EXTREMAL KÄHLER METRICS ON BLOWUPS

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ABSTRACT. Consider a compact Kähler manifold which either admits an extremal Kähler metric, or is a small deformation of such a manifold. We show that the blowup of the manifold at a point admits an extremal Kähler metric in Kähler classes making the exceptional divisor sufficiently small if and only if it is relatively K-stable, thus proving a special case of the Yau-Tian-Donaldson conjecture. We also give a geometric interpretation of what relative K-stability means in this case in terms of finite dimensional geometric invariant theory. This gives a complete solution to a problem addressed in special cases by Arezzo, Pacard, Singer and Székelyhidi. In addition, the case of a deformation of an extremal manifold proves the first non-trivial case of a general conjecture of Donaldson.

1. INTRODUCTION

A central goal of Kähler geometry is to understand the existence of canonical representatives of Kähler classes. The natural representatives are *constant scalar curvature Kähler (cscK) metrics* and more generally *extremal Kähler metrics* (or simply *extremal metrics*). Such metrics do not always exist, and the Yau-Tian-Donaldson conjecture states that the existence of cscK metrics should be equivalent to the algebro-geometric notion of *K-stability* [41, 39, 15]. Similarly the existence of extremal metrics should be equivalent to *relative K-stability* [32]. Despite significant progress, this conjecture is open in general, and even when the conjecture is known to hold, the geometric meaning of K-stability is typically not clear.

Rather than explore the general setting, we focus on a particular geometric case. One of the first constructions of cscK metrics is due to Arezzo-Pacard [2, 3], who examined the existence of cscK metrics on blowups of manifolds known to admit cscK metrics. In the absence of automorphisms of the starting manifold, they construct cscK metrics on the blowup using a gluing method. Perhaps the most interesting aspect of their work is that when the starting manifold admits automorphisms, there are obstructions to obtaining cscK metrics on the blowup related to stability of the blown-up point in the sense of geometric invariant theory. The problem was then generalised to the extremal setting by Arezzo-Pacard-Singer [4]. An important problem in the field has since been to characterise the existence of extremal metrics on the blowup through relative K-stability, in line with the Yau-Tian-Donaldson conjecture, and we refer to Pacard [25] and Székelyhidi [35] for surveys on this problem and for further context. As we discuss in more detail below, Székelyhidi has made significant progress on this problem, including a complete solution in the cscK case and in complex dimension at least three [34, 37]. We take a new approach to the problem, with which we provide a complete solution in general.

To state the main results we require some further notation. Consider a compact Kähler manifold X with complex structure J, together with a Kähler class α which admits an extremal metric ω. For a point $p \in X$, consider the blowup $\pi : \mathrm{Bl}_p X \to X$ endowed with the Kähler class $\alpha_\varepsilon = \pi^*\alpha - \varepsilon[E]$ with E the exceptional divisor; $\varepsilon > 0$ will be taken small. We fix a maximal compact torus $T \subset \mathrm{Aut}_0(X, \alpha)$ of the reduced automorphism group of X fixing ω, with complexification $T^{\mathbb{C}}$. We then consider a family of moment maps

$$A_\varepsilon \mu + B_\varepsilon \Delta\mu : X \to \mathfrak{t}_\varepsilon$$

for the T-action on X with respect to $A_\varepsilon \omega + B_\varepsilon \operatorname{Ric}\omega$, where $A_\varepsilon, B_\varepsilon$ are functions of ε defined explicitly in Corollary 6.9 with $A_\varepsilon > 0$ and with B_ε of strictly higher order in ε. Here we use a family of inner products $\langle \cdot, \cdot \rangle_\varepsilon$ to identify $\mathfrak{t}^*$ with its dual, leading to the dependence on ε. Letting $\mathfrak{t}_{p,\varepsilon}^\perp$ denote the orthogonal complement of the stabiliser of $\mathfrak{t}_p$ with respect to $\langle \cdot, \cdot \rangle_\varepsilon$, our main result is the following:

Theorem 1.1 (Stable case). *The following are equivalent:*

(i) $(\mathrm{Bl}_p X, \alpha_\varepsilon)$ *admits an extremal metric for all* $0 < \varepsilon \ll 1$*;*

(ii) $(\mathrm{Bl}_p X, \alpha_\varepsilon)$ *is relatively K-stable for all* $0 < \varepsilon \ll 1$*;*

(iii) for all $0 < \varepsilon \ll 1$ *and for all* $u \in \mathfrak{t}_{p,\varepsilon}^\perp$ *with Hamiltonian* H_u *with respect to* ω *we have*

$$A_\varepsilon H(q_u) + B_\varepsilon \Delta H(q_u) > 0,$$

with q_u *the specialisation of* p *under the flow of* Ju*;*

(iv) for all $0 < \varepsilon \ll 1$ *there is a point* $p_\varepsilon \in T^{\mathbb{C}}.p$ *with*

$$A_\varepsilon \mu(p_\varepsilon) + B_\varepsilon \Delta\mu(p_\varepsilon) \in \mathfrak{t}_{p_\varepsilon}.$$

An analogous statement holds characterising the existence of cscK metrics in terms of K-stability. Thus we have solved a special case of the Yau-Tian-Donaldson conjecture, and have also obtained a complete geometric understanding of what relative K-stability means in this setting in terms of geometric invariant theory and finite dimensional moment maps. In addition the inner products $\langle \cdot, \cdot \rangle_\varepsilon$ differ from the L^2-inner product on Hamiltonian vector fields on (X, ω) in a way that only depends on the properties of the function H at a fixed point of the $T^{\mathbb{C}}$-action, so the change in the inner product is finite dimensional. In the algebraic case, namely with X projective, $\alpha = c_1(L)$ and ε taken to be rational, *(iii)* of the above can be interpreted in terms of stability in the sense of geometric invariant theory for a Lie group $(T_{p,\varepsilon}^\perp)^{\mathbb{C}}$ associated with $(\mathfrak{t}_{p,\varepsilon}^\perp)^{\mathbb{C}}$ and with respect to the line bundle $A_\varepsilon L - B_\varepsilon K_X$, where again the change in inner product has an interpretation purely in terms of invariants often considered in geometric invariant theory. So *(iii)* can be seen as completely algebro-geometric in this case.

As mentioned above, Theorem 1.1 is due to Székelyhidi in the case (X, α) admits a cscK metric and has complex dimension at least three [37]. Székelyhidi's approach is to directly attack the problem analytically, by constructing very good "approximate solutions" to the cscK equation on the blowup, and most of the work goes into understanding these approximate solutions geometrically in terms of moment maps on X. He then shows that K-stability implies the required stability of the point p, which by general theory is equivalent to the appropriate moment map condition. The analysis and computations involved seem very difficult to extend to the extremal case, and although progress has been made by Datar [9], even the case of complex dimension two seems to be out of reach using Székelyhidi's strategy. On a technical level we also note that Theorem 1.1 "(*iv*) implies (*i*)" is stronger than [37, Theorem 1] as we allow the inner product to vary, and this is important in obtaining complete results in the extremal setting.

We use a new strategy in which we solve the problem in a different order. The main idea is to glue at the level of *almost complex structures* rather than Kähler metrics. This allows us to perform the argument in families of complex manifolds, and the key point is that we can then directly use the moment map interpretation of the scalar curvature [14, 17]. A finite dimensional argument then allows us to see relative K-stability directly as the obstruction, so our approach seems more natural in terms of the geometry of the scalar curvature operator. We then directly show that relative K-stability can be interpreted directly in terms of relative stability of the point p, and hence also in terms of the moment map criterion given in 1.1 (*iv*). Analytically the novelty in our approach is to understand the relevant estimates in weighted Hölder spaces in families of complex manifolds. Our approach relies on the analysis involved in the simplest, well-understood case of blowing up a fixed point of the automorphism group [4, 36, 34], but also involves considerable additional difficulties. Another technical advantage in comparison with Székelyhidi's work is that by gluing at the level of complex structures, all the relevant Hamiltonians on X have a natural lift to the blowup; Székelyhidi instead must choose a lift which is somewhat geometrically artificial.

The techniques we develop are strong enough to also prove the "semistable case", which has not been considered before. Work of Stoppa and Stoppa-Székelyhidi implies that if (X, α) is relatively K-unstable, then its blowup is also relatively K-unstable in the classes we consider and hence cannot admit an extremal metric [29, 30]. Thus with the "stable case" settled in Theorem 1.1, the only remaining case is that of a relatively K-semistable manifold. We now consider a Kähler manifold (X, α) which is *analytically relatively K-semistable*; this means that there is a $\mathrm{Aut}_0(X, \alpha)$-equivariant degeneration of (X, α) to an extremal Kähler manifold (X_0, α_0). As the name suggests, the condition implies relative K-semistability [16, 30, 11], and can also be seen as asking that (X, α) is a small equivariant deformation of an extremal manifold. We call (X_0, α_0) an *extremal degeneration* of (X, α). The condition can be seen as the natural analytic analogue of relative K-semistability.

The difference in the statement below in comparison with the stable case is the setup of the problem: instead of the action of the automorphism group (X, α) itself, we consider the action of a maximal torus $T^{\mathbb{C}} \subset \mathrm{Aut}_0(X_0, \alpha_0)$ on a space $\mathcal{X}$ built from the Kuranishi space of (X_0, α_0). Thus the specialisation of p under a vector field $Ju \in \mathfrak{t}_\varepsilon^\perp$ will no longer actually be a point on (X, α) itself. We also endow $\mathcal{X}$ with a relatively Kähler metric which is T-invariant. In this "semistable" setting we prove the following:

Theorem 1.2 (Semistable case). *The following are equivalent:*

(i) $(\mathrm{Bl}_p X, \alpha_\varepsilon)$ *admits an extremal metric for all* $0 < \varepsilon \ll 1$*;*
(ii) $(\mathrm{Bl}_p X, \alpha_\varepsilon)$ *is relatively K-stable for all* $0 < \varepsilon \ll 1$*;*
(iii) for all $0 < \varepsilon \ll 1$ *and for all* $u \in \mathfrak{t}_{p,\varepsilon}^\perp$ *with Hamiltonian* H_u *with respect to* ω *we have*

$$A_\varepsilon H(q_u) + B_\varepsilon \Delta H(q_u) > 0,$$

with q_u *the specialisation of* p *under the flow of* Ju*;*

Here the Hamiltonian and Laplacian are computed on the complex deformation of (X, α) on which q_u lies and J is the complex structure of $\mathcal{X}$.

We also obtain a characterisation of (*iii*), and hence the other conditions, in terms of finite dimensional moment maps on $\mathcal{X}$, analogously to (*iv*) of the "stable case". The statement is more technical and is given as Theorem 6.15. In the algebraic case (*iii*) can similarly be interpreted purely algebro-geometrically, and there is a corresponding version in the cscK case. An important consequence is a proof of the following conjecture of Donaldson in the first-nontrivial case:

Conjecture 1.3 (Donaldson). *Let (X, α) be a compact Kähler manifold. Then there is a collection of points $p_1, \ldots, p_r \in X$ such that the blowup $\mathrm{Bl}_{p_1,\ldots,p_r} X$ admits an extremal Kähler metric in some Kähler class.*

The conjecture is made by analogy with Taubes' work on anti-self dual metrics [38]. We prove the following special case.

Theorem 1.4. *Suppose (X, α) is an analytically relatively K-semistable manifold. Then there is a point $p \in X$ such that the blowup $(\mathrm{Bl}_p X, \alpha_\varepsilon)$ admits an extremal Kähler metric in some Kähler class.*

Without assuming relative K-semistability, one must consider more general Kähler classes than we consider and one needs a different technique. In our case we show that Donaldson's conjecture follows from Theorem 1.2 by showing that one can find a point satisfying condition (*iii*) of Theorem 1.2. The conjecture also predicts that one can blow up a suitably general collection of points to obtain an extremal metric, and we obtain version of this statement in our situation in the algebraic case.

An interesting aspect of our work is that we need to use more general degenerations than test configurations to prove Theorems 1.1 and 1.2; these are essentially the Kähler analogue of what were called $\mathbb{R}$-degenerations in [13]; that is finitely generated $\mathbb{R}$-indexed filtrations of the coordinate ring. Morally, test configurations correspond to "rational" degenerations, whereas in the general extremal, non-projective setting we must also include "irrational" degenerations. As we need only consider degenerations with smooth central fibre in our work, it is straightforward to generalise the theory of relative K-stability to this setting, but it is an interesting problem to understand the role played by these degenerations in the general theory. In two important special cases, namely when X is projective with α the first Chern class of an ample bundle, or when we consider cscK metrics rather than extremal metrics, we only need to consider test configurations.

We finally remark that while we have settled the problem when a single point is blown up, the question is also interesting when blowing up multiple points [2, 3], or when blowing up higher dimensional submanifolds [27], and precise results along the lines of Theorem 1.1 remain open in these settings. The reason our techniques do not apply in these cases is that the test configurations and $\mathbb{R}$-test configurations that one must consider may have singular central fibre. In the case when one blows up collections of points, the singularities are rather mild and it would be interesting, but likely very challenging, to generalise our approach to that setting.

Outline. We begin in Section 2 with preliminary material on extremal metrics and relative K-stability. Section 3 then describes the aspects of the space of almost complex structures that we require. We prove the main technical gluing result in Section 4, which we use in Section 5 to show that relative K-stability is the obstruction to the existence of extremal metrics on blowups. In particular, the equivalence between (*i*) and (*ii*) in Theorems 1.1 and 1.2 is proven in Section 5. Section 6 explains the geometric meaning of relative K-stability in this setting, showing that (*ii*) is equivalent to (*iii*) and (*iv*) in these results.

Acknowledgements. We thank Gábor Székelyhidi for a very helpful suggestion at the outset of this project. RD was funded by a Royal Society University Research Fellowship. LMS was funded by a Marie Skłodowska-Curie Individual Fellowship, funded from the European Union's Horizon2020 research and innovation programme under grant agreement No 101028041, and also by Villum Fonden Grant 0019098, while he was a member of Aarhus University. This work was completed while RD visited Newcastle University, and he thanks the department and Stuart Hall for their hospitality.

2. Extremal Kähler metrics and relative K-stability

2.1. **Extremal Kähler metrics.**

We wish to understand the existence of *extremal Kähler metrics*. A reference for the material of this section is Székelyhidi [36]. Thus let X be a compact Kähler manifold of dimension n and let α be a Kähler class on X. For any Kähler metric $\omega \in \alpha$, its *Ricci curvature* is defined by

$$\operatorname{Ric}\omega = -\frac{i}{2\pi} i\partial\bar{\partial}\log\omega^n,$$

while its *scalar curvature* is defined by

$$S(\omega) = \Lambda_\omega \operatorname{Ric}\omega.$$

The extremal condition also involves a class of functions on X called *holomorphy potentials*. These are functions H such that

$$\mathcal{D}h = \bar{\partial}\nabla^{1,0}H = 0,$$

where $\nabla^{1,0}H$ denotes the $(1,0)$-component of the gradient of H, taken with respect to the Riemannian metric induced by ω. Throughout we will denote

$$\overline{\mathfrak{h}} = \{H \in C^\infty(X) \mid \mathcal{D}H = 0\}.$$

Importantly, associated to each holomorphy potential H is a holomorphic vector field $\nabla^{1,0}h$. These vector fields can be characterised as the holomorphic vector fields on X that vanish somewhere, and we denote the space of such vector fields by $\mathfrak{h}$. Given a Kähler metric, there is an isomorphism between $\mathfrak{h}$ and the functions in $\overline{\mathfrak{h}}$ of average zero, and we will sometimes use $\mathfrak{h}$ to denote these functions instead.

Denoting by $\operatorname{Aut}_0(X)$ the connected component of the identity in the biholomorphism group of X, we further denote through the Lie algebra-Lie group correspondence

$$\operatorname{Aut}_0(X,\alpha) \subset \operatorname{Aut}_0(X)$$

the Lie subgroup associated with vector fields that vanish somewhere. This is sometimes called the reduced automorphism group of X, and in the case that $\alpha = c_1(L)$ for some ample line bundle L on X, corresponds to automorphisms which lift to L. Note that the group itself is actually independent of α.

Definition 2.1. We say that ω is an *extremal Kähler metric* if $S(\omega) \in \overline{\mathfrak{h}}$. If $S(\omega)$ is furthermore constant, we say that ω is a *constant scalar curvature Kähler* (cscK) metric.

Thus the extremal condition asks that $\nabla^{1,0}S(\omega)$ is a holomorphic vector field, called the *extremal vector field*.

2.2. **Relative K-stability.** Extremal Kähler metrics do no not always exist, and conjecturally their existence is characterised by the notion of *relative K-stability* of (X, α). As we are working in the transcendental setting, we in fact need a slightly stronger variant of relative K-stability. In the projective setting, with $\alpha = c_1(L)$ for some ample line bundle L, we only need the usual notion. But to avoid additional notational complexity, we simply refer to the notion used in our work as relative K-stability.

Relatively K-stability involves both a class of degenerations of (X, α), and associated numerical invariants. The standard class of degenerations are test configurations, but we will require a slightly more general notion.

Definition 2.2. An $\mathbb{R}$-*test configuration* for (X, α) a collection $\pi : (\mathcal{X}, \mathcal{A}) \to \mathbb{C}^r$ with

(i) $\mathcal{X}$ a complex manifold endowed with a $(\mathbb{C}^*)^r$-action;
(ii) $\mathcal{A}$ a $(1,1)$-class which is relatively Kähler and $(\mathbb{C}^*)^r$-invariant;
(iii) π a proper surjective $(\mathbb{C}^*)^r$-equivariant holomorphic submersion;
(iv) all fibres $(\mathcal{X}_b, \mathcal{A}_b)$ over $b \in (\mathbb{C}^*)^r \subset \mathbb{C}^r$ isomorphic to (X, α);
(v) a vector field $u \in \mathrm{Lie}(\mathbb{C}^*)^r$ whose flow takes each point to the origin.

When $r = 1$, we call $\pi : (\mathcal{X}, \mathcal{A}) \to \mathbb{C}$ a *test configuration*.

Here $\mathbb{C}^r$ is given the standard action of $(\mathbb{C}^*)^r$. The point of the extension is to allow irrational vector fields $u \in \mathrm{Lie}(\mathbb{C}^*)^r$, which as we will see is necessary when considering the existence of extremal Kähler metrics on Kähler manifolds. When the vector field is rational and hence generates a $\mathbb{C}^*$-action, one obtains an induced family over $\mathbb{C}$ and hence a test configuration. In the projective case, $\mathbb{R}$-test configuration can be encoded using $\mathbb{R}$-filtrations of the coordinate ring of (X, L); the reason we use the above geometric interpretation is the lack of a good analogue of such filtrations in the Kähler setting. We also note that $\mathbb{R}$-test configuration are essentially the same as what Inoue calls polyhedral test configurations [22] and what were originally called $\mathbb{R}$-degenerations in [13]; we use the language of Boucksom-Jonsson [5]. These degenerations seem to have first appeared in the work of Chen-Sun-Wang [8].

Remark 2.3. We have made a significant simplification in comparison with the usual theory of test configurations, in that we have assumed that $\pi : \mathcal{X} \to \mathbb{C}^r$ is a holomorphic submersion, which means that all of the fibres are smooth; this will be sufficient in our setting.

For a subgroup $G \subset \mathrm{Aut}_0(X, \alpha)$, we can ask for an $\mathbb{R}$-test configuration to be G-equivariant; this means that there is a G-action on $(\mathcal{X}, \mathcal{A})$ which fixes each fibre over $\mathbb{C}^r$, and induces the standard action on the general fibres $(\mathcal{X}_b, \mathcal{A}_b) \cong (X, \alpha)$ over $b \in (\mathbb{C}^*)^r \subset \mathbb{C}^r$. We will typically fix a maximal complex torus $T^{\mathbb{C}} \subset \mathrm{Aut}_0(X, \alpha)$ and consider test configurations which are $T^{\mathbb{C}}$-equivariant.

As $\pi : \mathcal{X} \to \mathbb{C}^r$ is a holomorphic submersion, we will be able to associate numerical invariants to the $\mathbb{R}$-test configuration using differential geometry. Consider the central fibre $(\mathcal{X}_0, \mathcal{A}_0)$ over $0 \in \mathbb{C}^r$, on which we have a holomorphic vector field u induced by the definition of an $\mathbb{R}$-test configuration. The flow of u induces a compact torus T of automorphisms of $(\mathcal{X}_0, \mathcal{A}_0)$, and we fix a Kähler metric ω_0 on

$\mathcal{X}_0$ which is invariant under this torus. In the case of a rational vector field the torus will simply be S^1, while in general we must take a higher rank torus defined by taking the closure of the flow of u. Denote by H_u an associated holomorphy potential for u, which exists since the class $\mathcal{A}_0$ is invariant under the action.

Definition 2.4. The *Futaki invariant* of u is given by

$$\mathrm{Fut}(u) = \int_{\mathcal{X}_0} H_u(\hat{S} - S(\omega_0))\omega_0^n,$$

while the *Donaldson-Futaki invariant* of $(\mathcal{X}, \mathcal{A})$ is given by

$$\mathrm{DF}(\mathcal{X}, \mathcal{A}) = \mathrm{Fut}(u).$$

Here $\hat{S}$ is the average scalar curvature. A classical result of Futaki states that this integral is actually independent of the choice of T-invariant Kähler metric $\omega_0 \in \mathcal{A}_0$. When u is rational, in the projective case the invariant can be computed purely algebro-geometrically [16], while in the Kähler case it can be computed as an intersection number over a compactification of $(\mathcal{X}, \mathcal{A})$ (which then also makes sense when $\mathcal{X} \to \mathbb{C}$ has singular central fibre) [12, 28].

We will also require an inner product on the space of vector fields, which has been provided by Futaki-Mabuchi [18]. Let u, v be two holomorphic vector fields on $\mathcal{X}_0$ which vanish somewhere, and with associated holomorphy potentials h_u, h_v. These holomorphy potentials are unique up to the addition of a constant, and we normalise so that H_u and H_v have integral zero over $\mathcal{X}_0$. Then we can define

$$\langle u, v\rangle = \int_{\mathcal{X}_0} H_u H_v \omega_0^n.$$

As the notation suggests, the integral is independent of choice of ω_0, and in the projective case it can also be computed purely algebro-geometrically.

Let us now fix a maximal complex torus $T^{\mathbb{C}}$ of automorphisms of (X, α) of rank m and consider a $T^{\mathbb{C}}$-equivariant $\mathbb{R}$-test configuration $(\mathcal{X}, \mathcal{A})$. Let $v_1, \ldots, v_m$ be a basis for $\mathfrak{t}^{\mathbb{C}} = \operatorname{Lie} T^{\mathbb{C}}$, so that on $(\mathcal{X}_0, \mathcal{A}_0)$ we have holomorphic vector fields $u, v_1, \ldots, v_m$, where u is the vector field induced on the $\mathbb{R}$-test configuration as above.

Definition 2.5. The *relative Donaldson-Futaki invariant* of $(\mathcal{X}, \mathcal{A})$ is defined to be

$$\mathrm{DF}_T(\mathcal{X}, \mathcal{A}) = \mathrm{DF}(\mathcal{X}, \mathcal{A}) - \sum_{j=1}^{m} \frac{\langle u, v_j\rangle}{\langle v_j, v_j\rangle} \mathrm{Fut}(v_j).$$

Here $\mathrm{Fut}(v_j)$ can be computed on either (X, α) or $(\mathcal{X}_0, \mathcal{A}_0)$ as the integrals agree on each [11, Propositions 3.3, 3.4]. If u is orthogonal to v_j for each j, then this is simply the Donaldson-Futaki invariant of $(\mathcal{X}, \mathcal{A})$ and one can view the invariant in general as the Donaldson-Futaki invariant of the projection of u orthogonal to $\mathfrak{t}^{\mathbb{C}}$. The definition is due to Székelyhidi in the projective setting [32].

Let us call an $\mathbb{R}$-test configuration a *product* $\mathbb{R}$-test configuration if $(\mathcal{X}_0, \mathcal{A}_0) \cong (X, \alpha)$, so that u is actually a holomorphic vector field on (X, α). Note that for a product $\mathbb{R}$-test configuration we have $\mathrm{DF}_T(\mathcal{X}, \mathcal{A}) = 0$.

Definition 2.6. We say that (X, α) is *relatively K-stable* if for all $T^{\mathbb{C}}$-equivariant $\mathbb{R}$-test configurations we have $\mathrm{DF}_T(\mathcal{X}, \mathcal{A}) \geq 0$, with equality if and only if $(\mathcal{X}, \mathcal{A})$ is a product.

If instead we ask for $\mathrm{DF}(\mathcal{X}, \mathcal{A}) \geq 0$, with equality precisely when $(\mathcal{X}, \mathcal{A})$ is a product, we say that (X, α) is *K-polystable*; this is the notion relevant to the existence of cscK metrics in the class α.

Remark 2.7. We emphasise that this is a different notion to what is usually called relative K-stability. The reason is twofold. In one sense, the condition is significantly weaker since, as explained in Remark 2.3, we only allow $\mathbb{R}$-test configurations with smooth central fibre, and in general one certainly must consider $\mathbb{R}$-test configurations with singular central fibre. The special geometry of our situation allows us to only require $\mathbb{R}$-test configurations with smooth central fibre.

In another sense, our definition is stronger than the usual one. The reason is that we include $\mathbb{R}$-test configurations, which do not appear in the prior definitions which only involve test configurations. It will be crucial to our arguments to include these more general degenerations, and this gives strong evidence that they are needed to understand the existence of extremal metrics on Kähler manifolds in full generality. As we discuss below, these seem unnecessary in the projective setting and it seems likely one can restrict to test configurations in that situation.

The notion of relative K-stability is motivated by concepts in finite dimensional geometric invariant theory, and Székelyhidi discusses the relevant parts of the theory in his work [37]. We note that he claims that one need only consider one-parameter subgroups (the analogue of test configurations, as opposed to $\mathbb{R}$-test configurations) in the analogous finite dimensional setting, but his discussion contains an error in his setup of the problem. The issue is in his claim that what he writes as $K_{T^\perp}$ is a closed subgroup of K [37, p. 929]; the reference he gives proves this only for rational inner products and the claim is false in general.

Theorem 2.8. *Suppose (X, α) admits an extremal metric. Then (X, α) is relatively K-stable.*

Proof. The required inequality for the relative Donaldson-Futaki invariant is proven to hold in [11, 37] for test configurations. The main point of the proof is to compute expansions of Futaki invariants and inner products for test configurations induced for appropriate blowups of (X, α). We will see in Section 5 that the necessary expansions hold in the more general setting of $\mathbb{R}$-test configurations (as we have assumed the central fibre is smooth), essentially by continuity of the quantities involved as one varies u, proving the result in general. □

In the projective setting, considering only test configurations, this result is due to Stoppa-Székelyhidi [30]. In the Kähler setting, the appropriate inequality for test configurations with possibly singular central fibre is proven in [11].

As well as manifolds that actually admit extremal metrics, we will also be interested in "strictly semistable" manifolds, in the following analytic sense.

Definition 2.9. We say that (X, α) is *analytically relatively K-semistable* if there is a $T^{\mathbb{C}}$-equivariant $\mathbb{R}$-test configuration for (X, α) such that the central fibre admits an extremal metric with extremal vector field induced by an element of $\mathrm{Lie}(T^{\mathbb{C}})$. We similarly say that (X, α) is *analytically K-semistable* if there is a $T^{\mathbb{C}}$-equivariant $\mathbb{R}$-test configuration such that the central fibre admits a cscK metric.

The equivariance hypotheses, with respect to a maximal torus $T^{\mathbb{C}} \subset \mathrm{Aut}(X, \alpha)$, will be important for our arguments, and in practice one expects the equivariance assumption to always be satisfied. Note that from [16, 11, 30], analytic relative

K-semistability implies relative K-semistability and the analogue is true also for K-semistability. In general a version of the Yau-Tian-Donaldson conjecture states that a K-semistable manifold admits a test configuration with K-polystable central fibre, which if smooth should then admit a cscK metric. In particular one should view analytic (relative) K-semistability as a smoothness hypothesis on the (relatively) K-polystable degeneration. By results of Brönnle [6] and Székelyhidi [33], small deformations of cscK manifolds are analytically K-semistable, and the same is true for deformations of extremal manifolds provided the extremal vector field extends to the deformation.

3. The space of almost complex structures

3.1. **Scalar curvature as a moment map.** We now describe some of the moment map theory of the scalar curvature operator, as initiated by Fujiki [17] and Donaldson [14]; a good reference is Gauduchon [19, Section 8]. Fix a compact $2n$-dimensional symplectic manifold (M,ω) and denote by $\mathcal{J}(M,\omega)$ the space of almost complex structures compatible with ω. The space $\mathcal{J}(M,\omega)$ is naturally an infinite dimensional manifold, with tangent space at a point $J\in\mathcal{J}(M,\omega)$ given by

$$T_J\mathcal{J}(M,\omega)=\{A:TM\to TM \mid AJ+JA=0 \text{ and } \omega(u,Av)=\omega(v,Au) \text{ for all } u,v\}.$$

There is a natural almost complex structure on the space $\mathcal{J}(M,\omega)$ obtained by sending $A\to JA$, and this almost complex structure is even formally integrable. We next use the natural Riemannian metric on $\mathcal{J}(M,\omega)$ defined by setting

$$\langle A,B\rangle_J=\int_M (A,B)_{g_J}\omega^n,$$

with g_J the Riemannian metric on M associated with ω and J, to define a Kähler metric Ω by

$$\Omega_J(u,v)=\langle Ju,v\rangle_J.$$

We will primarily be interested in the subspace of integrable almost complex structures, which we denote $\mathcal{J}^{\text{int}}(M,\omega)$ and which is a complex submanifold of $\mathcal{J}(M,\omega)$ [19, Proposition 8.1.1 (iii)].

Denote by $\mathcal{H}=\mathcal{H}(M,\omega)$ the space of Hamiltonian symplectomorphisms of (M,ω). By definition a Hamiltonian symplectomorphism is the time-one flow of a Hamiltonian vector field, and so there is a natural identification

$$\operatorname{Lie}\mathcal{H}=C_0^\infty(M,\omega),$$

where $C_0^\infty(M,\omega)$ denotes functions of integral zero. The action of $\mathcal{H}$ on $\mathcal{J}(M,\omega)$ leaves Ω invariant, and so we can ask for a moment map for the action.

For any $J\in\mathcal{J}(M,\omega)$, we denote by $S(J)$ the Hermitian scalar curvature of ω with respect to J; this is simply the usual scalar curvature when J is integrable, which will be the important case for us. Denote also $\hat{S}$ the average scalar curvature. We can consider the scalar curvature as a map

$$\mathcal{J}(M,\omega)\to\operatorname{Lie}\mathcal{H}^*$$

by identifying $\operatorname{Lie}\mathcal{H}$ with its dual using the L^2-inner product on functions. The key result we make use of is the following.

Theorem 3.1. [14, 17] *The scalar curvature operator*

$$J\to S(J)-\hat{S}$$

is a moment map for the action of $\mathcal{H}$ on $\mathcal{J}(M,\omega)$.

An important point of our work is to use this perspective to actually construct cscK and extremal metrics. If $\mathcal{H}_J$ denotes the stabiliser of a given complex structure J, then $\mathcal{H}_J$ is simply the isometry group of g_J. Denoting by $\mathfrak{h}_J$ its Lie algebra, the extremal condition reduces to asking that

$$S(J) - \hat{S} \in \mathfrak{h}_J.$$

As the problem we are interested in is a perturbation problem, we will need to understand how best to perturb complex structures using functions on M. The analogous linear question for Kähler metrics is standard: if ω is Kähler, then $\omega_t = \omega + it\partial\bar{\partial}\varphi$ is Kähler for all $t \in [0,1]$ provided $\varphi \in C^\infty(X)$ has sufficiently small C^2-norm. Given such a φ, we can define a new path of complex structures by using Moser's trick, as in [33, Section 2]. Write

$$\omega_t = \omega - tdJd\varphi = \omega - td\beta$$

for a fixed one-form β, and let v_t be the vector field dual to $-\beta$ via ω_t. Integrating these vector fields produces a one-parameter path of diffeomorphisms f_t for $t \in [0,1]$, and setting $F_\varphi = f_1$ we have $F_\varphi^*\omega_1 = \omega$. Thus F_φ^*J is a new complex structure compatible with ω, and this is how we perturb complex structures using functions.

As is always the case for perturbation problems, we will be later require the linearisation of the non-linear operator

$$\varphi \mapsto S(F_\varphi^*J)$$

of interest. Denoting by $\mathcal{D}^*$ the L^2-adjoint of the operator $\mathcal{D} = \bar{\partial}\nabla^{1,0}$ used in Section 2.1, the *Lichnerowicz operator* is defined to be $\mathcal{D}^*\mathcal{D}$. The following can, for example, be found in [33, Proof of Proposition 7].

Proposition 3.2. *The linearisation of the operator $\varphi \mapsto S(F_\varphi^*J)$ is the negative of the Lichnerowicz operator, i.e. the operator*

$$\varphi \mapsto -\mathcal{D}^*\mathcal{D}\varphi.$$

Remark 3.3. Although we have for simplicity explained this theory for smooth complex structures, the theory works equally well assuming less regularity, and we will later apply these results for C^k-complex structures for some large k.

3.2. **Kuranishi theory.** To understand relatively analytically K-semistable manifolds, we will require some results from deformation theory. Denote by (X,α) an analytically relatively K-semistable manifold, and let $X_0 = (M, J_0)$ denote its extremal equivariant degeneration with Kähler class α_0. Denote by T' a maximal compact subgroup of $\mathrm{Aut}_0(X,\alpha)$, so that there is an induced T'-action on (X_0,α_0) by the equivariance assumption on the degeneration of (X,α) to (X,α) (the notation T' will be helpful in Section 5). Denote also T a maximal compact torus of $\mathrm{Aut}_0(X_0,\alpha_0)$, so that $T' \subset T$ is a subtorus. We use the following version of Kuranishi theory, as used by Székelyhidi [33] and Inoue [21].

Theorem 3.4. *There exists a complex space B' and a holomorphic embedding*

$$\Psi : B' \to \mathcal{J}^{\mathrm{int}}(M,\omega)$$

such that:

(i) there is a holomorphic T-action on B' making Ψ a T-equivariant morphism;

(ii) there is a universal family $\pi : \mathcal{X}' \to B'$ with a T-action making π a T-equivariant holomorphic map;
(iii) the induced T-action on $\mathcal{X}'$ fixes each fibre of π;
(iv) the form ω induces a T-invariant relatively Kähler metric on $\mathcal{X}'$;
(v) the family $\mathcal{X}' \to B'$ is a T-equivariant versal deformation space.

A T-equivariant versal deformation space means the following. Consider an arbitrary family $\pi_Y : (Y, \alpha_Y) \to B_Y$ with fixed fibre $(Y_b, \alpha_b) \cong (X_0, \alpha_0)$, and suppose there is a holomorphic T-action on (Y, α_Y) which restricts to an action on each fibre. Then perhaps after shrinking to a smaller neighbourhood of b, there is a holomorphic map $p : B_Y \to B'$ such that $Y \cong p^*(\mathcal{X}, \alpha)$. Thus B' parametrises small T-equivariant deformations of (X_0, α_0).

An important aspect of the construction is that B' is naturally a complex subspace of a vector space (denoted $\tilde{H}^1$ by Székelyhidi) such that the T_0-action on B' is actually induced by a linear action of T on the vector space, with $\Psi(0)$ equaling J_0. We complexify T to $T^{\mathbb{C}}$, which then acts locally on $\pi' : \mathcal{X}' \to B'$, in the sense that we have an action for each element of $T^{\mathbb{C}}$ induced by sufficiently small elements of $\operatorname{Lie} T^{\mathbb{C}}$.

As by assumption (X, α) is analytically relatively K-semistable, we obtain a sequence of points $p_t \in B' \to 0$ such that the fibre of the Kuranishi family over p_t is isomorphic to (X, α). For any such p_t, with t sufficiently close to zero, there is point q_t in the closure of the orbit $\overline{T_0^{\mathbb{C}}.p_t}$ such that the fibre $(\mathcal{X}'_{q_t}, \alpha'_{q_t})$ over q_t admits an extremal metric.

By construction, the universal family $\mathcal{X}' \to B'$ consists of the fixed smooth manifold M, with almost complex structure on each fibre compatible with the symplectic form ω. Thus we have a natural action on (M, ω) induced by the action on (X_0, ω), and the induced T-action on $\mathcal{X}' = M \times B'$ is simply the product action. In particular, if $b \in B'$ denotes a point with fibre isomorphic to (X, α), then the stabiliser of p is isomorphic to T'. We now replace B' with the closure of the $T^{\mathbb{C}}$-orbit of p, which is a complex manifold as the action is induced by a linear action on a vector space. This produces a new family $\pi'' : \mathcal{X}'' \to B''$, with B'' now a complex manifold of dimension r given by the difference between the ranks of $T^{\mathbb{C}}$ and $T'^{\mathbb{C}}$. There is a $T'^{\mathbb{C}}$-action on $\mathcal{X}''$ fixing the fibres of π'' and modify our family again by considering the product family $\mathcal{X}'' \times \mathbb{C}^r \to B'' \times \mathbb{C}^r$. Note that the dimension of $B'' \times \mathbb{C}^r$ is equal to the rank of $T^{\mathbb{C}}$ by construction. We finally denote $\mathcal{X} = \mathcal{X}'' \times \mathbb{C}^r$ and $B = B'' \times \mathbb{C}^r$, which is our starting holomorphic submersion of interest. The general fibre of the induced family $\pi : (\mathcal{X}, \alpha_{\mathcal{X}}) \to B$ is isomorphic to (X, α), while there is a unique fixed point q of the $T^{\mathbb{C}}$-action with $(\mathcal{X}_q, \alpha_q)$ an extremal Kähler manifold.

4. The gluing argument

This section contains the main gluing argument of the paper. Our results will construct metrics whose scalar curvature lies in certain finite dimensional space of functions corresponding to the holomorphy potentials on the central fibre $(\operatorname{Bl}_q \mathcal{X}_0, \omega_\varepsilon)$. More precisely, we will construct a sequence of maps

$$\Psi_\varepsilon : B \to \mathcal{J}(\operatorname{Bl}_q M, \omega_\varepsilon)$$

such that

$$S(J_{\varepsilon,b}) - \hat{S}_\varepsilon \in \mathfrak{h}_\varepsilon,$$

where $\mathfrak{h}_\varepsilon$ denotes holomorphy potentials on the central fibre. These will not in general be holomorphic maps, but the image of B will nevertheless be symplectic, which will be enough to reduce to a finite dimensional moment map problem.

4.1. **The initial setup.** Before considering blowups, we begin by constructing the appropriate map

$$\Psi : B \to \mathcal{J}(M, \omega).$$

Kuranishi theory, as discussed in Section 3.2, will be used to produce the desired map on the B''-component. Our goal is to extend this to a map from B, retaining the crucial properties of the map from B''.

The extension will be performed in the following manner. We work first on the central fibre $\mathcal{X}_0$ (which also completely covers the "stable case", namely when $\mathcal{X}_0 = X$). From the construction of the family $\mathcal{X}_0 \times \mathbb{C}^r \to \mathbb{C}^r$, for each point $s \in \mathbb{C}^r$, there is a corresponding point $q_s \in \overline{T^{\mathbb{C}}.p} \subset \mathcal{X}_0$ which maps to s. The point q_0, which we denote q, on the central fibre $\mathcal{X}_0$ is the point with maximal stabiliser which we blow up. We will construct a diffeomorphism Φ of $M \times \mathbb{C}^r$, which is the identity in the $\mathbb{C}^r$ component, and such that $\varphi_s = \Phi|_{M\times\{s\}} : M \to M$ is a symplectomorphism such that the blowup in $\varphi_s(q)$ is biholomorphic to the blowup in q_s. This will then produce a new complex structure $J_s = \varphi_s^* J_0$; a further small perturbation of this will then be the desired complex structure.

Note that for s near the origin J_s is not equal to J_0. This follows from the fact that the point q_s is not fixed by the action of the full torus, in contrast to q_0. An extension of this observation will be used in showing that the map Ψ is an embedding.

To actually construct the diffeomorphism Φ, we use Moser's trick. We first we choose holomorphic normal coordinates $(z_1, \ldots, z_n)$ around p with respect to the complex structure J_0, taken in such a way that the $T^{\mathbb{C}}$-action is given by the standard linear torus action on the first r coordinates. These coordinates can be assumed to be defined on the disk of radius 2. We also fix a cutoff function $\chi(t)$ which equals 1 when $t \leq \frac{1}{2}$ and 0 when $t \geq 1$.

We now define a family of symplectomorphisms of M, parametrised by a neighbourhood of the origin in $\mathbb{C}^r$. Let $w_1, \ldots, w_r$ be the coordinates on $\mathbb{C}^r$. Write $z_j = x_j + iy_j$ and $w_j = u_j + iv_j$ and define a one-form β_w on M by

$$\beta_w = d\left(\chi(|z|)\sum_{j=1}^{r}(u_j x_j + v_j y_j)\right). \tag{4.1}$$

Note that the z_i are only defined locally near q, but since we are using a cutoff function, the form vanishes outside the disk of radius 1 and so extends trivally to the whole of M.

Moser's trick thus produces a family of exact symplectomorphisms $\varphi_s : M \to M$, parametrised by a neighbourhood of the origin $\mathbb{C}^r$. Since these are symplectomorphisms, the pullback complex structure $\varphi_s^* J_0$ is compatible with ω, hence inducing a Kähler metric. In addition since $\beta_0 = 0$, over $s = 0$ the map is just the identity. The key point of the following lemma is that, even if $q_s \neq \varphi_s(0)$, the associated blowups of $\mathcal{X}_0$ are biholomorphic.

Lemma 4.1. *We have*

$$\varphi_s(0) = (s, 0) + O(|s|^2)$$

for all sufficiently small s. Moreover, the blowup of X at $\varphi_s(0)$ is biholomorphic to the blowup in q_s.

Proof. If ω were the Euclidean symplectic form, the dual vector field to the 1-form $d\left(\chi(|z|)\sum_{j=1}^{r}(u_j x_j + v_j y_j)\right)$ would, in this region, be the vector field

$$u_j \frac{\partial}{\partial x_j} + v_j \frac{\partial}{\partial y_j}, \tag{4.2}$$

whose time-one flow sends the origin to $(s, 0)$. In reality, the symplectic form is an $O(|z|^2)$ perturbation of this, and this causes the perturbation in $\varphi_s(0)$. The blowups at the points q_s and $\varphi_s(0)$ are biholomorphic because they lie in the same $T^{\mathbb{C}}$-orbit. This follows from the fact the action is T-linear and that the vector field producing φ_s and the vector field given by Equation (4.2) are dual to the same 1-form, with respect to two T-invariant symplectic forms. □

We next make this compatible with the B'' direction by performing exactly the same procedure for the family over $\{b\} \times \mathbb{C}^r$, just as for $\{0\} \times \mathbb{C}^r$. In order to do so, we need to introduce appropriate coordinates with respect to the complex structures J_b for $b \in B''$, so that we can define the corresponding one-form β to use in Moser's trick.

The base B'' is a ball in a vector space, which we equip with a norm $\|\cdot\|$. That the complex structures J_b tend to the complex structure J_0 of the central fibre as b tends to 0 implies the bound

$$\|J_0 - J_b\| = O(\|b\|). \tag{4.3}$$

Recall that we have picked T-invariant holomorphic normal coordinates

$$z_j = x_j + iy_j$$

around q on $\mathcal{X}_0$, where $y_j = J(x_j)$. We define holomorphic coordinates around $\varphi_s(0)$ by setting

$$z_j^b = x_j + iy_j^b,$$

where

$$y_j^b = J_b(x_j).$$

After shrinking B'', we can assume this gives a holomorphic coordinate system with respect to J_b for all $b \in B''$ lying on the disk of radius 2

Using these coordinates near q, we can define the 1-forms using the Equation (4.1). The construction then goes through in exactly the same manner, and the bound given by Equation (4.3) holds for the full family over B.

Using this family of diffeomorphisms, we obtain a smooth map

$$\Psi : B \to \mathcal{J}(M, \omega),$$

extending the map $B'' \to \mathcal{J}(M, \omega)$ of Section 3.2.

Lemma 4.2. *The image $\Psi(B) \subset \mathcal{J}(M, \omega)$ is a symplectic submanifold of $\mathcal{J}(M, \omega)$.*

Proof. The only property which is not automatic is to show that the pullback of the form Ω on $\mathcal{J}(M, \omega)$ is actually nondegenerate. We are allowed to shrink B in our construction, so we will show that this holds at the linearised level at $s = 0$. This will then imply that there is a ball about the origin in $B'' \times \mathbb{C}^r$ on which the statement holds. The map is already known to be a holomorphic embedding in the B'' direction from Kuranishi theory, so it suffices to check the injectivity of the derivative in the $\mathbb{C}^r$ direction.

For tangent vectors w_1, w_2 in $\mathbb{C}^r$, we have

$$\Omega_{J_0}(\Psi_* w_1, \Psi_* w_2) = \int_M \langle J_0(\Psi_* w_1), \Psi_* w_2 \rangle_g \omega^n.$$

Since the symplectic form on the space of almost complex structures comes from a Riemannian metric, it suffices to show the positivity when $w_1 = w$ and $w_2 = iw$. In other words, it suffices to show that

$$\Omega_{J_0}(\Psi_* w, \Psi_* iw) > 0$$

for all $w \neq 0$.

Now,

$$\begin{aligned} \Psi_* w &= \frac{d}{dt}\bigg|_{t=0} (J_{tw}) \\ &= \frac{d}{dt}\bigg|_{t=0} (\varphi_{tw}^* J_0), \end{aligned}$$

where φ_{tw} is the diffeomorphism generating $\Psi(tw)$ as in Lemma 4.1. Thus we need to compare the flows of w and iw, infinitesimally. At the origin in $\mathbb{C}^r$, the vector field generating φ_w is the ω dual of $\beta_w = d\left(\chi(|z|)\sum_{j=1}^r (u_j x_j + v_j y_j)\right)$ and φ_{iw} is dual to

$$\begin{aligned} \beta_{iw} &= d\left(\chi(|z|)\sum_{j=1}^r (-v_j x_j + u_j y_j)\right) \\ &= J_0(\beta_w). \end{aligned}$$

Thus

$$\frac{d}{dt}\bigg|_{t=0} (\varphi_{tiw}^* J_0) = J_0 \frac{d}{dt}\bigg|_{t=0} (\varphi_{tw}^* J_0).$$

This implies that for $w \neq 0$

$$\Omega_{J_0}(\varphi_* w, \varphi_* iw) = \int_M \left| \frac{d}{dt}\bigg|_{t=0} (\varphi_{tw}^* J_0) \right|_g^2 \omega^n > 0,$$

which is what we wanted to show. □

Lemma 4.3. *The map Ψ is equivariant with respect to the T-actions on B and $\mathcal{J}(M, \omega)$.*

Proof. This is a direct consequence of the fact that for $\lambda \in T$ we have $\beta_{\lambda w} = \lambda^* \beta_w$ and torus-invariance of ω. □

It will be important to choose a good Kuranishi slice in our construction. The existence of this over B'' is given by a result of Székelyhidi and Brönnle.

Lemma 4.4 ([33, Proposition 7], [6, Section 3.2])**.** *One can assume that the map $B'' \to \mathcal{J}(M, \omega)$ given by $b \mapsto J_b$ is such that $S(J_b) - \hat{S} = H_b \in \mathfrak{h}$.*

Unfortunately, by applying the symplectomorphism to push q_s to q, we have ruined this property in the $\mathbb{C}^r$ direction, as the scalar curvature is pulled back under the symplectomorphism, and the symplectomorphism may not preserve the condition of lying in $\mathfrak{h}$. Thus, the above only holds over $B'' \times \{0\} \subset B$. We now wish to apply a further perturbation, in order to obtain the above property on the full family over B. The proof is a consequence of the implicit function theorem, in much the same way as the work of Székelyhidi and Brönnle [33, 6].

Proposition 4.5. *We may assume that the map $\Psi : B \to \mathcal{J}(M, \omega)$, which we denote*

$$b \mapsto J_b,$$

satisfies

$$S(J_b) - \hat{S} = H_b \in \mathfrak{h}.$$

This map retains the properties of Lemma 4.2 and Lemma 4.3.

Proof. Thus far we have associated a complex structure to each point $b \in B$ in such a way that B is a symplectic submanifold of $\mathcal{J}(M, \omega)$. For the duration of the proof we denote these complex structures by I_b, for notational clarity. We will perturb I_b using functions, as in Section 3.1. That is, we consider the map

$$f \mapsto S(F_f^* I_b) - \hat{S}.$$

The linearisation is the map$f \mapsto -\mathcal{D}^*\mathcal{D}f$, whose kernel consists of the holomorphy potentials with respect to the Kähler structure (ω, I_0). In particular, the linearisation is surjective modulo the holomorphy potentials $\mathfrak{h}$ on the central fibre $b = 0$, *uniformly* around the origin.

The implicit function theorem then implies that the map taking a f that is L^2-orthogonal to $\mathfrak{h}$ to the L^2-orthogonal projection away from $\mathfrak{h}$ of $S(F_f^* I_b) - \hat{S}$ can be inverted on appropriate Sobolev spaces. This can be achieved with uniform bounds on the inverse, in a sufficiently small neighbourhood of the origin in B. Moreover, from the construction of the complex structures I_b, it follows that there is a $C > 0$ such that for all $b = (b'', s)$,

$$\|S(I_{(b'',s)}) - S(I_{(b'',0)})\| \leq C\|s\|.$$

Since $S(I_{(b'',0)}) - \hat{S} \in \mathfrak{h}$ by Lemma 4.4, it follows that for all sufficiently small $b = (b'', s) \in B$, there exists a function f_b such that

$$S(F_{f_b}^* I_b) - \hat{S} \in \mathfrak{h}.$$

Thus we can take $J_b = F_{f_b}^* I_b$.

The equivariance property with respect to the torus action follows because we are always restricting to torus-invariant functions. The fact that the map is an embedding with symplectic image close to $0 \in B$ follows because these are open properties, and the map $b \mapsto J_b$ is a small perturbation of the map $b \mapsto I_b$, for which these properties hold. Thus $b \mapsto J_b$ has all the required properties. This completes the proof. □

4.2. **Weighted Hölder spaces.** We now define the weighted Hölder spaces relevant to our problem. This will be done by defining the spaces on the extremal central fibre $\mathcal{X}_0$, and then using the same definition on all fibres, by thinking of all the fibres as the same smooth manifold. The spaces on the central fibre are the same as used in previous works [2, 34].

4.2.1. *Weighted spaces on punctured manifolds.* Let $q \in \mathcal{X}_0$ be fixed under the action of a maximal torus $T^{\mathbb{C}}$ in its automorphism group, arising as the specialisation over $0 \in \mathbb{C}^r$ of moving a point p with trivial stabiliser around $\mathcal{X}_0$ through the action of $T^{\mathbb{C}}$. As above, we fix T-invariant holomorphic coordinates $z_1, \ldots, z_n$ around q which are normal with respect to the Kähler metric induced by ω, and assume that they exist at least in the disk D_2 of radius 2 about q. This can be assumed after scaling ω.

For $r > 0$, define $f_r^\delta : D_2 \setminus D_1 \to \mathbb{R}$ by

$$f_r^\delta(z) = r^{-\delta} f(rz).$$

Definition 4.6. The $C^{k,\alpha}_\delta$-*norm* on $M \setminus \{q\}$ is defined to be

$$\|f\|_{C^{k,\alpha}_\delta(M\setminus\{q\})} = \sup_{r\in(0,1)} \|f_r^\delta\|_{C^{k,\alpha}}(D_2 \setminus D_1) + \|f\|_{M\setminus D_1(q)}.$$

Remark 4.7. Note that we have used holomorphic normal coordinates with respect to the complex structure $\mathcal{X}_0$ to define this norm. We will use the same norm on the other holomorphic structures in the Kuranishi family near $\mathcal{X}_0$. Thus the norm for these holomorphic structures are *not* defined using holomorphic normal coordinates.

4.2.2. *Weighted spaces on the blowup.* We now consider the above issues, concerning symplectic embeddings into the space of almost complex structures, on the blowup. We first describe how we define the complex structure on $\mathrm{Bl}_q \mathcal{X}_b$, for which we will glue annuli with respect to the coordinates z_j to corresponding regions on the blowup. This means that we are gluing regions on the non-central fibres that are not actually annuli. We will however be able to uniformly compare them to annuli, which is the key to establish the basic properties of the weighted norms on the blowup.

We begin by discussing the central fibre. Here we have coordinates z around q which we will identify with coordinates $w = \varepsilon^{-1}z$ on the blowup. We are identifying the annulus $D_1 \setminus D_\varepsilon$ around q in $\mathcal{X}_0$ with the (preimage via the blowdown map of the) annulus $D_{\varepsilon^{-1}} \setminus D_1$ in $\mathrm{Bl}_0 \mathbb{C}^n$. In other words,

$$\mathrm{Bl}_q \mathcal{X}_0 = \mathcal{X}_0 \setminus D_\varepsilon(q) \bigcup_{D_1\setminus D_\varepsilon \sim \pi^{-1}(D_{\varepsilon^{-1}}\setminus D_1)} \pi^{-1}(D_{\varepsilon^{-1}}).$$

This induces the almost complex structure on the underlying smooth manifold $\mathrm{Bl}_q M$ of the blowup from the almost complex structure on M corresponding to the central fibre.

We also define a weighted norm on the blowup. Given a function $f : \mathrm{Bl}_q M \to \mathbb{R}$, we can define a function $f_\varepsilon^\delta : \mathrm{Bl}_0 D_1 \to \mathbb{R}$ by

$$f_\varepsilon^\delta(w) = \varepsilon^{-\delta} f(\varepsilon w).$$

Up to a rescaling depending on ε, this is the restriction of f to the preimage via the blowdown map of the ball of radius ε about q, pulled back to the preimage of a ball of fixed size.

Definition 4.8. The $C^{k,\alpha}_\delta$-*norm* on $\mathrm{Bl}_q M$ is defined to be

$$\|f\|_{C^{k,\alpha}_\delta(\mathrm{Bl}_q M)} = \|f_\varepsilon^\delta\|_{C^{k,\alpha}(\mathrm{Bl}_0 D_1)} + \sup_{r\in(\varepsilon,1)} \|f_r^\delta\|_{C^{k,\alpha}}(D_2 \setminus D_1) + \|f\|_{M\setminus D_1(q)}.$$

We also obtain almost complex structures on the blowup for all other $b \in B$. At the level of coordinates, this works in the same way: we identify the coordinates z_j^b with new coordinates $w_j^b = \varepsilon^{-1} z_j^b$ on the blowup of $\mathbb{C}^n$ in the origin. These new coordinates (and hence the almost complex structures) on $\mathrm{Bl}_0 \mathbb{C}^n$ are therefore obtained from the coordinates w in just as z_j^b was obtained from z_j. The difference is that we are still gluing over the annulus with respect to the original coordinates z_j and w_j. This means that we are no longer working on an annulus in the z_j^b and w_j^b coordinates.

Remark 4.9. We could have defined norms $\|\cdot\|_{C^{k,\alpha}(\mathrm{Bl}_q \mathcal{X}_b)}$ for $\mathrm{Bl}_q \mathcal{X}_b$ using the same definition as for $b = 0$, but with annuli with respect to the coordinates z_j^b, defined in Section 4.1. It is possible to show that there is a $C > 0$ such that for all $b \in B$,

$$C^{-1}\|\cdot\|_{C^{k,\alpha}_\delta(\mathrm{Bl}_q M)} \leq \|\cdot\|_{C^{k,\alpha}_\delta(\mathrm{Bl}_q \mathcal{X}_b)} \leq C\|\cdot\|_{C^{k,\alpha}_\delta(\mathrm{Bl}_q M)};$$

that is, these norms are uniformly equivalent to the norms we have defined. Such inequalities hold also for the setting of punctured manifolds.

The proof boils down to establishing a comparison property for annuli with respect to the different coordinate systems. More precisely, the above estimate follows from the fact that there is a $\lambda > 1$ such that

$$D_{\frac{1}{\lambda}r} \subseteq D^b_r \subset D_{\lambda r} \tag{4.4}$$

for all $r \leq 1$, where the notation D^b_r means the disk of radius r with respect to the coordinate system z^b. We will not use this, so omit the proof.

4.3. **The approximate solution.** We now construct an approximate solution to the extremal equation on the blowup. In the usual case, the approximate solution on the blowup is constructed by first matching the metric near the blown up point q with the Burns–Simanca metric on $\mathrm{Bl}_0 \mathbb{C}^n$ in the annular gluing region, the key being to work in holomorphic normal coordinates around q. We will follow the same procedure, where on the central fibre $\mathcal{X}_0$, we do precisely this, while on the other fibres we end up taking an appoximation.

It is important for our argument to work with a fixed symplectic form on every fibre of the family. A novel feature in our approach is therefore that we will be applying Moser's trick to pull back our approximate solutions to a fixed form − the form constructed on the central fibre. We remark that working on the level of almost complex structures has been considered by Vernier [40], but her method is rather different, in that it does not involve the use of Moser's trick. Ultimately, she is interested in applying the methods in a non-Kähler setting, but even if the starting manifold is Kähler, her gluing method would break the integrability of the almost complex structure. The application of Moser's trick allows us to glue the almost complex structures in a way that retains integrability.

We begin by recalling the description of the Burns–Simanca metric. This is a scalar-flat metric η on $\mathrm{Bl}_0 \mathbb{C}^n$ which is asymptotically flat. More precisely, we can write

$$\eta = i\partial\bar{\partial}\left(|w|^2 + \psi(w)\right)$$

in coordinates $w_1, \dots, w_n$ on the complement of the exceptional divisor in $\mathrm{Bl}_0 \mathbb{C}^n$, where $\psi = O(|w|^{4-2n})$ if $n \neq 2$ and $\psi = \log(|w|)$ if $n = 2$.

We first describe the gluing of metrics on the central fibre $\mathcal{X}_0$. In the holomorphic normal coordinates $z_1, \dots, z_n$ used in the definition of the weighted spaces above, the metric ω can be written

$$\omega = i\partial\bar{\partial}\left(|z|^2 + \varphi(z|)\right),$$

for a function φ which is $O(|z|^4)$. Note also that

$$\varepsilon^2\eta = i\partial\bar\partial\left(|z|^2 + \varepsilon^2\psi(\varepsilon^{-1}z)\right)$$

under the coordinate change $z = \varepsilon w$. We then interpolate between the two metrics on the level of potentials over an annular region

$$D_{2r_\varepsilon} \setminus D_{r_\varepsilon} = \{z : r_\varepsilon < |z| \leq 2r_\varepsilon\},$$

where $r_\varepsilon = \varepsilon^{\frac{2n-1}{2n+1}}$. This is done by first picking a smooth cut-off function χ which vanishes on $(-\infty, 1]$ and is equal to 1 on $[2, \infty)$. Let $\chi_1(z) = \chi(\frac{z}{r_\varepsilon})$ and $\chi_2 = 1-\chi_1$. Then define ω_ε to be

- ω on $\mathcal{X}_0 \setminus D_{2r_\varepsilon}$;
- $\varepsilon^2\eta$ on D_{r_ε};
- $i\partial\bar\partial\left(|z|^2 + \chi_1(z)\varphi(z) + \varepsilon^2\chi_2(z)\psi(\varepsilon^{-1}z)\right)$ on $D_{2r_\varepsilon} \setminus D_{r_\varepsilon}$.

We note the uniform bound

$$\|\chi_i\|_{C_0^{4,\alpha}} \leq c \tag{4.5}$$

for the cutoff functions, which follows directly from the definition [34, Equation 13].

We will now define a metric ω_ε^b on the fibre $\mathcal{X}_b$, which we in turn will pull back to ω_ε, altering the complex structure J_ε^b on $\mathrm{Bl}_q M$. This induced form will in general only be *relatively* Kähler – positivity will not be preserved in the B direction.

To glue the metrics, we first note that the functions z^b and $\varphi_b(z^b)$ glue together to a smooth functions ζ and $\Phi(\zeta)$ on $B \times D_2(z)$, where $D_2(z)$ denotes the disk of radius 2 in M, with respect to the coordinates z. We can then define $\omega_{\varepsilon,\mathcal{X}}$ on the full family by

- ω on $B \times (M \setminus D_{2r_\varepsilon})$;
- $\varepsilon^2\eta(\zeta)$ on $B \times D_{r_\varepsilon}$;
- $i\partial\bar\partial\left(|\zeta|^2 + \chi_1(z)\Phi(\zeta) + \varepsilon^2\chi_2(z)\psi(\varepsilon^{-1}\zeta)\right)$ on $(B \times D_{2r_\varepsilon}) \setminus (B \times D_{r_\varepsilon})$,

where $i\partial\bar\partial$ now denotes this operator on $\mathrm{Bl}_Z \mathcal{X}$, the family of blowups. Note that in the cut-off part, we still use the z-coordinate, so the interpolation happens in a fixed annulus on every fibre.

Fibrewise, one can view the above in the following way, with respect to the z^b-coordinates. We use the operator

$$i\partial\bar\partial_b = idJ_bd$$

to define a symplectic form $\omega_{\varepsilon,b}$ by

- ω on $\mathcal{X}_0 \setminus D_{2r_\varepsilon}$;
- $\varepsilon^2\eta(z^b)$ on D_{r_ε};
- $i\partial\bar\partial_b\left(|z^b|^2 + \chi_1(z)\varphi_b(z^b) + \varepsilon^2\chi_2(z)\psi(\varepsilon^{-1}z^b)\right)$ on $D_{2r_\varepsilon} \setminus D_{r_\varepsilon}$.

Here φ_b is the potential for ω in the coordinates z^b, and D_b^r denotes the ball of radius r with respect to these coordinates. Note that $\omega_{\varepsilon,b}$ makes sense only on the fibre, and if we glued these for all b, then we would only obtain a relatively closed form in general. However, this glued form would agree with $\omega_{\varepsilon,\mathcal{X}}$ on every fibre. The upshot is that there exists a globally closed form on $\mathrm{Bl}_Z \mathcal{X}$, whose restriction to every fibre takes the above form.

We start by comparing the two potentials φ and φ_b.

Lemma 4.10. *Let φ_b be the potential for ω in the coordinates z^b. Then, for any k and α, there exists a constant $C > 0$ such that*

$$\|\varphi - \varphi_b\|_{C_2^{k,\alpha}} \leq C \cdot |b|.$$

Proof. There are two types of contributions: from moving in the Kuranishi direction B'' and the $\mathbb{C}^r$ direction. We begin with the former, for which we have

$$\begin{aligned}\omega =& d\left(Jd\left(|z|^2 + \varphi(z)\right)\right) \\ =& d\left(J_b d\left(|z^b|^2 + \varphi(z^b)\right)\right).\end{aligned}$$

Recall that

$$z_j^b = x_j + iy_j^b$$

where

$$y_j^b = J_b(x_j) = (J_b - J)(x_j) + y_j,$$

and that $J_b - J = O(\|b\|)$. Then

$$|z_j^b|^2 = |z_j|^2 + 2(J_b - J_0)(x_j)y_j + |(J_b - J_0)(x_j)|^2,$$

so that

$$|\|z^b\|^2 - \|z\|^2| \leq C \cdot \|b\| \cdot \|z\|^2.$$

We therefore have that

$$\begin{aligned}d(J_b d(\varphi_b)) =& d\left(Jd\left(|z|^2 + \varphi(z)\right) - J_b d(|z^b|^2)\right) \\ =& d\left((J - J_b)d\left(|z|^2 + \varphi(z)\right) + J_b d(|z|^2 - |z^b|^2 + \varphi(z))\right)\end{aligned}$$

so that

$$d(J_b d(\varphi_b - \varphi)) = d\left((J - J_b)d\left(|z|^2 + \varphi(z)\right) + J_b d(|z|^2 - |z^b|^2)\right).$$

The term $(J - J_b)d\left(|z|^2 + \varphi(z)\right)$ is $O(|z| \cdot |b|)$, as is the term $J_b d(|z|^2 - |z^b|^2)$, by the above. Integrating and using that $\varphi_b(0) = 0$ for all b, and the same for the derivative, gives the result. Here we used that when integrating an $O(|z|^l \cdot |b|)$ term, we get an $O(|z|^{l+1} \cdot |b|)$ term.

For moving in the $\mathbb{C}^r$-direction, note that the we have pulled back normal coordinates around points q_s to give coordinates around q_0 through diffeomorphisms ψ_s. The almost complex structure in a fibre of the family is the pullback almost complex structure, and so the upshot is that we need to compare $\psi_s^* \varphi_s$ to φ.

Now, we are only comparing these for s sufficiently close to 0 (independently of ε). If s is small enough, we can view the coordinates at $s = 0$ as some (non-normal) coordinates around s, say in the ball of radius 1, for all s sufficiently close to 0. From this one can produce normal coordinates at q_s, using the standard proof that Kähler manifolds admit holomorphic normal coordinates at any given point (see for example [36, Proposition 1.16]). These are then $O(|z|^2)$ times a matrix of constants that are $O(|s|)$. The upshot is that the change in holomorphic normal coordinates is also $O(|z|^2 \cdot |s|)$. Following the same argument as above, we obtain the required bound on the potential.

The result for moving in a general direction in B then follows by first moving in the B'' direction, which gives the required comparison to the complex structure of J_0, but with a different blowup point, and then we move in the $\mathbb{C}^r$ direction, to compare to the central fibre. □

The following will be useful.

Lemma 4.11. *We can choose coordinates such that*

$$\varphi_b = O(|z|^4),$$

keeping the properties of Lemma 4.10.

Proof. The point is to alter the given coordinate systems to holomorphic normal coordinate systems, which we can do by virtue of ω being Kähler. From the standard proof of the fact that we can choose normal coordinates [36, Proposition 1.16], one sees that the perturbation of z^b required to achieve this is $O(|z^b|^2)$. Thus we can achieve this preserving the bound $|z^b|^2 = |z|^2 + O(|z|^2|b|)$, which was the key to establishing Lemma 4.10. □

We now pull the above back to the fixed symplectic form ω_ε constructed on the central fibre.

Proposition 4.12. *There exists diffeomorphisms $f_{\varepsilon,b} : \mathrm{Bl}_q M \to \mathrm{Bl}_q M$ such that $f_{b,\varepsilon}^*\omega_{\varepsilon,b} = \omega_\varepsilon$ and the complex structures $J_{\varepsilon,b} = f_{\varepsilon,b}^* J_b$ satisfy*

$$\|J_{\varepsilon,b} - J_b\|_{C_0^{k,\alpha}} = O(\varepsilon^2|b|). \tag{4.6}$$

Proof. The proof is based on Moser's trick, which guarantees the existence of such diffeomorphisms as the time-one flow of the vector field $\nu_{\varepsilon,b}$ satisfying

$$\omega_\varepsilon(\nu_{\varepsilon,b}, \cdot) = -d(\varphi_{\varepsilon,b}),$$

where $\varphi_{\varepsilon,b}$ is the Kähler potential for $\omega_{\varepsilon,b}$ with respect to $(\omega_\varepsilon, J_b)$. Thus we reduce to establishing similar estimates as in Lemma 4.10 for the more complicated ε-dependent forms on the blowup.

There are two parts, both of which are ε-dependent: the annular region, and the neighbourhood about the exceptional divisor. On the final region in the blowup, the region identified with the complement of a ball about the blown up point, the two Kähler forms are equal and consequently the complex structures produced from Moser's trick will be identical on this region too.

In comparison with ω, the form $\omega_{\varepsilon,b}$ satisfies

$$\omega_{\varepsilon,b} = \omega + dJ_b d\left(\chi_2(z)\left(\varphi_b(z^b) + \varepsilon^2\psi(\varepsilon^{-1}z^b)\right)\right)$$

on the annular region, so

$$\begin{aligned}\omega_{\varepsilon,b} - \omega_\varepsilon =& dJ_b d\left(\chi_2(z)\left(\varphi_b(z^b) + \varepsilon^2\psi(\varepsilon^{-1}z^b)\right)\right)\\ &- dJd\left(\chi_2(z)\left(\varphi(z) + \varepsilon^2\psi(\varepsilon^{-1}z)\right)\right)\\ =& dJ_b d\left(\chi_2(z)\varphi_b(z^b)\right) - dJd\left(\chi_2(z)\varphi(z)\right)\\ &+ dJ_b d\left(\chi_2(z)\varepsilon^2\psi(\varepsilon^{-1}z^b)\right) - dJd\left(\chi_2(z)\varepsilon^2\psi(\varepsilon^{-1}z)\right)\end{aligned}$$

and we deal with the two lines of the last expression separately. As can be seen above, this term is exact, and we estimate the corresponding 1-form.

The first line is given by

$$\begin{aligned}&J_b d\left(\chi_2(z)\varphi_b(z^b)\right) - Jd\left(\chi_2(z)\varphi(z)\right)\\ =& J_b d\left(\chi_2(z)\left(\varphi_b(z^b) - \varphi(z)\right)\right) + (J_b - J)d\left(\chi_2(z)\varphi(z)\right).\end{aligned}$$

By the uniform bound in Equation (4.5) for the cutoff functions and the multiplicative property of the weighted norms, it follows that we have to establish the bounds for $\varphi_b(z^b) - \varphi(z)$ and $(J_b - J)d\left(\varphi(z)\right)$.

The first of these is covered by Lemma 4.10, which says that

$$\|\varphi - \varphi_b\|_{C^{k,\alpha}_2} \leq C \cdot |b|.$$

Similarly, the function $\varphi \in C^{k,\alpha}_4$ is fixed and independent of b. Since $J_b - J = O(|b|)$, we therefore have the better bound

$$\|(J_b - J)d\left(\varphi(z)\right)\|_{C^{k,\alpha}_4} \leq C \cdot |b|.$$

Thus

$$\|J_b d\left(\chi_2(z)\varphi_b(z^b)\right) - Jd\left(\chi_2(z)\varphi(z)\right)\|_{C^{k,\alpha}_1} \leq C|b|,$$

where the drop in the weight comes from taking the derivative.

On the annular region, we have the comparison property

$$\|\cdot\|_{C^{k,\alpha}_0} \leq \varepsilon\|\cdot\|_{C^{k,\alpha}_1}$$

of the weighted norms. Thus, on the annulus, we have

$$\|J_b d\left(\chi_2(z)\varphi_b(z^b)\right) - Jd\left(\chi_2(z)\varphi(z)\right)\|_{C^{k,\alpha}_0} \leq C\varepsilon|b|,$$

which is the bound we wanted for $d(\varphi_{\varepsilon,b})$ (and hence for $\varphi_{\varepsilon,b}$), on the annular region.

For the part coming from the Burns–Simanca metric, we must estimate

$$\begin{aligned}&J_b d\chi_2(z)\left(\psi(\varepsilon^{-1}z^b)\right) - Jd\chi_2(z)\left(\psi(\varepsilon^{-1}z)\right)\\ =&J_b d\left(\chi_2(z)\left(\psi(\varepsilon^{-1}z^b) - \psi(\varepsilon^{-1}z)\right)\right) + (J_b - J)d\left(\chi_2(z)\psi(\varepsilon^{-1}z)\right).\end{aligned}$$

Again, we can ignore the contribution from the cutoff function.

Now, $\psi(w) = O(|w|^{4-2n})$, so that

$$\psi(\varepsilon^{-1}z^b) - \psi(\varepsilon^{-1}z) = O(\varepsilon^{2n-4} \cdot |b| \cdot |z|^{4-2n}),$$

which is $O(|b|)$ on the annular region, since $\varepsilon^{2n-4} \cdot |z|^{4-2n}$ is bounded independently of ε there. The above bound follows, since by the proof of Lemma 4.10, $\frac{|z^b|^2}{|z|^2} = 1 + f(b,z)$, where $f(b,z) = O(|b|)$, so that

$$\left(\frac{|z^b|^2}{|z|^2}\right)^{2-n} = \frac{1}{(1 + f(b,z))^{n-2}} = 1 + O(|b|).$$

When $n = 2$, $\psi(w) = \log(w)$, and we use the Taylor expansion of $\log(1+x)$ instead to obtain that

$$\begin{aligned}\psi(\varepsilon^{-1}z^b) - \psi(\varepsilon^{-1}z) =&\log\left(\frac{|z^b|^2}{|z|^2}\right)\\ =&O(|b|)\end{aligned}$$

The upshot is therefore that the norm of

$$J_b d\left(\chi_2(z)\varepsilon^2\psi(\varepsilon^{-1}z^b)\right) - Jd\left(\chi_2(z)\varepsilon^2\psi(\varepsilon^{-1}z)\right)$$

is uniformly bounded by $\varepsilon^2|b|$ on the region near the exceptional divisor. This is precisely the bound

$$\|J_b d\left(\chi_2(z)\varepsilon^2\psi(\varepsilon^{-1}z^b)\right) - Jd\left(\chi_2(z)\varepsilon^2\psi(\varepsilon^{-1}z)\right)\|_{C^{k,\alpha}_0} \leq C\varepsilon^2|b|$$

that we wanted to establish.

Finally, we need to estimate the difference of the forms on the region *on* the exceptional divisor (the above gives the estimate away from the exceptional divisor itself). The forms are given by the Kähler form of the Burns–Simanca metric, but with respect to *distinct* complex structures. However, since $J_b = J + O(|b|)$, it follows that that $\eta_b = \eta + O(|b|)$, too. As these are multiplied with ε^2, we have an $O(\varepsilon^2|b|)$ bound on the exceptional divisor, and this is precisely what is required on the exceptional divisor.

The potential $\varphi_{\varepsilon,b}$ is thus $O(\varepsilon^2|b|)$ in the $C^{k,\alpha}_0$-norm. It follows that we have a similar bound for $\nu_{\varepsilon,b}$, and so

$$f_{\varepsilon,b} - \mathrm{Id} = O(\varepsilon^2|b|),$$

again in the $C^{k,\alpha}_0$-norm. The result follows. □

Remark 4.13. From the above, we in particular obtain the estimate

$$\|J_{\varepsilon,b} - J_b\|_{C^{k,\alpha}_2} = O(|b|),$$

due to the general inequality

$$\|f\|_{C^{k,\alpha}_{\delta+2}} \leq \varepsilon^{-2}\|f\|_{C^{k,\alpha}_\delta} \tag{4.7}$$

for the weighted norm. Our result is stronger, however. Reversing the weights, one only has the inequality $\|f\|_{C^{k,\alpha}_\delta} \leq \|f\|_{C^{k,\alpha}_{\delta+2}}$ in general, due to functions supported away from a small neighbourhood of the exceptional divisor. In other words, the bound $\|\cdot\|_{C^{k,\alpha}_0} \leq \varepsilon^2\|\cdot\|_{C^{k,\alpha}_2}$ we used in the proof only holds on the annular region, not globally. It is because our forms ω_ε and $\omega_{\varepsilon,b}$ only differ near the exceptional divisor that we are able to obtain the improved bound (4.6). The fact that we have the bound (4.6) will be important for proving Proposition 4.16.

Next, we show that $(\mathrm{Bl}_{q_b}\mathcal{X}_b, \omega_{\varepsilon,b})$ – equivalently, $(\mathrm{Bl}_q M, J_{\varepsilon,b}, \omega_\varepsilon)$ – is approximately extremal. It is here that we use that we have chosen a good Kuranishi family before blowing up. It is important to lift the Hamiltonians in $\mathfrak{h}$ to the blowup $(\mathrm{Bl}_q M, \omega_\varepsilon)$. This can be done uniquely by letting the lift H_ε of $H \in \mathfrak{h}$ be the holomorphy potential with respect to the complex structure of the central fibre $\mathrm{Bl}_q \mathcal{X}_0$, of average zero with respect to ω_ε. That we can lift holomorphically uses q is a fixed point of the maximal torus acting on $\mathcal{X}_0$.

Proposition 4.14. *Let $g_{\varepsilon,b}$ be the Kähler metric associated to $(\mathrm{Bl}_{q_b}\mathcal{X}_b, \omega_\varepsilon)$ arising from the metric g_b on $\mathcal{X}_b$, where the scalar curvature of g_b equals $H_b \in \mathfrak{h}$. Let $H_{\varepsilon,b}$ be the corresponding Hamiltonian on $(\mathrm{Bl}_q M, \omega_\varepsilon)$. Then for any $\delta \leq 0$, there exists $C > 0$ such that*

$$\|S(g_{\varepsilon,b}) - H_{\varepsilon,b} - \hat{S}\|_{C^{k,\alpha}_\delta} \leq C \cdot r_\varepsilon^{-\delta}$$

for all $b \in B$.

Proof. We follow the proof of [36, Lemma 8.19] closely. Thus we assume $n > 2$, with the adaptations needed in the case explained in Remark 4.15. There are three regions to consider: the complement of D_{2r_ε}, the annulus $D_{2r_\varepsilon} \setminus D_{r_\varepsilon}$ and the region near the exceptional divisor, which can be identified with the blowup of D_{r_ε} in the origin.

On the complement of D_{2r_ε}, the complex structures J_b and $J_{b,\varepsilon}$ are the same, and consequently $\omega_{\varepsilon,b}$ and ω_ε are also the same. The Hamiltonian $H_{\varepsilon,b}$ is therefore equal to H_b in this region, and so $S(g_{\varepsilon,b}) - H_{\varepsilon,b} = 0$ in this region.

Next, we consider the annulus, where the metric is an $O(|z|^4)$ perturbation of the Euclidean metric. This follows by the bound of Lemma 4.11 for the potentials φ_b, using the argument of [36, Lemma 8.19]. This step requires the choice $\frac{n-1}{n}$

of exponent for $r_\varepsilon = \varepsilon^{\frac{n-1}{n}}$. Note that on the fibres, the expression of $\omega_{\varepsilon,b}$ is not exactly the same as doing fibrewise what is done on the central fibre: the difference is that we use the cut-off functions with respect to $|z|$, rather than $|z^b|$. However, by Equation (4.4), the two choices are mutually bounded, and so we obtain the required bound.

That the metric is an $O(|z|^4)$ perturbation of the Euclidean metric means that we can compare the curvature of $S(J_{\varepsilon,b})$ to that of the Euclidean metric, which vanishes, using the mean value theorem. This gives, as in [36, Lemma 8.19], the bound

$$\|S(J_{\varepsilon,b})\|_{C^{k,\alpha}_\delta} \leq C \cdot r_\varepsilon^{-\delta}.$$

We thus reduce to estimating $\|H_{\varepsilon,b}\|_{C^{k,\alpha}_\delta}$ and $\|\hat{S}\|_{C^{k,\alpha}_\delta}$ in the annular region. The latter is a constant, and so satisfies the required $O(r_\varepsilon^{-\delta})$ bound. But the situation is even better for this potential: since we have used torus invariant holomorphic coordinates at the blowup point on the central fibre, and the lift is defined using that coordinate system, the Hamiltonian is $O(|z|^2)$. Thus the bound on the annular region is better for this function on the annular region, than it would be for a constant. The upshot is that

$$\|H_{\varepsilon,b} + \hat{S}\|_{C^{k,\alpha}_\delta} \leq C \cdot r_\varepsilon^{-\delta}.$$

Finally, on D_{r_ε}, the metric is simply the Burns-Simanca metric, which is flat, so we again reduce to an estimate in B_{r_ε} of $H_{\varepsilon,b} + \hat{S}$, which works exactly as on the annulus. □

Remark 4.15. The above works when the dimension n is greater than 2. For the case $n = 2$, one needs to further match the metrics with the Burns-Simanca metric. In order to do this, one changes the Kähler form ω at the outset at higher order in ε, so that the difference between this metric and the Burns–Simanca metric has decay at higher order than the initial glued solution. This can be achieved, up to adding a holomorphy potential.

The matching at higher order with the Burns–Simanca metric allows one to obtain better approximate solutions, essentially because the subleading order term in the expression of the Burns-Simanca metric is harmonic with respect to the Euclidean metric. An improved approximate solution is needed to perform the perturbation, as the linearised operator has worse mapping properties and bounds when the dimension is 2, than in the higher dimensional case. The upshot is that after performing the matching of the metrics, we get the bound, for each $\delta \in (-1, 0)$, the bound

$$\|S(g_{\varepsilon,b}) - H_{\varepsilon,b} - \hat{S} - \varepsilon^2(c + \widetilde{H}_{\varepsilon,b})\|_{C^{k,\alpha}_\delta} \leq C \cdot r_\varepsilon^{-\delta}.$$

We refer to [36, Section 8.4 and Lemma 8.22] for the exact details in the change in the argument, in this case.

We end the section by proving that we obtain embeddings into the space of compatible almost complex structures on the blowup, analogously to Proposition 4.5 before blowing up. Recall that the basic setup is that we have consider a point $p \in X$ whose T-stabiliser is trivial, and identified p with the associated point in any general fibre. This produces a submanifold $Z = \overline{T^{\mathbb{C}}.p}$, which we blow up to obtain a new family $\mathrm{Bl}_Z \mathcal{X} \to B$ which admits a $T^{\mathbb{C}}$ action. As a smooth manifold, this is nothing but $\mathrm{Bl}_q M \times B$, where q is the specialisation of p at 0. We have endowed this family with the structure of a fixed symplectic manifold with varying almost complex structure, which we write as J_b for $b \in B$, and we wish to show that this map is a T-equivariant symplectic embedding.

Proposition 4.16. *The induced map*

$$\begin{aligned} \Psi'_\varepsilon : B &\to \mathcal{J}(\mathrm{Bl}_q M, \omega_\varepsilon), \\ b &\to J_{\varepsilon,b} \end{aligned}$$

is a T-equivariant embedding for all $\varepsilon > 0$ sufficiently small, whose image is a symplectic submanifold of $\mathcal{J}(\mathrm{Bl}_q M, \omega_\varepsilon)$.

Proof. First note that the maps Ψ'_ε are certainly T-equivariant, since all perturbations were defined through torus-invariant functions, and the original maps are equivariant, using Proposition 4.5. To see that the image of the map is a symplectic submanifold, we detail how the map is constructed. First, we start with the initial map of Proposition 4.5, and the image of this map is symplectic. To show this then holds on the blowup, we need an estimate as in Lemma 4.2 for the derivative of the map at the origin in B. This involves integrating over $\mathrm{Bl}_q M$ with respect to ω_ε. But this can be estimated uniformly by the contribution on M, as ω_ε approaches ω away from the exceptional divisor. Thus the image has the property of being symplectic in $\mathcal{J}(\mathrm{Bl}_q M, \omega_\varepsilon)$.

It remains to show the embedding property. Note that as the map on the Kuranishi part B'' of B was already an embedding before blowing up, we only need to consider the $\mathbb{C}^r$ part of B. As we are allowed to shrink B (though independently of ε only), it suffices to establish the uniform estimate

$$|d(\Psi'_\varepsilon)_{|b=0}| \geq C$$

for all ε. So, we want to compute the pushforward of a tangent vector $v \in T_0 B \cong \mathbb{C}^r$. This is the derivative of the map

$$t \mapsto J_{\varepsilon,tv},$$

where we map to endomorphisms of $T\,\mathrm{Bl}_q M$.

Recall that $J_{\varepsilon,b} = f^*_{\varepsilon,b} J_b$, where J_b was the lift at b of the complex structure to the blowup, and that we have an expansion

$$J_{\varepsilon,b} = (J_{\varepsilon,b} - J_b) + (J_b - J_0) + J_0.$$

Now, $J_b - J_0 = O(|b|)$, while $J_{\varepsilon,b} - J_b = O(\varepsilon^2|b|)$, by Proposition 4.12. Thus, it suffices to show the claim for the lifted complex structure J_b. Since $J_b - J_0 = O(|b|)$, we have an expansion $J_{tv} = J_0 + tI_v + O(t^2)$ for some endomorphism I_v of TM, and this is the derivative term we seek. But this term precisely comes from the base and is independent of ε, and so, by Proposition 4.5, the derivative is injective. This completes the proof. □

That the maps Ψ'_ε are symplectic embeddings produces the following:

Corollary 4.17. *A moment map for the T-action on B, with respect to the symplectic form induced from the embedding $\Psi'_\varepsilon : B \to \mathcal{J}(\mathrm{Bl}_q M, \omega_\varepsilon)$, is given by*

$$\mu(b) = \Pi_\varepsilon \circ (S(J_{\varepsilon,b}) - \hat{S})),$$

with Π_ε the L^2-projection onto $\mathfrak{h}_\varepsilon$, and with $\mathfrak{h}_\varepsilon$ identified with $\mathrm{Lie}\, T$.

4.4. **Perturbing the metrics.** We will need to perturb the Kähler metrics to produce actual extremal metrics on the blowup. We use the contraction mapping theorem for this, for which we need to understand the mapping properties of the linearised operator. The key is to relate these to the mapping properties of the linearised operator on the central fibre.

Proposition 4.18. *Let $L_{b,\varepsilon} = \mathcal{D}^*_{g_{\varepsilon,b}}\mathcal{D}_{g_{\varepsilon,b}}$. Then there exists a $C > 0$ such that*

$$\|L_{\varepsilon,b} - L_{\varepsilon,0}\|_{C^{k+4,\alpha}_{\delta} \to C^{k,\alpha}_{\delta-4}} \leq C|b|.$$

Proof. Recall that the Lichnerowicz operator is given by

$$L_{\varepsilon,b}(\varphi) = \Delta^2_{\varepsilon,b}(\varphi) + \langle \mathrm{Ric}(g_{\varepsilon,b}), i\partial\bar{\partial}_b(\varphi)\rangle,$$

where $\Delta_{\varepsilon,b}$ is the Laplacian associated to the metric $g_{\varepsilon,b}$. We deal with the Laplacian and Ricci terms separately.

We begin with the Laplacian term, which satisfies

$$\Delta_{\varepsilon,b}(\varphi)\omega^n_{\varepsilon} = ni\partial\bar{\partial}_{\varepsilon,b}(\varphi) \wedge \omega^{n-1}_{\varepsilon}.$$

By Proposition 4.12, $J_{\varepsilon,b} - J_b$ is bounded in operator norm between weighted spaces by a constant multiple of $\varepsilon^2|b|$. Similarly, $J_b - J_0$ is bounded by a constant multiple of $|b|$. The triangle inequality gives the bound

$$\|\Delta_{\varepsilon,b} - \Delta_{\varepsilon,0}\|_{C^{k+2,\alpha}_{\delta} \to C^{k,\alpha}_{\delta-2}} \leq C|b|.$$

We can then estimate $\Delta^2_{\varepsilon,b} - \Delta^2_{\varepsilon,0}$. We have

$$\begin{aligned}\|\Delta^2_{\varepsilon,b} - \Delta^2_{\varepsilon,0}\|_{C^{k+4,\alpha}_{\delta} \to C^{k,\alpha}_{\delta-4}} =& \|\Delta_{\varepsilon,b} \circ (\Delta_{\varepsilon,b} - \Delta_{\varepsilon,0}) + (\Delta_{\varepsilon,b} - \Delta_{\varepsilon,0}) \circ \Delta_{\varepsilon,0}\|_{C^{k+4,\alpha}_{\delta} \to C^{k,\alpha}_{\delta-4}} \\ \leq& \|\Delta_{\varepsilon,b} \circ (\Delta_{\varepsilon,b} - \Delta_{\varepsilon,0})\|_{C^{k+4,\alpha}_{\delta} \to C^{k,\alpha}_{\delta-4}} \\ &+ \|(\Delta_{\varepsilon,b} - \Delta_{\varepsilon,0}) \circ \Delta_{\varepsilon,0}\|_{C^{k+4,\alpha}_{\delta} \to C^{k,\alpha}_{\delta-4}} \\ \leq& C|b|,\end{aligned}$$

since the operators $\Delta_{\varepsilon,b}$ are uniformly bounded independently of ε and b.

Next, we consider the Ricci term. We use that

$$\mathrm{Ric}(g_{\varepsilon,b}) = -\frac{i}{2\pi}\partial\bar{\partial}_b \log \omega^n_{\varepsilon},$$

using the usual way of thinking of ω^n_{ε} as a metric on $\Lambda^n T\,\mathrm{Bl}_p\,\mathcal{X}_b$. Now,

$$i\partial\bar{\partial}_{\varepsilon,b}(\varphi) = i\partial\bar{\partial}_0(\varphi) + d(J_b - J_0)d(\varphi) + d(J_{\varepsilon,b} - J_b)d(\varphi).$$

Again, we can use properties of the difference of $i\partial\bar{\partial}_{\varepsilon,b}b$ and $i\partial\bar{\partial}_0$ exactly as with the Laplace operator above.

We therefore reduce the Ricci estimate to an estimate of the weighted norm of the volume form ω^n_{ε}. As the volume form and the norm do not depend on b, we can work the central fibre to prove the estimate. Away from the exceptional divisor, we have a fixed form, and so here we have a global bound independent of ε. In the annular region, we have a global bound in the 0-weighted norm, as the actual volume is a perturbation of the Euclidean one. Thus ω^n_{ε} is bounded in the annular region in δ-weighted norm by a constant times $\varepsilon^{-\delta}$, as the the smallest annular region in the estimate is $D_{2\varepsilon} \setminus D_{\varepsilon}$. In particular, we obtain a universal bound, since $\delta < 0$. Finally, over the exceptional divisor, we have a perturbation of the

Burns–Simanca volume form, and so also here we get a constant times the scaling factor. In summary, we obtain the bound

$$\|\omega_\varepsilon^n\|_{C^{k,\alpha}_\delta} \leq C.$$

Combining this with the remarks about the change in $i\partial\bar\partial$-operators gives the bound

$$\| \operatorname{Ric}(g_{\varepsilon,b}) - \operatorname{Ric}(g_{\varepsilon,0})\|_{C^{k+4,\alpha}_\delta} \leq C|b|.$$

Together with the multiplicative properties of the weighted norms, we therefore obtain that

$$\begin{aligned}
&\|\langle \operatorname{Ric}(g_{\varepsilon,b}), i\partial\bar\partial_{\varepsilon,b}(\varphi)\rangle - \operatorname{Ric}(g_{\varepsilon,0}), i\partial\bar\partial_0(\varphi)\rangle\|_{C^{k+4,\alpha}_\delta \to C^{k+2,\alpha}_{\delta-2}} \\
\leq &\|\langle \operatorname{Ric}(g_{\varepsilon,b}) - \operatorname{Ric}(g_{\varepsilon,0}), i\partial\bar\partial_{\varepsilon,b}(\varphi)\rangle\|_{C^{k+4,\alpha}_\delta \to C^{k+2,\alpha}_{\delta-2}} \\
&+ \|\langle \operatorname{Ric}(g_{\varepsilon,0}), (i\partial\bar\partial_{\varepsilon,b} - i\partial\bar\partial_0)(\varphi)\rangle\|_{C^{k+4,\alpha}_\delta \to C^{k+2,\alpha}_{\delta-2}} \\
\leq &C|b|.
\end{aligned}$$

Here we are again using the expansion of $i\partial\bar\partial_{\varepsilon,b} - i\partial\bar\partial_0$ as above.

Combining the Laplacian and Ricci bounds, we obtain the required bound on the Lichnerowicz operator. □

We now want to estimate the linearised operator to our equation. Recall that $\mathfrak{h}$ denotes the holomorphy potentials of average 0 on $\mathcal{X}_0$ with respect to ω. For $H \in \mathfrak{h}$, let H_ε denote the holomorphy potential of average 0 with respect to ω_ε of the lift of the vector field determined by H on the central fibre $\mathcal{X}_0$, to its blowup.

Proposition 4.19. *Let $P_{\varepsilon,b} : C^{k+4,\alpha}_\delta \times \mathfrak{h} \times \mathbb{R} \to C^{k,\alpha}_\delta$ be the operator*

$$(\varphi, H, c) \mapsto L_{\varepsilon,b}(\varphi) - H_\varepsilon - c.$$

Then for $\delta \in (4-2n, 0)$, $P_{\varepsilon,b}$ is surjective with right inverse $Q_{\varepsilon,b}$ satisfying

$$\|Q_{\varepsilon,b}\|_{C^{k+4,\alpha}_\delta \to C^{k,\alpha}_{\delta-4}} \leq C,$$

for some constant C, independent of ε and b, when $n > 2$. When $n = 2$, we instead have, for $\delta \in (-1, 0)$, the bound

$$\|Q_{\varepsilon,b}\|_{C^{k+4,\alpha}_\delta \to C^{k,\alpha}_{\delta-4}} \leq C\varepsilon^\delta,$$

provided b satisfies an estimate $|b| \leq C\varepsilon^\delta$, for a suitable constant C.

Proof. For the case $b = 0$, this is known by [34, Proposition 22]. The proof therefore reduces to an estimate on $P_{\varepsilon,b} - P_{\varepsilon,0}$. But since the the dependence of the map on $H \in \mathfrak{h}$ component is independent of b, this simply requires us to estimate the difference of the Lichnerowicz operators. Thus the above is a direct consequence of Proposition 4.18.

Note that in dimension 2, the additional requirement that b lying in a shrinking ball with respect to ε comes from the analogous bound from [34, Proposition 22] for $P_{\varepsilon,0}$. We then need b to be small with respect to ε for Proposition 4.18 to be applied to see $P_{\varepsilon,b}$ as a perturbation of $P_{\varepsilon,0}$. □

We now follow exactly the same steps as in [34]. First, we estimate the change in the linearised operator, upon perturbing the complex structure slightly. For a complex structure J, we will let L_J denote the Lichnerowicz operator of (J, ω_ε).

Proposition 4.20. [34, Proposition 20] *For any $\delta < 0$, there exists constants $c, C > 0$ such that if $\|f\|_{C_2^{k+4,\alpha}} < c$, then*

$$\|L_{F_f^* J_{\varepsilon,b}} - L_{J_{\varepsilon,b}}\|_{C_\delta^{k+4,\alpha} \to C_{\delta-4}^{k,\alpha}} < C\|f\|_{C_2^{k+4,\alpha}}.$$

The proof only uses estimates on the Riemannian metrics produced from $(J_{\varepsilon,b}, \omega_\varepsilon)$ and $(F_f^* J_{\varepsilon,b}, \omega_\varepsilon)$. It therefore remains the same as in [34], even though we are taking the point of view of fixing the symplectic form, whereas Székelyhidi fixes the complex structure.

A direct consequence of the above is that we now can estimate the non-linear part of the scalar curvature operator. More precisely, we have an expansion

$$S(F_f^* J_{\varepsilon,b}) = S(J_{\varepsilon,b}) + L_{J_{\varepsilon,b}}(f) + \mathcal{N}_{\varepsilon,b}(f), \tag{4.8}$$

for some non-linear operator $\mathcal{N}_{\varepsilon,b} : C_\delta^{k+4,\alpha} \to C_{\delta-4}^{k,\alpha}$, and we want an estimate for $\mathcal{N}_{\varepsilon,b}$. As in [34, Lemma 21], the lemma below follows from Proposition 4.20 and the mean value theorem.

Lemma 4.21. *For every $\delta < 0$, there exists constants $c, C > 0$ such that if f_1, f_2 are functions satisfying $\|f_i\|_{C_2^{k+4,\alpha}} \leq c$, then*

$$\|\mathcal{N}_{\varepsilon,b}(f_1) - \mathcal{N}_{\varepsilon,b}(f_2)\|_{C_{\delta-4}^{k+4,\alpha}} \leq C \cdot \left(\|f_1\|_{C_2^{k+4,\alpha}} + \|f_2\|_{C_2^{k+4,\alpha}}\right) \|f_1 - f_2\|_{C_\delta^{k+4,\alpha}}.$$

We now rephrase the equation we wish to solve as a fixed point problem. The equation we want to solve is

$$S(F_f^* J_{\varepsilon,b}) = H_\varepsilon + c$$

for some $f \in C_\delta^{k+4,\alpha}$, $H \in \mathfrak{h}$ and constant c. Using the expansion (4.8), we can then rewrite this as

$$L_{J_{\varepsilon,b}}(f) - H_\varepsilon - c = S(J_{\varepsilon,b}) - \mathcal{N}_{\varepsilon,b}(f) - H_{\varepsilon,b} - \hat{S}.$$

The term $h_{\varepsilon,b} + \hat{S}$, corresponding to the scalar curvature of the complex structure J_b before blowing up, is only subtracted to make $(0,0,0)$ an approximate solution. Note that the left hand side is simply the operator $P_{\varepsilon,b}$ of Proposition 4.19. Using the right-inverse $Q_{\varepsilon,b}$ of $P_{\varepsilon,b}$, we can then rephrase the equation as the fixed point problem

$$(f, H, c) = Q_{\varepsilon,b}\left(S(J_{\varepsilon,b}) - \mathcal{N}_{\varepsilon,b}(f) - H_{\varepsilon,b} - \hat{S}\right).$$

Remark 4.22. Here term $H_{\varepsilon,b}$ is the following. The initial scalar curvature equals $H_b \in H$ by Proposition 4.5, and $H_{\varepsilon,b}$ is the lift of H_b to the blowup. Thus it does not depend on the H component of (f, H, c). The right hand side is therefore independent of H and c, unlike in the similar step in [34]. The reason is that we are taking the point of view of fixing the symplectic form, and so the Hamiltonians do not depend on f.

Returning to our fixed point problem, let

$$\Phi_{\varepsilon,b} : C_\delta^{k+4,\alpha} \times \mathfrak{h} \times \mathbb{R} \to C_{\delta-4}^{k,\alpha}$$

denote the map

$$(f, H, c) \mapsto Q_{\varepsilon,b}\left(S(J_{\varepsilon,b}) - \mathcal{N}_{\varepsilon,b}(f) - H_{\varepsilon,b} - \hat{S}\right).$$

We now want to show that $\Phi_{\varepsilon,b}$ is a contraction on a suitable set, as in [34, Lemma 23].

Lemma 4.23. *Suppose $n > 2$ and that $\delta \in (4-2n, 0)$. There exists constants ε_0, λ such that for all $b \in B$ and for all $\varepsilon \in (0, \varepsilon_0)$ the operator $\Phi_{\varepsilon,b}$ is a contraction with constant $\frac{1}{2}$ on the set $\mathcal{U} \subset C^{k+4,\alpha}_\delta \times \mathfrak{h} \times \mathbb{R}$ defined by*

$$\mathcal{U} = \{(f, H, c) : \|f\|_{C^{k+4,\alpha}_\delta}, |H|, |c| < \lambda \varepsilon^{2-\delta}\}.$$

When $n = 2$, the same statement holds, for $\delta \in (-1, 0)$, on

$$\mathcal{U} = \{(f, H, c) : \|f\|_{C^{k+4,\alpha}_\delta}, |H|, |c| < \lambda \varepsilon^{\sigma}\},$$

where $\sigma > 0$ can be made arbitrarily small, provided we take $\delta < 0$ sufficiently close to 0. The estimate then holds for all $b \in B$ satisfying $|b| \leq C\varepsilon^\delta$, for some constant C.

Proof. Suppose first $n > 2$. Since $\mathcal{Q}_{\varepsilon,b}$ is an operator that is bounded independently of ε, we have to estimate

$$\|\mathcal{N}_{\varepsilon,b}(f_1) - \mathcal{N}_{\varepsilon,b}(f_2)\|_{C^{k,\alpha}_{\delta-4}},$$

for two functions $f_i \in C^{k+4,\alpha}_\delta$. Now, if $\|f_i\|_{C^{k+4,\alpha}_\delta} < \lambda\varepsilon^{2-\delta}$, then $\|f_i\|_{C^{k+4,\alpha}_2} < \lambda$. This uses the weight comparison inequality for the weighted norms on the blowup, as in Equation (4.7). Choosing $\lambda > 0$ sufficiently small, we can then apply Lemma 4.21, giving the bound

$$\|\mathcal{N}_{\varepsilon,b}(f_1) - \mathcal{N}_{\varepsilon,b}(f_2)\|_{C^{k,\alpha}_{\delta-4}} \leq 2\lambda C \cdot \|f_1 - f_2\|_{C^{k+4,\alpha}_\delta}$$

Reducing λ further if necessary, we make $2\lambda C < \frac{\|\mathcal{Q}_{\varepsilon,b}\|}{2}$ for all b and sufficiently small ε. Consequently,

$$\|\Phi_{\varepsilon,b}(f_1, H_1, c_1) - \Phi_{\varepsilon,b}(f_2, H_2, c_2)\|_{C^{k+4,\alpha}_\delta} < \frac{1}{2}\|f_1 - f_2\|_{C^{k+4,\alpha}_\delta}$$

which gives the result when $n > 2$. For $n = 2$, the term of interest $\|Q_{\varepsilon,b}\|$ is bounded by a constant multiple of ε^δ, and this gives the dependence on ε in the $\mathcal{U}$, when going through the same steps in this case. □

We are finally ready to show that we can solve the extremal equation, up to holomorphy potentials on the central fibre. Let $\overline{\mathfrak{h}}_\varepsilon$ denote the space of holomorphy potentials with respect to ω_ε on the central fibre $\mathcal{X}_0$.

Theorem 4.24. *If $n > 2$, then there exists an $\varepsilon_0 > 0$ and a $\kappa > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and for all $b \in B$ such that $|b| < \kappa$, there exists an $f = f_{\varepsilon,b}$ such that*

$$S(F_f^* J_{\varepsilon,b}) \in \overline{\mathfrak{h}}_\varepsilon.$$

When $n = 2$, then for any $\delta > 0$ sufficiently small the same holds for all $b \in B$ satisfying $|b| < \kappa\varepsilon^\delta$.

Proof. Assume first that $n > 2$. We follow the exact argument of [36, Proposition 8.20]. It follows from Proposition 4.14 that

$$\|\Phi_{\varepsilon,b}(0,0,0)\|_{C^{k+4,\alpha}_\delta} \leq C' \cdot \|S(g_{\varepsilon,b}) - h_{\varepsilon,b} - \hat{S}\|_{C^{k,\alpha}_{\delta-4}} \leq C \cdot r_\varepsilon^{4-\delta},$$

for some constant C' and C that are independent of ε and b. As $r_\varepsilon^{4-\delta} = \varepsilon^{(4-\delta)\cdot(1-\frac{1}{n})}$, we can, by picking δ sufficiently close to 0, pick $\varepsilon_0 > 0$ and $\kappa > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$ and $|b| < \kappa$, then $\|\Phi_{\varepsilon,b}(0,0,0)\|_{C^{k+4,\alpha}_\delta} \in \mathcal{U}$ and $\Phi_{\varepsilon,b}(\mathcal{U}) \subseteq \mathcal{U}$. For the latter we use also that $\Phi_{\varepsilon,b}$ is a contraction on $\mathcal{U}$ of a particular constant, namely

$\frac{1}{2}$. Since $\Phi_{\varepsilon,b}$ is a contraction on $\mathcal{U}$ which sends $\mathcal{U}$ to itself, the contraction mapping theorem guarantees the existence of a fixed point in $\mathcal{U}$. This completes the proof of the case $n > 2$. Note also that the family version of the contraction mapping theorem implies regularity properties of $f_{\varepsilon,b}$ in b.

When $n = 2$, the argument is similar, but changes slightly due to the worse bound in Lemma 4.23, and because we have to use the altered almost complex structure on the blowup, see Remark 4.15. We refer to [36, Section 8.4] for the exact changes in the argument for this case. □

In Proposition 4.16, we saw that the map $b \mapsto J_{\varepsilon,b}$ is an embedding for all $\varepsilon > 0$. It is important the map sending $b \in B$ to the complex structure appearing as the solution in Theorem 4.24 still retains this property, which we now show.

Proposition 4.25. *The induced map*

$$\Psi_\varepsilon : B \to \mathcal{J}(\mathrm{Bl}_q M, \omega_\varepsilon),$$
$$b \to F^*_{f_{\varepsilon,b}} J_{\varepsilon,b}$$

is a T-equivariant embedding.

Proof. The new solutions are obtained from using torus-invariant functions, and the family over B is still a holomorphic submersion. The embedding property is preserved because the new solutions are produced from functions lying in the open set $\mathcal{U}$ defined above, which in particular implies that the solutions are an $O(\varepsilon^2)$ perturbation of the approximate solutions, in the weight 0 norms. The openness of the embedding property then implies that the new perturbed map Ψ_ε is an embedding, too. □

5. K-stability as the sufficient condition

5.1. **Setting up the problem.** We next characterise the existence of extremal metrics in terms of relative K-stability. We assume that $\dim X > 2$, with the case $\dim X = 2$ covered in Section 5.3. We recall what we have proven thus far. We have a complex manifold B which is an open ball around the origin in a complex vector space. There is a local holomorphic action of a complex torus $T^{\mathbb{C}}$ on B, with $T^{\mathbb{C}}$ of rank $r = \dim B$. The action is linear and effective, so there is a unique dense orbit. We are interested in the orbit of a point $b \in B$ which has stabiliser T''. In the notation of Section 3.2, $T'' \subseteq T' \subseteq T$ is the subtorus fixing the point $p \in X$ we want to blow up. Denoting $\mathfrak{t}$ and $\mathfrak{t}''$ the Lie algebras of T and T'' respectively, we also set $w = \dim \mathfrak{t} - \dim \mathfrak{t}''$. In general we cannot obtain a splitting of tori $T = T''' \times T''$ for some T''', though this is possible at the level of Lie algebras.

The Lie algebra $\mathfrak{t}$ is endowed with a sequence of inner products $\langle \cdot, \cdot \rangle_\varepsilon$ depending on ε, and using these inner products to identify $\mathfrak{t}$ with its dual, we have a sequence of moment maps, which we write as

$$\mu_\varepsilon : B \to \mathfrak{t}_\varepsilon,$$

for the T-action on B with respect to a family of symplectic forms Ω_ε on B. We stress this point: normally one considers

$$\mu_\varepsilon : B \to \mathfrak{t}^*,$$

but we use the ε-dependent inner products to identify $\mathfrak{t}$ with its dual $\mathfrak{t}^*$. The inclusion of ε in the notation is to emphasise that we use the inner product $\langle \cdot, \cdot \rangle_\varepsilon$ on $\mathfrak{t}_\varepsilon$.

Concretely, the inner products $\langle\cdot,\cdot\rangle_\varepsilon$ are defined as follows. The Lie algebra $\mathfrak{t}$ can be identified with Hamiltonian functions on both (M,ω) for $\varepsilon=0$ and $(\mathrm{Bl}_q M,\omega_\varepsilon)$, continuing the notation of the previous section, since we have Hamiltonian actions of T on both spaces. For vector fields $u,v\in\mathfrak{t}$ the natural sequence of L^2-inner products is given for $\varepsilon>0$ by

$$\langle u,v\rangle_\varepsilon=\int_{\mathrm{Bl}_q M} H_{u,\varepsilon}H_{v,\varepsilon}\omega_\varepsilon^n,$$

where $H_{u,\varepsilon},H_{v,\varepsilon}$ are the associated Hamiltonians normalised to integrate to zero with respect to ω_ε^n. In the limiting case we obtain the analogous integral over M:

$$\langle u,v\rangle_0=\int_M H_{u,0}H_{v,0}\omega^n.$$

We consider a family of orthonormal bases $u_{\varepsilon,j}$ of $\mathfrak{t}_\varepsilon$ with respect to the inner product $\langle\cdot,\cdot\rangle_\varepsilon$, so that $u_{0,j}$ is orthonormal with respect to the limiting inner product $\langle u,v\rangle_0$. We often abuse notation slightly and write $h_{\varepsilon,u}\in\mathfrak{t}_\varepsilon$ to mean that $h_{\varepsilon,u}$ is a Hamiltonian function on B with respect to Ω_ε associated to some $u\in\mathfrak{t}_\varepsilon$. This defines $h_{\varepsilon,u}$ only up to a constant, but the normalisation will be given in Equation (5.1) below. Denote by $\mathfrak{t}''_\varepsilon$ the Lie subalgebra of $\mathfrak{t}_\varepsilon$ associated with T'', and $(\mathfrak{t}''_\varepsilon)^\perp$ its orthogonal complement. We can ensure that $h_{\varepsilon,1},\dots,h_{\varepsilon,w}$ is a basis for $(\mathfrak{t}''_\varepsilon)^\perp$ while $h_{\varepsilon,w+1},\dots,h_{\varepsilon,r}$ is a basis for $\mathfrak{t}''_\varepsilon$.

Remark 5.1. In general $u_{\varepsilon,j}$ cannot be chosen to be rational elements of $\mathfrak{t}$ in general, even for rational ε. Ultimately this is the reason why we must consider $\mathbb{R}$-test configurations rather than merely test configurations.

As we have pulled back the symplectic forms from $\mathcal{J}(\mathrm{Bl}_q M,\omega_\varepsilon)$, the Hamiltonians $h_{\varepsilon,j}$ are concretely given for $\varepsilon>0$ by

$$h_{\varepsilon,j}(b)=\int_{\mathrm{Bl}_q M} H_{\varepsilon,j}(S(J_{\varepsilon,b})-\hat{S}_\varepsilon)\omega_\varepsilon^n, \tag{5.1}$$

where $H_{\varepsilon,j}$ is the Hamiltonian for the T-action on $(\mathrm{Bl}_q M,\omega_\varepsilon)$ and $J_{\varepsilon,b}$ denotes the almost complex structure associated to b through the map

$$\Psi_\varepsilon: B\to\mathcal{J}(\mathrm{Bl}_q M,\omega_\varepsilon).$$

For $\varepsilon=0$ they are given by the corresponding integral over (M,ω).

We next choose holomorphic coordinates on B such that the action is diagonal with respect to the $u_{0,j}$. Denoting by $\rho(u_{0,j})\in T_0B$ the induced tangent through the action, we have an expansion

$$\Omega_\varepsilon(\rho(u_{0,j}),J\rho(u_{0,j}))=c_j+O(\varepsilon)$$

for some $c_j>0$, with J the endomorphism of T_0B arising from the complex structure. In particular in these coordinates the symplectic form Ω_ε admits an expansion

$$\Omega_\varepsilon=\sum_{j=1}^r\left(c_j+O(|z|)+O(\varepsilon)\right)idz_j\wedge d\bar{z}_j.$$

By rescaling the z_j linearly, in these coordinates we can then assume that $c_j=1$ and so the Hamiltonian for $u_{\varepsilon,j}$ satisfies

$$h_{\varepsilon,j}=h_{\varepsilon,j}(0)+|z_j|^2+O(\varepsilon)+O(|z|^3).$$

Indeed, and as we shall discuss in more detail in Lemma 5.3, the linear terms vanishes by the Hamiltonian condition since $u_{\varepsilon,j}$ vanishes at the origin, while our choice of coordinates ensures the quadratic term is of the claimed form.

Since the action on B is linear, perhaps shrinking B we can assume B takes the form

$$B = B' \times B'',$$

where T'' acts effectively on B'' and trivially on B'. Note that B'' may be different than the notation of Section 3.2. Recall that we are interested in a point $b \in B$ whose stabiliser is given by T'', so that $b = (b', 0) \in B' \times B''$, and none of the B'-coordinates of b vanish. Thus the points $\tilde{b} \in B$ such that $(\mathrm{Bl}_q M, \Psi_\varepsilon(J_{\tilde{b},\varepsilon}), \alpha_\varepsilon)$ is isomorphic to $(\mathrm{Bl}_p X, \alpha_\varepsilon)$ are characterised as those $\tilde{b}$ that have stabiliser T'', or equivalently such that the B''-coordinates of $\tilde{b}$ vanish, while none of the B'-coordinates vanish.

Lemma 5.2. *The almost complex structure $J_{b,\varepsilon}$ corresponds to an extremal metric on $(\mathrm{Bl}_q M, \omega_\varepsilon)$ provided*

$$h_\varepsilon(b) = 0$$

for all $h_\varepsilon \in (\mathfrak{t}''_\varepsilon)^\perp$. In particular it is enough to show this for a basis of $(\mathfrak{t}''_\varepsilon)^\perp$.

Proof. The moment map μ_ε is given by

$$\mu_\varepsilon = S(J_{\varepsilon,b}) - \hat{S}_\varepsilon \in \mathfrak{t}_\varepsilon,$$

from our construction of the almost complex structures $J_{\varepsilon,b}$. For a vector field $u \in \mathfrak{t}_\varepsilon$, let us write $H_{u,\varepsilon}$ for the associated Hamiltonian with respect to ω_ε on $\mathrm{Bl}_p M$. By definition the condition that

$$S(J_{\varepsilon,b}) - \hat{S}_\varepsilon) \in \mathfrak{t}_\varepsilon$$

means that

$$\int_{\mathrm{Bl}_q M} H(S(J_{\varepsilon,b}) - \hat{S}_\varepsilon)\omega_\varepsilon^n = 0$$

for all H which are L^2-orthogonal to the space of Hamiltonians of vector fields lying in $\mathfrak{t}$ with respect to ω_ε. To obtain an extremal metric, we need to show that $S(J_{\varepsilon,b}) - \hat{S}_\varepsilon$ actually lies in $\mathfrak{t}''_\varepsilon$, which requires us to show in addition that

$$\int_{\mathrm{Bl}_q M} H_{u,\varepsilon}(S(J_{\varepsilon,b}) - \hat{S}_\varepsilon)\omega_\varepsilon^n = 0$$

when $u \in (\mathfrak{t}''_\varepsilon)^\perp$, since the decomposition $\mathfrak{t}_\varepsilon = (\mathfrak{t}''_\varepsilon)^\perp \oplus \mathfrak{t}''_\varepsilon$ is an L^2-orthogonal decomposition. But by definition of the Hamiltonians h_ε this asks that $h_\varepsilon(b) = 0$ for all $h_\varepsilon \in (\mathfrak{t}''_\varepsilon)^\perp$, proving the result. □

Thus we need to find a point $b \in B' \times \{0\}$ such that $h_{\varepsilon,j}(b) = 0$ for all $j = 1, \dots, w$, inside the open dense orbit within $B' \times \{0\}$. General moment map theory provides the following.

Lemma 5.3. *For all j, the functions $h_{\varepsilon,j}$ satisfy the following properties:*

(i) for any $\varepsilon \geq 0$, the value $h_{\varepsilon,j}(z)$ is in independent of z provided the j^{th}-coordinate of z is zero;
(ii) at any point z with j^{th}-coordinate zero, the differential $dh_{\varepsilon,j}$ evaluated at z vanishes.

Proof. (i) There is a dense orbit inside the locus of such z which correspond to biholomorphic complex manifolds. For each z lying in this locus, the value $h_{\varepsilon,j}(z)$ is the simply the Futaki invariant of the holomorphic vector field $u_{\varepsilon,j}$ on the associated Kähler manifold. Thus the statement follows from the classical fact that the Futaki invariant is independent of choice of Kähler metric. But then by continuity of the function, $h_{\varepsilon,j}(z)$ must take the same value for all such z.

(ii) Since any such z is a fixed point of the vector field $u_{\varepsilon,j}$, by the Hamiltonian condition

$$dh_{\varepsilon,j} = -\iota_{u_{\varepsilon,j}}\Omega_\varepsilon$$

the differential must be zero as the vector field $u_{\varepsilon,j}$ vanishes at z. □

The way in which our stability hypothesis enters is as follows.

Lemma 5.4. *Suppose* $(\mathrm{Bl}_p X, \alpha_\varepsilon)$ *is relatively K-stable for all* $0 < \varepsilon \ll 1$. *Then for all* $j = 1, \dots, w$ *and for all* $0 < \varepsilon \ll 1$ *we have*

$$h_{\varepsilon,j}(z) < 0$$

provided the j^{th}*-coordinate of* z *is zero and* $z \in B' \times \{0\}$.

Proof. This again follows by definition of the $h_{\varepsilon,j}$, which is given as

$$h_{\varepsilon,j}(b) = \int_{\mathrm{Bl}_q M} H_{\varepsilon,j}(S(J_{\varepsilon,b}) - \hat{S}_\varepsilon)\omega_\varepsilon^m. \tag{5.2}$$

Since by Lemma 5.3 (i), the value of $h_{\varepsilon,j}(z)$ for whose j^{th}-coordinate is zero is actually independent of such z, we may assume that the $z_1, \dots, z_w$-coordinates are non-zero, except for the z_j-coordinate. Thus z is a limit under the flow of $u_{j,\varepsilon}$ of a point $b \in B' \times \{0\}$. As explained by Székelyhidi, any element $u \in \mathfrak{t}$ generates a test configuration if it is rational [33, p. 1088], and in general one obtains an $\mathbb{R}$-test configuration. To see this one restricts the universal family over B to the closure of the complexified orbit; that is, one takes the closure of the torus, and then takes the closure of the orbit of the complexification. The central fibre is then simply the complex manifold associated to z, on which $u_{\varepsilon,j}$ corresponds to a holomorphic vector field. Again then by definition of $h_{\varepsilon,j}(z)$ and the definition of relative K-stability, we have $h_{\varepsilon,j}(z) < 0$. □

5.2. **Solving the problem.** Here we construct extremal metrics under our hypothesis of relative K-stability, by solving a general problem in symplectic geometry. We summarise the setup and properties proven in the previous section to clarify what is needed to solve the problem.

We have an open ball B in a vector space containing the origin, which admits a linear, diagonal, effective and holomorphic action of a torus $T^{\mathbb{C}}$, such that the only fixed point is the origin. We are interested in the orbit of a point $b \in B$ with stabiliser $T'' \subset T$, and we denote by $\mathfrak{t}'', \mathfrak{t}$ the Lie algebras of T'', T respectively. We can write $B = B' \times B''$ such that T'' acts trivially on B' and effectively on B''. Thus to ask that b has stabiliser T'' asks that the B''-coordinates of b vanish, while none of the B'-coordinates vanish.

The Lie algebra $\mathfrak{t}$ admits a sequence of inner products $\langle \cdot, \cdot \rangle_\varepsilon$, along with a family of T-invariant symplectic forms Ω_ε, with moment maps

$$\mu_\varepsilon : B \to \mathfrak{t}_\varepsilon.$$

We again stress that we use the inner products $\langle \cdot, \cdot \rangle_\varepsilon$ to identify $\mathfrak{t}^*$ with its dual, which we denote $\mathfrak{t}_\varepsilon$ via this identification. We then have an orthogonal complement $(\mathfrak{t}''_\varepsilon)^\perp$ to $\mathfrak{t}''_\varepsilon$, which may not generate a closed subtorus of T.

We choose a collection $h_{\varepsilon,1}, \dots, h_{\varepsilon,w}$ of Hamiltonians associated to a basis of $(\mathfrak{t}''_\varepsilon)^\perp$, and normalise in such a way that

$$h_{\varepsilon,j} = h_{\varepsilon,j}(0) + |z_j|^2 + O(\varepsilon) + O(|z|^3),$$

where the linear term vanishes by the Hamiltonian condition as in Lemma 5.3.

Theorem 5.5. *Suppose that:*

(i) for all $j = 1, \dots, w$ and for all $\varepsilon \ll 0$ we have

$$h_{\varepsilon,j}(z) < 0$$

provided the j^{th}-coordinate of z is zero and $z \in B' \times \{0\}$;

(ii) for any $\varepsilon \geq 0$, the value $h_{\varepsilon,j}(z)$ is in independent of z provided the j^{th}-coordinate of z is zero;

Then for all $0 < \varepsilon \ll 1$ there is a sequence of points $b_\varepsilon \to 0$ such that

$$\mu_\varepsilon(b_\varepsilon) \in \mathfrak{t}''_\varepsilon,$$

and in addition none of the B'-coordinates of b_ε vanish.

Remark 5.6. A special case of Theorem 5.5 is proved in [10, Section 3.5] and our proof is a generalisation of the one given there. There are two new difficulties we encounter. Primarily, we are interested in the case that the stabiliser of b is non-trivial, and especially in the extremal case, whereas only the case of trivial stabiliser is considered there. Secondly, we will later be interested in the case $\dim X = 2$, which is dealt with in Section 5.3, and in which the radius in B on which we actually have a symplectic form is shrinking with ε. Both of these lead us to be more direct in our arguments, in comparison to [10, Section 3.5] where an equivariant version of the Darboux theorem was applied to simplify the problem.

We begin by examining more carefully the Taylor series expansions of the Hamiltonians.

Corollary 5.7. *The Hamiltonian $h_{\varepsilon,j}$ admits an expansion of the form*

$$h_{\varepsilon,j}(z) = h_{\varepsilon,j}(0) + |z_j|^2 + O(|z_j|^3) + O(\varepsilon |z_j|^2).$$

Here the Hamiltonian *can* depend on other variables z_i, for example the term $|z_j^3 z_i|$ is $O(|z_j|^3)$. But a term such as $\varepsilon |z_i|^2$ cannot appear for $i \neq j$.

Proof. Part (i) of Lemma 5.3 implies that any term not involving z_j must be constant. Thus there can only be terms which are constant, linear in z_j or quadratic in z_j (of course, there can be additonal dependence on other variables). The constant term is simply $h_{\varepsilon,j}(0)$, while the linear term in z_j must vanish by the second part of Lemma 5.3. The quadratic term is of the claimed form due to our choice of coordinates, proving the result. □

We next produce approximate solutions to the problem, which is the key step of the argument.

Proposition 5.8. *For any l, there is a point b_l such that*

$$h_{\varepsilon,j}(b_l) = O(\varepsilon^{l+1}) \text{ for } j = 1, \dots, r.$$

Proof. We choose the sequence explicitly by solving a sequence of linear problems, using an inductive strategy. The first step is to remove the lowest order error. First expand for each j

$$h_{\varepsilon,j}(0) = c_j\varepsilon^{p_j} + O(\varepsilon^{p_j+1}).$$

We choose

$$b_{1,j} = \lambda_{1,j}t_{1,j} \text{ with } \lambda_{1,j}^2 = -c_j \text{ and } t_{1,j}^2 = \varepsilon^{p_j},$$

with $\lambda_{1,j} > 0$, which is possible since $c_j < 0$. Thus we choose

$$\lambda_{1,j} = (-c_j)^{\frac{1}{2}} \qquad t_{1,j} = \varepsilon^{p_j/2}.$$

This forms the first step in the approximate solution. The approximate solution stays within B as $\varepsilon \to 0$ as $p_j > 0$.

In the higher order solutions, iterate this process. We consider the smallest of the values

$$e = \min_j \left\{ \frac{p_j}{2} \right\},$$

and note that

$$h_{\varepsilon,j}(b_1) = O(\varepsilon^{p_j+e}).$$

We fix the smallest order at which we have an error, namely the smallest value of $p_j + e$, and correct at this order first. The key point is that if we consider that j and expand

$$h_{\varepsilon,j}(b_1) = \psi_{2,j}\varepsilon^{p_j+e} + O(\varepsilon^{p_j+2e}),$$

we see that if we consider replacing

$$b_{1,j} \to b_{1,j} + \lambda_{2,j}\varepsilon^{(p_j+e)/2}$$

then choosing

$$\lambda_{2,j} = -\frac{\psi_{2,j}}{\lambda_{1,j}}$$

removes the $O(\varepsilon^{p_j+e})$-error. Proceeding inductively produces the result. □

From here, the proof of Theorem 5.5 is the same as [10, Section 3.5]. Having constructed approximate solutions to all orders, one uses the quantitive inverse function theorem to obtain genuine solutions. The main step is achieve a good understanding of the linearised operator, which can be understood directly in this finite dimensional setting, and so we refer to [10, Section 3.5] for further details.

More explicitly, the quantitive inverse function theorem produces points b_ε such that $h_{\varepsilon,j}(b_\varepsilon) = 0$ for all $j = 1, \dots, w$, as perturbations of the approximate solutions. Note that the b_ε constructed is *a priori* may have some vanishing B'-coordinate. But in that case, supposing that the j^{th}-coordinate of b_ε vanishes, we must have $h_{\varepsilon,j}(b_\varepsilon) < 0$ by hypothesis (i) of Theorem 5.5, contradicting the fact that $h_{\varepsilon,j}(b_\varepsilon) = 0$ by construction.

Summing up, applying Theorem 5.5 to the setting of blowups, we have proven part of our main result:

Corollary 5.9. *If the blowup* $(\mathrm{Bl}_p X, \alpha_\varepsilon)$ *is relatively K-stable for all* $0 < \varepsilon \ll 1$, *then* α_ε *admits extremal metrics for all* $0 < \varepsilon \ll 1$.

Proof. Lemma 5.2 implies that the existence of a point b_ε with $\mu_\varepsilon(b_\varepsilon) \in \mathfrak{t}''_\varepsilon$ implies the existence of an extremal metric in α_ε, while Lemmas 5.4 and 5.3 show that the hypotheses of Theorem 5.5 hold under the assumption of relatively K-stable for all $0 < \varepsilon \ll 1$. Thus the existence of a extremal metrics is a consequence of Theorem 5.5. □

5.3. **Complex surfaces.** We next turn to case of complex surfaces, which requires only a little adaptation. In this situation the gluing argument shows that there is a $c > 0$ such that for *any* $\kappa > 0$, if we denote by $B_\varepsilon \subset B$ the ball of radius $c\varepsilon^\kappa$ with $c > 0$, the map

$$\Psi_\varepsilon : B_\varepsilon \hookrightarrow \mathcal{J}(\mathrm{Bl}_q M, \omega_\varepsilon)$$

is an equivariant embedding, and produces a family of symplectic forms Ω_ε on B_ε. So B_ε is actually shrinking as $\varepsilon \to 0$. But it is easy to see that our results apply to this situation, provided one chooses κ correctly. Indeed, the first approximate solutions satisfy

$$(b_{\varepsilon,1})_j = \lambda_{1,j}^{1/2} \varepsilon^{p_j/2},$$

where $p_j > 0$. Thus for $0 < \varepsilon \ll 1$ we have $b_{\varepsilon,1} \in B_\varepsilon$ provided $\kappa < p_j/2$ for all j. But similarly, having chosen such a κ, the approximate solutions satisfy $b_{\varepsilon,l} \in B_\varepsilon$ for all l when $0 < \varepsilon \ll 1$, and a standard part of the inverse function theorem states that the genuine solutions are small perturbations of the approximate solutions, so we obtain $b_{\varepsilon,\infty} \in B_\varepsilon$ for all $0 < \varepsilon \ll 1$ as well.

Corollary 5.10. *Suppose* $(\mathrm{Bl}_p X, \alpha_\varepsilon)$ *is a relatively K-stable complex surface for all* $0 < \varepsilon \ll 1$*. Then* α_ε *admits extremal metrics for all* $0 < \varepsilon \ll 1$*.*

5.4. **Test configurations.** The reason we need to use $\mathbb{R}$-test configurations in our argument is that we have chosen an L^2-orthonormal basis of Hamiltonians. In the algebraic case, provided ε is rational the L^2-inner product is rational (we see this explicitly in Proposition 6.2 below), meaning we can choose a rational orthogonal basis. Thus we can pick a rational basis for the one-parameter subgroups, and hence only need to consider test configurations. Similarly in the cscK case, it is not actually necessary to use any particular basis, and so one can again use a rational basis. Thus we obtain the following two corollaries:

Corollary 5.11. *Suppose* $\alpha = c_1(L)$ *for some ample line bundle* L *on* X*, and* $(\mathrm{Bl}_p X, L - \varepsilon E)$ *is relatively K-stable for all* $0 < \varepsilon \ll 1$ *rational, with relative K-stability meant only with respect to test configurations. Then* $c_1(L - \varepsilon E)$ *admits extremal metrics for all* $0 < \varepsilon \ll 1$*.*

Note that our result only gives the existence of extremal metrics for rational values of ε in the above, but the fact that the set of Kähler classes admitting an extremal metric is open [23, Theorem A] also gives the result for irrational values of ε.

Corollary 5.12. *Suppose* $(\mathrm{Bl}_p X, \alpha_\varepsilon)$ *is K-polystable for all* $0 < \varepsilon \ll 1$*, with K-polystability meant only with respect to test configurations. Then* α_ε *admits cscK metrics for all* $0 < \varepsilon \ll 1$*.*

6. A geometric interpretation of K-stability

6.1. **Expanding the Donaldson-Futaki invariants.** We next turn to the algebro-geometric aspect of our arguments. In Section 5 the objects of interest were $\mathbb{R}$-test configurations induced on blowups. Recalling the construction, let $\pi : (\mathcal{X}, \mathcal{A}) \to B$ be an $\mathbb{R}$-test configuration for (X, α) which is also a holomorphic submersion. Fixing a point $p \in X$ we obtain a submanifold $Z = \overline{T^{\mathbb{C}}.p}$, where $T^{\mathbb{C}}$ is the torus involved in the definition of an $\mathbb{R}$-test configuration. We then blow Z up to obtain $(\mathrm{Bl}_Z \mathcal{X}, \mathcal{A}_\varepsilon)$, where $\mathcal{A}_\varepsilon = \mathcal{A} - \varepsilon^2[\mathcal{E}]$ and $\mathcal{E}$ denotes the exceptional divisor of the blowup. The following construction is due to Stoppa for test configurations [29]; we note that it applies in our more general situation.

Lemma 6.1. $(\mathrm{Bl}_Z \mathcal{X}, \mathcal{A}_\varepsilon) \to B$ *admits the structure of an* $\mathbb{R}$*-test configuration for* $(\mathrm{Bl}_p X, \alpha_\varepsilon)$*. In addition the map* $\mathrm{Bl}_Z \mathcal{X} \to B$ *is a holomorphic submersion.*

Proof. Since Z is fixed by the $T^{\mathbb{C}}$-action by construction, there is an induced $T^{\mathbb{C}}$-action on the blowup $(\mathrm{Bl}_Z \mathcal{X}, \mathcal{A}_\varepsilon)$. We can thus take the same element $u \in \mathrm{Lie}\, T^{\mathbb{C}}$ involved in the definition of the $\mathbb{R}$-test configuration $(\mathcal{X}, \mathcal{A})$ as the vector field inducing $(\mathrm{Bl}_Z \mathcal{X}, \mathcal{A}_\varepsilon)$.

The fibre of the induced map $\mathrm{Bl}_Z \mathcal{X} \to B$ over a point b is simply the blowup $\mathrm{Bl}_{p_b} \mathcal{X}_b$, where p_b denotes the point associated to p on the fibre over $b \in B$, meaning we have a holomorphic submersion. This also demonstrates that the fibres over $T^{\mathbb{C}} \subset B$ are all isomorphic to $(\mathrm{Bl}_p X, \alpha_\varepsilon)$, proving everything required. □

Our next goal will be to calculate the relative Donaldson-Futaki invariant of these $\mathbb{R}$-test configurations. This only involves the vector field u on the central fibre $\mathrm{Bl}_q \mathcal{X}_0$, and so to ease notation we simply replace $\mathcal{X}_0$ with X, α_0 with α and p_0 with p, so that the vector field u vanishes at p. We will first consider the case of test configurations, as an approximation argument will then produce the result in general.

To understand the relative Donaldson-Futaki invariant of the blow up, we must understand how both the Donaldson-Futaki invariant and the inner product vary with ε. For the Donaldson-Futaki invariant, this has already been fully understood by Székelyhidi [34, 37] using the strategy of Stoppa [29]. As we will need to extend these arguments to also allow understanding of the inner product, and since one needs a new idea in considering the inner product, we go through the arguments.

We first consider the projective case, with L an ample line bundle such that $c_1(L) = \alpha$, and such that ε is rational, as we will be able to reduce to this case. While we are interested in the blowup $(\mathrm{Bl}_p X, L - \varepsilon^2 E)$, we begin by recalling how to express the various invariants algebraically on (X, L); we will use the corresponding formulas for the blowup to obtain the required results.

The $\mathbb{C}^*$-action on (X, L) induces a $\mathbb{C}^*$-action on the vector spaces $V_k = H^0(X, kL)$ for all $k > 0$. Standard results of Donaldson imply that the Donaldson-Futaki invariant and the inner product can be understood from these $\mathbb{C}^*$-actions in the following manner. Suppose the $\mathbb{C}^*$-action on V_k diagonalises as $(t^{\lambda_1}, \ldots, t^{\lambda_{N_k}})$, with $N_k = \dim V_k$. Consider the three functions $a(k), b(k)$ and $c(k)$ defined by

$$a(k) = \dim V_k, \qquad b(k) = \sum_{j=1}^{N_k} \lambda_j, \qquad c(k) = \sum_{j=1}^{N_k} \lambda_j^2.$$

Then these are polynomials with rational coefficients of degree $n, n+1, n+2$ respectively for $k \gg 0$, which we write

$$\begin{aligned} a(k) &= a_0 k^n + a_1 k^{n-1} + O(k^{n-2}), \\ b(k) &= b_0 k^{n+1} + b_1 k^n + O(k^{n-1}), \\ c(k) &= c_0 k^{n+2} + O(k^{n-1}). \end{aligned}$$

For a general vector space V of dimension N with a $\mathbb{C}^*$-action, we also use the notation $\operatorname{wt} V = \sum_{j=1}^N \lambda_j$ for the *total weight*, and $\operatorname{wt}^2 V = \sum_{j=1}^N \lambda_j^2$ for the *squared weight*, so that $a(k) = \dim V_k, b(k) = \operatorname{wt} V_k$ and $c(k) = \operatorname{wt}^2 V_k$.

Proposition 6.2. [15, 16] *The Futaki invariant and norm are given by*

$$F(u) = 4\frac{b_0 a_1 - b_1 a_0}{a_0}, \qquad \|u\|_2^2 = \frac{c_0 a_0 - b_0^2}{a_0^2}.$$

The polarisation identity

$$2\langle u, v\rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2$$

will ultimately allow us to understand the inner products by understanding the norms, so we only consider the norms for the moment.

Let us now write $a_0(\varepsilon), b_0(\varepsilon)$ and $c_0(\varepsilon)$ for the corresponding numerical invariants calculated with respect to the induced $\mathbb{C}^*$-action on the vector spaces $H^0(\operatorname{Bl}_p X, k(L - \varepsilon E))$, where we only consider k, ε such that $k\varepsilon$ is an integer. In order to understand their dependence on ε, we also consider the $\mathbb{C}^*$-action on the one-dimensional vector space L_p and on the n-dimensional vector space $T_p^* X$.

Lemma 6.3. *Let h be a Hamiltonian on X for the vector field u. Then*

$$\begin{aligned} \operatorname{wt} L_p &= -H(p), \\ \operatorname{wt} T_p^* X &= -\Delta H(p), \\ \operatorname{wt}^2 T_p^* X &= \langle \operatorname{Hess}(H), \operatorname{Hess}(H)\rangle(p), \end{aligned}$$

where $\operatorname{Hess}(H)$ *denotes the Hessian.*

Proof. The fact that $\operatorname{wt} L_p = -H(p)$ is standard, while the additional observation that $\operatorname{wt} T_p^* X = -\Delta H(p)$ is due to Székelyhidi [34, Lemma 28], and follows from the fact that the action on $T_p X$ is induced by the Hessian of H at p. The same reasoning shows that $\operatorname{wt}^2 T_p^* X = \langle \operatorname{Hess}(H), \operatorname{Hess}(H)\rangle(p)$. □

Proposition 6.4. *We have expansions*

$$a_0(\varepsilon) = a_0 - \frac{\varepsilon^{2n}}{n!},$$

$$a_1(\varepsilon) = a_1 - \frac{\varepsilon^{2n-2}}{2(n-1)!},$$

$$b_0(\varepsilon) = b_0 + \frac{\varepsilon^{2n}}{n!} H(p) + \frac{\varepsilon^{2n+2}}{(n+1)!}\Delta H(p),$$

$$b_1(\varepsilon) = b_1 + \frac{\varepsilon^{n-1}}{2(n-2)!} H(p) + \frac{(n-2)\varepsilon^{2n+2}}{2n!}\Delta H(p),$$

$$c_0(\varepsilon) = c_0 - \varepsilon^{2n}\frac{H(p)^2}{n!} - \varepsilon^{2n+2}\frac{2H(p)\Delta H(p)}{(n+1)!} - \varepsilon^{2n+4}\frac{\langle \operatorname{Hess}(H), \operatorname{Hess}(H)\rangle(p) + H(p)^2}{(n+2)!}.$$

Note that $H(p)^2 = \mathrm{wt}^2 L_p$, since L_p is one-dimensional.

Proof. The expansions of $a_0(\varepsilon), a_1(\varepsilon), b_0(\varepsilon)$ and $b_1(\varepsilon)$ are due to Székelyhidi [34, Lemma 28], refining work of Stoppa [29], and the starting point of our approach is the same as in their work.

Denoting by $\mathcal{I}_p$ the ideal sheaf of the point p, the isomorphism

$$H^0(\mathrm{Bl}_p X, k(L-\varepsilon^2 E)) \cong H^0(X, kL \otimes \mathcal{I}_{k\varepsilon^2 p})$$

allows us to reduce to understanding the action on $H^0(X, kL \otimes \mathcal{I}_{k\varepsilon^2 p})$. Here, as above, we only consider k, ε^2 such that $k\varepsilon^2$ is an integer. The short exact sequence

$$0 \to kL \otimes \mathcal{I}_{k\varepsilon^2 p} \to kL \to \mathcal{O}_{k\varepsilon^2 p} \otimes kL_p \to 0$$

induces for $k \gg 0$ a short exact sequence

$$0 \to H^0(X, kL \otimes \mathcal{I}_{k\varepsilon^2 p}) \to H^0(X, kL) \to \mathcal{O}_{k\varepsilon^2 p} \otimes kL_p \to 0,$$

where we think of the latter as a vector space and where, slightly abusively, kL_p denotes the fibre of $L^{\otimes k}$ over p. By equivariance of the short exact sequence, it is enough to understand the action on the vector space $\mathcal{O}_{k\varepsilon^2 p} \otimes kL|_p$.

Since the vector space kL_p is one-dimensional, the weight on $\mathcal{O}_{k\varepsilon^2 p} \otimes kL_p$ is given by

$$\mathrm{wt}(\mathcal{O}_{k\varepsilon^2 p} \otimes kL_p) = \mathrm{wt}(\mathcal{O}_{k\varepsilon^2 p}) + k\,\mathrm{wt}(L_p)\dim \mathcal{O}_{k\varepsilon^2 p},$$

while similarly the square weight is given by

$$\mathrm{wt}^2(\mathcal{O}_{k\varepsilon^2 p} \otimes kL_p) = \mathrm{wt}^2(\mathcal{O}_{k\varepsilon^2 p}) + k^2\,\mathrm{wt}\, L_p^2 \dim \mathcal{O}_{k\varepsilon^2 p} + 2k\,\mathrm{wt}\, L_p\,\mathrm{wt}(\mathcal{O}_{k\varepsilon^2 p}).$$

We next turn to the action on $\mathcal{O}_{k\varepsilon^2 p}$. Setting $k\varepsilon^2 = l$, similarly to Székelyhidi we think of $\mathcal{O}_{lp}$ as the space of $(l-1)$-jets of functions at p, so that

$$\mathcal{O}_{lp} = \mathbb{C} \oplus T_p^*X \oplus \ldots \oplus S^{l-1}T_p^*X.$$

Denoting $V = T_p^*X$, the dimension and total weight satisfy

$$\dim S^jV = \binom{n+j-1}{j}, \qquad \mathrm{wt}\, S^jV = \binom{n+j-1}{j-1}\mathrm{wt}\, V,$$

which gives

$$\dim \mathcal{O}_{lp} = \sum_{j=0}^{l-1} \dim S^jV = \binom{n+l-1}{n} = \frac{1}{n!}\left(l^n + \frac{n(n-1)}{2}l^{n-1}\right) + O(l^{n-2}),$$

$$\mathrm{wt}\,\mathcal{O}_{lp} = \binom{n+l-1}{n+1}\mathrm{wt}\, V = \frac{\mathrm{wt}\, V}{(n+1)!}\left(l^{n+1} + \frac{(n+1)(n-2)}{2}l^n\right) + O(l^{n-1}).$$

Summing these formulae reproduces the formulae for $a_0(\varepsilon), a_1(\varepsilon), b_0(\varepsilon), b_1(\varepsilon)$.

There seems to be no analogous formula for $\mathrm{wt}^2(\mathcal{O}_{lp})$ in terms of $\mathrm{wt}^2 V$, so we use a more geometric argument to calculate its leading order term in l. Consider the $(n-1)$-dimensional variety $\mathbb{P}(V)$ with its induced $\mathbb{C}^*$-action. If we denote

$$e(j) = \mathrm{wt}^2(H^0(\mathbb{P}(V), \mathcal{O}(j)) = e_0 j^{n+1} + O(j),$$

then we know that $e(j)$ is a polynomial for all $j \gg 0$ with leading order term

$$e_0 = \int_{\mathbb{P}(V)} H_u^2 \omega_{FS}^{n-1}$$

where if we diagonalise so that the action is given by $(t^{\gamma_1}, \dots, t^{\gamma_n})$ the function H_u is given by

$$H_u = \frac{\sum_{j=1}^n \gamma_j |z_j|^2}{|z|^2}.$$

We can explicitly calculate this integral (see for example [26, Proposition 3.1.1]), giving for $i \neq j$

$$\int_{\mathbb{P}(V)} \frac{|z_i|^2 |z_j|^2}{|z|^4} \omega_{FS}^{n-1} = \frac{1}{(n+1)!}, \qquad \int_{\mathbb{P}(V)} \frac{|z_i|^4}{|z|^4} \omega_{FS}^{n-1} = \frac{2}{(n+1)!},$$

and so

$$e_0 = \frac{1}{(n+1)!} (\operatorname{wt}^2 V + (\operatorname{wt} V)^2).$$

As we are interested in the asymptotics of $\operatorname{wt}^2(\mathcal{O}_{lp})$, we may assume that $e(j)$ is actually a polynomial for all j. Then we see that

$$\sum_{j=0}^{l-1} e(j) = e_0 \sum_{j=0}^{l-1} j^{n+1} + O(l^{n+1}) = \frac{e_0}{n+2} l^{n+2} + O(l^{n+1}),$$

so

$$\operatorname{wt}^2(\mathcal{O}_{k\varepsilon^2 p}) = k^{n+2} \varepsilon^{2n+4} \frac{(\operatorname{wt}^2 V + (\operatorname{wt} V)^2)}{(n+2)!} + O(k^{n+1}).$$

It follows that

$$\begin{aligned} \operatorname{wt}^2(\mathcal{O}_{k\varepsilon^2 p} \otimes kL_p) &= \operatorname{wt}^2(\mathcal{O}_{k\varepsilon^2 p}) + k^2 \operatorname{wt} L_p^2 \dim \mathcal{O}_{k\varepsilon^2 p} + 2k \operatorname{wt} L_p \operatorname{wt}(\mathcal{O}_{k\varepsilon^2 p}), \\ &= k^{n+2} \left(\varepsilon^{2n+4} \frac{(\operatorname{wt}^2 V + (\operatorname{wt} V)^2)}{(n+2)!} + \frac{\operatorname{wt} L_p^2}{n!} \varepsilon^{2n} + \frac{2 \operatorname{wt} L_p \operatorname{wt} V}{(n+1)!} \varepsilon^{2n+2} \right) + O(k^{n+1}). \end{aligned}$$

Finally this means that

$$c_0(\varepsilon) = c_0 - \varepsilon^{2n} \frac{\operatorname{wt} L_p^2}{n!} - \varepsilon^{2n+2} \frac{2 \operatorname{wt} L_p \operatorname{wt} V}{(n+1)!} - \varepsilon^{2n+4} \frac{(\operatorname{wt}^2 V + (\operatorname{wt} V)^2)}{(n+2)!},$$

which using Lemma 6.3 proves the result. □

Remark 6.5. The value $\Delta H(p)$ is simply the weight of the $\mathbb{C}^*$-action on the fibre of the line bundle K_X over p, and hence has a natural interpretation in terms of geometric invariant theory. The reason is that if H is a Hamiltonian for a vector field with respect to ω, then ΔH is a Hamiltonian with respect to the Ricci curvature $\operatorname{Ric} \omega \in c_1(-K_X)$ (in the sense that the Hamiltonian condition is formally satisfied; $\operatorname{Ric} \omega$ may not be Kähler), so that Lemma 6.3 applies, see for example [34, Lemma 28].

In order to understand the inner product, suppose we have two commuting $\mathbb{C}^*$-actions on (X, L) fixing p and hence inducing actions λ and γ on $(\operatorname{Bl}_p X, L - \varepsilon E)$, and let their Hamiltonians be H_λ and H_γ with respect to ω. The key invariant in defining the inner product is defined as follows. Diagonalise the two one-parameter subgroups as $(t^{\lambda_1}, \dots, t^{\lambda_{N_{k,\varepsilon}}})$ and $(t^{\gamma_1}, \dots, t^{\gamma_{N_{k,\varepsilon}}})$ respectively, then define d_0 by

$$\sum_{j=1}^{N_{k,\varepsilon}} \lambda_j \gamma_j = d_0(\varepsilon) k^{n+2} + O(k^{n+1}).$$

The *inner product* is then defined to be

$$\langle \lambda, \gamma \rangle = \frac{d_0 a_0 - b_{0,\lambda} b_{0,\gamma}}{a_0^2};$$

this agrees with the inner product of Hamiltonians normalised to integrate to zero. The following is an immediate consequence.

Corollary 6.6. *We have*

$$\begin{aligned} d_0(\varepsilon) = d_0 - \varepsilon^{2n} \frac{H_\lambda(p) H_\gamma(p)}{n!} &- \varepsilon^{2n+2} \frac{H\lambda(p)\Delta H_\gamma(p) + H_\gamma(p) \Delta h_\lambda(p)}{(n+1)!} \\ &- \varepsilon^{2n+4} \frac{\langle \mathrm{Hess}(H_\lambda), \mathrm{Hess}(H_\gamma)\rangle(p) + H_\lambda(p) H_\gamma(p)}{(n+2)!}, \end{aligned}$$

where d_0 denotes the corresponding term computed on (X, L).

We next show that one can reduce to the projective, rational case.

Proposition 6.7. *The formulae of Corollaries 6.8 and 6.6 hold for arbitrary holomorphic vector fields on compact Kähler manifolds.*

Proof. The idea of the reduction to the projective case is due to Székelyhidi [37, Proposition 35]. Indeed, when viewing these invariants as integrals rather than algebro-geometric invariants through Proposition 6.2, all calculations are local around the exceptional divisor, and our definition of the metric ω_ε is that it is a glued-in copy of the Burns-Simanca metric on $\mathrm{Bl}_0 \mathbb{C}^n$. Since we use the same metric around in the exceptional divisor in both the projective and non-projective settings, the formulae in the projective case imply those in the general case.

By continuity all of these formulae also extend to the case that u is irrational. For example, in calculating the expansion of the norms a term of interest is

$$\int_{\mathrm{Bl}_p X} H_{u,\varepsilon}^2 \omega_\varepsilon^n,$$

which is clearly continuous in u. □

This allows the calculation of the Futaki invariant on blowups, which is straightforward from Proposition 6.4.

Corollary 6.8. [34, Corollary 29] *The Futaki invariant is a quotient of polynomials in ε which has the following expansion.*

(i) In general, we have

$$F_\varepsilon(u) = F(u) - \frac{\varepsilon^{2n-2}}{2(n-2)!} H(p) - \frac{\varepsilon^n}{n!} \left(\frac{2n-4}{2} \Delta H(p) - \frac{a_1}{a_0} H(p) \right) + O(\varepsilon^{n+1}).$$

(ii) Suppose in addition $n = 2$ and $a_1 \neq 0$. Then

$$F_\varepsilon(u) = F(u) - \frac{\varepsilon^2}{2} H(p) + \frac{\varepsilon^4 a_1}{2a_0} H(p) + \frac{\varepsilon^6}{2a_0} \left(\frac{a_1}{3} \Delta H(p) - \frac{H(p)}{2} \right) + O(\varepsilon^8).$$

(iii) Suppose $n = 2$ and $a_1 = 0$. Then

$$F_\varepsilon(u) = F(u) - \frac{\varepsilon^2}{2} H(p) - \frac{\varepsilon^6}{4a_0} H(p) + \frac{\varepsilon^8}{12a_0} \Delta H(p) + O(\varepsilon^{10}).$$

Moreover, if $H(p) = \Delta H(p) = 0$, then $F_\varepsilon(u)$ vanishes identically.

Note that while the expansion in ε is not actually finite, this is only caused by the fact that the Futaki invariant is given by a *quotient* of polynomials in ε.

One can similarly expand the inner product $\langle\cdot,\cdot\rangle_\varepsilon$ through Corollary 6.6 and Proposition 6.4, though it does not seem illuminating to explicitly write the resulting inner product. What is important is that, returning to our notation that (X_0, α_0) is the central fibre on which p degenerates to a fixed point p_0 of the $T^{\mathbb{C}}$-action on X_0, we have a sequence of inner products $\langle\cdot,\cdot\rangle_\varepsilon$ on the Lie algebra $\mathfrak{t}$. We emphasise that while the $\langle\cdot,\cdot\rangle_\varepsilon$ are *not* defined purely in terms of invariants of the vector field and Hamiltonian at the fixed point p_0, the manner in which $\langle\cdot,\cdot\rangle_\varepsilon$ differs from $\langle\cdot,\cdot\rangle_0$ *is* purely through invariants of the vector field and Hamiltonian at the fixed point p_0. The expansion of $\langle\cdot,\cdot\rangle_\varepsilon$ in ε is again not finite, but this is simply because it is also a quotient of polynomials in ε.

Denote by $\mathfrak{t}_\varepsilon^\perp$ the orthogonal complement of the stabiliser $\mathfrak{t}' \subset \mathfrak{t}$ with respect to the inner product $\langle u, v\rangle_\varepsilon$. Summarising, what we have proven is the following:

Corollary 6.9. *Suppose the blowup* $(\mathrm{Bl}_p X, \alpha_\varepsilon)$ *is relatively K-stable for all* $\varepsilon > 0$ *with respect to all* $\mathbb{R}$*-test configurations induced by* $T^{\mathbb{C}}$*. Then for all* $u \in \mathfrak{t}_\varepsilon^\perp$ *we have*

$$A_\varepsilon H(q_u) + B_\varepsilon \Delta H(q_u) > 0$$

for ε *sufficiently small, where* $A_\varepsilon > 0$ *and* B_ε *depend only on* ε *and topological invariants of* (X, α)*, and these quantities are all calculated on* $\mathcal{X}_0$*. Here* q_u *is the specialisation of* p *under the flow of* u*.*

Here $A_\varepsilon, B_\varepsilon$ can be computed explicitly from Corollary 6.8, with B_ε of strictly higher order in ε. Note that in the case that (X, α) itself is extremal, the specialisation q_u is also a point on X itself, while in the cscK case, the condition simply asks that for all ε sufficiently small the function $H(q_u) + \varepsilon\Delta H(q_u)$ is strictly positive, since B_ε is of higher order in ε. The reason these quantities are calculated on the central fibre is that the algebro-geometric quantities are themselves computed on the central fibre.

Remark 6.10. That the expansions of Corollaries 6.8 and 6.6 hold for irrational vector fields also shows that the existence of an extremal metric implies relative K-stability, following the same argument as [37, 11], which itself was the Kähler analogue of arguments of Stoppa-Székelyhidi [30].

6.2. **Applying the Kempf-Ness Theorem.** We next use the calculation of the Futaki invariants and inner products to relate relative K-stability to finite dimensional geometric invariant theory, which will complete the proof of our main result. We will continue with the notation of the previous section, especially Corollary 6.9. We first consider the case that (X, α) itself admits an extremal metric; the semistable case will be dealt with in Section 6.2.1.

Let $\mu : X \to \mathfrak{t}_\varepsilon$ be the moment map with respect to the T-action with respect to ω, where we identify $\mathfrak{t}$ with $\mathfrak{t}^*$ using the inner product $\langle\cdot,\cdot\rangle_\varepsilon$. Denote also $\Delta\mu$ the Laplacian of the moment map, where we recall from Remark 6.5 that this means taking the Laplacian of the associated Hamiltonians and can be interpreted as the moment map with respect to $\mathrm{Ric}\,\omega$.

Theorem 6.11. *There is a point* $p_\varepsilon \in T^{\mathbb{C}}.p$ *with*

$$A_\varepsilon \mu(p_\varepsilon) + B_\varepsilon \Delta\mu(p_\varepsilon) \in \mathfrak{t}_{p_\varepsilon}$$

if and only if for all $u \in \mathfrak{t}_\varepsilon^\perp$ *we have*

$$A_\varepsilon H(q_u) + B_\varepsilon \Delta H(q_u) > 0,$$

with q_u *the specialisation of* p *under the flow of* Ju *and* $\mathfrak{t}_{p_\varepsilon}$ *the Lie algebra of the stabiliser of* p_ε*.*

Proof. This is proven in exactly the same way as the usual Kempf-Ness theorem. That is, one fixes a basis of $\mathfrak{t}_\varepsilon^\perp$ and shows that the hypothesis $A_\varepsilon h(q_u) + B_\varepsilon \Delta h(q_u) > 0$ implies the existence of a zero of $A_\varepsilon H + B_\varepsilon \Delta H$ in $T^{\mathbb{C}}.p$; iterating over each basis element and using commutativity of the vector fields implies that there is a point $p_\varepsilon \in T^{\mathbb{C}}.p$ such that the projection of $A_\varepsilon \mu + B_\varepsilon \Delta\mu$ orthogonal to $\mathfrak{t}_{p_\varepsilon}$ vanishes. But this simply means that $A_\varepsilon \mu + B_\varepsilon \Delta\mu \in \mathfrak{t}_{p_\varepsilon}$, giving the result. □

Remark 6.12. In the usual Kempf-Ness theorem, one only needs to consider rational one-parameter subgroups, due to the Hilbert-Mumford criterion. In our case we do not actually have a closed Lie group corresponding to $\mathfrak{t}_\varepsilon^\perp$, so we cannot apply the Hilbert-Mumford criterion. We note that Székelyhidi claims that one need only consider rational one-parameter subgroups in a general setting of which Theorem 6.11 is a special case. However there is an error in his assertion that the Lie group associated with $\mathfrak{t}_\varepsilon^\perp$ is closed, which is important in his proofs [37, p. 929]; the result he uses requires the inner product to be rational. In the algebraic case, or the case in which one seeks a zero of the moment map, only rational one-parameter subgroups are needed, however, due to rationality of the inner products and Luna's slice theorem respectively.

In the algebraic case, for a rational u generating a $\mathbb{C}^*$-action, the value $A_\varepsilon h(q_u) + B_\varepsilon \Delta h(q_u)$ is simply the weight of the $\mathbb{C}^*$-action at q_u associated to the line bundle $A_\varepsilon L - B_\varepsilon K_X$, giving an interpretation in terms of classical geometric invariant theory. By rationality of the inner products, the subgroup $\mathfrak{t}_\varepsilon^\perp$ corresponds to a closed Lie subgroup $K_\varepsilon^\perp \subset K$, with complexification $G_\varepsilon^\perp \subset G = \mathrm{Aut}_0(X_0, \alpha_0)$ [31, Lemma 1.3.2]. The following is then immediate from the Hilbert-Mumford criterion [24, Theorem 2.1].

Corollary 6.13. *Suppose* X *is projective with* $\alpha = c_1(L)$ *the first Chern class of an ample line bundle. Then there is a point* $p_\varepsilon \in T^{\mathbb{C}}.p$ *with*

$$A_\varepsilon \mu + B_\varepsilon \Delta\mu \in \mathfrak{t}_{p_\varepsilon}$$

if and only if for all $0 < \varepsilon \ll 1$ *the point* p *is stable in the sense of geometric invariant theory with respect to the action of* $G_\varepsilon^\perp$ *and with respect to the polarisation* $A_\varepsilon L - B_\varepsilon K_X$*.*

We can now complete the proof of Theorem 1.1, which we restate:

Theorem 6.14. *The following are equivalent:*

(i) $(\mathrm{Bl}_p X, \alpha_\varepsilon)$ *admits an extremal metric for all* $0 < \varepsilon \ll 1$*;*
(ii) $(\mathrm{Bl}_p X, \alpha_\varepsilon)$ *is relatively K-stable for all* $0 < \varepsilon \ll 1$*;*
(iii) for all $u \in \mathfrak{t}_\varepsilon^\perp$ *we have*

$$A_\varepsilon H(q_u) + B_\varepsilon \Delta H(q_u) > 0,$$

with q_u *the specialisation of* p *under the flow of* u *and for all* $0 < \varepsilon \ll 1$*;*
(iv) for all $0 < \varepsilon \ll 1$ *there is a point* $p_\varepsilon \in T^{\mathbb{C}}.p$ *with*

$$A_\varepsilon \mu + B_\varepsilon \Delta\mu \in \mathfrak{t}_{p_\varepsilon}.$$

Proof. We have already established the equivalence of (i) and (ii) for the class of $\mathbb{R}$-test configurations used in our work, so in particular (ii) implies relative K-stability for all $\mathbb{R}$-test configurations. Corollary 6.9 shows the equivalence between (ii) and (iii), while Theorem 6.11 shows the equivalence of (iii) and (iv). □

In addition Corollary 6.13 allows one to view (iii) and (iv) purely in terms of stability in the sense of classical geometric invariant theory in the case that X is projective with $\alpha = c_1(L)$. Similarly Corollary 5.12 implies that in the cscK case, in (ii) only test configurations are needed, and in (iii) only rational vector fields generating one-parameter subgroups are needed.

6.2.1. *The semistable case.* We finally turn to the semistable case of our main results. We recall the setup, which is the same as throughout. The space $\mathcal{X}$ built from the Kuranishi space admits a holomorphic T-action, and admits the structure of a holomorphic submersion $\mathcal{X} \to B$. Here B is an open neighbourhood of the origin in a vector space, which itself admits a T-action making $\mathcal{X} \to B$ a T-equivariant morphism.

The key difference in comparison with the stable case is that $\mathcal{X}$ is only endowed with a relatively Kähler metric η. This form is induced by the symplectic form ω on M from the map $B \to \mathcal{J}(M, \omega)$; this map itself induces the family $\mathcal{X} \to B$. Denote by ρ the curvature of the metric induced by η on the relative anti-canonical class $-K_{\mathcal{X}/B}$, so that ρ restricts on each fibre to the Ricci curvature of the restriction of η.

Any vector field $u \in \mathfrak{t}$ induces a Hamiltonian function H_u with respect to η. The Hamiltonian is meant in a formal sense, as η may not be positive. We normalise such that H_u has integral zero over $\mathcal{X}_0$. The relevant function for ρ is the vertical Laplacian of h_u, which is formally a Hamiltonian with respect to ρ much as the Laplacian of a Hamiltonian is the Hamiltonian with respect to the Ricci curvature. Here we recall that the vertical Laplacian is given for $x \in \mathcal{X}_b$ by

$$(\Delta_V f)(x) = \Delta_{\eta|_{\mathcal{X}_b}} f|_{\mathcal{X}_b},$$

i.e. it is the Laplacian computed on the relevant fibre. Then our input from Corollary 6.9 is that for all $u \in \mathfrak{t}^{\perp}_{p,\varepsilon}$ we have

$$A_\varepsilon H(q_u) + B_\varepsilon \Delta_V H(q_u) > 0,$$

with q_u the specialisation of p under the flow of Ju and for all $0 < \varepsilon \ll 1$.

To apply the Kempf-Ness theorem we must make η a Kähler metric. Since B is an open subset of a vector space, we take the flat metric ω_{Euc} on B, so that perhaps after shrinking B there is a $k \gg 0$ such that $\eta + k\omega_{\mathrm{Euc}}$ is Kähler (where we pull back ω_{Euc} to $\mathcal{X}$). We note that a choice of flat metric depends on a choice of coordinates, and we will shortly exploit this ambiguity. This changes nothing on the central fibre $\mathcal{X}_0$ as ω_{Euc} is trivial on any fibre. What changes instead is the moment map. It is most transparent to work in coordinates, so we pick coordinates on B such that the torus action is diagonal. Then a Hamiltonian for the action on the j^{th}-coordinate on B is given by $b \to |b_j|^2$, and this defines a moment map μ_{Euc}.

A Kempf-Ness argument as in Theorem 6.11 applied to the metric $A_\varepsilon \eta + B_\varepsilon \rho + k\omega_{\mathrm{Euc}}$ implies that there is a $p_\varepsilon \in T^{\mathbb{C}}.p$ with

$$A_\varepsilon \mu(p_\varepsilon) + B_\varepsilon \Delta_V \mu(p_\varepsilon) + k\mu_{\mathrm{Euc}}(b_\varepsilon) \in \mathfrak{t}_{p_\varepsilon} \tag{6.1}$$

if and only if for all $0 < \varepsilon \ll 1$ and for all $u \in \mathfrak{t}_{p,\varepsilon}^{\perp}$ with Hamiltonian H_u with respect to ω we have

$$A_\varepsilon H(q_u) + B_\varepsilon \Delta_V H(q_u) > 0,$$

with q_u the specialisation of p under the flow of Ju as before. The condition of Equation 6.1 can be rephrased due to to the explicit nature of the moment maps; supposing $p_\varepsilon \in \mathcal{X}_{b_\varepsilon}$, we have

$$\mu_{\mathrm{Euc}}(p_\varepsilon) = \mathrm{diag}(|b_{\varepsilon,j}|^2) \in \mathfrak{t}_\varepsilon.$$

Thus the condition is equivalent to

$$A_\varepsilon \mu(p_\varepsilon) + B_\varepsilon \Delta_V \mu(p_\varepsilon) + k\,\mathrm{diag}(|b_{\varepsilon,j}|^2) \in \mathfrak{t}_{p_\varepsilon}.$$

In fact one can take any inner product on B, since we could in have considered the form $\sum k_j dz_j \wedge d\bar{z}_j$ instead of $k\omega_{\mathrm{Euc}}$, provided the k_j are sufficiently large. So for any such choice, the following follows in the same way as Theorem 6.14:

Theorem 6.15. *The following are equivalent:*

(i) $(\mathrm{Bl}_p X, \alpha_\varepsilon)$ *admits an extremal metric for all* $0 < \varepsilon \ll 1$*;*
(ii) $(\mathrm{Bl}_p X, \alpha_\varepsilon)$ *is relatively K-stable for all* $0 < \varepsilon \ll 1$*;*
(iii) for all $0 < \varepsilon \ll 1$ *and for all* $u \in \mathfrak{t}_{p,\varepsilon}^{\perp}$ *with Hamiltonian* H_u *with respect to* ω *we have*

$$A_\varepsilon H(q_u) + B_\varepsilon \Delta_V H(q_u) > 0,$$

with q_u *the specialisation of* p *under the flow of* Ju *and for all* $0 < \varepsilon \ll 1$*;*
(iv) for all $0 < \varepsilon \ll 1$ *there is a point* $p_\varepsilon \in T^{\mathbb{C}}.p$ *with*

$$A_\varepsilon \mu(p_\varepsilon) + B_\varepsilon \Delta_V \mu(p_\varepsilon) + k\mu_{\mathrm{Euc}}(p_\varepsilon) \in \mathfrak{t}_{p_\varepsilon}.$$

7. Applications

We end the paper with an application of our results to an analytically relatively K-semistable manifold. While Theorem 6.15 abstractly characterises the points which can be blown up in such a way that the blowup admit an extremal metric, it is also interesting to ask if a point p satisfying the criteria of that result actually exists. We give a positive answer to this question:

Theorem 7.1. *Let* (X, α) *be an analytically relatively K-semistable manifold. Then there is a point* $p \in X$ *such that there exists an extremal metric on* $\mathrm{Bl}_p X$ *in* α_ε *for all sufficiently small* $\varepsilon > 0$*.*

Proof. We fix a maximal torus $T^{\mathbb{C}} \subset \mathrm{Aut}(X, \alpha)$. It suffices to find a point $p \in X$ such that, if $q \in X_0$ denotes the limit of p under the $T^{\mathbb{C}}$-action, then $H(q) < 0$ for all Hamiltonians H associated to $\mathfrak{t} = \mathrm{Lie}\, T$. Here our Hamiltonians are normalised to have integral zero over X_0 as before.

We first construct the appropriate point $q \in X_0$. We begin with a single $\mathbb{C}^*$-action on (X_0, α_0), where the problem is to find a fixed point $q \in X_0$ such that $H(q) < 0$ with H the associated Hamiltonian. The existence of such a point can be seen as follows. Take a point $q' \in X_0$ such that $H(q') < 0$; the existence of such a point follows from the fact that H is non-trivial. Next, consider the gradient flow of the norm squared of the moment map beginning at q'. This flow decreases H and converges to a fixed point q'' of the $\mathbb{C}^*$-action with $H(q'') < 0$. Standard properties of this flow can be found, for example, in Georgoulas-Robbin-Salamon [20, Chapter 3]. Iterating the construction over a basis of the torus produces a point q with $H(q) < 0$ for all Hamiltonians.

We now consider the problem on X itself. We follow the constructions of Section 3, so that we consider X and X_0 as the same smooth manifold M but with varying complex structure. The space $\mathcal{X} = M \times B$ is the space constructed there from the Kuranishi space, and the T-action on $\mathcal{X}$ is induced by the actions on B and M. Since $q \in M \times \{0\}$ is a fixed point of the T-action on $M \times B$, taking any point $(q, b) \in M \times B$ with $b \neq 0$ produces a point $p \in X$ whose limit is q. □

It follows from the proof that q can be taken to be fixed under the $T''^{\mathbb{C}}$-action of X.

Remark 7.2. A related result is proven in [29, 12]; the proofs given there apply also when X_0 is singular. The problem considered there is in a sense opposite, however, in that there one instead wishes to find a point on which the Hamiltonians take positive values at the associated specialisations; this requires different techniques.

Theorem 7.1 can be improved in the projective setting.

Corollary 7.3. *Suppose X is projective with $\alpha = c_1(L)$. Then the locus of points $U \subset X$ such that $(\mathrm{Bl}_p X, L_\varepsilon)$ admits an extremal metric for all $\varepsilon > 0$ sufficiently small is Zariski open inside the fixed locus of $T'^{\mathbb{C}}$.*

Proof. This is a consequence of Zariski openness of the stable locus in geometric invariant theory. More precisely, as in Corollary 6.13, denote by $G_\varepsilon^\perp \subset G$ the Lie subgroup associated to $\mathfrak{t}_\varepsilon^\perp$. By the algebraic interpretation of Theorem 6.15, the condition needed for the blowup $(\mathrm{Bl}_p X, L_\varepsilon)$ to admit an extremal metric for ε sufficiently small is that the GIT weight for each one-parameter subgroup of $G_\varepsilon^\perp$ is strictly negative. Denoting by $W \subset X$ the fixed locus of $T^{\mathbb{C}}$, by the Hilbert-Mumford criterion this asks that the point is GIT stable as a point of W, and the result now follows from Zariski openness of the stable locus inside W. □

Note that Theorem 7.1 can be applied to give many new examples of manifolds admitting extremal metrics. Indeed, there are now known many explicit examples of strictly K-semistable Fano threefolds, that admit a degeneration to a K-polystable Fano (see [1] and the references therein). In order to apply our construction, the central fibre of such a degeneration needs to be smooth. This holds for certain members of the family 1.10 of the Mori–Mukai list of smooth Fano threefolds, which is the family that includes the Mukai–Umemura manifold. One can find other examples for instance in the families 2.24, 3.10, 3.13 and 4.13 of the Mori–Mukai list. Some of these are infinite families to which the construction applies.

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