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## Central and convolution Herz-Schur multipliers

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# Central and convolution Herz-Schur multipliers 

Andrew McKee, Reyhaneh Pourshahami, Ivan G. Todorov and Lyudmila Turowska


#### Abstract

In this paper we obtain descriptions of central operator-valued Schur and Herz-Schur multipliers, akin to a classical characterisation due to Grothéndieck, that reveals a close link between central (linear) multipliers and bilinear multipliers into the trace class. Restricting to dynamical systems where a locally compact group acts on itself by translation, we identify their convolution multipliers as the right completely bounded multipliers, in the sense of Junge-Neufang-Ruan, of a canonical quantum group associated with the underlying group. We provide characterisations of contractive idempotent operator-valued Schur and Herz-Schur multipliers. Exploiting the link between Herz-Schur multipliers and multipliers on transformation groupoids, we provide a combinatorial characterisation of groupoid multipliers that are contractive and idempotent.


## CONTENTS

1. Introduction 1
2. Preliminaries 3
3. Central multipliers 12
4. Convolution multipliers 25
5. Idempotent multipliers 33

References 40

## 1. Introduction

Schur multipliers originated in the work of Schur on the Hadamard entrywise product of matrices in the early twentieth century. These are complexvalued functions, defined on the Cartesian product $X \times Y$ of two measure spaces $(X, \mu)$ and $(Y, \nu)$ that give rise to completely bounded maps on the space $\mathcal{K}$ of all compact operators from $L^{2}(X, \mu)$ into $L^{2}(Y, \nu)$, acting by pointwise multiplication on the integral kernels of the operators from the Hilbert-Schmidt class.

[^0]A concrete description of these objects, which has found numerous applications thereafter, was given by Grothéndieck in his Resumé [14]. Since then, Schur multipliers have played a significant role in operator theory, the theory of Banach spaces, the theory of operator spaces, and have been linked to perturbation theory through the concept of double operator integrals (see [8, 24] and the references therein).

The theory of Herz-Schur, or completely bounded, multipliers of the Fourier algebra of a locally compact group originated in the work of Herz [17], where they were viewed as a generalisation of Fourier-Stieltjes transforms. Similarly to Schur multipliers, Herz-Schur multipliers are complex-valued functions, this time defined on a locally compact group $G$, that give rise to completely bounded maps on the reduced $C^{*}$-algebra $C_{r}^{*}(G)$ of $G$, acting by pointwise multiplication on its subalgebra $L^{1}(G)$. An important development in the subject were the works of Gilbert and of Bożejko and Fendler [5], showing that the Herz-Schur multipliers on the locally compact group $G$ can be isometrically identified with the space of all Schur multipliers on $G \times G$ of Toeplitz type. Haagerup [15] pioneered the use of Herz-Schur multipliers to study the approximation properties of operator algebras (see also [6]).

Recently, several generalisations of Schur and Herz-Schur multipliers to the 'operator-valued’ case have appeared: Bédos and Conti [2, 3] introduced multipliers of a $C^{*}$-dynamical system based on a Hilbert module version of the Fourier-Stieltjes algebra, and applied these techniques to study $C^{*}$-crossed products while, in [28], three of the present authors defined Schur and Herz-Schur multipliers with values in the space of all completely bounded maps on a $C^{*}$ algebra and obtained a version of the Bożejko-Fendler correspondence. The use of multiplier techniques to study reduced crossed products, following Haagerup's work, has been furthered by Skalski and three of the present authors in [27], by the first author in [26], and by the first and the fourth authors in [29].

In this paper we consider special cases of the multipliers defined in [28]. We define central Schur and Herz-Schur multipliers in Definition 3.2 and Definition 3.8 , respectively. They are associated with completely bounded maps on a $C^{*}$-algebra $A$ that are multiplication operators by elements of the centre of the multiplier algebra of $A$, and are one of the most common type of multipliers that appear in specific circumstances. A special case of particular importance arises when $A$ is abelian. Given a central Herz-Schur multiplier of the $C^{*}$ - dynamical system $(A, G, \alpha)$, the corresponding completely bounded map on the crossed product is an $A$-bimodule map. Such maps were considered by Dong and Ruan [9] in their study of the Hilbert module Haagerup property of crossed products. Exploiting the fact that commutative (unital) $C^{*}$-algebras are algebras of continuous functions on compact topological spaces, we identify the central Schur and Herz-Schur multipliers with scalar-valued functions on three and two variables, respectively. This allows us to identify a close link, that seems to have remained unnoticed until now, between central multipliers and
the bilinear Schur multipliers into the trace class, introduced and characterised by Coine, Le Merdy and Sukochev in [8] (see also [24]).

A $C^{*}$-dynamical system of particular importance is $\left(C_{0}(G), G, \beta\right)$, where $G$ is a locally compact group, $C_{0}(G)$ is the $C^{*}$-algebra of all continuous functions on $G$ vanishing at infinity, and $\beta$ is the left translation action of $G$ on $C_{0}(G)$. The second main class of maps we are concerned with are the convolution multipliers of $\left(C_{0}(G), G, \beta\right)$ introduced in [28]. We answer [28, Question 6.6], identifying the Herz-Schur multipliers of the latter dynamical system with the right multipliers of a canonical quantum group associated with $G$; in the case where $G$ is abelian, we show that these multipliers coincide with the elements of the Fourier-Stieltjes algebra $B(G \times \Gamma)$, where $\Gamma$ is the dual group of $G$.

Finally, we investigate when the special classes of multipliers considered in this paper give rise to idempotent completely bounded maps. The general study of idempotent Herz-Schur multipliers goes back to Cohen [7], who characterised all idempotent elements of the measure algebra $M(G)$. In [18], Host generalised Cohen's characterisation by identifying the general form of idempotents in $B(G)$, for any locally compact group $G$, while Katavolos and Paulsen in [22] and Stan in [41] gave characterisations of contractive idempotent Schur multipliers and contractive idempotent Herz-Schur multipliers respectively, based on a combinatorial 3 -of- 4 property. In this paper, we use the 3 -of- 4 property to obtain characterisations of various classes of central idempotent Schur multipliers and idempotent Herz-Schur multipliers of dynamical systems.

The paper is organised as follows. Section 2 contains background material, including a review of crossed products and multipliers as introduced in [28]. The section also includes some preliminary results that will be needed later. In Section 3 we define central Schur $A$-multipliers, and present a characterisation of the central Schur $C_{0}(Z)$-multipliers, followed by a similar characterisation of central Schur $A$-multipliers for an arbitrary $C^{*}$-algebra $A$. After introducing central Herz-Schur multipliers, we characterise the central Herz-Schur ( $A, G, \alpha$ )-multipliers, the central Herz-Schur $\left(C_{0}(Z), G, \alpha\right)$-multipliers, as well as their canonical positive cones. Convolution multipliers are considered in Section 5, first in the abelian and then in the general case. Therein, we also investigate idempotent multipliers within the classes of central and convolution multipliers from Section 3 and Section 4.

## 2. Preliminaries

Throughout this paper, we make the following standing separability assumptions: unless otherwise stated, we consider only separable $C^{*}$-algebras, separable Hilbert spaces and second-countable locally compact groups. These assumptions allow us to consider multipliers defined on standard measure spaces. However, we note that the results remain valid for the case of discrete spaces with counting measure, in which case the separability assumptions above can be dropped.

### 2.1. General background.

2.1.1. Measure spaces. We fix for the whole paper standard measure spaces $(X, \mu)$ and $(Y, v)$; this means that there exist locally compact, metrisable, complete, separable topologies on $X$ and $Y$ (called admissible topologies), with respect to which $\mu$ and $\nu$ are regular Borel $\sigma$-finite measures. The direct products $X \times Y$ and $Y \times X$ are equipped with the corresponding product measures. We use standard notation for the $L^{p}$ spaces over $(X, \mu)$ and $(Y, v)(p=1,2, \infty)$; we will also consider (not necessarily countable) sets equipped with counting measure, in which case we write $\ell^{p}(X)$ in place of $L^{p}(X)$.

Given a Banach space $B$, the space $L^{p}(X, B)(p=1,2)$ is the space of (equivalence classes of) Bochner $p$-integrable functions from $X$ to $B$ with respect to $\mu$; each of these spaces contains the algebraic tensor product $C_{c}(X) \odot B$ as a dense subspace. The identification $L^{2}(X, \mathcal{H}) \cong L^{2}(X) \otimes \mathcal{H}$ will be used frequently; here, and in the sequel, we denote by $\mathcal{L} \otimes \mathcal{H}$ Hilbertian tensor product of Hilbert spaces $\mathcal{L}$ and $\mathcal{H}$. We refer to Williams [43, Appendix B.I.4] for further details.

Let $\mathcal{B}(\mathcal{H}, \mathcal{L})$ be the space of all bounded linear operators from $\mathcal{H}$ into $\mathcal{L}$; we write as usual $\mathcal{B}(\mathcal{H})=\mathcal{B}(\mathcal{H}, \mathcal{H})$. For a weak*-closed subspace $M \subseteq \mathcal{B}(\mathcal{H}, \mathcal{L})$ we let $L^{\infty}(X, M)$ denote the space of (equivalence classes of) bounded functions $f: X \rightarrow M$ such that, for each $x \in X$ and $\xi \in L^{2}(X, \mathcal{H}), \eta \in L^{2}(X, \mathcal{L})$, the functions $x \mapsto f(x)(\xi(x))$ and $x \mapsto f(x)^{*}(\eta(x))$ are weakly measurable as functions from $X$ to $\mathcal{L}$ and from $X$ to $\mathcal{H}$, respectively. We equip $L^{\infty}(X, M)$ with the norm $\|f\|:=\operatorname{esssup}_{x \in X}\|f(x)\|$ and identify each $f \in L^{\infty}(X, M)$ with the operator $D_{f}$ from $L^{2}(X, \mathcal{H})$ to $L^{2}(X, \mathcal{L})$ given by $\left(D_{f} \xi\right)(x)=f(x) \xi(x)$. See Takesaki [42, Section IV.7] for details. We write $L^{\infty}(X, \mathcal{H})$ for the space of (equivalence classes of) bounded weakly measurable $\mathcal{H}$-valued functions on $X$.

Since we have a standing second-countability assumption for locally compact groups (except when we specify a discrete group) our groups are metrisable as topological spaces, and are hence standard measure spaces when equipped with left Haar measure.
2.1.2. Operator spaces. Consider (concrete) operator spaces $V \subseteq \mathcal{B}(\mathcal{H})$ and $W \subseteq \mathcal{B}(\mathcal{L})$. The norm-closed spatial tensor product of $V$ and $W$ will be written $V \otimes W$, while if $V$ and $W$ are weak*-closed, their weak*-spatial tensor product will be denoted $V \bar{\otimes} W$. The operator space projective tensor product $V \widehat{\otimes} W$ satisfies the canonical completely isometric identifications $(V \widehat{\otimes} W)^{*}=$ $\mathrm{CB}\left(V, W^{*}\right)=\mathrm{CB}\left(W, V^{*}\right)[10$, Corollary 7.1.5]; if $M$ and $N$ are von Neumann algebras, $V=M_{*}$ and $W=N_{*}$, then $(V \widehat{\otimes} W)^{*}=M \bar{\otimes} N$, up to a complete isometry [10, Theorem 7.2.4]. For $u \in M_{n}(V \odot W)$ let $\|u\|_{h}=\inf \{\|a\|\|b\|\}$, where the infimum is taken over all integers $p$, and all matrices $a \in M_{n, p}(V)$ and $b \in M_{p, n}(W)$, such that $u_{i, j}=\sum_{k} a_{i, k} \otimes b_{k, j}$; the Haagerup tensor product $V \otimes^{h} W$ is the completion of the operator space $V \odot W$ in $\|\cdot\|_{h}$; see [10, Chapter 9] for further details.

For an index set $I$, we will write $C_{I}^{\omega}(V)$ for the operator space of families $\left(x_{i}\right)_{i \in I} \subseteq V$ such that the sums $\sum_{i \in J} x_{i}^{*} x_{i}$ are uniformly bounded over all finite
sets $J \subseteq I$; equivalently, $C_{I}^{\omega}(M)=\ell^{2}(I)_{c} \bar{\otimes} M$, where $\ell^{2}(I)_{c}$ denotes $\ell^{2}(I)$, equipped with the column operator space structure. Similarly, $R_{I}^{\omega}(V)$ denotes the operator space of families $\left(x_{i}\right)_{i \in I} \subseteq V$ such that the sums $\sum_{i \in J} x_{i} x_{i}^{*}$ are uniformly bounded over all finite sets $J \subseteq I$; equivalently, $R_{I}^{\omega}(M)=\ell^{2}(I)_{r} \bar{\otimes} M$, where $\ell^{2}(I)_{r}$ denotes $\ell^{2}(I)$, equipped with the row operator space structure. Further details on the row and column spaces can be found in [10] and [36]. If $V$ and $W$ are dual operator spaces then their weak* Haagerup tensor product will be written $V \otimes^{\omega^{*} h} W$; a typical element $u \in V \otimes^{\omega^{*} h} W$ is $u=\sum_{i \in I} f_{i} \otimes g_{i}$, where $I$ is some cardinal, $f=\left(f_{i}\right)_{i \in I} \in R_{I}^{\omega}(V)$ and $g=\left(g_{i}\right)_{i \in I} \in C_{I}^{\omega}(W)$; see [4] for further details.
2.1.3. The trace and Hilbert-Schmidt classes. Let $\mathcal{H}$ and $\mathcal{L}$ denote Hilbert spaces. We write $\mathcal{K}(\mathcal{H}, \mathcal{L})$ (resp. $\left.\mathcal{S}_{1}(\mathcal{H}, \mathcal{L})\right)$ for the compact (resp. trace class) operators from $\mathcal{H}$ to $\mathcal{L}$ and use the simplified notation $\mathcal{K}(\mathcal{H}):=\mathcal{K}(\mathcal{H}, \mathcal{H})$, etc. The space $\mathcal{S}_{1}(\mathcal{H}, \mathcal{L})$ is equipped with the norm $\|T\|_{1}:=\operatorname{tr}(|T|)$. Recall that, via trace duality, we have isometric identifications

$$
\mathcal{S}_{1}(\mathcal{H}, \mathcal{L}) \cong \mathcal{K}(\mathcal{L}, \mathcal{H})^{*} \quad \text { and } \quad \mathcal{B}(\mathcal{L}, \mathcal{H}) \cong \mathcal{S}_{1}(\mathcal{H}, \mathcal{L})^{*}
$$

The space of Hilbert-Schmidt operators $T: \mathcal{H} \rightarrow \mathcal{L}$, with the norm $\|T\|_{2}:=$ $\left(\operatorname{tr}\left(T^{*} T\right)\right)^{1 / 2}$, will be denoted $\mathcal{S}_{2}(\mathcal{H}, \mathcal{L})$. These spaces will often appear with $\mathcal{H}=L^{2}(X, \mu)$ and $\mathcal{L}=L^{2}(Y, \nu)$, in which case we will write $\mathcal{S}_{1}(X, Y), \mathcal{S}_{2}(X)$, etc.
2.1.4. Crossed products. Let $A$ be a $C^{*}$-algebra, viewed as a subalgebra of $B\left(\mathcal{H}_{A}\right)$, where $\mathcal{H}_{A}$ denotes the Hilbert space of the universal representation of $A$. Let $G$ be a locally compact group with modular function $\Delta$, equipped with left Haar measure $m_{G}$, and $\alpha: G \rightarrow \operatorname{Aut}(A)$ be a group homomorphism which is continuous in the point-norm topology, i.e. for all $a \in A$ the map $s \mapsto \alpha_{s}(a)$ is continuous from $G$ to $A$; we say $(A, G, \alpha)$ is a $C^{*}$-dynamical system. The space $L^{1}(G, A)$ is a Banach $*$-algebra when equipped with the product $\times$ given by

$$
(f \times g)(t):=\int_{G} f(s) \alpha_{s}\left(g\left(s^{-1} t\right)\right) d s, \quad f, g \in L^{1}(G, A), t \in G
$$

the involution $*$ defined by

$$
f^{*}(s):=\Delta(s)^{-1} \alpha_{s}\left(f\left(s^{-1}\right)^{*}\right), \quad f \in L^{1}(G, A), s \in G,
$$

and the $L^{1}$-norm $\|f\|_{1}:=\int_{G}\|f(s)\| d s$. These definitions also give a $*$-algebra structure on $C_{c}(G, A)$, which is a dense $*$-subalgebra of $L^{1}(G, A)$. Given a faithful representation $\theta: A \rightarrow \mathcal{B}\left(\mathcal{H}_{\theta}\right)$, we define new representations of $A$ and $G$ on $L^{2}\left(G, \mathcal{H}_{\theta}\right)$ as follows:

$$
\begin{gathered}
\pi^{\theta}: A \rightarrow \mathcal{B}\left(L^{2}\left(G, \mathcal{H}_{\theta}\right)\right) ;\left(\pi^{\theta}(a) \xi\right)(t):=\theta\left(\alpha_{t^{-1}}(a)\right)(\xi(t)), \\
\lambda^{\theta}: G \rightarrow \mathcal{B}\left(L^{2}\left(G, \mathcal{H}_{\theta}\right)\right) ;\left(\lambda_{t}^{\theta} \xi\right)(s):=\xi\left(t^{-1} s\right),
\end{gathered}
$$

for all $a \in A, s, t \in G, \xi \in L^{2}\left(G, \mathcal{H}_{\theta}\right)$. Then $\lambda^{\theta}$ is a (strongly continuous) unitary representation of $G$ and

$$
\pi^{\theta}\left(\alpha_{t}(a)\right)=\lambda_{t}^{\theta} \pi^{\theta}(a)\left(\lambda_{t}^{\theta}\right)^{*}, \quad a \in A, t \in G
$$

The pair $\left(\pi^{\theta}, \lambda^{\theta}\right)$ is thus a covariant representation of $(A, G, \alpha)$ and therefore gives rise to a $*$-representation $\pi^{\theta} \rtimes \lambda^{\theta}: L^{1}(G, A) \rightarrow B\left(L^{2}\left(G, \mathcal{H}_{\theta}\right)\right)$ given by

$$
\left(\pi^{\theta} \rtimes \lambda^{\theta}\right)(f):=\int_{G} \pi^{\theta}(f(s)) \lambda_{s}^{\theta} d s, \quad f \in L^{1}(G, A)
$$

The reduced crossed product $A \rtimes_{\alpha, r} G$ of $A$ by $G$ is independent of the choice of the faithful representation $\theta$ and is defined as the closure of $\left(\pi^{\theta} \rtimes \lambda^{\theta}\right)\left(L^{1}(G, A)\right)$ in the operator norm of $B\left(L^{2}\left(G, \mathcal{H}_{\theta}\right)\right)$; if we want to emphasise the representation $\theta$ of $A$ was used, we will write $A \rtimes_{\alpha, \theta} G$. In Section 4 we will use the weak* closure $A \rtimes_{\alpha, r}^{w^{*}} G$ of $A \rtimes_{\alpha, r} G$. In what follows we will often simplify our notation by omitting the superscript $\theta$. More on reduced crossed products can be found in Pedersen [34, Chapter 7], and Williams [43].
2.2. Multipliers. We will use some well-known results on classical Schur and Herz-Schur multipliers, as well as results from [28]. We recall some definitions and results required later.
2.2.1. Schur multipliers. Let $(X, \mu)$ and $(Y, v)$ be standard measure spaces. We say $E \subseteq X \times Y$ is marginally null if there exist null sets $M \subseteq X$ and $N \subseteq Y$ such that $E \subseteq(M \times Y) \cup(X \times N)$. Two measurable sets $E, F \subseteq X \times Y$ are called marginally equivalent if their symmetric difference is marginally null; we say that two functions $\varphi, \psi: X \times Y \rightarrow \mathbb{C}$ are marginally equivalent if they are equal up to a marginally null set. A measurable set $E \subseteq X \times Y$ is called $\omega$-open if it is marginally equivalent to a set of the form $\cup_{k \in \mathbb{N}} I_{k} \times J_{k}$, where $I_{k} \subseteq X$ and $J_{k} \subseteq Y$ are measurable, $k \in \mathbb{N}$. The collection of $\omega$-open subsets of $X \times Y$ is a pseudo-topology on $X \times Y$ - it is closed under finite intersections and countable unions; see [11, Section 3]. A function $h: X \times Y \rightarrow \mathbb{C}$ is called $\omega$-continuous [11] if $h^{-1}(U)$ is $\omega$-open for every open set $U \subseteq \mathbb{C}$.

Let $\mathcal{H}$ be a separable Hilbert space and $A \subseteq \mathcal{B}(\mathcal{H})$ be a separable $C^{*}$-algebra. With any $k \in L^{2}(Y \times X, A)$, one can associate an element

$$
T_{k} \in \mathcal{B}\left(L^{2}(X, \mathcal{H}), L^{2}(Y, \mathcal{H})\right)
$$

with $\left\|T_{k}\right\| \leq\|k\|_{2}$, by letting

$$
\left(T_{k} \xi\right)(y):=\int_{X} k(y, x)(\xi(x)) d x, \quad \xi \in L^{2}(X, \mathcal{H}), y \in Y
$$

The linear space of all such operators is denoted by $\mathcal{S}_{2}(X, Y ; A)$ and is norm dense in $\mathcal{K}\left(L^{2}(X), L^{2}(Y)\right) \otimes A$; we equip it with the operator space structure arising from this inclusion. Note that if $A=\mathbb{C}$ then the map $k \rightarrow T_{k}$ is an isometric identification of $L^{2}(Y \times X)$ and $\mathcal{S}_{2}(X, Y)$.

If $B$ is a(nother) $C^{*}$-algebra we write $\operatorname{CB}(A, B)$ for the space of completely bounded maps from $A$ to $B$ and set $\operatorname{CB}(A)=\operatorname{CB}(A, A)$. We say that $\varphi: X \times$
$Y \rightarrow \mathrm{CB}(A, B)$ is pointwise-measurable if $(x, y) \mapsto \varphi(x, y)(a) \in B$ is weakly measurable for each $a \in A$. If $\varphi: X \times Y \rightarrow \mathrm{CB}(A)$ is a bounded, pointwisemeasurable function, we define $\varphi \cdot k \in L^{2}(Y \times X, A)$ by

$$
(\varphi \cdot k)(y, x):=\varphi(x, y)(k(y, x)), \quad(y, x) \in Y \times X
$$

Let $S_{\varphi}$ denote the bounded linear map on $\mathcal{S}_{2}(X, Y ; A)$ given by

$$
S_{\varphi}\left(T_{k}\right):=T_{\varphi \cdot k}, \quad k \in L^{2}(Y \times X, A) .
$$

Definition 2.1. A bounded, pointwise-measurable function $\varphi: X \times Y \rightarrow$ $\operatorname{CB}(A)$ is called a Schur A-multiplier if $S_{\varphi}$ is a completely bounded map on $\mathcal{S}_{2}(X, Y ; A)$. We denote the space of such functions by $\mathfrak{S}(X, Y ; A)$ and endow it with the norm $\|\varphi\|_{\Im(X, Y ; A)}:=\left\|S_{\varphi}\right\|_{\text {cb }}$ (we write $\|\varphi\|_{\Im}$ when $X, Y$ and $A$ are clear from context).
This definition does not depend on the faithful $*$-representation of $A$ on a separable Hilbert space [28, Proposition 2.3].
Theorem 2.2. [28, Theorem 2.6] Let $A \subseteq \mathcal{B}(\mathcal{H})$ be a separable $C^{*}$-algebra and $\varphi: X \times Y \rightarrow \mathrm{CB}(A)$ a bounded, pointwise-measurable function. The following are equivalent:
(i) $\varphi$ is a Schur A-multiplier;
(ii) there exist a separable Hilbert space $\mathcal{H}_{\rho}$, a non-degenerate $*$-representation $\rho: A \rightarrow \mathcal{B}\left(\mathcal{H}_{\rho}\right)$, and $\mathcal{V} \in L^{\infty}\left(X, \mathcal{B}\left(\mathcal{H}, \mathcal{H}_{\rho}\right)\right), \mathcal{W} \in L^{\infty}\left(Y, \mathcal{B}\left(\mathcal{H}, \mathcal{H}_{\rho}\right)\right)$ such that

$$
\varphi(x, y)(a)=\mathcal{W}(y)^{*} \rho(a) \mathcal{V}(x), \quad a \in A
$$

for almost all $(x, y) \in X \times Y$.
Moreover, if these conditions hold then we may choose $\mathcal{V}$ and $\mathcal{W}$ so that

$$
\|\varphi\| \Im=\underset{x \in X}{\operatorname{esssup}}\|\mathcal{V}(x)\| \underset{y \in Y}{\operatorname{esssup}}\|\mathcal{W}(y)\| .
$$

Note that the definitions and theorems make sense in the case $X, Y$ are discrete spaces with counting measures, in which case we do not need to assume separability.

When discussing Schur $A$-multipliers we shall always assume without mentioning that $A$ is separable unless $X$ and $Y$ are discrete spaces with counting measures in which case $A$ can be arbitrary.

In the case where $A=\mathbb{C}$, Schur $A$-multipliers reduce to classical (measurable) Schur multipliers [35]. The elements $\sum_{i=1}^{\infty} f_{i} \otimes g_{i}$ of the projective tensor product $\mathcal{S}_{1}(Y, X)=L^{2}(X, \mu) \widehat{\otimes} L^{2}(Y, \mu)$ (where we assume $\sum_{i=1}^{\infty}\left\|f_{i}\right\|^{2}<\infty$ and $\left.\sum_{i=1}^{\infty}\left\|g_{i}\right\|^{2}<\infty\right)$ can be identified with functions $\sum_{i=1}^{\infty} f_{i}(x) g_{i}(y)$ on $X \times Y$, well-defined up to a marginally null set [1]; under this identification, Schur multipliers coincide with the multipliers of $\mathcal{S}_{1}(Y, X)$.

Given $a \in L^{\infty}(X, \mu)$, let $M_{a}$ be the operator on $L^{2}(X, \mu)$ defined by

$$
\left(M_{a} \xi\right)(x):=a(x) \xi(x), \quad x \in X
$$

Let $\mathcal{D}_{X}=\left\{M_{a}: a \in L^{\infty}(X, \mu)\right\}$ and define $\mathcal{D}_{Y}$ analogously. By a well-known result of Haagerup [16] (see also[4]), there is a completely isometric weak*homeomorphism between the algebra of weak*-continuous, completely bounded $\mathcal{D}_{Y}, \mathcal{D}_{X}$-bimodule maps on $\mathcal{B}\left(L^{2}(X), L^{2}(Y)\right)$ and the weak* Haagerup tensor product $\mathcal{D}_{Y} \otimes^{\omega^{*} h} \mathcal{D}_{X}$ [4]; this homeomorphism sends $\sum_{k=1}^{\infty} b_{k} \otimes a_{k} \in \mathcal{D}_{Y} \otimes^{\omega^{*} h}$ $\mathcal{D}_{X}$ to the map

$$
T \mapsto \sum_{k=1}^{\infty} b_{k} T a_{k}
$$

on $\mathcal{B}\left(L^{2}(X), L^{2}(Y)\right)$. Note that $\mathcal{D}_{Y} \otimes^{\omega^{*} h} \mathcal{D}_{X}$ can be viewed as a space of (equivalence classes of) functions, and each of these functions belongs to $\mathfrak{S}(X, Y)$. Theorem 2.2 can be specialised as follows in the scalar-valued case.

Theorem 2.3. Let $\varphi \in L^{\infty}(X \times Y)$. The following are equivalent:
(i) $\varphi \in \mathbb{S}(X, Y)$ and $\|\varphi\|_{\subseteq} \leq C$;
(ii) there exists sequences $\left(a_{k}\right)_{k=1}^{\infty} \subseteq L^{\infty}(X, \mu)$ and $\left(b_{k}\right)_{k=1}^{\infty} \subseteq L^{\infty}(Y, \nu)$ with

$$
\underset{x \in X}{\operatorname{esssup}} \sum_{k=1}^{\infty}\left|a_{k}(x)\right|^{2} \leq C \quad \text { and } \quad \operatorname{esssup} \sum_{y \in Y}^{\infty}\left|b_{k=1}(y)\right|^{2} \leq C
$$

such that

$$
\varphi(x, y)=\sum_{k=1}^{\infty} a_{k}(x) b_{k}(y) \quad \text { for almost all }(x, y) \in X \times Y
$$

(iii) there exist a separable Hilbert space $\mathcal{H}$ and weakly measurable functions $v$ : $X \rightarrow \mathcal{H}, w: Y \rightarrow \mathcal{H}$, such that

$$
\underset{x \in X}{\operatorname{esssup}}\|v(x)\| \leq \sqrt{C}, \quad \underset{y \in Y}{\operatorname{esssup}}\|w(y)\| \leq \sqrt{C}
$$

and

$$
\varphi(x, y)=\langle v(x), w(y)\rangle, \quad \text { for almost all }(x, y) \in X \times Y ;
$$

(iv) $\left\|T_{\varphi \cdot k}\right\| \leq C\left\|T_{k}\right\|$ for all $k \in L^{2}(Y \times X)$.

We remark that if $X$ and $Y$ are discrete spaces with counting measures the theorem holds true with possibly uncountable families $\left(a_{k}\right)$ and $\left(b_{k}\right)$.
2.2.2. Herz-Schur multipliers. Let $G$ be a locally compact second countable group, $\mathrm{vN}(G)$ (resp. $C_{r}^{*}(G)$ ) be its von Neumann algebra (resp. reduced $C^{*}$-algebra) and $A(G)$ be the Fourier algebra of $G$ [12]. Let $A$ be a separable $C^{*}$-algebra. A bounded function $F: G \rightarrow \mathrm{CB}(A)$ will be called pointwisemeasurable if, for every $a \in A$, the map $s \mapsto F(s)(a)$ is a weakly measurable function from $G$ into $A$. Suppose that the function $F: G \rightarrow \mathrm{CB}(A)$ is bounded and pointwise-measurable, and define

$$
(F \cdot f)(s):=F(s)(f(s)), \quad f \in L^{1}(G, A), s \in G .
$$

Since $F$ is pointwise-measurable, $F \cdot f$ is weakly measurable, and $\|F \cdot f\|_{1} \leq$ $\sup _{s \in G}\|F(s)\|\|f\|_{1}\left(f \in L^{1}(G, A)\right)$; hence $F \cdot f \in L^{1}(G, A)$ for every $f \in$ $L^{1}(G, A)$.

Definition 2.4. A bounded, pointwise-measurable function $F: G \rightarrow \mathrm{CB}(A)$ will be called a $\operatorname{Herz}-\operatorname{Schur}(A, G, \alpha)$-multiplier if the map $S_{F}$ on $(\pi \rtimes \lambda)\left(L^{1}(G, A)\right)$, given by

$$
S_{F}((\pi \rtimes \lambda)(f)):=(\pi \rtimes \lambda)(F \cdot f),
$$

is completely bounded.
If $F$ is a Herz-Schur $(A, G, \alpha)$-multiplier, we continue to denote by $S_{F}$ the corresponding extension to a completely bounded map on $A \rtimes_{\alpha, r} G$.

Definition 2.4 is independent of the faithful representation of $A$ [28, Remark 3.2(ii)]. We note that the set of all Herz-Schur $(A, G, \alpha)$-multipliers is an algebra with respect to the pointwise operations; we denote it by $\mathbb{S}(A, G, \alpha)$ and endow it with the norm $\|F\|_{\mathrm{HS}}:=\left\|S_{F}\right\|_{\mathrm{cb}}$.

The definition makes sense when $G$ is an arbitrary discrete group. In this case we can drop the separability assumption on $A$.

In what follows we shall always consider $C^{*}$-dynamical systems $(A, G, \alpha)$ where either $G$ is second countable and $A$ is separable or $G$ is discrete in which case $A$ can be arbitrary.

Given a function $F: G \rightarrow \mathrm{CB}(A)$, define $\mathcal{N}(F): G \times G \rightarrow \mathrm{CB}(A)$ by letting

$$
\mathcal{N}(F)(s, t)(a):=\alpha_{t^{-1}}\left(F\left(t s^{-1}\right)\left(\alpha_{t}(a)\right)\right), \quad s, t \in G, a \in A
$$

Observe that if $F$ is pointwise-measurable then so is $\mathcal{N}(F)$. The following result [28, Theorem 3.5] relates Schur $A$-multipliers and Herz-Schur ( $A, G, \alpha$ )multipliers, generalising a classical transference result of Bożejko-Fendler [5].

Theorem 2.5. Let $(A, G, \alpha)$ be a $C^{*}$-dynamical system and $F: G \rightarrow \mathrm{CB}(A) a$ bounded, pointwise-measurable function. The following are equivalent:
(i) F is a Herz-Schur (A, G, $\alpha$ )-multiplier;
(ii) $\mathcal{N}(F)$ is a Schur A-multiplier.

Moreover, if the above conditions hold then $\|F\|_{\mathrm{HS}}=\|\mathcal{N}(F)\| \Subset$.
The Schur $A$-multipliers $\varphi$ of the form $\varphi=\mathcal{N}(F)$ will be called $\alpha$-invariant. We note that a different definition was given in [28] (see [28, Definition 3.14]), but by [28, Theorem 3.18], it agrees with the one adopted here.

In the case where $A=\mathbb{C}$ and the action is trivial, Herz-Schur $(A, G, \alpha)$ multipliers coincide with the classical Herz-Schur multipliers of $G$ [6], that is, with the functions $u: G \rightarrow \mathbb{C}$ such that $u A(G) \subseteq A(G)$ and the map

$$
m_{u}: A(G) \rightarrow A(G) ; m_{u}(v):=u v, \quad v \in A(G)
$$

is completely bounded. Here we equip $A(G)$ with the operator space structure, arising from the identification $A(G)^{*}=\mathrm{vN}(G)$ [12, Chapitre 3]. The space of classical Herz-Schur multipliers of $G$ will be denoted by $\mathrm{M}_{\mathrm{cb}} A(G)$. We note that
if $u \in \mathrm{M}_{\mathrm{cb}} A(G)$ then the restriction $S_{u}:=\left.m_{u}^{*}\right|_{C_{r}^{*}(G)}$ is a completely bounded map satisfying [6]

$$
S_{u}: C_{r}^{*}(G) \rightarrow C_{r}^{*}(G) ; S_{u}(\lambda(f))=\lambda(u f), \quad f \in L^{1}(G) .
$$

2.3. Preliminary results. In this subsection, we give several technical results on Schur and Herz-Schur multipliers that will be needed in the sequel. The equivalence between (i) and (iii) in the next proposition was given, in the scalarvalued case, in [22, Theorem 7].

Proposition 2.6. Let $\mathcal{H}$ be a separable Hilbert space, $A \subseteq \mathcal{B}(\mathcal{H})$ a separable $C^{*}$-algebra and $\varphi: X \times Y \rightarrow \mathrm{CB}(A)$ a bounded, pointwise-measurable function. The following are equivalent:
(i) $\varphi(x, y)=0$ for almost all $(x, y) \in X \times Y$;
(ii) $S_{\varphi}=0$.

If $\varphi$ is a Schur $A$-multiplier of the form $\varphi(x, y)(a)=\mathcal{W}(y)^{*} \rho(a) \mathcal{V}(x), a \in A$, as in Theorem 2.2, then these conditions are equivalent to:
(iii) $\varphi(x, y)=0$ for marginally almost all $(x, y) \in X \times Y$.

Proof. (i) $\Longrightarrow$ (ii) Let $T_{k} \in \mathcal{S}_{2}(X, Y ; A)$. If $\varphi(x, y)=0$ for almost all $(x, y) \in$ $X \times Y$ then $\varphi \cdot k=0$ almost everywhere, for every $k \in L^{2}(Y \times X, A)$, and hence $S_{\varphi}\left(T_{k}\right)=T_{\varphi \cdot k}=0$ for every $k \in L^{2}(Y \times X, A)$.
(ii) $\Longrightarrow$ (i) Suppose $S_{\varphi}=0$ and let $k \in L^{2}(Y \times X, A)$. We have $S_{\varphi}\left(T_{k}\right)=$ $T_{\varphi \cdot k}=0$, so we conclude that $\varphi \cdot k=0$ almost everywhere by [28, Lemma 2.1]. We claim that $\varphi(x, y)=0$ for almost all $(x, y) \in X \times Y$. Indeed, let $\left\{e_{i}\right\}_{i \in \mathbb{N}}$ be a dense subset of $\mathcal{H}, \xi \in L^{2}(X)$ and $\eta \in L^{2}(Y)$; then

$$
\begin{align*}
& \left\langle S_{\varphi}\left(T_{k}\right)\left(\xi \otimes e_{i}\right), \eta \otimes e_{j}\right\rangle=\int_{Y}\left\langle S_{\varphi}\left(T_{k}\right)\left(\xi \otimes e_{i}\right)(y),\left(\eta \otimes e_{j}\right)(y)\right\rangle d y  \tag{1}\\
= & \int_{Y}\left\langle\int_{X}(\varphi \cdot k)(y, x)\left(\xi \otimes e_{i}\right)(x) d x,\left(\eta \otimes e_{j}\right)(y)\right\rangle d y \\
= & \int_{Y} \int_{X}\left\langle\varphi(x, y)(k(y, x)) e_{i}, e_{j}\right\rangle \xi(x) \overline{\eta(y)} d x d y .
\end{align*}
$$

Fix $a \in A$, choose $w \in L^{2}(Y \times X)$, and let $k(y, x)=w(y, x) a$. Then (1) implies

$$
\int_{Y} \int_{X}\left\langle\varphi(x, y)(a) e_{i}, e_{j}\right\rangle w(y, x) \xi(x) \overline{\eta(y)} d x d y=0
$$

Since $\varphi(x, y)(a)$ is a bounded operator, we conclude that $\left\langle\varphi(x, y)(a) e_{i}, e_{j}\right\rangle=0$ almost everywhere for all $i, j \in \mathbb{N}$. Hence $\varphi(x, y)=0$ almost everywhere by the separability of $A$ and the continuity of $\varphi(x, y)$ as a map on $A$.

Now suppose that $\varphi$ is a Schur $A$-multiplier.
(iii) $\Longrightarrow$ (i) is trivial.
(i) $\Longrightarrow$ (iii) Assume that the set

$$
R:=\{(x, y) \in X \times Y: \varphi(x, y) \neq 0\}
$$

is null. Let $A_{0}$ and $\mathcal{H}_{0}$ be countable dense subsets of $A$ and $\mathcal{H}$ respectively; then

$$
\begin{aligned}
R^{c}=\{(x, y): \varphi(x, y)=0\} & =\bigcap_{a \in A_{0}, \xi, \eta \in \mathcal{H}_{0}}\{(x, y):\langle\varphi(x, y)(a) \xi, \eta\rangle=0\} \\
& =\bigcap_{a \in A_{0},, \xi, \eta \in \mathcal{H}_{0}}\{(x, y):\langle\rho(a) \mathcal{V}(x) \xi, \mathcal{W}(y) \eta\rangle=0\} .
\end{aligned}
$$

It is easily seen that a function of the form $(x, y) \mapsto\langle\alpha(x), \beta(y)\rangle$, where $\alpha \in$ $L^{\infty}\left(X, \mathcal{H}_{\rho}\right)$ and $\beta \in L^{\infty}\left(Y, \mathcal{H}_{\rho}\right)$, is $\omega$-continuous; thus, the set

$$
\{(x, y):\langle\alpha(x), \beta(y)\rangle \neq 0\}
$$

is $\omega$-open. It follows that the set

$$
\bigcup_{a \in A_{0}, \xi, \eta \in \mathcal{H}_{0}}\{(x, y):\langle\rho(a) \mathcal{V}(x) \xi, \mathcal{W}(y) \eta\rangle \neq 0\}
$$

is $\omega$-open. Hence there are families $A_{n} \subseteq X, B_{n} \subseteq Y$ of measurable sets such that $R$ is marginally equivalent to $\cup_{n=1}^{\infty} A_{n} \times B_{n}$. Since $(\mu \times \nu)(R)=0$ we have $\mu\left(A_{n}\right) \nu\left(B_{n}\right)=0$ for each $n$. Let

$$
N_{1}:=\bigcup_{\nu\left(B_{n}\right) \neq 0} A_{n} \quad \text { and } \quad N_{2}:=\bigcup_{\mu\left(A_{n}\right) \neq 0} B_{n} .
$$

We have that $\mu\left(N_{1}\right)=0, \nu\left(N_{2}\right)=0$ and $R$ that is marginally equivalent to a subset of $N_{1} \times Y \cup X \times N_{2}$; thus, $R$ is marginally null.

The next lemma contains a completely isometric version of the main transference result of [28, Section 3].

Lemma 2.7. Let $(A, G, \alpha)$ be a $C^{*}$-dynamical system. The map $\mathcal{N}$ is a completely isometric algebra homomorphism from the space of $\operatorname{Herz-Schur}(A, G, \alpha)-$ multipliers to the space of Schur A-multipliers on $G \times G$.

Proof. Fix $n \in \mathbb{N}$ and $\operatorname{Herz-Schur}(A, G, \alpha)$-multipliers $F_{i, j}, 1 \leq i, j \leq n$. Since $\left(S_{F_{i, j}}\right)_{i, j}$ is an element of $\mathrm{CB}\left(A \rtimes_{\alpha, r} G, M_{n}\left(A \rtimes_{\alpha, r} G\right)\right)$ there exist a representation $\rho: A \rtimes_{\alpha, r} G \rightarrow \mathcal{B}\left(\mathcal{H}_{\rho}\right)$ and operators $V, W: L^{2}(G, \mathcal{H}) \rightarrow \mathcal{H}_{\rho}$ such that $\left(S_{F_{i, j}}\right)_{i, j}=W^{*} \rho(\cdot) V$ and $\|V\|\|W\|=\left\|\left(S_{F_{i, j}}\right)_{i, j}\right\|_{\mathrm{cb}}$. Take $a \in A$ and $r \in G$. Arguing as in the proof of [28, Theorem 3.8] we obtain representations $\rho_{A}$ and $\rho_{G}$, of $A$ and $G$ respectively, such that

$$
\left(\pi\left(F_{i, j}(t)(a)\right) \lambda_{r}\right)_{i, j}=\left(S_{F_{i, j}}\left(\pi(a) \lambda_{r}\right)\right)_{i, j}=W^{*} \rho_{A}(a) \rho_{G}(r) V
$$

Define

$$
\mathcal{V}(s):=\rho_{G}\left(s^{-1}\right) V \lambda_{s} \quad \text { and } \quad \mathcal{W}(t):=\rho_{G}\left(t^{-1}\right) W \lambda_{t},
$$

so that $\sup _{s \in G}\|\mathcal{V}(s)\| \sup _{t \in G}\|\mathcal{W}(t)\|=\|V\|\|W\|=\left\|\left(S_{F_{i, j}}\right)_{i, j}\right\|_{c b}$. Calculations as in the proof of [28, Theorem 3.8] show that

$$
\left(\mathcal{N}\left(F_{i, j}\right)(s, t)(a)\right)_{i, j}=\mathcal{W}(t)^{*} \rho_{A}(a) \mathcal{V}(s)
$$

almost everywhere, so

$$
\left\|\left(S_{\mathcal{N}\left(F_{i, j}\right)}\right)_{i, j}\right\|_{\mathrm{cb}} \leq \sup _{s \in G}\|\mathcal{V}(s)\| \sup _{t \in G}\|\mathcal{W}(t)\|=\|V\|\|W\|=\left\|\left(S_{F_{i, j}}\right)_{i, j}\right\|_{\mathrm{cb}}
$$

In the converse direction, note that $\left(S_{F_{i, j}}\right)_{i, j}$ is the restriction of $\left(S_{\mathcal{N}\left(F_{i, j}\right)}\right)_{i, j}$ to $M_{n}\left(A \rtimes_{\alpha, r} G\right)$, so $\left.\left\|\left(S_{F_{i, j}}\right)_{i, j}\right\|_{\mathrm{cb}} \leq \|\left(S_{\mathcal{N}\left(F_{i, j}\right)}\right)\right)_{i, j} \|_{\mathrm{cb}}$. Thus $F \mapsto \mathcal{N}(F)$ is a complete isometry. The homomorphism claim is trivial.

## 3. Central multipliers

Let $(X, \mu)$ and $(Y, \nu)$ be standard measure spaces. We denote for brevity by $\mathcal{B}$ (resp. $\mathcal{K}$ ) the space $\mathcal{B}\left(L^{2}(X, \mu), L^{2}(Y, \nu)\right)$ (resp. $\mathcal{K}\left(L^{2}(X, \mu), L^{2}(Y, \nu)\right)$ ). Throughout this section $A$ denotes a separable $C^{*}$-algebra, acting non-degenerately on a separable Hilbert space $\mathcal{H}$. The multiplier algebra of $A$ will be written $\mathcal{M}(A)$ and identified with the idealiser of $A$ in $\mathcal{B}(\mathcal{H})$ :

$$
\mathcal{M}(A)=\{c \in \mathcal{B}(\mathcal{H}): c a, a c \in A \text { for all } a \in A\}
$$

As usual, we denote by $Z(B)$ the centre of the $C^{*}$-algebra $B$.
The following is immediate, and will be used several times in the sequel.
Remark 3.1. Let $B \subseteq A$ be a $C^{*}$-subalgebra, and $\varphi: X \times Y \rightarrow \mathrm{CB}(A)$ be a Schur $A$-multiplier. Suppose that $\varphi(x, y)$ leaves $B$ invariant for almost all $(x, y)$, and let $\varphi_{B}: X \times Y \rightarrow \mathrm{CB}(B)$ be the map given by $\varphi_{B}(x, y)(b):=\varphi(x, y)(b)(b \in B$, $(x, y) \in X \times Y)$. Then $\varphi_{B}$ is a Schur $B$-multiplier and $\left\|\varphi_{B}\right\|_{\mathfrak{S}} \leq\|\varphi\|_{\mathfrak{S}}$.

### 3.1. Central Schur multipliers.

Definition 3.2. A Schur $A$-multiplier $\varphi \in \mathbb{S}(X, Y ; A)$ will be called central if there exists a family $\left(a_{x, y}\right)_{(x, y) \in X \times Y} \subseteq Z(\mathcal{M}(A))$ such that

$$
\begin{equation*}
\varphi(x, y)(a)=a_{x, y} a, \quad a \in A \tag{2}
\end{equation*}
$$

Remark 3.3. Let $\varphi \in \mathbb{S}(X, Y ; A)$ be a central Schur $A$-multiplier.
i. The family $\left(a_{x, y}\right)_{(x, y) \in X \times Y}$ associated to $\varphi$ in Definition 3.2 is unique up to a set of zero product measure.
ii. If $\left(a_{x, y}\right)_{(x, y) \in X \times Y}$ is associated to $\varphi$ as in Definition 3.2 then the map $X \times Y \rightarrow$ $Z(\mathcal{M}(A)),(x, y) \mapsto a_{x, y}$, is weakly measurable.

Let $A$ be a commutative $C^{*}$-algebra, and assume that $A=C_{0}(Z)$, where $Z$ is a locally compact Hausdorff space. The standing separability assumption implies that $Z$ is second-countable, and hence metrisable. Since $C_{0}(Z)$ is separable it has a faithful state, so the associated Radon measure $m$ on $Z$ has full support.

Let $C_{0}(Z, B)$ be the space of all continuous functions from $Z$ into a normed space $B$ vanishing at infinity. We write $\mathcal{K}=\mathcal{K}\left(L^{2}(X), L^{2}(Y)\right)$ and note that, up to a canonical $*$-isomorphism,

$$
\begin{equation*}
\mathcal{K} \otimes C_{0}(Z)=C_{0}(Z, \mathcal{K}) \tag{3}
\end{equation*}
$$

The algebraic tensor product $L^{2}(Y \times X) \odot C_{0}(Z)$ can thus be viewed as a (dense) subspace of both the space $\mathcal{K} \otimes C_{0}(Z)$ and the space $C_{0}(Z, \mathcal{K})$.

Let $\varphi \in \mathbb{S}\left(X, Y ; C_{0}(Z)\right)$ be a central Schur $C_{0}(Z)$-multiplier, associated with a family $\left(a_{x, y}\right)_{(x, y) \in X \times Y} \subseteq C_{b}(Z)$ as in Definition 3.2; we view $\varphi$ as a scalar-valued function on $X \times Y \times Z$ by letting

$$
\varphi(x, y, z)=a_{x, y}(z), \quad x \in X, y \in Y, z \in Z
$$

By definition, $\varphi$ is a bounded, measurable function on $X \times Y \times Z$ which is continuous in the $Z$-variable. On the other hand, suppose $\varphi: X \times Y \times Z \rightarrow \mathbb{C}$ is a bounded measurable function, continuous in the $Z$-variable. Then $(x, y) \mapsto$ $\varphi(x, y, \cdot) a(\cdot) \in C_{0}(Z)$ is weakly measurable for each $a \in C_{0}(Z)$. Indeed, the function $(x, y) \mapsto \delta_{z}(\varphi(x, y)(a))=\varphi(x, y, z) a(z)$ is measurable for each $z \in Z$ (here $\delta_{z}$ denotes the point mass measure at $z \in Z$ ). As any $m \in M(Z)=C_{0}(Z)^{*}$ is the weak* limit of linear combinations of point mass measures, we conclude that the function $(x, y) \mapsto m(\varphi(x, y)(a))$ is measurable for all $m \in M(Z)$. We thus identify the central Schur $C_{0}(Z)$-multipliers with bounded measurable functions $\varphi: X \times Y \times Z \rightarrow \mathbb{C}$, continuous in the $Z$-variable. For each $z \in Z$, let $\varphi_{z}: X \times Y \rightarrow \mathbb{C}$ be given by $\varphi_{z}(x, y)=\varphi(x, y, z)$; clearly, $\varphi_{z}$ is a measurable function for each $z \in Z$.

We recall some terminology from [8] that will be used in the sequel. Let $\varphi \in L^{\infty}(X \times Y \times Z)$ and associate with it a bounded bilinear map

$$
\Lambda_{\varphi}: \mathcal{S}_{2}(Y, Z) \times \mathcal{S}_{2}(X, Y) \rightarrow \mathcal{S}_{2}(X, Z) ; \Lambda_{\varphi}\left(T_{h}, T_{k}\right):=T_{\varphi(h * k)}
$$

where $k \in L^{2}(Y \times X), h \in L^{2}(Z \times Y)$ and

$$
\varphi(h * k)(z, x):=\int_{Y} \varphi(x, y, z) h(z, y) k(y, x) d y, \quad(x, z) \in X \times Z .
$$

By [8, Corollary 10] the norm $\left\|\Lambda_{\varphi}\right\|$ of $\Lambda_{\varphi}$ as a bilinear map, where the spaces $\mathcal{S}_{2}(Y, Z)$ and $\mathcal{S}_{2}(X, Y)$ are equipped with their Hilbert-Schmidt norm, is equal to $\|\varphi\|_{\infty}$. We say that $\varphi$ is an operator $\mathcal{S}_{1}$-multiplier if $\Lambda_{\varphi}$ maps $\mathcal{S}_{2}(Y, Z) \times$ $\mathcal{S}_{2}(X, Y)$ into $\mathcal{S}_{1}(X, Z)$. The following characterisation of operator $\mathcal{S}_{1}$-multipliers was obtained in [8]:

Theorem 3.4. Let $\varphi: X \times Y \times Z \rightarrow \mathbb{C}$ be a bounded measurable function. The following are equivalent:
(i) the function $\varphi$ is an operator $\mathcal{S}_{1}$-multiplier;
(ii) there exist a Hilbert space $\mathcal{L}$ and weakly measurablefunctions $v: X \times Z \rightarrow \mathcal{L}$, $w: Y \times Z \rightarrow \mathcal{L}$, satisfying

$$
\operatorname{esssup}_{(x, z) \in X \times Z}\|v(x, z)\|<\infty, \underset{(y, z) \in Y \times Z}{\operatorname{esssup}}\|w(y, z)\|<\infty,
$$

such that

$$
\begin{equation*}
\varphi(x, y, z)=\langle v(x, z), w(y, z)\rangle, \quad \text { almost all }(x, y, z) \in X \times Y \times Z . \tag{4}
\end{equation*}
$$

Moreover, if these conditions hold then

$$
\|\varphi\| \Im=\operatorname{esssup}_{(x, z) \in X \times Z}\|v(x, z)\| \operatorname{esssup}_{(y, z) \in Y \times Z}\|w(y, z)\| .
$$

In Theorem 3.6, we relate operator $\mathcal{S}_{1}$-multipliers to central multipliers. We first include a lemma. If $\mathcal{E}$ is an operator space then we identify $C_{0}(Z) \odot \mathcal{E}$ with a dense subspace of the minimal tensor product $C_{0}(Z) \otimes \mathcal{E}$ (and equip it with the operator space structure arising from this inclusion), and its elements - with continuous functions from $Z$ into $\mathcal{E}$. If $\mathcal{E}$ is in addition an operator system, we equip the algebraic tensor product $C_{0}(Z) \odot \mathcal{E}$ with the operator system structure arising from its inclusion in $C_{0}(Z) \otimes \mathcal{E}$.

Lemma 3.5. Let $Z$ be a locally compact Hausdorff space and $\mathcal{E}$ be an operator space. Let $\Phi_{z}: \mathcal{E} \rightarrow \mathcal{E}$ be a linear map, $z \in Z$, and $\Phi: C_{0}(Z) \odot \mathcal{E} \rightarrow C_{0}(Z) \otimes \mathcal{E}$ a linear map defined by

$$
\Phi(a \otimes T)(z)=a(z) \Phi_{z}(T), \quad z \in Z .
$$

The following are equivalent:
(i) $\Phi$ is completely bounded;
(ii) $\Phi_{z}$ is completely bounded for every $z \in Z$ and $\sup _{z \in Z}\left\|\Phi_{z}\right\|_{\mathrm{cb}}<\infty$. Moreover, if these conditions are fulfilled then $\|\Phi\|_{\mathrm{cb}}=\sup _{z \in Z}\left\|\Phi_{z}\right\|_{\mathrm{cb}}$.

Assume that $\mathcal{E}$ is an operator system. The following are equivalent:
(i) $\Phi$ is completely positive;
(ii') $\Phi_{z}$ is completely positive for every $z \in Z$.
Proof. (i) $\Longrightarrow$ (ii) Fix $z \in Z$ and note that, if $a \in C_{0}(Z)$ has norm one and $a(z)=1$ then

$$
\Phi_{z}(T)=\left(\delta_{z} \otimes \mathrm{id}\right)(\Phi(a \otimes T)), \quad T \in \mathcal{E}
$$

It follows that $\Phi_{z}$ is completely bounded and

$$
\begin{equation*}
\sup _{z \in Z}\left\|\Phi_{z}\right\|_{\mathrm{cb}} \leq\|\Phi\|_{\mathrm{cb}} . \tag{5}
\end{equation*}
$$

(ii) $\Longrightarrow$ (i) We identify $M_{n}\left(C_{0}(Z) \odot \mathcal{E}\right)$ with a subspace of $C_{0}\left(Z, M_{n}(\mathcal{E})\right)$ in the canonical way. Let $\left(h_{i, j}\right)_{i, j} \in M_{n}\left(C_{0}(Z) \odot \mathcal{E}\right)$. The claim is immediate from the fact that

$$
\Phi^{(n)}\left(\left(h_{i, j}\right)_{i, j}\right)(z)=\left(\Phi\left(h_{i, j}\right)(z)\right)_{i, j}=\left(\Phi_{z}\left(h_{i, j}(z)\right)\right)_{i, j} .
$$

It remains to note the reverse inequality in (5); it follows by the fact that, if $a \in C_{0}(Z)$ has norm one and $a(z)=1$ then $\left\|\Phi_{z}^{(n)}(T)\right\| \leq\left\|\Phi^{(n)}(a \otimes T)\right\|$, for every $T \in M_{n}(\mathcal{E})$.

Now assume that $\mathcal{E}$ is an operator system.
( $\mathrm{i}^{\prime}$ ) $\Longrightarrow$ (ii') follows as the implication (i) $\Longrightarrow$ (ii), by choosing the function $a$ to be in addition positive.
(ii') $\Longrightarrow$ (i') follows similarly to the implication (ii) $\Longrightarrow$ (i), by taking into account that a matrix $\left(h_{i, j}\right)_{i, j}$ belongs to the positive cone of $M_{n}\left(C_{0}(Z) \odot \mathcal{E}\right)$ if and only if $\left(h_{i, j}(z)\right)_{i, j} \in M_{n}^{+}$for every $z \in Z$.
Theorem 3.6. Let $\varphi: X \times Y \times Z \rightarrow \mathbb{C}$ be a bounded measurable function, continuous in the $Z$-variable. The following are equivalent:
(i) $\varphi$ is a central Schur $C_{0}(Z)$-multiplier;
(ii) the function $\varphi_{z}$ is a Schur multiplier for every $z \in Z$, and the map $D_{\varphi}$ : $C_{0}(Z, \mathcal{K}) \rightarrow C_{0}(Z, \mathcal{K})$ given by

$$
D_{\varphi}(h)(z)=S_{\varphi_{z}}(h(z)), \quad z \in Z,
$$

is completely bounded;
(iii) the function $\varphi_{z}$ is a Schur multiplier for every $z \in Z$, and

$$
\sup _{z \in Z}\left\|\varphi_{z}\right\| \subseteq<\infty
$$

(iv) the function $\varphi$ is an operator $\mathcal{S}_{1}$-multiplier.

If these conditions hold then $\|\varphi\|_{\mathfrak{S}}=\sup _{z \in Z}\left\|\varphi_{z}\right\| \Subset$.
Proof. (i) $\Longleftrightarrow$ (ii) Let $\varphi$ be a central Schur $C_{0}(Z)$-multiplier. We fix a measure $m \in M(Z)$ so that the representation of $C_{0}(Z)$ on $L^{2}(Z, m)$, given by $a \mapsto M_{a}$, where

$$
\left(M_{a} \xi\right)(z):=a(z) \xi(z), \quad a \in C_{0}(Z), \xi \in L^{2}(Z, m), z \in Z
$$

is faithful. By [28, Proposition 2.3], we may identify $C_{0}(Z)$ with its image in $\mathcal{B}\left(L^{2}(Z)\right)$, so we abuse notation by writing $a$ in place of $M_{a}$. We recall that the map $S_{\varphi}$ extends to a completely bounded map on $\mathcal{K} \otimes C_{0}(Z)$. We observe that, when the identification (3) is made, we have that the map $S_{\varphi}$ (which is defined as a transformation on $\mathcal{K} \otimes C_{0}(Z)$ ) is identified with $D_{\varphi}$. Indeed, if $k \in L^{2}(Y \times X)$ and $a \in C_{0}(Z)$ then

$$
S_{\varphi}(k \otimes a)(z)=(\varphi(\cdot, \cdot, z) \cdot k) a(z)=D_{\varphi}(k \otimes a)(z), \quad z \in Z
$$

The equivalence now follows.
(ii) $\Longleftrightarrow$ (iii) is immediate from Lemma 3.5.
(i) $\Longrightarrow$ (iv) Define a map $\psi: f \mapsto \psi_{f}$, on $L^{1}(Z)$ by letting

$$
\psi_{f}(x, y):=\int_{Z} \varphi(x, y, z) f(z) d z, \quad(x, y) \in X \times Y
$$

We will show that $\psi_{f}$ belongs to $L^{\infty}(X) \otimes^{\omega^{*} h} L^{\infty}(Y)$ and has norm at most $\|\varphi\|_{\subseteq}$. Take $f \in C_{c}(Z), k \in L^{2}(Y \times X)$, and $a \in C_{0}(Z)$ with $\|a\|=1$ and $a(z)=1$ for all $z \in \operatorname{supp}(f)$. Writing $f=f_{1} f_{2}, f_{1}, f_{2} \in L^{2}(Z),\|f\|_{1}=\left\|f_{1}\right\|_{2}\left\|f_{2}\right\|_{2}$, for $\xi \in L^{2}(X)$ and $\eta \in L^{2}(Y)$, we have

$$
\begin{aligned}
\left|\left\langle S_{\psi_{f}}\left(T_{k}\right) \xi, \eta\right\rangle\right| & =\left|\int_{X \times Y}\left(\int_{Z} \varphi(x, y, z) f(z) d z\right) k(y, x) \xi(x) \overline{\eta(y)} d x d y\right| \\
& =\left|\left\langle S_{\varphi}\left(T_{k \otimes a}\right)\left(\xi \otimes f_{1}\right), \eta \otimes f_{2}\right\rangle\right| \\
& \leq\|\varphi\| \Im\left\|T_{k \otimes a}\right\|\|\xi\|_{2}\left\|f_{1}\right\|_{2}\|\eta\|_{2}\left\|f_{2}\right\|_{2} \\
& \leq\|\varphi\| \Im\left\|T_{k}\right\|\|f\|_{1}\|\xi\|_{2}\|\eta\|_{2} .
\end{aligned}
$$

Thus the $\operatorname{map} S_{\psi_{f}}$ is bounded in the operator norm, implying that $\psi_{f}$ is a Schur multiplier with $\left\|\psi_{f}\right\|_{\Im} \leq\|\varphi\|_{\Im}\|f\|_{1}$. It follows from the density of $C_{c}(Z)$ in $L^{1}(Z)$ that $\psi$ is a bounded map, with $\|\psi\| \leq\|\varphi\|_{\Im}$; we view $\psi$ as taking values
in $L^{\infty}(X) \otimes^{\omega^{*} h} L^{\infty}(Y)$ using the standard identification of this tensor product with the Schur multipliers on $X \times Y$.

By standard operator space identifications (see [8] and [24]), we have

$$
\psi \in \mathcal{B}\left(L^{1}(Z), L^{\infty}(X) \otimes^{w^{*} h} L^{\infty}(Y)\right) \cong L^{\infty}(Z) \bar{\otimes}\left(L^{\infty}(X) \otimes^{\omega^{*} h} L^{\infty}(Y)\right)
$$

where $\varphi \in L^{\infty}(X \times Y \times Z)$ is the corresponding element in $L^{\infty}(Z) \bar{\otimes}\left(L^{\infty}(X) \otimes^{w^{*} h}\right.$ $L^{\infty}(Y)$ ). Condition (iv) now follows by [8, Theorem 19] and Theorem 3.4.
(iv) $\Longrightarrow$ (i) Let $v$ and $w$ be the functions arising as in Theorem 3.4, and $M \subseteq X \times Y \times Z$ be a set with $(\mu \times \nu \times m)\left(M^{c}\right)=0$, such that (4) holds for all $(x, y, z) \in M$. Set $M_{x, y}=\{z:(x, y, z) \in M\}$ and $N=\left\{(x, y): m\left(M_{x, y}^{c}\right)=0\right\} ;$ it is clear that $(\mu \times \nu)\left(N^{c}\right)=0$. Write $\mathcal{W}(y): L^{2}(Z) \rightarrow \mathcal{L} \otimes L^{2}(Z)$ and $\mathcal{V}(x):$ $L^{2}(Z) \rightarrow \mathcal{L} \otimes L^{2}(Z)$ for the maps, given by

$$
(\mathcal{V}(x) \xi)(z):=v(x, z) \xi(z) \text { and }(\mathcal{W}(y) \xi)(z):=w(y, z) \xi(z), \quad \xi \in L^{2}(Z)
$$

we have

$$
\begin{aligned}
& \underset{x \in X}{\operatorname{esssup}}\|\mathcal{V}(x)\|=\underset{(x, z) \in X \times Z}{\operatorname{esssup}}\|v(x, z)\|<\infty, \\
& \underset{y \in Y}{\operatorname{esssup}}\|\mathcal{W}(y)\|=\underset{(y, z) \in Y \times Z}{\operatorname{esssup}}\|w(y, z)\|<\infty .
\end{aligned}
$$

For $a \in C_{0}(Z), \xi, \eta \in L^{2}(Z)$ and $(x, y) \in N$, we have

$$
\begin{aligned}
\left\langle\mathcal{W}(y)^{*}\left(I \otimes M_{a}\right) \mathcal{V}(x) \xi, \eta\right\rangle & =\left\langle\left(I \otimes M_{a}\right) \mathcal{V}(x) \xi, \mathcal{W}(y) \eta\right\rangle \\
& =\int_{Z} a(z)\langle v(x, z), w(y, z)\rangle \xi(z) \overline{\eta(z)} d m(z) \\
& =\int_{Z} a(z) \varphi(x, y, z) \xi(z) \overline{\eta(z)} d m(z) .
\end{aligned}
$$

It follows that, if $(x, y) \in N$ then

$$
\mathcal{W}(y)^{*}\left(I \otimes M_{a}\right) \mathcal{V}(x)=M_{\varphi_{x, y} a}, \quad a \in C_{0}(Z)
$$

(here $\varphi_{x, y}$ is the function on $Z$ given by $\varphi_{x, y}(z)=\varphi(x, y, z)$ ). By [28, Theorem 2.6], $\varphi$ is a Schur $C_{0}(Z)$-multiplier which is clearly central, and

$$
\|\varphi\| \subseteq \leq \operatorname{esssup}_{x \in X}\|\mathcal{V}(x)\| \underset{y \in Y}{\operatorname{esssup}}\|\mathcal{W}(y)\|=\underset{z \in Z}{\operatorname{esssup}}\left\|\varphi_{z}\right\| \subseteq
$$

Finally, from the proof of (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii), equation (5), and the estimate in (iv) $\Longrightarrow$ (i) we have $\|\varphi\|_{\Im}=\sup _{z \in Z}\left\|\varphi_{z}\right\|_{\Im}$.

In the next result we assume that $A$ acts non-degenerately on a separable Hilbert space $\mathcal{H}$, and we identify the elements of the centre $Z(\mathcal{M}(A))$ of $A$ with completely bounded maps on $A$ acting by operator multiplication.

Corollary 3.7. Let $\varphi: X \times Y \rightarrow Z(\mathcal{M}(A))$ be a pointwise-measurable function, and assume that $\overline{Z(A) A}=A$. The following are equivalent:
(i) $\varphi$ is a central Schur A-multiplier;
(ii) there exist an index set I and operators $V \in C_{I}^{\omega}\left(L^{\infty}\left(X, Z(A)^{\prime \prime}\right)\right)$ and $W \in$ $C_{I}^{\omega}\left(L^{\infty}\left(Y, Z(A)^{\prime \prime}\right)\right)$, such that

$$
\varphi(x, y)=\sum_{i \in I} W_{i}(y)^{*} V_{i}(x), \quad \text { for almost all }(x, y) \in X \times Y
$$

Moreover, if $\varphi: X \times Y \rightarrow Z(\mathcal{N}(A))$ is weakly measurable then the above conditions are equivalent to:
(iii) $\varphi$ is a central Schur $B$-multiplier for any $C^{*}$-algebra $B \subseteq \mathcal{B}(\mathcal{H})$ with $Z(A) \subseteq$ $Z(B)$.
If the conditions hold we may choose $V, W$ such that

$$
\|\varphi\|_{\Im}=\|V\|_{C_{I}^{\omega}\left(L^{\infty}\left(X, Z(A)^{\prime \prime}\right)\right)}\|W\|_{C_{I}^{\omega}\left(L^{\infty}\left(Y, Z(A)^{\prime \prime}\right)\right)}
$$

where $\|\varphi\|_{\Im}$ is the norm of the Schur multiplier in either (i) or (iii).
Proof. Since $\overline{Z(A) A}=A$, the algebra $Z(A)$ is non-degenerate and $Z(A)^{\prime \prime}=$ $\overline{Z(A)}^{w}$, where the latter closure is in the weak operator topology.
(i) $\Longrightarrow$ (ii) By Remark 3.1, $\varphi$ is a Schur $Z(A)$-multiplier. Following the proof of Theorem 3.6, and using the identification $Z(A) \cong C_{0}(Z)$ and $Z(A)^{\prime \prime} \cong$ $L^{\infty}(Z, m)$, for some measure space ( $Z, m$ ), we identify $\varphi$ with an element of $L^{\infty}(Z, m) \bar{\otimes}\left(L^{\infty}(X) \otimes^{w^{*} h} L^{\infty}(Y)\right)$. Using [8], we see that there exist an index set $I$ and two families $\left(V_{i}\right)_{i \in I},\left(W_{i}\right)_{i \in I}$, where $V_{i}: X \rightarrow Z(A)^{\prime \prime}$ and $W_{i}: Y \rightarrow Z(A)^{\prime \prime}$ are measurable functions satisfying

$$
\operatorname{esssup}_{x \in X}\left\|\sum_{i \in I} V_{i}(x)^{*} V_{i}(x)\right\|<\infty \quad \text { and } \quad \underset{y \in Y}{\operatorname{esssup}}\left\|\sum_{i \in I} W_{i}(y)^{*} W_{i}(y)\right\|<\infty,
$$

such that $\varphi(x, y)=\sum_{i \in I} W_{i}(y)^{*} V_{i}(x)$ almost everywhere on $X \times Y$ (the series converges weakly) and

$$
\begin{equation*}
\|\varphi\|_{\subseteq(X, Y ; Z(A))}=\underset{x \in X}{\operatorname{esssup}}\left\|\sum_{i \in I} V_{i}(x)^{*} V_{i}(x)\right\| \operatorname{esssup}_{y \in Y}\left\|\sum_{i \in I} W_{i}(y)^{*} W_{i}(y)\right\| . \tag{6}
\end{equation*}
$$

(ii) $\Longrightarrow$ (i) For $a \in A$, we have

$$
\begin{equation*}
\varphi(x, y)(a)=\sum_{i \in I} W_{i}(y)^{*} V_{i}(x) a=\sum_{i \in I} W_{i}(y)^{*} a V_{i}(x)=\mathcal{W}^{*}(y) \rho(a) \mathcal{V}(x), \tag{7}
\end{equation*}
$$

where $\mathcal{V}(x):=\left(V_{i}(x)\right)_{i \in I}, \mathcal{W}(y):=\left(W_{i}(y)\right)_{i \in I}$ and $\rho(a):=\operatorname{id}_{\ell^{2}(I)} \otimes a$. By [28, Theorem 2.6] $\varphi$ is a Schur $A$-multiplier, and it is clearly central.
(ii) $\Longrightarrow$ (iii) The assumption implies that $(x, y) \mapsto \varphi(x, y)(b) \in B$ is weakly measurable for all $b \in B$, so it makes sense to speak of $\varphi$ being a Schur $B$ multiplier. Now the same proof as that of the implication (ii) $\Longrightarrow$ (i) can be applied.
(iii) $\Longrightarrow$ (i) is trivial.

For the norm equality observe that $\|\varphi\|_{\Im_{(X, Y ; B)}} \geq\|\varphi\|_{\Im_{(X, Y ; Z(A))}}$ while, by (7), we have

$$
\begin{aligned}
\|\varphi\| \Subset(X, Y ; B) & \leq \underset{x \in X}{\operatorname{esssup}}\|\mathcal{V}(x)\| \operatorname{esssup}_{y \in Y}^{\operatorname{ess}}\|\mathcal{W}(y)\| \\
& =\operatorname{esssup}_{x \in X}^{\operatorname{essup}}\left\|\sum_{i \in I} V_{i}(x)^{*} V_{i}(x)\right\| \underset{y \in Y}{\operatorname{esssup}}\left\|\sum_{i \in I} W_{i}(y)^{*} W_{i}(y)\right\| \\
& =\|V\|_{C_{I}^{\omega}\left(L^{\infty}\left(X, Z(A)^{\prime \prime}\right)\right)}\|W\|_{C_{I}^{\omega}\left(L^{\infty}\left(Y, Z(A)^{\prime \prime}\right)\right)} .
\end{aligned}
$$

The equality follows by combining this with (6).
We remark that the results of this subsection and the rest of the section remain true when $X$ and $Y$ are discrete spaces with counting measures, $Z$ is an arbitrary (not necessarily second countable) locally compact Hausdorff space and $A$ is an arbitrary (not necessarily separable ) $C^{*}$-algebra.
3.2. Central Herz-Schur multipliers. In this subsection, similarly to Theorem 3.6, we characterise central Herz-Schur multipliers, a natural invariant version of central Schur multipliers, which we now introduce.

Definition 3.8. Let $(A, G, \alpha)$ be a $C^{*}$-dynamical system. A Herz-Schur $(A, G, \alpha)$ multiplier $F$ will be called central if there exists a family $\left(a_{r}\right)_{r \in G} \subseteq Z(\mathcal{M}(A))$ such that

$$
F(r)(a)=a_{r} a, \quad a \in A, r \in G .
$$

Proposition 3.9. Let $A$ be a $C^{*}$-algebra such that $\overline{Z(A) A}=A,(A, G, \alpha)$ be a $C^{*}$-dynamical system, $\left(a_{r}\right)_{r \in G}$ be a family in $Z(\mathcal{M}(A))$ and suppose that the map $F: G \rightarrow \mathrm{CB}(A)$, given by $F(r)(a)=a_{r} a$, is pointwise-measurable. The following are equivalent:
(i) $F$ is a central Herz-Schur $(Z(A), G, \alpha)$-multiplier;
(ii) $F$ is a central Herz-Schur ( $A, G, \alpha$ )-multiplier;
(iii) there exist $V, W \in C_{I}^{\omega}\left(L^{\infty}\left(G, Z(A)^{\prime \prime}\right)\right)$ such that

$$
\alpha_{t^{-1}}\left(a_{t s^{-1}}\right)=\sum_{i \in I} W_{i}(t)^{*} V_{i}(s), \quad \text { for almost all }(s, t) \in G \times G .
$$

Moreover, $V$ and $W$ may be chosen so that

$$
\|F\|_{\mathrm{HS}}=\|V\|_{C_{I}^{\omega}\left(L^{\infty}\left(G, Z(A)^{\prime \prime}\right)\right)}\|W\|_{C_{I}^{\omega}\left(L^{\infty}\left(G, Z(A)^{\prime \prime}\right)\right)}
$$

where $\|F\|_{\mathrm{HS}}$ refers to the norm of $F$ in either (i) or (ii).
Proof. (i) $\Longrightarrow$ (ii) By [28, Theorem 3.8] $\mathcal{N}(F)$ is a Schur $Z(A)$-multiplier; it is clearly central. Using the assumption $\overline{Z(A) A}=A$ we observe that $Z(A)$ acts non-degenerately on any Hilbert space where $A$ acts non-degenerately, so by Corollary 3.7 we have that $\mathcal{N}(F)$ is a central Schur $A$-multiplier. Applying again [28, Theorem 3.8], we obtain that $F$ is a central Herz-Schur $(A, G, \alpha)$-multiplier.
(ii) $\Longrightarrow$ (i) Immediate from [28, Theorem 3.8] and Remark 3.1.
(i) $\Longrightarrow$ (iii) By $[28$, Theorem 3.8] $\mathcal{N}(F)$ is a central Schur $Z(A)$-multiplier, and for $a \in A$ and $s, t \in G$,

$$
\mathcal{N}(F)(s, t)(a)=\alpha_{t^{-1}}\left(a_{t s^{-1}}\right) a, \quad a \in A .
$$

By Corollary 3.7(ii), there exist $V, W \in C_{I}^{\omega}\left(L^{\infty}\left(G, Z(A)^{\prime \prime}\right)\right)$ such that

$$
\alpha_{t^{-1}}\left(a_{t s^{-1}}\right) a=\sum_{i \in I} W_{i}(t)^{*} a V_{i}(s)=\sum_{i \in I} W_{i}(t)^{*} V_{i}(s) a \quad \text { almost everywhere. }
$$

Since this holds for every $a \in A$ and $A \subseteq \mathcal{B}(\mathcal{H})$ is separable and non-degenerate, we conclude that

$$
\alpha_{t^{-1}}\left(a_{t s^{-1}}\right)=\sum_{i \in I} W_{i}(t)^{*} V_{i}(s),
$$

for almost all $(s, t) \in G \times G$.
(iii) $\Longrightarrow$ (i) For $a \in A$ and almost all $s, t \in G$ we have

$$
\mathcal{N}(F)(s, t)(a)=\alpha_{t^{-1}}\left(a_{t s^{-1}}\right) a=\sum_{i \in I} W_{i}(t)^{*} a V_{i}(s)=\mathcal{W}(t)^{*} \rho(a) \mathcal{V}(s),
$$

where $\rho(a):=\operatorname{id}_{\ell^{2}(I)} \otimes a, \mathcal{V}(s):=\left(V_{i}(s)\right)_{i \in I}$ and $\mathcal{W}(t):=\left(W_{i}(t)\right)_{i \in I}$. Therefore $F$ is a Herz-Schur $(Z(A), G, \alpha)$-multiplier by [28, Theorem 3.8].

Since $\mathcal{N}$ is an isometry, the norm equality follows from the norm equality in Theorem 3.7.

A central Herz-Schur $\left(C_{0}(Z), G, \alpha\right)$-multiplier $F: G \rightarrow \mathrm{CB}\left(C_{0}(Z)\right.$ ), associated with a family $\left(a_{r}\right)_{r \in G} \subseteq C_{b}(Z)$, may be identified with a bounded measurable function, continuous in the $Z$-variable, given by

$$
F: G \times Z \rightarrow \mathbb{C} ; F(r, z)=a_{r}(z), \quad r \in G, z \in Z ;
$$

conversely, if $F: G \times Z \rightarrow \mathbb{C}$ is a bounded measurable function, continuous in the $Z$-variable, then the associated function $F: G \rightarrow \mathrm{CB}\left(C_{0}(Z)\right)$ is bounded and pointwise-measurable. In the sequel, if $Z$ is a locally compact Hausdorff space and $\left(C_{0}(Z), G, \alpha\right)$ is a $C^{*}$-dynamical system, we let $(z, t) \rightarrow z t$ be the mapping from $Z \times G$ into $Z$ that satisfies the condition $f(z t)=\alpha_{t}(f)(z), z \in Z$, $t \in G$. The mapping is jointly continuous and satisfies $z(s t)=(z s) t$ for all $z \in Z$ and $s, t \in G$.

Corollary 3.10. Let $\left(C_{0}(Z), G, \alpha\right)$ be a $C^{*}$-dynamical system, and $F: G \times Z \rightarrow \mathbb{C}$ a bounded measurable function, continuous in the $Z$-variable. The following are equivalent:
(i) $F$ is a central Herz-Schur $\left(C_{0}(Z), G, \alpha\right)$-multiplier;
(ii) there exist a Hilbert space $\mathcal{L}$ and weakly measurable bounded functions $v, w$ : $G \times Z \rightarrow \mathcal{L}$ such that

$$
F\left(t s^{-1}, z t^{-1}\right)=\langle v(s, z), w(t, z)\rangle \quad \text { almost all }(s, t, z) \in G \times G \times Z .
$$

Moreover, $\|F\|_{\text {HS }}=\operatorname{esssup}_{(s, x) \in G \times Z}\|v(s, x)\| \operatorname{esssup}_{(t, y) \in G \times Z}\|w(t, y)\|$.

Proof. Immediate from Proposition 3.9 by taking $\mathcal{L}:=\ell^{2}(I)$,

$$
v(s, x)_{i}:=\left(V_{i}(s)\right)(x) \quad \text { and } \quad w(t, y)_{i}:=\left(W_{i}(t)\right)(y), \quad s, t \in G, x, y \in Z
$$

3.3. Positive central multipliers. Positive Schur $A$-multipliers, in the case of sets equipped with the counting measure, were studied in [27] (see [27, Definition 2.3] and [27, Theorem 2.6]). Here we extend this by considering arbitrary standard measure spaces and identifying corresponding versions of the previous results.

Definition 3.11. Let $A$ be a $C^{*}$-algebra. A Schur $A$-multiplier $\varphi: X \times X \rightarrow$ $\mathrm{CB}(A)$ is called positive if $S_{\varphi}$ is completely positive.

Before giving a completely positive version of Theorem 3.6, we include a lemma. Since $L^{\infty}(X) \otimes^{w^{*} h} L^{\infty}(X)=\left(L^{1}(X) \otimes^{h} L^{1}(X)\right)^{*}$, every Schur multiplier $\varphi$ on $X \times X$ gives rise to a canonical bilinear $\operatorname{map} F_{\varphi}: L^{1}(X) \times L^{1}(X) \rightarrow \mathbb{C}$. As usual, we write $F_{\varphi}^{(n, n)}$ for the corresponding amplification, a bilinear map from $M_{n}\left(L^{1}(X)\right) \times M_{n}\left(L^{1}(X)\right)$ into $M_{n}$.
Lemma 3.12. Let $(X, \mu)$ be a standard measure space and $\varphi \in L^{\infty}(X) \otimes^{w^{*} h}$ $L^{\infty}(X)$ be a positive Schur multiplier. If $T=\left(f_{i, j}\right)_{i, j=1}^{n} \in M_{n}\left(L^{1}(X)\right)$ and $T^{*}=$ $\left(\overline{f_{j, i}}\right)_{i, j=1}^{n}$ then $F_{\varphi}^{(n, n)}\left(T, T^{*}\right) \in M_{n}^{+}$.

Proof. Note that, if $\varphi$ is a positive Schur multiplier, by virtue of [16], one may write $\varphi=\sum_{i=1}^{\infty} a_{i} \otimes \overline{a_{i}}$, where $\left(a_{i}\right)_{i=1}^{\infty}$ is a bounded row operator with entries in $L^{\infty}(X)$. It thus suffices to prove the statement in the case where $\varphi=a \otimes \bar{a}$, for some $a \in L^{\infty}(X)$. However, then we have

$$
F_{\varphi}^{(n, n)}\left(T, T^{*}\right)=\left(\sum_{k=1}^{n}\left\langle f_{i, k}, a\right\rangle\left\langle\overline{f_{j, k}}, \bar{a}\right\rangle\right)_{i, j=1}^{n}=\sum_{k=1}^{n}\left(\left\langle f_{i, k}, a\right\rangle \overline{\left\langle f_{j, k}, a\right\rangle}\right)_{i, j=1}^{n},
$$

and the conclusion follows.
Theorem 3.13. Let $\varphi: X \times X \times Z \rightarrow \mathbb{C}$ be a bounded measurable function, continuous in the $Z$-variable. The following are equivalent:
(i) $\varphi$ is a positive central Schur $C_{0}(Z)$-multiplier;
(ii) there exist a Hilbert space $\mathcal{L}$ and an essentially bounded, weakly measurable function $v: X \times Z \rightarrow \mathcal{L}$ such that $\varphi(x, y, z)=\langle v(x, z), v(y, z)\rangle$ for almost all $(x, y, z) \in X \times X \times Z$;
(iii) for each $z \in Z$ the function $\varphi_{z}$ is a positive Schur multiplier, and

$$
\sup _{z \in Z}\left\|\varphi_{z}\right\| \varsigma<\infty
$$

Moreover, if the space $X$ is discrete and $\mu$ is the counting measure the above conditions are equivalent to:
(iv) for any $x_{1}, \ldots, x_{n} \in X$ and $z \in Z$ the matrix $\left(\varphi\left(x_{i}, x_{j}, z\right)\right)_{i, j}$ is positive in $M_{n}$.

Proof. (i) $\Longrightarrow$ (ii) Suppose that $\varphi$ is a positive central Schur $C_{0}(Z)$-multiplier. We have seen in the proof of Theorem 3.6 that $\varphi \in L^{\infty}(Z) \bar{\otimes}\left(L^{\infty}(X) \otimes^{w^{*} h}\right.$ $L^{\infty}(X)$ ). With $\varphi$ we associate the completely bounded bilinear $\operatorname{map} \Phi_{\varphi}: L^{1}(X) \times$ $L^{1}(X) \rightarrow L^{\infty}(Z)$ given by

$$
\Phi_{\varphi}((f, g))(h)=\langle\varphi, h \otimes(f \otimes g)\rangle, \quad f, g \in L^{1}(X), h \in L^{1}(Z)
$$

We obtain

$$
\begin{align*}
\Phi_{\varphi}((f, g))(h) & =\iiint_{X \times X \times Z} \varphi(x, y, z) h(z) f(x) g(y) d x d y d z \\
& =\int_{Z}\left(\int_{X \times X} \varphi_{z}(x, y) f(x) g(y) d x d y\right) h(z) d z \tag{8}
\end{align*}
$$

and

$$
\Phi_{\varphi}((f, g))(z)=\int_{X \times X} \varphi_{z}(x, y) f(x) g(y) d x d y \quad \text { almost everywhere. }
$$

Set

$$
\Phi_{\varphi_{z}}((f, g))=\Phi_{\varphi}((f, g))(z), \quad z \in Z
$$

By Lemma 3.5, $\varphi_{z}$ is a positive Schur multiplier and, by Lemma 3.12,

$$
\Phi_{\varphi_{z}}^{(n, n)}\left(\left(\left(f_{i, j}\right),\left(f_{i, j}^{*}\right)\right)\right) \in M_{n}^{+}
$$

for any $\left(f_{i, j}\right) \in M_{n}\left(L^{1}(X)\right)$. By [40, Theorem 4.4, Remark 4.5(iii)], there exists a family $\left(\psi_{i}\right)_{i \in \Lambda} \subseteq \operatorname{CB}\left(L^{1}(X), L^{\infty}(Z)\right)$ such that $\left\|\sum_{i \in I}\left|\psi_{i}(a)\right|^{2}\right\|_{\infty} \leq C\|a\|_{1}^{2}$, $a \in L^{1}(X)$, for some constant $C>0$, and

$$
\Phi_{\varphi}((a, b))=\sum_{i \in \Lambda} \psi_{i}(a) \psi_{i}\left(b^{*}\right)^{*}, \quad a, b \in L^{1}(X)
$$

Identifying each $\psi_{i}$ with an element $\psi_{i}$ of $L^{\infty}(X \times Z)$ via

$$
\psi_{i}(f)(h)=\int_{X} \int_{Z} \psi_{i}(x, z) f(x) h(z) d x d z, \quad f \in L^{1}(X), h \in L^{1}(Z),
$$

letting $\mathcal{L}=\ell^{2}(\Lambda)$ and $v(x, z):=\left(\psi_{i}(x, z)\right)_{i \in \Lambda}$ gives (ii).

$$
\begin{aligned}
& \text { (ii) } \Longrightarrow \text { (i) Define } \\
& \mathcal{V}(x): L^{2}(Z) \rightarrow \mathcal{L} \otimes L^{2}(Z) ;(\mathcal{V}(x) \xi)(z):=v(x, z) \xi(z), \quad \xi \in L^{2}(Z) .
\end{aligned}
$$

Then

$$
\varphi(x, y)(a)=\mathcal{V}(y)^{*}\left(\operatorname{id} \otimes M_{a}\right) \mathcal{V}(x), \quad a \in C_{0}(Z)
$$

for almost all $(x, y)$ (see the proof of Theorem 3.6 (iv) $\Longleftrightarrow$ (i)). Therefore $\varphi$ is a central Schur $C_{0}(Z)$-multiplier, and (as in the proof of [28, Theorem 2.6]) writing $\rho$ for the representation $a \mapsto \mathrm{id} \otimes M_{a}$ of $C_{0}(Z)$ on $\mathcal{L} \otimes L^{2}(Z)$ we have

$$
S_{\varphi}(T)=\mathcal{V}^{*}(\operatorname{id} \otimes \rho)(T) \mathcal{V}, \quad T \in \mathcal{K}\left(L^{2}(X)\right) \otimes C_{0}(Z)
$$

Hence $S_{\varphi}$ is completely positive.
(i) $\Longleftrightarrow$ (iii) follows from the following two facts: (a) since $\varphi$ is a Schur $C_{0}(Z)$ multiplier, we have that $S_{\varphi}(K)(z)=S_{\varphi_{z}}(K(z)), z \in Z$ for any $K \in C_{0}(Z, \mathcal{K})$, and
(b) an element $K \in C_{0}(Z, \mathcal{K})$ is positive if and only if $K(z) \geq 0$ as an operator in $\mathcal{K}$ for all $z \in Z$.

Now assume that $\mu$ is the counting measure on the discrete space $X$. Observe that (iv) is equivalent to ( $\varphi\left(x_{i}, x_{j}\right)$ ) being a positive element of $M_{n}\left(C_{0}(Z)\right)$.
(i) $\Longrightarrow$ (iv) Let $x_{1}, \ldots, x_{n} \in X$. By [27, Theorem 2.6], the matrix $\left(\varphi\left(x_{i}, x_{j}\right)(a)\right)$ $\in M_{n}\left(C_{0}(Z)\right)$ is positive when $a \in C_{0}(Z)$ is positive. For a fixed $z_{0} \in Z$, let $a \in C_{0}(Z)$ be such that $a\left(z_{0}\right)=1$. It follows that $\left(\varphi\left(x_{i}, x_{j}, z_{0}\right)\right)_{i, j} \in M_{n}^{+}$.
(iv) $\Longrightarrow$ (i) For a positive $\left(a_{i, j}\right) \in M_{n}\left(C_{0}(Z)\right)$, the matrix $\left(\varphi\left(x_{i}, x_{j}\right)\left(a_{i, j}\right)\right)$ is the Schur product of $\left(\varphi\left(x_{i}, x_{j}\right)\right)$ and $\left(a_{i, j}\right)$ in $M_{n}\left(C_{0}(Z)\right)$. Since (iv) ensures the positivity of $\left(\varphi\left(x_{i}, x_{j}\right)\right)$, and the Schur product of two positive matrices over a commutative $C^{*}$-algebra is positive, (i) follows from [27, Theorem 2.6].

In the next corollary we assume $A$ acts nondegenerately on a separable Hilbert space $\mathcal{H}$.
Corollary 3.14. Let $\varphi: X \times X \rightarrow Z(\mathcal{N}(A)) \subseteq \mathrm{CB}(A)$ be a pointwise-measurable function, and assume that $\overline{Z(A) A}=A$. The following are equivalent:
(i) $\varphi$ is a positive central Schur A-multiplier;
(ii) there exist an index set I and $V \in C_{I}^{\omega}\left(L^{\infty}\left(X, Z(A)^{\prime \prime}\right)\right)$ such that

$$
\varphi(x, y)=\sum_{i \in I} V_{i}(y)^{*} V_{i}(x), \quad \text { for almost all }(x, y) \in X \times Y
$$

Moreover, if $\varphi: X \times X \rightarrow Z(\mathcal{M}(A))$ is weakly measurable then the above conditions are equivalent to:
(iii) $\varphi$ is a positive central Schur B-multiplier for any $C^{*}$-algebra $B \subseteq \mathcal{B}(\mathcal{H})$ with $Z(A) \subseteq Z(B)$.

Proof. Follows from Theorem 3.13 in the same way as Corollary 3.7 follows from Theorem 3.6.

We recall the following definition from [27].
Definition 3.15. A Herz-Schur $(A, G, \alpha)$-multiplier $F: G \rightarrow \mathrm{CB}(A)$ is called completely positive if $S_{F}$ is completely positive on $A \rtimes_{\alpha, r} G$.

Theorem 3.16. Let $(A, G, \alpha)$ be a $C^{*}$-dynamical system such that $\overline{Z(A) A}=A$, and $F: G \rightarrow Z(\mathcal{M}(A))$ be a pointwise-measurable function. The following are equivalent:
(i) $F$ is a completely positive central Herz-Schur $(Z(A), G, \alpha)$-multiplier;
(ii) $F$ is a completely positive central Herz-Schur $(A, G, \alpha)$-multiplier;
(iii) $\mathcal{N}(F)$ is a positive central Schur $Z(A)$-multiplier;
(iv) $\mathcal{N}(F)$ is a positive central Schur $A$-multiplier.

Proof. (ii) $\Longrightarrow$ (iv) Assume that $F: G \rightarrow Z(\mathcal{M}(A))$ is a positive central HerzSchur $(A, G, \alpha)$-multiplier. By the proof of [28, Theorem 3.8], using the Stinespring dilation theorem in place of the Haagerup-Paulsen-Wittstock theorem, we have $S_{F}(T)=V^{*} \rho(T) V, T \in A \rtimes_{\alpha, r} G$. The representation $\rho \circ(\pi \rtimes \lambda)$ of the
full crossed product $A \rtimes_{\alpha} G$ has the form $\rho_{A} \rtimes \rho_{G}$, where $\left(\rho_{A}, \rho_{G}\right)$ is a covariant pair. Let $\mathcal{V}(s):=\rho_{G}\left(s^{-1}\right) V \lambda_{s}$; as in [28, page 408], we have $\mathcal{N}(F)(s, t)(a)=$ $\mathcal{V}(t)^{*} \rho_{A}(a) \mathcal{V}(s)$, so $S_{\mathcal{N}(F)}=\mathcal{V}^{*}\left(\rho_{A} \otimes \mathrm{id}\right)(\cdot) \mathcal{V}$ is completely positive. Therefore $\mathcal{N}(F)$ is a positive Schur $A$-multiplier, and it is clearly central.
(iv) $\Longrightarrow$ (ii) As in the proof of [28, Theorem 3.8], we have $S_{F}=\left.S_{\mathcal{N}(F)}\right|_{A \rtimes_{\alpha, G} G}$, so $S_{F}$ is completely positive.
(iv) $\Longrightarrow$ (iii) Follows from Remark 3.1.
(iii) $\Longrightarrow$ (iv) Let $\mathcal{N}(F)$ be a positive central Schur $Z(A)$-multiplier. Following the proof of the implication (i) $\Longrightarrow$ (ii) of Corollary 3.7 and applying [40, Remark 4.5(iii)], we see that there exists an index set $I$ and an essentially bounded function $V \in C_{I}^{\omega}\left(L^{\infty}\left(G, Z(A)^{\prime \prime}\right)\right)$ such that $\mathcal{N}(F)(s, t)=\sum_{i \in I} V_{i}(t)^{*} V_{i}(s)$ almost everywhere on $G \times G$ (the series converges weakly). Hence for $a \in A$ and $s, t \in G$ we have

$$
\mathcal{N}(F)(s, t)(a)=\sum_{i \in I} V_{i}(t)^{*} V_{i}(s) a=\sum_{i \in I} V_{i}(t)^{*} a V_{i}(s)=\mathcal{V}(t)^{*} \rho(a) \mathcal{V}(s),
$$

where $\mathcal{V}(r):=\left(V_{i}(r)\right)_{i \in I}$ and $\rho(a)=\mathrm{id} \otimes a$. As in the proof of the implication (ii) $\Longrightarrow$ (i) of [28, Theorem 2.6], it follows that $S_{\mathcal{N}(F)}=\mathcal{V}^{*}(\mathrm{id} \otimes \rho)(\cdot) \mathcal{V}$ is completely positive, so $\mathcal{N}(F)$ is a positive central Schur $A$-multiplier.
(i) $\Longleftrightarrow$ (iii) This is a special case of (ii) $\Longleftrightarrow$ (iv).

Using Theorem 3.13 , similarly to Corollary 3.10, one can obtain the following description of completely positive central Herz-Schur $\left(C_{0}(Z), G, \alpha\right)$-multipliers.

Corollary 3.17. Let $\left(C_{0}(Z), G, \alpha\right)$ be a $C^{*}$-dynamical system, and $F: G \times Z \rightarrow \mathbb{C}$ a measurable function, continuous in the $Z$-variable. The following are equivalent:
(i) $F$ is a completely positive central Herz-Schur $\left(C_{0}(Z), G, \alpha\right)$-multiplier;
(ii) there exist a Hilbert space $\mathcal{L}$ and a weakly measurable function $v: G \times Z \rightarrow \mathcal{L}$ such that $F\left(t s^{-1}, x t^{-1}\right)=\langle v(s, x), v(t, x)\rangle$ almost everywhere on $G \times G \times Z$.
3.4. Connections with other types of multipliers. Let $Z$ be a locally compact Hausdorff space, equipped with an action of a locally compact group $G$; thus, we are given a map $Z \times G \rightarrow Z,(x, s) \rightarrow x s$, jointly continuous and such that $x(s t)=(x s) t$ for all $x \in Z$ and all $s, t \in G$. We consider the crossed product $C_{0}(Z) \rtimes_{\alpha, r} G$, where $\alpha$ is the corresponding action of $G$ on $C_{0}(Z)$. The set $\mathcal{G}=Z \times G$ is a groupoid, where the set $\mathcal{G}^{2}$ of composable pairs is given by

$$
\mathcal{G}^{2}=\left\{\left[\left(x_{1}, t_{1}\right),\left(x_{2}, t_{2}\right)\right]: x_{2}=x_{1} t_{1}\right\},
$$

and if $\left[\left(x_{1}, t_{1}\right),\left(x_{2}, t_{2}\right)\right] \in \mathcal{G}^{2}$, the product $\left(x_{1}, t_{1}\right) \cdot\left(x_{2}, t_{2}\right)$ is defined to be $\left(x_{1}, t_{1} t_{2}\right)$, while the inverse $(x, t)^{-1}$ of $(x, t)$ is defined to be $\left(x t, t^{-1}\right)$. The domain and range maps are given by

$$
d((x, t)):=(x, t)^{-1} \cdot(x, t)=(x t, e), \quad r((x, t)):=(x, t) \cdot(x, t)^{-1}=(x, e) .
$$

The unit space $\mathcal{G}_{0}$ of the groupoid, which is by definition equal to the common image of the maps $d$ and $r$, can therefore be canonically identified with $X$. We refer to [37] for background on groupoids (see also [28, Section 5.2]).

Let $\psi: Z \times G \rightarrow \mathbb{C}$ be a bounded continuous function. Let $F_{\psi}(s) \in C B\left(C_{0}(Z)\right)$ be given by $F_{\psi}(s)(f)(x):=\psi(x, s) f(x), f \in C_{0}(Z), s \in G$. In [28, Section 5] it was shown that such a function $\psi$ is a $\operatorname{Herz-Schur}\left(C_{0}(Z), G, \alpha\right)$-multiplier if and only if $\psi$ is a completely bounded multiplier of the Fourier algebra of $\mathcal{G}$ in the sense of Renault [38]. In the terminology of this paper such functions $\psi$ are central Herz-Schur $\left(C_{0}(Z), G, \alpha\right)$-multipliers. The following is therefore immediate from [28, Proposition 5.3] and Corollary 3.10.

Corollary 3.18. Let $\left(C_{0}(Z), G, \alpha\right)$ be a $C^{*}$-dynamical system, and write $\mathcal{G}$ for the underlying groupoid. Let $\psi: Z \times G \rightarrow \mathbb{C}$ be a bounded continuous function and write $F_{\psi}(r)(f)(x):=\psi(x, r) f(x), f \in C_{0}(Z)$. The following are equivalent:
(i) $F_{\psi}$ is a central Herz-Schur $\left(C_{0}(Z), G, \alpha\right)$-multiplier;
(ii) $\psi$ is a completely bounded multiplier of the Fourier algebra of $\mathcal{G}$;
(iii) there exist a Hilbert space $\mathcal{L}$ and essentially bounded functions $v, w: G \times Z \rightarrow$ $\mathcal{L}$ such that

$$
\psi\left(x t^{-1}, t s^{-1}\right)=\langle v(s, x), w(t, x)\rangle, \quad s, t \in G, \text { almost all } x \in X .
$$

If the conditions hold then we can choose $v$ and $w$ such that

$$
\|\psi\|_{\mathrm{HS}}=\operatorname{esssup}_{(s, x) \in G \times Z}\|v(s, x)\| \operatorname{esssup}_{(t, x) \in G \times Z}\|w(t, x)\| \text {. }
$$

We next link central multipliers to the multipliers studied by Dong-Ruan in [9]. Let $(A, G, \alpha)$ be a $C^{*}$-dynamical system with $A$ unital and $G$ discrete. Dong-Ruan define a function $h: G \rightarrow A$ to be a multiplier with respect to $\alpha$ if there is an $A$-bimodule map $\Phi$ on $A \rtimes_{\alpha, r} G$ such that $\Phi\left(\lambda_{r}\right)=\lambda_{r} \pi(h(r))$. The $A$-bimodule requirement forces $h(r) \in Z(A)$ for all $r \in G$. Hence $\Phi=S_{F}$ for the central $(A, G, \alpha)$ multiplier given by $F(r)(a)=h(r) a$.

In [9, Section 6], the authors use the fact that classical (positive) Schur multipliers on a discrete group $G$ give rise to (positive) central Herz-Schur multipliers of $\left(\ell^{\infty}(G), G, \beta\right)$ (here $\beta$ denotes the left translation action). This connection is also utilised by Ozawa [33]. We formalise this connection in the next proposition.

Proposition 3.19. Let $G$ be a discrete group. Consider a function $\varphi: G \times G \rightarrow \mathbb{C}$ and a family $a=\left(a_{r}\right)_{r \in G} \subseteq C_{b}(G)$. Define

$$
a_{r}^{\varphi}(p):=\varphi\left(r^{-1} p^{-1}, p^{-1}\right) \quad \text { and } \quad \varphi_{a}(s, t):=a_{t s^{-1}}\left(t^{-1}\right) .
$$

The assignments $\varphi \mapsto a^{\varphi}$ and $a \mapsto \varphi_{a}$ are mutual inverses, and give a one-toone correspondence between the classical Schur multipliers and the central HerzSchur $\left(C_{0}(G), G, \beta\right)$-multipliers. This bijection is an isometric algebra isomorphism which preserves positivity.

Proof. It is easy to check that $\varphi_{a^{\varphi}}=\varphi$ and $a^{\varphi_{a}}=a$ and that these assignments are linear and multiplicative.

Now suppose that $a=\left(a_{r}\right)_{r \in G}$ is a central Herz-Schur $\left(C_{0}(G), G, \beta\right)$-multiplier. By Corollary 3.10, there exist a Hilbert space $\mathcal{L}$ and weakly measurable functions $v, w: G \times G \rightarrow \mathcal{L}$, such that

$$
\varphi_{a}(s, t)=a_{t s^{-1}}\left(t^{-1}\right)=\langle v(s, e), w(t, e)\rangle, \quad s, t \in G .
$$

It follows from [5] that $\varphi_{a}$ is a Schur multiplier and $\left\|\varphi_{a}\right\|_{\Im} \leq\|a\|_{\mathrm{HS}}$.
Conversely, suppose $\varphi: G \times G \rightarrow \mathbb{C}$ is a Schur multiplier, and take $v, w$ : $G \rightarrow \mathcal{H}$ are such that $\varphi(s, t)=\langle v(s), w(t)\rangle$ and

$$
\|\varphi\| \subseteq=\sup _{s \in G}\|v(s)\| \sup _{t \in G}\|w(t)\| .
$$

Then, for $s, t, x \in G$,

$$
a_{t s^{-1}}^{\varphi}\left(x t^{-1}\right)=\varphi\left(s t^{-1} t x^{-1}, t x^{-1}\right)=\varphi\left(s x^{-1}, t x^{-1}\right)=\left\langle v\left(s x^{-1}\right), w\left(t x^{-1}\right)\right\rangle .
$$

Therefore, by Corollary 3.10, $a^{\varphi}=\left(a_{r}^{\varphi}\right)_{r \in G}$ is a central Herz-Schur $\left(C_{0}(G), G, \alpha\right)$ multiplier with $\left\|a^{\varphi}\right\|_{\mathrm{HS}} \leq\|\varphi\|_{\subseteq}$.

If $a$ is a positive central multiplier (resp. $\varphi$ is a positive Schur multiplier) then applying Corollary 3.17, taking $v=w$ in the above calculations, shows $\varphi_{a}$ (resp. $a^{\varphi}$ ) is also positive.

## 4. Convolution multipliers

In this section, we give a characterisation of Herz-Schur convolution multipliers first studied in [28, Section 6]. We will use the notion of a Herz-Schur $\theta$-multiplier of a $C^{*}$-dynamical system $(A, G, \alpha)$, introduced in [28, Definition 3.3]. Let $\theta: A \rightarrow \mathcal{B}\left(\mathcal{H}_{\theta}\right)$ be a faithful representation of (the separable $C^{*}$ algebra) $A$ on the separable Hilbert space $\mathcal{H}_{\theta}$, and let $\left(\pi^{\theta}, \lambda^{\theta}\right)$ be the regular covariant pair associated to this representation (see Subsection 2.1.4). A function $F: G \rightarrow \mathrm{CB}(A)$ will be called a Herz-Schur $\theta$-multiplier of $(A, G, \alpha)$ if the map

$$
\pi^{\theta}(a) \lambda_{r}^{\theta} \mapsto \pi^{\theta}(F(r)(a)) \lambda_{r}^{\theta}
$$

extends to a completely bounded, weak*-continuous map on $A \rtimes_{\alpha, \theta}^{w^{*}} G$. As before we assume that $G$ is either second countable or discrete.
4.1. Abelian case. Let $G$ be an abelian locally compact group equipped with a Haar measure and $\Gamma$ be its dual group. We denote by $\lambda^{\Gamma}$ the left regular representation on $L^{2}(\Gamma)$. We shall identify each element $s \in G$ with a character on $\Gamma$, and use $\beta$ to denote the natural action of $G$ on $C_{r}^{*}(\Gamma)$ by letting

$$
\beta_{s}\left(\lambda^{\Gamma}(f)\right):=\lambda^{\Gamma}(s f), \quad s \in G, f \in L^{1}(\Gamma) ;
$$

thus, $\left(C_{r}^{*}(\Gamma), G, \beta\right)$ is a $C^{*}$-dynamical system.
Given a bounded measurable function $\psi: G \times \Gamma \rightarrow \mathbb{C}$ and $t \in G$ (resp. $x \in$ $\Gamma$ ), let the function $\psi_{t}: \Gamma \rightarrow \mathbb{C}\left(\right.$ resp. $\left.\psi^{x}: G \rightarrow \mathbb{C}\right)$ be given by $\psi_{t}(y):=\psi(t, y)$ (resp. $\psi^{x}(s):=\psi(s, x)$ ). We call $\psi$ admissible if $\psi_{t} \in B(\Gamma)$ for every $t \in G$ and
$\sup _{t}\left\|\psi_{t}\right\|_{B(\Gamma)}<\infty$. Assuming that $\psi$ is admissible, let $F_{\psi}(t): C_{r}^{*}(\Gamma) \rightarrow C_{r}^{*}(\Gamma)$ be the map given by

$$
F_{\psi}(t)\left(\lambda^{\Gamma}(g)\right)=\lambda^{\Gamma}\left(\psi_{t} g\right), \quad g \in L^{1}(\Gamma)
$$

We define the Herz-Schur convolution multipliers of $G$ to be the elements of the set

$$
\mathbb{S}_{\text {conv }}(G)=\left\{\psi: G \times \Gamma \rightarrow \mathbb{C}: \psi \text { is admissible and } F_{\psi}\right. \text { is }
$$

$$
\text { a Herz-Schur } \left.\left(C_{r}^{*}(\Gamma), G, \beta\right) \text {-multiplier }\right\}
$$

and write

$$
\begin{aligned}
\mathfrak{S}_{\mathrm{conv}}^{\mathrm{id}}(G)=\{\psi: G \times \Gamma \rightarrow \mathbb{C} & : \psi \text { is admissible and } F_{\psi} \text { is } \\
& \text { a Herz-Schur id-multiplier of } \left.\left(C_{r}^{*}(\Gamma), G, \beta\right)\right\} .
\end{aligned}
$$

Here we write id for the canonical representation of $C_{r}^{*}(\Gamma)$ on $L^{2}(\Gamma)$. Clearly, the space $\widetilde{S}_{\text {conv }}(G)$ is an algebra with respect to the operations of pointwise addition and multiplication, and $\mathbb{S}_{\text {conv }}^{\mathrm{id}}(G)$ is a subalgebra of $\mathbb{S}_{\text {conv }}(G)$. For $\psi \in$ $\mathfrak{S}_{\text {conv }}(G)$, let $\|\psi\|_{\mathrm{HS}}=\left\|F_{\psi}\right\|_{\mathrm{HS}}$, and use $S_{\psi}$ to denote the map $S_{F_{\psi}}$.

We identify an elementary tensor $u \otimes h$, where $u \in B(G)$ and $h \in B(\Gamma)$, with the function $(s, x) \rightarrow u(s) h(x), s \in G, x \in \Gamma$. Let $\mathfrak{F}(B(G), B(\Gamma))$ be the complex vector space of all separately continuous functions $\psi: G \times \Gamma \rightarrow \mathbb{C}$ such that, for every $s \in G$ (resp. $x \in \Gamma$ ), the function $\psi_{s}: \Gamma \rightarrow \mathbb{C}$ (resp. $\psi^{x}: G \rightarrow \mathbb{C}$ ) belongs to $B(\Gamma)$ (resp. $B(G))$. By [28, Section 6], we have the following inclusions:

$$
B(G) \odot B(\Gamma) \subseteq \mathfrak{S}_{\mathrm{conv}}^{\mathrm{id}}(G) \subseteq \mathfrak{F}(B(G), B(\Gamma))
$$

We now answer [28, Question 6.6] by identifying $\mathbb{S}_{\text {conv }}^{\text {id }}(G)$.
Theorem 4.1. Let $G$ be a locally compact abelian group and $\psi: G \times \Gamma \rightarrow \mathbb{C}$ be an admissible function. The following are equivalent:
(i) $\psi \in \mathbb{S}_{\mathrm{conv}}^{\mathrm{id}}(G)$;
(ii) $\psi \in B(G \times \Gamma)$.

The identification is an isometric algebra homomorphism.
Proof. (i) $\Longrightarrow$ (ii) Let $\psi \in \mathfrak{S}_{\mathrm{conv}}^{\mathrm{id}}(G)$ and let $F_{\psi}: G \rightarrow \mathrm{CB}\left(C_{r}^{*}(\Gamma)\right)$ be the corresponding Herz-Schur multiplier of $\left(C_{r}^{*}(\Gamma), G, \beta\right)$. By [28, Theorem 3.8], $\mathcal{N}\left(F_{\psi}\right)(s, t)$ is a Schur $C_{r}^{*}(\Gamma)$-multiplier and hence there exist a Hilbert space $\mathcal{H}_{\rho}$, operators $\mathcal{V}, \mathcal{W} \in L^{\infty}\left(G, \mathcal{B}\left(L^{2}(\Gamma), \mathcal{H}_{\rho}\right)\right)$, a continuous unitary representation $\rho: \Gamma \rightarrow \mathcal{B}\left(\mathcal{H}_{\rho}\right)$ and a subset $N \subseteq G \times G$ with $\left(m_{G} \times m_{G}\right)(N)=0$, such that

$$
\mathcal{N}\left(F_{\psi}\right)(s, t)\left(\lambda^{\Gamma}(f)\right)=\mathcal{W}(t)^{*} \rho(f) \mathcal{V}(s), \quad f \in L^{1}(\Gamma)
$$

for all $(s, t) \notin N$, and

$$
\begin{equation*}
\|\psi\|_{\mathfrak{S}}=\underset{s \in G}{\operatorname{esssup}}\|\mathcal{V}(s)\| \underset{t \in G}{\operatorname{esssup}}\|\mathcal{W}(t)\| \tag{9}
\end{equation*}
$$

As

$$
\mathcal{N}\left(F_{\psi}\right)(s, t)\left(\lambda^{\Gamma}(f)\right)=\beta_{t^{-1}}\left(F_{\psi}\left(t s^{-1}\right)\left(\beta_{t}\left(\lambda^{\Gamma}(f)\right)\right)\right)=\lambda^{\Gamma}\left(\psi_{t s^{-1}} f\right)
$$

we obtain

$$
\lambda^{\Gamma}\left(\psi_{t s^{-1}} f\right)=\mathcal{W}(t)^{*} \rho(f) \mathcal{V}(s), \quad f \in L^{1}(\Gamma)
$$

for all $(s, t) \notin N$. As $\psi_{t s^{-1}} \in B(\Gamma)$, we have that $\psi_{t s^{-1}}$ is a completely bounded multiplier of $A(\Gamma)$, and the map $S_{\psi_{t s^{-1}}}$ can be extended to a weak*-continuous linear operator on $\mathrm{vN}(\Gamma)$; we have

$$
\psi\left(t s^{-1}, x\right) \lambda_{x}^{\Gamma}=\mathcal{W}(t)^{*} \rho(x) \mathcal{V}(s), \quad x \in \Gamma,(s, t) \notin N .
$$

Thus, for $\xi \in L^{2}(\Gamma)$ with $\|\xi\|_{2}=1$, we have

$$
\begin{aligned}
\psi\left(t s^{-1}, x y^{-1}\right)\langle\xi, \xi\rangle & =\left\langle\lambda_{x^{-1}}^{\Gamma} \mathcal{W}(t)^{*} \rho(x) \rho(y)^{*} \mathcal{V}(s) \lambda_{y}^{\Gamma} \xi, \xi\right\rangle \\
& =\left\langle\rho(y)^{*} \mathcal{V}(s) \lambda_{y}^{\Gamma} \xi, \rho(x)^{*} \mathcal{W}(t) \lambda_{x}^{\Gamma} \xi\right\rangle
\end{aligned}
$$

Letting $v(s, y):=\rho(y)^{*} \mathcal{V}(s) \lambda_{y}^{\Gamma} \xi$ and $w(t, x):=\rho(x)^{*} \mathcal{W}(t) \lambda_{x}^{\Gamma} \xi$, we obtain

$$
\psi\left((t, x)(s, y)^{-1}\right)=\langle v(s, y), w(t, x)\rangle, \quad(s, t) \notin N
$$

By [5], $\psi$ is equal almost everywhere to a completely bounded multiplier of $A(G \times \Gamma)$, and hence to an element $u \in B(G \times \Gamma)$ [21, Theorem 5.1.8]. To see that $\psi(t, x)=u(t, x)$ for all $(t, x)$, for each $t \in G$ we let

$$
N_{t}=\{x \in \Gamma: \psi(t, x)=u(t, x)\} .
$$

By Fubini's Theorem, the set $\left\{t \in G: m_{\Gamma}\left(N_{t}^{c}\right)>0\right\}$ has measure zero, that is, for almost all $t \in G$, we have that $\psi(t, x)=u(t, x)$ almost everywhere. As $\psi$ is separately continuous, the last equality holds for all $x \in \Gamma$. Using again the separate continuity of $\psi$ we obtain that $\psi(t, x)=u(t, x)$ for all $(t, x)$. Furthermore, by (9),

$$
\begin{aligned}
\|\psi\|_{B(G \times \Gamma)} & \leq \operatorname{esssup}_{(s, y) \in G \times \Gamma}\left\|\rho(y)^{*} \mathcal{V}(s) \lambda_{y}^{\Gamma} \xi\right\| \operatorname{esssup}_{(t, x) \in G \times \Gamma}\left\|\rho(x)^{*} \mathcal{W}(t) \lambda_{x}^{\Gamma} \xi\right\| \\
& \leq \underset{s \in G}{\operatorname{esssup}}\|\mathcal{V}(s)\| \underset{t \in G}{\operatorname{esssup}}\|\mathcal{W}(t)\|=\|\psi\| \Subset .
\end{aligned}
$$

(ii) $\Longrightarrow$ (i) Assume that $\psi \in B(G \times \Gamma)$. By [5], there exist a Hilbert space $\mathcal{H}$ and continuous $v, w: G \times \Gamma \rightarrow \mathcal{H}$ such that

$$
\psi\left(t s^{-1}, x y^{-1}\right)=\langle v(s, y), w(t, x)\rangle, \quad s, t \in G, x, y \in \Gamma
$$

and

$$
\|\psi\|_{B(G \times \Gamma)}=\sup _{(s, y)}\|v(s, y)\| \sup _{(t, x)}\|w(t, x)\|
$$

Choose an orthonormal basis $\left\{e_{i}\right\}_{i \in I}$ in $\mathcal{H}$ and let $v_{i}(s, y):=\left\langle v(s, y), e_{i}\right\rangle$ and $w_{i}(t, x):=\left\langle e_{i}, w(t, x)\right\rangle$. Then

$$
\psi\left(t s^{-1}, x y^{-1}\right)=\sum_{i \in I} v_{i}(s, y) w_{i}(t, x), \quad s, t \in G, x, y \in \Gamma .
$$

Let $S$ be the completely bounded operator on $\mathcal{B}\left(L^{2}(G \times \Gamma)\right)$, given by $S(T):=$ $\sum_{i \in I} M_{w_{i}} T M_{v_{i}}$. Clearly,

$$
\begin{equation*}
\|S\|_{\mathrm{cb}}=\|\psi\|_{B(G \times \Gamma)} . \tag{10}
\end{equation*}
$$

To complete the proof, it suffices to show that the restriction of the operator $S$ to $C_{r}^{*}(\Gamma) \rtimes_{\beta, \text { id }}^{w^{*}} G$ is given by

$$
\begin{equation*}
S\left(\pi^{\mathrm{id}}\left(\lambda_{x}^{\Gamma}\right) \lambda_{s}^{\mathrm{id}}\right)=\pi^{\mathrm{id}}\left(\psi(s, x) \lambda_{x}^{\Gamma}\right) \lambda_{s}^{\mathrm{id}} \tag{11}
\end{equation*}
$$

First note that

$$
\begin{equation*}
\left(\pi^{\mathrm{id}}\left(\lambda_{x}^{\Gamma}\right) \xi\right)(t)=\beta_{t^{-1}}\left(\lambda_{x}^{\Gamma}\right) \xi(t)=\overline{t(x)} \lambda_{x}^{\Gamma} \xi(t), \quad \xi \in L^{2}\left(G, L^{2}(\Gamma)\right) . \tag{12}
\end{equation*}
$$

Writing $v_{i}(t)(\cdot)$ and $w_{i}(t)(\cdot)$ for $v_{i}(t, \cdot)$ and $w_{i}(t, \cdot)$, respectively, for $t \in G$ and $y \in \Gamma$, and fixing $\xi, \eta \in L^{2}\left(G, L^{2}(\Gamma)\right)$, we have

$$
\begin{aligned}
& \left\langle S\left(\pi^{\mathrm{id}}\left(\lambda_{x}^{\Gamma}\right) \lambda_{s}^{\mathrm{id}}\right) \xi, \eta\right\rangle \\
= & \sum_{i \in I}\left\langle M_{w_{i}} \pi^{\mathrm{id}}\left(\lambda_{x}^{\Gamma}\right) \lambda_{s}^{\mathrm{id}} M_{v_{i}} \xi, \eta\right\rangle \\
= & \sum_{i \in I} \int\left(M_{w_{i}(t)} \overline{t(x)} \lambda_{x}^{\Gamma} M_{v_{i}\left(s^{-1} t\right)} \xi\left(s^{-1} t\right)\right)(y) \overline{\eta(t, y)} d t d y \\
= & \sum_{i \in I} \int w_{i}(t, y) v_{i}\left(s^{-1} t, x^{-1} y\right) \overline{t(x)} \xi\left(s^{-1} t, x^{-1} y\right) \overline{\eta(t, y)} d t d y \\
= & \int \psi\left(t t^{-1} s, y y^{-1} x\right) \overline{t(x)} \xi\left(s^{-1} t, x^{-1} y\right) \overline{\eta(t, y)} d t d y \\
= & \int \psi(s, x)\left(\pi^{\mathrm{id}}\left(\lambda_{x}^{\Gamma}\right) \lambda_{s}^{\mathrm{id}} \xi\right)(t, y) \overline{\eta(t, y)} d t d y .
\end{aligned}
$$

Together with (12), this establishes (11). In addition,

$$
\|\psi\|_{\mathfrak{S}}=\left\|\left.S\right|_{C_{r}^{*}(\Gamma) \searrow_{\beta, \mathrm{d}}^{\omega *} G}\right\|_{\mathrm{cb}} \leq\|S\|_{\mathrm{cb}}=\|\psi\|_{B(G \times \Gamma)},
$$

which together with (10) gives the desired equality.
To see that the identification is multiplicative, observe that if $\psi, \chi \in \mathbb{S}_{\mathrm{conv}}^{\mathrm{id}}(G)$ then $S_{F_{\psi}} S_{F_{\chi}}=S_{F_{\psi \chi}}$.

In Theorem 4.4 below we will show that the identification in Theorem 4.1 is in fact a complete isometry.
4.2. General case. Now let $G$ be an arbitrary locally compact group. In order to define convolution multipliers, we replace $C_{r}^{*}(\Gamma)$ with the quantum group dual of $C_{r}^{*}(G)$, namely $C_{0}(G)$, equipped with its natural action of $G$. Similarly we replace $B(\Gamma)$ by $M(G)$, the Banach algebra of all complex-valued Radon measures on $G$ with the convolution multiplication, given by

$$
(\mu * \nu)(f):=\int_{G} \int_{G} f(s t) d \mu(s) d \nu(t), \quad f \in C_{0}(G), \mu, \nu \in M(G) .
$$

We identify $L^{1}(G)$ with the norm-closed ideal in $M(G)$ consisting of absolutely continuous measures with respect to left Haar measure. We have that $L^{1}(G)$
is an $M(G)$-bimodule in the natural way. Using the identification $L^{1}(G)^{*}=$ $L^{\infty}(G)$, we arrive at an $M(G)$-bimodule structure on $L^{\infty}(G)$, given by

$$
\langle\mu \cdot f, h\rangle=\langle f, h * \mu\rangle \quad \text { and } \quad\langle f \cdot \mu, h\rangle=\langle f, \mu * h\rangle,
$$

for $h \in L^{1}(G), f \in L^{\infty}(G), \mu \in M(G)$. In particular,

$$
(\mu \cdot f)(s)=\int_{G} f(s t) d \mu(t) \quad \text { and } \quad(f \cdot \mu)(t)=\int_{G} f(s t) d \mu(s) .
$$

Let $\rho$ be the right regular representation of $G$ on $L^{2}(G)$; thus,

$$
\left(\rho_{s} \xi\right)(t)=\Delta(s)^{1 / 2} \xi(t s)
$$

For $\mu \in M(G)$, define a bounded linear operator $\theta(\mu)(a), a \in \mathcal{B}\left(L^{2}(G)\right)$, by

$$
\theta(\mu)(a):=\int_{G} \rho_{s} a \rho_{s}^{*} d \mu(s)
$$

By [31, Theorem 3.2] (see also [30, Theorem 4.5]), the map $\theta$ above is a weak*weak* continuous completely isometric homomorphism from $M(G)$ to the space $\mathrm{CB}^{\sigma}\left(\mathcal{B}\left(L^{2}(G)\right)\right)$ of all completely bounded weak* continuous linear maps on $\mathcal{B}\left(L^{2}(G)\right)$ and $\|\theta(\mu)\|_{\mathrm{cb}}=\|\theta(\mu)\|=\|\mu\|$. We have

$$
\theta(\mu)(f)=\mu \cdot f \in L^{\infty}(G), \quad f \in L^{\infty}(G)
$$

Moreover, $\theta(\mu)$ is a $v \mathrm{~N}(G)$-bimodule map.
For each $t \in G$, let $\beta_{t}: L^{\infty}(G) \rightarrow L^{\infty}(G)$ be given by $\beta_{t}(f):=\lambda_{t}^{G} f \lambda_{t^{-1}}^{G}=f_{t}$, where $f_{t}(x)=f\left(t^{-1} x\right)$. Then

$$
\begin{equation*}
\beta_{t} \circ \theta(\mu)=\theta(\mu) \circ \beta_{t}, \quad t \in G . \tag{13}
\end{equation*}
$$

For $\Lambda=\left\{\mu_{t}\right\}_{t \in G} \subseteq M(G)$, define $F_{\Lambda}: G \rightarrow \mathrm{CB}\left(C_{0}(G)\right)$ by

$$
F_{\Lambda}(t)(f):=\theta\left(\mu_{t}\right)(f), \quad t \in G, f \in C_{0}(G)
$$

Definition 4.2. A family $\Lambda=\left\{\mu_{t}\right\}_{t \in G} \subseteq M(G)$ is called a convolution multiplier if $F_{\Lambda}$ is a Herz-Schur $\left(C_{0}(G), G, \beta\right)$-multiplier.
If $\Lambda=\left\{\mu_{t}\right\}_{t \in G}$ is a convolution multiplier, we set $\|\Lambda\|_{\mathrm{HS}}=\left\|F_{\Lambda}\right\|_{\mathrm{HS}}$.
Let id denote the representation of $C_{0}(G)$ on $L^{2}(G)$ by multiplication operators and $\mathfrak{S}_{\text {conv }}^{\text {id }}(G)$ be the collection of families $\Lambda=\left\{\mu_{t}\right\}_{t \in G} \subseteq M(G)$ such that $F_{\Lambda}$ is a Herz-Schur id-multiplier of $\left(C_{0}(G), G, \beta\right)$, endowed with the algebra structure coming from pointwise operations on the maps $F_{\Lambda}$. When $G$ is abelian, the identifications $C_{0}(G) \equiv C_{r}^{*}(\Gamma)$ and $M(G) \equiv B(\Gamma)$ show that this usage of the notation $\mathfrak{S}_{\text {conv }}^{\text {id }}(G)$ agrees with that from Subsection 4.1.

Consider the operator space projective tensor product

$$
L^{1}(G) \hat{\otimes} A(G)=\left(L^{\infty}(G) \bar{\otimes} \mathrm{vN}(G)\right)_{*} .
$$

We note that, when equipped with the product given on elementary tensors by

$$
(f \otimes u)(g \otimes v)=(f * g) \otimes(u v), \quad f, g \in L^{1}(G), u, v \in A(G),
$$

the operator space $L^{1}(G) \widehat{\otimes} A(G)$ is a completely contractive Banach algebra. A $\operatorname{map} T \in \mathcal{B}\left(L^{1}(G) \widehat{\otimes} A(G)\right)$ will be called a right multiplier of $L^{1}(G) \widehat{\otimes} A(G)$ if

$$
T(a b)=a T(b), \quad a, b \in L^{1}(G) \widehat{\otimes} A(G)
$$

If, in addition, $T$ is completely bounded, we write $T \in \mathrm{M}_{\mathrm{cb}}^{r}\left(L^{1}(G) \widehat{\otimes} A(G)\right)$, and call $T$ a right completely bounded multiplier of $L^{1}(G) \widehat{\otimes} A(G)$. When $G$ is abelian we have the identifications

$$
\mathrm{M}_{\mathrm{cb}}^{r}\left(L^{1}(G) \widehat{\otimes} A(G)\right)=\mathrm{M}_{\mathrm{cb}}(A(\Gamma \times G))=B(\Gamma \times G)
$$

Our goal is to generalise Theorem 4.1, identifying $\mathbb{S}_{\text {conv }}^{\mathrm{id}}(G)$ with the space of right completely bounded multipliers $\mathrm{M}_{\mathrm{cb}}^{r}\left(L^{1}(G) \widehat{\otimes} A(G)\right)$.

If $M$ is any of the von Neumann algebras $L^{\infty}(G), \mathrm{vN}(G)$ or $L^{\infty}(G) \bar{\otimes} \mathrm{vN}(G)$, $T \in M$ and $f \in M_{*}$, we write $f \cdot T$ and $T \cdot f \in M$ for the operators given by

$$
\langle f \cdot T, g\rangle:=\langle T, g f\rangle, \quad\langle T \cdot f, g\rangle:=\langle T, f g\rangle, \quad g \in M_{*},
$$

where $\langle\cdot, \cdot\rangle$ is the pairing between $M$ and $M_{*}$. We recall [12] that the support of $T \in \operatorname{vN}(G)$ is the closed set of all $t \in G$ such that $u \cdot T \neq 0$ whenever $u \in A(G)$ and $u(t) \neq 0$.

Lemma 4.3. If $T \in \mathrm{M}_{\mathrm{cb}}^{r}\left(L^{1}(G) \widehat{\otimes} A(G)\right)$ then there exists a unique family $\left\{\mu_{t}\right\}_{t \in G} \subseteq$ $M(G)$ such that

$$
T^{*}\left(f \otimes \lambda_{t}^{G}\right)=\theta\left(\mu_{t}\right)(f) \otimes \lambda_{t}^{G}, \quad f \in L^{\infty}(G), t \in G
$$

Proof. Let $f_{1}, f_{2} \in L^{1}(G), a_{1}, a_{2} \in A(G)$. The equality

$$
T\left(\left(f_{1} \otimes a_{1}\right)\left(f_{2} \otimes a_{2}\right)\right)=\left(f_{1} \otimes a_{1}\right) T\left(f_{2} \otimes a_{2}\right)
$$

implies that, if $g \in L^{\infty}(G)$ then

$$
\begin{equation*}
\left\langle T^{*}\left(g \otimes \lambda_{t}^{G}\right),\left(f_{1} \otimes a_{1}\right)\left(f_{2} \otimes a_{2}\right)\right\rangle=a_{1}(t)\left\langle T^{*}\left(g \cdot f_{1} \otimes \lambda_{t}^{G}\right), f_{2} \otimes a_{2}\right\rangle \tag{14}
\end{equation*}
$$

Taking the limit along an approximate identity $\left\{f_{\alpha}\right\}_{\alpha \in \mathbb{A}}$ of $L^{1}(G)$, we obtain

$$
\begin{equation*}
\left\langle T^{*}\left(g \otimes \lambda_{t}^{G}\right), f_{2} \otimes a_{1} a_{2}\right\rangle=\left\langle a_{1}(t) T^{*}\left(g \otimes \lambda_{t}^{G}\right), f_{2} \otimes a_{2}\right\rangle \tag{15}
\end{equation*}
$$

For $\omega \in L^{1}(G)$, let $R_{\omega}: L^{\infty}(G) \bar{\otimes} \mathrm{vN}(G) \rightarrow \mathrm{vN}(G)$ be the slice map, defined by

$$
\left\langle R_{\omega}(S), a\right\rangle:=\langle S, \omega \otimes a\rangle, \quad S \in L^{\infty}(G) \bar{\otimes} \mathrm{vN}(G), a \in A(G)
$$

After taking a limit along an approximate unit for $L^{1}(G)$, equation (15) implies that

$$
a_{1} \cdot R_{\omega}\left(T^{*}\left(g \otimes \lambda_{t}^{G}\right)\right)=a_{1}(t) R_{\omega}\left(T^{*}\left(g \otimes \lambda_{t}^{G}\right)\right)
$$

It follows that $R_{\omega}\left(T^{*}\left(g \otimes \lambda_{t}^{G}\right)\right) \in \mathrm{vN}(G)$ has support in $\{t\}$. By [12, Théorème 4.9], $R_{\omega}\left(T^{*}\left(g \otimes \lambda_{t}^{G}\right)\right)=c(\omega, t) \lambda_{t}^{G}$ for some constant $c(\omega, t)$ and

$$
R_{\omega}\left(T^{*}\left(g \otimes \lambda_{t}^{G}\right)\left(1 \otimes \lambda_{t^{-1}}^{G}\right)\right) \in \mathbb{C} I
$$

By [23], $T^{*}\left(g \otimes \lambda_{t}^{G}\right)\left(1 \otimes \lambda_{t^{-1}}^{G}\right) \in L^{\infty}(G) \otimes \mathbb{C} I$ and hence $T^{*}\left(g \otimes \lambda_{t}^{G}\right)=g_{t} \otimes \lambda_{t}^{G}$ for some $g_{t} \in L^{\infty}(G)$.

The map $\Phi_{t}: g \mapsto g_{t}$ is completely bounded, normal, and $T^{*}\left(g \otimes \lambda_{t}^{G}\right)=$ $\Phi_{t}(g) \otimes \lambda_{t}^{G}, t \in G$. By (14),

$$
\Phi_{t}\left(g \cdot f_{1}\right)=\Phi_{t}(g) \cdot f_{1}, \quad f_{1} \in L^{1}(G)
$$

showing that $\left(\Phi_{t}\right)_{*}\left(f_{1} * f_{2}\right)=f_{1} *\left(\left(\Phi_{t}\right)_{*}\left(f_{2}\right)\right)$. Thus $\left(\Phi_{t}\right)_{*}$ is a right completely bounded multiplier of $L^{1}(G)$. By [31, Theorem 3.2] (see also [30, Theorem 4.5]), there exists $\left\{\mu_{t}\right\}_{t \in G}$ such that $\Phi_{t}(g)=\theta\left(\mu_{t}\right)(g)$.

In what follows we will speak of a family $\Lambda=\left\{\mu_{t}\right\}_{t \in G} \subseteq M(G)$ being a convolution multiplier or a (completely bounded) right multiplier. For a right multiplier $\Lambda$ of $L^{1}(G) \widehat{\otimes} A(G)$ we denote by $R_{\Lambda}$ the mapping on $L^{\infty}(G) \bar{\otimes} \mathrm{vN}(G)$, given by

$$
\begin{equation*}
R_{\Lambda}\left(f \otimes \lambda_{r}^{G}\right):=\theta\left(\mu_{r}\right)(f) \otimes \lambda_{r}^{G} \tag{16}
\end{equation*}
$$

Theorem 4.4. Let $\Lambda=\left\{\mu_{t}\right\}_{t \in G} \subseteq M(G)$. The following are equivalent:
(i) $\Lambda \in \mathbb{S}_{\text {conv }}^{\text {id }}(G)$;
(ii) $\Lambda \in \mathrm{M}_{\mathrm{cb}}^{r}\left(L^{1}(G) \hat{\otimes} A(G)\right)$.

The identification $R_{\Lambda} \mapsto S_{F_{\Lambda}}$ is a completely isometric algebra isomorphism.
Proof. (i) $\Longrightarrow$ (ii) We identify $C_{0}(G) \rtimes_{\beta, i d}^{w^{*}} G$ with the von Neumann algebra crossed product $L^{\infty}(G) \rtimes_{\beta}^{\mathrm{VN}} G$, and let $\Lambda=\left\{\mu_{t}\right\}_{t \in G}$ be a convolution multiplier. For $f \in L^{\infty}(G)$, using (13) we have

$$
\begin{aligned}
\mathcal{N}\left(F_{\Lambda}\right)(s, t)(f) & =\beta_{t^{-1}}\left(F_{\Lambda}\left(t s^{-1}\right)\left(\beta_{t}(f)\right)\right) \\
& =\beta_{t^{-1}}\left(\theta\left(\mu_{t s^{-1}}\right)\left(\beta_{t}(f)\right)\right)=\theta\left(\mu_{t s^{-1}}\right)(f)
\end{aligned}
$$

Following similar arguments as in the proof of [28, Theorem 3.8], we obtain that there exist a normal $*$-representation $\rho$ of $L^{\infty}(G)$ on $\mathcal{H}_{\rho}$ and

$$
\mathcal{V}, \mathcal{W} \in L^{\infty}\left(G, \mathcal{B}\left(L^{2}(G), \mathcal{H}_{\rho}\right)\right)
$$

such that

$$
\theta\left(\mu_{t s^{-1}}\right)(f)=\mathcal{W}^{*}(t) \rho(f) \mathcal{V}(s)
$$

and $\|\Lambda\|_{\mathfrak{\Im}}=\operatorname{esssup}_{s \in G}\|\mathcal{V}(s)\| \operatorname{esssup}_{t \in G}\|\mathcal{W}(t)\|$.
Define a map $R_{\Lambda}: L^{\infty}(G) \bar{\otimes} \mathrm{vN}(G) \rightarrow \mathcal{B}\left(L^{2}(G) \otimes L^{2}(G)\right)$ by

$$
R_{\Lambda}\left(f \otimes \lambda_{t}^{G}\right):=W^{*}\left(\rho(f) \otimes \lambda_{t}^{G}\right) V
$$

where $V, W \in \mathcal{B}\left(L^{2}\left(G, \mathcal{H}_{\rho} \otimes L^{2}(G)\right)\right)$ are given by $(V \xi)(t)=\mathcal{V}(t) \xi(t),(W \xi)(t)=$ $\mathcal{W}(t) \xi(t)$. Then

$$
\begin{aligned}
R_{\Lambda}\left(f \otimes \lambda_{t}^{G}\right) \xi(s)=\mathcal{W}^{*}(s) \rho(f) \mathcal{V}\left(t^{-1} s\right) \xi\left(t^{-1} s\right) & =\theta\left(\mu_{s\left(s^{-1} t\right)}\right)(f) \xi\left(t^{-1} s\right) \\
& =\left(\theta\left(\mu_{t}\right)(f) \otimes \lambda_{t}^{G} \xi\right)(s)
\end{aligned}
$$

In particular, $R_{\Lambda}\left(f \otimes \lambda_{t}^{G}\right) \in L^{\infty}(G) \bar{\otimes} \mathrm{vN}(G)$, and hence $R_{\Lambda}$ is a normal completely bounded map on $L^{\infty}(G) \bar{\otimes} \mathrm{vN}(G)$. Moreover, if $f_{1}, f_{2} \in L^{1}(G), a_{1}, a_{2} \in$
$A(G), g \in L^{\infty}(G)$, and $\left(R_{\Lambda}\right)_{*}$ is the predual of $R_{\Lambda}$, we have

$$
\begin{aligned}
\langle g & \left.\otimes \lambda_{t}^{G},\left(R_{\Lambda}\right)_{*}\left(\left(f_{1} \otimes a_{1}\right)\left(f_{2} \otimes a_{2}\right)\right)\right\rangle=\left\langle\theta\left(\mu_{t}\right)(g) \otimes \lambda_{t}^{G}, f_{1} * f_{2} \otimes a_{1} a_{2}\right\rangle \\
& =\left\langle\mu_{t} \cdot g, f_{1} * f_{2}\right\rangle\left\langle\lambda_{t}^{G}, a_{1} a_{2}\right\rangle=\left\langle g,\left(f_{1} * f_{2}\right) * \mu_{t}\right\rangle\left\langle a_{1} \cdot \lambda_{t}^{G}, a_{2}\right\rangle \\
& =\left\langle g \cdot f_{1}, f_{2} * \mu_{t}\right\rangle\left\langle a_{1} \cdot \lambda_{t}^{G}, a_{2}\right\rangle=\left\langle\mu_{t} \cdot\left(g \cdot f_{1}\right), f_{2}\right\rangle\left\langle a_{1}(t) \lambda_{t}^{G}, a_{2}\right\rangle \\
& =\left\langle R_{\Lambda}\left(g \cdot f_{1} \otimes a_{1}(t) \lambda_{t}^{G}\right), f_{2} \otimes a_{2}\right\rangle=\left\langle g \cdot f_{1} \otimes a_{1}(t) \lambda_{t}^{G},\left(R_{\Lambda}\right)_{*}\left(f_{2} \otimes a_{2}\right)\right\rangle \\
& =\left\langle g \otimes \lambda_{t}^{G},\left(f_{1} \otimes a_{1}\right)\left(R_{\Lambda}\right)_{*}\left(f_{2} \otimes a_{2}\right)\right\rangle,
\end{aligned}
$$

i.e.

$$
\left(R_{\Lambda}\right)_{*}\left(\left(f_{1} \otimes a_{1}\right)\left(f_{2} \otimes a_{2}\right)\right)=\left(f_{1} \otimes a_{1}\right)\left(R_{\Lambda}\right)_{*}\left(f_{2} \otimes a_{2}\right) .
$$

Hence $\left(R_{\Lambda}\right)_{*}(a b)=a\left(R_{\Lambda}\right)_{*}(b)$ for any $a, b \in L^{1}(G) \widehat{\otimes} A(G)$ and therefore $\left(R_{\Lambda}\right)_{*}$ is a right completely bounded multiplier of $L^{1}(G) \widehat{\otimes} A(G)$. In addition,

$$
\begin{equation*}
\left\|R_{\Lambda}\right\|_{\mathrm{cb}} \leq \underset{s \in G}{\operatorname{esssup}}\|\mathcal{V}(s)\| \operatorname{esssup}_{t \in G}\|\mathcal{W}(t)\|=\|\Lambda\| \Subset . \tag{17}
\end{equation*}
$$

(ii) $\Longrightarrow$ (i) Assume now that $\Lambda=\left\{\mu_{t}\right\}_{t \in G} \in \mathrm{M}_{\mathrm{cb}}^{r}\left(L^{1}(G) \widehat{\otimes} A(G)\right.$ ), i.e. the $\operatorname{map} f \otimes \lambda_{t}^{G} \mapsto \theta\left(\mu_{t}\right)(f) \otimes \lambda_{t}^{G}$ extends to a normal right $L^{1}(G) \widehat{\otimes} A(G)$-modular completely bounded map $R_{\Lambda}$ on $L^{\infty}(G) \bar{\otimes} \mathrm{vN}(G)$. By [20, Proposition 4.3], there exists a unique $\mathrm{vN}(G) \bar{\otimes} L^{\infty}(G)$-bimodule map $\widetilde{R_{\Lambda}} \in \mathrm{CB}^{\sigma}\left(\mathcal{B}\left(L^{2}(G \times G)\right)\right)$ such that $\left.\widetilde{R_{\Lambda}}\right|_{L^{\infty}(G) \otimes} \overline{\mathrm{vN}(G)}{ }=R_{\Lambda}$ and $\left\|\widetilde{R_{\Lambda}}\right\|_{\mathrm{cb}}=\left\|R_{\Lambda}\right\|_{\mathrm{cb}}$. We have, in particular,

$$
\begin{equation*}
\widetilde{R_{\Lambda}}\left(g \otimes f \lambda_{t}^{G}\right)=\theta\left(\mu_{t}\right)(g) \otimes f \lambda_{t}^{G}, \quad f, g \in L^{\infty}(G) . \tag{18}
\end{equation*}
$$

Note that $L^{2}(G \times G) \equiv L^{2}\left(G, L^{2}(G)\right)$ and let $\pi: L^{\infty}(G) \rightarrow \mathcal{B}\left(L^{2}(G \times G)\right)$ be the ${ }^{*}$-representation, given by

$$
\pi(f) \xi(t)=\beta_{t^{-1}}(f)(\xi(t)), \quad \xi \in L^{2}(G \times G), f \in L^{\infty}(G) .
$$

Let $f \in L^{\infty}(G)$ and note that $\pi(f) \in L^{\infty}(G \times G)$. Thus, there exists a net $\left\{\omega_{\alpha}\right\}_{\alpha \in \mathbb{A}} \subseteq \operatorname{span}\left\{g \otimes h: g, h \in L^{\infty}(G)\right\}$, with $\omega_{\alpha} \rightarrow_{\alpha \in \mathbb{A}} \pi(f)$ in the weak* topology. Write $\omega_{\alpha}=\sum_{i=1}^{n_{\alpha}} g_{i, \alpha} \otimes h_{i, \alpha}$. Using (13) and (18), we have

$$
\left(\widetilde{R_{\Lambda}}\left(\pi(f)\left(1 \otimes \lambda_{r}^{G}\right)\right)=\lim _{\alpha \in \mathbb{A}} \sum_{i=1}^{n_{\alpha}} \theta\left(\mu_{r}\right)\left(g_{i, \alpha}\right) \otimes h_{i, \alpha} \lambda_{r}^{G}=\pi\left(\theta\left(\mu_{r}\right)(f)\right)\left(1 \otimes \lambda_{r}^{G}\right) .\right.
$$

Since

$$
\left(\widetilde{{F_{F_{\Lambda}}}}\left(\pi(f)\left(1 \otimes \lambda_{t}^{G}\right)\right)=\left(\pi\left(\theta\left(\mu_{t}\right)(f)\right)\left(1 \otimes \lambda_{t}^{G}\right),\right.\right.
$$

the restriction of $\widetilde{R_{\Lambda}}$ to the crossed product $C_{0}(G) \rtimes_{\beta, r} G$ coincides with $S_{F_{\Lambda}}$, implying the converse statement. Note, in addition, that

$$
\begin{equation*}
\left\|S_{F_{\Lambda}}\right\|_{\mathrm{cb}} \leq\left\|\widetilde{R_{\Lambda}}\right\|_{\mathrm{cb}}=\left\|R_{\Lambda}\right\|_{\mathrm{cb}} . \tag{19}
\end{equation*}
$$

By (17), $\left\|R_{\Lambda}\right\|_{\mathrm{cb}} \leq\left\|S_{F_{\Lambda}}\right\|_{\mathrm{cb}}$, and together with (19) this shows that $\left\|R_{\Lambda}\right\|_{\mathrm{cb}}=$ $\left\|S_{F_{\Lambda}}\right\|_{\mathrm{cb}}$. Moreover, by Lemma 2.7 the $\operatorname{map} F_{\Lambda} \mapsto \mathcal{N}\left(F_{\Lambda}\right)$ is a complete isometry, and by [20, Proposition 4.3] the map $R_{\Lambda} \mapsto \widetilde{R_{\Lambda}}$ is a complete isometry, therefore the norm inequalities hold on all matrix levels, implying that the identification $S_{F_{\Lambda}} \mapsto R_{\Lambda}$ is a complete isometry.

The homomorphism claim follows from Lemma 2.7 and the fact that the identification in [20, Proposition 4.3] is a homomorphism.

We observe that the product of the convolution multipliers $\Lambda=\left\{\mu_{t}\right\}_{t \in G}$ and $\Xi=\left\{\nu_{t}\right\}_{t \in G}$ is given by $\Lambda \Xi=\left\{\mu_{t} * \nu_{t}\right\}_{t \in G}$. We write $\mathbb{S}_{\text {cent }}(A, G, \alpha)$ for the central Herz-Schur ( $A, G, \alpha$ )-multipliers.

Proposition 4.5. We have $\mathbb{S}_{\text {conv }}(G) \cap \mathbb{S}_{\text {cent }}\left(C_{0}(G), G, \beta\right)=\mathrm{M}_{\mathrm{cb}} A(G)$.
Proof. Suppose that $F: G \rightarrow \mathrm{CB}\left(C_{0}(G)\right)$ is a central multiplier which is also a convolution multiplier. Then for each $r \in G$ there is $a_{r} \in C_{b}(G)$ such that $F(r)(a)=a_{r} a$. Also, since $F$ is a convolution multiplier, by (13) $F(r)$ satisfies

$$
\beta_{t}(F(r)(a))=F(r)\left(\beta_{t}(a)\right), \quad r, t \in G, a \in C_{0}(G)
$$

Combining these two identities, and allowing $a$ to vary, gives $a_{r}(s t)=a_{r}(t)$ for all $s, t \in G$, so $a_{r}$ is a scalar multiple of the identity. The conclusion follows from [28, Proposition 4.1].

## 5. Idempotent multipliers

Given standard measure spaces $(X, \mu)$ and $(Y, v)$, a well-known open problem asks for the identification of the idempotent Schur multipliers on $X \times Y$. A characterisation of the contractive idempotent Schur multipliers, based on a combinatorial argument, combined with an observation of Livshitz [25], was given by Katavolos-Paulsen in [22].

In a similar vein, for a general locally compact group $G$, there is no known characterisation of the idempotent Herz-Schur multipliers. Some partial results are known: the idempotent measures in $M(G)$ of norm one were characterised by Greenleaf [13] - a measure $\mu$ has the properties $\mu * \mu=\mu$ and $\|\mu\|=1$ if and only if $\mu=\gamma m_{H}$, where $m_{H}$ is the Haar measure on a compact subgroup $H$ and $\gamma$ is a character of $H$. Such $\mu$ is positive if and only if $\gamma$ above is equal to 1. Dually, the idempotent elements of $B(G)$ were characterised by Host [18]; using Host's method, Ilie and Spronk [19] characterised contractive idempotents - a function $u \in B(G)$ has the properties $u^{2}=u$ and $\|u\|=1$ if and only if $u=\chi_{C}$, where $C$ is an open coset of $G$. Such $u$ is positive if and only if $C$ is a subgroup of $G$. Stan [41] extended this characterisation to norm one idempotent elements of $\mathrm{M}_{\mathrm{cb}} A(G)$.

In this section we use the aforementioned results of Katavolos-Paulsen and Stan to study the idempotent central and the idempotent convolution multipliers.
5.1. Central idempotent multipliers. We fix standard measure spaces $(X, \mu)$ and $(Y, v)$ and a separable, non-degenerate $C^{*}$-algebra $A \subseteq \mathcal{B}(\mathcal{H})$. Suppose $\varphi \in L^{\infty}(X \times Y)$ is an idempotent Schur multiplier, so the map $k \mapsto \varphi \cdot k$ on $L^{2}(Y \times X)$ gives rise to a bounded idempotent $\operatorname{map} S_{\varphi}$ on the space of compact operators; we have that $\varphi^{2}(x, y) k(y, x)=\varphi(x, y) k(y, x)$ almost everywhere for
all $k \in L^{2}(Y \times X)$, which implies that $\varphi^{2}=\varphi$. By [22, Proposition 11], $\varphi=\chi_{E}$ almost everywhere for some $\omega$-open and $\omega$-closed $E \subseteq X \times Y$.

Recall from [22] that a subset $E \subseteq X \times Y$ is said to have the 3-of-4 property provided that given any distinct pair of points $x_{1} \neq x_{2}$ in $X$ and any pair of distinct pairs $y_{1} \neq y_{2}$ in $Y$, whenever 3 of the 4 ordered pairs $\left(x_{i}, y_{j}\right)$ belong to $E$ then the fourth one also belongs to $E$.

For a subset $W \subseteq C \times Z$, where $C$ is a set (which will below be equal to either $X$ or $Y$ ), and an element $z \in Z$, we write $W_{z}=\{t \in C:(t, z) \in W\}$. The following result generalises [22, Theorem 10].

Proposition 5.1. Let $(X, \mu)$ and $(Y, \nu)$ be standard measure spaces and $Z$ a locally compact Hausdorff space. Let $\varphi: X \times Y \times Z \rightarrow \mathbb{C}$ be a measurable function, continuous in the $Z$-variable. The following are equivalent:
(i) $\varphi$ is a contractive idempotent central Schur $C_{0}(Z)$-multiplier;
(ii) for each $z \in Z$, there exist families $\left(A_{i}^{z}\right)_{i \in \mathbb{N}}$ and $\left(B_{i}^{z}\right)_{i \in \mathbb{N}}$ of pairwise disjoint measurable subsets of $X$ and $Y$, respectively, such that

$$
\varphi(x, y, z)=\sum_{i=1}^{\infty} \chi_{A_{i}^{z}}(x) \chi_{B_{i}^{z}}(y)
$$

almost everywhere.
Proof. (i) $\Longrightarrow$ (ii) By Theorem 3.6, $\varphi_{z}$ is a contractive idempotent Schur multiplier for every $z \in Z$. By [22, Theorem 10], there exist families $\left(A_{i}^{z}\right)_{i=1}^{\infty}$ and $\left(B_{i}^{z}\right)_{i=1}^{\infty}$ of pairwise disjoint measurable subsets of $X$ and $Y$, respectively, such that $\varphi_{z}(x, y)=\sum_{i=1}^{\infty} \chi_{A_{i}^{z}}(x) \chi_{B_{i}^{z}}(y)$ almost everywhere.
(ii) $\Longrightarrow$ (i) By [22, Theorem 10], $\varphi_{z}$ is a contractive idempotent Schur multiplier for every $z \in Z$; thus, by Theorem $3.6, \varphi$ is a central $C_{0}(Z)$-multiplier, which is easily seen to be idempotent. Since each $\varphi_{z}$ is contractive we have $\varphi$ is contractive by Theorem 3.6.

Remark 5.2. The statement holds when the standard measure spaces are replaced by discrete spaces $X$ and $Y$ with counting measures, but in this case the families $\left(A_{i}^{z}\right)_{i},\left(B_{i}^{z}\right)_{i}$ might be uncountable if $X$ or $Y$ is uncountable. In this case (i) is also equivalent to $\varphi=\chi_{W}$, where $W_{z}$ has the 3-of-4 property for each $z \in Z$, see [22, Lemma 2].

Let $Z$ be a locally compact Hausdorff space equipped with an action $\alpha$ of a locally compact group $G$. In the subsequent results, we view the set $Z \times G$ as a groupoid as in Section 3.4. We provide a combinatorial characterisation of the contractive central Herz-Schur $\left(C_{0}(Z), G, \alpha\right)$-multipliers. It is easy to see that in this case $\psi(x, t)=\chi_{V}(x, t)$ for some subset $V \subseteq Z \times G$. Theorem 5.3 generalises the result of Stan [41, Theorem 3.3].

Theorem 5.3. Assume that $V \subseteq Z \times G$ is a subset that is both closed and open. The following are equivalent:
(i) $F_{\chi_{V}}$ is a contractive central Herz-Schur $\left(C_{0}(Z), G, \alpha\right)$-multiplier;
(ii) if $(x, t),(x, s),\left(x r, r^{-1} s\right) \in V$ then $\left(x r, r^{-1} t\right) \in V$; equivalently, if $(x, t)$, $(y, s),(z, p) \in V$ and the product $(z, p)(y, s)^{-1}(x, t)$ is well defined then $(z, p)(y, s)^{-1}(x, t) \in V$.
In particular, if $V=Z \times A$ for some $A \subseteq G$ then $A$ is an open coset of $G$.
Proof. Let

$$
W=\left\{(x, s, t) \in Z \times G \times G:\left(x t^{-1}, t s^{-1}\right) \in V\right\} .
$$

By Corollary 3.18, $F_{\chi_{V}}$ is a Herz-Schur $\left(C_{0}(Z), G, \alpha\right)$-multiplier if and only if the map $\mathcal{N}\left(F_{\chi_{V}}\right)$, given by

$$
\mathcal{N}\left(F_{\chi_{V}}\right)(s, t)(a)(x)=\chi_{V}\left(x t^{-1}, t s^{-1}\right) a(x)=\chi_{W}(x, s, t) a(x),
$$

is a Schur $C_{0}(Z)$-multiplier.
We first show that condition (ii) is equivalent to $W_{z}:=\{(s, t) \in G \times G$ : $(z, s, t) \in W\}$ having the 3 -of-4 property for all $z \in Z$. Suppose that $\left(z, t_{1}, s_{1}\right)$, $\left(z, t_{1}, s_{2}\right)$ and $\left(z, t_{2}, s_{2}\right) \in W$, which is equivalent to $\left(z t_{1}^{-1}, t_{1} s_{1}^{-1}\right),\left(z t_{1}^{-1}, t_{1} s_{2}^{-1}\right)$, $\left(z t_{2}^{-1}, t_{2} s_{2}^{-1}\right) \in V$. Writing $z t_{1}^{-1}=x, t_{1} s_{1}^{-1}=t, t_{1} s_{2}^{-1}=s$ and $t_{1} t_{2}^{-1}=r$, we get $z t_{2}^{-1}=x r, t_{2} s_{1}^{-1}=r^{-1} t$ and $t_{2} s_{2}^{-1}=r^{-1} s$ and hence $(x, t),(x, s),\left(x r, r^{-1} s\right) \in$ $V$. The condition $\left(z, t_{2}, s_{1}\right) \in W$ is equivalent to $\left(x r, r^{-1} t\right) \in V$, giving the statement. We note that $(z, p)(y, s)^{-1}(x, t)=(z, p)\left(y s, s^{-1}\right)(x, t)$ is well defined if and only if $y=x$ and $z=x s p^{-1}$; letting $r=s p^{-1}$, we have $(z, p)=\left(x r, r^{-1} s\right)$. We have shown that condition (ii) is equivalent to the 3-of-4 property for each $W_{z}$.

Assume first that $G$ is a locally compact second countable group and hence ( $G, m_{G}$ ) is a standard measure space.
(i) $\Longrightarrow$ (ii) If (i) holds then $\mathcal{N}\left(F_{\chi_{V}}\right)$ is a contractive idempotent Schur $C_{0}(Z)$ multiplier. By Theorem 3.6, $\varphi_{z}=\chi_{W_{z}}$ is a contractive idempotent Schur multiplier for each $z \in Z$. By [22, Theorem 10], there exist countable collections $\left\{I_{m}\right\}$ and $\left\{J_{m}\right\}$ of mutually disjoint Borel subsets of $G$, such that, if $E=\cup_{m} I_{m} \times J_{m}$, then $\chi_{W_{z}}=\chi_{E}$ almost everywhere.

As $\chi_{W_{z}}$ is continuous and hence $\omega$-continuous and $\chi_{E}$ is $\omega$-continuous, by [39, Lemma 2.2], $\chi_{W_{z}}=\chi_{E}$ marginally almost everywhere. Hence there exists a null set $N_{z}$ such that $\chi_{W_{z}}=\chi_{E}$ on $N_{z}^{c} \times N_{z}^{c}$. In particular, $W_{z} \cap\left(N_{z}^{c} \times N_{z}^{c}\right)$ has the 3-of-4 property. To see that the whole $W_{z}$ has the property, take $t_{1}, t_{2}, s_{1}, s_{2}$ such that $\left(t_{1}, s_{1}\right),\left(t_{1}, s_{2}\right),\left(t_{2}, s_{2}\right) \in W_{z}$, but some of $t_{1}, s_{1}, t_{2}, s_{2}$ belong to $N_{z}$. Using the fact that $W_{z}$ is open and $m\left(N_{z}\right)=0$ we can find sequences $\left(t_{1}^{n}\right)_{n},\left(s_{1}^{n}\right)_{n},\left(t_{2}^{n}\right)_{n}$, $\left(s_{2}^{n}\right)_{n}$ of elements in $N_{z}^{c}$ such that $\left(t_{1}^{n}, s_{1}^{n}\right),\left(t_{1}^{n}, s_{2}^{n}\right),\left(t_{2}^{n}, s_{2}^{n}\right) \in W_{z}$ and $t_{i}^{n} \rightarrow t_{i}$, $s_{i}^{n} \rightarrow s_{i}, i=1,2$. Hence $\left(t_{2}^{n}, s_{1}^{n}\right) \in W_{z}$, and as $1=\chi_{W_{z}}\left(t_{2}^{n}, s_{1}^{n}\right) \rightarrow \chi_{W_{z}}\left(t_{2}, s_{1}\right)$, we obtain that $\left(t_{2}, s_{1}\right) \in W_{z}$. Hence (ii) holds.
(ii) $\Longrightarrow$ (i) As $W_{z}$ is open and hence $\omega$-open, $W_{z}$ is marginally equivalent to a countable union of Borel rectangles. Hence $W_{z} \cap\left(N_{z}^{c} \times N_{z}^{c}\right)=\cup_{m=1}^{\infty} A_{m}^{z} \times B_{m}^{z}$, where $m_{G}\left(N_{z}\right)=0$ and each $A_{m}^{z} \times B_{m}^{z}$ is Borel. By [22, Lemma 2] and the second paragraph in the proof, $W_{z}$ and hence $W_{z} \cap\left(N_{z}^{c} \times N_{z}^{c}\right)$ has the 3-of-4 property for each $z \in Z$ and there exist families $\left\{X_{i}^{z}\right\}_{i \in I}$ and $\left\{Y_{i}^{z}\right\}_{i \in I}$ of pairwise disjoint sets of $G$, such that $W_{z} \cap\left(N_{z}^{c} \times N_{z}^{c}\right)=\cup_{i \in I} X_{i}^{z} \times Y_{i}^{z}$. Arguing as in the proof of
[22, Theorem 10] one shows that the index set $I$ can be chosen countable and each $X_{i}^{z} \times Y_{i}^{z}$ is a Borel rectangle. Hence $\chi_{W_{z}}$ is a contractive Schur multiplier. By Proposition $5.1 \chi_{W}$ is a contractive idempotent central Schur multiplier, so $\chi_{V}$ is a contractive idempotent central Herz-Schur $\left(C_{0}(Z), G, \alpha\right)$-multiplier.

If $G$ is discrete, the statement follows from Remark 5.2. Finally, if $V=Z \times A$ then $\chi_{V}(x, t)=\chi_{A}(t)$ which is a Herz-Schur $\left(C_{0}(Z), G, \alpha\right)$-multiplier if and only if $\chi_{A}$ is a Herz-Schur multiplier. It is of norm at most 1 if and only if $A$ is an open coset of $G$.
Remark 5.4. It follows from Theorem 5.3 that if $F_{\chi_{V}}$ is a contractive HerzSchur $\left(C_{0}(Z), G, \alpha\right)$-multiplier and the points

$$
(x, t), \quad r((x, t))=(x, e), \quad d((x, t))=(x t, e)
$$

all belong to $V$ then $(x, t)^{-1}=\left(x t, t^{-1}\right) \in V$. Moreover, if $(x, t), d((x, t))=$ $(x t, e)$ and $(x t, s) \in V$ then $(x, t)(x t, s)=(x, t s) \in V$.

The following corollary is an immediate consequence of Remark 5.4.
Corollary 5.5. With the notation of Theorem 5.3, assume that $\mathcal{G}_{0} \subseteq V$. We have that $F_{\chi_{V}}$ is a contractive Herz-Schur $\left(C_{0}(Z), G, \alpha\right)$-multiplier if and only if $V$ is a subgroupoid of $\mathcal{G}$.
5.2. Positive central idempotent multipliers. The following description of positive contractive Schur multipliers can be obtained in a similar manner to [22, Theorem 10], and we omit its proof.
Proposition 5.6. Let $(X, \mu)$ be a standard measure space and $E \subseteq X \times X$. The following are equivalent:
(i) $\chi_{E}$ is a positive contractive Schur multiplier;
(ii) $E$ is equivalent to a subset of the form $\cup_{m=1}^{\infty} I_{m} \times I_{m}$ with respect to product measure, where $\left\{I_{m}\right\}_{m=1}^{\infty}$ is a collection of disjoint Borel subset of $X$.
Remark 5.7. The standard measure space ( $X, \mu$ ) can be replaced by discrete space $X$ with counting measure. In this case the collection of disjoint subsets of $X$ might be uncountable.

The following positive version of Proposition 5.1 and its discrete version can be proved using similar ideas, and we omit the detailed argument.
Proposition 5.8. Let $(X, \mu)$ and $(Y, \nu)$ be standard measure spaces and $Z$ a locally compact Hausdorff space. Let $\varphi: X \times Y \times Z \rightarrow \mathbb{C}$ be a measurable function which is continuous in the $Z$-variable. The following are equivalent:
(i) $\varphi$ is a positive contractive idempotent central Schur $C_{0}(Z)$-multiplier;
(ii) for each $z \in Z$, there exists a family $\left(A_{i}^{z}\right)_{i}$ of pairwise disjoint measurable subsets of $X$, such that $\varphi(x, y, z)=\sum_{i=1}^{\infty} \chi_{A_{i}^{z}}(x) \chi_{A_{i}^{z}}(y)$ almost everywhere.
Proposition 5.1 and the transference theorem of [28] give an implicit characterisation of the positive central idempotent Herz-Schur $\left(C_{0}(Z), G, \alpha\right)$-multipliers. In Theorem 5.9 below, we give a more direct description of the positive central idempotent Herz-Schur multipliers of norm not exceeding 1.

Theorem 5.9. Let $\left(C_{0}(Z), G, \alpha\right)$ be a $C^{*}$-dynamical system and $V \subseteq Z \times G$ be a closed and open subset. The following are equivalent:
(i) $F_{\chi_{V}}$ is a positive, contractive Herz-Schur $\left(C_{0}(Z), G, \alpha\right)$-multiplier;
(ii) $V$ is a subgroupoid of $Z \times G$.

Proof. We will prove the theorem for $G$ a locally compact second countable group. The case when $G$ is discrete can be treated in a similar but simpler way. (i) $\Longrightarrow$ (ii) Let

$$
W=\left\{(z, s, t) \in Z \times G \times G:\left(z t^{-1}, t s^{-1}\right) \in V\right\} .
$$

If $F_{\chi_{V}}$ is a positive contractive Herz-Schur $\left(C_{0}(Z), G, \alpha\right)$-multiplier then the function $\mathcal{N}\left(F_{\chi_{V}}\right)$, given by $\mathcal{N}\left(F_{\chi_{V}}\right)(s, t)(a)(z)=\chi_{W}(z, s, t) a(z)$, is a positive Schur $C_{0}(Z)$-multiplier. By Theorem 3.13, $\chi_{W_{z}}$ is a positive Schur multiplier for each $z \in Z$. Note also that, as it is continuous, it is $\omega$-continuous. Using [39, Lemma 2.2], we see that there exist a weakly measurable function $v_{z}: G \rightarrow \ell^{2}$ and a null set $N_{z} \subseteq G$ such that

$$
\chi_{W_{z}}(s, t)=\left\langle v_{z}(s), v_{z}(t)\right\rangle, \quad s, t \notin N_{z}
$$

Let $(x, t) \in V$; as in Remark 5.4, it suffices to show that $(x, e)$ and $(x t, e) \in V$. Assume that ( $x, e) \notin V$, and note that

$$
\chi_{V}(x, e)=\chi_{V}\left((x t) t^{-1}, t t^{-1}\right)=\chi_{W}(x t, t, t) \text { and } \chi_{V}(x, t)=\chi_{W}(x t, e, t) .
$$

If $t \notin N_{x t}$ and $e \notin N_{x t}$ then

$$
\chi_{W}(x t, t, t)=\left\|v_{x t}(t)\right\|_{2}^{2}=0 \text { and } \chi_{W}(x t, e, t)=\left\langle v_{x t}(e), v_{x t}(t)\right\rangle=0,
$$

giving a contradiction. If one or both of $e$ or $t$ are in $N_{x t}$, say $t \in N_{x t}$ bute $\notin N_{x t}$, then, as $m\left(N_{x t}\right)=0$ there exists a sequence $s_{n} \notin N_{x t}$ such that $s_{n} \rightarrow t$. As $\chi_{W}$ is continuous, we obtain

$$
\left\|v_{x t}\left(s_{n}\right)\right\|_{2}^{2}=\chi_{W}\left(x t, s_{n}, s_{n}\right) \rightarrow \chi_{W}(x t, t, t)=0
$$

while

$$
\left\langle v_{x t}(e), v_{x t}\left(s_{n}\right)\right\rangle=\chi_{W}\left(x t, e, s_{n}\right) \rightarrow \chi_{W}(x t, e, t),
$$

forcing $\chi_{W}(x t, e, t)=0$, a contradiction. The other cases are treated similarly. To see that $(x t, e) \in V$ observe that

$$
\chi_{V}(x t, e)=\chi_{W}\left(x, t^{-1}, t^{-1}\right) \text { and } \chi_{V}(x, t)=\chi_{W}\left(x, t^{-1}, e\right)
$$

and apply similar analysis.
(ii) $\Longrightarrow$ (i) Let now $V$ be an open subgroupoid. Arguing as in the proof of Theorem 5.3 we see that $W_{z}$ has the 3 -of-4 property for each $z \in Z$. Moreover, if $(x, s, t) \in W$ we have that $\left(x t^{-1}, t s^{-1}\right) \in V$ and hence $r\left(x t^{-1}, t s^{-1}\right)=\left(x t^{-1}, e\right) \in$ $V$ and $d\left(x t^{-1}, t s^{-1}\right)=\left(x s^{-1}, e\right) \in V$, implying $(x, t, t) \in W$ and $(x, s, s) \in$ $W$. Therefore the projections $W_{z}^{1}$ and $W_{z}^{2}$ of $W_{z}$ on the first and the second coordinates are equal and $\left\{(s, s): s \in W_{z}^{1}\right\} \subseteq W_{z}$. It follows easily now that for each $z \in Z$ there exists disjoint sets $\left\{X_{t}^{z}\right\}_{t \in T}$ such that $W_{z}=\cup_{t \in T} X_{t}^{z} \times X_{t}^{z}$. Arguing as in [22, Theorem 10], there is a Borel subset $N_{z}, m_{G}\left(N_{z}\right)=0$ such that $\left(X_{t}^{z} \cap N_{z}^{c}\right) \times\left(X_{t}^{z} \cap N_{z}^{c}\right)$ is a Borel rectangle and $W_{z} \cap\left(N_{z}^{c} \times N_{z}^{c}\right)$ is a countable
union of $\left(X_{t}^{z} \cap N_{z}^{c}\right) \times\left(X_{t}^{z} \cap N_{z}^{c}\right)$. By Proposition $5.6 \chi_{W_{z}}$ is a positive contractive Schur multiplier. Therefore $\chi_{W}$ is a positive contractive Schur $C_{0}(Z)$-multiplier by Theorem 3.13.
5.3. Idempotent convolution multipliers. We next provide some examples of idempotent convolution multipliers. The following is immediate from Theorem 4.1 and [19, Theorem 2.1].

Corollary 5.10. Suppose $G$ is an abelian locally compact group and $W \subseteq G \times \Gamma$ is a measurable set, such that $\chi_{W} \in \mathfrak{S}_{\mathrm{conv}}^{\mathrm{id}}$. Then $\left\|\chi_{W}\right\|_{\mathfrak{S}} \leq 1$ if and only if $W$ is an open coset of $G \times \Gamma$.

It is clear that if $G$ is abelian, and $C$ and $D$ are open cosets of $G$ and $\Gamma$ respectively, then $C \times D$ is an open coset of $G \times \Gamma$ and therefore $\chi_{C \times D}$ is an idempotent convolution multiplier of norm 1 by Corollary 5.10. The following example shows that not all idempotent convolution multipliers of norm 1 are of this product form.

Example 5.11. Consider the abelian group $G=\mathbb{R} \times \mathbb{Z}_{2}$, and note that $G$ is isomorphic to its dual group $\Gamma$. Define

$$
H:=\{(a, 0, b, 0),(c, 1, d, 1): a, b, c, d \in \mathbb{R}\}
$$

It is clear that $H$ is an open subgroup of $G \times \Gamma$, but $H$ cannot be written as a product of subgroups of $G$ and $\Gamma$.

Remark 5.12. Let $G$ be an abelian locally compact group; by Theorem 4.1, a contractive idempotent Herz-Schur convolution multiplier, say $F$, corresponds to a characteristic function $\chi_{W}$, for an open coset $W \subseteq G \times \Gamma$. In the following, we show more precisely how the family $(F(r))_{r \in G} \subseteq \mathrm{CB}\left(C_{r}^{*}(\Gamma)\right)$ arises. Suppose that $W=x H$ for an open subgroup $H$ of $G \times \Gamma$ and $x \in G \times \Gamma$. Let $v$ be the representation of $G \times \Gamma$ on $\ell^{2}((G \times \Gamma) / H)$, given by $\nu(z) \delta_{y H}:=\delta_{z y H}(z, y \in$ $G \times \Gamma),\left\{\delta_{y H}\right\}_{y}$ be the standard orthonormal basis in $\ell^{2}((G \times \Gamma) / H)$ ), and write $\bar{\nu}$ for the unitary representation $\gamma \mapsto \nu(e, \gamma)$ of $\Gamma$. For $r \in G$, let $u_{r} \in B(\Gamma)$ be the function given by

$$
u_{r}: \Gamma \rightarrow \mathbb{C} ; u_{r}(\gamma):=\left\langle\bar{v}(\gamma) \delta_{(r, e) H}, \delta_{W}\right\rangle
$$

Then

$$
\begin{aligned}
S_{\chi_{W}}\left(\lambda_{\gamma}^{\Gamma} \otimes \lambda_{r}^{G}\right) & =\chi_{W}(r, \gamma)\left(\lambda_{\gamma}^{\Gamma} \otimes \lambda_{r}^{G}\right)=\left\langle\delta_{(r, \gamma) H}, \delta_{W}\right\rangle\left(\lambda_{\gamma}^{\Gamma} \otimes \lambda_{r}^{G}\right) \\
& =\left\langle\nu(r, \gamma) \delta_{H}, \delta_{W}\right\rangle\left(\lambda_{\gamma}^{\Gamma} \otimes \lambda_{r}^{G}\right) \\
& =\left\langle\bar{\nu}(\gamma) \delta_{(r, e) H}, \delta_{W}\right\rangle\left(\lambda_{\gamma}^{\Gamma} \otimes \lambda_{r}^{G}\right) \\
& =u_{r}(\gamma) \lambda_{\gamma}^{\Gamma} \otimes \lambda_{r}^{G},
\end{aligned}
$$

so the idempotent element $\chi_{W} \in B(G \times \Gamma)$ corresponds to the Herz-Schur convolution multiplier $F(r):=u_{r}$.

It is immediate from Host's theorem that if $G$ is a connected locally compact group then $B(G)$ does not have non-trivial idempotent elements. We observe that this extends to idempotent convolution multipliers on abelian groups. Indeed, let $\psi$ be an idempotent convolution multiplier of the dynamical system ( $\left.C_{r}^{*}(\Gamma), G, \beta\right)$ and write $\psi=\chi_{W}$ for some $W \subseteq G \times \Gamma$. For $x \in \Gamma$ and $s \in G$, let

$$
W^{x}:=\{t \in G:(t, x) \in W\} \quad \text { and } \quad W_{s}:=\{y \in \Gamma:(s, y) \in W\} .
$$

Proposition 5.13. Let $\psi=\chi_{W} \in \mathbb{S}_{\text {conv }}^{\mathrm{id}}(G)$ and $\|\psi\|_{\mathfrak{S}} \leq 1$. Then $W^{x}$ (resp. $W_{s}$ ) is an open coset of $G$ (resp. $\Gamma$ ) for all $x \in \Gamma$ (resp. $s \in G$ ).

Proof. Since for any $x \in \Gamma, s \in G$, we have $\psi^{x}=\chi_{W^{x}}$ and $\psi_{s}=\chi_{W_{s}}$, the statement follows from [19, Theorem 2.1], as $\psi^{x} \in B(G)$ and $\psi_{s} \in B(\Gamma)$.

If $\psi=\chi_{W} \in \mathbb{S}_{\text {conv }}^{\text {id }}(G)$ is contractive, as $\psi$ is separately continuous, we obtain that $W_{s}=W_{s^{\prime}}$ if $s$ and $s^{\prime}$ are in the same connected component of $G$. Similarly, we have $W^{x}=W^{x^{\prime}}$ for $x, x^{\prime}$ in the same connected component of $\Gamma$. This implies the following corollary.

Corollary 5.14. If the group $G$ (resp. Г) is connected then any contractive idempotent multiplier $\psi \in \mathbb{S}_{\mathrm{conv}}^{\mathrm{id}}(G)$ is given by $\psi=1 \otimes \chi_{A}\left(\right.$ resp. $\left.\psi=\chi_{A} \otimes 1\right)$, where $A$ is an open coset of $\Gamma$ (resp. G).

In particular, we have that $C_{r}^{*}(\mathbb{R}) \rtimes_{\beta, r} \mathbb{R}$ has no non-trivial idempotent HerzSchur convolution multipliers, and any idempotent Herz-Schur convolution multiplier of $C(\mathbb{T}) \rtimes_{\beta, r} \mathbb{Z}$ is given by $\chi_{A} \otimes 1$, where $A$ is a coset of $\mathbb{Z}$.

Example 5.15. Let $G$ be a locally compact group. Since $\mathrm{M}_{\mathrm{cb}} L^{1}(G)=M(G)$, we have that $\gamma m_{H} \otimes \chi_{C} \in \mathrm{M}_{\mathrm{cb}}\left(L^{1}(G) \widehat{\otimes} A(G)\right)$, where $C$ is an open coset of $G, H$ is a compact subgroup and $\gamma$ is a character of $H$. The corresponding convolution multiplier $\Lambda=\left(\mu_{t}\right)_{t \in G}$ is given by $\mu_{t}=\chi_{C}(t) \gamma m_{H}$. In fact, if $R$ is the completely bounded map

$$
R(f \otimes g)=\left(\left(\gamma m_{H}\right) * f\right) \otimes \chi_{C} g, \quad f \in L^{1}(G), g \in A(G),
$$

then

$$
R^{*}\left(h \otimes \lambda_{t}\right)=\theta\left(\gamma m_{H}\right)(h) \otimes \chi_{C} \lambda_{t}=\theta\left(\gamma m_{H}\right) h \otimes \chi_{C}(t) \lambda_{t} .
$$

Remark 5.16. For a (not necessarily abelian) locally compact group $G$ the algebra $C_{0}(G \times \hat{G}):=C_{0}(G) \otimes C_{r}^{*}(G)$ can be considered as a quantum group with the comultiplication induced from comultiplications of the factors $C_{0}(G)$ and $C_{r}^{*}(G)$. In [32] the authors give a characterisation of contractive idempotent functionals on $C^{*}$-quantum groups in terms of compact quantum subgroups and group-like unitaries of the subgroup. It would be interesting to use this characterisation to describe contractive convolution multipliers in the nonabelian case. At present, however, a lack of examples of compact quantum subgroups of $C_{0}(G \times \hat{G})$ impedes the application of the results of [32] to convolution multipliers.

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## References

[1] Arveson, William. Operator algebras and invariant subspaces. Ann. of Math. (2) 100 (1974), 433-532. MR0365167 (51 \#1420), Zbl 0334.46070, doi: 10.2307/1970956. 7
[2] Bédos, Erik; Conti, Roberto. Fourier series and twisted $C^{*}$-crossed products. J. Fourier Anal. Appl. 21 (2015), no. 1, 32-75. MR3302101, Zbl 1328.46055, doi: 10.1007/s00041-014-9360-3. 2
[3] Bédos, Erik; Conti, Roberto. The Fourier-Stieltjes algebra of a $C^{*}$-dynamical system. Internat. J. Math. 27 (2016), no. 6, 1650050, 50 pp. MR3516977, Zbl 1362.46068, doi: 10.1142/S0129167X16500506. 2
[4] Blecher, David P.; Smith, Roger R. The dual of the Haagerup tensor product. J. London Math. Soc. (2) 45 (1992), no. 1, 126-144. MR1157556 (93h:46078), Zbl 0712.46029, doi: $10.1112 / \mathrm{jlms} / \mathrm{s} 2-45.1 .126 .5,8$
[5] Bożejko, MAREK; Fendler, Gero. Herz-Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact group. Boll. Un. Mat. Ital. A (6) 3 (1984), no. 2, 297-302. MR753889 (86b:43009), Zbl 0564.43004. 2, 9, 25, 27
[6] De Cannière, Jean; HaAGerup, UfFe. Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups. Amer. J. Math. 107 (1985), no. 2, 455-500. MR784292 (86m:43002), Zbl 0577.43002, doi: 10.2307/2374423. 2, 9, 10
[7] Cohen, Paul J. On a conjecture of Littlewood and idempotent measures. Amer. J. Math. 82 (1960), 191-212. MR133397 (24 \#A3231), Zbl 0099.25504, doi: 10.2307/2372731. 3
[8] Coine, Clément; Le Merdy, Christian; Sukochev, Fedor. When do triple operator integrals take value in the trace class? Preprint, 2017. To appear in Annales Institut Fourier. arXiv:1706.01662. 2, 3, 13, 16, 17
[9] Dong, Zhe; Ruan, Zhong-Jin. A Hilbert module approach to the Haagerup property. Integral Equations Operator Theory 73 (2012), no. 3, 431-454. MR2945214, Zbl 1263.46043, doi: 10.1007/s00020-012-1979-3. 2, 24
[10] Effros, Edward G.; Ruan, Zhong-Jin. Operator spaces. London Mathematical Society Monographs. New series, 23. The Clarendon Press, Oxford University Press, New York, 2000. xvi+363 pp. ISBN: 0-19-853482-5. MR1793753 (2002a:46082), Zbl 0969.46002. 4, 5
[11] Erdos, John A.; Katavolos, Aristides; Shulman, Victor S. Rank one subspaces of bimodules over maximal abelian selfadjoint algebras. J. Funct. Anal. 157 (1998), no. 2, 554587. MR1638277 (99f:47054), Zbl 0977.47062, doi: 10.1006/jfan.1998.3274. 6
[12] Eymard, Pierre. L'algèbre de Fourier d'un groupe localement compact. Bull. Soc. Math. France 92 (1964), 181-236. MR228628 (37 \#4208), Zbl 0169.46403, doi: 10.24033/bsmf.1607. 8, 9, 30
[13] Greenleaf, Frederick P. Norm decreasing homomorphisms of group algebras. Pacific J. Math. 15 (1965), 1187-1219. MR0194911 (33 \#3117), Zbl 0136.11402, doi: 10.2140/pjm.1965.15.1187. 33
[14] Grothendieck, Alexander. Résumé de la théorie métrique des produits tensoriels topologiques. Bol. Soc. Mat. São Paulo 8 (1953), 1-79. MR94682 (20 \#1194), Zbl 1019.46038. 2
[15] HAAGERUP, UfFe. An example of a nonnuclear $C^{*}$-algebra, which has the metric approximation property. Invent. Math. 50 (1978/79), no. 3, 279-293. MR520930 (80j:46094), Zbl 0408.46046, doi: 10.1007/BF01410082. 2
[16] HaAGERUP, UFFE. Decomposition of completely bounded maps on operator algebras. Unpublished manuscript, 1980. 8, 20
[17] HERz, CARL. Une généralisation de la notion de transformée de Fourier-Stieltjes. Ann. Inst. Fourier (Grenoble) 24 (1974), no. 3, xiii, 145-157. MR425511 (54 \#13466), Zbl 0287.43006, doi: 10.5802/aif.522. 2
[18] Host, Bernard. Le théorème des idempotents dans $B(G)$. Bull. Soc. Math. France 114 (1986), no. 2, 215-223. MR860817 (88b:43003), Zbl 0606.43002, doi: 10.24033/bsmf.2055. 3, 33
[19] Ilie, MONica; Spronk, Nico. Completely bounded homomorphisms of the Fourier algebras. J. Funct. Anal. 225 (2005), no. 2, 480-499. MR2152508 (2006f:43005), Zbl 1077.43004, doi: 10.1016/j.jfa.2004.11.011. 33, 38, 39
[20] Junge, Marius; Neufang, Matthias; Ruan, Zhong-Jin. A representation theorem for locally compact quantum groups. Internat. J. Math. 20 (2009), no. 3, 377-400. MR2500076 (2010c:46128), Zbl 1194.22003, doi: 10.1142/S0129167X09005285. 32, 33
[21] Kaniuth, Eberhard; Lau, Anthony To-Ming. Fourier and Fourier-Stieltjes algebras on locally compact groups. Mathematical Surveys and Monographs, 231. American Mathematical Society, Providence, R.I., 2018. xi+306 pp. ISBN: 978-0-8218-5365-8. MR3821506, Zbl 1402.43001, doi: 10.1090/surv/231. 27
[22] Katavolos, Aristides; Paulsen, Vern I. On the ranges of bimodule projections. Canad. Math. Bull. 48 (2005), no. 1, 97-111. MR2118767 (2005j:46037), Zbl 1085.46039, doi: 10.4153/CMB-2005-009-4. 3, 10, 33, 34, 35, 36, 37
[23] Kraus, Jon. The slice map problem for $\sigma$-weakly closed subspaces of von Neumann algebras. Trans. Amer. Math. Soc. 279 (1983), no. 1, 357-376. MR704620 (85e:46036), Zbl 0525.46036, doi: 10.2307/1999389. 30
[24] Le Merdy, Christian; Todorov, Ivan G.; Turowska, Lyudmila. Bilinear operator multipliers into the trace class. J. Funct. Anal. 279 (2020), no. 7, 108649, 40 pp. MR4107808, Zbl 1465.46057, doi: 10.1016/j.jfa.2020.108649. 2, 3, 16
[25] Livshits, Leo. A note on 0-1 Schur multipliers. Linear Algebra Appl. 222 (1995), 15-22. MR1332920 (96d:15040), Zbl 0828.15014, doi: 10.1016/0024-3795(93)00268-5. 33
[26] McKee, Andrew. Weak amenability for dynamical systems. Studia Math. 258 (2021), no. 1, 53-70. MR4214353, Zbl 1455.37007, doi: 10.4064/sm200227-20-7. 2
[27] McKee, Andrew; Skalski, Adam; Todorov, Ivan G.; Turowska, lyudmila. Positive Herz-Schur multipliers and approximation properties of crossed products. Math. Proc. Cambridge Philos. Soc. 165 (2018), no. 3, 511-532. MR3860401, Zbl 1412.46075, doi: 10.1017/S0305004117000639. 2, 20, 22
[28] McKee, Andrew; Todorov, Ivan G.; Turowska, Lyudmila. Herz-Schur multipliers of dynamical systems. Adv. Math. 331 (2018), 387-438. MR3804681, Zbl 1400.46046, doi: 10.1016/j.aim.2018.04.002. 2, 3, 6, 7, 9, 10, 11, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26, 31, 33, 36
[29] McKee, Andrew; Turowska, Lyudmila. Exactness and SOAP of crossed products via Herz-Schur multipliers. J. Math. Anal. Appl. 496 (2021), no. 2, 124812, 16 pp. MR4189017, Zbl 1459.42014, doi: 10.1016/j.jmaa.2020.124812. 2
[30] NEUFANG, MATtHIAS. Abstrakte harmonische analyse und modulhomomorphismen über von Neumann algebren. PhD thesis, University of Saarland (2000). 29, 31
[31] Neufang, Matthias; Ruan, Zhong-Jin; Spronk, Nico. Completely isometric representations of $M_{\mathrm{cb}} A(G)$ and UCB $(\hat{G})$. Trans. Amer. Math. Soc. 360 (2008), no. 3, 1133-1161. MR2357691 (2008h:22007), Zbl 1142.22002, doi: 10.1090/S0002-9947-07-03940-2. 29, 31
[32] Neufang, Matthias; Salmi, Pekka; Skalski, Adam; Spronk, Nico. Contractive idempotents on locally compact quantum groups. Indiana Univ. Math. J. 62 (2013), no. 6, 19832002. MR3205538, Zbl 1305.46061, doi: 10.1512/iumj.2013.62.5178. 39
[33] Ozawa, Narutaka. Amenable actions and exactness for discrete groups. C. R. Acad. Sci. Paris Sér. I Math. 330 (2000), no. 8, 691-695. MR1763912 (2001g:22007), Zbl 0953.43001, doi: 10.1016/S0764-4442(00)00248-2. 24
[34] Pedersen, Gert K. $C^{*}$-algebras and their automorphism groups. London Mathematical Society Monographs, 14. Academic Press, Inc., London-New York, 1979. ix+416 pp. ISBN: 0-12-549450-5. MR548006 (81e:46037), Zbl 0416.46043. 6
[35] Peller, Vladimir V. Hankel operators in the theory of perturbations of unitary and selfadjoint operators. Funct. Anal. Appl. 19 (1985), 111-123; English translation of the original. Funktsional. Anal. i Prilozhen. 19 (1985), no. 2, 37-51, 96. MR0800919 (87e:47029), Zbl 0587.47016. 7
[36] PISIER, GILLES. Introduction to operator space theory. London Mathematical Society Lecture Note Series, 294. Cambridge University Press, Cambridge, 2003. viii+478 pp. ISBN: 0-521-81165-1. MR2006539 (2004k:46097), Zbl 1093.46001, doi: 10.1017/CBO9781107360235. 5
[37] Renault, Jean. A groupoid approach to $C^{*}$-algebras. Lecture Notes in Mathematics, 793. Springer, Berlin, 1980. ii+160 pp. ISBN: 3-540-09977-8. MR584266 (82h:46075), Zbl 0433.46049. 23
[38] Renault, Jean. The Fourier algebra of a measured groupoid and its multipliers. J. Funct. Anal. 145 (1997), no. 2, 455-490. MR1444088 (99a:43004), Zbl 0874.43003, doi: 10.1006/jfan.1996.3039. 24
[39] Shulman, Viktor S.; Todorov, Ivan G.; Turowska, Lyudmila. Closable multipliers. Integral Equations Operator Theory 69 (2011), no. 1, 29-62. MR2749447 (2012c:47195), Zbl 1251.47037, doi: 10.1007/s00020-010-1819-2. 35, 37
[40] Sinclair, Allan M.; Smith, Roger R. Factorization of completely bounded bilinear operators and injectivity. J. Funct. Anal. 157 (1998), no. 1, 62-87. MR1637933 (99i:46047), Zbl 0930.46050, doi: 10.1006/jfan.1998.3260. 21, 23
[41] Stan, ANA-MARIA Popa. On idempotents of completely bounded multipliers of the Fourier algebra $A(G)$. Indiana Univ. Math. J. 58 (2009), no. 2, 523-535. MR2514379 (2010g:46079), Zbl 1232.43002, doi: 10.1512/iumj.2009.58.3452. 3, 33, 34
[42] TAKESAKI, MASAMICHI. Theory of operator algebras. I. Encyclopaedia of Mathematical Sciences, 124. Operator Algebras and Non-commutative Geometry, 5. Springer-Verlag, Berlin, 2002. xx+415 pp. ISBN: 3-540-42248-X. MR1873025 (2002m:46083), Zbl 0990.46034. 4
[43] Williams, Dana P. Crossed products of $C^{*}$-algebras. Mathematical Surveys and Monographs, 134. American Mathematical Society, Providence, RI, 2007. xvi+528 pp. ISBN: 978-0-8218-4242-3; 0-8218-4242-0. MR2288954 (2007m:46003), Zbl 1119.46002, doi: 10.1090/surv/134. 4, 6
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