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On the squashed seven-sphere operator spectrum

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Abstract: We derive major parts of the eigenvalue spectrum of the operators on the squashed seven-sphere that appear in the compactification of eleven-dimensional supergravity. These spectra determine the mass spectrum of the fields in AdS_4 and are important for the corresponding $\mathcal{N} = 1$ supermultiplet structure. This work is a continuation of the work in [1] where the complete spectrum of irreducible isometry representations of the fields in AdS_4 was derived for this compactification. Some comments are also made concerning the G_2 holonomy and its implications on the structure of the operator equations on the squashed seven-sphere.

Keywords: Flux compactifications, M-Theory

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1 Introduction

The purpose of this paper is to continue the work in [1] and derive the eigenvalue spectrum of some of the operators that determine the masses and supermultiplet structure of the entire Kaluza-Klein spectrum of the squashed seven-sphere compactification of eleven-dimensional supergravity. The list of previously known eigenvalue spectra, containing Δ_0, $\not{D}_{1/2}$ and Δ_1, is in this paper extended by those of Δ_2, Δ_3 and Δ_L while $\not{D}_{3/2}$ remains to be done. Note that parts of the spectrum of the Lichnerowicz operator Δ_L are derived below but they were quoted already in [2].[1]

With the intention to keep this paper as brief as possible we refer the reader to the review [2] for a Kaluza-Klein background on the problem and for some of the necessary details and conventions needed in the derivations below. For the full structure of irreducible isometry representations on AdS_4 stemming from the squashed seven-sphere compactification we refer to [1]. The latter paper summarises in a few pages the most relevant information from [2] in particular several tables that are spread out in various chapters of [2]. In this

[1] See ref. [198], unpublished work by Nilsson and Pope.

spirit we present below just the most crucial formulas that are needed to define the problem and derive the spectra. In the appendix we collect a number of useful octonionic identities and other formulas, as well as some for our purposes crucial Weyl tensor calculations.

The coset description of the squashed seven-sphere is

$$G/H = \mathrm{Sp}_2 \times \mathrm{Sp}_1^C/\mathrm{Sp}_1^A \times \mathrm{Sp}_1^{B+C}, \tag{1.1}$$

where, in order to define the denominator, Sp_2 is split into $\mathrm{Sp}_1^A \times \mathrm{Sp}_1^B$ and Sp_1^{B+C} is the diagonal subgroup of Sp_1^B and Sp_1^C. The G irreps specifying the mode functions Y of a general Fourier expansion on G/H are denoted $(p, q; r)$ [2] and the entire Kaluza-Klein irrep spectrum is derived and tabulated in terms of *cross diagrams* in [1]. The present paper will fill in some gaps in our knowledge of the eigenvalue spectrum of the relevant operators on the squashed seven-sphere but some are still missing. The remaining issues that we need to address to complete the eigenvalue spectra will be elaborated upon in a forthcoming publication [3].

The interest in deriving complete spectra in various Kaluza-Klein compactifications has recently been revived due to the discovery of new powerful versions of the embedding tensor technique. However, these methods can be applied directly only when the vacuum is an extremum of a maximal gauged supergravity theory in AdS_4 which can be lifted to ten or eleven dimensions, see for instance [4, 5] and references therein.[2] The point we want to emphasise here is that, due to the *space invaders scenario* [2, 6], the squashed seven-sphere solution is not of this kind and it therefore seems clear that other methods are required to obtain the AdS_4 spectrum in this case.

There are also potential applications of this work in the context of the swampland program,[3] see for instance [7–9] and references therein. This particular connection will be addressed elsewhere.

In the next section we very briefly review the background of the problem and the method that is used in this paper to derive the eigenvalue spectrum of operators on coset manifolds. The method is explained in many places, e.g. [2, 10], and in section 3 we first apply it to obtain the spectra of the spin 0 and 1 operators giving straightforwardly the well known results cited in [2]. Already when applied to the spin 1/2 Dirac operator complications arise which get further pronounced when we subsequently turn to more and more complicated operators. A summary of the obtained eigenvalues is provided in the Conclusions, together with comments on some of the remaining issues. Some technical aspects needed in the derivations below are explained in the appendix.

2 Compactification on the seven-sphere

The Fourier expansion technique that we will apply is, following the general strategy explained in [10] and summarised in [2], based on the *coset master equation* for the Spin(n) covariant derivative on a n-dimensional coset space G/H:

$$\nabla_a Y + \frac{1}{2} f^{bc}{}_a \,\Sigma_{bc} Y = -T_a Y. \tag{2.1}$$

[2] We are grateful to Oscar Varela for a clarification on this issue.

[3] We are grateful to M.J. Duff for raising some interesting aspects of this question that eventually led to this work.

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The mode functions Y have two suppressed indices: one corresponding to the Spin(n) tangent space irrep of the field that is being Fourier expanded[4] and one that specifies the mode, that is a G irrep. This equation is derived in [2] from group elements of the isometry group G. In the conventions used there the Lie algebra of the group G has generators T_A, satisfying $[T_A, T_B] = f_{AB}{}^C T_C$, which are divided into $T_{\bar{a}}$ of the subgroup H and T_a in the complement of H in G. Thus, if the Lie algebra of G is reductive it splits as follows:

$$[T_{\bar{a}}, T_{\bar{b}}] = f_{\bar{a}\bar{b}}{}^{\bar{c}} T_{\bar{c}}, \;\; [T_{\bar{a}}, T_b] = f_{\bar{a}b}{}^c T_c, \quad [T_a, T_b] = f_{ab}{}^{\bar{c}} T_{\bar{c}} + f_{ab}{}^c T_c. \tag{2.2}$$

The indices $a, b, \ldots$ are vector indices in the tangent space of G/H and ∇_a in (2.1) is an Spin(n) covariant derivative with a torsion free spin connection while the Σ_{ab} are the generators of Spin(n) in the representation relevant for the operator equation we are solving. Note that it is only the structure constants $f_{ab}{}^c$ that appear in the coset master equation (2.1). For symmetric spaces like the round seven-sphere $f_{ab}{}^c = 0$. Furthermore, the second algebraic relation in (2.2) defines the imbedding of H in the tangent space group Spin(n).

For symmetric coset spaces the coset master equations thus reads $\nabla_a Y = -T_a Y$ and the eigenvalue spectrum for the operators on G/H are rather easily derived. The complications arising in the squashed seven-sphere case therefore stem entirely from the structure constant dependent term in (2.1). As first shown in [11] for the squashed seven-sphere they are given by the structure constants of the octonions a_{abc}. In the normalisation used in [2] they read

$$f_{abc} = -\frac{1}{\sqrt{5}} a_{abc} = -\frac{2}{3} m \, a_{abc}. \tag{2.3}$$

Since octonions will play a key role in the rest of this paper we have listed some useful octonionic identities in the appendix. Furthermore, in the last expression for these structure constants we have introduced the scale parameter m arising from the ansatz $F_{\mu\nu\rho\sigma} = 3m\epsilon_{\mu\nu\rho\sigma}$ in the compactification of eleven-dimensional supergravity. This is useful since we are dealing with dimensionful quantities, like the covariant derivatives and Hodge-de Rham operators. The conventions used in [2] correspond to setting

$$m^2 = \frac{9}{20}. \tag{2.4}$$

We will assume $m > 0$ and also use $m = \frac{3}{2\sqrt{5}}$ as for f_{abc} above. The squashed seven-sphere discussed in this work has the orientation that gives rise to $\mathcal{N} = 1$ supersymmetry in AdS_4 after compactification. This fact will be used below when we discuss the corresponding supermultiplets and the SO$(3,2)$ irreps entering these multiplets.

With this preparation we can readily attack the eigenvalue problems $\Delta_p Y_p = \kappa_p^2 Y_p$ where the Hodge-de Rham[5] operator on p-forms is defined as $\Delta_p = \mathrm{d}\delta + \delta\mathrm{d}$. Here d is the exterior derivative and $\delta = (-1)^p \star \mathrm{d} \star$ its adjoint. These operators act on forms according to

$$(\mathrm{d}Y)_{a_1 \ldots a_p} = p \nabla_{[a_1} Y_{a_2 \ldots a_p]}, \quad (\delta Y)_{a_1 \ldots a_p} = -\nabla^b Y_{b a_1 \ldots a_p}, \tag{2.5}$$

[4]Note, however, that the whole matrix of the G group element is involved in this equation.
[5]This operator generalises the Laplace-Beltrami operator to p-forms with $p > 0$.

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where brackets here and in the following are weighted antisymmetrisations, $Y_{[a_1\ldots a_p]} = \frac{1}{p!}(Y_{a_1\ldots a_p} + \text{permutations with sign})$. The explicit form of Δ_p can for all operators considered in this paper be expressed using the Riemann tensor, R_{abcd}, and the d'Alembertian $\Box \equiv \nabla^a\nabla_a$. On the squashed seven-sphere, which is an Einstein space with $R = 42m^2$,

$$R_{ab}{}^{cd} = W_{ab}{}^{cd} + 2m^2\delta^{cd}_{ab}, \tag{2.6}$$

where W_{abcd} is the Weyl tensor given in the appendix.

The main use of the coset master equation (2.1) will be to replace derivatives by group-theoretical algebraic data. In particular, for the squashed sphere, by squaring (2.1) we obtain an expression for $\Box$ which can then be used to replace $\Box$ by algebraic data plus terms linear in ∇_a. This will be clear below.

3 Eigenvalue spectra on the squashed seven-sphere

In this section we start by briefly reviewing some of the results previously obtained with the coset space techniques mentioned above. These results are then generalised to the more complicated operators for which we present a number of new eigenvalues.

3.1 Review of the method applied to forms of rank 0 and 1

The method outlined above becomes rather trivial when applied to 0-forms. In this case we want to solve the eigenvalue equation

$$\Delta_0 Y = \kappa_0^2 Y, \tag{3.1}$$

where the positive semi-definite Hodge-de Rham operator on 0-forms is $\Delta_0 = -\Box$. Thus, following [2], the spectrum is obtained directly by squaring the coset master equation (2.1):

$$\Delta_0 Y = -\Box Y = -T_a T_a Y = (C_G - C_H)Y = \kappa_0^2 Y, \tag{3.2}$$

where the two second order Casimir operators for $G = \mathrm{Sp}_2 \times \mathrm{Sp}_1^C$ and $H = \mathrm{Sp}_1^A \times \mathrm{Sp}_1^{B+C}$ have eigenvalues

$$C_G = C(p,q) + 3C^C(r) = \frac{1}{2}(p^2 + 2q^2 + 4q + 6q + 2pq) + \frac{3}{4}r(r+2), \tag{3.3}$$

and

$$C_H = 2C^A(s) + \frac{6}{5}C^{B+C}(t) = \frac{1}{2}\,s(s+2) + \frac{3}{10}\,t(t+2). \tag{3.4}$$

Here (p,q) are Dynkin labels for Sp_2 irreps and r, s, t those for the irreps of the various Sp_1 groups that occur in the squashed seven-sphere coset description.

The whole problem of deriving the eigenvalue spectrum of harmonics is reduced to determining first which H irreps are contained in the Spin(7) irrep of Y (trivial for 0-forms) and then finding all G irreps that when split into H irreps contain any of the H irreps found in the Spin(7) irrep of Y. Some partial results on the spectrum of G irreps are obtained in [2, 12] by breaking up the spectrum on the round sphere according to $\mathrm{Spin}(8) \to \mathrm{Sp}_2 \times \mathrm{Sp}_1$

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while the entire spectrum of all operators is derived directly from the squashed sphere coset in [1] using the coset method just described.

For 0-forms $C_H = 0$ and the spectrum therefore becomes $\kappa_0^2 = C_G$. However, as explained above, we should introduce the scale parameter m. In view of this we can use $m^2 = \frac{9}{20}$ to write the eigenvalues as follows:[6]

$$\Delta_0 = \kappa_0^2 = \frac{m^2}{9} 20\, C_G(p, q; r). \tag{3.5}$$

As an example how this result is used let us consider the spectrum in the graviton sector:[7] In units of m the SO(3, 2) irrep defining energy $E_0(spin)$ is [14–16]

$$E_0(2^+) = \frac{3}{2} + \frac{1}{2}\sqrt{\frac{M_2^2}{m^2} + 9} = \frac{3}{2} + \frac{1}{2}\sqrt{\frac{20C_G}{9} + 9} = \frac{3}{2} + \frac{1}{6}\sqrt{20C_G + 81}, \tag{3.6}$$

where we have used the fact that the mass-square operator for spin 2 fields in AdS_4 is $M_2^2 = \Delta_0$. We will occasionally refer to results like this as being of *square-root form*, here a $\sqrt{81}$-form. The reason for this is that all the fields in a supermultiplet must be of the same square-root form.

We now turn to the 1-form case and review the result and the derivation in [2].[8] In this case the structure constant term in the coset master equation comes in and complicates the calculation somewhat. As for the 0-form modes we write out the Δ_1 eigenvalue equation explicitly:

$$\Delta_1 : \quad \Delta_1 Y_a = -\Box Y_a + R_a{}^b Y_b = -\Box Y_a + 6m^2 Y_a = \kappa_1^2 Y_a. \tag{3.7}$$

We want to use the square of the coset master equation to eliminate the $\Box$ term: inserting $f_{abc} = -\frac{2}{3} m a_{abc}$ into (2.1) gives $\nabla_a Y_b - \frac{m}{3} a_{abc} Y_c = -T_a Y_b$ which when squared yields

$$G/H : \quad \Box Y_a + \frac{2m}{3} a_{abc} \nabla_b Y_c + \frac{m}{3} (\nabla_b a_{abc}) Y_c - \frac{m^2}{9} a_{abc} a_{bcd} Y_d = T_b T_b Y_a. \tag{3.8}$$

Using the octonionic identities $\nabla_a a_{bcd} = m c_{abcd}$ and $a_{abe} a^{cd}{}_e = 2\delta_{ab}^{cd} + c_{ab}{}^{cd}$ (see the appendix) together with $T_a T_a = -(C_G - C_H)$, the G/H equation above simplifies to

$$-\Box Y_a - \frac{2m}{3} a_{abc} \nabla_b Y_c + 6m^2 Y_a = C_G Y_a. \tag{3.9}$$

Here we have also used the fact that the irrep **7** of SO(7) splits into $(1, 1) \oplus (0, 2)$ of $H = Sp_1^A \times Sp_1^{B+C}$ and that $C_H = \frac{12}{5}$ for both of these H irreps. So eliminating the $\Box$ term from the Δ_1 and G/H equations above gives

$$(\kappa_1^2 - C_G) Y_a = \frac{2m}{3} a_{abc} \nabla_b Y_c. \tag{3.10}$$

In order to extract the eigenvalues κ_1^2 from this equation we will have to square it. Let us define the operator

$$D Y_a \equiv a_{abc} \nabla_b Y_c. \tag{3.11}$$

[6] We will in the rest of this paper display eigenvalues either as κ_p^2 or, equivalently, as Δ_p.

[7] As in [1] we will use the notation of [13] which includes the parity of the field, as in, e.g., 2^+.

[8] For a different derivation see [17].

Taking the square of D then goes as follows:

$$D^2 Y_a = a_{abc}\nabla_b a_{cde}\nabla_d Y_e = a_{abc}(\nabla_b a_{cde})\nabla_d Y_e + a_{abc}a_{cde}\nabla_b\nabla_d Y_e. \tag{3.12}$$

Using identities from the appendix this equation becomes

$$D^2 Y_a = m a_{abc}c_{bcde}\nabla_d Y_e + (2\delta^{de}_{ab} + c_{ab}{}^{de})\nabla_b\nabla_d Y_e = 4mDY_a + (6m^2 - \Box)Y_a. \tag{3.13}$$

Here we have also used the Ricci identity on 1-forms $[\nabla_a, \nabla_b]Y_c = R_{abc}{}^d Y_d$ and the fact that the Riemann tensor term in the computation of D^2 above drops due to its contraction with c_{eabc}. Then since the last term in the equation for $D^2 Y_a$ above is just Δ_1, the equation can be written

$$D^2 Y_a - 4mDY_a - \kappa_1^2 Y_a = 0. \tag{3.14}$$

If we now use $DY_a = \frac{3}{2m}(\kappa_1^2 - C_G)Y_a$ and $D^2 Y_a = (\frac{3}{2m}(\kappa_1^2 - C_G))^2 Y_a$ we get the final result for the 1-form eigenvalues

$$\Delta_1 = \frac{m^2}{9}\left(20C_G + 14 \pm 2\sqrt{20C_G + 49}\right) = \frac{m^2}{9}\left(\sqrt{20C_G + 49} \pm 1\right)^2 - 4m^2. \tag{3.15}$$

The last form of the answer will be useful later.

3.2 Spin 1/2 by squaring

Before turning our attention to forms of rank two and three, and after that second rank symmetric tensors, we will check that our methods are able to produce the known result for the Dirac operator acting on spin 1/2 modes. The spectrum of $\not{D}_{1/2} \equiv -i\not\nabla$ was derived in [12] by a different method based on the fibre bundle description of the squashed seven-sphere. The virtue of the method in [12] is that one can follow the eigenvalues as one turns the squashing parameter from the round sphere value to the squashed Einstein space value. This nice feature is unfortunately lacking for the methods used in this paper.

To apply our present techniques we start by squaring the Dirac operator in order to obtain a situation that is similar to the one for the Hodge-de Rham operators on p-forms. Using $\Gamma^a\nabla_a\Gamma^b\nabla_b\psi = (\Box + \frac{1}{2}\Gamma^{ab}[\nabla_a, \nabla_b])\psi$ we find

$$-i\not\nabla\psi = \lambda\psi \Rightarrow \left(-\Box + \frac{R}{4}\right)\psi = \lambda^2\psi. \tag{3.16}$$

It is now possible to use the G_2 structure to split the tangent space spinor irrep into G_2 irreps as $\mathbf{8} \to \mathbf{7} \oplus \mathbf{1}$, or in terms of indices $A = (a, 8)$. Then by defining two spinors η and η_a we can expand a general Dirac spinor as follows

$$\psi = V^a\eta_a + f\,\eta, \tag{3.17}$$

where V^a is a vector field and f a scalar field on the seven-sphere. While η is the standard Killing spinor with components $\eta_B = \delta^8_B$, η_a is defined by $\eta_a \equiv -i\Gamma_a\eta$ which implies the its explicit form is $(\eta_a)_B = \delta_{aB}$. These spinors are linearly independent and satisfy

$$\nabla_a\eta = \frac{m}{2}\eta_a, \quad \nabla_a\eta_b = -\frac{m}{2}\delta_{ab}\eta + \frac{m}{2}a_{abc}\eta_c. \tag{3.18}$$

These equations imply

$$\Box\eta = -\frac{7m^2}{4}\eta, \quad \Box\eta_a = -\frac{7m^2}{4}\eta_a, \tag{3.19}$$

from which we obtain the equations

$$\Box(f\eta) = \left(\Box f - \frac{7m^2}{4}f\right)\eta + m(\nabla^a f)\eta_a, \tag{3.20}$$

$$\Box(V^a\eta_a) = \left(\Box V^a - \frac{7m^2}{4}V^a\right)\eta_a + m(\nabla^a V^b)a_{abc}\eta_c. \tag{3.21}$$

Consider now the coset master equation for Dirac spinor modes. Squaring it gives

$$\Box\psi - \frac{m}{6}a_{abc}\Gamma_{ab}\nabla_c\psi - \frac{7m^2}{12}\psi + \frac{m^2}{144}c_{abcd}\Gamma_{abcd}\psi = T_aT_a\psi. \tag{3.22}$$

So eliminating the $\Box$ term and inserting $R = 42m^2$ we find that the equation we need to solve reads

$$-\lambda^2\psi - \frac{m}{6}a_{abc}\Gamma_{ab}\nabla_c\psi + \frac{119m^2}{12}\psi + \frac{m^2}{144}c_{abcd}\Gamma_{abcd}\psi = T_aT_a\psi. \tag{3.23}$$

In order to get the last term on the l.h.s. in a nice form we introduce the projection operators for the G_2 split $\psi = \psi_1 + \psi_7$ corresponding to $\mathbf{8} \to \mathbf{1} \oplus \mathbf{7}$. They are (see appendix for more details)

$$P_1^s = \frac{1}{8}\left(1 - \frac{1}{24}c_{abcd}\Gamma_{abcd}\right), \quad P_7^s = 1 - P_1 = \frac{1}{8}\left(7 + \frac{1}{24}c_{abcd}\Gamma_{abcd}\right). \tag{3.24}$$

In terms of these projectors the sum of the last two terms on the l.h.s. above becomes

$$\frac{119}{12}\psi + \frac{1}{144}c_{abcd}\Gamma_{abcd}\psi = \frac{35}{4}P_1^s\psi + \frac{121}{12}P_7^s\psi. \tag{3.25}$$

Then using the octonionic version of the seven-dimensional gamma matrices given in the appendix, some algebra gives

$$a_{abc}\Gamma_{ab}\nabla_c\psi = (-6\nabla_aV^a - 21mf)\eta + (2a^{abc}\nabla_aV_b - 9mV^c + 6(\nabla^c f))\eta_c. \tag{3.26}$$

Inserting these results into (3.23) we get an equation that can be hit by the projectors P_1^s and P_7^s leading to the two equations

$$\lambda^2 f = \frac{20}{9}m^2C_Gf + \frac{35}{4}m^2f + m\left(\nabla_aV^a + \frac{7}{2}mf\right), \tag{3.27}$$

and

$$\lambda^2 V_a = \frac{20}{9}m^2\left(C_G - \frac{12}{5}\right)V_a + \left(\frac{121}{12} + \frac{3}{2}\right)m^2V_a - \frac{1}{3}ma_{abc}\nabla_bV_c - m\nabla_af. \tag{3.28}$$

To solve these equations we either have $f = 0$ or $f \neq 0$. In the former case the first equation gives $\nabla^aV_a = 0$ which implies that the second equation for V_a can be analysed in exactly

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the same way as for the 1-forms discussed previously. Not surprisingly the result is also of $\sqrt{49}$-form and reads

$$\lambda^2 = m^2\left(\frac{1}{6} \pm \frac{1}{3}\sqrt{20C_G + 49}\right)^2. \tag{3.29}$$

Turning to the latter case, that is $f \neq 0$, we start by taking the divergence of the second equation and use the fact that $\Box f = -\frac{20m^2}{9}C_G f$ to find

$$\left(\frac{20}{9}m^2C_G + \frac{25}{4}m^2 - \lambda^2\right)(\nabla_a V^a) = -\frac{20}{9}m^3 C_G\, f. \tag{3.30}$$

Now the system of two equations for the functions f and $\nabla^a V_a$ has the solutions

$$\lambda^2 = m^2\left(\frac{1}{2} \pm \frac{1}{3}\sqrt{20C_G + 81}\right)^2. \tag{3.31}$$

These two sets of eigenvalues are consistent with the known result from [12]. However, in that paper there is no sign ambiguity since there one obtains λ instead of λ^2. This problem is easily eliminated by applying our present result to the spin 2^+ supermultiplet.

This completes the review. We now turn to 2-forms where the results are new.

3.3 2-forms

In terms of the Hodge-de Rham operator on 2-forms, the equation to be solved is

$$\Delta_2 Y_{ab} = -\Box Y_{ab} - 2R_{acbd}Y^{cd} - 2R_{[a}{}^{c}Y_{b]c} = \kappa_2^2 Y_{ab}. \tag{3.32}$$

Using (2.6) for the Riemann tensor gives the following equation for the box operator

$$\Delta_2: \quad \Box Y_{ab} = -\kappa_2^2 Y_{ab} - 2W_{acbd}Y^{cd} + 10m^2 Y_{ab}. \tag{3.33}$$

As for the previous cases the next step is to express the box operator in terms of algebraic objects on the coset manifold. The master coset formula (2.1) for 2-forms reads

$$\nabla_a Y_{bc} = -T_a Y_{bc} - f_a{}^d{}_{[b}Y_{c]d}, \tag{3.34}$$

which can be written

$$\tilde{\nabla}_a Y_{bc} = -T_a Y_{bc}, \tag{3.35}$$

by indroducing the "G_2-derivative" $\tilde{\nabla}_a$ defined on 2-forms by

$$\tilde{\nabla}_a Y_{bc} \equiv \nabla_a Y_{bc} + \frac{2m}{3}a_{a[b}{}^{d}Y_{c]d}. \tag{3.36}$$

This derivative is "G_2" in the sense that it satisfies $\tilde{\nabla}_a a_{bcd} = 0$ and $\tilde{\nabla}_a c_{bcde} = 0$. This step is not crucial here but we will find more implications later of the presence of the G_2 holonomy so we will have reason to return to this derivative then. (For more details about G_2 in this context, see for instance [18].) To get an algebraic expression for the box operator we can now square the master coset equation, $\tilde{\nabla}_a Y_{bc} = -T_a Y_{bc}$, to get $\tilde{\Box} Y_{ab} = T_c T_c Y_{ab}$. This equation can also be written as

$$G/H: \quad \tilde{\Box} Y_{ab} = \left(\Box - \frac{10}{9}m^2\right)Y_{ab} + \frac{4}{3}m a_{cd[a}\nabla_{|c}Y_{d|b]} - \frac{2}{9}m^2 c_{abcd}Y_{cd} = T_c T_c Y_{ab}. \tag{3.37}$$

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Finally, to get the equation that needs to be solved to find the eigenvalues we just eliminate the box operator from (3.33) and (3.37). Using the projectors defined in the appendix gives the following useful form of the resulting equation

$$\left(\kappa_2^2 - 12m^2 + T_c T_c\right) Y_{ab} = -W_{abcd} Y^{cd} - \frac{4}{3} m^2 (P_{14} Y)_{ab} + \frac{4}{3} m \left(a_{cd[a} \tilde{\nabla}_{|c} Y_{d|b]}\right), \qquad (3.38)$$

which, again, is not entirely algebraic due to the appearance of the operator $\tilde{\nabla}_a$ in the last term. This also happened in the case of the 1-form where it was easy to handle by a squaring procedure. This trick is a bit harder to apply in the case of 2-forms (as well as for the other operators to be discussed later) as will become clear shortly. Compared to the 1-form case there is also a new feature here namely the presence of the Weyl tensor in one of the terms which will cause additional complications. The approach used in this paper to deal with the Weyl tensor terms, in this and the other cases discussed below, is explained in the appendix.

Thus there are two new issues when trying to solve the 2-form equation (3.38): the Weyl tensor term and the $\tilde{\nabla}$ term which now tends to lead to symmetrised derivatives when squared. To deal with the former issue we recall how $T_a T_a$ can be expressed in terms of the Casimir operators for the groups involved: $T_a T_a = -(C_G - C_H)$. This, however, has implications for the how the spectrum is organised in terms of towers. To see this we use the tangent space decomposition $SO(7) \to G_2 \to Sp_1^A \times Sp_1^{B+C}$ which makes it possible to read off the relevant decompositions directly from the McKay and Patera tables [19]. In the case of the 2-form the composition reads, see [2] or the summary in [1],

$$\mathbf{21} \to \mathbf{7} \oplus \mathbf{14} \to ((\mathbf{1},\mathbf{1}) \oplus (\mathbf{0},\mathbf{2})) \oplus ((\mathbf{0},\mathbf{2}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{2},\mathbf{0})). \qquad (3.39)$$

This means that the towers will be tabulated according to their H irrep which will thus obscure the G_2 structure of the spectrum. As is explained in the appendix this problem is automatically eliminated once the Weyl tensor term is analysed and the result combined with the result from the C_H term. In fact, adding these two terms gives $\frac{24}{5} Y_{ab} = \frac{32}{3} m^2 Y_{ab}$ for all the H irreps in the G_2 irrep **14**. Recall that the corresponding answer for the irrep **7** is $\frac{12}{5} = \frac{16}{3} m^2$.

Using this insight from the appendix, namely that the sum $C_H + Weyl$ has a common value on all H irreps in the decomposition of each G_2 irrep arising from the $SO(7)$ representation **21**, it becomes possible to split the 2-form equation into two by acting on it with the projectors P_7 and P_{14} and insert the respective values of $C_H + Weyl$. We find, using the definitions $Y^{(7)}_{ab} \equiv (P_7 Y)_{ab}$ and $Y_a \equiv a_{abc} Y^{(7)}_{bc}$,

$$\mathbf{7}: \qquad \kappa_2^2 Y_a = C_G Y_a + \left(12 + \frac{4}{3} - \frac{16}{3}\right) m^2 Y_a - \frac{4m}{3} (a_{abc} \nabla_b Y_c), \qquad (3.40)$$

$$\mathbf{14}: \quad \kappa_2^2 Y^{(14)}_{ab} = C_G Y^{(14)}_{ab} + \left(12 - \frac{4}{3} - \frac{32}{3}\right) m^2 Y^{(14)}_{ab} + \frac{2m}{3} (P_{14})^{cd}_{ab} (\nabla_c Y_d). \qquad (3.41)$$

Note that the last term in the second equation contains Y_a and thus seems to mix with the first equation. However, the structure of the derivative terms as a 2 by 2 matrix shows that

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this term will have no role in determining the eigenvalues for the 2-form as will be clear below. To proceed we write these two equations as

$$\mathbf{7}: \qquad \kappa_2^2 Y_a = \frac{m^2}{9}(20C_G + 72)Y_a - \frac{4}{3}m(a_{abc}\nabla_b Y_c), \tag{3.42}$$

$$\mathbf{14}: \quad \kappa_2^2 Y_{ab}^{(14)} = \frac{m^2}{9}\, 20C_G Y_{ab}^{(14)} + \frac{2}{3}m(P_{14}\nabla Y)_{ab}. \tag{3.43}$$

Consider first the possibility to take the square of the first equation above, the one for Y_a. However, although this will give rise to a calculation quite similar to the one for the 1-form, there is one important difference. While the 1-form is transverse (i.e., divergence free) this is not the case for Y_a coming from the 2-form. Therefore one needs to analyse the two equations for the 2-form in stages:

1) Either $Y_a = 0$ or $Y_a \neq 0$,

2) when $Y_a \neq 0$ then either $\nabla_a Y_a = 0$ or $\nabla_a Y_a \neq 0$.

So, when $Y_a = 0$ the equation left to solve is the **14**-equation with the derivative term set to zero. This equation therefore gives the eigenvalues

$$\Delta_2 = \frac{m^2}{9}\, 20\, C_G. \tag{3.44}$$

According to the characterisation mention above, this result leads to an energy E_0 of the $\sqrt{9}$-form. This fits nicely with a supermultiplet containing spins $(1^+, 1/2, 1/2, 0^+)$ with masses related to the operators $(\Delta_2, \not{D}_{3/2}, \not{D}_{3/2}, \Delta_L)$ since we know from ref. [198] in [2] that Δ_L also has an eigenvalue leading to a $\sqrt{9}$-form. One can check that the corresponding values of E_0 work out as they should in relation to supersymmetry. This eigenvalue of Δ_L leading to the $\sqrt{9}$-form will be derived in detail later in this section.

We now turn to the cases with $Y_a \neq 0$, namely $\nabla_a Y_a = 0$ and $\nabla_a Y_a \neq 0$. Clearly, when $\nabla_a Y_a = 0$ the calculation will follow the one for 1-forms very closely. Indeed, the result is also of $\sqrt{49}$-form and reads

$$\Delta_2 = \frac{m^2}{9}\left(\sqrt{20C_G + 49} \pm 2\right)^2 - m^2. \tag{3.45}$$

Although being of $\sqrt{49}$-form, this 2-form eigenvalue formula differs from the one for 1-forms given in [2], and also derived above,

$$\Delta_1 = \frac{m^2}{9}\left(20C_G + 14 \pm 2\sqrt{20C_G + 49}\right) = \frac{m^2}{9}\left(\sqrt{20C_G + 49} \pm 1\right)^2 - 4m^2. \tag{3.46}$$

The differences are partly compensated for by the different relations between the Hodge-de Rham operators and the respective M^2 operators [2]:

$$M^2(1^+) = \Delta_2, \tag{3.47}$$

$$M^2(1^-) = \Delta_1 + 12m^2 \pm 6m\sqrt{\Delta_1 + 4m^2} = \left(\sqrt{\Delta_1 + 4m^2} \pm 3m\right)^2 - m^2. \tag{3.48}$$

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If the Δ_1 eigenvalues are inserted into this formula for $M^2(1^-)$ we see that the last term, that is $-4m^2$ in Δ_1, is really required to have a chance for supersymmetry to work in a supermultiplet containing both 1^+ and 1^- fields. In fact, the field content of the spin 3/2 supermultiplets has this property and requires $E_0(1^+) = E_0(1^-) \pm 1$. To verify this relation we need the energy formula for spin 1 unitary SO(3, 2) irreps

$$E_0(1^\pm) = \frac{3}{2} + \frac{1}{2}\sqrt{\frac{M^2}{m^2} + 1}. \tag{3.49}$$

This expression gives the energy values

$$E_0(1^{-(1),(2)}) = \frac{3}{2} \pm \frac{3}{2} \mp \frac{1}{6} + \frac{1}{6}\sqrt{20C_G + 49}, \tag{3.50}$$

where the first $\pm$ sign refers to the two towers of spin $1^{-(1),(2)}$ fields in AdS_4 supergravity theory. Comparing these to

$$E_0(1^+) = \frac{3}{2} \pm \frac{1}{3} + \frac{1}{6}\sqrt{20C_G + 49}, \tag{3.51}$$

one finds (using also the spectrum of the spin 3/2 fields in these supermultiplets) that it is possible to eliminate the sign ambiguities in the spectra of Δ_1 and Δ_2. For more details see [3].

Finally, when $\nabla_a Y_a \neq 0$ we take the divergence of the **7** projected part of the equation and use $a_{abc}\nabla_a\nabla_b Y_c = 0$ to obtain a simple equation giving the eigenvalue

$$\kappa_2^2 = \frac{m^2}{9}(20C_G + 72) \tag{3.52}$$

This completes the analysis of the 2-form eigenvalue spectrum. A list of the obtained eigenvalues can be found in the Conclusions.

3.4 3-forms

We now turn to 3-forms leaving the discussion of the Lichnerowicz modes to the next subsection. The reason for doing the analysis in this order is that we will obtain some equations in the 3-form case that can be applied also for Lichnerowicz.

The equation for 3-forms that we wish to solve reads

$$\Delta_3 Y_{abc} = -\Box Y_{abc} + 6R^d{}_{[ab}{}^e Y_{c]de} + 3R_{[a}{}^d Y_{bc]d} = \kappa_3^2 Y_{abc}. \tag{3.53}$$

Inserting the Riemann and Ricci tensors of the squashed seven-sphere this equation becomes

$$\Delta_3: \quad \Box Y_{abc} = -\kappa_3^2 Y_{abc} + 6W^d{}_{[ab}{}^e Y_{c]de} + 12m^2 Y_{abc}. \tag{3.54}$$

Squaring the coset master equation for 3-forms gives

$$G/H: \quad -(C_G - C_H)Y_{abc} = \Box Y_{abc} - 2m a_{d[a}{}^e \tilde{\nabla}_{|d|} Y_{bc]e} + \frac{m^2}{3}(4Y_{abc} + 2c_{[ab}{}^{de} Y_{c]de}), \tag{3.55}$$

where for later convenience we have used the G_2 derivative

$$\tilde{\nabla}_d Y_{abc} \equiv \nabla_d Y_{abc} - m a_{d[a}{}^e Y_{bc]e}. \tag{3.56}$$

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Note that this implies that the divergence

$$\tilde{\nabla}^a Y_{abc} = \nabla^a Y_{abc} - \frac{m}{3}(a_{aa}{}^e Y_{bce} + 2a_{a[b}{}^e Y_{c]ae}) = \frac{2m}{3} a_{[b}{}^{de} Y_{c]de}. \tag{3.57}$$

Eliminating the box-operator term from the above equations gives

$$\kappa_3^2 Y_{abc} = C_G Y_{abc} - (C_H Y_{abc} - 6W^d{}_{[ab}{}^e Y_{c]de}) + \frac{m^2}{3}(40Y_{abc} + 2c_{[ab}{}^{de} Y_{c]de}) - 2m a_{d[a}{}^e \tilde{\nabla}^d Y_{bc]e}. \tag{3.58}$$

To proceed we decompose the 3-form into G_2 irreps as $\mathbf{35} = \mathbf{1} \oplus \mathbf{7} \oplus \mathbf{27}$. By defining $Y \equiv a_{abc} Y_{abc}$ and $Y_a \equiv c_{abcd} Y_{bcd}$ we can split the 3-form as follows

$$Y_{abc} = Y^{(1)}_{abc} + Y^{(7)}_{abc} + Y^{(27)}_{abc} = \frac{1}{42} a_{abc} Y - \frac{1}{24} c_{abcd} Y_d + (P_{27} Y)_{abc}, \tag{3.59}$$

where we have utilised the 3-form projectors defined in the appendix. The term $C_H Y_{abc} - 6W^d{}_{[ab}{}^e Y_{c]de}$ can be written (see appendix)

$$C_H Y_{abc} - 6W^d{}_{[ab}{}^e Y_{c]de} = -\frac{12}{5}\left(\frac{1}{24} c_{abcd} Y_d\right) + \frac{28}{5}(P_{27} Y)_{abc}. \tag{3.60}$$

Apart from this there are two more problematic terms, the c_{abcd}-term and the derivative term that will force us to square also this equation.

However, before doing that we will follow the strategy in the 2-form case and start by splitting the equation into G_2 pieces. The singlet term $\mathbf{1}$ is obtained by contracting all three indices by a_{abc} which gives

$$\kappa_3^2 Y = C_G Y + \frac{m^2}{3}(40Y + 2a_{abc} c_{[ab}{}^{de} Y_{c]de}) - 2m a_{abc} a_{d[a}{}^e \tilde{\nabla}^d Y_{bc]e}. \tag{3.61}$$

Cleaning up the structure constant factors this equation becomes

$$\mathbf{1}: \quad \kappa_3^2 Y = (C_G + 16m^2) Y + 2m \tilde{\nabla}_a Y_a. \tag{3.62}$$

Turning to the $\mathbf{7}$ part, we find after some algebra

$$\mathbf{7}: \quad \kappa_3^2 Y_a = (C_G + 6m^2) Y_a - \frac{12m}{7} \nabla_a Y + \frac{m}{3} a_{abc} \nabla_b Y_c + 2m a_{bcd} \nabla_b Y^{(27)}_{cda}. \tag{3.63}$$

When we now come to the last part, the $\mathbf{27}$, it will require some new steps that will relate it to the metric and the Lichnerowicz equation. This connection is in fact already indicated by the last term in the $\mathbf{7}$ equation just discussed: $a_{bcd} \nabla_b Y^{(27)}_{cda}$. This expression suggests that the 2-index tensor $a_{acd} Y^{(27)}_{bcd}$ will be useful as we now elaborate upon (see [18] for a closely related discussion). Let us define a 2-index tensor from the $\mathbf{27}$ part of the 3-form by

$$\tilde{Y}_{ab} \equiv a_{acd} Y^{(27)}_{bcd}, \tag{3.64}$$

where we use a tilde to avoid confusing it with the antisymmetric 2-form discussed previously. Clearly, the symmetric and traceless part of $\tilde{Y}_{ab}$ is also in the irrep $\mathbf{27}$. However, a nice and indeed very useful fact about this definition of $\tilde{Y}_{ab}$ is that it is automatically symmetric

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and traceless. The tracelessness follows immediately from the identity $a_{abc}c_{abcd} = 0$, or, using the 3-form projectors in the appendix, from $P_1 P_{27} = 0$, while the vanishing of its antisymmetric part follows from $a_{abc}\tilde{Y}_{bc} = 0$ and $c_{abcd}\tilde{Y}_{cd} = 0$.[9] These two results can be checked as follows:

$$a_{abc}\tilde{Y}_{bc} = a_{abc}a_{bde}Y^{(27)}_{cde} = -(2\delta^{de}_{ac} + c_{acde})Y^{(27)}_{cde} = -c_{acde}Y^{(27)}_{cde} = 0, \tag{3.65}$$

where we have used $P_7 P_{27} = 0$ in the last step, and

$$c_{abcd}\tilde{Y}_{cd} = c_{abcd}a_{cef}Y^{(27)}_{def} = -6a_{[ab}{}^{[e}\delta^{f]}_{d]}Y^{(27)}_{def} = 4a_{[a}{}^{de}Y^{(27)}_{b]de} = 0. \tag{3.66}$$

Here the very last equality is a consequence of the identity

$$a_{abc}(P_{27})_{bcd}{}^{efg} = a_{(a}{}^{[ef}\delta^{g]}_{d)} - \frac{1}{7}\delta_{ad}a^{efg}. \tag{3.67}$$

Clearly, also $a_{abc}(P_{27})_{abc}{}^{efg} = 0$ again showing that the trace of $\tilde{Y}_{ab}$ vanishes.

An alternative, and perhaps more direct, way to see that $\tilde{Y}_{ab}$ is symmetric and traceless is to just insert the expression for the P_{27} projector which gives

$$a_{abc}Y^{(27)}_{bcd} = \frac{1}{2}(a_{abc}Y_{bcd} + a_{dbc}Y_{bca}) - \frac{1}{7}\delta_{ad}Y. \tag{3.68}$$

Finally, for the definition $\tilde{Y}_{ab} \equiv a_{acd}Y^{(27)}_{bcd}$ to be useful we need to give also the inverse relation:[10]

$$Y^{(27)}_{abc} = \frac{3}{4}a_{d[ab}\tilde{Y}_{c]d}. \tag{3.69}$$

The decomposition of the 3-form therefore takes the following simple form in terms of $Y, Y_a, \tilde{Y}_{ab}$, respectively in irreps $\mathbf{1}, \mathbf{7}, \mathbf{27}$,

$$Y_{abc} = \frac{1}{42}a_{abc}Y - \frac{1}{24}c_{abcd}Y_d + \frac{3}{4}a_{d[ab}\tilde{Y}_{c]d}. \tag{3.70}$$

With these preliminary results at hand it is now a rather straightforward matter to project the equation (3.58) onto its $\mathbf{27}$ component. One finds

$$\begin{aligned}\kappa_3^2\tilde{Y}_{ab} = &\left(C_G - \frac{m^2}{3}\left(40 - \frac{4}{3}\right)\right)\tilde{Y}_{ab} - \left(C_H\delta^{cd}_{(ab)} + 2W_{acbd}\right)\tilde{Y}_{cd} \\ &- \frac{2}{3}ma_{cd(a}\tilde{\nabla}^c\tilde{Y}_{b)}{}^d - \frac{m}{3}\left(\tilde{\nabla}_{(a}Y_{b)} - \frac{1}{7}\delta_{ab}\tilde{\nabla}^c Y_c\right).\end{aligned} \tag{3.71}$$

Again we can replace the second bracket by its common eigenvalue on all H irreps in the G_2 irrep $\mathbf{27}$ as explained in the appendix, that is with $\frac{28}{5} = \frac{112m^2}{9}$.

We now insert this into the $\mathbf{27}$ equation and sum up what we have found so far:

$$\mathbf{1}: \quad \kappa_3^2 Y = (C_G + 16m^2)Y + 2m\tilde{\nabla}_a Y_a, \tag{3.72}$$

$$\mathbf{7}: \quad \kappa_3^2 Y_a = (C_G + 6m^2)Y_a - \frac{12m}{7}\nabla_a Y + \frac{m}{3}a_{abc}\nabla_b Y_c + 2ma_{bcd}\nabla_b Y^{(27)}_{cda}. \tag{3.73}$$

$$\mathbf{27}: \quad \kappa_3^2\tilde{Y}_{ab} = \left(C_G + \frac{4m^2}{9}\right)\tilde{Y}_{ab} - \frac{2}{3}ma_{cd(a}\tilde{\nabla}^c\tilde{Y}_{b)}{}^d - \frac{m}{3}\left(\tilde{\nabla}_{(a}Y_{b)} - \frac{1}{7}\delta_{ab}\tilde{\nabla}^c Y_c\right). \tag{3.74}$$

[9] Note that these two conditions can be combined to the identity since $\delta^{cd}_{ab} = \frac{1}{2}(a_{abe}a^{cde} - c_{ab}{}^{cd})$.

[10] Here one may use the fact that $Y^{(27)}_{abc} = (P_{27}Y^{(27)})_{abc}$ implies the identity $Y^{(27)}_{abc} = -\frac{3}{2}c_{de[ab}Y^{(27)}_{c]de}$.

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Note that the term $2ma_{bcd}\nabla_b Y^{(27)}_{cda}$ appearing in the **7** equation can be replaced by terms containing only Y and Y_a. This relation is derived from the gauge condition $\nabla^a Y_{abc} = 0$ as follows:

$$\nabla^a (Y^1_{abc} + Y^7_{abc} + Y^{27}_{abc}) = 0. \tag{3.75}$$

Contracting this equation with a_{dbc} and using $\nabla_a a_{bcd} = mc_{abcd}$ we get

$$\nabla^a a_{dbc}(Y^1_{abc} + Y^7_{abc} + Y^{27}_{abc}) - mc_{adbc}(Y^1_{abc} + Y^7_{abc} + Y^{27}_{abc}) = 0. \tag{3.76}$$

However, since c_{abcd} projects onto Y_a we can use the relations

$$a_{dbc}Y^1_{abc} = \frac{1}{42}a_{dbc}a_{abc}Y = \frac{1}{7}\delta_{da}Y, \quad a_{dbc}Y^7_{abc} = -\frac{1}{24}a_{dbc}c_{abce}Y_e = -\frac{1}{6}a_{dab}Y_b, \quad a_{dbc}Y^{27}_{abc} = \tilde{Y}_{da}, \tag{3.77}$$

to rewrite the above 3-form gauge condition as

$$\nabla^a \tilde{Y}_{ab} + \frac{1}{7}\nabla_b Y - \frac{1}{6}a_{bcd}\nabla_c Y_d + mY_b = 0. \tag{3.78}$$

This equation will be used when we summarise the system of equations to be solved below.

Finally, we show that the last bracket in the **27** equation will not play a role in the analysis of the spectrum. This can be seen by writing the three equations above in matrix form as

$$\kappa_3^2 \begin{pmatrix} Y^1 \\ Y^7 \\ Y^{27} \end{pmatrix} = \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & E & F \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^7 \\ Y^{27} \end{pmatrix} \tag{3.79}$$

The eigenvalues must be the roots of the equation $\det(X - \kappa_3^2 \mathbf{1}) = 0$, where X is the matrix appearing above. Since the element E does not enter this equation we can proceed and solve the three equations in two steps, first the two coupled equations for Y and Y_a

$$\mathbf{1}: \quad \kappa_3^2 Y = (C_G + 16m^2)Y + 2m\nabla_a Y_a, \tag{3.80}$$

$$\mathbf{7}: \quad \kappa_3^2 Y_a = (C_G + 4m^2)Y_a - 2m\nabla_a Y + \frac{2m}{3}a_{abc}\nabla_b Y_c, \tag{3.81}$$

and then the single equation involving only $\tilde{Y}_{ab}$

$$\mathbf{27}: \quad \kappa_3^2 \tilde{Y}_{ab} = \left(C_G + \frac{4m^2}{9}\right)\tilde{Y}_{ab} - \frac{2m}{3}a_{cd(a}\tilde{\nabla}^c \tilde{Y}_{b)}{}^d. \tag{3.82}$$

We start by solving the first two coupled equations. There are two distinct cases: either 1) $Y = 0$ or 2) $Y \neq 0$.

Case 1). Setting $Y = 0$ in the **1** equation gives $\nabla_a Y_a = 0$ which implies that the **7** equation is of exactly of the same type as the equation solved previously for 1-forms, but with different coefficients. However, one has to pay attention to the fact that the eigenvalues dealt with here are for 3-forms, not 1-forms, so solving the present equation for

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Y_a involves some new steps. This fact will become clear directly when squaring the operator $(DY)_a \equiv a_{abc}\nabla_b Y_c$:

$$
\begin{aligned}
(DDY)_a :&= a_{abc}\nabla_b a_{cde}\nabla_d Y_e = a_{abc}a_{cde}\nabla_b\nabla_d Y_e + m a_{abc}c_{bcde}\nabla_d Y_e \\
&= (2\delta^{de}_{ab} + c_{abde})\nabla_b\nabla_d Y_e + 4m a_{ade}\nabla_d Y_e.
\end{aligned} \tag{3.83}
$$

Since the c-term vanishes this equation simplifies to

$$
(DDY)_a = 4m(DY)_a - \Box Y_a + 6m^2 Y_a. \tag{3.84}
$$

At this point the $\Box Y_a$ term must be related to the 3-form eigenvalue κ_3^2. Using $Y = 0$ and $\nabla^a Y_{abc} = 0$, we find that

$$
\begin{aligned}
\Box Y_a &= \Box c_{abcd} Y_{bcd} = c_{abcd}\Box Y_{bcd} - 2m^2 Y_a = \\
&= c_{abcd}\left(-\kappa_3^2 Y_{bcd} + 6W^e{}_{[bc}{}^f Y_{d]ef} + 12m^2 Y_{bcd}\right) - 2mY_a = -(\kappa_3^2 - 10m^2)Y_a,
\end{aligned} \tag{3.85}
$$

where we have also used $c_{abcd}W^e{}_{[bc}{}^f Y_{d]ef} = W_{adef}Y_{def} = 0$ (where the first equality follows from $P_7^{(2)}W = 0$). The $(DDY)_a$ equation above then reads

$$
(DDY)_a = 4m(DY)_a + (\kappa_3^2 - 4m^2)Y_a. \tag{3.86}
$$

Inserting the expression for $(DY)_a$ from the eigenvalue equation **7** above we find

$$
(\kappa_3^2 - C_G - 4m^2)^2 - \frac{8m^2}{3}(\kappa_3^2 - C_G - 4m^2) - \frac{4m^2}{9}(\kappa_3^2 - 4m^2) = 0, \tag{3.87}
$$

which has the following two solutions (inserting $1 = \frac{20}{9}m^2$ in front of C_G)

$$
\Delta_3 = \frac{m^2}{9}\left(\sqrt{20C_G + 49} \pm 1\right)^2. \tag{3.88}
$$

These eigenvalues are obtained for $\Delta_3 = Q^2$ so it is gratifying to find its eigenvalues come out as a square.

Case 2). Now we turn to the second case where $Y \neq 0$. Then either $\nabla_a Y_a = 0$ or $\nabla_a Y_a \neq 0$. The former case gives a simple equation for the eigenvalue directly from the **1** equation. However, taking the divergence of the **7** equation tells us that $\Box Y = 0$ so the **1** equation reduces to

$$
\kappa_3^2 = 16m^2, \tag{3.89}
$$

which is the eigenvalue of the singlet constant mode $Y_{abc} = a_{abc}$ as is easily verified. So let us turn to the latter case with $\nabla_a Y_a \neq 0$. Taking the divergence of the **7** equation gives

$$
\Box Y = \frac{1}{2m}(-\kappa_3^2 + C_G + 4m^2)\nabla_a Y_a. \tag{3.90}
$$

Inserting this equation back into the **1** equation, recalling that $\Box Y = -C_G Y$, gives

$$
(\kappa_3^2 - C_G - 16m^2)(\kappa_3^2 - C_G - 4m^2) = \frac{9}{5}C_G. \tag{3.91}
$$

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This equation has the following solutions (inserting again $1 = \frac{20}{9}m^2$ in front of C_G):

$$\Delta_3 = \frac{m^2}{9}\left(\sqrt{20C_G + 81} \pm 3\right)^2 . \tag{3.92}$$

Note that the single eigenvalue found above, $\kappa_3^2 = 16m^2$, belongs to the plus branch.

We now turn to the **27** part of the 3-form equation

$$\mathbf{27}: \quad \kappa_3^2 \tilde{Y}_{ab} = \left(C_G + \frac{4m^2}{9}\right)\tilde{Y}_{ab} + \frac{2m}{3} a_{c(a}{}^d \tilde{\nabla}^c \tilde{Y}_{|d|b)}. \tag{3.93}$$

The derivative term requires as usual a squaring of the whole equation. The G_2 derivative on $\tilde{Y}_{ab}$ is given by

$$(D\tilde{Y})_{ab} \equiv 2a_{cd(a}\tilde{\nabla}^c \tilde{Y}_{b)}{}^d = \tilde{\nabla}^c\left(a_{ca}{}^d \tilde{Y}_{db} + a_{cb}{}^d \tilde{Y}_{ad}\right), \tag{3.94}$$

and computing its square, using the fact that $\tilde{\nabla}_a$ is zero on both a_{abc} and c_{abcd}, gives

$$(DD\tilde{Y})_{ab} \equiv \tilde{\nabla}^e\tilde{\nabla}^f\left(a_{ea}{}^c a_{fc}{}^d \tilde{Y}_{db} + a_{eb}{}^c a_{fc}{}^d \tilde{Y}_{ad} + a_{ea}{}^c a_{fb}{}^d \tilde{Y}_{cd} + a_{eb}{}^c a_{fa}{}^d \tilde{Y}_{cd}\right). \tag{3.95}$$

Clearly, the last two terms in this expression contain a symmetric combination of the two G_2 covariant derivatives. Such terms can be a real obstacle to carrying through the calculation but we will see below that there is a trick that can be used to eliminate this issue. Before applying this trick we simplify the other terms to get

$$(DD\tilde{Y})_{ab} = -2\tilde{\Box}\tilde{Y}_{ab} + 2\tilde{\nabla}^c\tilde{\nabla}_{(a}\tilde{Y}_{b)c} - 2c_{cde(a}\tilde{\nabla}^c\tilde{\nabla}^d\tilde{Y}_{b)}{}^e + 2a_{e(a}{}^c a_{|f|b)}{}^d \tilde{\nabla}^e\tilde{\nabla}^f\tilde{Y}_{cd}. \tag{3.96}$$

We now address the four terms on the r.h.s. of this equation starting with last one. The trick used to deal with the symmetric combination of derivatives in this term is to reintroduce the 3-form via $Y^{(27)}_{abc} = \frac{3}{4}a_{d[ab}\tilde{Y}_{c]d}$ temporarily giving

$$a_{e(a}{}^c a_{|f|b)}{}^d \tilde{\nabla}^e\tilde{\nabla}^f\tilde{Y}_{cd} = a_{e(a}{}^c \tilde{\nabla}^e\tilde{\nabla}^f a_{|f|b)}{}^d \tilde{Y}_{cd} = a_{e(a}{}^c \tilde{\nabla}^e\tilde{\nabla}^f\left(4Y^{(27)}_{|f|b)c} - a_{b)c}{}^d \tilde{Y}_{fd} - a_{|cf|}{}^d \tilde{Y}_{b)d}\right). \tag{3.97}$$

The first term in the last expression is just the G_2 covariant divergence of the 3-form which can be seen to satisfy

$$\tilde{\nabla}^a Y^{(27)}_{abc} = \nabla^a Y^{(27)}_{abc} = \nabla^a Y_{abc} = 0, \tag{3.98}$$

where in the first equality we have used the fact that $\tilde{Y}_{ab}$ is symmetric and in the second that $\nabla^a Y^{(1)}_{abc} = \nabla^a Y^{(7)}_{abc} = 0$. Thus $\nabla^a Y_{abc} = 0$ is implies $\tilde{\nabla}^a\tilde{Y}_{ab} = 0$ which gives the following much simpler form of the expression above

$$a_{e(a}{}^c a_{|f|b)}{}^d \tilde{\nabla}^e\tilde{\nabla}^f\tilde{Y}_{cd} = -a_{e(a}{}^c \tilde{\nabla}^e\tilde{\nabla}^f a_{|cf|}{}^d \tilde{Y}_{b)d} = -\tilde{\Box}\tilde{Y}_{ab} + \tilde{\nabla}^c\tilde{\nabla}_{(a}\tilde{Y}_{b)c} + c_{(a}{}^{cde}\tilde{\nabla}^c\tilde{\nabla}^d\tilde{Y}_{b)}{}^e. \tag{3.99}$$

Inserting this result into the above expression for $(DD\tilde{Y})_{ab}$ we get

$$(DD\tilde{Y})_{ab} = 4\left(-\tilde{\Box}\tilde{Y}_{ab} + \tilde{\nabla}^c\tilde{\nabla}_{(a}\tilde{Y}_{b)c} - c_{cde(a}\tilde{\nabla}^c\tilde{\nabla}^d\tilde{Y}_{b)}{}^e\right). \tag{3.100}$$

Thus we have managed to eliminate the symmetric combination of covariant derivatives that has caused a bit of headache until now. The terms in the last formula for $(DD\tilde{Y})_{ab}$

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can be dealt with rather easily as we now show. The first term is simply the coset master equation squared, i.e.,

$$\tilde{\Box}\tilde{Y}_{ab} = -(C_G - C_H)\tilde{Y}_{ab}. \tag{3.101}$$

The other two terms both involve the commutator of two covariant derivatives (since $\tilde{\nabla}^a\tilde{Y}_{ab} = 0$). So we need to compute

$$[\tilde{\nabla}_c, \tilde{\nabla}_d]\tilde{Y}_{ab} = \tilde{\nabla}_c\left(\nabla_d\tilde{Y}_{ab} - \frac{m}{3}a_{da}{}^e\tilde{Y}_{eb} - \frac{m}{3}a_{db}{}^e\tilde{Y}_{ae}\right) - (c \leftrightarrow d). \tag{3.102}$$

Expanding out the terms we find

$$\begin{aligned} \nabla_c\nabla_d\tilde{Y}_{ab} - \frac{m}{3}a_{cd}{}^e\nabla_e\tilde{Y}_{ab} - \frac{m}{3}a_{ca}{}^e\nabla_d\tilde{Y}_{eb} - \frac{m}{3}a_{cb}{}^e\nabla_d\tilde{Y}_{ae} \\ - \frac{m}{3}a_{da}{}^e\tilde{\nabla}_c\tilde{Y}_{eb} - \frac{m}{3}a_{db}{}^e\tilde{\nabla}_c\tilde{Y}_{ae} - (c \leftrightarrow d), \end{aligned} \tag{3.103}$$

and finally

$$\begin{aligned} [\nabla_c, \nabla_d]\tilde{Y}_{ab} - \frac{2m}{3}a_{cd}{}^e\nabla_e\tilde{Y}_{ab} - \frac{2m^2}{9}\Big(a_{[c|a|}{}^e a_{d]e}{}^f\tilde{Y}_{fb} + a_{[c|a|}{}^e a_{d]b}{}^f\tilde{Y}_{ef} \\ + a_{[c|b|}{}^e a_{d]a}{}^f\tilde{Y}_{fe} + a_{[c|b|}{}^e a_{d]e}{}^f\tilde{Y}_{af}\Big). \end{aligned} \tag{3.104}$$

Thus, all single derivative terms cancel except one. Noting also that the two non-derivative terms in the middle of the bracket cancel the commutator becomes

$$[\tilde{\nabla}_c, \tilde{\nabla}_d]\tilde{Y}_{ab} = [\nabla_c, \nabla_d]\tilde{Y}_{ab} - \frac{2m}{3}a_{cd}{}^e\nabla_e\tilde{Y}_{ab} - \frac{4m^2}{9}\, a_{[c}{}^{(a|e|}a_{d]e}{}^{|f|}\tilde{Y}_f{}^{b)}. \tag{3.105}$$

Simplifying the non-derivative terms finally gives

$$[\tilde{\nabla}_c, \tilde{\nabla}_d]\tilde{Y}_{ab} = [\nabla_c, \nabla_d]\tilde{Y}_{ab} - \frac{2m}{3}a_{cd}{}^e\nabla_e\tilde{Y}_{ab} + \frac{4m^2}{9}\left(\delta^{[c}_{(a}\tilde{Y}_{b)}{}^{d]} - c_{cd(a}{}^e\tilde{Y}_{b)e}\right). \tag{3.106}$$

Since we are interested in expressing the right hand side in terms of the D operator defined above using $\tilde{\nabla}_a$ we rewrite the last equation as

$$[\tilde{\nabla}_c, \tilde{\nabla}_d]\tilde{Y}_{ab} = [\nabla_c, \nabla_d]\tilde{Y}_{ab} - \frac{2m}{3}a_{cd}{}^e\tilde{\nabla}_e\tilde{Y}_{ab} - \frac{4m^2}{9}\left(\delta^{[c}_{(a}\tilde{Y}_{b)}{}^{d]} + 2c_{cd(a}{}^e\tilde{Y}_{b)e}\right). \tag{3.107}$$

To get the final expression that will be useful here we replace the commutator on the right hand side by the Riemann tensor. This gives

$$[\tilde{\nabla}_c, \tilde{\nabla}_d]\tilde{Y}_{ab} = 2W_{cd(a}{}^e\tilde{Y}_{b)e} - \frac{2m}{3}a_{cd}{}^e\tilde{\nabla}_e\tilde{Y}_{ab} + \frac{32m^2}{9}\left(\delta^{[c}_{(a}\tilde{Y}_{b)}{}^{d]} - \frac{1}{4}c_{cd(a}{}^e\tilde{Y}_{b)e}\right). \tag{3.108}$$

The first term we need to compute in the $DD\tilde{Y}$ equation contains the contracted expression $[\tilde{\nabla}^b, \tilde{\nabla}_d]\tilde{Y}_{ab}$. Setting $b = c$ in the last equation above we get

$$[\tilde{\nabla}^c, \tilde{\nabla}_a]\tilde{Y}_{bc} = W^c{}_{ab}{}^e\tilde{Y}_{ce} - \frac{2m}{3}a^c{}_a{}^e\tilde{\nabla}_e\tilde{Y}_{bc} + \frac{56m^2}{9}\tilde{Y}_{ab}. \tag{3.109}$$

The second term in the $DD\tilde{Y}$ we need is the one with a contraction of the commutator with the c symbol:

$$c_{cd(a}{}^f[\tilde{\nabla}_c, \tilde{\nabla}_d]\tilde{Y}_{b)f} = -2W_{(a}{}^e{}_{b)}{}^f\tilde{Y}_{ef} + \frac{8m}{3}a_{(a}{}^{ef}\tilde{\nabla}^e\tilde{Y}_{b)}{}^f + \frac{112m^2}{9}\tilde{Y}_{ab}. \tag{3.110}$$

Inserting the two results in (3.109) and (3.110) into (3.100) we find

$$\frac{1}{4}(DD\tilde{Y})_{ab} = -\tilde{\Box}\tilde{Y}_{ab} + \frac{m}{3}(D\tilde{Y})_{ab} + \frac{112m^2}{9}\tilde{Y}_{ab} - 2W_{(a}{}^e{}_{b)}{}^f\tilde{Y}_{ef}. \tag{3.111}$$

Replacing the G_2 covariant box with Casimirs gives

$$\frac{1}{4}(DD\tilde{Y})_{ab} = C_G Y_{ab} + \frac{m}{3}(D\tilde{Y})_{ab} + \frac{112m^2}{9}\tilde{Y}_{ab} - \left(C_H\tilde{Y}_{ab} + 2W_{(a}{}^e{}_{b)}{}^f\tilde{Y}_{ef}\right). \tag{3.112}$$

Then from the appendix we know that the last bracket gives $\frac{112m^2}{9}$ which implies the amazingly simple equation

$$(DD\tilde{Y})_{ab} - \frac{4m}{3}(D\tilde{Y})_{ab} - 4C_G\tilde{Y}_{ab} = 0. \tag{3.113}$$

In view of the eigenvalue equation (3.93), we may express the "solution" to the last equation as

$$D\tilde{Y} = \frac{2m}{3} \pm 2\sqrt{C_G + \frac{m^2}{9}} = \frac{2m}{3}\left(1 \pm \sqrt{20C_G + 1}\right). \tag{3.114}$$

Then by replacing $(D\tilde{Y})_{ab}$ with the expression coming from the 3-form (3.93), we find

$$\mathbf{27}: \quad \kappa_3^2 = \frac{m^2}{9}\left(20C_G + 2 \pm 2\sqrt{20C_G + 1}\right). \tag{3.115}$$

Since κ_3^2 is the eigenvalue of $\Delta_3 = Q^2$ this must be a square. Indeed, it can be written

$$\mathbf{27}: \quad \Delta_3 = \frac{m^2}{9}\left(\sqrt{20C_G + 1} \pm 1\right)^2. \tag{3.116}$$

3.5 Lichnerowicz

When we now turn to the transverse and traceless metric modes h_{ab} on the squashed seven-sphere we can take advantage of the results obtained in the previous case of the 3-form. To see how to do this let us write out the Lichnerowicz equation explicitly

$$\Delta_L: \quad \Delta_L h_{ab} = -\Box h_{ab} - 2W_{acbd}h^{cd} + 14m^2 h_{ab} = \kappa_L^2 h_{ab}. \tag{3.117}$$

The coset master equation reads in this case

$$\nabla_a h_{bc} + \frac{2m}{3}a_{ad(b}h_{c)d} = -T_a h_{bc}, \tag{3.118}$$

which when squared gives

$$G/H: \quad \Box h_{ab} + \frac{4m}{3}a_{cd(a}\nabla^c h_{b)}{}^d - \frac{14m^2}{9}h_{ab} = T_cT_c h_{ab}. \tag{3.119}$$

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Eliminating the box operator from the above Δ_L and G/H equations and using $T_cT_c = -(C_H - C_H)$ gives

$$\kappa_L^2 h_{ab} = C_H h_{ab} - (C_H h_{ab} + 2W_{acbd}h^{cd}) + \left(14 + \frac{14}{9}\right) m^2 h_{ab} + \frac{4m}{3} a_{cd(a}\tilde{\nabla}^c h_{b)}{}^d, \tag{3.120}$$

where we have used the G_2 covariant derivative

$$\tilde{\nabla}_a h_{bc} = \nabla_a h_{bc} - \frac{2m}{3} a_{a(b}{}^d h_{c)}{}^d. \tag{3.121}$$

As in the previous cases we now use the result for the Weyl tensor term from the appendix, i.e., that $C_H + 2W$ acting on the different H irreps in **27** gives the universal value $\frac{28}{5} = \frac{112}{5}m^2$. This gives us the rather simple equation

$$\kappa_L^2 h_{ab} = \frac{m^2}{9}(20C_G + 28)h_{ab} + \frac{4m}{3} a_{cd(a}\tilde{\nabla}^c h_{b)}{}^d. \tag{3.122}$$

There is one crucial difference between the Lichnerowicz modes and the 3-form modes analysed in the last subsection: transversality of the metric modes does not imply that the associated 3-form $Y_{abc} \equiv \frac{4}{3} a_{d[ab}h_{c]d}$ is transverse (recall the result $\tilde{\nabla}^a \tilde{Y}_{ab} = 0$ derived in the previous subsection). Thus when squaring $(Dh)_{ab} \equiv 2a_{cd(a}\tilde{\nabla}^c h_{b)}{}^d$ we cannot simply use (3.113) since this equation was derived with the assumption of a transverse $Y^{(27)}_{abc}$. The change is however relatively small; we only need to add the term in (3.97) proportional to $\nabla^a Y^{(27)}_{abc}$ to proceed. We find

$$(DDh)_{ab} - \frac{4m}{3}(Dh)_{ab} - 4C_G h_{ab} - 8a_{cd(a}\tilde{\nabla}^c \tilde{\nabla}^e Y^{(27)}_{|ed|b)} = 0. \tag{3.123}$$

To deal with the last term we apply the D operator yet another time. To simplify the computation we introduce the notation $Y_{ab} \equiv \tilde{\nabla}^e Y_{eab} = \nabla^e Y_{eab}$ where $\nabla^a Y_{ab} = 0$ and define $(DY)_{ab} \equiv 2a_{cd(a}\tilde{\nabla}^c Y_{|d|b)} = 2a_{cd(a}\nabla^c Y_{|d|b)}$. It is then immediately found that

$$(DDY)_{ab} = 2a_{cd(a}\tilde{\nabla}^c (DY)_{|d|b)} = -\frac{14m}{3}(DY)_{ab} + 2a_{cd(a}\nabla^c (DY)_{|d|b)} \tag{3.124}$$

Expanding out the nontrivial last term on the right side gives

$$2a_{cd(a}\nabla^c (DY)_{|d|b)} = 2a_{cd(a}\nabla^c a_{|efd}\nabla^e Y_{f|b)} + 2a_{cd(a}\nabla^c a_{|ef|b)}\nabla^e Y_{fd}\,. \tag{3.125}$$

We start by analysing the first term in (3.125). When the first covariant derivative passes a_{efd} we need to compute

$$\begin{aligned} a_{cd}{}^{(a} a_{efd}\nabla_c\nabla_e Y_f{}^{b)} &= -2\delta^{c(a}_{ef}\nabla_c\nabla_e Y_f{}^{b)} - c_{cef}{}^{(a}\nabla_c\nabla_e Y_f{}^{b)} \\ &= \nabla_f \nabla^{(a} Y_f{}^{b)} = 6mY^{(ab)} + R^{f(ab)g}Y_{fg} \\ &= 0 \end{aligned} \tag{3.126}$$

In the second equality we have used the antisymmetry of Y_{ab} and the fact that the 2-form projector P_7 vanishes when acting on the Weyl tensor (which implies that $c_{ab}{}^{cd} \propto \delta^{cd}_{ab}$). The last equality is true due to the antisymmetry of Y_{ab} together with the symmetrisation (ab). When the covariant derivative acts on a_{efg} we find

$$ma_{cd}{}^{(a}c_{cefd}\nabla^e Y_f{}^{b)} = 4ma_{ef}{}^{(a}\nabla^{|e|}Y_f{}^{b)} = 2m(DY)_{ab}. \tag{3.127}$$

We conclude that

$$2a_{cd(a}\nabla^c a_{|efd}\nabla^e Y_{f|b)} = 4m(DY)_{ab}. \tag{3.128}$$

We then turn to the second term in (3.125). Here the covariant derivative can act in two different ways. First, moving the derivative past a_{efb} gives the expression

$$a_{cd}{}^{(a}a_{ef}{}^{b)}\nabla^c\nabla^e Y_{fd} = \frac{1}{2}a_{cd}{}^{(a}a_{ef}{}^{b)}[\nabla^c,\nabla^e]Y_{fd}\,. \tag{3.129}$$

The action of the antisymmetrized covariant derivatives will give two terms which are essentially identical and we only show how to deal with the first one. Using the squashed sphere Riemann tensor it follows that

$$\begin{aligned} a_{cd}{}^{(a}a_{ef}{}^{b)}R^{ce}{}_{fg}Y^g{}_d &= 2m^2\delta^{ce}_{fg}\,a_{cd}{}^{(a}a_{ef}{}^{b)}Y^g{}_d + a_{cd}{}^{(a}a_{ef}{}^{b)}W^{ce}{}_{fg}Y^g{}_d \\ &= m^2 a_{fd}{}^{(a}a_{gf}{}^{b)}Y^g{}_d \\ &= 0, \end{aligned} \tag{3.130}$$

where we have used that $a_{efb}W^{ce}{}_{fg} = -\frac{1}{2}a_{efb}W^{cg}{}_{ef} = 0$ (see the appendix). The other term vanishes in exactly the same way. Finally, we have a term coming from the covariant derivative hitting a_{efb}. After some algebra, this term becomes

$$2ma_{cd}{}^{(a}c_{cef}{}^{b)}\nabla^e Y_{fd} = 2ma_{cd}{}^{(a}\nabla_c Y_d{}^{b)} = 2m(DY)_{ab}, \tag{3.131}$$

and so $a_{cd}{}^{(a}a_{ef}{}^{b)}\nabla^c\nabla^e Y_{fd} = 2m(DY)_{ab}$. Putting these results together we find that

$$(DDY)_{ab} = \left(6m - \frac{14m}{3}\right)(DY)_{ab} = \frac{4m}{3}(DY)_{ab}. \tag{3.132}$$

Having calculated the action of D on $(DY)_{ab}$ we can apply D on (3.123) and then subtracting the previous equation to eliminate the $(DY)_{ab}$ terms. From this procedure we find the third order equation

$$\left(D - \frac{4m}{3}\right)\left(D^2 h - \frac{4m}{3}Dh - 4C_G h\right) = 0. \tag{3.133}$$

"Solving" for $(Dh)_{ab}$ and plugging the result back into (3.122) gives the following three different eigenvalues:

$$\Delta_L = \frac{m^2}{9}(20C_G + 36), \quad \Delta_L = \frac{m^2}{9}\left(20C_G + 32 \pm 4\sqrt{20C_G + 1}\right). \tag{3.134}$$

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4 Conclusions

Let us summarise what we know so far about the spectrum of operators on the squashed seven-sphere including the new results for Δ_2 and Δ_3 obtained in this paper. The previously known eigenvalues for Δ_0, $\not{D}_{1/2}$ [12], Δ_1 [17], and Δ_L [2] are

$$\Delta_0 = \frac{m^2}{9} 20C_G, \tag{4.1}$$

$$\not{D}_{1/2} = -\frac{m}{2} \pm \frac{m}{3}\sqrt{20C_G + 81}, \tag{4.2}$$

$$\not{D}_{1/2} = \frac{m}{6} \pm \frac{m}{3}\sqrt{20C_G + 49}, \tag{4.3}$$

$$\Delta_1 = \frac{m^2}{9}\left(20C_G + 14 \pm 2\sqrt{20C_G + 49}\right) = \frac{m^2}{9}\left(\sqrt{20C_G + 49} \pm 1\right)^2 - 4m^2, \tag{4.4}$$

$$\Delta_L = \frac{m^2}{9}(20C_G + 36), \tag{4.5}$$

$$\Delta_L = \frac{m^2}{9}\left(20C_G + 32 \pm 4\sqrt{20C_G + 1}\right) = \frac{m^2}{9}\left(\sqrt{20C_G + 1} \pm 2\right)^2 + 3m^2, \tag{4.6}$$

while the new ones obtained in this paper are

$$\Delta_2 = \frac{m^2}{9}(20C_G + 72), \tag{4.7}$$

$$\Delta_2 = \frac{m^2}{9}\left(20C_G + 44 \pm 4\sqrt{20C_G + 49}\right) = \frac{m^2}{9}\left(\sqrt{20C_G + 49} \pm 2\right)^2 - m^2, \tag{4.8}$$

$$\Delta_2 = \frac{m^2}{9} 20C_G, \tag{4.9}$$

$$\Delta_3 = \frac{m^2}{9}\left(\sqrt{20C_G + 49} \pm 1\right)^2, \tag{4.10}$$

$$\Delta_3 = \frac{m^2}{9}\left(\sqrt{20C_G + 81} \pm 3\right)^2, \tag{4.11}$$

$$\Delta_3 = \frac{m^2}{9}\left(\sqrt{20C_G + 1} \pm 1\right)^2. \tag{4.12}$$

Here it is appropriate to make some comments on the limitations of the obtained results. First, we have so far no results for the eigenvalues of the spin 3/2 operator $\not{D}_{3/2}$, although some can easily be extracted from supersymmetry. Secondly, for Δ_2 and Δ_L we seem to lack some eigenvalues. This is indicated by the degeneracies in the cross diagrams for these two operators derived in [1], as well as by supersymmetry. For Δ_3 there may already be too many available eigenvalues but if one is supposed to pick only one sign when removing the square (as done for $\not{D}_{1/2}$) one is instead lacking two eigenvalues. These problematic features are partly due to the fact that although we have extensive knowledge of the eigenvalue spectra from the list above, the method applied here does not provide direct information how to associate these eigenvalues[11] with the cross diagrams of [1]. These and other issues will be elaborated upon in a forthcoming publication [3].

[11] By assuming that the eigenvalues obtained at this point fill out entire supermultiplets the spin 3/2 eigenvalues can be derived although it is, as explained here, not clear which supermultiplets these are.

The results of this paper clearly demonstrate the important role of weak G_2 holonomy when solving the eigenvalue equations. There are, however, deeper issues in the context of holonomy and string/M theory that might be interesting to study in relation to the squashed seven-sphere, for instance the notion of generalised holonomy discussed in [20].

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A Octonions: conventions and some identities

The octonions satisfy a non-associative algebra defined by the totally antisymmetric structure constants a_{abc} $(a, b, c, \ldots = 0, 1, \ldots, 6)$:

$$a_{abc} = 1 \text{ for } abc = 456, 041, 052, 063, 162, 135, 243. \tag{A.1}$$

By splitting $a = (0, i, \hat{i})$ (with $i = 1, 2, 3, \hat{i} = 4, 5, 6 = \hat{1}, \hat{2}, \hat{3}$) they can be written more compactly as

$$a_{0i\hat{j}} = -\delta_{ij}, \quad a_{ij\hat{k}} = -\epsilon_{ijk}, \quad a_{\hat{i}\hat{j}\hat{k}} = \epsilon_{ijk}. \tag{A.2}$$

We define its dual, denoted c_{abcd}, by

$$c_{abcd} = \frac{1}{6}\epsilon_{abcdefg}\, a_{efg}, \tag{A.3}$$

where the epsilon tensor is totally antisymmetric with $\epsilon_{0123456} = 1$.

Sometimes it is convenient to write the gamma matrices Γ_a in seven dimensions in terms of the structure constants defined above and a Killing spinor η satisfying

$$\nabla_a \eta = -\frac{i}{2} m \Gamma_a \eta, \quad \bar{\eta}\eta = 1. \tag{A.4}$$

The conventions used here are

$$a_{abc} = i\bar{\eta}\Gamma_{abc}\eta, \quad c_{abcd} = -\bar{\eta}\Gamma_{abcd}\eta, \tag{A.5}$$

and

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}, \quad -i\Gamma_{abcdefg} = \epsilon_{abcdefg}\mathbf{1}. \tag{A.6}$$

These gamma matrices are 8×8 matrices with spinor indices $A, B, \ldots$ taking the eight values $A = (a, 8)$ etc. The consistency of these conventions can then be verified by explicitly writing out the gamma matrices in terms of the structure constants as follows:[12]

$$(\Gamma_a)_B{}^C: \quad (\Gamma_a)_b{}^c = i a_{abc}, \qquad (\Gamma_a)_b{}^8 = i\delta_{ab}, \quad (\Gamma_a)_8{}^b = -i\delta_{ab}. \tag{A.7}$$

$$(\Gamma_{ab})_B{}^C: \quad (\Gamma_{ab})^{cd} = 2\delta^{cd}_{ab} - c_{ab}{}^{cd}, \quad (\Gamma_{ab})_c{}^8 = a_{abc}. \tag{A.8}$$

[12]Note that the indices for both vectors, $a, b, c, \ldots$, and spinors, $A, B, C, \ldots$, are raised and lowered by a unit matrix which means that equations may appear with indices in the wrong up or down position.

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Note that all seven gamma matrices are antisymmetric and imaginary. With these definitions one can check that $\bar\eta\,\Gamma_{abc}\,\eta = -ia_{abc}$ as stipulated above. However, this calculation requires some octonionic structure constant identities that are seldom given in the literature. For the convenience of the reader we list the identities used in this paper below:

$$a_{abe}\,a^{cde} = 2\delta^{cd}_{ab} + c_{ab}{}^{cd}, \tag{A.9}$$

$$c_{abc}{}^{g}c^{defg} + a_{abc}a^{def} = 6\delta^{def}_{abc} + 9\delta^{[d}_{[a}c_{bc]}{}^{ef]}, \tag{A.10}$$

$$c_{abcf}a^{def} = 6a_{[ab}{}^{[d}\delta^{e]}_{c]}. \tag{A.11}$$

These identities may be verified directly by inserting the actual values of the constants, or by using the above expressions in terms of the Killing spinor η and the gamma matrices. The latter approach requires the Fierz formula

$$\Gamma_a\eta\bar\eta\Gamma_a = \mathbf{1} - \eta\bar\eta. \tag{A.12}$$

Various contractions of indices in these identities lead directly to a number of additional identities that are used frequently in the main text. A quite extensive list of identities is presented in [18].

Using the Killing spinor equation we furthermore find the derivative identities

$$\nabla_a a_{bcd} = -\frac{m}{2}\bar\eta(\Gamma_a\Gamma_{bcd} - \Gamma_{bcd}\Gamma_a)\eta = -m\bar\eta(\Gamma_{abcd})\eta = mc_{abcd}, \tag{A.13}$$

$$\nabla_a c_{bcde} = -\frac{im}{2}\bar\eta(\Gamma_a\Gamma_{bcde} - \Gamma_{bcde}\Gamma_a)\eta = -4im\delta_{a[b}\bar\eta(\Gamma_{cde]})\eta = -4m\delta_{a[b}a_{cde]}. \tag{A.14}$$

B Projection operators

B.1 2-forms

$$(P_7)_{ab}{}^{cd} = \frac{1}{6}a_{ab}{}^{e}a^{cde} = \frac{1}{6}(2\delta^{cd}_{ab} + c_{ab}{}^{cd}), \tag{B.1}$$

$$(P_{14})_{ab}{}^{cd} = \frac{1}{6}(4\delta^{cd}_{ab} - c_{ab}{}^{cd}), \tag{B.2}$$

which implies the useful relation

$$c_{ab}{}^{cd} = 2(2P_7 - P_{14})_{ab}{}^{cd}. \tag{B.3}$$

B.2 3-forms

$$(P_1)_{abc}{}^{def} = \frac{1}{42}a_{abc}a^{def}, \tag{B.4}$$

$$(P_7)_{abc}{}^{def} = \frac{1}{24}c_{abc}{}^{g}c^{defg} = \frac{1}{24}\left(6\delta^{def}_{abc} + 9\delta^{[d}_{[a}c_{bc]}{}^{ef]} - a_{abc}a^{def}\right), \tag{B.5}$$

$$(P_{27})_{abc}{}^{def} = (1 - P_1 - P_7)_{abc}{}^{def} = \frac{1}{56}\left(42\delta^{def}_{abc} - 21\delta^{[d}_{[a}c_{bc]}{}^{ef]} + a_{abc}a^{def}\right). \tag{B.6}$$

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B.3 Spin 1/2

The purpose of the projection operators in this case is to split the SO(7) spinor representation into two G_2 irreps as follows $\mathbf{8} \to \mathbf{1} \oplus \mathbf{7}$:

$$P_1^s = \frac{1}{8}\left(\mathbf{1} - \frac{1}{24} c_{abcd}\Gamma^{abcd}\right), \tag{B.7}$$

$$P_7^s = \mathbf{1} - P_1^s = \frac{1}{8}\left(7 \cdot \mathbf{1} + \frac{1}{24} c_{abcd}\Gamma^{abcd}\right). \tag{B.8}$$

While they trivially sum to the unit matrix, the fact that they both square to themselves and are orthogonal to each other require some algebra to show. In fact, all these properties follow directly if we show that $(P_1^s)^2 = P_1^s$. This is done as follows

$$(P_1^s)^2 = \frac{1}{64}\left(\mathbf{1} - \frac{1}{12} c_{abcd}\Gamma^{abcd} + \frac{1}{(24)^2} c_{abcd} c_{efgh}\Gamma^{abcd}\Gamma^{efgh}\right) = P_1^s. \tag{B.9}$$

The key calculation in the last step is to expand $\Gamma^{abcd}\Gamma^{efgh}$ in the gamma basis $\Gamma^{(n)}$, $n = 0, \ldots, 7$, in seven dimensions:

$$\Gamma^{abcd}\Gamma_{efgh} = -16\delta^{[a}_{[e}\Gamma^{bcd]}{}_{fgh]} - 72\delta^{[ab}_{[ef}\Gamma^{cd]}{}_{gh]} + 96\delta^{[abc}_{[efg}\Gamma^{d]}{}_{h]} + 24\delta^{abcd}_{efgh}. \tag{B.10}$$

However, when contracted with $c_{abcd}c_{efgh}$ the first and third terms vanish since they are antisymmetric under interchange of the two index sets $abcd$ and $efgh$. The remaining two terms are easily computed using the two identities for c_{abcd}: $c^{abcd}c_{efcd} = 8\delta^{ab}_{ef} + 2c^{ab}{}_{ef}$ and $c^{abcd}c_{abcd} = 168$. $(P_1^s)^2 = P_1^s$ then follows directly.

C The role of the Weyl tensor

The Weyl tensor of the squashed seven-sphere is given in [2]. The explicit expression given there can be written in a more compact form in terms of 't Hooft symbols. By setting $\alpha = (0, 1, 2, 3) = (0, i)$ and introducing the 't Hooft symbols by

$$\eta^k_{\alpha\beta}: \ \ \eta^k_{ij} = \epsilon_{ijk}, \ \ \eta^k_{0j} = -\eta^k_{j0} = -\delta_{jk}, \tag{C.1}$$

$$\bar{\eta}^i_{\alpha\beta}: \ \ \bar{\eta}^k_{ij} = \epsilon_{ijk}, \ \ \bar{\eta}^k_{0j} = -\bar{\eta}^k_{j0} = \delta_{jk}, \tag{C.2}$$

the Weyl tensor can be explicitly expressed as follows

$$W_{\alpha\beta}{}^{\gamma\delta} = \frac{4}{5}\delta^{\gamma\delta}_{\alpha\beta}, \ W_{\hat{i}\hat{j}}{}^{\hat{k}\hat{l}} = \frac{8}{5}\delta^{kl}_{ij}, \ W_{\alpha\beta}{}^{\hat{k}\hat{l}} = \frac{2}{5}\bar{\eta}^m_{\alpha\beta}\epsilon^{mkl}, \ W_{\alpha}{}^{\hat{j}}{}_{\gamma}{}^{\hat{l}} = \frac{1}{5}\bar{\eta}^m_{\alpha\gamma}\epsilon^{mjl} - \frac{2}{5}\delta_{\alpha\gamma}\delta^{jl}. \tag{C.3}$$

The demonstration that the squashed seven-sphere has holonomy G_2 relies on the fact that defining $W_{ab} \equiv \frac{1}{4}W_{ab}{}^{cd}\Gamma_{cd}$ we find

$$W_{0i} = \frac{1}{5}\left(\Gamma_{0i} + \frac{1}{2}\epsilon_{ijk}\Gamma_{\hat{j}\hat{k}}\right), \tag{C.4}$$

$$W_{ij} = \frac{1}{5}\left(\Gamma_{ij} + \Gamma_{\hat{i}\hat{j}}\right), \tag{C.5}$$

$$W_{i\hat{j}} = \frac{1}{5}\left(-\Gamma_{i\hat{j}} - \frac{1}{2}\Gamma_{j\hat{i}} + \frac{1}{2}\delta_{ij}\Gamma_{k\hat{k}} - \frac{1}{2}\epsilon_{ijk}\Gamma_{0\hat{k}}\right), \tag{C.6}$$

$$W_{0\hat{i}} = -\frac{1}{5}\left(\Gamma_{0\hat{i}} + \frac{1}{2}\epsilon_{ijk}\Gamma_{j\hat{k}}\right), \tag{C.7}$$

$$W_{\hat{i}\hat{j}} = \frac{1}{5}\left(2\Gamma_{\hat{i}\hat{j}} + \Gamma_{ij} + \epsilon_{ijk}\Gamma_{0k}\right), \tag{C.8}$$

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while the remaining components are not independent but instead given by

$$W_{i\hat{i}} = 0, \quad W_{0\hat{i}} = \epsilon_{ijk} W_{j\hat{k}}, \;\; W_{\hat{i}\hat{j}} = W_{ij} + \epsilon_{ijk} W_{0k}. \tag{C.9}$$

Thus only 14 components are linearly independent which was shown in [2] to result in G_2 holonomy. An immediate consequence, as can be easily verified using the explicit expressions, is that

$$a_{abc} W_{bc}{}^{de} = 0\,. \tag{C.10}$$

To obtain identities of this kind it is convenient to express the octonionic structure constants in terms of 't Hooft symbols. The non-zero components are then (recall that $a = (\alpha, \hat{i})$ and $\hat{i} = (\hat{1}, \hat{2}, \hat{3}) = (4, 5, 6)$):

$$a_{abc}\!: \quad a_{\alpha\beta\hat{k}} = -\bar{\eta}^k_{\alpha\beta}, \qquad\qquad a_{\hat{i}\hat{j}\hat{k}} = \epsilon_{ijk}, \tag{C.11}$$

$$c_{abcd}: \quad c_{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta\gamma\delta} = \bar{\eta}^k_{[\alpha\beta}\bar{\eta}^k_{\gamma\delta]}, \quad c_{\alpha\beta\hat{i}\hat{j}} = -\bar{\eta}^k_{\alpha\beta}\epsilon_{ijk}, \tag{C.12}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is antisymmetric with $\epsilon_{0123} = 1$.

In our present context we will for instance use these results to express the 2-form modes $Y^{(14)}_{ab}$ in a way that makes its projections onto the H irreps clear, i.e., the decomposition

$$G_2 \to \mathrm{Sp}_1^A \times \mathrm{Sp}_1^{B+C}: \quad \mathbf{14} \to (1,3) \oplus (2,0) \oplus (0,2). \tag{C.13}$$

To this end we will parametrise the $\mathrm{Spin}(4) \times \mathrm{SU}(2)$ subgroup of $G = \mathrm{Sp}_2 \times \mathrm{Sp}_1^C$ using

$$\Gamma^a = (\Gamma^\alpha, \Gamma^{\hat{i}}): \qquad \Gamma^\alpha = \begin{pmatrix} 0 & (\sigma^\alpha)_{A\dot{B}} \\ (\bar{\sigma}^\alpha)^{\dot{A}B} & 0 \end{pmatrix} \otimes \delta^{\dot{A}}_{\dot{B}}, \quad \Gamma^{\hat{i}} = \mathbf{1}_{4\times 4} \otimes (\sigma^{\hat{i}})^{\dot{A}}{}_{\dot{B}}, \tag{C.14}$$

where

$$\sigma^\alpha \equiv (-i\mathbf{1}, \sigma^i), \;\; \bar{\sigma}^\alpha \equiv (i\mathbf{1}, \sigma^i). \tag{C.15}$$

Note the dot notation on the indices in the second tensor factor. This reflects the fact that we are using the diagonal subgroup of Sp_1^B and Sp_1^C where the former is a subgroup of Sp_2 when split into $\mathrm{Sp}_1^A \times \mathrm{Sp}_1^B$ according to the first tensor factor.

The 't Hooft symbols $(\eta^m_{\alpha\beta}, \bar{\eta}^m_{\alpha\beta})$ introduced above can also be defined from the Pauli matrices by

$$\sigma^{\alpha\beta} \equiv i\eta^m_{\alpha\beta}\sigma^m, \quad \bar{\sigma}^{\alpha\beta} \equiv i\bar{\eta}^m_{\alpha\beta}\sigma^m. \tag{C.16}$$

C.1 2-form modes

From the above expressions we immediately obtain the projectors onto irreps (s,t) of $H = \mathrm{Sp}_1^A \times \mathrm{Sp}_1^{B+C}$

$$\left(P^{ab}_{\mathbf{14}\to(2,0)} Y_{ab}\right)_A{}^B = (\sigma^{\alpha\beta} Y_{\alpha\beta})_A{}^B, \tag{C.17}$$

$$\left(P^{ab}_{\mathbf{14}\to(0,2)} Y_{ab}\right)^{\dot{A}}{}_{\dot{B}} = \left(\bar{\sigma}^{\alpha\beta} Y_{\alpha\beta} + 2\sigma^{\hat{i}\hat{j}} Y_{\hat{i}\hat{j}}\right)^{\dot{A}}{}_{\dot{B}}, \tag{C.18}$$

$$\left(P^{ab}_{\mathbf{14}\to(1,3)} Y_{ab}\right)_{A(\dot{B}\dot{C}\dot{D})} = (\sigma^\alpha)_{A(\dot{B}}\sigma^{\hat{j}}_{\dot{C}\dot{D})} Y_{\alpha\hat{j}}. \tag{C.19}$$

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H-irrep	H Casimir	Weyl eigenvalue	Sum
$(2,0)$	4	$\frac{4}{5}$	$\frac{24}{5}$
$(0,2)$	$\frac{12}{5}$	$\frac{12}{5}$	$\frac{24}{5}$
$(1,3)$	6	$-\frac{6}{5}$	$\frac{24}{5}$

Table 1. Weyl tensor eigenvalues for the H irreps of the G_2 **14**-part of the 2-form.

As two examples of how to use these projectors we compute the Weyl tensor eigenvalue of the $(2,0)$ and $(0,2)$ irreps. For $(2,0)$

$$P^{ab}_{\mathbf{14}\to(2,0)}W_{abcd}Y_{cd}=\sigma^{\alpha\beta}W_{\alpha\beta cd}Y_{cd}=\sigma^{\alpha\beta}\left(\frac{4}{5}\delta^{\gamma\delta}_{\alpha\beta}Y_{\gamma\delta}+\frac{2}{5}\bar{\eta}^{m}_{\alpha\beta}\epsilon^{mkl}Y_{\hat{k}\hat{l}}\right)=\frac{4}{5}\sigma^{\alpha\beta}Y_{\alpha\beta}, \tag{C.20}$$

showing that the sought for eigenvalue is $\frac{4}{5}$. Note that in the last equality we used the fact that $\sigma^{\alpha\beta}\bar{\eta}^{m}_{\alpha\beta}=0$. For $(0,2)$ we find

$$2\sigma^{\hat{i}\hat{j}}W_{\hat{i}\hat{j}cd}Y^{cd}=2\sigma^{\hat{i}\hat{j}}\left(\frac{8}{5}Y_{\hat{i}\hat{j}}+\frac{2}{5}\bar{\eta}^{m}_{\alpha\beta}\epsilon^{m\hat{i}\hat{j}}Y^{\alpha\beta}\right)=\frac{8}{5}\left(2\sigma^{\hat{i}\hat{j}}Y_{\hat{i}\hat{j}}+\bar{\sigma}^{\alpha\beta}Y_{\alpha\beta}\right) \tag{C.21}$$

$$\bar{\sigma}^{\alpha\beta}W_{\alpha\beta ab}Y^{ab}=\bar{\sigma}^{\alpha\beta}\left(\frac{4}{5}Y_{\alpha\beta}+\frac{2}{5}\bar{\eta}^{m}_{\alpha\beta}\epsilon^{mij}Y_{\hat{i}\hat{j}}\right)=\frac{4}{5}\left(\bar{\sigma}^{\alpha\beta}Y_{\alpha\beta}+2\sigma^{\hat{i}\hat{j}}Y_{\hat{i}\hat{j}}\right). \tag{C.22}$$

Then adding together (C.21) and (C.22) implies $P_{\mathbf{14}\to(0,2)}WY=\frac{12}{5}P_{\mathbf{14}\to(0,2)}Y$. Repeating this for the last case one finds the values given in table 1 together with the corresponding values of the H Casimir operator C_H.

Note that the two eigenvalues do not individually respect the G_2 holonomy but that their sums do. Similarly, we know from before that the Weyl tensor does not enter the equations for the G_2 **7**-part of the 2-form but that the H irreps both give $\frac{12}{5}$ which thus automatically respects G_2.

C.2 3-form and metric modes: the 27 irrep

Here we need two more projectors, namely[13]

$$\left(P^{ab}_{\mathbf{27}\to(2,2)}Y_{ab}\right)_{(A(\dot{B}C)\dot{D})}=(\bar{\sigma}^{\alpha})_{(A(\dot{B}}(\bar{\sigma}^{\beta})_{C)\dot{D})}Y_{\alpha\beta}, \tag{C.23}$$

$$\left(P^{ab}_{\mathbf{27}\to(0,4)}Y_{ab}\right)_{(\dot{A}\dot{B}\dot{C}\dot{D})}=(\bar{\sigma}^{\hat{i}})_{(\dot{A}\dot{B}}(\bar{\sigma}^{\hat{j}})_{\dot{C}\dot{D})}Y_{\hat{i}\hat{j}}. \tag{C.24}$$

As an example how to get the Weyl eigenvalues we consider the second projector. The calculation works as follows:

$$\left(P^{ab}_{\mathbf{27}\to(0,4)}W_{acbd}h^{cd}\right)=\sigma^{\hat{i}}\sigma^{\hat{j}}W_{\hat{i}c\hat{j}d}h^{cd}=\sigma^{\hat{i}}\sigma^{\hat{j}}\left(\frac{8}{5}\delta^{\hat{i}\hat{k}}_{\hat{j}\hat{l}}h^{\hat{k}\hat{l}}+\frac{1}{5}\bar{\eta}^{m}_{\alpha\beta}\epsilon_{m\hat{i}\hat{j}}h^{\alpha\beta}-\frac{2}{5}\delta_{\hat{i}\hat{j}}h_{\alpha\alpha}\right). \tag{C.25}$$

The term containing the 't Hooft symbol vanishes since $\bar{\eta}^{m}_{\alpha\beta}$ is antisymmetric in the two lower indices. The remaining terms simplify immediately to

$$\left(P^{ab}_{\mathbf{27}\to(0,4)}W_{acbd}h^{cd}\right)=-\frac{4}{5}(\sigma^{\hat{i}}\sigma^{\hat{j}})_{(0,4)}h_{\hat{i}\hat{j}}, \tag{C.26}$$

[13] In the first formula below the use of the symmetrisation brackets is not optimal: the two pairs of dotted and undotted indices are symmetrised independently of each other.

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H-irrep	H Casimir	2 × Weyl eigenvalue	Sum
$(0,0)$	0	$\frac{28}{5}$	$\frac{28}{5}$
$(2,2)$	$\frac{32}{5}$	$-\frac{4}{5}$	$\frac{28}{5}$
$(0,4)$	$\frac{36}{5}$	$-\frac{8}{5}$	$\frac{28}{5}$
$(1,1)$	$\frac{12}{5}$	$\frac{16}{5}$	$\frac{28}{5}$
$(1,3)$	6	$-\frac{2}{5}$	$\frac{28}{5}$

Table 2. Weyl tensor eigenvalues for the H irreps of the G_2 **27**-part of the 3-form.

where we also used that tracing over the hatted indices on $\sigma^{\hat{i}}\sigma^{\hat{j}}$ gives zero since this expression is in the irrep **5** of SU(2) given by the indices $(\dot{A}\dot{B}\dot{C}\dot{D})$.

Again we find the striking result that all H irrep pieces a G_2 irrep produce the same value for $C_H + 2 \times Weyl$, this time for the **27**.

In the main text we found the expression $C_H Y_{abc} - 6W^d{}_{[ab}{}^e Y_{c]de}$. Using the explicit form of the projection operators we can verify that $W^d{}_{[ab}{}^e Y_{c]de} = W^d{}_{[ab}{}^e Y^{(27)}_{c]de}$. Applying $(\tilde{P}^{27})_{ab}{}^{fgh} \equiv a_{acd}(P^{27})_{cdb}{}^{fgh}$ gives

$$(\tilde{P}^{27})_{ab}{}^{fgh} W^d{}_{fg}{}^e Y^{(27)}_{gde} = -\frac{1}{3} W_{(a}{}^c{}_{b)}{}^d \tilde{Y}_{cd}. \tag{C.27}$$

Using the eigenvalues listed in table 2 it follows that

$$(\tilde{P}^{27})_{ab}{}^{fgh} \left(C_H Y_{abc} - 6W^d{}_{[ab}{}^e Y_{c]de} \right) = \left(C_H \tilde{Y}_{ab} + 2W_{(a}{}^c{}_{b)}{}^e \tilde{Y}_{ce} \right) = \frac{28}{5} \tilde{Y}_{ab}. \tag{C.28}$$

Going back to the 3-form notation and using that C_H has eigenvalue $\frac{12}{5}$ on **7** we find

$$C_H Y_{abc} - 6W^d{}_{[ab}{}^e Y_{c]de} = -\frac{1}{10} c_{abcd} Y_d + \frac{28}{5} Y_{abc}. \tag{C.29}$$

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