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International Journal of Mathematics
(2022) 2250017 (46 pages)

DOI: 10.1142/S0129167X22500173

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Received 23 September 2021
Revised 21 October 2021
Accepted 10 December 2021
Published 19 January 2022

Communicated by Yasuyuki Kawahigashi

To 75th birthday of our teacher Yurii S. Samoilenko

We consider the C^*-algebra $\mathcal{E}_{n,m}^q$, which is a q-twist of two Cuntz–Toeplitz algebras. For the case $|q| < 1$, we give an explicit formula which untwists the q-deformation showing that the isomorphism class of $\mathcal{E}_{n,m}^q$ does not depend on q. For the case $|q| = 1$, we give an explicit description of all ideals in $\mathcal{E}_{n,m}^q$. In particular, we show that $\mathcal{E}_{n,m}^q$ contains a

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unique largest ideal $\mathcal{M}_q$. We identify $\mathcal{E}^q_{n,m}/\mathcal{M}_q$ with the Rieffel deformation of $\mathcal{O}_n \otimes \mathcal{O}_m$ and use a K-theoretical argument to show that the isomorphism class does not depend on q. The latter result holds true in a more general setting of multiparameter deformations.

Keywords: Cuntz–Toeplitz algebra; Rieffel's deformation; q-deformation; Fock representation; K-theory.

Mathematics Subject Classification 2020: 46L05, 46L35, 46L80, 46L65, 47A67, 81R10

1. Introduction

Since the early 1980s, a wide study of non-classical models of mathematical physics, quantum group theory and noncommutative probability (see e.g. [5, 21, 23, 38, 40, 56]) gave rise to a number of papers on operator algebras generated by various deformed commutation relations [6, 34, 39], a prominent example being the irrational rotation algebra, also called the non-commutative torus [48]. A major question for such objects is whether deformations exist and how they relate to the original object.

Other objects of studies, closely related to the ones mentioned above, are C^*-algebras generated by isometries, such as Cuntz algebras, extensions of non-commutative tori and their multiparameter generalizations, see, [9, 10, 16, 45, 46]. The problems of classification of representations, existence of faithful (universal) representation, as well as the study of structure of corresponding C^*-algebra and dependence of C^*-isomorphism classes on parameter of the deformations are among the central ones in this area.

In our paper, we work with a certain class of C^*-algebras generated by isometries subject to deformed commutation relations. We study the structure of these C^*-algebras, present their faithful representations and show that some of the algebras we deal with are independent of the deformation parameter of deformation. The relation with Rieffel deformation of tensor products, see e.g. [30], has been significant for our studies.

1.1. *Context*: *Wick algebras*

Let us put our work into a broader context. A general approach to the study of such deformed commutation relations has been provided by the framework of quadratic $*$-algebras allowing Wick ordering (Wick algebras), see [29]. It includes, among others, deformations of canonical commutation relations of quantum mechanics, some quantum groups and quantum homogeneous spaces, see e.g. [22, 35, 47, 53]. On the other hand, one can consider Wick algebras as deformations of Cuntz–Toeplitz algebras, see [10, 17, 29].

For $\{T^{kl}_{ij},\ i,j,k,l = \overline{1,d}\} \subset \mathbb{C}$, $T^{kl}_{ij} = \overline{T}^{lk}_{ji}$, the Wick algebra $W(T)$ is the $*$-algebra generated by elements a_j, a_j^*, $j = \overline{1,d}$ subject to the relations

$$a_i^* a_j = \delta_{ij}\mathbf{1} + \sum_{k,l=1}^{d} T^{kl}_{ij} a_l a_k^*.$$

It depends [29] on the so-called operator of coefficients T, given as follows. Let $\mathcal{H} = \mathbb{C}^d$ and $e_1, \ldots, e_d$ be the standard orthonormal basis, then

$$T \colon \mathcal{H}^{\otimes 2} \to \mathcal{H}^{\otimes 2}, \quad Te_k \otimes e_l = \sum_{i,j=1}^{d} T_{ik}^{lj} e_i \otimes e_j.$$

It is a nontrivial and central problem in the theory of Wick algebras to determine whether a Fock representation $\pi_{F,T}$ of a Wick algebra exists, see [6, 26, 29] for some sufficient conditions: for instance, it exists, if T is braided, i.e. $(\mathbf{1} \otimes T)(T \otimes \mathbf{1})(\mathbf{1} \otimes T) = (T \otimes \mathbf{1})(\mathbf{1} \otimes T)(\mathbf{1} \otimes T)$, and if $\|T\| \leq 1$; moreover, if $\|T\| < 1$ then $\pi_{F,T}$ is a faithful representation of $W(T)$.

Another important question concerns the stability of isomorphism classes of the universal C^*-envelope[a] $\mathcal{W}(T) = C^*(W(T))$. It was conjectured in [28]:

Conjecture 1.1. If T is self-adjoint, braided and $||T|| < 1$, then $\mathcal{W}(T) \simeq \mathcal{W}(0)$.

In particular, the authors of [28] have shown that the conjecture holds for the case $||T|| < \sqrt{2} - 1$, for more results on the subject see [17, 31].

In the case $T = 0$ and $d = \dim \mathcal{H} = 1$, the Wick algebra $W(0)$ is generated by a single isometry s, its universal C^*-algebra exists and is isomorphic to the C^*-algebra generated by the unilateral shift, and the Fock representation is faithful. The ideal $\mathcal{I}$ in $\mathcal{E}$, generated by $\mathbf{1} - ss^*$ is isomorphic to the algebra of compact operators and $\mathcal{E}/\mathcal{I} \simeq C(S^1)$, see [9]. When $d \geq 2$, the enveloping universal C^*-algebra exists and it is called the Cuntz–Toeplitz algebra $\mathcal{O}_d^{(0)}$. It is isomorphic to $C^*(\pi_{F,d}(W(0)))$, so the Fock representation of $\mathcal{O}_d^{(0)}$ is faithful, see [10]. Furthermore, the ideal $\mathcal{I}$ generated by $1 - \sum_{j=1}^{d} s_j s_j^*$ is the unique largest ideal in $\mathcal{O}_d^{(0)}$. It is isomorphic to the algebra of compact operators on $\mathcal{F}_d$. The quotient $\mathcal{O}_d^{(0)}/\mathcal{I}$ is called the Cuntz algebra $\mathcal{O}_d$. It is nuclear (as well as $\mathcal{O}_d^{(0)}$), simple and purely infinite, see [10] for more details.

1.2. *Our objects of interest*: *The C^*-algebras $\mathcal{E}_{n,m}^q$*

In this paper, we study the C^*-algebras $\mathcal{E}_{n,m}^q$ generated by Wick algebras $WE_{n,m}^q$ with the operator of coefficients T given by

$$Tu_1 \otimes u_2 = 0, \quad Tv_1 \otimes v_2 = 0, \quad u_1, u_2 \in \mathbb{C}^n,\ v_1, v_2 \in \mathbb{C}^m,$$

$$Tu \otimes v = qv \otimes u, \quad Tv \otimes u = \overline{q} u \otimes v, \quad u \in \mathbb{C}^n,\ v \in \mathbb{C}^m,$$

[a]Recall that given a $*$-algebra A, we denote by $\mathcal{A} = C^*(A)$ its universal C^*-algebra, if it exists, i.e. if the set $\mathsf{Rep}\, A$ of bounded $*$-representations of A is non-empty and

$$\sup_{\pi \in \mathsf{Rep}\, A} \|\pi(a)\| < \infty$$

for any $a \in A$. The universal C^*-algebra $\mathcal{A}$ is determined by the universal property: there exists a $*$-homomorphism $\theta \colon A \to \mathcal{A}$ such that for any C^*-algebra $\mathcal{B}$ and $*$-homomorphism $\beta \colon A \to \mathcal{B}$, there exists a unique $*$-homomorphism $\tilde{\beta} \colon \mathcal{A} \to \mathcal{B}$, such that $\beta = \tilde{\beta} \circ \theta$.

for $\mathcal{H} = \mathbb{C}^n \oplus \mathbb{C}^m$, $|q| \leq 1$. Note, that T satisfies the braid relation and $||T|| = |q| \leq 1$ for any $n, m \in \mathbb{N}$. In particular, the Fock representation $\pi_{F,q}$ exists for $|q| \leq 1$ and is faithful on $WE^q_{n,m}$ for $|q| < 1$, see the above discussion on general Wick algebras. The C^*-algebra $\mathcal{E}^q_{n,m}$ is generated by isometries $\{s_j\}_{j=1}^n$, and $\{t_r\}_{r=1}^m$, satisfying commutation relations of the following form:

$$\begin{aligned} s_i^* s_j &= 0, \quad 1 \leq i \neq j \leq n, \\ t_r^* t_s &= 0, \quad 0 \leq r \neq s \leq m, \\ s_j^* t_r &= q t_r s_j^*, \quad 0 \leq j \leq n, \ 0 \leq r \leq m. \end{aligned} \tag{1.1}$$

They are related to the C^*-algebras of deformed canonical commutation relations $\mathcal{G}_{\{q_{ij}\}}$, $q_{ij} = \overline{q}_{ji}$, $|q_{ij}| \leq 1$, $i, j \in \{1, \ldots, d\}$, $-1 < q_{ii} < 1$, given by

$$a_i^* a_j = \delta_{ij} \mathbf{1} + q_{ij} a_j a_i^*, \quad i, j = \overline{1, d}. \tag{1.2}$$

In one degree of freedom ($d = 1$), $\mathcal{G}_q$ exists for $q \in [-1, 1)$ and $\mathcal{G}_q \simeq \mathcal{E}$ for any $q \in (-1, 1)$, see [28]. See also [2, 38] for more on this algebra. The C^*-algebra $\mathcal{G}_{q,d}$ of quon commutation relations with d degrees of freedom was introduced and studied in [5, 21, 23, 56] and one has $\mathcal{G}_{q,d} \simeq \mathcal{O}_d^0$, for $q < \sqrt{2} - 1$. The above multiparameter version of quons was considered in [6, 39, 40]. For $|q_{ij}| < \sqrt{2} - 1$ we get $\mathcal{G}_{\{q_{ij}\}} \simeq \mathcal{O}_d^0$. Further related version have been studied in [4, 37, 46].

1.3. $\mathcal{E}^q_{n,m}$ *in the case* $n = m = 1$

In the case $n = 1$, $m = 1$, $WE^q_{1,1}$ is generated by isometries s_1, s_2 subject to the relations

$$s_1^* s_2 = q s_2 s_1^*.$$

It is easy to see that its universal C^*-algebra $\mathcal{E}^q_{1,1}$ exists for any $|q| \leq 1$.

If $|q| < 1$, the main result of [27] states that $\mathcal{E}^q_{1,1} \simeq \mathcal{E}^{(0)}_{1,1} = \mathcal{O}_2^{(0)}$ for any $|q| < 1$. In particular the Fock representation of $\mathcal{E}^q_{1,1}$ is faithful. Notice that the C^*-algebra $\mathcal{E}^q_{1,1}$ was the only known family of Wick algebras where Conjecture 1.1 holds (in this case $\|T\| < 1$ if and only if $|q| < 1$). In particular, even the isomorphism between C^*-algebra generated by three q-commuting isometries and $\mathcal{O}_3^0$ for all $|q| < 1$ is still not established.

The case $|q| = 1$ was studied in [32, 46, 54]. Here, the additional relation

$$s_2 s_1 = q s_1 s_2$$

holds in $\mathcal{E}^q_{1,1}$. It was shown that $\mathcal{E}^q_{1,1}$ is nuclear for any $|q| = 1$. Let $\mathcal{M}_q$ be the ideal generated by the projections $1 - s_1 s_1^*$ and $1 - s_2 s_2^*$. Then $\mathcal{E}^q_{1,1}/\mathcal{M}_q \simeq \mathcal{A}_q$, where $\mathcal{A}_q$ is the non-commutative torus, see [48],

$$\mathcal{A}_q = C^*(u_1, u_2 \,|\, u_1^* u_1 = u_1 u_1^* = \mathbf{1}, \ u_2^* u_2 = u_2 u_2^* = \mathbf{1}, \ u_2^* u_1 = q u_1 u_2^*).$$

If q is not a root of unity, then the corresponding non-commutative torus $\mathcal{A}_q$ is simple and $\mathcal{M}_q$ is the unique largest ideal in $\mathcal{E}_{1,1}^q$. Let us stress that unlike the case $|q| < 1$, the C^*-isomorphism class of $\mathcal{E}_{1,1}^q$ is "unstable" with respect to q. Namely, $\mathcal{E}_{1,1}^{q_1} \simeq \mathcal{E}_{1,1}^{q_2}$ if and only if $\mathcal{A}_{q_1} \simeq \mathcal{A}_{q_2}$, see [32, 46, 54].

One can consider another higher-dimensional analog of $\mathcal{E}_{1,1}^q$. For a set $\{q_{ij}\}_{i,j=1}^d$ of complex numbers such that $|q_{ij}| \le 1$, $q_{ij} = \overline{q}_{ji}$, $q_{ii} = 1$, and $d > 2$, one can consider a C^*-algebra $\mathcal{E}_{\{q_{ij}\}}$, generated by s_j, s_j^*, $j = \overline{1,d}$ subject to the relations

$$s_j^* s_j = 1, \quad s_i^* s_j = q_{ij} s_j s_i^*.$$

The case $|q_{ij}| < 1$ was considered in [36], where it was proved that $\mathcal{E}_{\{q_{ij}\}}$ is nuclear and the Fock representation is faithful. It turned out that the fixed point C^*-subalgebra of $\mathcal{E}_{\{q_{ij}\}}$ with respect to the canonical action of $\mathbb{T}^d$ is an AF-algebra and is independent of $\{q_{ij}\}$. However, the conjecture that $\mathcal{E}_{\{q_{ij}\}} \simeq \mathcal{E}_{\{0\}}$ remains open.

The case $|q_{ij}| = 1$ was studied in [16, 32, 46]. It was shown that $\mathcal{E}_{\{q_{ij}\}}$ is nuclear for any such family $\{q_{ij}\}$ and the Fock representation is faithful.

Let us note, see [46], that the C^*-algebra $\mathcal{E}_{\{q_{ij}\}}$ with $|q_{ij}| = 1$ is isomorphic to the C^*-algebra $\mathcal{G}_{\{q_ij\}}$ determined by deformed quons. In particular, this isomorphism implies that the Fock realization of $\mathcal{G}_{\{q_{ij}\}}$ is faithful, so the Fock representation can be considered as the universal representation of $\mathcal{G}_{\{q_ij\}}$. We stress also that apart the case $|q_{ij}| < 1$ the C^*-isomorphism class of $\mathcal{E}_{\{q_{ij}\}}$ with $|q_{ij}| = 1$ depends on the C^*-isomorphism class of the non-commutative torus $\mathbf{T}_{\{qij\}}$,

$$\mathbf{T}_{\{q_{ij}\}} = C^*(u_i,\ u_i^* = u_i^{-1},\ u_j u_i = q_{ij} u_j u_i),$$

and is unstable at any point.

1.4. $\mathcal{E}_{n,m}^q$ *in the case* $n, m \ge 2$

In our paper, we focus on the study of $\mathcal{E}_{n,m}^q$ with $n, m \ge 2$ (see [43] for the case $n = 1$, $m \ge 2$) and we also consider a multiparameter case. In the one parameter case, the analysis is separated into two conceptually different cases, $|q| < 1$ and $|q| = 1$.

If $|q| < 1$, we show that $\mathcal{E}_{n,m}^q \simeq \mathcal{E}_{n,m}^0 = \mathcal{O}_{n+m}^{(0)}$, where the latter is the Cuntz–Toeplitz algebra with $n + m$ generators.

For the case $|q| = 1$, we stress out that $\mathcal{E}_{n,m}^q$ is isomorphic to the Rieffel deformation of $\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}$ implying in particular the nuclearity of $\mathcal{E}_{n,m}^q$. We show that it contains a unique largest ideal $\mathcal{M}_q$, and we consider the quotient $\mathcal{O}_n \otimes_q \mathcal{O}_m := \mathcal{E}_{n,m}^q / \mathcal{M}_q$, in fact, even for a multiparameter $\Theta = (q_{ij})$ with $|q_{ij}| = 1$, denoting the object $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$. We show that $\mathcal{O}_n \otimes_q \mathcal{O}_m$ and $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ are simple and purely infinite. We use Kirchberg–Philips's classification theorem, see [33, 44], to get one of our main results, namely

$$\mathcal{O}_n \otimes_q \mathcal{O}_m \simeq \mathcal{O}_n \otimes \mathcal{O}_m \quad \text{and} \quad \mathcal{O}_n \otimes_\Theta \mathcal{O}_m \simeq \mathcal{O}_n \otimes \mathcal{O}_m$$

for any $q, q_{ij} \in \mathbb{C}$, $|q| = |q_{ij}| = 1$. Next, we show that the isomorphism class of $\mathcal{M}_q$ is independent of q and consider $\mathcal{E}_{n,m}^q$ as an (essential) extension of $\mathcal{O}_n \otimes \mathcal{O}_m$ by $\mathcal{M}_q$ and study the corresponding Ext group. In particular, if $\gcd(n-1, m-1) = 1$, this group is zero. Thus in this case, $\mathcal{E}_{n,m}^q$ and $\mathcal{E}_{n,m}^1$ both determine the zero class in $\mathsf{Ext}(\mathcal{O}_n \otimes_q \mathcal{O}_m, \mathcal{M}_q)$.

We stress that unlike the case of extensions by compacts, one cannot immediately deduce that two trivial essential extensions are isomorphic. So, the problem of an isomorphism $\mathcal{E}_{n,m}^q \simeq \mathcal{E}_{n,m}^1$ remains open.

1.5. *Relation with deformed CCR*

Finally, we present the relation of $\mathcal{E}_{n,m}$, $n, m \geq 2$ with multi-component commutation relations, see [4, 14, 37].

Take $k \in (0,1)$ and $q \in \mathbb{C}$, $|q| = 1$. Construct $\mathcal{H} = \mathbb{C}^n \oplus \mathbb{C}^m$, n, $m \geq 2$ and define $T\colon \mathcal{H}^{\otimes 2} \to \mathcal{H}^{\otimes 2}$ as follows:

$$Tu_1 \otimes u_2 = k\, u_2 \otimes u_1, \quad \text{if either } u_1, u_2 \in \mathbb{C}^n \quad \text{or} \quad u_1, u_2 \in \mathbb{C}^m$$

$$Tu \otimes v = q\, v \otimes u, \quad \text{if } u \in \mathbb{C}^n, \ v \in \mathbb{C}^m.$$

Denote the corresponding Wick algebra by $WE_{n,m}^{q,k}$ and its universal C^*-algebra by $\mathcal{E}_{n,m}^{q,k}$. This C^*-algebra is generated by s_j, t_r, $j = \overline{1,n}$, $r = \overline{1,m}$, subject to the relations

$$\begin{aligned} s_i^* s_j &= \delta_{ij}(\mathbf{1} + k\, s_j s_i^*), \\ t_r^* t_l &= \delta_{rl}(\mathbf{1} + k\, t_l t_r^*), \\ s_j^* t_r &= q\, t_r s_j^*. \end{aligned} \tag{1.3}$$

Notice that in $\mathcal{E}_{n,m}^{q,k}$ with $|q| = 1$, $k \in (0,1)$, the relations

$$t_r s_j = q\, s_j t_r, \quad j = \overline{1,n}, \ r = \overline{1,m}, \tag{1.4}$$

hold as well. Indeed, for $B_{jr} = t_r s_j - q\, s_j t_r$ we have $B_{jr}^* B_{jr} = k^2 B_{jr} B_{jr}^*$ and $B_{jr} = 0$.

Relations (1.3), (1.4) can be regarded as an example of system considered in [4, 14] in the case of finite count of degrees of freedom.

One can show, [46], that $\mathcal{E}_{n,m}^{q,k} \simeq \mathcal{E}_{n,m}^q$ for $k \in (0,1)$. Hence, in particular, one of our main results says that the C^*-algebra generated by (1.3) is an extension of $\mathcal{O}_n \otimes \mathcal{O}_m$. One more important corollary of the isomorphism is that the Fock representation of relations (1.3), (1.4) is faithful.

Notice that for $k = \pm 1$ we get a discrete analogue of commutation relations for generalized statistics introduced in [37].

2. The case $|q| < 1$

We start with some lemmas. Let Λ_n denote the set of all words in alphabet $\{\overline{1,n}\}$. For any non-empty $\mu = (\mu_1, \ldots, \mu_k)$, and a family of elements $b_1, \ldots, b_n$, we denote by b_μ the product $b_{\mu_1} \cdots b_{\mu_k}$; we also put $b_\emptyset = \mathbf{1}$. In this section, we assume that any word μ belongs to Λ_n.

Lemma 2.1. *Let* $Q = \sum_{i=1}^n s_i s_i^*$, *then*

$$\sum_{|\mu|=k} s_\mu Q s_\mu^* = \sum_{|\nu|=k+1} s_\nu s_\nu^*.$$

Proof. Straightforward. □

Lemma 2.2. *For any* $x \in \mathcal{E}_{n,m}^q$ *one has*

$$\left\| \sum_{|\mu|=k} s_\mu x s_\mu^* \right\| \le \|x\|.$$

Proof. (1) First prove the claim for positive x. In this case one has $0 \le x \le \|x\|\mathbf{1}$. Hence $0 \le s_\mu x s_\mu^* \le \|x\| s_\mu s_\mu^*$, and

$$\left\| \sum_{|\mu|=k} s_\mu x s_\mu^* \right\| \le \|x\| \cdot \left\| \sum_{|\mu|=k} s_\mu s_\mu^* \right\|.$$

Note that $s_\mu^* s_\lambda = \delta_{\mu\,\lambda}$, $\mu, \lambda \in \Lambda_n$, $|\mu| = |\lambda| = k$, implying that $\{s_\mu s_\mu^* \mid |\mu| = k\}$ form a family of pairwise orthogonal projections. Hence $\|\sum_{|\mu|=k} s_\mu s_\mu^*\| = 1$, and the statement for positive x is proved.

(2) For any $x \in \mathcal{E}_{n,m}^q$. write $A = \sum_{|\mu|=k} s_\mu x s_\mu^*$, then $A^* = \sum_{|\mu|=k} s_\mu x^* s_\mu^*$ and

$$A^* A = \sum_{|\mu|=k} s_\mu x^* x s_\mu^*.$$

Then by the proved above

$$\|A\|^2 = \|A^* A\| \le \|x^* x\| = \|x\|^2. \qquad \square$$

Construct $\widetilde{t}_l = (\mathbf{1} - Q) t_l$, $l = \overline{1, m}$.

Lemma 2.3. *The following commutation relations hold*:

$$s_i^* \widetilde{t}_l = 0, \quad i = \overline{1, n}, \quad l = \overline{1, m},$$
$$\widetilde{t}_r^* \widetilde{t}_l = 0, \quad l \ne r \quad l, r = \overline{1, m},$$
$$\widetilde{t}_r^* \widetilde{t}_r = \mathbf{1} - |q|^2 Q > 0, \quad r = \overline{1, m}.$$

Proof. We have $s_i^*(\mathbf{1} - Q) = 0$, implying that $s_i^* \widetilde{t}_l = 0$ for any $i = \overline{1, n}$, and $l = \overline{1, m}$.

Further,

$$\widetilde{t}_r^*\widetilde{t}_l = t_r^*(\mathbf{1}-Q)t_l = t_r^*t_l - \sum_{i=1}^n t_r^*s_is_i^*t_l = \delta_{rl}\mathbf{1} - \sum_{i=1}^n |q|^2 s_it_r^*t_ls_i^*$$
$$= \delta_{rl}(\mathbf{1}-|q|^2Q). \qquad \square$$

Proposition 2.1. *For any* $r=\overline{1,m}$, *one has*

$$t_r = \sum_{k=0}^{\infty}\sum_{|\mu|=k} q^k s_\mu \widetilde{t}_r s_\mu^*.$$

In particular, the family $\{s_i, \widetilde{t}_r,\ i=\overline{1,n},\ r=\overline{1,m}\}$ *generates* $\mathcal{E}_{n,m}^q$.

Proof. Put $M_k^r = \sum_{|\mu|=k} q^k s_\mu\widetilde{t}_r s_\mu^*$, $k\in\mathbb{Z}_+$. Then

$$M_0^r = t_r - Qt_r = t_r - \sum_{|\mu|=1} s_\mu s_\mu^* t_r,$$

and

$$M_k^r = \sum_{|\mu|=k} q^k s_\mu(\mathbf{1}-Q)t_rs_\mu^* = \sum_{|\mu|=k} s_\mu(\mathbf{1}-Q)s_\mu^*t_r$$
$$= \sum_{|\mu|=k} s_\mu s_\mu^* t_r - \sum_{|\mu|=k+1} s_\mu s_\mu^* t_r.$$

Then

$$S_N^r = \sum_{k=0}^N M_k^r = t_r - \sum_{|\mu|=N+1} s_\mu s_\mu^* t_r = t_r - q^{N+1}\sum_{|\mu|=N+1} s_\mu t_r s_\mu^*.$$

Since $\|\sum_{|\mu|=N+1} s_\mu t_r s_\mu^*\| \le \|t_r\| = 1$ one has that $S_N^r \to t_r$ in $\mathcal{E}_{n,m}^q$ as $N\to\infty$. $\square$

Suppose that $\mathcal{E}_{n,m}^q$ is realized by Hilbert space operators. Consider the left polar decomposition $\widetilde{t}_r = \widehat{t}_r\cdot c_r$, where $c_r^2 = \widetilde{t}_r^*\widetilde{t}_r = \mathbf{1}-|q|^2Q > 0$, implying that $\widehat{t}_r$ is an isometry and

$$\widehat{t}_r = \widetilde{t}_r c_r^{-1} \in \mathcal{E}_{n,m}^q, \quad r=\overline{1,m}.$$

Lemma 2.4. *The following commutation relations hold*:

$$s_i^*\widehat{t}_r = 0, \quad i=\overline{1,n}, \quad r=\overline{1,m},$$
$$\widehat{t}_r^*\widehat{t}_l = \delta_{rl}\mathbf{1}, \quad r,l=\overline{1,m}.$$

Proof. Indeed, for any $i=\overline{1,n}$, and $r=\overline{1,m}$. one has

$$s_i^*\widehat{t}_r = s_i^*\widetilde{t}_r\, c_r^{-1} = 0,$$

and

$$\widehat{t}_r^*\widehat{t}_l = c_r^{-1}\widetilde{t}_r^*\widetilde{t}_l c_r^{-1} = 0, \quad r \neq l. \qquad \square$$

Summing up the results stated above, we get the following theorem.

Theorem 2.1. *Let* $\widehat{t}_r = (\mathbf{1} - Q)t_r(\mathbf{1} - |q|^2Q)^{-\frac{1}{2}}$, $r = \overline{1,m}$. *Then the family* $\{s_i, \widehat{t}_r\}_{i=1}^{n}{}_{r=1}^{m}$ *generates* $\mathcal{E}_{n,m}^q$, *and*

$$s_i^* s_j = \delta_{ij}\mathbf{1}, \quad \widehat{t}_r^*\widehat{t}_l = \delta_{rl}\mathbf{1}, \quad s_i^*\widehat{t}_r = 0, \quad i, j = \overline{1,n}, \quad r, l = \overline{1,m}.$$

Proof. It remains to note that $\widetilde{t}_r = \widehat{t}_r(1 - |q|^2Q)^{\frac{1}{2}}$, so $\widetilde{t}_r \in C^*(\widehat{t}_r, Q)$, so by Proposition 2.1 the elements s_i, $\widehat{t}_r$, $i = \overline{1,n}$, $r = \overline{1,m}$, generate $\mathcal{E}_{n,m}^q$. $\square$

Corollary 2.1. *Denote by* v_i, $i = \overline{1,n+m}$, *the isometries generating* $\mathcal{E}_{n,m}^0 = \mathcal{O}_{n+m}^{(0)}$. *Then Theorem* 2.1 *implies that the correspondence*

$$v_i \mapsto s_i, \quad i = \overline{1,n}, \quad v_{n+r} \mapsto \widehat{t}_r, \quad r = \overline{1,m},$$

extends uniquely to a surjective homomorphism $\varphi\colon \mathcal{E}_{n,m}^0 \to \mathcal{E}_{n,m}^q$.

Our next aim is to construct the inverse homomorphism $\psi\colon \mathcal{E}_{n,m}^q \to \mathcal{E}_{n,m}^0$. To do it, put

$$\widetilde{Q} = \sum_{i=1}^{n} v_i v_i^* \quad \widetilde{w}_r = v_{n+r}(1 - |q|^2\widetilde{Q})^{\frac{1}{2}}, \quad r = \overline{1,m}.$$

Then $\widetilde{w}_r^*\widetilde{w}_r = 1 - |q|^2\widetilde{Q}$, and $\widetilde{w}_r^*\widetilde{w}_l = 0$ if $r \neq l$, $r, l = \overline{1,m}$. Construct

$$w_r = \sum_{k=0}^{\infty}\sum_{|\mu|=k} q^k v_\mu \widetilde{w}_r v_\mu^*, \quad r = \overline{1,m},$$

where μ runs over Λ_n, and set as above $v_\mu = v_{\mu_1}\cdots v_{\mu_k}$. Note that the series above converges with respect to norm in $\mathcal{E}_{n,m}^0$.

Lemma 2.5. *The following commutation relations hold*:

$$w_r^* w_l = \delta_{rl}\mathbf{1}, \quad v_i^* w_r = q w_r v_i^*, \quad i = \overline{1,n}, \quad r, l = \overline{1,m}.$$

Proof. First, we note that $v_i^*\widetilde{w}_r = 0$, and $\widetilde{w}_r^* v_i = 0$ for any $i = \overline{1,n}$, and $j = \overline{1,m}$, implying that

$$v_\delta^*\widetilde{w}_r = 0, \quad \widetilde{w}_r^* v_\delta = 0, \quad \text{for any nonempty } \delta \in \Lambda_n, \quad r = \overline{1,m}.$$

Let $|\lambda| \neq |\mu|$, $\lambda, \mu \in \Lambda_n$. If $|\lambda| > |\mu|$, then $\lambda = \widehat{\lambda}\gamma$ with $|\lambda| = |\mu|$ and $v_\lambda^* v_\mu = \delta_{\widehat{\lambda}\mu} v_\gamma^*$. Otherwise $\mu = \widehat{\mu}\beta$, $|\widehat{\mu}| = |\lambda|$ and $v_\lambda^* v_\mu = \delta_{\lambda\widehat{\mu}} v_\beta$. So, if $|\lambda| > |\mu|$ one has

$$v_\lambda \widetilde{w}_r^* v_\lambda^* v_\mu \widetilde{w}_r v_\mu^* = \delta_{\widehat{\lambda}\mu} v_\lambda \widetilde{w}_r^* v_\gamma^* \widetilde{w}_r v_\mu = 0,$$

and if $|\mu| > |\lambda|$, then

$$v_\lambda \widetilde{w}_r^* v_\lambda^* v_\mu \widetilde{w}_r v_\mu^* = \delta_{\lambda\widehat{\mu}} v_\lambda \widetilde{w}_r^* v_\beta \widetilde{w}_r v_\mu = 0.$$

Since $v_\mu^* v_\lambda = \delta_{\mu\lambda}\mathbf{1}$, if $|\mu| = |\lambda|$, one has

$$\begin{aligned} w_r^* w_r &= \lim_{N\to\infty} \left(\sum_{k=0}^{N} \sum_{|\lambda|=k} |q|^k v_\lambda \widetilde{w}_r^* v_\lambda^* \right) \cdot \left(\sum_{l=0}^{N} \sum_{|\mu|=l} |q|^l v_\mu \widetilde{w}_r v_\mu^* \right) \\ &= \lim_{N\to\infty} \sum_{k,l=0}^{N} \sum_{|\lambda|=k,|\mu|=l} |q|^{k+l} v_\lambda \widetilde{w}_r^* v_\lambda^* v_\mu \widetilde{w}_r v_\mu^* \\ &= \lim_{N\to\infty} \sum_{k=0}^{N} \sum_{|\lambda|,|\mu|=k,} |q|^{2k} v_\lambda \widetilde{w}_r^* v_\lambda^* v_\mu \widetilde{w}_r v_\mu^* \\ &= \lim_{N\to\infty} \sum_{k=0}^{N} \sum_{|\mu|=k} |q|^{2k} v_\mu \widetilde{w}_r^* \widetilde{w}_r v_\mu^* = \lim_{N\to\infty} \sum_{k=0}^{N} \sum_{|\mu|=k} |q|^{2k} v_\mu (\mathbf{1} - |q|^2 \widetilde{Q}^2) v_\mu^* \\ &= \lim_{N\to\infty} \sum_{k=0}^{N} \left(\sum_{|\mu|=k} |q|^{2k} v_\mu v_\mu^* - \sum_{|\mu|=k+1} |q|^{2k+2} v_\mu v_\mu^* \right) \\ &= \lim_{N\to\infty} \left(\mathbf{1} - |q|^{2N+2} \sum_{|\mu|=N+1} v_\mu v_\mu^* \right) = \mathbf{1}. \end{aligned}$$

Since $\widetilde{w}_r^* \widetilde{w}_l = 0$, $r \neq l$, the same arguments as above imply that $w_r^* w_l = 0$, $r \neq l$.

For any non-empty $\mu \in \Lambda_n$ write $\sigma(\mu) = \emptyset$ if $|\mu| = 1$, and $\sigma(\mu) = (\mu_2, \ldots, \mu_k)$ if $|\mu| = k > 1$. Further, for any $i = \overline{1,n}$, $r = \overline{1,m}$ one has

$$\begin{aligned} v_i^* w_r &= \sum_{k=0}^{\infty} \sum_{|\mu|=k} q^k s_i^* v_\mu \widetilde{w}_r v_\mu^* \\ &= v_i^* \widetilde{w}_r + \sum_{k=1}^{\infty} \sum_{|\mu|=k} q^k \delta_{i\mu_1} v_{\sigma(\mu)} \widetilde{w}_r v_{\sigma(\mu)}^* v_i^* \\ &= q \sum_{k=0}^{\infty} \sum_{|\mu|=k} q^k v_\mu \widetilde{w}_r v_\mu^* v_i^* = q w_r v_i^*. \end{aligned}$$

□

Lemma 2.6. *For any $r = \overline{1,m}$, one has $\widetilde{w}_r = (\mathbf{1} - \widetilde{Q}) w_r$.*

Proof. First note that $(\mathbf{1} - \widetilde{Q}) v_i = 0$, $i = \overline{1,n}$, implies that

$$(\mathbf{1} - \widetilde{Q}) v_\mu = 0, \quad |\mu| \in \Lambda_n, \quad \mu \neq \emptyset.$$

Then

$$(\mathbf{1}-\widetilde{Q})w_r = (\mathbf{1}-\widetilde{Q})\left(\sum_{k=0}\sum_{|\mu|=k} q^k v_\mu \widetilde{w}_r v_\mu^*\right)$$
$$= (\mathbf{1}-\widetilde{Q})\widetilde{w}_r + \sum_{k=1}\sum_{|\mu|=k} q^k(\mathbf{1}-\widetilde{Q})v_\mu \widetilde{w}_r v_\mu^* = (\mathbf{1}-\widetilde{Q})\widetilde{w}_r.$$

To complete the proof it remains to note that $\widetilde{Q}v_{n+r} = 0$, $r = \overline{1,m}$. So,

$$\widetilde{Q}\widetilde{w}_r = \widetilde{Q}v_{n+r}(\mathbf{1}-|q|^2\widetilde{Q})^{\frac{1}{2}} = 0. \qquad \square$$

Theorem 2.2. *Let v_i, $i = \overline{1,n}+m$, be the isometries generating $\mathcal{E}_{n,m}^0$, and $\widetilde{Q} = \sum_{i=1}^n v_i v_i^*$. Put*

$$\widetilde{w}_r = v_{n+r}(\mathbf{1}-|q|^2\widetilde{Q})^{\frac{1}{2}} \quad \textit{and} \quad w_r = \sum_{k=0}\sum_{|\mu|=k} q^k v_\mu \widetilde{w}_r v_\mu^*.$$

Then

$$v_i^* v_j = \delta_{ij}\mathbf{1}, \quad w_r^* w_l = \delta_{rl}\mathbf{1}, \quad v_i^* w_r = q w_r v_i^*, \quad i,j = \overline{1,n}, \ \ r,l = \overline{1,m}.$$

Moreover, the family $\{v_i, w_r\}_{i=1}^{n}{}_{r=1}^{m}$ generates $\mathcal{E}_{n,m}^0$.

Proof. We need to prove only the last statement of the theorem. We have

$$v_{n+r} = \widetilde{w}_r(\mathbf{1}-|q|^2\widetilde{Q})^{-\frac{1}{2}} = (\mathbf{1}-\widetilde{Q})w_r(\mathbf{1}-|q|^2\widetilde{Q})^{-\frac{1}{2}} \in C^*(w_r, v_i, \ i = \overline{1,n}).$$

Hence v_i, w_r, $i = \overline{1,n}$, $r = \overline{1,m}$, generate $\mathcal{E}_{n,m}^0$. $\square$

Corollary 2.2. *The statement of Theorem* 2.2 *and the universal property of $\mathcal{E}_{n,m}^q$ imply the existence of a surjective homomorphism $\psi\colon \mathcal{E}_{n,m}^q \to \mathcal{E}_{n,m}^0$ defined by*

$$\psi(s_i) = v_i, \quad \psi(t_r) = w_r, \quad i = \overline{1,n}, \quad r = \overline{1,m}.$$

Now, we are ready to formulate the main result of this section.

Theorem 2.3. *For any $q \in \mathbb{C}$, $|q| < 1$, one has an isomorphism $\mathcal{E}_{n,m}^q \simeq \mathcal{E}_{n,m}^0$.*

Proof. In Theorem 2.1, we constructed the surjective homomorphism $\varphi\colon \mathcal{E}_{n,m}^0 \to \mathcal{E}_{n,m}^q$ defined by

$$\varphi(v_i) = s_i, \quad \varphi(v_{n+r}) = \widehat{t}_r, \quad i = \overline{1,n}, \quad r = \overline{1,m}.$$

Show that $\psi\colon \mathcal{E}_{n,m}^q \to \mathcal{E}_{n,m}^0$ from Corollary 2.2 is the inverse of φ. Indeed, the equalities $\psi(s_i) = v_i$, $i = \overline{1,n}$, imply that

$$\psi(\mathbf{1}-Q) = \mathbf{1}-\widetilde{Q}.$$

Then, since $\psi(t_r) = w_r$, we get

$$\psi(\widetilde{t}_r) = \psi((\mathbf{1} - Q)t_r) = (\mathbf{1} - \widetilde{Q})w_r = \widetilde{w}_r, \quad r = \overline{1, m},$$

and

$$\psi(\widehat{t}_r) = \psi(\widetilde{t}_r(\mathbf{1} - |q|^2 Q)^{-\frac{1}{2}}) = \widetilde{w}_r(\mathbf{1} - |q|^2\widetilde{Q})^{-\frac{1}{2}} = v_{n+r}, \quad r = \overline{1, m}.$$

So, $\psi\varphi(v_i) = \psi(s_i) = v_i$, $\psi\varphi(v_{n+r}) = \psi(\widehat{t}_r) = v_{n+r}$, $i = \overline{1, n}$, $r = \overline{1, m}$, and

$$\psi\varphi = id_{\mathcal{E}^0_{n,m}}.$$

Show that $\varphi\psi = id_{\mathcal{E}^q_{n,m}}$. Indeed,

$$\varphi(\widetilde{w}_r) = \varphi(v_{n+r}(\mathbf{1} - |q|^2\widetilde{Q})^{\frac{1}{2}}) = \widehat{t}_r(\mathbf{1} - |q|^2 Q)^{\frac{1}{2}} = \widetilde{t}_r, \quad r = \overline{1, d}.$$

Then for any $r = \overline{1, m}$, one has

$$\varphi(w_r) = \sum_{k=0} \sum_{|\mu|=k} q^k \varphi(v_\mu)\varphi(\widetilde{w}_r)\varphi v_\mu^* = \sum_{k=0} \sum_{|\mu|=k} q^k s_\mu \widetilde{t}_r s_\mu^* = t_r.$$

So, $\varphi\psi(s_i) = \varphi(v_i) = s_i$, $\varphi\psi(t_r) = \varphi(w_r) = t_r$, $i = \overline{1, n}$, $r = \overline{1, m}$. □

3. The Case $|q| = 1$

In this section, we discuss the case $|q| = 1$. Notice that for $|q| = 1$, formula (1.1) implies $t_j s_i = q s_i t_j$, $i = \overline{1, n}$, $j = \overline{1, m}$. Indeed, one has just put $B_{ij} = t_j s_i - q s_i t_j$ and check that $B_{ij}^* B_{ij} = 0$, $i \neq j$.

3.1. *Auxiliary results*

In this subsection, we collect some general facts about C^*-dynamical systems, crossed products and Rieffel deformations which we will use in our considerations.

3.1.1. *Fixed point subalgebras*

First, we recall how properties of a fixed point subalgebra of a C^*-algebra with an action of a compact group are related to properties of the whole algebra.

Definition 3.1. Let $\mathcal{A}$ be a C^*-algebra with an action γ of a compact group G. A fixed point subalgebra $\mathcal{A}^\gamma$ is a subset of all $a \in \mathcal{A}$ such that $\gamma_g(a) = a$ for all $g \in G$.

Notice that for every action of a compact group G on a C^*-algebra $\mathcal{A}$ one can construct a faithful conditional expectation $E_\gamma : \mathcal{A} \to \mathcal{A}^\gamma$ onto the fixed point subalgebra, given by

$$E_\gamma(a) = \int_G \gamma_g(a) d\lambda,$$

where λ is the Haar measure on G.

A homomorphism $\varphi\colon\mathcal{A}\to\mathcal{B}$ between C^*-algebras with actions α and β of a compact group G is called equivariant if

$$\varphi\circ\alpha_g=\beta_g\circ\varphi\quad\text{for any } g\in G.$$

3.1.2. *Crossed products*

Given a locally compact group G and a C^*-algebra $\mathcal{A}$ with a G-action α, consider the full crossed product C^*-algebra $\mathcal{A}\rtimes_\alpha G$, see [55]. One has two natural embeddings into the multiplier algebra $M(\mathcal{A}\rtimes_\alpha G)$,

$$i_{\mathcal{A}}\colon\mathcal{A}\to M(\mathcal{A}\rtimes_\alpha G),\quad i_G\colon G\to M(\mathcal{A}\rtimes_\alpha G),$$

$$(i_{\mathcal{A}}(a)f)(s)=af(s),\quad (i_G(t)f)(s)=\alpha_t(f(t^{-1}s)),\quad t,s\in G,\ a\in\mathcal{A},$$

for $f\in C_c(G,\mathcal{A})$.

Remark 3.1. Obviously, $i_G(s)$ is a unitary element of $M(\mathcal{A}\rtimes_\alpha G)$ for any $s\in G$. Recall that i_G determines the following homomorphism denoted also by i_G

$$i_G\colon C^*(G)\to M(\mathcal{A}\rtimes_\alpha G)$$

defined by

$$i_G(f)=\int_G f(s)i_G(s)d\lambda(s),$$

where λ is the left Haar measure on G.

Notice that for any $g\in C_c(G,\mathcal{A})$ one has

$$(i_G(f)g)(t)=f\cdot_\alpha g,$$

where $\cdot_\alpha$ denotes the product in $\mathcal{A}\rtimes_\alpha G$. In particular, when $\mathcal{A}$ is unital we can identify $i_G(f)$ with $f\cdot_\alpha\mathbf{1}_{\mathcal{A}}$, and in fact i_G maps $C^*(G)$ into $\mathcal{A}\rtimes_\alpha G$. Also notice that

$$i_G(t)i_{\mathcal{A}}(a)i_G(t)^{-1}=i_{\mathcal{A}}(\alpha_t(a))\in M(\mathcal{A}\rtimes_\alpha G).$$

If φ is an equivariant homomorphism between C^*-algebras $\mathcal{A}$ with a G-action α and $\mathcal{B}$ with a G-action β, then one can define the homomorphism

$$\varphi\rtimes G\colon\mathcal{A}\rtimes_\alpha G\to\mathcal{B}\rtimes_\beta G,\quad(\varphi\rtimes G)(f)(t)=\varphi(f(t)),\quad f\in C_c(G,\mathcal{A}).$$

Let $\mathcal{A}$ be a unital C^*-algebra with G-action α. Then $\iota_{\mathcal{A}}\colon\mathbb{C}\to\mathcal{A}$,

$$\iota_{\mathcal{A}}(\lambda)=\lambda\mathbf{1}_{\mathcal{A}},$$

is an equivariant homomorphism, where G acts trivially on $\mathbb{C}$. Since $\mathbb{C}\rtimes G=C^*(G)$, one has that

$$\iota_{\mathcal{A}}\rtimes G\colon C^*(G)\to\mathcal{A}\rtimes_\alpha G.$$

In fact, in this case we have

$$\iota_{\mathcal{A}}\rtimes G=i_G,\tag{3.1}$$

where $i_G\colon C^*(G) \to \mathcal{A} \rtimes_\alpha G$ is described in Remark 3.1. Indeed, for any $g \in G_c(G, \mathcal{A})$ one has

$$\begin{aligned}(i_G(f) \cdot_\alpha g)(s) &= \int_G f(t)\alpha_t(g(t^{-1}s))dt \\ &= \int_G f(t)1_A\alpha_t(g(t^{-1}s))dt \\ &= ((f(\cdot)1_{\mathcal{A}}) \cdot_\alpha g)(s) = ((\iota_{\mathcal{A}} \rtimes G)(f) \cdot_\alpha g)(s),\end{aligned}$$

implying $i_G(f) = (\iota_{\mathcal{A}} \rtimes G)(f)$ for any $f \in C^*(G)$.

3.1.3. *Rieffel's deformation*

In what follows, we recall some basic facts on Rieffel's deformations. Given a C^*-algebra $\mathcal{A}$ equipped with an action α of $\mathbb{R}^n$ and a skew symmetric matrix $\Theta \in M_n(\mathbb{R})$, one can construct the Rieffel deformation of $\mathcal{A}$, denoted by $\mathcal{A}_\Theta$, see [1, 49]. In particular, the elements $a \in \mathcal{A}$ such that $x \mapsto \alpha_x(a) \in C^\infty(\mathbb{R}^n, \mathcal{A})$ form a dense subset $\mathcal{A}_\infty$ in $\mathcal{A}_\Theta$ and for any $a, b \in \mathcal{A}_\infty$ their product in $\mathcal{A}_\Theta$ is given by the following oscillatory integral (see [49]):

$$a \cdot_\Theta b := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \alpha_{\Theta(x)}(a)\alpha_y(b)e^{2\pi i\langle x,y\rangle}dxdy, \tag{3.2}$$

where $\langle\cdot,\cdot\rangle$ is a scalar product in $\mathbb{R}^n$. The mapping $\alpha_x^\Theta\colon a \to \alpha_x(a)$, $a \in \mathcal{A}_\infty$, extends naturally to an action α^Θ of $\mathbb{R}^n$ on $\mathcal{A}_\Theta$.

Given an equivariant $*$-homomorphism ϕ between C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ with actions of $\mathbb{R}^n$, one can define a $*$-homomorphism $\phi_\Theta\colon \mathcal{A}_\Theta \to \mathcal{B}_\Theta$, which is also equivariant with respect to the induced actions. Moreover, ϕ_Θ is injective if and only if ϕ is injective, see e.g. [30].

The next result follows directly from [30, Lemma 3.5].

Proposition 3.1. *The mapping* $\mathrm{id}\colon \mathcal{A} \to (\mathcal{A}_\Theta)_{-\Theta}$ *is an equivariant $*$-isomorphism.*

In what follows, we will be interested in periodic actions of $\mathbb{R}^n$, i.e. we assume that α is an action of $\mathbb{T}^n$. Given a character $\chi \in \widehat{\mathbb{T}}^n \simeq \mathbb{Z}^n$, consider

$$\mathcal{A}_\chi = \{a \in \mathcal{A} : \alpha_z(a) = \chi(z)a \text{ for every } z \in \mathbb{T}^n\}.$$

Then

$$\mathcal{A} = \overline{\bigoplus_{\chi\in\mathbb{Z}^n} \mathcal{A}_\chi},$$

where some terms could be equal to zero, and $\mathcal{A}_{\chi_1} \cdot \mathcal{A}_{\chi_2} \subset \mathcal{A}_{\chi_1+\chi_2}$, $\mathcal{A}_\chi^* = \mathcal{A}_{-\chi}$. So, $\mathcal{A}_\chi$, $\chi \in \mathbb{Z}^n$, can be treated as homogeneous components of $\mathbb{Z}^n$-grading on $\mathcal{A}$.

For a periodic action α of $\mathbb{R}^n$ on a C^*-algebra $\mathcal{A}$ and a skew-symmetric matrix $\Theta \in M_n(\mathbb{R})$, construct the Rieffel deformation $\mathcal{A}_\Theta$. Notice that all homogeneous elements belong to $\mathcal{A}_\infty$. Apply formula (3.2) to $a \in \mathcal{A}_p$, $b \in \mathcal{A}_q$:

$$\begin{aligned} a \cdot_\Theta b &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{2\pi i\langle \Theta(x), p\rangle} a e^{2\pi i \langle y, q\rangle} b e^{2\pi i\langle x, y\rangle} dx\, dy \\ &= a \cdot b \int_{\mathbb{R}^n} e^{2\pi i\langle y,q\rangle} \int_{\mathbb{R}^n} e^{2\pi i\langle x, -\Theta(p)\rangle} e^{2\pi i\langle x,y\rangle} dx\, dy \\ &= a\cdot b\int_{\mathbb{R}^n} e^{2\pi i \langle y,q\rangle}\delta_{y-\Theta(p)}\, dy \\ &= e^{2\pi i\langle\Theta(p),q\rangle} a\cdot b. \end{aligned}$$

Thus, given $a \in \mathcal{A}_p$ and $b \in \mathcal{A}_q$ one has

$$a \cdot_\Theta b = e^{2\pi i\langle \Theta(p), q\rangle} a \cdot b. \tag{3.3}$$

Remark 3.2. Notice that $\mathcal{A}_\Theta$ also possesses a $\mathbb{Z}^n$-grading such that $(\mathcal{A}_\Theta)_p = \mathcal{A}_p$ for every $p \in \mathbb{Z}^n$. Due to (3.3), we have $a \cdot_\Theta b = a \cdot b$ for any $a, b \in \mathcal{A}_{\pm p}$, $p \in \mathbb{Z}^n$. Indeed, for any skew symmetric $\Theta \in M_n(\mathbb{R}^n)$ and $p \in \mathbb{Z}^n$, one has $\langle \Theta\, p\, , \pm p\rangle = 0$. The involution on $(\mathcal{A}_\Theta)_p$ coincides with the involution on $\mathcal{A}_p$.

Consider a C^*-dynamical system $(\mathcal{A}, \mathbb{T}^n, \alpha)$, and its covariant representation (π, U) on a Hilbert space $\mathcal{H}$. For any $p \in \mathbb{Z}^n \simeq \widehat{\mathbb{T}}^n$, put

$$\mathcal{H}_p = \{h \in \mathcal{H} \mid U_t h = e^{2\pi i\langle t,p\rangle} h\}.$$

Then $\mathcal{H} = \bigoplus_{p\in\mathbb{Z}^n} \mathcal{H}_p$ (see [55]).

Proposition 3.2 ([8, Theorem 2.8]). *Let (π, U) be a covariant representation of $(\mathcal{A}, \mathbb{T}^n, \alpha)$ on a Hilbert space $\mathcal{H}$. Then one can define a representation π_Θ of $\mathcal{A}_\Theta$ as follows*:

$$\pi_\Theta(a)\xi = e^{2\pi i\langle\Theta(p),q\rangle}\pi(a)\xi,$$

for every $\xi \in \mathcal{H}_q$, $a \in \mathcal{A}_p$, $p, q \in \mathbb{Z}^n$. Moreover, π_Θ is faithful if and only if π is faithful.

It is known that Rieffel's deformation can be embedded into $M(\mathcal{A} \rtimes_\alpha \mathbb{R}^n)$, but for the periodic actions we have an explicit description of this embedding.

Proposition 3.3 ([52, Lemma 3.1.1]). *The following mapping defines an embedding*:

$$i_{\mathcal{A}_\Theta} : \mathcal{A}_\Theta \to M(\mathcal{A} \rtimes_\alpha \mathbb{R}^n), \quad i_{\mathcal{A}_\Theta}(a_p) = i_{\mathcal{A}}(a_p) i_{\mathbb{R}^n}(-\Theta(p)),$$

where $p \in \mathbb{Z}^n$ and a_p is homogeneous of degree p.

Proposition 3.4 (30, Proposition 3.2; 52, Sec. 3.1). *Let $(\mathcal{A}, \mathbb{R}^n, \alpha)$ be a C^*-dynamical system with periodic α and unital $\mathcal{A}$. Put $\mathcal{A}_\Theta$ to be the Rieffel deformation* *of* $\mathcal{A}$*. There exist a periodic action* α^{Θ} *of* $\mathbb{R}^n$ *on* $\mathcal{A}_{\Theta}$ *and an isomorphism* $\Psi : \mathcal{A}_{\Theta} \rtimes_{\alpha^{\Theta}} \mathbb{R}^n \to \mathcal{A} \rtimes_{\alpha} \mathbb{R}^n$ *such that the following diagram is commutative*:

$$\begin{array}{ccc} & C^*(\mathbb{R}^n) \simeq C_0(\mathbb{R}^n) & \\ {\scriptstyle i_{\mathbb{R}^n}} \swarrow & & \searrow {\scriptstyle i_{\mathbb{R}^n}} \\ \mathcal{A}_{\Theta} \rtimes_{\alpha^{\Theta}} \mathbb{R}^n & \xrightarrow{\quad \Psi \quad} & \mathcal{A} \rtimes_{\alpha} \mathbb{R}^n \end{array}$$

Namely, $\alpha^{\Theta}(a) = \alpha(a)$ holds for any $a \in \mathcal{A}_p$, $p \in \mathbb{Z}^n$. Then it is easy to verify that $i_{\mathcal{A}_{\Theta}} : \mathcal{A}_{\Theta} \to M(\mathcal{A} \rtimes_{\alpha} \mathbb{R}^n)$ with $i_{\mathbb{R}^n} : \mathbb{R}^n \to M(\mathcal{A} \rtimes_{\alpha} \mathbb{R}^n)$ determine a covariant representation of $(\mathcal{A}_{\Theta}, \mathbb{R}^n, \alpha^{\Theta})$ in $M(\mathcal{A} \rtimes_{\alpha} \mathbb{R}^n)$. Hence, by the universal property of crossed product we get the corresponding homomorphism

$$\Psi : \mathcal{A}_{\Theta} \rtimes_{\alpha^{\Theta}} \mathbb{R}^n \to M(\mathcal{A} \rtimes_{\alpha} \mathbb{R}^n).$$

In fact, the range of Ψ coincides with $\mathcal{A} \rtimes_{\alpha} \mathbb{R}^n$ and Ψ defines an isomorphism

$$\Psi : \mathcal{A}_{\Theta} \rtimes_{\alpha^{\Theta}} \mathbb{R}^n \to \mathcal{A} \rtimes_{\alpha} \mathbb{R}^n, \tag{3.4}$$

see [30, 52] for more detailed considerations.

The following propositions shows that Rieffel's deformation inherits properties of the non-deformed counterpart.

Proposition 3.5 ([30, Theorem 3.10]). *A* C^**-algebra* $\mathcal{A}_{\Theta}$ *is nuclear if and only if* $\mathcal{A}$ *is nuclear.*

Proposition 3.6 ([30, Theorem 3.13]). *For a* C^**-algebra* $\mathcal{A}$ *one has*

$$K_0(\mathcal{A}_{\Theta}) = K_0(\mathcal{A}) \quad \textit{and} \quad K_1(\mathcal{A}_{\Theta}) = K_1(\mathcal{A}).$$

3.1.4. *Rieffel's deformation of a tensor product*

In this part, we apply Rieffel's deformation procedure to a tensor product of two nuclear unital C^*-algebras equipped with an action of $\mathbb{T}$.

Let $\mathcal{A}$, $\mathcal{B}$ be C^*-algebras with actions α and β of $\mathbb{T}$. Then there is a natural action $\alpha \otimes \beta$ of $\mathbb{T}^2$ on $\mathcal{A} \otimes \mathcal{B}$ defined as

$$(\alpha \otimes \beta)_{\varphi_1, \varphi_2}(a \otimes b) = \alpha_{\varphi_1}(a) \otimes \beta_{\varphi_2}(b).$$

Consider the induced gradings on $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{A} = \bigoplus_{p_1 \in \mathbb{Z}} \mathcal{A}_{p_1}, \quad \mathcal{B} = \bigoplus_{p_2 \in \mathbb{Z}} \mathcal{B}_{p_2}.$$

Then the corresponding grading on $\mathcal{A} \otimes \mathcal{B}$ is

$$\mathcal{A} \otimes \mathcal{B} := \bigoplus_{(p_1, p_2)^t \in \mathbb{Z}^2} \mathcal{A}_{p_1} \otimes \mathcal{B}_{p_2}.$$

In particular, $a \otimes \mathbf{1} \in (\mathcal{A} \otimes \mathcal{B})_{(p_1, 0)^t}$ and $\mathbf{1} \otimes b \in (\mathcal{A} \otimes \mathcal{B})_{(0, p_2)^t}$, where $a \in \mathcal{A}_{p_1}$ and $b \in \mathcal{B}_{p_2}$.

Given $q = e^{2\pi i\varphi_0}$, consider

$$\Theta_q = \begin{pmatrix} 0 & \dfrac{\varphi_0}{2} \\ -\dfrac{\varphi_0}{2} & 0 \end{pmatrix}. \tag{3.5}$$

One can see that the Rieffel deformation $(\mathcal{A} \otimes \mathcal{B})_{\Theta_q}$ is in fact the universal C^*-algebra $\mathcal{A} \otimes_{\Theta_q} \mathcal{B}$, generated by all homogeneous elements

$$a \in \mathcal{A}_{p_1}, \quad b \in \mathcal{B}_{p_2}, \quad p_1, p_2 \in \mathbb{Z},$$

subject to the relations

$$ba = e^{2\pi i\varphi_0 p_1 p_2} ab. \tag{3.6}$$

In what follows, we present more precise formulation and elementary prove of this result. The discussion on universality properties for much more general deformation of tensor product can be found in [41].

First, construct $*$-algebra $\mathcal{A} \hat{\otimes}_{\Theta_q} \mathcal{B}$,

$$\mathcal{A} \hat{\otimes}_{\Theta_q} \mathcal{B} = \mathbb{C}\langle\, a \in \mathcal{A}_{p_1},\ b \in \mathcal{B}_{p_2} \mid ba = e^{2\pi i\varphi_0 p_1 p_2} ab,\ p_1, p_2 \in \mathbb{Z}\,\rangle$$

It is easy to see that correspondence $\tilde{\eta}$, determined by

$$\tilde{\eta}(a) = a \otimes \mathbf{1}, \quad \tilde{\eta}(b) = \mathbf{1} \otimes b, \quad a \in \mathcal{A}_{p_1}, \quad b \in \mathcal{B}_{p_2}, \quad p_1, p_2 \in \mathbb{Z},$$

extends to a $*$-algebra homomorphism $\tilde{\eta}\colon \mathcal{A} \hat{\otimes}_{\Theta_q} \mathcal{B} \to (\mathcal{A} \otimes \mathcal{B})_{\Theta_q}$. Indeed, let $e_1 = (1,0)^t$, $e_2 = (0,1)^t$. Then for $a \in \mathcal{A}_{p_1}$, $b \in \mathcal{B}_{p_2}$ one has

$$\tilde{\eta}(a) \cdot_{\Theta_q} \tilde{\eta}(b) = (a \otimes \mathbf{1}) \cdot_{\Theta_q} (\mathbf{1} \otimes b) = e^{2\pi i\varphi_0 \langle p_1 \Theta_q e_1\, ,\, p_2 e_2\rangle} a \otimes b = e^{-\pi i\varphi_0 p_1 p_2} a \otimes b$$

and

$$\tilde{\eta}(b) \cdot_{\Theta_q} \tilde{\eta}(a) = (\mathbf{1} \otimes b) \cdot_{\Theta_q} (a \otimes \mathbf{1}) = e^{2\pi i\varphi_0 \langle p_2 \Theta_q e_2\, ,\, p_1 e_1\rangle} a \otimes b = e^{\pi i\varphi_0 p_1 p_2} a \otimes b.$$

Hence

$$\tilde{\eta}(b) \cdot_{\Theta_q} \tilde{\eta}(a) = e^{2\pi i\varphi_o p_1 p_2} \tilde{\eta}(a) \cdot_{\Theta_q} \tilde{\eta}(b).$$

In particular,

$$\tilde{\eta}(ab) = e^{-\pi\varphi_0 p_1 p_2} a \otimes b, \quad a \in \mathcal{A}_{p_1}, \quad b \in \mathcal{B}_{p_2},\ \ p_1, p_2 \in \mathbb{Z}.$$

Since $(\mathcal{A} \otimes \mathcal{B})_{\Theta_q}$ is a C^*-algebra, the set $\mathsf{Rep}\, \mathcal{A} \hat{\otimes}_{\Theta_q} \mathcal{B}$ is non-empty. Further, take any $\pi \in \mathsf{Rep}\, \mathcal{A} \hat{\otimes}_{\Theta_q} \mathcal{B}$. Consider its restriction, $\pi_{|\mathcal{A}}$ to $\mathcal{A}$. Then for any $a \in \mathcal{A}_{p_1}$ one has

$$\|\pi(a)\| = \|\pi_{|\mathcal{A}}(a)\| \le \|a\|_{\mathcal{A}}.$$

Analogously, $\|\pi(b)\| \le \|b\|_{\mathcal{B}}$, $b \in B_{p_2}$. Hence for any $x \in \mathcal{A} \hat{\otimes}_{\Theta_q} \mathcal{B}$ one has

$$\sup_{\pi \in \mathsf{Rep}\, \mathcal{A} \hat{\otimes}_{\Theta_q} \mathcal{B}} \|\pi(x)\| < \infty$$

and the universal C^*-algebra $\mathcal{A} \otimes_{\Theta_q} \mathcal{B} := C^*(\mathcal{A} \hat{\otimes}_{\Theta_q} \mathcal{B})$ exists.

Notice that one has the natural $\mathbb{T}^2$ action γ on $\mathcal{A} \otimes_{\Theta_q} \mathcal{B}$ determined by

$$\gamma_{\varphi_1,\varphi_2}(a \cdot b) = e^{2\pi i(p_1\varphi_1 + p_2\varphi_2)} a \cdot b, \quad a \in \mathcal{A}_{p_1}, \quad b \in \mathcal{B}_{p_2}, \ \ p_1, p_2 \in \mathbb{Z}.$$

Proposition 3.1 implies the following result.

Theorem 3.1. *One has a* $\mathbb{T}^2$*-equivariant* $*$*-isomorphism*

$$\mathcal{A} \otimes_{\Theta_q} \mathcal{B} \simeq (\mathcal{A} \otimes \mathcal{B})_{\Theta_q}.$$

Proof. The following argument is due to a private communication by P. Kaszprak.

By the universal property we extend the homomorphism

$$\tilde{\eta}\colon \mathcal{A} \hat{\otimes}_{\Theta_q} \mathcal{B} \to (\mathcal{A} \otimes \mathcal{B})_{\Theta_q}$$

to surjective homomorphism

$$\eta\colon \mathcal{A} \otimes_{\Theta_q} \mathcal{B} \to (\mathcal{A} \otimes \mathcal{B})_{\Theta_q}.$$

Evidently, η is equivariant with respect to $\mathbb{T}^2$ actions on $\mathcal{A} \otimes_{\Theta_q} \mathcal{B}$ and $(\mathcal{A} \otimes \mathcal{B})_{\Theta_q}$ described above.

To show the injectivity of η, we construct homomorphism

$$\psi\colon \mathcal{A} \otimes \mathcal{B} \to (\mathcal{A} \otimes_{\Theta_q} \mathcal{B})_{-\Theta_q},$$

defined as follows. For homogeneous $a \in \mathcal{A}_{p_1}$, $b \in \mathcal{B}_{p_2}$, put

$$\psi(a \otimes \mathbf{1}) = a \in (\mathcal{A} \otimes_{\Theta_q} \mathcal{B})_{-\Theta_q}, \quad \psi(\mathbf{1} \otimes b) = b \in (\mathcal{A} \otimes_{\Theta_q} \mathcal{B})_{-\Theta_q}.$$

If $a \in \mathcal{A}_{p_1}$ and $b \in \mathcal{B}_{p_2}$, then

$$\psi(a \otimes \mathbf{1}) \cdot_{-\Theta_q} \psi(\mathbf{1} \otimes b) = e^{\pi i \varphi_0 p_1 p_2} ab = e^{-\pi i \varphi_0 p_1 p_2} ba = \psi(\mathbf{1} \otimes b) \cdot_{-\Theta_q} \psi(a \otimes \mathbf{1}).$$

Here, we use the relation $ab = e^{-2\pi i \varphi_0 p_1 p_2} ba$ which holds in $\mathcal{A} \otimes_{\Theta_q} \mathcal{B}$.

Due to the universal property of tensor product, ψ extends to a surjective homomorphism from $\mathcal{A} \otimes \mathcal{B}$ to $(\mathcal{A} \otimes_{\Theta_q} \mathcal{B})_{-\Theta_q}$.

Recall that the equivariant homomorphism

$$\eta\colon \mathcal{A} \otimes_{\Theta_q} \mathcal{B} \to (\mathcal{A} \otimes \mathcal{B})_{\Theta_q}$$

is injective if and only if the induced homomorphism

$$\eta_{-\Theta_q}\colon (\mathcal{A} \otimes_{\Theta_q} \mathcal{B})_{-\Theta_q} \to ((\mathcal{A} \otimes \mathcal{B})_{\Theta_q})_{-\Theta_q}$$

determined by

$$\eta_{-\Theta_q}(a) = a \otimes \mathbf{1}, \quad \eta_{-\Theta_q}(b) = \mathbf{1} \otimes b, \quad a \in \mathcal{A}_{p_1}, \quad b \in \mathcal{B}_{p_2}, \ \ p_1, p_2 \in \mathbb{Z}_+,$$

is injective. Recall also that, due to functorial properties of Rieffel deformation the identity mapping

$$a \otimes b \mapsto a \otimes b, \quad a \in \mathcal{A}_{p_1}, \quad b \in \mathcal{B}_{p_2}, \quad p_1, p_2 \in \mathbb{Z},$$

extends to isomorphism $((\mathcal{A} \otimes \mathcal{B})_{\Theta_q})_{-\Theta_q} \simeq \mathcal{A} \otimes \mathcal{B}$.

So, one has the following commutative diagram:

$$\begin{array}{ccc} (\mathcal{A} \otimes_{\Theta_q} \mathcal{B})_{-\Theta_q} & \xrightarrow{\eta_{-\Theta_q}} & ((\mathcal{A} \otimes \mathcal{B})_{\Theta_q})_{-\Theta_q} \\ \psi \uparrow & \diagup\!\!\diagup & \\ \mathcal{A} \otimes \mathcal{B} & & \end{array}$$

Since ψ is surjective, $\eta_{-\Theta_q}$ is injective. □

3.2. *Nuclearity of* $\mathcal{E}^q_{n,m}$

The nuclearity of $\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}$ and Proposition 3.5 immediately imply the following corollary.

Corollary 3.1. *The* C^**-algebra* $\mathcal{E}^q_{n,m}$ *is nuclear for any* $q \in \mathbb{C}$, $|q| = 1$.

The nuclearity of $\mathcal{E}^q_{n,m}$ can also be shown using more explicit arguments. One can use the standard trick of untwisting the q-deformation in the crossed product, which clarifies informally the nature of isomorphism (3.4). Namely, for $q = e^{2\pi i\varphi_0}$ consider the action α_q of $\mathbb{Z}$ on $\mathcal{E}^q_{n,m}$ defined on the generators as

$$\alpha_q^k(s_j) = e^{\pi ik\varphi_0} s_j, \quad \alpha_q^k(t_r) = e^{-\pi ik\varphi_0} t_r, \quad j = 1, \dots, n, \quad r = 1, \dots, m, \ \ k \in \mathbb{Z}.$$

Denote by the same symbol the similar action on $\mathcal{E}^1_{n,m} \simeq \mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}$. Here, we denote by $\widetilde{s}_j$ and $\widetilde{t}_r$ the generators of $\mathcal{E}^1_{n,m}$.

Proposition 3.7. *For any* $\varphi_0 \in [0, 1)$, *one has an isomorphism* $\mathcal{E}^q_{n,m} \rtimes_{\alpha_q} \mathbb{Z} \simeq \mathcal{E}^1_{n,m} \rtimes_{\alpha_q} \mathbb{Z}$.

Proof. Recall that $\mathcal{E}^1_{n,m} \rtimes_{\alpha_q} \mathbb{Z}$ is generated as a C^*-algebra by elements $\widetilde{s}_j$, $\widetilde{t}_r$ and a unitary u, such that the following relations satisfied:

$$u\widetilde{s}_j u^* = e^{i\pi\varphi_0}\, \widetilde{s}_j, \quad u\widetilde{t}_r u^* = e^{-i\pi\varphi_0}\, \widetilde{t}_r, \quad j = \overline{1,n}, \ \ r = \overline{1,m}.$$

Put $\widehat{s}_j = \widetilde{s}_j\, u$ and $\widehat{t}_r = \widetilde{t}_r\, u$. Obviously, $\widehat{s}_j$, $\widehat{t}_r$ and u generate $\mathcal{E}^1_{n,m} \rtimes_{\alpha_q} \mathbb{Z}$. Further,

$$\widehat{s}_j^*\widehat{s}_k = \delta_{jk}\mathbf{1}, \quad \widehat{t}_r^*\widehat{t}_l = \delta_{rl}\mathbf{1}$$

and

$$\widehat{s}_j\widehat{t}_r = \widetilde{s}_j u\widetilde{t}_r u = e^{-i\pi\varphi_0}\widetilde{s}_j\widetilde{t}_r u^2 = e^{-i\pi\varphi_0}\widetilde{t}_r\widetilde{s}_j u^2 = e^{-2\pi i\varphi_0}\widetilde{t}_r u\widetilde{s}_j u = \overline{q}\, \widehat{s}_j\widehat{t}_r.$$

In a similar way we get $\widehat{s}_j^*\widehat{t}_r = q\widehat{t}_r\widehat{s}_j^*$, $j = \overline{1,n}$, $r = \overline{1,m}$. Finally,

$$u\widehat{s}_j u^* = e^{i\pi\varphi_0}\widehat{s}_j, \quad u\widehat{t}_r u^* = e^{-i\pi\varphi_0}\widehat{t}_r.$$

Hence the correspondence

$$s_j \mapsto \widehat{s}_j, \quad t_j \mapsto \widehat{t}_j, \quad u \mapsto u,$$

determines a homomorphism $\Phi_q\colon \mathcal{E}_{n,m}^q \rtimes_{\alpha_q} \mathbb{Z} \to \mathcal{E}_{n,m}^1 \rtimes_{\alpha_q} \mathbb{Z}$. The inverse is constructed evidently. □

Let us show the nuclearity of $\mathcal{E}_{n,m}^q$ again. Indeed, $\mathcal{E}_{n,m}^1 = \mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}$ is nuclear. Then so is the crossed product $\mathcal{E}_{n,m}^1 \rtimes_{\alpha_q} \mathbb{Z}$. Then due to the isomorphism above, $\mathcal{E}_{n,m}^q \rtimes_{\alpha_q} \mathbb{Z}$ is nuclear, implying the nuclearity of $\mathcal{E}_{n,m}^q$, see [3].

3.3. *Fock representation of* $\mathcal{E}_{n,m}^q$

In this part, we study the Fock representation of $\mathcal{E}_{n,m}^q$.

First of all let us stress out that according to Theorem 3.1 we can identify $\mathcal{E}_{n,m}^q$ with $(\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)})_{\Theta_q}$. In particular, we use this isomorphism below to show that Fock representation of $\mathcal{E}_{n,m}^q$ is faithful.

Definition 3.2. The Fock representation of $\mathcal{E}_{n,m}^q$ is the unique up to unitary equivalence irreducible $*$-representation π_F^q determined by the action on vacuum vector Ω, $||\Omega|| = 1$,

$$\pi_F^q(s_j^*)\Omega = 0, \quad \pi_F^q(t_r^*)\Omega = 0, \quad j = \overline{1,n}, \;\; r = \overline{1,m}.$$

Denote by $\pi_{F,n}$ the Fock representation of $\mathcal{O}_n^{(0)} \subset \mathcal{E}_{n,m}^q$ acting on the space

$$\mathcal{F}_n = \mathcal{T}(\mathcal{H}_n) = \mathbb{C}\Omega \oplus \bigoplus_{d=1}^{\infty} \mathcal{H}_n^{\otimes d}, \quad \mathcal{H}_n = \mathbb{C}^n,$$

by formulas

$$\pi_{F,n}(s_j)\Omega = e_j, \quad \pi_{F,n}(s_j)e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_d} = e_j \otimes e_{i_1} \otimes e_{i_2} \cdots \otimes e_{i_d},$$

$$\pi_{F,n}(s_j^*)\Omega = 0, \quad \pi_{F,n}(s_j^*)e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_d} = \delta_{ji_1} e_{i_2} \otimes \cdots \otimes e_{i_d}, \quad d \in \mathbb{N},$$

where $e_1, \ldots, e_n$ is the standard orthonormal basis of $\mathcal{H}_n$. Notice that $\pi_{F,n}$ is the unique irreducible faithful representations of $\mathcal{O}_n^{(0)}$, see for example [28].

In what follows, we give an explicit formula for $\pi_F^q(s_j)$, $\pi_F^q(t_r)$. Consider the Fock representations $\pi_{F,n}$ and $\pi_{F,m}$ of $*$-subalgebras $C^*(\{s_1, \ldots, s_n\}) = \mathcal{O}_n^{(0)} \subset \mathcal{E}_{n,m}^q$ and $C^*(\{t_1, \ldots, t_m\}) = \mathcal{O}_m^{(0)} \subset \mathcal{E}_{n,m}^q$, respectively. Denote by $\Omega_n \in \mathcal{F}_n$ and $\Omega_m \in \mathcal{F}_m$ the corresponding vacuum vectors.

Theorem 3.2. *The Fock representation* π_F^q *of* $\mathcal{E}_{n,m}^q$ *acts on the space* $\mathcal{F} = \mathcal{F}_n \otimes \mathcal{F}_m$ *as follows*:

$$\pi_F^q(s_j) = \pi_{F,n}(s_j) \otimes d_m(q^{-\frac{1}{2}}), \quad j = \overline{1,n},$$

$$\pi_F^q(t_r) = d_n(q^{\frac{1}{2}}) \otimes \pi_{F,m}(t_r), \quad r = \overline{1,m},$$

where $d_k(\lambda)$ *acts on* $\mathcal{F}_k$, $k = n, m$ *by*

$$d_k(\lambda)\Omega_k = \Omega_k, \quad d_k(\lambda)X = \lambda^l X, \quad X \in \mathcal{H}_k^{\otimes l}, \quad l \in \mathbb{N}.$$

Proof. It is a direct calculation to verify that the operators defined above satisfy the relations of $\mathcal{E}^q_{n,m}$. Since $\pi_{F,k}$ is irreducible on $\mathcal{F}_k$, $k = m, n$, the representation π^q_F is irreducible on $\mathcal{F}_n \otimes \mathcal{F}_m$. Finally put $\Omega = \Omega_n \otimes \Omega_m$, then obviously

$$\pi^q_F(s_j^*)\Omega = 0, \quad \text{and} \quad \pi^q_F(t_r^*)\Omega = 0, \quad j = \overline{1,n}, r = \overline{1,m}$$

Thus π^q_F is the Fock representation of $\mathcal{E}^q_{n,m}$. □

Remark 3.3. In some cases, it will be more convenient to present the operators of the Fock representation of $\mathcal{E}^q_{n,m}$ in one of the alternative forms

$$\pi^q_F(s_j) = \pi_{F,n}(s_j) \otimes \mathbf{1}_{\mathcal{F}_m}, \quad j = \overline{1,n},$$
$$\pi^q_F(t_r) = d_n(q) \otimes \pi_{F,m}(t_r), \quad r = \overline{1,m},$$

or

$$\pi^q_F(s_j) = \pi_{F,n}(s_j) \otimes d_m(q^{-1}), \quad j = \overline{1,n},$$
$$\pi^q_F(t_r) = \mathbf{1}_{\mathcal{F}_n} \otimes \pi_{F,m}(t_r), \quad r = \overline{1,m},$$

which are obviously unitary equivalent to the one presented in the statement above.

The next step we show that in fact π^q_F can be obtained applying the construction form Proposition 3.2 to the representation $\pi_{F,n} \otimes \pi_{F,m}$ of $\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}$. To this end construct the family of unitary operators $\{U_{\varphi_1,\varphi_2},\ \varphi_1, \varphi_2 \in [0, 2\pi)\}$ acting on $\mathcal{F}_n \otimes \mathcal{F}_m$ as follows:

$$U_{\varphi_1,\varphi_2}(\xi_1 \otimes \xi_2) = e^{2\pi i(\varphi_1 p_1 + \varphi_2 p_2)}\, \xi_1 \otimes \xi_2, \quad \xi_1 \in \mathcal{H}_n^{\otimes p_1},$$
$$\xi_2 \in \mathcal{H}_m^{\otimes p_2}, \quad p_1, p_2 \in \mathbb{Z}_+.$$

The pair $(\pi_{F,n} \otimes \pi_{F,m},\ U_{\varphi_1,\varphi_2})$ determines a covariant representation of $(\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}, \mathbb{T}^2, \alpha)$, where as above

$$\alpha_{\varphi_1,\varphi_2}(s_j \otimes \mathbf{1}) = e^{2\pi i \varphi_1}(s_j \otimes \mathbf{1}), \quad \alpha_{\varphi_1,\varphi_2}(\mathbf{1} \otimes t_r) = e^{2\pi i \varphi_2}(\mathbf{1} \otimes t_r).$$

Notice that for $p = (p_1, p_2)^t \in \mathbb{Z}_+^2$, the subspace $\mathcal{H}_n^{\otimes p_1} \otimes \mathcal{H}_m^{\otimes p_2}$ is the $(p_1, p_2)^t$-homogeneous component of $\mathcal{F}$ related to the action of U_{φ_1,φ_2}, and $(\mathcal{F})_p = \{0\}$ for any $p \in \mathbb{Z}^2 \setminus \mathbb{Z}_+^2$.

Recall also that $\widehat{s}_j = s_j \otimes \mathbf{1}$ is contained in $e_1 = (1, 0)^t$-homogeneous component and $\widehat{t}_r = \mathbf{1} \otimes t_r$ is in $e_2 = (0, 1)^t$-homogeneous component of $\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}$ with respect to α. Now one can apply Proposition 3.2. Namely, given $\xi = \xi_1 \otimes \xi_2 \in \mathcal{H}_n^{\otimes p_1} \otimes \mathcal{H}_m^{\otimes p_2}$ one gets

$$(\pi_{F,n} \otimes \pi_{F,m})_{\Theta_q}(\widehat{s}_j)\, \xi = e^{2\pi i \langle \Theta_q\, e_1,\, p\rangle}\, \pi_{F,n} \otimes \pi_{F,m}(\widehat{s}_j)\, \xi$$
$$= \pi_{F,n}(s_j)\xi_1 \otimes e^{-\pi i\, p_2\, \varphi_0}\, \xi_2 = (\pi_{F,n}(s_j) \otimes d_m(q^{-\frac{1}{2}}))\, \xi,$$

and

$$(\pi_{F,n}\otimes\pi_{F,m})_{\Theta_q}(\widehat{t_r})\,\xi = e^{2\pi i\langle\Theta_q\, e_2,\, p\rangle}\,\pi_{F,n}\otimes\pi_{F,m}(\widehat{t_r})\xi$$
$$= e^{\pi i\, p_1\,\varphi_0}\,\xi_1\otimes\pi_{F,m}(t_r)\,\xi_2 = (d_n(q^{\frac{1}{2}})\otimes\pi_{F,m}(t_r))\,\xi.$$

Notice that for any $j=\overline{1,n}$, and $r=\overline{1,m}$,

$$(\pi_{F,n}\otimes\pi_{F,m})_{\Theta_q}(\widehat{s}_j^*)\Omega = 0,\quad (\pi_{F,n}\otimes\pi_{F,m})_{\Theta_q}(\widehat{t}_r^*)\Omega = 0.$$

So, we have shown that $\pi_F^q = (\pi_{F,n}\otimes\pi_{F,m})_{\Theta_q}$.

Since $\pi_{F,n}\otimes\pi_{F,m}$ is faithful representation of $\mathcal{O}_n^{(0)}\otimes\mathcal{O}_m^{(0)}$ we immediately get the following result.

Theorem 3.3. *The Fock representation π_F^q of $\mathcal{E}_{m,n}^q$ is faithful.*

We finish this part by an analog of the well-known Wold decomposition theorem for a single isometry. Recall that

$$Q=\sum_{j=1}^{n} s_j s_j^*,\quad P=\sum_{r=1}^{m} t_r t_r^*.$$

Theorem 3.4 (Generalized Wold decomposition). *Let $\pi\colon\mathcal{E}_{n,m}^q\to\mathbb{B}(\mathcal{H})$ be a $*$-representation. Then*

$$\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3\oplus\mathcal{H}_4,$$

where each $\mathcal{H}_j$, $j=1,2,3,4$, is invariant with respect to π, and for $\pi_j=\pi\restriction_{\mathcal{H}_j}$ one has

- $\mathcal{H}_1=\mathcal{F}\otimes\mathcal{K}$ *for some Hilbert space* $\mathcal{K}$, *and* $\pi_1=\pi_F^q\otimes\mathbf{1}_{\mathcal{K}}$;
- $\pi_2(\mathbf{1}-Q)=0$, $\pi_2(\mathbf{1}-P)\neq 0$;
- $\pi_3(\mathbf{1}-P)=0$, $\pi_3(\mathbf{1}-Q)\neq 0$;
- $\pi_4(\mathbf{1}-Q)=0$, $\pi_4(\mathbf{1}-P)=0$;

where any of $\mathcal{H}_j$, $j=1,2,3,4$, could be zero.

Proof. We will use the fact that any representation of $\mathcal{O}_n^{(0)}$ is a direct sum of a multiple of the Fock representation and a representation of $\mathcal{O}_n$.

So, restrict π to $\mathcal{O}_n^{(0)}\subset\mathcal{E}_{n,m}^q$, and decompose $\mathcal{H}=\mathcal{H}_F\oplus\mathcal{H}_F^\perp$, where

$$\pi(\mathbf{1}-Q)_{|\mathcal{H}_F^\perp}=0,$$

and $\pi(\mathcal{O}_n^{(0)})_{|\mathcal{H}_F}$ is a multiple of the Fock representation. Denote

$$S_j:=\pi(s_j)\restriction_{\mathcal{H}_F},\quad Q:=\pi(Q)\restriction_{\mathcal{H}_F}.$$

Put $S_\emptyset:=\mathbf{1}_{\mathcal{H}_F}$ and $S_\lambda:=S_{\lambda_1}\cdots S_{\lambda_k}$ for any non-empty $\Lambda_n\ni\lambda=(\lambda_1,\ldots,\lambda_k)$. Since

$$\mathcal{H}_F=\bigoplus_{\lambda\in\Lambda_n} S_\lambda(\ker Q),$$

it is invariant with respect to $\pi(t_r)$, $\pi(t_r^*)$, $r = \overline{1,m}$. Indeed, $t_r Q = Q t_r$ in $\mathcal{E}_{n,m}^q$, implying the invariance of $\ker Q$ with respect to $\pi(t_r)$ and $\pi(t_r^*)$. Denote $\ker Q$ by $\mathcal{G}$ and $T_r := \pi(t_r)\restriction_{\mathcal{G}}$. Then

$$\pi(t_r) S_\lambda \xi = q^{|\lambda|} S_\lambda \pi(t_r)\xi = q^{|\lambda|} S_\lambda T_r \xi, \quad \xi \in \mathcal{G}.$$

Thus $\mathcal{H}_F \simeq \mathcal{F}_n \otimes \mathcal{G}$ with

$$\pi(s_j)\restriction_{\mathcal{H}_F} = \pi_{F,n}(s_j) \otimes \mathbf{1}_{\mathcal{G}}, \quad \pi(t_r)\restriction_{\mathcal{H}_F} = d_n(q) \otimes T_r, \quad j = \overline{1,n}, \ r = \overline{1,m},$$

where the family $\{T_r\}$ determines a $*$-representation $\widetilde{\pi}$ of $\mathcal{O}_m^{(0)}$ on $\mathcal{G}$.

Further, decompose $\mathcal{G}$ as $\mathcal{G} = \mathcal{G}_F \oplus \mathcal{G}_F^\perp$ into an orthogonal sum of subspaces invariant with respect to $\widetilde{\pi}$, where $\mathcal{G}_F = \mathcal{F}_m \otimes \mathcal{K}$,

$$\widetilde{\pi}_{\mathcal{G}_F}(t_r) = \pi_{F,m}(t_r) \otimes \mathbf{1}_{\mathcal{K}}, \quad r = \overline{1,m}, \quad \text{and} \quad \widetilde{\pi}\restriction_{\mathcal{G}_F^\perp} (\mathbf{1} - P) = 0.$$

Thus $\mathcal{H}_F = (\mathcal{F}_n \otimes \mathcal{F}_m \otimes \mathcal{K}) \oplus (\mathcal{F}_n \otimes \mathcal{G}_F^\perp)$ and

$$\pi_{\mathcal{H}_F}(s_j) = (\pi_{F,n}(s_j) \otimes \mathbf{1}_{\mathcal{F}_m} \otimes \mathbf{1}_K) \oplus (\pi_{F,n}(s_j) \otimes \mathbf{1}_{\mathcal{G}_F^\perp}), \quad j = \overline{1,n},$$

$$\pi_{\mathcal{H}_F}(t_r) = (d_n(q) \otimes \pi_{F,m}(t_r) \otimes \mathbf{1}_K) \oplus (d_n(q) \otimes \widetilde{\pi}_{|\mathcal{G}_F^\perp}(t_r)), \quad r = \overline{1,m}.$$

Put $\mathcal{H}_1 = \mathcal{F}_n \otimes \mathcal{F}_m \otimes \mathcal{K} = \mathcal{F} \otimes \mathcal{K}$ and notice that that $\pi\restriction_{\mathcal{H}_1} = \pi_F^q \otimes \mathbf{1}_{\mathcal{K}}$, see Remark 3.3. Put $\mathcal{H}_3 = \mathcal{F}_n \otimes \mathcal{G}_F^\perp$ and $\pi_3 = \pi\restriction_{\mathcal{H}_3}$ i.e.

$$\pi_3(s_j) = \pi_{F,n}(s_j) \otimes \mathbf{1}_{\mathcal{G}_F^\perp}, \quad \pi_3(t_r) = d_n(q) \otimes \widetilde{\pi}_{|\mathcal{G}_F^\perp}(t_r), \quad j = \overline{1,n}, \ r = \overline{1,m}.$$

Evidently, $\pi_3(\mathbf{1} - P) = 0$ and $\pi_3(\mathbf{1} - Q) \neq 0$.

Finally, applying similar arguments to the invariant subspace $\mathcal{H}_F^\perp$ one can show that there exists a decomposition

$$\mathcal{H}_F^\perp = \mathcal{H}_2 \oplus \mathcal{H}_4$$

into the orthogonal sum of invariant subspaces, where

- $\mathcal{H}_2 = \mathcal{F}_m \otimes \mathcal{L}$ and

  $$\pi_2(s_j) := \pi\restriction_{\mathcal{H}_2}(s_j) = d_m(\overline{q}) \otimes \widehat{\pi}(s_j), \quad \pi_2(t_r) := \pi\restriction_{\mathcal{H}_2}(t_r) = \pi_{F,m}(t_r) \otimes \mathbf{1}_{\mathcal{L}},$$

  for a representation $\widehat{\pi}$ of $\mathcal{O}_n$. Evidently, $\pi_2(\mathbf{1} - Q) = 0$, $\pi_2(\mathbf{1} - P) \neq 0$.
- For $\pi_4 := \pi\restriction_{\mathcal{H}_4}$ one has

  $$\pi_4(\mathbf{1} - Q) = 0, \quad \pi_4(\mathbf{1} - P) = 0.$$ □

3.4. *Ideals in* $\mathcal{E}_{n,m}^q$

In this part, we give a complete description of ideals in $\mathcal{E}_{n,m}^q$, and prove their independence on the deformation parameter q.

For

$$Q = \sum_{j=1}^{n} s_j s_j^*, \quad P = \sum_{r=1}^{m} t_r t_r^*.$$

we consider two-sided ideals, $\mathcal{M}_q$ generated by $1-P$ and $1-Q$, $\mathcal{J}_1^q$ generated by $1-Q$, $\mathcal{J}_2^q$ generated by $\mathbf{1}-P$, and $\mathcal{J}_q$ generated by $(\mathbf{1}-Q)(\mathbf{1}-P)$. Evidently,

$$\mathcal{J}_q = \mathcal{J}_1^q \cap \mathcal{J}_2^q = \mathcal{J}_1^q \cdot \mathcal{J}_2^q.$$

In what follows, we will show that any ideal in $\mathcal{E}_{n,m}^q$ coincides with the one listed above.

To clarify the structure of $\mathcal{J}_1^q$, $\mathcal{J}_2^q$ and $\mathcal{J}_q$, we use the construction of twisted tensor product of a certain C^*-algebra with the algebra of compact operators $\mathbb{K}$, see [54]. We give a brief review of the construction, adapted to our situation.

Recall that the C^*-algebra $\mathbb{K}$ can be considered as a universal C^*-algebra generated by a closed linear span of elements $e_{\mu\nu}$, $\mu, \nu \in \Lambda_n$ subject to the relations

$$e_{\mu_1\nu_1} e_{\mu_2\nu_2} = \delta_{\mu_2\nu_1} e_{\mu_1\nu_2}, \quad e^*_{\mu_1\nu_1} = e_{\nu_1\mu_1}, \quad \nu_i, \mu_i \in \Lambda_n,$$

here $e_\emptyset := e_{\emptyset\emptyset}$ is a minimal projection.

Definition 3.3. Let $\mathcal{A}$ be a C^*-algebra,

$$\alpha = \{\alpha_\mu,\ \mu \in \Lambda_n\} \subset \mathsf{Aut}(\mathcal{A}), \quad \text{where } \alpha_\emptyset = \mathrm{id}_{\mathcal{A}},$$

and $e_{\mu\nu}$, $\mu, \nu \in \Lambda_n$ be the generators of $\mathbb{K}$ specified above. Construct the universal C^*-algebra

$$\langle \mathcal{A}, \mathbb{K}\rangle_\alpha = C^*(a \in \mathcal{A}, e_{\mu\nu} \in \mathbb{K} \,|\, a e_{\mu\nu} = e_{\mu\nu}\alpha_\nu^{-1}(\alpha_\mu(a)).$$

We define $\mathcal{A} \otimes_\alpha \mathbb{K}$ as a subalgebra of $\langle \mathcal{A}, \mathbb{K}\rangle_\alpha$ generated by ax, $a \in \mathcal{A} \subset \langle \mathcal{A}, \mathbb{K}\rangle_\alpha$, $x \in \mathbb{K} \subset \langle \mathcal{A}, \mathbb{K}\rangle_\alpha$.

Notice that $\langle \mathcal{A}, \mathbb{K}\rangle_\alpha$ exists for any C^*-algebra $\mathcal{A}$ and family $\alpha \subset \mathsf{Aut}(\mathcal{A})$, see [54].

Remark 3.4. (1) Let $x_\mu = e_{\mu\emptyset}$. Then $a x_\mu = x_\mu \alpha_\mu(a)$, $a x_\mu^* = x_\mu^* \alpha_\mu^{-1}(a)$, $a \in \mathcal{A}$, compare with [54].

(2) For any $a \in \mathcal{A}$ one has $e_{\mu\nu} a = \alpha_\mu^{-1}(\alpha_\nu(a)) e_{\mu\nu}$ implying that

$$(a e_{\mu\nu})^* = \alpha_\mu^{-1}(\alpha_\nu(a)) e_{\nu\mu}.$$

(3) For any $a_1, a_2 \in \mathcal{A}$ one has $(a_1 e_{\mu_1\nu_1})(a_2 e_{\mu_2\nu_2}) = \delta_{\nu_1\mu_2} a_1 \alpha_{\mu_1}^{-1}(\alpha_{\mu_2}(a_2)) e_{\mu_1\nu_2}$.

Proposition 3.8 ([54]). *Let $\mathcal{A}$ be a C^*-algebra and*

$$\alpha = \{\alpha_\mu,\ \mu \in \Lambda_n\} \subset \mathsf{Aut}(\mathcal{A}) \quad \textit{with } \alpha_\emptyset = \mathrm{id}_{\mathcal{A}}.$$

Then the correspondence

$$a e_{\mu\nu} \mapsto \alpha_\mu(a) \otimes e_{\mu\nu}, \quad a \in \mathcal{A},\ \mu, \nu \in \Lambda_n$$

extends by linearity and continuity to an isomorphism

$$\Delta_\alpha \colon \mathcal{A} \otimes_\alpha \mathbb{K} \to \mathcal{A} \otimes \mathbb{K},$$

where Δ_α^{-1} *is constructed via the correspondence*

$$a \otimes e_{\mu\nu} \mapsto \alpha_\mu^{-1}(a) e_{\mu\nu}, \quad a \in \mathcal{A}, \ \mu, \nu \in \Lambda_n.$$

Remark 3.5. For $x_\mu = e_{\mu\emptyset}$, $\mu \in \Lambda_n$ one has, see [54],

$$\Delta_\alpha(a x_\mu) = \alpha_\mu(a) \otimes x_\mu, \quad \Delta_\alpha(a x_\mu^*) = a \otimes x_\mu^*.$$

The following functorial property of $\otimes_\alpha \mathbb{K}$ can be derived easily. Consider

$$\alpha = (\alpha_\mu)_{\mu \in \Lambda_n} \subset \mathsf{Aut}(\mathcal{A}), \quad \beta = (\beta_\mu)_{\mu \in \Lambda_n} \subset \mathsf{Aut}(\mathcal{B}).$$

Suppose $\varphi : \mathcal{A} \to \mathcal{B}$ is equivariant, i.e. $\varphi(\alpha_\mu(a)) = \beta_\mu(\varphi(a))$ for any $a \in \mathcal{A}$ and $\mu \in \Lambda_n$. Then one can define the homomorphism

$$\varphi \otimes_\alpha^\beta : \mathcal{A} \otimes_\alpha \mathbb{K} \to \mathcal{B} \otimes_\beta \mathbb{K}, \quad \varphi \otimes_\alpha^\beta (ak) = \varphi(a) k, \quad a \in \mathcal{A}, \quad k \in \mathbb{K},$$

making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{A} \otimes_\alpha \mathbb{K} & \xrightarrow{\varphi \otimes_\alpha^\beta} & \mathcal{B} \otimes_\beta \mathbb{K} \\ \downarrow \Delta_\alpha & & \downarrow \Delta_\beta \\ \mathcal{A} \otimes \mathbb{K} & \xrightarrow{\varphi \otimes \mathrm{id}_\mathbb{K}} & \mathcal{B} \otimes \mathbb{K} \end{array} \tag{3.7}$$

Namely, it is easy to verify that

$$(\Delta_\beta^{-1} \circ (\varphi \otimes \mathrm{id}_\mathbb{K}) \circ \Delta_\alpha)(a e_{\mu\nu}) = \varphi(a) e_{\mu\nu} = \varphi \otimes_\alpha^\beta (a e_{\mu\nu}), \quad a \in A, \ \mu, \nu \in \Lambda_n.$$

An important consequence of the commutativity of the diagram above is exactness of the functor $\otimes_\alpha \mathbb{K}$. Let

$$\beta = (\beta_\mu)_{\mu \in \Lambda_n} \subset \mathsf{Aut}(\mathcal{B}), \quad \alpha = (\alpha_\mu)_{\mu \in \Lambda_n} \subset \mathsf{Aut}(\mathcal{A}), \quad \gamma = (\gamma_\mu)_{\mu \in \Lambda_n} \subset \mathsf{Aut}(\mathcal{C})$$

and consider a short exact sequence

$$0 \longrightarrow \mathcal{B} \xrightarrow{\varphi_1} \mathcal{A} \xrightarrow{\varphi_2} \mathcal{C} \longrightarrow 0,$$

where φ_1, φ_2 are equivariant homomorphisms. Then one has the following short exact sequence:

$$0 \longrightarrow \mathcal{B} \otimes_\beta \mathbb{K} \xrightarrow{\varphi_1 \otimes_\beta^\alpha} \mathcal{A} \otimes_\alpha \mathbb{K} \xrightarrow{\varphi_2 \otimes_\alpha^\gamma} \mathcal{C} \otimes_\gamma \mathbb{K} \longrightarrow 0.$$

Now, we are ready to study the structure of the ideals $\mathcal{I}_1^q, \mathcal{I}_2^q, \mathcal{I}_q \subset \mathcal{E}_{n,m}^q$. We start with $\mathcal{I}_1^q$. Notice that

$$\mathcal{I}_1^q = c.l.s. \{ t_{\mu_2} t_{\nu_2}^* s_{\mu_1} (\mathbf{1} - Q) s_{\nu_1}^*, \ \mu_1, \nu_1 \in \Lambda_n, \ \mu_2, \nu_2 \in \Lambda_m \}.$$

Put $E_{\mu_1 \nu_1} = s_{\mu_1} (\mathbf{1} - Q) s_{\nu_1}^*$, $\mu_1, \nu_1 \in \Lambda_n$. Then $E_{\mu_1 \nu_1}$ satisfy the relations for matrix units generating $\mathbb{K}$. Moreover, $c.l.s. \{ E_{\mu\nu}, \ \mu, \nu \in \Lambda_n \}$ is an ideal in $\mathcal{O}_n^{(0)}$ isomorphic to $\mathbb{K}$.

Consider the family $\alpha^q = (\alpha_\mu)_{\mu\in\Lambda_n} \subset \mathsf{Aut}(\mathcal{O}_m^{(0)})$ defined as

$$\alpha_\mu(t_r) = q^{|\mu|}t_r, \quad \alpha_\mu(t_r^*) = q^{-|\mu|}t_r^*, \quad \mu \in \Lambda_n,\ \ r = \overline{1,m}.$$

Proposition 3.9. *The correspondence* $ae_{\mu\nu} \mapsto aE_{\mu\nu}$, $a \in \mathcal{O}_m^{(0)}$, $\mu,\nu \in \Lambda_n$, *extends to an isomorphism*

$$\Delta_{q,1} \colon \mathcal{O}_m^{(0)} \otimes_{\alpha^q} \mathbb{K} \to \mathcal{I}_1^q.$$

Proof. We note that for any $\mu_1,\nu_1 \in \Lambda_n$ and $\mu_2,\nu_2 \in \Lambda_m$ one has

$$t_{\mu_2}t_{\nu_2}^*E_{\mu_1\nu_1} = q^{(|\nu_1|-|\mu_1|)(|\mu_2|-|\nu_2|)}E_{\mu_1\nu_1}t_{\mu_2}t_{\nu_2}^* = E_{\mu_1\nu_1}\alpha_{\nu_1}^{-1}(\alpha_{\mu_1}(t_{\mu_2}t_{\nu_2}^*)).$$

Thus, due to the universal property of $\langle \mathcal{O}_m^{(0)}, \mathbb{K}\rangle_{\alpha^q}$, the correspondence

$$ae_{\mu\nu} \mapsto aE_{\mu\nu}$$

determines a surjective homomorphism $\Delta_{q,1} \colon \mathcal{O}_m^{(0)} \otimes_{\alpha^q} \mathbb{K} \to \mathcal{I}_1^q$.

It remains to show that $\Delta_{q,1}$ is injective. Since the Fock representation of $\mathcal{E}_{n,m}^q$ is faithful, we can identify $\mathcal{I}_1^q$ with $\pi_F^q(\mathcal{I}_1^q)$. It will be convenient for us to use the following form of the Fock representation, see Remark 3.3:

$$\pi_F^q(s_j) = \pi_{F,n}(s_j) \otimes \mathbf{1}_{\mathcal{F}_m} := S_j \otimes \mathbf{1}_{\mathcal{F}_m},\ \ j = \overline{1,n},$$

$$\pi_F^q(t_r) = d_n(q) \otimes \pi_{F,m}(t_r) := d_n(q) \otimes T_r,\ \ r = \overline{1,m}.$$

In particular, for any $\mu_1,\nu_1 \in \Lambda_n,\ \mu_2,\nu_2 \in \Lambda_m$

$$\pi_F^q(t_{\mu_2}t_{\nu_2}^*E_{\mu_1\nu_1}) = d_n(q^{|\mu_2|-|\nu_2|})S_{\mu_1}(\mathbf{1} - Q)S_{\nu_1} \otimes T_{\mu_2}T_{\nu_2}^*.$$

Consider $\Delta_{q,1} \circ \Delta_{\alpha^q}^{-1} \colon \mathcal{O}_m^{(0)} \otimes \mathbb{K} \to \pi_F^q(\mathcal{I}_1^q)$. We intend to show that

$$\Delta_{q,1} \circ \Delta_{\alpha^q}^{-1} = \pi_F^1,$$

where π_F^1 is the restriction of the Fock representation of $\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}$ to $\mathbb{K} \otimes \mathcal{O}_m^{(0)}$, and $\mathbb{K}$ is generated by $E_{\mu\nu}$ specified above. Notice that the family

$$\{t_{\mu_2}t_{\nu_2}^* \otimes E_{\mu_1\nu_1},\ \mu_1,\nu_1 \in \Lambda_n,\ \mu_2,\nu_2 \in \Lambda_m\}$$

generates $\mathcal{O}_m^{(0)} \otimes \mathbb{K}$. Then

$$\Delta_{\alpha^q}^{-1}(t_{\mu_2}t_{\nu_2}^* \otimes E_{\mu_1\nu_1}) = \alpha_{\mu_1}^{-1}(t_{\mu_2}t_{\nu_2}^*)e_{\mu_1\nu_1} = q^{-|\mu_1|(|\mu_2|-|\nu_2|)}t_{\mu_2}t_{\nu_2}^*e_{\mu_1\nu_1},$$

and

$$\begin{aligned}
&\Delta_{q,1} \circ \Delta_{\alpha^q}^{-1}(t_{\mu_2}t_{\nu_2}^* \otimes E_{\mu_1\nu_1})\\
&\quad= q^{-|\mu_1|(|\mu_2|-|\nu_2|)}\pi_F^q(t_{\mu_2}t_{\nu_2}^*E_{\mu_1\nu_1})\\
&\quad= q^{-|\mu_1|(|\mu_2|-|\nu_2|)}d_n(q^{|\mu_2|-|\nu_2|})S_{\mu_1}(\mathbf{1} - Q)S_{\nu_1}^* \otimes T_{\mu_2}T_{\nu_2}^*\\
&\quad= q^{-|\mu_1|(|\mu_2|-|\nu_2|)}q^{|\mu_1|(|\mu_2|-|\nu_2|)}S_{\mu_1}d_n(q^{|\mu_2|-|\nu_2|})(\mathbf{1} - Q)S_{\nu_1}^* \otimes T_{\mu_2}T_{\nu_2}^*\\
&\quad= S_{\mu_1}(\mathbf{1} - Q)S_{\nu_1}^* \otimes T_{\mu_2}T_{\nu_2}^* = \pi_F^1(E_{\mu_1\nu_1} \otimes t_{\mu_2}t_{\nu_2}^*),
\end{aligned}$$

where we used relations $d_n(\lambda)S_j = \lambda S_j d_n(\lambda)$, $j = \overline{1,n}$, $\lambda \in \mathbb{C}$, and the obvious fact that

$$d_n(\lambda)(\mathbf{1} - Q) = \mathbf{1} - Q.$$

To complete the proof we recall that π_F^1 is a faithful representation of $\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)}$, so its restriction to $\mathbb{K} \otimes \mathcal{O}_m^{(0)}$ is also faithful, implying the injectivity of Δ_q. $\square$

Remark 3.6. Note that any $\mathbb{T}$-action on the C^*-algebra of compact operators is inner. Hence, [41, Corollary 5.16 and Example 5.17] imply that any twisted tensor product with the compact operators is isomorphic to the usual tensor product. This gives a short proof of the above proposition, though without an explicit formula for the isomorphism.

Remark 3.7. Evidently, $\mathcal{I}_q$ is a closed linear span of the family

$$\{\, t_{\mu_2}(1-P)t_{\nu_2}^* s_{\mu_1}(\mathbf{1} - Q)s_{\nu_1}^*,\ \mu_1, \nu_1 \in \Lambda_n,\ \mu_2, \nu_2 \in \Lambda_m\} \subset \mathcal{I}_1^q.$$

Moreover, *c.l.s.*$\{t_{\mu_2}(1-P)t_{\nu_2}^*,\ \mu_2, \nu_2 \in \Lambda_m\} = \mathbb{K} \subset \mathcal{O}_m^{(0)}$. It is easy to see that

$$\alpha_\mu(t_{\mu_2}(1-P)t_{\nu_2}^*) = q^{|\mu|(|\mu_2| - |\nu_2|)} t_{\mu_2}(1-P)t_{\nu_2}^*,$$

so every $\alpha_\mu \in \alpha^q$ can be regarded as an element of $\mathsf{Aut}(\mathbb{K})$.

A moment reflection and Proposition 3.9 give the following corollary:

Proposition 3.10. *Restriction of* $\Delta_{q,1}$ *to* $\mathbb{K} \otimes_{\alpha^q} \mathbb{K} \subset \mathcal{O}_m^{(0)} \otimes_{\alpha^q} \mathbb{K}$ *gives an isomorphism*

$$\Delta_{q,1} \colon \mathbb{K} \otimes_{\alpha^q} \mathbb{K} \to \mathcal{I}_q.$$

To deal with $\mathcal{I}_2^q$, we consider the family $\beta^q = \{\beta_\mu,\ \mu \in \Lambda_m\} \subset \mathsf{Aut}(\mathcal{O}_n^{(0)})$ defined as

$$\beta_\mu(s_j) = q^{-|\mu|} s_j, \quad \beta_\mu(s_j^*) = q^{|\mu|} s_j^*, \quad j = \overline{1,n}.$$

Proposition 3.11. *One has an isomorphism* $\Delta_{q,2} \colon \mathcal{O}_n^{(0)} \otimes_{\beta^q} \mathbb{K} \to \mathcal{I}_2^q$.

Obviously, $\Delta_{q,2}$ induces the isomorphism $\mathbb{K} \otimes_{\beta^q} \mathbb{K} \simeq \mathcal{I}_q$, where the first term is an ideal in $\mathcal{O}_n^{(0)}$ and the second in $\mathcal{O}_m^{(0)}$, respectively.

Write

$$\varepsilon_n \colon \mathbb{K} \to \mathcal{O}_n^{(0)}, \quad \varepsilon_m \colon \mathbb{K} \to \mathcal{O}_m^{(0)},$$

for the canonical embeddings and

$$q_n \colon \mathcal{O}_n^{(0)} \to \mathcal{O}_n, \quad q_m \colon \mathcal{O}_m^{(0)} \to \mathcal{O}_m,$$

for the quotient maps. Let also

$$\varepsilon_{q,j} \colon \mathcal{I}_q \to \mathcal{I}_j^q, \quad j = 1, 2,$$

be the embeddings and

$$\pi_{q,j} \colon \mathcal{J}_j^q \to \mathcal{J}_j^q/\mathcal{J}_q, \quad j = 1, 2,$$

the quotient maps. Notice also that the families $\alpha^q \subset \mathsf{Aut}(\mathcal{O}_m^{(0)})$, $\beta^q \subset \mathsf{Aut}(\mathcal{O}_n^{(0)})$ determine families of automorphisms of $\mathcal{O}_m$ and $\mathcal{O}_n$, respectively, also denoted by α^q and β^q.

Theorem 3.5. *One has the following isomorphism of extensions*:

$$\begin{CD}
0 @>>> \mathcal{J}_q @>{\varepsilon_{q,1}}>> \mathcal{J}_1^q @>{\pi_{q,1}}>> \mathcal{J}_1^q/\mathcal{J}_q @>>> 0\\
@. @VV{\Delta_{\alpha^q}\circ\Delta_{q,1}^{-1}}V @VV{\Delta_{\alpha^q}\circ\Delta_{q,1}^{-1}}V @VV{\simeq}V @.\\
0 @>>> \mathbb{K}\otimes\mathbb{K} @>{\varepsilon_m\otimes\mathrm{id}_{\mathbb{K}}}>> \mathcal{O}_m^0\otimes\mathbb{K} @>{q_m\otimes\mathrm{id}_{\mathbb{K}}}>> \mathcal{O}_m\otimes\mathbb{K} @>>> 0
\end{CD}$$

and

$$\begin{CD}
0 @>>> \mathcal{J}_q @>{\varepsilon_{q,2}}>> \mathcal{J}_2^q @>{\pi_{q,2}}>> \mathcal{J}_2^q/\mathcal{J}_q @>>> 0\\
@. @VV{\Delta_{\beta^q}\circ\Delta_{q,2}^{-1}}V @VV{\Delta_{\beta^q}\circ\Delta_{q,2}^{-1}}V @VV{\simeq}V @.\\
0 @>>> \mathbb{K}\otimes\mathbb{K} @>{\varepsilon_n\otimes\mathrm{id}_{\mathbb{K}}}>> \mathcal{O}_n^0\otimes\mathbb{K} @>{q_n\otimes\mathrm{id}_{\mathbb{K}}}>> \mathcal{O}_n\otimes\mathbb{K} @>>> 0
\end{CD}$$

Proof. Indeed, each row in diagram (3.8) below is exact and every non-dashed vertical arrow is an isomorphism. The bottom left and bottom right squares are commutative due to (3.7). The top left square is commutative due to the arguments in the proof of Proposition 3.9 combined with Remark 3.7. Hence there exists a unique isomorphism

$$\Phi_{q,1} \colon \mathcal{J}_1^q/\mathcal{J}_q \to \mathcal{O}_m \otimes_{\alpha^q} \mathbb{K},$$

making the diagram (3.8) commutative

$$\begin{CD}
0 @>>> \mathcal{J}_q @>{\varepsilon_{q,1}}>> \mathcal{J}_1^q @>{\pi_{q,1}}>> \mathcal{J}_1^q/\mathcal{J}_q @>>> 0\\
@. @VV{\Delta_{q,1}^{-1}}V @VV{\Delta_{q,1}^{-1}}V @VV{\Phi_{q,1}}V @.\\
0 @>>> \mathbb{K}\otimes_{\alpha^q}\mathbb{K} @>{\varepsilon_m\otimes_{\alpha_q}^{\alpha^q}}>> \mathcal{O}_m^0\otimes_{\alpha^q}\mathbb{K} @>{q_m\otimes_{\alpha_q}^{\alpha^q}}>> \mathcal{O}_m\otimes_{\alpha^q}\mathbb{K} @>>> 0\\
@. @VV{\Delta_{\alpha^q}}V @VV{\Delta_{\alpha^q}}V @VV{\Delta_{\alpha^q}}V @.\\
0 @>>> \mathbb{K}\otimes\mathbb{K} @>{\varepsilon_m\otimes\mathrm{id}_{\mathbb{K}}}>> \mathcal{O}_m^0\otimes\mathbb{K} @>{q_m\otimes\mathrm{id}_{\mathbb{K}}}>> \mathcal{O}_m\otimes\mathbb{K} @>>> 0
\end{CD} \tag{3.8}$$

The proof for $\mathcal{J}_2^q$ is similar. □

The following lemma follows from the fact that $\mathcal{M}_q = \mathcal{J}_1^q + \mathcal{J}_2^q$.

Lemma 3.1.

$$\mathcal{M}_q/\mathcal{J}_q \simeq \mathcal{J}_1^q/\mathcal{J}_q \oplus \mathcal{J}_2^q/\mathcal{J}_q \simeq \mathcal{O}_m \otimes \mathbb{K} \oplus \mathcal{O}_n \otimes \mathbb{K}.$$

Theorem 3.5 implies that $\mathcal{I}_q, \mathcal{I}_1^q, \mathcal{I}_2^q$ are stable C^*-algebras. It follows from [50, Proposition 6.12], that an extension of a stable C^*-algebra by $\mathbb{K}$ is also stable. Thus, Lemma 3.1 implies immediately the following important corollary.

Corollary 3.2. *For any $q \in \mathbb{C}$, $|q| = 1$, the C^*-algebra $\mathcal{M}_q$ is stable.*

Denote the Calkin algebra by Q. Recall that for C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ the isomorphism

$$\mathsf{Ext}(\mathcal{A} \oplus \mathcal{B}, \mathbb{K}) \simeq \mathsf{Ext}(\mathcal{A}, \mathbb{K}) \oplus \mathsf{Ext}(\mathcal{B}, \mathbb{K})$$

is given as follows. Let

$$\iota_1 : \mathcal{A} \to \mathcal{A} \oplus \mathcal{B}, \quad \iota_1(a) = (a, 0), \quad \iota_2 : \mathcal{B} \to \mathcal{A} \oplus \mathcal{B}, \quad \iota_2(b) = (0, b).$$

For a Busby invariant $\tau : \mathcal{A} \oplus \mathcal{B} \to Q$ define

$$\mathsf{F} : \mathsf{Ext}(\mathcal{A} \oplus \mathcal{B}, \mathbb{K}) \to \mathsf{Ext}(\mathcal{A}, \mathbb{K}) \oplus \mathsf{Ext}(\mathcal{B}, \mathbb{K}), \quad \mathsf{F}(\tau) = (\tau \circ \iota_1, \tau \circ \iota_2).$$

It can be shown, see [24], that F determines a group isomorphism.

Remark 3.8. Consider an extension

$$0 \longrightarrow \mathcal{B} \longrightarrow \mathcal{E} \longrightarrow \mathcal{A} \longrightarrow 0. \tag{3.9}$$

Let $i : \mathcal{B} \to M(\mathcal{B})$ be the canonical embedding. Define $\beta : \mathcal{E} \to M(\mathcal{B})$, to be the unique map such that

$$\beta(e)i(b) = i(eb), \quad \text{for every } b \in \mathcal{B}, \ e \in \mathcal{E}.$$

Then the Busby invariant τ is the unique map which makes the diagram commute.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{B} & \xrightarrow{i} & M(\mathcal{B}) & \longrightarrow & M(\mathcal{B})/\mathcal{B} & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle\tau} & & \\
0 & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{A} & \longrightarrow & 0
\end{array}$$

We will use both notations $[\mathcal{E}]$ and $[\tau]$ in order to denote the class of the extension (3.9) in $\mathsf{Ext}(\mathcal{A}, \mathcal{B})$.

Let $[\mathcal{M}_q] \in \mathsf{Ext}(\mathcal{I}_1^q/\mathcal{I}_q \oplus \mathcal{I}_2^q/\mathcal{I}_q, \mathcal{I}_q)$, $[\mathcal{I}_1^q] \in \mathsf{Ext}(\mathcal{I}_1^q/\mathcal{I}_q, \mathcal{I}_q)$, $[\mathcal{I}_2^q] \in \mathsf{Ext}(\mathcal{I}_2^q/\mathcal{I}_q, \mathcal{I}_q)$, respectively, be the classes of the following extensions:

$$0 \to \mathcal{I}_q \to \mathcal{M}_q \to \mathcal{I}_1^q/\mathcal{I}_q \oplus \mathcal{I}_2^q/\mathcal{I}_q \to 0,$$

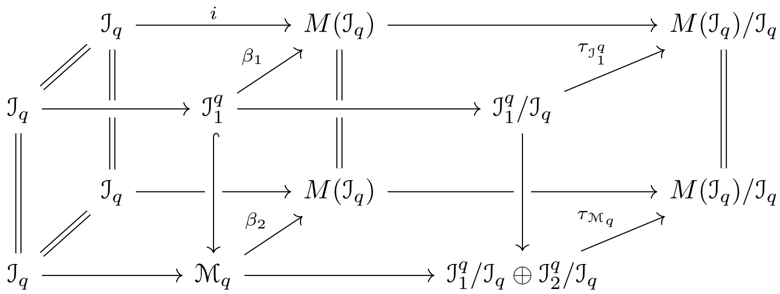
$$0 \to \mathcal{I}_q \to \mathcal{I}_1^q \to \mathcal{I}_1^q/\mathcal{I}_q \to 0,$$

$$0 \to \mathcal{I}_q \to \mathcal{I}_2^q \to \mathcal{I}_2^q/\mathcal{I}_q \to 0.$$

Lemma 3.2.

$$[\mathcal{M}_q] = ([\mathcal{I}_1^q], [\mathcal{I}_2^q]) \in \mathsf{Ext}(\mathcal{I}_1^q/\mathcal{I}_q, \mathcal{I}_q) \oplus \mathsf{Ext}(\mathcal{I}_2^q/\mathcal{I}_q, \mathcal{I}_q) \simeq \mathsf{Ext}(\mathcal{I}_1^q/\mathcal{I}_q \oplus \mathcal{I}_2^q/\mathcal{I}_q, \mathcal{I}_q).$$

Proof. Consider the following morphism of extensions:



Here

$$\beta_1 \colon \mathcal{I}_1^q \to M(\mathcal{I}_q), \quad \beta_2 \colon \mathcal{M}_q \to M(\mathcal{I}_q),$$

are homomorphisms introduced in Remark 3.8, the vertical arrow

$$j_1 \colon \mathcal{I}_1^q \hookrightarrow \mathcal{M}_q$$

is the inclusion, and the vertical arrow

$$\iota_1 : \mathcal{I}_1^q/\mathcal{I}_q \to \mathcal{I}_1^q/\mathcal{I}_q \oplus \mathcal{I}_2^q/\mathcal{I}_q$$

has the form $\iota_1(x) = (x, 0)$.

Notice that for every $b \in \mathcal{I}_q$ and $x \in \mathcal{I}_1^q$ one has

$$(\beta_2 \circ j_1)(x)i(b) = i(j_1(x)b) = i(xb) = \beta_1(x)i(b).$$

By the uniqueness of β_1, we get $\beta_2 \circ j_1 = \beta_1$. Thus, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{I}_1^q & \xrightarrow{\beta_1} & M(\mathcal{I}_q) \\ \big\downarrow & & \big\| \\ \mathcal{M}_q & \xrightarrow{\beta_2} & M(\mathcal{I}_q) \end{array}$$

Further, Remark 3.8 implies that for Busby invariants $\tau_{\mathcal{I}_1^q}$ and $\tau_{\mathcal{M}_q}$ the squares below are commutative

$$\begin{array}{ccc} M(\mathcal{I}_q) & \longrightarrow & M(\mathcal{I}_q)/\mathcal{I}_q \\ {\scriptstyle\beta_1}\big\uparrow & & {\scriptstyle\tau_{\mathcal{I}_1^q}}\big\uparrow \\ \mathcal{I}_1^q & \longrightarrow & \mathcal{I}_1^q/\mathcal{I}_q \end{array} \qquad \begin{array}{ccc} M(\mathcal{I}_q) & \longrightarrow & M(\mathcal{I}_q)/\mathcal{I}_q \\ {\scriptstyle\beta_2}\big\uparrow & & {\scriptstyle\tau_{\mathcal{M}_q}}\big\uparrow \\ \mathcal{M}_q & \longrightarrow & \mathcal{I}_1^q/\mathcal{I}_q \oplus \mathcal{I}_2^q/\mathcal{I}_q \end{array}$$

Hence the square

$$\begin{array}{ccc} \mathcal{I}_1^q/\mathcal{I}_q & \xrightarrow{\tau_{\mathcal{I}_1^q}} & M(\mathcal{I}_q)/\mathcal{I}_q \\ \downarrow{\iota_1} & & \| \\ \mathcal{I}_1^q/\mathcal{I}_q \oplus \mathcal{I}_2^q/\mathcal{I}_q & \xrightarrow{\tau_{\mathcal{M}_q}} & M(\mathcal{I}_q)/\mathcal{I}_q \end{array}$$

is also commutative. Thus, $\tau_{\mathcal{I}_1^q} = \tau_{\mathcal{M}_q} \circ \iota_1$. By the same arguments we get $\tau_{\mathcal{I}_2^q} = \tau_{\mathcal{M}_q} \circ \iota_2$, where

$$\iota_2 \colon \mathcal{I}_2^q/\mathcal{I}_q \to \mathcal{I}_1^q/\mathcal{I}_q \oplus \mathcal{I}_2^q/\mathcal{I}_q, \quad \iota_2(y) = (0, y).$$

Thus

$$[\tau_{\mathcal{M}_q}] = ([\tau_{\mathcal{M}_q} \circ \iota_1], [\tau_{\mathcal{M}_q} \circ \iota_2]) = ([\tau_{\mathcal{I}_1^q}], [\tau_{\mathcal{I}_2^q}]). \qquad \square$$

In the following theorem, we give a description of all ideals in $\mathcal{E}_{n,m}^q$.

Theorem 3.6. *Any ideal* $J \subset \mathcal{E}_{n,m}^q$ *coincides with one of* $\mathcal{I}_q$, $\mathcal{I}_1^q$, $\mathcal{I}_2^q$, $\mathcal{M}_q$.

Proof. First, we notice that $\mathcal{I}_1^q/\mathcal{I}_q \simeq \mathcal{O}_m \otimes \mathbb{K}$, $\mathcal{I}_2^q/\mathcal{I}_q \simeq \mathcal{O}_n \otimes \mathbb{K}$ are simple. Hence for any ideal $\mathcal{J}$ such that $\mathcal{I}_q \subseteq \mathcal{J} \subseteq \mathcal{I}_1^q$ or $\mathcal{I}_q \subseteq \mathcal{J} \subseteq \mathcal{I}_2^q$, one has $\mathcal{J} = \mathcal{I}_q$, or $\mathcal{J} = \mathcal{I}_1^q$, or $\mathcal{J} = \mathcal{I}_2^q$.

Further, using the fact that $\mathcal{M}_q = \mathcal{I}_1^q + \mathcal{I}_2^q$ and $\mathcal{I}_q = \mathcal{I}_1^q \cap \mathcal{I}_2^q$ we get

$$\mathcal{M}_q/\mathcal{I}_1^q \simeq \mathcal{I}_2^q/\mathcal{I}_q \simeq \mathcal{O}_n \otimes \mathbb{K}.$$

So if $\mathcal{I}_1^q \subseteq \mathcal{J} \subseteq \mathcal{M}_q$, then again either $\mathcal{J} = \mathcal{I}_1^q$ or $\mathcal{J} = \mathcal{M}_q$. Obviously, the same result holds for $\mathcal{I}_2^q \subseteq \mathcal{J} \subseteq \mathcal{M}_q$.

In what follows, see Proposition 4.2, we show that $\mathcal{E}_{n,m}^q/\mathcal{M}_q$ is simple and purely infinite. In particular, $\mathcal{M}_q$ contains any ideal in $\mathcal{E}_{n,m}^q$, see Corollary 4.2.

Let $\mathcal{J} \subset \mathcal{E}_{n,m}^q$ be an ideal and π be a representation of $\mathcal{E}_{n,m}^q$ such that $\ker \pi = \mathcal{J}$. Notice that the Fock component π_1 in the Wold decomposition of π is zero. Thus, by Theorem 3.4

$$\pi = \pi_2 \oplus \pi_3 \oplus \pi_4, \tag{3.10}$$

and $\mathcal{J} = \ker \pi = \ker \pi_2 \cap \ker \pi_3 \cap \ker \pi_4$. Let us describe these kernels. Suppose that the component π_2 is non-zero. Since $\pi_2(\mathbf{1} - Q) = 0$ and $\pi_2(\mathbf{1} - P) \neq 0$, we have

$$\mathcal{I}_1^q \subseteq \ker \pi_2 \subsetneq \mathcal{M}_q,$$

implying $\ker \pi_2 = \mathcal{I}_1^q$. Using the same arguments, one can deduce that if the component π_3 is non-zero, then $\ker \pi_3 = \mathcal{I}_2^q$, and if π_4 is non-zero, then $\ker \pi_4 = \mathcal{M}_q$.

Finally, if in (3.10) π_2 and π_3 are non-zero then $\mathcal{J} = \ker \pi = \mathcal{I}_q$. If either $\pi_2 \neq 0$ and $\pi_3 = 0$ or $\pi_3 \neq 0$ and $\pi_2 = 0$, then either $\mathcal{J} = \mathcal{I}_1^q$ or $\mathcal{J} = \mathcal{I}_2^q$. In the case $\pi_2 = 0$ and $\pi_3 = 0$ one has $\mathcal{J} = \ker \pi_4 = \mathcal{M}_q$. $\square$

Corollary 3.3. *All ideals in $\mathcal{E}^q_{n,m}$ are essential. The ideal $\mathcal{I}^q$ is the unique minimal ideal.*

In particular, the extension

$$0 \to \mathcal{I}_q \to \mathcal{M}_q \to \mathcal{I}^q_1/\mathcal{I}_q \oplus \mathcal{I}^q_2/\mathcal{I}_q \to 0$$

is essential. Indeed, the ideal $\mathbb{K} = \mathcal{I}_q \subset \mathcal{E}^q_{n,m}$ is the unique minimal ideal. Since an ideal of an ideal in a C^*-algebra is an ideal in the whole algebra, $\mathcal{I}_q$ is the unique minimal ideal in $\mathcal{M}_q$, thus it is essential in $\mathcal{M}_q$.

The following proposition is a corollary of Voiculescu's Theorem, see [3, Theorem 15.12.3].

Proposition 3.12. *Let $\mathcal{E}_1, \mathcal{E}_2$ be two essential extensions of a nuclear C^*-algebra $\mathcal{A}$ by $\mathbb{K}$. If $[\mathcal{E}_1] = [\mathcal{E}_2] \in \mathsf{Ext}(\mathcal{A}, \mathbb{K})$ then $\mathcal{E}_1 \simeq \mathcal{E}_2$.*

Theorem 3.7. *For any $q \in \mathbb{C}$, $|q| = 1$, one has $\mathcal{M}_q \simeq \mathcal{M}_1$.*

Proof. By Theorem 3.5, $[\mathcal{I}^q_1] \in \mathsf{Ext}(\mathcal{O}_m \otimes \mathbb{K}, \mathbb{K})$, and $[\mathcal{I}^q_2] \in \mathsf{Ext}(\mathcal{O}_n \otimes \mathbb{K}, \mathbb{K})$ do not depend on q. By Lemma 3.2, $[\mathcal{M}_q]$ does not depend on q. Thus, by Corollary 3.3 and Proposition 3.12, $\mathcal{M}_q \simeq \mathcal{M}_1$. □

4. The Multiparameter Case

We now turn to the C^*-algebra $\mathcal{O}_n \otimes_q \mathcal{O}_m := \mathcal{E}^q_{n,m}/\mathcal{M}_q$. Our goal is to show that it is isomorphic to $\mathcal{O}_n \otimes \mathcal{O}_m$. More generally, we show that even multiparameter twists of $\mathcal{O}_n \otimes \mathcal{O}_m$ are impossible. This is very specific for the Cuntz algebra. Indeed, recall that the tensor product of $C(S^1) \otimes C(S^1)$ *does* admit twists, the famous rotation algebra A_q being the result. Also twists of tensor products of Toeplitz algebras have been considered [16, 54]. However, the tensor product of Cuntz algebras may not be twisted as we will see.

Our proof uses deep theory from the classification of C^*-algebras: We use Kirchberg's seminal result from the 1990's [33], stating that two unital, separable, nuclear, simple, purely infinite C^*-algebras are isomorphic if and only if they have the same K-groups.

Note that twists of C^*-algebras A with Cuntz algebras $\mathcal{O}_m$ were first considered by Cuntz in 1981 [12], see also the PhD thesis of Neubüser from 2000 [42]. We reformulate Cuntz's definition in terms of universal C^*-algebras later. Let us now focus on the case $A = \mathcal{O}_n$.

Definition 4.1. Let $2 \leq n, m < \infty$ and let $\Theta = (q_{ij})$ be a matrix with $q_{ij} \in S^1$ being scalars of absolute value one, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. We define the twist of two Cuntz algebras $\mathcal{O}_n$ and $\mathcal{O}_m$ as the universal C^*-algebra $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ generated by isometries $s_1, \ldots, s_n$ and $t_1, \ldots, t_m$ with $\sum_{i=1}^n s_i s_i^* = \sum_{j=1}^m t_j t_j^* = 1$ and $s_i t_j = q_{ij} t_j s_i$ for all i and j.

Observe that for $q_{ij} = 1$ for all i and j, we obtain $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m = \mathcal{O}_n \otimes \mathcal{O}_m$, while the choice $q_{ij} = \bar{q}$ yields the one parameter deformation $\mathcal{O}_n \otimes_q \mathcal{O}_m := \mathcal{E}^q_{n,m}/\mathcal{M}_q$, which can be seen as follows.

We have $s_i t_j = q_{ij} t_j s_i$ for all i and j if and only if $s_i^* t_j = \bar{q}_{ij} t_j s_i^*$ for all i and j. Indeed, assuming the former relations and using $t_k^* t_i = \delta_{ik}$, we infer

$$s_i^* t_j = \sum_k t_k t_k^* s_i^* t_j = \sum_k \bar{q}_{ik} t_k s_i^* t_k^* t_j = \bar{q}_{ij} t_j s_i^*.$$

Conversely, fixing i and j and assuming $s_i t_j^* = \bar{q}_{ij} t_j^* s_i$, we obtain

$$(s_i t_j - q_{ij} t_j s_i)^*(s_i t_j - q_{ij} t_j s_i) = 1 - \bar{q}_{ij} s_i^* t_j^* s_i t_j - q_{ij} t_j^* s_i^* t_j s_i + 1 = 0.$$

Moreover, note that we may view $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ as a twisted tensor product in the following sense. Putting Cuntz's definition from [12] to a language of universal C^*-algebras, we may say that given a unital C^*-algebra A and automorphisms $\alpha_1, \ldots, \alpha_m$ of A satisfying $\alpha_i \circ \alpha_j = \alpha_j \circ \alpha_i$, we may define a twisted tensor product $A \times_{(\alpha_1,\ldots,\alpha_m)} \mathcal{O}_m$ as the universal unital C^*-algebra generated by all elements $a \in A$ (including the relations of A) and isometries $t_1, \ldots, t_m$ such that $\sum_k t_k t_k^* = 1$ and $t_j a = \alpha_j(a) t_j$, for all $j = 1, \ldots, m$ and $a \in A$. We observe that $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ is such a twisted tensor product, if we put $\alpha_j(s_i) := \bar{q}_{ij} s_i$ for $A = \mathcal{O}_n = C^*(s_1, \ldots, s_n)$.

Let us quickly show that $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ may be represented concretely on a Hilbert space. First, represent $\mathcal{O}_m$ on $\ell^2(\mathbb{N}_0)$ by $\hat{T}_j e_k := e_{\beta_j(k)}$, where

$$\beta_1, \ldots, \beta_m : \mathbb{N}_0 \to \mathbb{N}_0$$

are injective functions with disjoint ranges such that the union of their ranges is all of $\mathbb{N}_0$. Second, choose any representation $\sigma : \mathcal{O}_n \to L(\ell^2(\mathbb{N}_0))$ mapping $s_i \mapsto \hat{S}_i$. Third, let $\tilde{S}$ be the unilateral shift on $\ell^2(\mathbb{Z})$. Fourthly, define the diagonal unitaries U_i on $\ell^2(\mathbb{N}_0 \times \mathbb{Z})$ by $U_i e_{m,n} := \zeta_i(m,n) e_{m,n}$, where the scalars $\zeta_i(m,n)$ with $|\zeta_i(m,n)| = 1$ obey the inductive rule $\zeta_i(\beta_j(m), n+1) = q_{ij} \zeta_i(m,n)$. Finally, $\pi : \mathcal{O}_n \otimes_\Theta \mathcal{O}_m \to B(\ell^2(\mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{Z}))$ with

$$\pi(s_i) := \hat{S}_i \otimes U_i, \quad \pi(t_j) := \mathrm{id} \otimes \hat{T}_j \otimes \tilde{S}$$

exists by the universal property.

4.1. $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ *is nuclear*

Let us now begin with collecting the ingredients for an application of Kirchberg's Theorem. The first step is to verify that $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ is nuclear. For the one parameter deformation $\mathcal{O}_n \otimes_q \mathcal{O}_m$, this is a consequence of Corollary 3.1. For the multiparameter deformations, we use the following lemma by Rosenberg, which is actually a statement about crossed products with the semigroup $\mathbb{N}_0$ by the endomorphism $b \mapsto sbs^*$ of B. Recall that the bootstrap class $\mathcal{N}$ (also called UCT class) is a class of nuclear C^*-algebras which is closed under many operations; see [3, Chap. IX, Sec. 22.3] for more on $\mathcal{N}$.

Lemma 4.1 ([51, Theorem 3]). *Let A be a unital C^*-algebra, and let $B \subset A$ be a nuclear C^*-subalgebra containing the unit of A. Let $s \in A$ be an isometry such* *that $sBs^* \subset B$ and $A = C^*(B, s)$, i.e. let A be generated by B and s. Then A is nuclear. Moreover, if B is in the bootstrap class $\mathcal{N}$, so is A.*

Let us denote by s_μ the product $s_{\mu_1} \dots s_{\mu_k}$, where $\mu = (\mu_1, \dots, \mu_k)$ is a multi-index of length $|\mu| = k$ and $\mu_1, \dots, \mu_k \in \{1, \dots, n\}$. Given $k \in \mathbb{N}_0$, denote by $\mathcal{G}_k$ the C^*-subalgebra of $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ defined as

$$\mathcal{G}_k := C^*(s_\mu x s_\nu^* \mid x \in \mathcal{O}_m, |\mu| = |\nu| = k) \subset \mathcal{O}_n \otimes_\Theta \mathcal{O}_m.$$

Here, we denote by $\mathcal{O}_m$ the C^*-subalgebra $\mathcal{G}_0 = C^*(t_1, \dots, t_m) \subset \mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ (which is isomorphic to $\mathcal{O}_m$, of course). The closed union of all $\mathcal{G}_k$ is denoted by B_0, so

$$B_0 := C^*(s_\mu x s_\nu^* \mid x \in \mathcal{O}_m, |\mu| = |\nu| \in \mathbb{N}_0) \subset \mathcal{O}_n \otimes_\Theta \mathcal{O}_m.$$

Proposition 4.1. *We have the following*:

(a) $\mathcal{G}_k \subset \mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ *is nuclear and in* $\mathcal{N}$ *for all* $k \in \mathbb{N}_0$.
(b) $B_0 \subset \mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ *is nuclear and in* $\mathcal{N}$.
(c) $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ *is nuclear and in* $\mathcal{N}$.

Proof. For (a), we show that $\mathcal{G}_k$ is isomorphic to $M_{n^k}(\mathbb{C}) \otimes \mathcal{O}_m$, where $M_{n^k}(\mathbb{C})$ denotes the algebra of $n^k \times n^k$ matrices with complex entries. For doing so, let $k > 0$ and write $M_{n^k}(\mathbb{C}) \otimes \mathcal{O}_m$ as the universal C^*-algebra generated by elements $e_{\mu\nu}$ for multi-indices μ, ν in $\{1, \dots, n\}$ of length k, together with isometries $t_1, \dots, t_m$ such that $\sum_k t_k t_k^* = 1$ and $e_{\mu\nu} e_{\rho\sigma} = \delta_{\nu\rho} e_{\mu\sigma}$, $e_{\mu\nu}^* = e_{\nu\mu}$ and $e_{\mu\nu} t_i = t_i e_{\mu\nu}$.

It is then easy to see that the elements $\hat{e}_{\mu\nu} := s_\mu s_\nu^*$ and $\hat{t}_i := \sum_{|\rho|=k} s_\rho t_i s_\rho^*$ in $\mathcal{G}_k$ fulfill the relations of $M_{n^k}(\mathbb{C}) \otimes \mathcal{O}_m$. As this C^*-algebra is simple, $\mathcal{G}_k$ is isomorphic to $M_{n^k}(\mathbb{C}) \otimes \mathcal{O}_m$. Now, $\mathcal{O}_m$ is in $\mathcal{N}$ and the bootstrap class is closed under tensoring with matrix algebras, so $M_{n^k}(\mathbb{C}) \otimes \mathcal{O}_m$ is in $\mathcal{N}$.

As for (b), note that we have $\mathcal{G}_k \subset \mathcal{G}_{k+1}$ for all k. Indeed, check that $s_\mu t_i s_\nu^* = \sum_k \bar{q}_{ki} s_\mu s_k t_i s_k^* s_\nu^* \in \mathcal{G}_{k+1}$. We may then view B_0 as the inductive limit of the system $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \dots$. As $\mathcal{N}$ is closed under inductive limits, we get the result by (a).

Finally, for (c), put $B_{j+1} := C^*(B_j, s_{j+1}) \subset \mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ for $j = 0, \dots, n-1$. We prove inductively that B_{j+1} is in $\mathcal{N}$ and that $B_{j+1} = C^*(B_0, s_1, \dots, s_{j+1})$, using Lemma 4.1. By induction hypothesis (or by (b) in the case $j = 0$), the C^*-algebra B_j is nuclear and in $\mathcal{N}$, it is unital with the same unit as B_{j+1} and it is contained in B_{j+1}. Therefore, to apply Rosenberg's lemma, we only have to check, that $s_{j+1} B_j s_{j+1}^* \subset B_j = C^*(B_0, s_1, \dots, s_j)$.

For this, let $s_\mu x s_\nu^*$ be in B_0 for some $x \in \mathcal{O}_m$ and $|\mu| = |\nu| \in \mathbb{N}_0$. Then $s_{j+1} s_\mu x s_\nu^* s_{j+1}^* \in B_0 \subset B_j$. We conclude $s_{j+1} B_0 s_{j+1}^* \subset B_j$. For $k = 1, \dots, j$, we have $s_{(j+1,k)} s_{(k,j+1)}^* \in B_0$, where $(j+1, k)$ and $(k, j+1)$ denote multi-indices of length two. Thus:

$$s_{j+1}s_k s_{j+1}^* = s_{j+1}s_k s_{j+1}^* s_k^* s_k = (s_{(j+1,k)}s_{(k,j+1)}^*)s_k \in B_j.$$

Thus, if $x = x_1 \dots x_k$ is a product of elements $x_l \in B_0 \cup \{s_1, s_1^*, \dots, s_j, s_j^*\}$, for $l = 1, \dots, k$, we have

$$s_{j+1}xs_{j+1}^* = (s_{j+1}x_1 s_{j+1}^*) \dots (s_{j+1}x_k s_{j+1}^*) \in B_j.$$

This proves $s_{j+1}C^*(B_0, s_1, \dots, s_j)s_{j+1}^* \subset B_j$ and we are done.

Since $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m = B_n$, we obtain the stated result. □

4.2. $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ *is purely infinite*

Next, we are going to show that $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ is purely infinite. Recall, that a unital C^*-algebra A is called purely infinite, if for all nonzero elements $x \in A$, there are $a, b \in A$ such that $axb = 1$. In particular, A is then simple.

In our proof, we follow Cuntz's work on $\mathcal{O}_n$ from 1977 [10] and we adapt it to $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$. The idea is to find a faithful expectation ϕ on $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$, such that an element $0 \neq x \in \mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ (or rather an element y close to x^*x) is mapped to a self-adjoint complex valued matrix with controlled norm. By linear algebra, we then can find a minimal projection e, which projects onto a one-dimensional subspace corresponding to the largest eigenvalue of this matrix: its norm. We then choose a unitary u transforming this projection e into $S_1^r S_1'^r (S_1'^*)^r (S_1^*)^r$. Moreover, we implement ϕ locally by an isometry $w \in \mathcal{O}_n \otimes_\Theta \mathcal{O}_m$. Using all these elements in $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$, we can define $z \in A$, such that $zyz^* = 1$. This implies that zx^*xz^* is invertible and we finish the proof by putting $a := b^*x^*$ and $b := z^*(zx^*xz^*)^{-\frac{1}{2}}$. Then $axb = 1$.

Let us now work out the details. We define the following subsets of $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$:

- $\mathcal{F}_{k,l} := \operatorname{span}\{s_\mu t_{\mu'} t_{\nu'}^* s_\nu^* \mid |\mu| = |\nu| = k, \, |\mu'| = |\nu'| = l\}$ for $k, l \in \mathbb{N}_0$
- $\mathcal{F}_{\mathbb{N}_0,\mathbb{N}_0} := \overline{\operatorname{span}}\{s_\mu t_{\mu'} t_{\nu'}^* s_\nu^* \mid |\mu| = |\nu|, |\mu'| = |\nu'|\} = \overline{\bigcup_{k,l \in \mathbb{N}_0} \mathcal{F}_{k,l}}$
- $\mathcal{F}_{\mathbb{N}_0,\bullet} := \overline{\operatorname{span}}\{s_\mu t_{\mu'} t_{\nu'}^* s_\nu^* \mid |\mu| = |\nu|\}$
- $\mathcal{F}_{\bullet,\mathbb{N}_0} := \overline{\operatorname{span}}\{s_\mu t_{\mu'} t_{\nu'}^* s_\nu^* \mid |\mu'| = |\nu'|\}$

Lemma 4.2. *The above subsets $\mathcal{F}_{k,l}$, $\mathcal{F}_{\mathbb{N}_0,\mathbb{N}_0}$, $\mathcal{F}_{\mathbb{N}_0,\bullet}$, $\mathcal{F}_{\bullet,\mathbb{N}_0}$ are C^*-subalgebras of $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ and we have $\mathcal{F}_{k,l} = \cup_{i \le k, j \le l} \mathcal{F}_{i,j}$. Actually, $\mathcal{F}_{k,l}$ is isomorphic to $M_{n^k m^l}(\mathbb{C})$.*

Proof. Using the well-known relations $t_\mu^* t_\nu = \delta_{\mu\nu}$ for multi-indices $|\mu| = |\nu|$ and similarly for $s_{\mu'}$, we infer that all these subsets are in fact *-subalgebras. Moreover, all of these *-subalgebras are closed. For $\mathcal{F}_{k,l}$ this follows from the fact that the elements $t_\mu s_{\mu'} s_{\nu'}^* t_\nu^*$ satisfy the relations of matrix units and hence $\mathcal{F}_{k,l}$ is isomorphic to $M_{n^k m^l}(\mathbb{C})$. Finally, let $s_\mu t_{\mu'} t_{\nu'}^* s_\nu^* \in \mathcal{F}_{i,j}$ for $i \le k, j \le l$ and write 1 as the sum P of all projections $s_\delta t_{\epsilon'} t_{\epsilon'}^* s_\delta^*$ with $|\delta| = k - i$ and $|\epsilon'| = l - j$. Then $s_\mu t_{\mu'} t_{\nu'}^* s_\nu^* = s_\mu t_{\mu'} P t_{\nu'}^* s_\nu^* \in \mathcal{F}_{k,l}$. □

Next, we are going to construct a faithful expectation for $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$, similar to the one for $\mathcal{O}_n$ or the rotation algebra A_q. Recall that for a unital C^*-algebra A, a unital, linear, positive map $\phi : A \to B$ is an expectation, if $B \subset A$ is a C^*-subalgebra and $\phi^2 = \phi$. It is faithful, if it maps non-zero positive elements to non-zero positive elements.

Let $\rho_{\zeta,\xi} : \mathcal{O}_n \otimes_\Theta \mathcal{O}_m \to \mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ be the automorphisms mapping $s_i \mapsto \zeta s_i$ and $t_j \mapsto \xi t_j$, where $\zeta, \xi \in \mathbb{C}$ are scalars of absolute value one. Put, for $x \in \mathcal{O}_n \otimes_\Theta \mathcal{O}_m$:

$$\phi_1(x) := \int_0^1 \rho_{e^{2\pi i t},1}(x)\mathrm{dt},$$

$$\phi_2(x) := \int_0^1 \rho_{1,e^{2\pi i t}}(x)\mathrm{dt}.$$

Lemma 4.3. *The maps* $\phi_1 : \mathcal{O}_n \otimes_\Theta \mathcal{O}_m \to \mathcal{F}_{\mathbb{N}_0,\bullet}$, $\phi_2 : \mathcal{O}_n \otimes_\Theta \mathcal{O}_m \to \mathcal{F}_{\bullet,\mathbb{N}_0}$ *and* $\phi := \phi_1 \circ \phi_2 = \phi_2 \circ \phi_1 : \mathcal{O}_n \otimes_\Theta \mathcal{O}_m \to \mathcal{F}_{\mathbb{N}_0,\mathbb{N}_0}$ *are faithful expectations.*

Proof. The map $(\zeta, \xi) \mapsto \rho_{\zeta,\xi}(x)$ is continuous in norm, for all $x \in \mathcal{O}_n \otimes_\Theta \mathcal{O}_m$, as $\rho_{\zeta,\xi}(s_\mu t_{\mu'} t^*_{\nu'} s^*_\nu) = \zeta^{|\mu|-|\nu|}\xi^{|\mu'|-|\nu'|} s_\mu t_{\mu'} t^*_{\nu'} s^*_\nu$. Thus, ϕ_1 and ϕ_2 are well-defined as limits of sums $\frac{1}{N}\sum_{k=1}^N \rho_{e^{2\pi i\theta_k},1}(x)$ for partitions $\theta_1, \ldots, \theta_N$ of the unit interval (likewise for ϕ_2). They are unital, linear, positive and faithful as may be deduced immediately from this approximation by finite sums.

Applying ϕ_1 to $s_\mu t_{\mu'} t^*_{\nu'} s^*_\nu$ (where μ, μ', ν and ν' are of arbitrary length) yields:

$$\begin{aligned}\phi_1(s_\mu t_{\mu'} t^*_{\nu'} s^*_\nu) &= \left(\int_0^1 e^{2\pi i(|\mu|-|\nu|)t}\mathrm{dt}\right) s_\mu t_{\mu'} t^*_{\nu'} s^*_\nu \\ &= \begin{cases} s_\mu t_{\mu'} t^*_{\nu'} s^*_\nu & \text{for } |\mu| = |\nu|, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Therefore, since the span of the words $s_\mu t_{\mu'} t'^*_{\nu'} s^*_\nu$ is dense in $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$, we get $\phi_1^2 = \phi_1$ and the image of ϕ_1 is $\mathcal{F}_{\mathbb{N}_0,\bullet}$, similarly for ϕ_2.

The composition of two faithful expectations is again a unital, linear, positive and faithful map. If they commute, we even get an expectation. This is the case for ϕ_1 and ϕ_2, due to the following computation:

$$\phi_1 \circ \phi_2(s_\mu t_{\mu'} t^*_{\nu'} s^*_\nu) = \begin{cases} s_\mu t_{\mu'} t^*_{\nu'} s^*_\nu & \text{for } |\mu| = |\nu| \text{ and } |\mu'| = |\nu'|, \\ 0 & \text{otherwise.}\end{cases}$$

The same holds for $\phi_2 \circ \phi_1(s_\mu t_{\mu'} t^*_{\nu'} s^*_\nu)$, thus $\phi_1 \circ \phi_2 = \phi_2 \circ \phi_1$ and we are done. □

Let us now implement ϕ locally by an isometry using a standard trick.

Lemma 4.4. *For all* $k, l \in \mathbb{N}_0$, *there exists an isometry* $w \in \mathcal{O}_n \otimes_\Theta \mathcal{O}_m$, *which commutes with all elements in* $\mathcal{F}_{k,l}$, *such that* $\phi(y) = w^* y w$ *for all* $y \in \mathrm{span}\{s_\mu t_{\mu'} t^*_{\nu'} s^*_\nu \,|\, |\mu|, |\nu| \leq k \text{ and } |\mu'|, |\nu'| \leq l\}$.

Proof. Put $s_\gamma := s_1^{2k} s_2 t_1^{2l} t_2$ and define w by

$$w := \sum_{|\delta|=k, |\epsilon'|=l} s_\delta t_{\epsilon'} s_\gamma t_{\epsilon'}^* s_\delta^*$$

Then, w is an isometry. We have $w s_\mu t_{\mu'} = s_\mu t_{\mu'} s_\gamma$ and $t_{\nu'}^* s_\nu^* w = s_\gamma t_{\nu'}^* s_\nu^*$, for all μ, μ', ν, ν' with $|\mu| = |\nu| = k$ and $|\mu'| = |\nu'| = l$. We conclude $w s_\mu t_{\mu'} t_{\nu'}^* s_\nu^* = s_\mu t_{\mu'} t_{\nu'}^* s_\nu^* w$, thus w commutes with $\mathcal{F}_{k,l}$.

Now, let $y = s_\mu t_{\mu'} t_{\nu'}^* s_\nu^*$ be a word with $|\mu|, |\nu| \leq k$ and $|\mu'|, |\nu'| \leq l$. If $|\mu| = |\nu|$ and $|\mu'| = |\nu'|$, then $y \in \mathcal{F}_{k,l}$ by Lemma 4.2, thus we have $w^* y w = w^* w y = y = \phi(y)$. In the case of $|\mu| \neq |\nu|$ or $|\mu'| \neq |\nu'|$, we have $w^* y w = 0 = \phi(y)$. □

Proposition 4.2. *$\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ is purely infinite.*

Proof. Let $0 \neq x \in \mathcal{O}_n \otimes_\Theta \mathcal{O}_m$. Under the faithful expectation ϕ, the non-zero, positive element $x^* x$ is mapped to the non-zero, positive element $\phi(x^* x)$ of norm 1, if suitably scaled. Since the linear span $\mathcal{S}$ of all elements $s_\mu t_{\mu'} t_{\nu'}^* s_\nu^*$, where μ, μ', ν, ν' are multi-indices of arbitrary length, is dense in $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$, there is a self-adjoint element $y \in \mathcal{S}$, such that $\|x^* x - y\| < \frac{1}{4}$. We conclude that $\|\phi(y)\| > \frac{3}{4}$, as $1 = \|\phi(x^* x)\| \leq \|\phi(x^* x - y)\| + \|\phi(y)\| < \frac{1}{4} + \|\phi(y)\|$.

Let r be the maximal length of the multi-indices of the summands of y from its presentation as an element in $\mathcal{S}$. By Lemma 4.4, there is an isometry $w \in \mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ which commutes with $\mathcal{F}_{r,r}$ such that $\phi(y) = w^* y w$.

By Lemma 4.2, we can view $\phi(y) \in \mathcal{F}_{r,r} \cong M_{n^r m^r}(\mathbb{C})$ as a self-adjoint matrix. Thus, there is a minimal projection $e \in \mathcal{F}_{r,r}$, such that $e\phi(y) = \phi(y)e = \|\phi(y)\| e > \frac{3}{4} e$. There is also a unitary $u \in \mathcal{F}_{r,r}$ with $ueu^* = s_1^r t_1^r (t_1^*)^r (s_1^*)^r$, transforming one minimal projection into another.

Put $z := \|\phi(y)\|^{-\frac{1}{2}} (t_1^*)^r (s_1^*)^r u e w^*$. Then $\|z\| \leq \|\phi(y)\|^{-\frac{1}{2}} < \frac{2}{\sqrt{3}}$ and we have

$$z y z^* = \|\phi(y)\|^{-1} (t_1^*)^r (s_1^*)^r u e \phi(y) e u^* s_1^r t_1^r = (t_1^*)^r (s_1^*)^r u e u^* s_1^r t_1^r = 1$$

Hence $z x^* x z^*$ is invertible:

$$\|1 - z x^* x z^*\| = \|z(y - x^* x) z^*\| < \frac{4}{3} \cdot \frac{1}{4} = \frac{1}{3} < 1$$

Finally, put $a := b^* x^*$ and $b := z^* (z x^* x z^*)^{-\frac{1}{2}}$. Then $axb = 1$. □

4.3. *K-groups of* $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$

Finally, we calculate the K-groups of $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ building on the work by Cuntz [12]. Recall, that Cuntz gave a slightly different definition for the twist of a C^*-algebra A with $\mathcal{O}_m$ in his paper: Let $A \subset B(H)$ be a unital C^*-algebra, $\alpha_1, \dots, \alpha_m$ be pairwise commuting automorphisms of A and let $u_1, \dots, u_m \in B(H)$ be pairwise commuting unitaries implementing the automorphisms by $\alpha_i = \mathrm{Ad}(u_i)$. We denote by $\mathcal{U} := (u_1, \dots, u_m)$ the tupel of the unitaries and by $A \times_{\mathcal{U}} \mathcal{O}_m$ the C^*-subalgebra of $B(H)\otimes\mathcal{O}_m$ generated by all elements $a\otimes 1$ for $a\in A$ together with $u_1\otimes S_1,\ldots,$ $u_m\otimes S_m$.

Proposition 4.3 ([12, Theorem 1.5]). *In the situation as above, the following 6term sequence is exact*:

$$\begin{array}{ccccc} K_0(A) & \overset{\mathrm{id}-\sum\alpha_{i*}^{-1}}{\longrightarrow} & K_0(A) & \longrightarrow & K_0(A\times_{\mathcal{U}}\mathcal{O}_m) \\ \uparrow & & & & \downarrow \\ K_1(A\times_{\mathcal{U}}\mathcal{O}_m) & \longleftarrow & K_1(A) & \overset{\mathrm{id}-\sum\alpha_{i*}^{-1}}{\longleftarrow} & K_1(A) \end{array}$$

Here $K_j(A)\to K_j(A\times_{\mathcal{U}}\mathcal{O}_m)$ for $j=0,1$ is the map induced by the natural inclusion of A into $A\times_{\mathcal{U}}\mathcal{O}_m$ via $a\mapsto a\otimes 1$.

We now get an according 6-term exact sequence for $\mathcal{O}_n\otimes_\Theta\mathcal{O}_m$.

Corollary 4.1. *The following sequence in K-theory is exact*:

$$0\to K_1(\mathcal{O}_n\otimes_\Theta\mathcal{O}_m)\to K_0(\mathcal{O}_n)\overset{(m-1)}{\to}K_0(\mathcal{O}_n)\to K_0(\mathcal{O}_n\otimes_\Theta\mathcal{O}_m)\to 0$$

Hence, the K-groups of $\mathcal{O}_n\otimes_\Theta\mathcal{O}_m$ are independent from the parameter Θ.

Proof. In our situation, we put $A:=\mathcal{O}_n$ and $\alpha_j(s_i)=\bar{q}_{ij}s_i$. We may represent $\mathcal{O}_n\rtimes_{\alpha_1}\mathbb{Z}\rtimes_{\alpha_2}\mathbb{Z}\ldots\rtimes_{\alpha_m}\mathbb{Z}$ concretely on some Hilbert space by some representation π which yields unitaries $u_1,\ldots,u_m$ implementing the automorphisms $\alpha_1,\ldots,\alpha_m$. We may thus form $\mathcal{O}_n\times_{\mathcal{U}}\mathcal{O}_m$ in the sense of Cuntz and we obtain a *-homomorphism $\sigma:\mathcal{O}_n\otimes_\Theta\mathcal{O}_m\to\mathcal{O}_n\times_{\mathcal{U}}\mathcal{O}_m$ mapping $s_i\mapsto\pi(s_i)\otimes 1$ and $t_j\mapsto u_j\otimes S_j$. As $\mathcal{O}_n\otimes_\Theta\mathcal{O}_m$ is purely infinite, it is simple in particular. Hence, σ is an isomorphism.

We now apply Proposition 4.3 to $\mathcal{O}_n\otimes_\Theta\mathcal{O}_m\cong\mathcal{O}_n\times_{\mathcal{U}}\mathcal{O}_m$. Since the automorphisms α_i are homotopic to the identity, we get $\alpha_{i*}=\mathrm{id}$ for all $i=1,\ldots,m$ on the level of K-theory. Thus the map $\mathrm{id}-\sum\alpha_{i*}^{-1}$ is multiplication by $(m-1)$ on $K_0(\mathcal{O}_n)=\mathbb{Z}/(n-1)\mathbb{Z}$. As $K_1(\mathcal{O}_n)=0$, we end up with the exact sequence of our assertion.

Now, the map $K_0(\mathcal{O}_n)\overset{(m-1)}{\to}K_0(\mathcal{O}_n)$ is independent from Θ and so are its kernel $K_1(\mathcal{O}_n\otimes_\Theta\mathcal{O}_m)$ and its image $K_0(\mathcal{O}_n\otimes_\Theta\mathcal{O}_m)$. Note that the isomorphisms of $K_0(\mathcal{O}_n\otimes_\Theta\mathcal{O}_m)$ and $K_0(\mathcal{O}_n\otimes_{\Theta'}\mathcal{O}_m)$ for different parameters Θ and Θ' map units to units. □

Remark 4.1. We may use the Künneth formula to compute the K-theory of $\mathcal{O}_m\otimes\mathcal{O}_n$ explicitly. Recall that $\mathcal{O}_m\otimes\mathcal{O}_n$ satisfies the UCT (see Proposition 4.1). The Künneth formula for K-theory, see [3, Theorem 23.1.3], gives the following short exact sequences, $j\in\mathbb{Z}_2$,

$$\begin{aligned}0&\to\bigoplus_{i\in\mathbb{Z}_2}K_i(\mathcal{O}_n)\otimes_{\mathbb{Z}}K_{i+j}(\mathcal{O}_m)\to K_j(\mathcal{O}_n\otimes\mathcal{O}_m)\\&\to\bigoplus_{i\in\mathbb{Z}_2}\mathrm{Tor}_1^{\mathbb{Z}}(K_i(\mathcal{O}_n),K_{i+j+1}(\mathcal{O}_m))\to 0.\end{aligned}$$

Let $d = \gcd(n-1, m-1)$. It is a well-known fact in homological algebra, see [18], that for an abelian group A

$$\mathrm{Tor}_1^{\mathbb{Z}}(A, \mathbb{Z}/d\mathbb{Z}) \simeq \mathrm{Ann}_A(d) = \{a \in A \mid da = 0\}.$$

In particular,

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \simeq \mathbb{Z}/\gcd(n,m)\mathbb{Z}.$$

Recall that, see [11]

$$K_0(\mathcal{O}_n) = \mathbb{Z}/(n-1)\mathbb{Z}, \quad K_1(\mathcal{O}_n) = 0.$$

Hence, for $\mathcal{O}_n \otimes \mathcal{O}_m$, one has the following short exact sequences:

$$0 \to \mathbb{Z}/(n-1)\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/(m-1)\mathbb{Z} \to K_0(\mathcal{O}_n \otimes \mathcal{O}_m) \to 0 \to 0,$$

$$0 \to 0 \to K_1(\mathcal{O}_n \otimes \mathcal{O}_m) \to \mathbb{Z}/d\mathbb{Z} \to 0.$$

This implies

$$K_0(\mathcal{O}_n \otimes \mathcal{O}_m) \simeq \mathbb{Z}/d\mathbb{Z}, \quad K_1(\mathcal{O}_n \otimes \mathcal{O}_m) \simeq \mathbb{Z}/d\mathbb{Z}.$$

4.4. $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ *is isomorphic to* $\mathcal{O}_n \otimes \mathcal{O}_m$

We may now put together all ingredients in order to apply Kirchberg's Theorem [33] which is as follows.

Proposition 4.4. *Let A and B be unital, separable, nuclear, simple and purely infinite C^*-algebras in the bootstrap class $\mathcal{N}$ with $K_j(A) \cong K_j(B)$ for $j = 0, 1$ (with matching units in the case of $j = 0$). Then $A \cong B$.*

Theorem 4.1. *For any parameter Θ, the C^*-algebra $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ is isomorphic to $\mathcal{O}_m \otimes \mathcal{O}_n$. Hence we may not twist the tensor product of two Cuntz algebras in this sense.*

Proof. By Proposition 4.1, $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ is nuclear and in $\mathcal{N}$ and by Proposition 4.2, it is purely infinite and simple. By Corollary 4.1, the K-groups of $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ and $\mathcal{O}_m \otimes \mathcal{O}_n$ coincide. Thus, by Kirchberg's Theorem, $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ is isomorphic to $\mathcal{O}_m \otimes \mathcal{O}_n$. □

Remark 4.2. The above result on $\mathcal{O}_n \otimes_\Theta \mathcal{O}_m$ has been part of the fourth author's PhD thesis from 2011 and it has not been published in any other paper.

4.5. *The isomorphism* $\mathcal{O}_n \otimes_q \mathcal{O}_m \simeq \mathcal{O}_n \otimes \mathcal{O}_m$. *The Rieffel deformation approach*

In this section, we return to one parameter case of $\mathcal{O}_n \otimes_q \mathcal{O}_m$, and present an approach to the proof of isomorphism

$$\mathcal{O}_n \otimes_q \mathcal{O}_m \simeq \mathcal{O}_n \otimes \mathcal{O}_m, \quad |q| < 1,$$

which is based on the properties of Rieffel deformation.

In [20], the authors have shown that for every C^*-algebra $\mathcal{A}$ with an action α of $\mathbb{R}$, there exists a KK-isomorphism $t_\alpha \in KK_1(\mathcal{A}, \mathcal{A} \rtimes_\alpha \mathbb{R})$. This t_α is a generalization of the Connes–Thom isomorphisms for K-theory. In what follows, we will denote by $\circ : KK(\mathcal{A}, \mathcal{B}) \times KK(\mathcal{B}, \mathcal{C}) \to KK(\mathcal{A}, \mathcal{C})$ the Kasparov product, and by $\boxtimes$: $KK(\mathcal{A}, \mathcal{B}) \times KK(\mathcal{C}, \mathcal{D}) \to KK(\mathcal{A} \otimes \mathcal{C}, \mathcal{B} \otimes \mathcal{D})$ the exterior tensor product. Given a homomorphism $\phi : \mathcal{A} \to \mathcal{B}$, put $[\phi] \in KK(\mathcal{A}, \mathcal{B})$ to be the induced KK-morphism. For more details see [3, 25].

We list some properties of t_α that will be used below.

(1) Inverse of t_α is given by $t_{\widehat{\alpha}}$, where $\widehat{\alpha}$ is the dual action.
(2) If $\mathcal{A} = \mathbb{C}$ with the trivial action of $\mathbb{R}$, then the corresponding element

$$t_1 \in KK_1(\mathbb{C}, C_0(\mathbb{R})) \simeq \mathbb{Z}$$

is the generator of the group.
(3) Let $\phi : (\mathcal{A}, \alpha) \to (\mathcal{B}, \beta)$ be an equivariant homomorphism. Then the following diagram commutes in KK-theory:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{t_\alpha} & \mathcal{A} \rtimes_\alpha \mathbb{R} \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi \rtimes \mathbb{R}} \\ \mathcal{B} & \xrightarrow{t_\beta} & \mathcal{B} \rtimes_\beta \mathbb{R} \end{array}$$

(4) Let β be an action of $\mathbb{R}$ on $\mathcal{B}$. For the action $\gamma = \mathrm{id}_\mathcal{A} \otimes \beta$ on $\mathcal{A} \otimes \mathcal{B}$ we have

$$t_\gamma = \mathbf{1}_\mathcal{A} \boxtimes t_\beta.$$

Further, we will need the following version of classification result by Kirchberg and Philips:

Theorem 4.2 ([33, Corollary 4.2.2]). *Let $\mathcal{A}$ and $\mathcal{B}$ be separable nuclear unital purely infinite simple C^*-algebras, and suppose that there exists an invertible element $\eta \in KK(\mathcal{A}, \mathcal{B})$, such that $[\iota_\mathcal{A}] \circ \eta = [\iota_\mathcal{B}]$, where $\iota_\mathcal{A} : \mathbb{C} \to \mathcal{A}$ is defined by $\iota_\mathcal{A}(1) = \mathbf{1}_\mathcal{A}$, and $\iota_\mathcal{B} : \mathbb{C} \to \mathcal{B}$ is defined by $\iota_\mathcal{B}(1) = \mathbf{1}_\mathcal{B}$. Then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.*

Notice that the conditions of the theorem above does not require C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ to be in the bootstrap class $\mathcal{N}$.

Theorem 4.3. *The C^*-algebras $\mathcal{O}_n \otimes_q \mathcal{O}_m$ and $\mathcal{O}_n \otimes \mathcal{O}_m$ are isomorphic for any $|q| = 1$.*

Proof. Throughout the proof, we will distinguish between the actions of $\mathbb{T}^2$ on $\mathcal{O}_n \otimes \mathcal{O}_m$ and on $\mathcal{O}_n \otimes_q \mathcal{O}_m$, denoting the latter by α^q. As shown above, the both algebras are separable nuclear unital simple and purely infinite.

Further, Proposition 3.4, and the isomorphism $\mathcal{O}_n \otimes_q \mathcal{O}_m \simeq (\mathcal{O}_n \otimes \mathcal{O}_m)_{\Theta_q}$ yield the isomorphism

$$\Psi : (\mathcal{O}_n \otimes \mathcal{O}_m) \rtimes_\alpha \mathbb{R}^2 \to (\mathcal{O}_n \otimes_q \mathcal{O}_m) \rtimes_{\alpha^q} \mathbb{R}^2.$$

Decompose the crossed products as follows:

$$(\mathcal{O}_n \otimes \mathcal{O}_m) \rtimes_\alpha \mathbb{R}^2 \simeq (\mathcal{O}_n \otimes \mathcal{O}_m) \rtimes_{\alpha_1} \mathbb{R} \rtimes_{\alpha_2} \mathbb{R},$$
$$(\mathcal{O}_n \otimes_q \mathcal{O}_m) \rtimes_{\alpha^q} \mathbb{R}^2 \simeq (\mathcal{O}_n \otimes_q \mathcal{O}_m) \rtimes_{\alpha_1^q} \mathbb{R} \rtimes_{\alpha_2^q} \mathbb{R}.$$

Define

$$t_\alpha = t_{\alpha_1} \circ (\mathbf{1}_{C_0(\mathbb{R})} \boxtimes t_{\alpha_2}) \in KK(\mathcal{O}_n \otimes \mathcal{O}_m, (\mathcal{O}_n \otimes \mathcal{O}_m) \rtimes_\alpha \mathbb{R}^2),$$
$$t_{\alpha^q} = t_{\alpha_1^q} \circ (\mathbf{1}_{C_0(\mathbb{R})} \boxtimes t_{\alpha_2^q}) \in KK(\mathcal{O}_n \otimes_q \mathcal{O}_m, (\mathcal{O}_n \otimes_q \mathcal{O}_m) \rtimes_{\alpha^q} \mathbb{R}^2),$$

Then

$$\eta = t_{\alpha^q} \circ [\Psi] \circ t_\alpha^{-1} \in KK(\mathcal{O}_n \otimes_q \mathcal{O}_m, \mathcal{O}_n \otimes \mathcal{O}_m)$$

is a KK-isomorphism. The property $[\iota_{\mathcal{O}_n \otimes_q \mathcal{O}_m}] \circ \eta = [\iota_{\mathcal{O}_n \otimes \mathcal{O}_m}]$ follows from the commutativity of the following diagram:

$$\begin{array}{ccccccc}
\mathbb{C} & \xrightarrow{t_1 \circ (\mathbf{1}_{C_0(\mathbb{R})} \boxtimes t_1)} & & C_0(\mathbb{R}^2) & & \xrightarrow{(\mathbf{1}_{C_0(\mathbb{R})} \boxtimes t_1)^{-1} \circ t_1^{-1}} & \mathbb{C} \\
\downarrow \iota_{\mathcal{O}_n \otimes_q \mathcal{O}_m} & & \swarrow \iota_{\mathcal{O}_n \otimes_q \mathcal{O}_m \rtimes \mathbb{R}^2} & & \searrow \iota_{\mathcal{O}_n \otimes \mathcal{O}_m \rtimes \mathbb{R}^2} & & \downarrow \iota_{\mathcal{O}_n \otimes \mathcal{O}_m} \\
\mathcal{O}_n \otimes_q \mathcal{O}_m & \stackrel{t_{\alpha^q}}{\to} & (\mathcal{O}_n \otimes_q \mathcal{O}_m) \rtimes_{\alpha^q} \mathbb{R}^2 & \xrightarrow{\Psi} & (\mathcal{O}_n \otimes \mathcal{O}_m) \rtimes_\alpha \mathbb{R}^2 & \stackrel{t_\alpha^{-1}}{\to} & \mathcal{O}_n \otimes \mathcal{O}_m.
\end{array}$$

□

Remark 4.3. The proof presented above was obtained by the first author independently of the proof of multiparameter case.

4.6. *Computation of* Ext *for* $\mathcal{E}_{n,m}^q$

Let us finish with some remarks on $\mathcal{E}_{n,m}^q$. Firstly, the simplicity of $\mathcal{O}_n \otimes_q \mathcal{O}_m$ implies that $\mathcal{M}_q \subset \mathcal{E}_{n,m}^q$ is the largest ideal.

Corollary 4.2. *The ideal $\mathcal{M}_q \subset \mathcal{E}_{n,m}^q$ is the unique largest ideal.*

Proof. Let $\eta\colon \mathcal{E}_{n,m}^q \to \mathcal{O}_n \otimes_q \mathcal{O}_m$ be the quotient homomorphism. Suppose that $\mathcal{J} \subset \mathcal{E}_{n,m}^q$ is a two-sided $*$-ideal. Due to the simplicity of $\mathcal{O}_n \otimes_q \mathcal{O}_m$ we have that either $\eta(\mathcal{J}) = \{0\}$ and $\mathcal{J} \subset \mathcal{M}_q$, or $\eta(\mathcal{J}) = \mathcal{O}_n \otimes_q \mathcal{O}_m$. In the latter case, $\mathbf{1} + x \in \mathcal{J}$ for a certain $x \in \mathcal{M}_q$. For any $0 < \varepsilon < 1$, choose $N_\varepsilon \in \mathbb{N}$, such that for

$$x_\varepsilon = \sum_{\substack{\varepsilon_1,\varepsilon_2 \in \{0,1\}, \\ \varepsilon_1+\varepsilon_2 \neq 0}} \sum_{\substack{\mu_1,\mu_2 \in \Lambda_n, \\ |\mu_j| \le N_\varepsilon}} \sum_{\substack{\nu_1,\nu_2 \in \Lambda_m, \\ |\nu_j| \le N_\varepsilon}} \Psi^{(\varepsilon_1,\varepsilon_2)}_{\mu_1,\mu_2 \nu_1 \nu_2} s_{\mu_1} t_{\nu_1} (\mathbf{1} - P)^{\varepsilon_1} (\mathbf{1} - Q)^{\varepsilon_2} t_{\nu_2}^* s_{\mu_2^*} \in \mathcal{M}_q$$

one has $\|x - x_\varepsilon\| < \varepsilon$. Notice that for any $\mu \in \Lambda_n$, $\nu \in \Lambda_m$ with $|\mu|, |\nu| > N_\varepsilon$ one has $s_\mu^* t_\nu^* x_\varepsilon = 0$.

Fix $\mu \in \Lambda_n$ and $\nu \in \Lambda_m$, $|\mu| = |\nu| > N_\varepsilon$, then

$$y_\varepsilon = s_\mu^* t_\nu^* (\mathbf{1} - x) t_\nu s_\mu = \mathbf{1} - s_\mu^* t_\nu^* (x - x_\varepsilon) t_\nu s_\mu \in \mathcal{J}.$$

Thus $\|s_\mu^* t_\nu^* (x - x_\varepsilon) t_\nu s_\mu\| < \varepsilon$ implies that y_ε is invertible, so $\mathbf{1} \in \mathcal{J}$. □

Second, we may show that $\mathsf{Ext}(\mathcal{O}_n \otimes_q \mathcal{O}_m, \mathcal{M}_q) = 0$ if $\gcd(n-1, m-1) = 1$. To this end we compute first the K-theory of $\mathcal{M}_q$.

Theorem 4.4. *Let* $d = \gcd(n-1, m-1)$. *Then*

$$K_0(\mathcal{M}_q) \simeq \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}, \ \ K_1(\mathcal{M}_q) \simeq 0.$$

Proof. The isomorphism $\mathcal{E}^q_{n,m} \simeq (\mathcal{O}_n^{(o)} \otimes \mathcal{O}_m^{(0)})_{\Theta_q}$, Proposition 3.6, and [11, Proposition 3.9], imply that

$$K_0(\mathcal{E}^q_{n,m}) = K_0((\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)})_{\Theta_q}) = K_0(\mathcal{O}_n^0 \otimes \mathcal{O}_m^0) = \mathbb{Z},$$

$$K_1(\mathcal{E}^q_{n,m}) = K_1((\mathcal{O}_n^{(0)} \otimes \mathcal{O}_m^{(0)})_{\Theta_q}) = K_1(\mathcal{O}_n^0 \otimes \mathcal{O}_m^0) = 0.$$

Applying the 6-term exact sequence for

$$0 \to \mathbb{K} \to \mathcal{M}_q \to \mathcal{O}_n \otimes \mathbb{K} \oplus \mathcal{O}_m \otimes \mathbb{K} \to 0,$$

we get

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & K_0(\mathcal{M}_q) & \longrightarrow & \mathbb{Z}/(n-1)\mathbb{Z} \oplus \mathbb{Z}/(m-1)\mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(\mathcal{M}_q) & \longleftarrow & 0 \end{array}$$

Then $K_1(\mathcal{M}_q) = 0$, and elementary properties of finitely generated abelian groups imply that

$$K_0(\mathcal{M}_q) = \mathbb{Z} \oplus \mathsf{Tors},$$

where Tors is a direct sum of finite cyclic groups.

Further, the following exact sequence:

$$0 \to \mathcal{M}_q \to \mathcal{E}^q_{n,m} \to \mathcal{O}_n \otimes_q \mathcal{O}_m \to 0$$

gives

$$\begin{array}{ccccc} K_0(\mathcal{M}_q) & \xrightarrow{p} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/d\mathbb{Z} \\ {\scriptstyle i}\uparrow & & & & \downarrow \\ \mathbb{Z}/d\mathbb{Z} & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

The map $p : K_0(\mathcal{M}_q) \simeq \mathbb{Z} \oplus \mathsf{Tors} \to \mathbb{Z}$ has form $p = (p_1, p_2)$, where

$$p_1 : \mathbb{Z} \to \mathbb{Z}, \quad p_2 : \mathsf{Tors} \to \mathbb{Z}.$$

Evidently, $p_2 = 0$, and $p \neq 0$ implies that $\ker p_1 = \{0\}$. Thus,

$$\ker p = \mathsf{Tors} = \mathrm{Im}(i) \simeq \mathbb{Z}/d\mathbb{Z}.$$ □

Theorem 4.5. *Let* $d = \gcd(n-1, m-1) = 1$. *Then* $\mathsf{Ext}(\mathcal{O}_n \otimes_q \mathcal{O}_m, \mathcal{M}_q) = 0$.

Proof. Recall that for nuclear C^*-algebras $\mathsf{Ext}(\mathcal{A}, \mathcal{B}) \simeq KK_1(\mathcal{A}, \mathcal{B})$.

We use the UCT sequence

$$0 \to \bigoplus_{i\in\mathbb{Z}_2} \mathrm{Ext}^1_{\mathbb{Z}}(K_i(\mathcal{A}), K_i(\mathcal{B})) \to KK_1(\mathcal{A}, \mathcal{B}) \to \bigoplus_{i\in\mathbb{Z}_2} \mathrm{Hom}(K_i(\mathcal{A}), K_{i+1}(\mathcal{B})).$$

for $\mathcal{A} = \mathcal{O}_n \otimes_q \mathcal{O}_m$ and $\mathcal{B} = \mathcal{M}_q$.

Since $K_0(\mathcal{A}) = K_1(\mathcal{A}) = \mathbb{Z}/d\mathbb{Z}$ and $K_0(\mathcal{B}) = \mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$, $K_1(\mathcal{B}) = 0$, one has

$$\mathrm{Hom}(K_0(\mathcal{A}), K_1(\mathcal{B})) = 0, \quad \mathrm{Hom}(K_1(\mathcal{A}), K_0(\mathcal{B})) = \mathbb{Z}/d\mathbb{Z},$$

and, see [18],

$$\mathrm{Ext}^1_{\mathbb{Z}}(K_0(\mathcal{A}), K_0(\mathcal{B})) = \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}, \quad \mathrm{Ext}^1_{\mathbb{Z}}(K_1(\mathcal{A}), K_1(\mathcal{B})) = 0.$$

Hence the following sequence is exact:

$$0 \to \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z} \to KK_1(\mathcal{O}_n \otimes_q \mathcal{O}_m, \mathcal{M}_q) \to \mathbb{Z}/d\mathbb{Z} \to 0. \qquad \square$$

By Theorem 4.5, for the case of $\gcd(n-1, m-1) = 1$ one can immediately deduce that extension classes of

$$0 \to \mathcal{M}_q \to \mathcal{E}^q_{n,m} \to \mathcal{O}_n \otimes_q \mathcal{O}_m \to 0,$$

and

$$0 \to \mathcal{M}_1 \to \mathcal{E}^1_{n,m} \to \mathcal{O}_n \otimes \mathcal{O}_m \to 0,$$

coincide in $\mathsf{Ext}(\mathcal{O}_n \otimes \mathcal{O}_m, \mathcal{M}_1)$ and are trivial. These extensions are essential, however in general case one does not have an immediate generalization of Proposition 3.12. Thus, the study of the problem whether $\mathcal{E}^q_{n,m} \simeq \mathcal{E}^1_{n,m}$ would require further investigations, see [13, 19].

Acknowledgments

The work on the paper was initiated during the visit of V. Ostrovskyi, D. Proskurin and R. Yakymiv to Chalmers University of Technology. We appreciate the working atmosphere and stimulating discussions with Prof. Lyudmila Turowska and Prof. Magnus Goffeng. We are grateful to Prof. Pawel Kasprzak for discussions on properties of Rieffels deformation and to Prof. Eugene Lytvynov for discussions on generalized statistics and his kind hospitality during the visit of D. Proskurin to Swansea University. We also indebted to Prof. K. Iusenko for helpful comments and remarks. M. Weber has been supported by *SFB-TRR 195* and the *DFG*.

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