequent time points $0 < \tau_1 < \tau_2 \ldots$. Here $\mathbb{N}$ is the set of non-negative integers, and we make no assumptions about the times between changes. Let $\mathscr{F}_n$ be the sigma-algebra of all events up to and including the n-th change - i.e. really *all* events, not only population size changes - and introduce a *carrying capacity* $K > 0$, the population size where reproduction turns conditionally subcritical. More precisely:

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Assumption 1

$$\mathbb{E}[Z_{n+1}|\mathscr{F}_n] \le Z_n, \quad \text{if } \ Z_n \ge K. \tag{1}$$

Further,

Assumption 2 There is no resurrection or immigration but, otherwise, a change is a change in population size:

$$Z_n = 0 \Rightarrow Z_{n+1} = 0, \tag{2}$$

$$Z_n > 0 \Rightarrow Z_{n+1} \neq Z_n. \tag{3}$$

Assumption 3 Non-extinct populations, smaller than the carrying capacity, run a definite risk of decreasing:

$$\exists \epsilon > 0; \forall n \in \mathbb{N}, 0 < Z_n < K \Rightarrow \mathbb{P}(0 \le Z_{n+1} < Z_n|\mathscr{F}_n] \ge \epsilon. \tag{4}$$

Then:

Theorem 1 *Under the three assumptions given, the population must die out: with probability 1, $Z_n = 0$ eventually.*

The original paper (Jagers and Zuyev 2020) had a stronger third assumption, *viz.* that, whatever the population history, there must be a definite, strictly positive risk that the population size decreases by exactly one unit at the next change. This is not unnatural and can be interpreted as a possibility that a change involves no reproduction but merely the death of one individual. But it turns out to be unnecessary.

2 The proof

Like the original proof, this starts from stopping times $\nu_1, \nu_2, \ldots$ and $\mu_1, \mu_2, \ldots$, the former denoting the times of successive visits to the integer interval $[0, K)$, the latter the subsequent first hittings of levels $\ge K$. More precisely,

$$\nu_1 := \inf\{n \in \mathbb{N}; Z_n < K\},$$

and for $k = 1, 2, \ldots,$

$$\mu_k := \inf\{n \in \mathbb{N}; n > \nu_k \text{ and } Z_n \ge K\}, \nu_{k+1} := \inf\{n \in \mathbb{N}; n > \mu_k \text{ and } Z_n < K\}.$$

As was noted, $\nu_1 < \infty$, whereas the μ_k constitute an increasing sequence, possibly hitting infinity. Clearly, $\nu_k < \infty, \mu_k = \infty$ means that the population dies out at or after ν_k, without ever reaching K again. Also for any k, $\mu_k < \infty \Rightarrow \nu_{k+1} < \infty$. Proceeding like in the original paper, note that

$$Z_n \to 0 \Leftrightarrow \exists n \in \mathbb{N}; Z_n = 0 \Leftrightarrow \exists k; \mu_k = \infty,$$

and

$$\mathbb{P}(\exists k; \mu_k = \infty) = \lim_{k\to\infty} \mathbb{P}(\mu_k = \infty) = 1 - \lim_{k\to\infty} \mathbb{P}(\mu_k < \infty).$$

But

$$\mathbb{P}(\mu_k < \infty) = \mathbb{P}(\mu_k < \infty, \nu_k < \infty) = \mathbb{E}[\mathbb{P}(\mu_k < \infty | \mathscr{F}_{\nu_k}); \nu_k < \infty.]$$

For short, write

$$D_n := \{Z_n \leq (Z_{n-1} - 1)^+\}$$

for the event that the n-th change is a decrease, provided $Z_{n-1} > 0$ (and of course the population remains extinct if $Z_{n-1} = 0$). By Assumption 3, $Z_n < K$ implies that

$$\begin{aligned}\mathbb{P}(\cap_{j=1}^{K} D_{n+j} | \mathscr{F}_n) &= \mathbb{E}[\mathbb{P}(D_{n+K} | \mathscr{F}_{n+K-1}; \cap_{j=1}^{K-1} D_{n+j} | \mathscr{F}_n] \\ &\geq \epsilon \mathbb{P}(\cap_{j=1}^{K-1} D_{n+j} | \mathscr{F}_n) \geq \ldots \geq \epsilon^K.\end{aligned}$$

Since $Z_n < K$ implies that $Z_{n+K} = 0$ on the set

$$\cap_{j=1}^{K} D_{n+j},$$

and the population size never crosses the carrying capacity, we can conclude that

$$\begin{aligned}\mathbb{P}(\mu_k = \infty) &= 1 - \mathbb{P}(\mu_k < \infty) \\ &\geq 1 - (1 - \epsilon^K)\mathbb{P}(\mu_{k-1} < \infty) \geq \ldots \geq 1 - (1 - \epsilon^K)^k \to 1.\end{aligned}$$

The theorem follows.

Reference

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