Canonicity and homotopy canonicity for cubical type theory

(article starts on next page)

CANONICITY AND HOMOTOPY CANONICITY
FOR CUBICAL TYPE THEORY

THIERRY COQUAND [a], SIMON HUBER [a], AND CHRISTIAN SATTLER [b]

[a] University of Gothenburg, Sweden
 e-mail address: coquand@chalmers.se, simonh@fripost.org

[b] Chalmers University of Technology, Sweden
 e-mail address: sattler@chalmers.se

ABSTRACT. Cubical type theory provides a constructive justification of homotopy type theory. A crucial ingredient of cubical type theory is a path lifting operation which is explained computationally by induction on the type involving several non-canonical choices. We present in this article two canonicity results, both proved by a sconing argument: a *homotopy canonicity* result, every natural number is *path equal* to a numeral, even if we take away the equations defining the lifting operation on the type structure, and a *canonicity* result, which uses these equations in a crucial way. Both proofs are done internally in a presheaf model.

INTRODUCTION

This article is a contribution to the analysis of the computational content of the univalence axiom [Voe14] (and higher inductive types). In previous work [ABC+21, BCH14, CCHM18, CHM18, OP16], various *presheaf models* of this axiom have been described in a constructive metatheory. In this formalism, the notion of fibrant type is stated as a refinement of the path lifting operation where one not only provides one of the endpoints but also a partial lift (for a suitable notion of partiality). This generalized form of path lifting operation is a way to state a homotopy extension property, which was recognized very early (see e.g. [Eil39]) as a key for an abstract development of algebraic topology. The axiom of univalence is then captured by a suitable *equivalence extension operation* (the "glueing" operation), which expresses that we can extend a partially defined equivalence of a given total codomain to a total equivalence. These presheaf models suggest possible extensions of type theory where we manipulate higher dimensional objects [ABC+21, CCHM18]. One can define a notion of reduction and prove canonicity for this extension [Hub19]: any closed term of type N (natural number) is convertible to a numeral. There are however several *non-canonical* choices when defining the path lifting operation by induction on the type, which produce different notion of convertibility.[1] A natural question is how *essential* these non-canonical choices are: can it be that a closed term of type N, defined without use of such non-canonical reduction rules,

---

Key words and phrases: cubical type theory, univalence, canonicity, sconing, Artin glueing.

[1]For instance, the definition of this operation for "glue" types is different in [CCHM18] and [OP16].

becomes convertible to 0 for one choice and 1 for another? The main result of this article, the *homotopy canonicity theorem*, implies that this cannot be the case: the value of a term is independent of these non-canonical choices. Homotopy canonicity states that, even without providing reduction rules for path lifting operations at type formers, we *still* have that any closed term of type N is *path equal* to a numeral. (We cannot hope to have convertibility anymore with these path lifting constants.) We can then see this numeral as the "value" of the given term.

Our proof of homotopy canonicity can be seen as a proof-relevant extension of the *reducibility* or *computability* method, going back to the work of Gödel [Göd58] and Tait [Tai67]. It is however best expressed in an *algebraic* setting. We first define a general notion of model, called *cubical category with families*, defined as a category with families [Dyb96] with certain special operations internal to presheaves over a category $\mathcal{C}$ (such as a cube category) with respect to the parameters of an *interval* $\mathbb{I}$ and a *cofibration classifier* $\mathbb{F}$. In this article, we will work with models of the cubical type theory described by [CCHM18, OP16]. However, our methods apply equally well to other versions of cubical type theories that can be presented in a similar setting, for example [ABC$^+$21].

We describe the term model and how to re-interpret the cubical presheaf models as cubical categories with families. The computability method can then be expressed as a general operation (called "scoining") which applied to an arbitrary model $\mathcal{M}$ produces a new model $\mathcal{M}^*$ with a strict morphism $\mathcal{M}^* \to \mathcal{M}$. Homotopy canonicity is obtained by applying this general operation to the initial model, which we conjecture to be the term model. This construction associates to a (for simplicity, closed) type A a predicate A' on the closed terms $|A|$ and each closed term u of A a proof u' of $A'\,u$. The main rules in the closed case are summarized in Figure 1.

We explain next how a similar method can be used to prove *canonicity* (or "strict" canonicity) when we add computation rules of filling at type formers (using as primitive the operation of *composition*). Here, every closed term of type N is (strictly) equal (as opposed to path equal) to a numeral. This was originally proved by [Hub19]. The main advantage of the present approach is that we don't need to define an auxiliary reduction relation, and that it is independent of the exact choice of the equational presentation of cubical type theory.

Some extensions and variations are then described:

- Our development extends uniformly to identity types and higher inductive types (using the methods of [CHM18]) (Subsections 5.1 and 5.2).
- Our development applies equally to the case where one treats univalence instead of glue types as primitive (Appendix C.1). We expect that a similar scoining argument (glueing along a global sections functor to simplicial sets) works to establish homotopy canonicity for the initial split univalent simplicial tribe in the setting of Joyal [Joy17].
- Assuming excluded middle, a version of the simplicial set model [KL12] forms an instance of our development, and distributive lattice cubical type theory interprets in it (Appendix D).

Using our technique, one may also reprove canonicity for ordinary Martin-Löf type theory with inductive families in a reduction-free way.

Shulman [Shu15] proves homotopy canonicity for homotopy type theory with a truncatedness assumption using the scoining technique. This proof was one starting point for the present work.

Two models of type theory and scoining. Since Martin Hofmann's work [Hof97], it is known how to interpret *extensional* type theory with a hierarchy of universes $\mathcal{U}_0, \mathcal{U}_1, \ldots$ in

$$\Pi(A, B)'(w) \;=\; \textstyle\prod_{u:|A|} \prod_{u':A'\, u} B'\, u\, u'\, (\mathtt{app}(w, u))$$

$$\Sigma(A, B)'(w) \;=\; \textstyle\sum_{u':A'\, (\mathtt{fst}(w))} B'\, (\mathtt{fst}(w))\, u'\, (\mathtt{snd}(w))$$

$$\mathtt{Path}(A, a_0, a_1)'(w) \;=\; \mathtt{Path}_{\lambda_i\, A'\, i\, (\mathtt{ap}(w, i))}\, a_0'\, a_1'$$

$$\mathtt{Glue}_c(A, \psi \mapsto (B, w))'(v) \;=\; \mathsf{Glue}\, \big(A'\, (\mathtt{app}(\mathtt{unglue}, v)))\, \big[\psi \mapsto (B'\, v, (w'.1\, v, \ldots))\big]$$

Figure 1: These are the main rules for the computability predicate component in the sconing models for homotopy canonicity and canonicity in the case of the global context. The component relating to fibrancy differs between the two cases.

any presheaf model. As explained in [CCHM18, OP16, Coq18], in some class of presheaf models, parametrised by two presheaves $\mathbb{I}$ (representing an abstract interval) and $\mathbb{F}$ (the cofibration classifier), it is possible to define, as an internal model inside this presheaf model, a model of type theory with a hierarchy of universes $\mathcal{U}_0^{\mathsf{fib}}, \mathcal{U}_1^{\mathsf{fib}}, \ldots$ satisfying the *univalence* axiom. Both models are carried out in a *constructive* metalanguage. In particular, the second model provides a *computational* interpretation of univalence.

This model of univalence is a model of *cubical type theory* where each type has a *filling operation*. Univalence is then a *theorem* and not an axiom of cubical type theory. This filling operation is defined by induction on the type, using a more primitive *composition operation*.

The basic scheme for a canonicity proof that we follow here is to associate by induction on a type A a computability predicate A' on the (internal) set $|A|$ of closed elements of this type. As explained in [Coq19], this so-called sconing interpretation for canonicity goes back to Gödel's notion of *computability* predicates [Göd58], with the crucial feature here that these predicates are now proof-relevant. This scheme works as well for cubical type theory if we use a metalanguage with an interval object.

We now explain in general terms and by example the differences between the *homotopy canonicity* and the *canonicity* proofs. For the *homotopy canonicity* proof, we will have $A' : |A| \to \mathcal{U}_n^{\mathsf{fib}}$, while for the *canonicity* proof, we will have $A' : |A| \to \mathcal{U}_n$ with a separate component tracking computability of composition.

For the type of natural number $\mathbb{N}$, for the *canonicity* proof, we define $\mathbb{N}'\, t$ to be $\sum_{k:\mathbb{N}} t =_{|\mathbb{N}|} \mathsf{S}^k(0)$, where $t =_{|\mathbb{N}|} \mathsf{S}^k(0)$ is (strict) equality, and $\mathbb{N}$ is the constant presheaf of natural numbers. In this case, $\mathbb{N}'$ is not a fibrant family over $|N|$. For *homotopy canonicity*, we have to define $\mathbb{N}'\, t$ as a *fibrant* family over $|N|$ (see Subsection 3.2.4).[2]

One key step in both arguments is in ensuring that the *filling operation* is a *computable* operation. This is solved in very different ways for the two theories. For the *homotopy canonicity* proof, where $A' : |A| \to \mathcal{U}_n^{\mathsf{fib}}$, we can prove directly that the filling operation is computable without needing information on how the filling operation behaves at individual type formers. For the *canonicity* proof, where $A' : |A| \to \mathcal{U}_n$, the filling operation is defined in terms of a more primitive composition operation and we prove by induction on the type that this composition operation is computable.

---

[2]This fibrant family is not simply obtained by replacing the equality $t =_{|\mathbb{N}|} \mathsf{S}^k(0)$ by a path, as this would not model the β-equality of the eliminator in the successor case. Instead, we should view $\mathbb{N}'$ in the case of canonicity as an indexed inductive set and then replace it by a fibrant indexed inductive set for the case of homotopy canonicity.

As in [Coq19], we think that this interpretation is best described in an *algebraic* way, using what is essentially a generalized algebraic presentation of type theory. The difference with [Coq19] is that the notion of generalized algebraic theory we are using is now developed internally to a presheaf model with an interval $\mathbb{I}$ and cofibration classifier $\mathbb{F}$.

Setting. We work in a constructive set theory (as presented e.g. in [Acz99]) with a sufficiently long cumulative hierarchy of Grothendieck universes. However, our constructions are not specific to this setting and can be replayed in other constructive metatheories such as extensional type theory. In Appendix D, we assume classical logic for the discussion of models in simplicial sets.

1. Cubical categories with families

We first recall the notion of categories with families (cwf) [Dyb96] equipped with Π- and Σ-types, universes, and natural number types. This notion can be interpreted in any presheaf model. In that setting, we can consider new operations. A *cubical cwf* will be such a cwf in a presheaf model with extra operations that make use of an interval object $\mathbb{I}$ and a cofibration classifier $\mathbb{F}$ as introduced in [CHM18, OP16].

1.1. **Categories with families.** Categories with families form an algebraic notion of model of type theory. In order to simplify the treatment of universes, we define them in a stratified manner where instead of a single presheaf of types, we specify a filtration of presheaves of "small" types.[3] The length of the filtration is not essential: we have chosen $\omega + 1$ so that we may specify constructions just at the top level.

A *category with families* (cwf) consists of the following data.

- We have a category of *contexts* Con and *substitutions* $\mathsf{Hom}(\Delta, \Gamma)$ from Δ to Γ in Con. The identity substitution on Γ in Con is written id, and the composition of δ in $\mathsf{Hom}(\Theta, \Delta)$ and σ in $\mathsf{Hom}(\Delta, \Gamma)$ is written $\sigma\delta$.
- We have a presheaf Type of *types* over the category of contexts. The action of σ in $\mathsf{Hom}(\Delta, \Gamma)$ on a type A over Γ is written $A\sigma$. We have a cumulative sequence of sub-presheaves Type_n of *types of level n* of Type where n is a natural number.
- We have a presheaf Elem of *elements* over the category of elements of Type, i.e. a set $\mathsf{Elem}(\Gamma, A)$ for A in $\mathsf{Type}(\Gamma)$ with $a\sigma$ in $\mathsf{Elem}(\Delta, A\sigma)$ for a in $\mathsf{Elem}(\Gamma, A)$ and σ in $\mathsf{Hom}(\Delta, \Gamma)$ satisfying evident laws.
- We have a terminal context 1, with the unique element of $\mathsf{Hom}(\Gamma, 1)$ written $()$.
- Given A in $\mathsf{Type}(\Gamma)$, we have a *context extension* $\Gamma.A$. There is a *projection* p in $\mathsf{Hom}(\Gamma.A, \Gamma)$ and a *generic term* q in $\mathsf{Elem}(\Gamma.A, A\mathsf{p})$. Given σ in $\mathsf{Hom}(\Delta, \Gamma)$, A in $\mathsf{Type}(\Gamma)$, and a in $\mathsf{Elem}(\Delta, A\sigma)$ we have a *substitution extension* (σ, a) in $\mathsf{Hom}(\Delta, \Gamma.A)$. These operations satisfy $\mathsf{p}(\sigma, a) = \sigma$, $\mathsf{q}(\sigma, a) = a$, and $(\mathsf{p}\sigma, \mathsf{q}\sigma) = \sigma$. Thus, every element of $\mathsf{Hom}(\Delta, \Gamma.A)$ is uniquely of the form (σ, a) with σ and a as above.

---

[3] We note that this, some might say, non-algebraic aspect of the definition does not interfere with the otherwise algebraic character and that subsets could in principle be replaced by injections. Indeed, one can even relax the requirement that $\mathsf{Type}_n \to \mathsf{Type}$ is a monomorphism, at the cost of making it more tedious to state coherence of type formers under lifting and level coercion (if desired). One can also give a version where there is no top-level presheaf of types Type. None of these variations impact what we do in this article.

We introduce some shorthand notation related to substitution. Given σ in $\mathsf{Hom}(\Delta, \Gamma)$ and A in $\mathsf{Type}(\Gamma)$, we write $\sigma^+ = (\sigma\mathsf{p}, \mathsf{q})$ in $\mathsf{Hom}(\Delta.A\sigma, \Gamma.A)$. Given a in $\mathsf{Elem}(\Gamma, A)$, we write $[a] = (\mathsf{id}, a)$ in $\mathsf{Hom}(\Gamma, \Gamma.A)$. Thus, given B in $\mathsf{Type}_n(\Gamma.A)$ and a in $\mathsf{Elem}(\Gamma, A)$, we have $B[a]$ in $\mathsf{Type}(\Gamma)$. Given furthermore b in $\mathsf{Elem}(\Gamma.A, B)$, we have $b[a]$ in $\mathsf{Elem}(\Gamma, B[a])$. We extend this notation to several arguments: given a_i in $\mathsf{Elem}(\Gamma, A_i)$ for $1 \le i \le k$, we write $[a_1, \dots, a_k]$ for $[a_k][a_{k-1}\mathsf{p}] \cdots [a_1\mathsf{p}\dots\mathsf{p}]$ in $\mathsf{Hom}(\Gamma, \Gamma.A_1.\dots.A_k)$.

Note that we could take a different equational presentation. For instance, the presentation in [Ehr88] takes as primitive the operations σ^+ and $[u]$ and *defines* then (σ, u) as a derived operation $(\sigma, u) = \sigma^+[u]$. It is a strength of the present approach to canonicity proof to be independent of this choice.

Given a cwf as above, we define what it means to have the following type formers. In addition to the specified laws, all specified operations are furthermore required to be stable under substitution in the evident manner.

- **Dependent products.** For A in $\mathsf{Type}(\Gamma)$ and B in $\mathsf{Type}(\Gamma.A)$, we have $\Pi(A, B)$ in $\mathsf{Type}(\Gamma)$, of level n if A and B are. Given b in $\mathsf{Elem}(\Gamma.A, B)$, we have the *abstraction* $\lambda(b)$ in $\mathsf{Elem}(\Gamma, \Pi(A, B))$. Given c in $\mathsf{Elem}(\Gamma, \Pi(A, B))$ and a in $\mathsf{Elem}(\Gamma, A)$, we have the *application* $\mathtt{app}(c, a)$ in $\mathsf{Elem}(\Gamma, B[a])$. These operations satisfy

$$\mathtt{app}(\lambda(b), a) = b[a],$$
$$\lambda(\mathtt{app}(c\mathsf{p}, \mathsf{q})) = c.$$

  Given A and B in $\mathsf{Type}(\Gamma)$ we write $A \to B$ for $\Pi(A, B\mathsf{p})$.

- **Dependent sums.** For A in $\mathsf{Type}(\Gamma)$ and B in $\mathsf{Type}(\Gamma.A)$, we have $\Sigma(A, B)$ in $\mathsf{Type}(\Gamma)$, of level n if A and B are. Given a in $\mathsf{Elem}(\Gamma, A)$ and b in $\mathsf{Elem}(\Gamma, B[a])$, we have the *pairing* $\mathtt{pair}(a, b)$ in $\mathsf{Elem}(\Gamma, \Sigma(A, B))$. Given c in $\mathsf{Elem}(\Gamma, \Sigma(A, B))$, we have the *first projection* $\mathtt{fst}(c)$ in $\mathsf{Elem}(\Gamma, A)$ and *second projection* $\mathtt{snd}(c)$ in $\mathsf{Elem}(\Gamma, B[\mathtt{fst}(c)])$. These operations satisfy

$$\mathtt{fst}(\mathtt{pair}(a, b)) = a,$$
$$\mathtt{snd}(\mathtt{pair}(a, b)) = b,$$
$$\mathtt{pair}(\mathtt{fst}(c), \mathtt{snd}(c)) = c.$$

Thus, every element of $\mathsf{Elem}(\Gamma, \Sigma(A, B))$ is uniquely of the form $\mathtt{pair}(a, b)$ with a and b as above.

  Given A and B in $\mathsf{Type}(\Gamma)$ we write $A \times B$ for $\Sigma(A, B\mathsf{p})$.

- **Universes.** We have $\mathtt{U}_n$ in $\mathsf{Type}_{n+1}(\Gamma)$ and an isomorphism $\mathsf{Type}_n(\Gamma) \cong \mathsf{Elem}(\Gamma, \mathtt{U}_n)$, naturally in Γ.[4]

- **Natural numbers.** We have $\mathtt{N}$ in $\mathsf{Type}_0(\Gamma)$ with *zero* $\mathtt{0}$ in $\mathsf{Elem}(\Gamma, \mathtt{N})$ and *successor* $\mathtt{S}(n)$ in $\mathsf{Elem}(\Gamma, \mathtt{N})$ for n in $\mathsf{Elem}(\Gamma, \mathtt{N})$. Given P in $\mathsf{Type}(\Gamma.\mathtt{N})$, z in $\mathsf{Elem}(\Gamma, P[\mathtt{0}])$, s in $\mathsf{Elem}(\Gamma.\mathtt{N}.P, P(\mathsf{p}, \mathtt{S}(\mathsf{q}))\mathsf{p})$, and $n : \mathsf{Elem}(\Gamma, \mathtt{N})$, we have the *elimination* $\mathtt{natrec}(P, z, s, n)$ in $\mathsf{Elem}(\Gamma, P[n])$ with

$$\mathtt{natrec}(P, z, s, \mathtt{0}) = z,$$
$$\mathtt{natrec}(P, z, s, \mathtt{S}(n)) = s[n, \mathtt{natrec}(P, z, s, n)].$$

A *structured cwf* is a cwf with type formers as above.

---

[4]This presents Tarski-style universes. For Russell-style universes, we would additionally demand that this isomorphism is an identity.

A (strict) morphism $\mathcal{M} \to \mathcal{N}$ of cwfs is defined in the evident manner and consists of a functor $F\colon \mathsf{Con}_{\mathcal{M}} \to \mathsf{Con}_{\mathcal{N}}$ and natural transformations $u\colon \mathsf{Type}_{\mathcal{M}} \to \mathsf{Type}_{\mathcal{N}}F$ and $v\colon \mathsf{Elem}_{\mathcal{M}} \to \mathsf{Elem}_{\mathcal{N}}(F, u)$ such that v restricts to types of level n and the terminal context and context extension is preserved strictly. A morphism $\mathcal{M} \to \mathcal{N}$ of structured cwfs additionally preserves the operations of the above type formers. We obtain a category of structured cwfs.

1.2. Internal language of presheaves.

For the rest of the article, we fix a category $\mathcal{C}$ in the lowest Grothendieck universe. As in [ABC+21, OP16, LOPS18], we will use the language of extensional type theory (with subtypes) to describe constructions in the presheaf topos over $\mathcal{C}$.

In the interpretation of this language, a context is a presheaf A over $\mathcal{C}$, a type B over A is a presheaf over the category of elements of A, and an element of B is a section. A *global type* is a type in the global context, i.e. a presheaf over $\mathcal{C}$. Similarly, a *global element* of a global type is a section of that presheaf.

For elements x and y of a type A, we have the equality type $x =_A y$, satisfying reflection (we allow ourselves to omit the subscript A if it is evident from the context). Given a dependent type B over a type A, we think of B as a family of types $B\,a$ indexed by elements a of A. We have the usual dependent sum $\sum_{a:A} B\,a$ and dependent product $\prod_{a:A} B\,a$, with projections of $s : \sum_{a:A} B\,a$ written $s.1 : A$ and $s.2 : B\,s.1$, and application of $f : \prod_{a:A} B\,a$ to $a : A$ written $f\,a$. We have also the categorical pairing $\langle f, g \rangle : X \to \sum_{a:A} B\,a$ given $f : X \to A$ and $g : \prod_{x:X} B\,(f\,a)$ and other commonly used notations. The hierarchy of Grothendieck universes in the ambient set theory gives rise to a cumulative hierarchy $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_\omega$ of universes à la Russell. We model propositions as subtypes of a fixed type 1 with unique element tt. This implies that logically equivalent propositions are equal. We have subuniverses $\Omega_i \subseteq \mathcal{U}_i$ of propositions for $i \in \{0, 1 \ldots, \omega\}$.

When working in this internal language, we refer to the types as "sets" to avoid ambiguity with the types of (internal) cwfs we will be considering.

1.3. Cubical categories with families.

We now work internally to presheaves over $\mathcal{C}$. We assume the following:

- an *interval* $\mathbb{I} : \mathcal{U}_0$ with *endpoints* $0, 1 : \mathbb{I}$,
- an *cofibration classifier* consisting of $\mathbb{F} : \mathcal{U}_0$ with a monomorphism $[-] : \mathbb{F} \to \Omega_0$.[5]

As in [CHM18, OP16], a *partial element* of a set T is given by an element φ in $\mathbb{F}$ and a function $[\varphi] \to T$. We say that a total element v of T extends such a partial element φ, u if we have $[\varphi] \to u\,\mathsf{tt} = v$. (Note that the last equation make sense because $[\varphi] = \mathsf{tt}$ as soon as $[\varphi]$ is inhabited).

Given $A : \mathbb{I} \to \mathcal{U}_\omega$, we write $\mathsf{hasFill}(A)$ for the set of operations taking as inputs φ in $\mathbb{F}$, $b \in \{0, 1\}$, and a partial section u in $\prod_{i:\mathbb{I}} [\varphi] \vee (i = b) \to A\,i$ and producing an extension of u to a total section in $\prod_{i:\mathbb{I}} A\,i$. Given a set X and $Y : X \to \mathcal{U}_\omega$, we write $\mathsf{Fill}(X, Y)$ for the set of *filling structures* on Y, producing an element of $\mathsf{hasFill}(Y \circ x)$ for x in $\mathbb{I} \to X$. Given s in $\mathsf{Fill}(X, Y)$ and x, φ, b, u as above, we write $s(x, \varphi, b, u)$ for the resulting total section in $\prod_{i:\mathbb{I}} Y\,(x\,i)$.

---

[5]The requirement that $[-]$ is mono is not essential and can be relaxed. However, this comes at the cost of making later conditions on $\mathbb{F}$ more tedious to state.

We now interpret the definitions of Subsection 1.1 in the internal language of the presheaf topos. A *cubical cwf* is a structured cwf denoted as before that additionally has the following *cubical operations and type formers.* Again, all specified operations are required to be stable under substitution.

- **Filling operation.** We have `fill` in $\mathsf{Fill}(\mathsf{Type}(\Gamma), \lambda_A \mathsf{Elem}(\Gamma, A))$ for Γ in Con. Let us spell out stability under substitution: given $A \colon \mathbb{I} \to \mathsf{Type}(\Gamma)$, φ in $\mathbb{F}$, $b \in \{0, 1\}$, u in $\prod_{i:\mathbb{I}} [\varphi] \vee (i = b) \to \mathsf{Elem}(\Gamma, A\, i)$, and σ in $\mathsf{Hom}(\Delta, \Gamma)$ and $r : \mathbb{I}$, we have

$$(\mathtt{fill}(A, \varphi, b, u)\, r)\sigma = \mathtt{fill}(\lambda_i\, (A\, i)\sigma, \varphi, b, \lambda_{i,x}\, (u\, i\, x)\sigma)\, r.$$

Note that we do not include computation rules for `fill` at type formers. This corresponds to our decision to treat `fill` as a non-canonical operation.

- **Dependent path types.** Given A in $\mathbb{I} \to \mathsf{Type}(\Gamma)$ with a_b in $\mathsf{Elem}(\Gamma, A\, b)$ for $b \in \{0, 1\}$, we have $\mathtt{Path}(A, a_0, a_1)$ in $\mathsf{Type}(\Gamma)$, of level n if A is. Given u in $\prod_{i:\mathbb{I}} \mathsf{Elem}(\Gamma, A\, i)$, we have the *path abstraction* $\langle\rangle(u)$ in $\mathsf{Elem}(\Gamma, \mathtt{Path}(A, u\, 0, u\, 1))$. Given p in $\mathsf{Elem}(\Gamma, \mathtt{Path}(A, a_0, a_1))$ and i in $\mathbb{I}$, we have the *path application* $\mathtt{ap}(p, r)$ in $\mathsf{Elem}(\Gamma, A\, i)$. These operations satisfy the laws

$$\mathtt{ap}(p, b) = a_b,$$
$$\mathtt{ap}(\langle\rangle(u), i) = u\, i,$$
$$\langle\rangle(\lambda_i\, \mathtt{ap}(p, i)) = p.$$

Thus, every element of $\mathsf{Elem}(\Gamma, \mathtt{Path}(A, a_0, a_1))$ is uniquely of the form $\langle\rangle(u)$ with u in $\prod_{i:\mathbb{I}} \mathsf{Elem}(\Gamma, A\, i)$ such that $u\, 0 = a_0$ and $u\, 1 = a_1$.

Using path types, we define $\mathtt{isContr}_c(A)$ in $\mathsf{Type}(\Gamma)$ for A in $\mathsf{Type}(\Gamma)$ as well as $\mathtt{isEquiv}_c$ in $\mathsf{Type}(\Gamma.A \to B)$ and $\mathtt{Equiv}_c$ in $\mathsf{Type}(\Gamma)$ for A, B in $\mathsf{Type}(\Gamma)$ as in [CCHM18]. (We use a subscript here and for some other notions to distinguish them from analogous notions defined later in a different setting in Subsection 2.2.) These notions are used in the following type former, which extends any partially defined equivalence (given total codomain) to a totally defined function.

- **Glue types.** Given A in $\mathsf{Type}(\Gamma)$, φ in $\mathbb{F}$, T in $[\varphi] \to \mathsf{Type}(\Gamma)$, and

$$e : [\varphi] \to \mathsf{Elem}(\Gamma, \mathtt{Equiv}_c(T\, \mathtt{tt}, A)),$$

we have the *glueing* $\mathtt{Glue}_c(A, \varphi, T, e)$ in $\mathsf{Type}(\Gamma)$, equal to T on $[\varphi]$ and of level n if A and T are. We have $\mathtt{unglue}$ in $\mathsf{Elem}(\Gamma, \mathtt{Glue}_c(A, \varphi, T, e) \to A)$ such that $\mathtt{unglue} = \mathtt{fst}(e)\, \mathtt{tt}$ on $[\varphi]$. Given a in $\mathsf{Elem}(\Gamma, A)$ and t in $[\varphi] \to \mathsf{Elem}(\Gamma, T)$ such that $\mathtt{app}(\mathtt{fst}(e)\, \mathtt{tt}, t\, \mathtt{tt}) = a$ on $[\varphi]$, we have $\mathtt{glue}(a, t)$ in $\mathsf{Elem}(\Gamma, \mathtt{Glue}_c(A, \varphi, T, e))$ equal to t on $[\varphi]$. These operations satisfy

$$\mathtt{app}(\mathtt{unglue}, \mathtt{glue}(a, t)) = a,$$
$$\mathtt{glue}(\mathtt{app}(\mathtt{unglue}, u), \lambda_x\, u) = u.$$

Thus, every element of $\mathsf{Elem}(\Gamma, \mathtt{Glue}_c(A, \varphi, T, e))$ is uniquely of the form $\mathtt{glue}(a, t)$ with a and t as above.

The notion of morphism of structured cwfs lifts to an evident notion of morphism of cubical cwfs. We obtain, internally to presheaves over $\mathcal{C}$, a category of cubical cwfs. We now lift this category of cubical cwfs from the internal language to the ambient theory by interpreting it in the global context: externally, a cubical cwf (relative to the chosen

base category $\mathcal{C}$, interval $\mathbb{I}$, and cofibration classifier $\mathbb{F}$) consists of a presheaf Con over $\mathcal{C}$, a presheaf Type over the category of elements of Con, etc.

Remark 1.1. Fix a cubical cwf as above. Assume that $\mathbb{I}$ has a connection algebra structure and that $\mathbb{F}$ forms a sublattice of Ω_0 that contains the interval endpoint inclusions. As in [CCHM18], it is then possible in the above context of the glue type former to construct an element of $\mathsf{Elem}(\Gamma, \mathtt{isEquiv}_c[\mathtt{unglue}])$. From this, one derives an element of $\mathsf{Elem}(\Gamma, \mathtt{iUnivalence}_n)$ where

$$\mathtt{iUnivalence}_n = \Pi(\mathsf{U}_n, \mathtt{isContr}_c(\Sigma(\mathsf{U}_n, \mathtt{Equiv}_c(\mathsf{q}, \mathsf{qp}))))$$

for $n \geq 0$, i.e. univalence is provable. One may also show that the path type applied to constant families $\mathbb{I} \to \mathsf{Type}(\Gamma)$ interprets the rules of identity types of Martin-Löf with the computation rule for the eliminator J replaced by a propositional equality. Thus, we obtain an interpretation of univalent type theory with identity types with propositional computation in any cubical cwf.

1.4. Computational cubical categories with families.

In this subsection, we consider a variation of the notion of cubical categories with families where we replace the filling operation by a composition operation, and where we add *computation* rules for this composition operation. This version is the one used for (strict) canonicity in Section 6. The computation rules are needed since the proof of canonicity follows closely the constructive justification of cubical type theory. For this justification, we also have to replace the *filling* operation by a *composition operation*. We show then that we can define a filling operation from a composition operation and we define the composition operation on types structurally. Since the canonicity argument, like the one in [Coq19], follows closely the structure of the constructive justification of the model of univalence, we need to start from the composition operation instead.

In order to simplify the notations, we assume here that the interval $\mathbb{I}$ also has a reversal operation, like in [CCHM18, CHM18]. This assumption is not necessary (for instance, as noted in [CCHM18], and indeed as we did in Subsection 1.3, we can avoid the reverse operation at the cost of carrying around an external boolean parameter) but it simplifies the presentation slightly.

Given $A : \mathbb{I} \to \mathcal{U}_\omega$, we write $\mathsf{hasComp}(A)$ for the set of operations taking as inputs φ in $\mathbb{F}$ with a partial section u in $\prod_{i:\mathbb{I}} [\varphi] \vee (i = 0) \to A\,i$ and producing an element in $A\,1$ which is equal to $u\,1\,\mathtt{tt}$ on φ. Given a set X and $Y : X \to \mathcal{U}_\omega$, we write $\mathsf{Comp}(X, Y)$ for the set of *composition structures* on Y, producing an element of $\mathsf{hasComp}(Y \circ x)$ for x in $\mathbb{I} \to X$. Given s in $\mathsf{Comp}(X, Y)$ and x, φ, u as above, we write $s(x, \varphi, u)$ for the resulting element in $Y(x\,1)$.

We now change the definition of cubical cwf in two ways to obtain our notion of *computational cubical cwf*.

First, we replace the *filling* operation by a *composition operation*. We have $\mathtt{comp}$ in $\mathsf{Comp}(\mathsf{Type}(\Gamma), \lambda_A \mathsf{Elem}(\Gamma, A))$ for Γ in Con together with stability under substitution: given $A : \mathbb{I} \to \mathsf{Type}(\Gamma)$, φ in $\mathbb{F}$, u in $\prod_{i:\mathbb{I}} [\varphi] \vee (i = 0) \to \mathsf{Elem}(\Gamma, A\,i)$, and σ in $\mathsf{Hom}(\Delta, \Gamma)$ and $r : \mathbb{I}$, we have

$$\mathtt{comp}(A, \varphi, u)\sigma = \mathtt{comp}(\lambda_i\,(A\,i)\sigma, \varphi, \lambda_{i,x}\,(u\,i\,x)\sigma).$$

The filling operation is now a *derived* operation. We define

$$\mathtt{fill}(A, \varphi, u)\,r = \mathtt{comp}(A_r, \varphi, u_r)$$

where $A_r\, i = A\,(i \wedge r)$ and $u_r\, i = u\,(i \wedge r)$.

Second, we add suitable *computation rules* (equalities) for this composition operation, structurally over types, following the computation rules in [CCHM18, CHM18]. We give the details here for two representative examples.

- For dependent sums, we add the computation rule

$$\mathtt{comp}(\lambda_i\, \Sigma(A\, i, B\, i), \psi, w) = \mathtt{pair}(u, v)$$

  where $u = \tilde{u}\, 1$ and $v = \mathtt{comp}(\lambda_i\,(B\, i)[\tilde{u}\, i], \psi, \lambda_{i,x}\, \mathtt{snd}(w\, i\, x))$ using

$$\tilde{u} = \mathtt{fill}(A, \psi, \lambda_{i,x}\, \mathtt{fst}(w\, i\, x)).$$

- For natural numbers, we add the computation rules

$$\mathtt{comp}(\lambda_i\, \mathtt{N}, \psi, \lambda_{i,x}\, \mathtt{0}) = \mathtt{0}$$
$$\mathtt{comp}(\lambda_i\, \mathtt{N}, \psi, \lambda_{i,x}\, \mathtt{S}(v\, i\, x)) = \mathtt{S}(\mathtt{comp}(\lambda_i\, \mathtt{N}, \psi, b, v)).$$

2. Two examples of cubical cwfs

In this section we give two examples of cubical cwfs: a term model and a particular cubical cwfs formulated in a constructive metatheory, the latter with extra assumptions on $\mathbb{I}$ and $\mathbb{F}$.

2.1. **Term model.** We sketch how to give a cubical cwf $\mathcal{T}$ built from syntax, and refer the reader to Appendix A for more details. All our judgments will be indexed by an object X of $\mathcal{C}$ and given a judgment $\Gamma \vdash_X \mathcal{J}$ and $f\colon Y \to X$ in $\mathcal{C}$ we get $\Gamma f \vdash_Y \mathcal{J} f$. Here, f acts on expressions as an implicit substitution, while for substitutions on object variables we will use explicit substitutions.

The forms of judgment are:

$$\Gamma \vdash_X \qquad \Gamma \vdash_X A \qquad \Gamma \vdash_X A = B \qquad \Gamma \vdash_X t : A \qquad \Gamma \vdash_X t = u : A \qquad \sigma : \Delta \to_X \Gamma$$

The main rules are given in the appendix. This then induces a cubical cwf $\mathcal{T}$ by taking, say, the presheaf of contexts at stage X to be equivalence classes of Γ for $\Gamma \vdash_X$ where the equivalence relation is judgmental equality.

Some rules are a priori infinitary, but in some cases (such as the one considered in [CCHM18]) it is possible to present the rules in a finitary way.

This formal system expresses the laws of cubical cwfs in rule form. It defines the *term model*. Following [Str91, PV07] developed in an intuitionistic framework, we conjecture that this can be interpreted in an arbitrary cubical cwf in the usual way:

Conjecture 2.1. With chosen parameters $\mathcal{C}, \mathbb{I}, \mathbb{F}$, the cubical cwf $\mathcal{T}$ is initial in the category of cubical cwfs.

However, our canonicity result is orthogonal to this conjecture: It is a result about the initial model, without need for an explicit description of this model as a term model.

2.2. **Developments in presheaves over $\mathcal{C}$.** We now assume that $\mathbb{I}$ and $\mathbb{F}$ satisfy the axioms presented in [OP16, Coq18]. We briefly recall them for the reader's convenience. The subobject $\mathbb{F}$ of Ω_0 should define a *dominance* and be closed under disjunction. The subobject classified by the map $[-] \colon \mathbb{F} \to \Omega_0$ should be levelwise decidable. The interval $\mathbb{I}$ should have two distinct global elements 0 and 1 and connections. The interval endpoint inclusions should be cofibrations (i.e., the equalities to 0 and 1 are coded by elements of $\mathbb{F}$) and cofibrations should be closed under universal quantification over $\mathbb{I}$. Finally, the interval $\mathbb{I}$ should be *tiny*, i.e., the exponential functor $(-)^{\mathbb{I}}$ should have a right adjoint R.[6] This is for example the case if $\mathcal{C}$ has finite products and $\mathbb{I}$ is representable.

Most of the reasoning will be done in the internal language of the presheaf topos. At certain points however, we need to consider the set of global sections of a global type F; we denote this by $\Box F$. We stress that statements involving $\Box$ are external, not to be interpreted in the internal language. Crucially, the adjunction $(-)^{\mathbb{I}} \dashv R$ cannot be made internal [LOPS18].

We write $\widehat{\mathcal{C}}$ for the category of presheaves over $\mathcal{C}$. The right adjoint R is determined by an isomorphism

$$\widehat{\mathcal{C}}(A, RX) \simeq \widehat{\mathcal{C}}(A^{\mathbb{I}}, X)$$

natural in A and X. Using cocontinuity in X, we may equivalently restrict to $A = \mathsf{y}(I)$ where y denotes the Yoneda embedding and I is in $\mathcal{C}$. Then the isomorphism becomes

$$(RX)(I) \simeq \widehat{\mathcal{C}}(\mathsf{y}(I)^{\mathbb{I}}, X)$$

natural in I and X. We may modify the given right adjoint R so that this isomorphism becomes an equality. By our smallness assumptions on $\mathcal{C}$ and $\mathbb{I}$, we have that $y(I)^{\mathbb{I}}$ lives in the lowest Grothendieck universe in our hierarchy. It follows that R restricts to an operation on $\mathcal{U}_n$ for $n \geq 0$.[7]

Pseudofunctorially in a presheaf A, the adjunction $(-)^{\mathbb{I}} \dashv R$ descends to an adjunction between categories of families over A and $A^{\mathbb{I}}$. We record what we need from this in the rest of our development.

Lemma 2.2. *Let A be a global set and B a global family over $A^{\mathbb{I}}$. Then we have a global family $B_{\mathbb{I}}$ over A with a bijection of global elements*

$$\Box(\textstyle\prod_{A'} B_{\mathbb{I}} \circ f) \simeq \Box(\textstyle\prod_{(A')^{\mathbb{I}}} B \circ f^{\mathbb{I}})$$

natural in global $f : A' \to A$.

The construction $(-)_{\mathbb{I}}$ may be chosen so that:

(1) *if B is valued in $\mathcal{U}_n$ for $n \geq 0$, then so is $B_{\mathbb{I}}$,*
(2) *the induced isomorphism $(B \circ f^{\mathbb{I}})_{\mathbb{I}} \simeq B_{\mathbb{I}} \circ f$ is an identity.*

Proof. Let $\eta_A \colon A \to R(A^{\mathbb{I}})$ be the unit of the adjunction at A. We define $B_{\mathbb{I}}(a)$ as the fiber of $R((-).1) \colon R(\sum_{A^{\mathbb{I}}} B) \to R(A^{\mathbb{I}})$ over $\eta_A(a)$. Global sections of $\prod_{A'} B_{\mathbb{I}} \circ f$ are dotted maps

---

[6]This is not part of the axioms in [OP16], but it implies connectivity of $\mathbb{I}$, the first axiom in [OP16], since left adjoints preserve colimits.

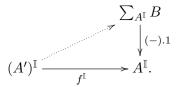[7]Without our modification of R, this would only be true up to isomorphism.

making the following diagram commute:

$$
\begin{array}{ccc}
 & & R(\textstyle\sum_{A^{\mathbb{I}}} B) \\
 & \nearrow & \downarrow {\scriptstyle R((-).1)} \\
A' \xrightarrow{\;f\;} A & \xrightarrow{\;\;\eta_A\;\;} & R(A^{\mathbb{I}}).
\end{array}
$$

Global sections of $\prod_{(A')^{\mathbb{I}}} B \circ f$ are dotted maps making the following diagram commute:

$$
\begin{array}{ccc}
 & & \textstyle\sum_{A^{\mathbb{I}}} B \\
 & \nearrow & \downarrow {\scriptstyle (-).1} \\
(A')^{\mathbb{I}} & \xrightarrow{\;\;f^{\mathbb{I}}\;\;} & A^{\mathbb{I}}.
\end{array}
$$

Under transposition of the adjunction, the two are in bijection, naturally in A'.

Recall the equivalence between maps into a presheaf and families over that presheaf. Under this equivalence, we can regard $(-)^{\mathbb{I}}$ as a functor from global families over A to global families over $A^{\mathbb{I}}$. The above discussion then shows that $(-)_{\mathbb{I}}$ is a right adjoint of $(-)^{\mathbb{I}}$.

Let examine the values of the presheaf $B_{\mathbb{I}}$ over the category of elements of A. By Yoneda and the natural bijection we have just verified, $B_{\mathbb{I}}(I, a)$ is naturally isomorphic to the set of sections of the restriction of B along $a^{\mathbb{I}} \colon \mathsf{y}(I)^{\mathbb{I}} \to A^{\mathbb{I}}$. By our smallness assumptions on $\mathcal{C}$ and $\mathbb{I}$, this is in $\mathcal{U}_n$ if B is valued in $\mathcal{U}_n$ (irrespective of the size of A). As in our discussion on size preservation of R, we may modify the definition of $(-)_{\mathbb{I}}$ so that the above isomorphism becomes an identity. This validates (1) and (2). □

In the above statement, the given bijection may be reduced to the case where f is an identity: $\Box(\prod_{A^{\mathbb{I}}} B) \simeq \Box(\prod_A B_{\mathbb{I}})$. The cost to pay is that the isomorphism $B_{\mathbb{I}} \circ f \simeq (B \circ f^{\mathbb{I}})_{\mathbb{I}}$ with appropriate coherence becomes primitive (non-derived) data.

We are going to apply Lemma 2.2 in two different instances. The first instance occurs in the following subsection and is used to build internal universes of fibrant sets, which will be needed to prove homotopy canonicity. The second (and more complex) instance occurs in in Subsection 6.1 and is used for interpreting types in the sconing model in the proof of canonicity.

2.2.1. *Fibrant presheaves.* Recall the global family $\mathsf{hasFill} : \mathcal{U}_\omega^{\mathbb{I}} \to \mathcal{U}_\omega$ from Subsection 1.3. Applying Lemma 2.2 to $\mathsf{hasFill}$, we obtain global $\mathsf{C} : \mathcal{U}_\omega \to \mathcal{U}_\omega$ such that naturally in a global set X with global $Y : X \to \mathcal{U}_\omega$, global elements of $\prod_{x:X^{\mathbb{I}}} \mathsf{hasFill}(Y \circ x)$ are in bijection with global elements of $\prod_{x:X} \mathsf{C}(Y\,x)$. Given a global set X and global $Y : X \to \mathcal{U}_\omega$, we thus have a logical equivalence (maps back and forth)

$$
\Box \mathsf{Fill}(X, Y) \longleftrightarrow \Box \textstyle\prod_{x:X} \mathsf{C}(Y\,x) \tag{2.1}
$$

natural in X.[8]

---

[8]We record only the logical equivalence instead of an isomorphism so that it will be easier to apply our constructions in situations where the right adjoint R fails to exist such as Appendix D. Naturality is only used at one point below, for the forward map, to construct suitable elements of C applied to glueings.

Note that C descends to $\mathsf{C} : \mathcal{U}_n \to \mathcal{U}_n$ for $n \geq 0$. We write $\mathcal{U}_i^{\mathsf{fib}} = \sum_{A:\mathcal{U}_i} \mathsf{C}(A)$ for $i \in \{0, 1, \ldots, \omega\}$; we call $\mathcal{U}_i^{\mathsf{fib}}$ a universe of *fibrant sets*. Now set $X = \mathcal{U}_\omega^{\mathsf{fib}}$ and $Y(A, c) = A$ in (2.1). We trivially have $\Box \prod_{x:X} \mathsf{C}(Y\,x)$, thus get

$$\mathsf{fill} : \mathsf{Fill}(\mathcal{U}_\omega^{\mathsf{fib}}, \lambda_{(A,c)}\, A). \tag{2.2}$$

This is essentially the counit of the adjunction defining C. Note that [LOPS18] use modal extensions of type theory to perform this reasoning internal to presheaves over $\mathcal{C}$.

Remark 2.3. Internally, a map $\mathsf{Fill}(X, Y) \to \prod_{x:X} \mathsf{C}(Y\,x)$ does not generally exist for a set X and $Y : X \to \mathcal{U}_\omega$ as for $X = 1$ one would derive a filling structure for any "homogeneously fibrant" set, which is impossible (see [OP16, Remark 5.9]). However, from (2.2) we get a map $\prod_{x:X} \mathsf{C}(Y\,x) \to \mathsf{Fill}(X, Y)$ natural in X using closure of filling structures under substitution (see below).

More examples of the interplay between *internal* and *external* reasoning involving elements of $\mathsf{C}(X)$ will occur in Subsubsection 2.2.2 when we reason that closure of Fill under various type formers, proven internally, transfers to corresponding closure properties of C, proven externally.

2.2.2. *Some general constructions.* We recall some constructions of [CCHM18, OP16] in the internal language.

- Given $A : \mathbb{I} \to \mathcal{U}_\omega$ and $a_b : A_b$ for $b \in \{0, 1\}$, *dependent paths* $\mathsf{Path}_A\, a_0\, a_1$ are the set of maps $p : \prod_{i:\mathbb{I}} A\,i$ such that $p\,0 = a_0$ and $p\,1 = a_1$. We use the same notation for non-dependent paths.
- For $A : \mathcal{U}_\omega$, we have a set $\mathsf{isContr}(A)$ of witnesses of *contractibility*, defined using paths.
- Given $A, B : \mathcal{U}_\omega$ with $f : A \to B$, we have the set $\mathsf{isEquiv}(f)$ with elements witnessing that f is an *equivalence*, defined using contractibility of homotopy fibers. We write $\mathsf{Equiv}(A, B) = \sum_{f:A\to B} \mathsf{isEquiv}(f)$.
- Given $A : \mathcal{U}_\omega$, $\varphi : \mathbb{F}$, $B : [\varphi] \to \mathcal{U}_\omega$, and $e : [\varphi] \to (B\,\mathsf{tt} \to A)$, the *glueing* $\mathsf{Glue}\,A\,[\varphi \mapsto (B, e)]$ consists of elements $\mathsf{glue}\,a\,[\varphi \mapsto b]$ with $a : A$ and $b : [\varphi] \to B$ such that $e.1\,(b\,\mathsf{tt}) = a$ on $[\varphi]$ and is defined in such a way that

$$\mathsf{Glue}\,A\,[\varphi \mapsto (B, e)] = T\,\mathsf{tt},$$
$$\mathsf{glue}\,a\,[\varphi \mapsto b] = b\,\mathsf{tt}$$

  on $[\varphi]$. We have a projection $\mathsf{unglue} : \mathsf{Glue}\,A\,[\varphi \mapsto (B, e)] \to A$.

These operations are valued in $\mathcal{U}_n$ if their inputs are. We further recall from [CCHM18, OP16] basic facts about filling structures in the internal language.

- Filling structures are closed under substitution: given $f : X' \to X$ and $Y : X \to \mathcal{U}_\omega$, any element of $\mathsf{Fill}(X, Y)$ induces an element of $\mathsf{Fill}(X', Y \circ f)$, naturally in X'.
- Filling structures are closed under exponentiation: given sets S, X and $Y : X \to \mathcal{U}_\omega$, any element of $\mathsf{Fill}(X, Y)$ induces an element of

$$\mathsf{Fill}(X^S, \lambda_x \prod_{s:S} Y(x\,s)),$$

  naturally in S.

- Filling structures are closed under $\Pi, \Sigma, \mathsf{Path}$. For example, for dependent products, this means the following. Given $A : \Gamma \to \mathcal{U}_\omega$ with $\mathsf{Fill}(\Gamma, A)$ and $B : \prod_{\rho:\Gamma} A\,\rho \to \mathcal{U}_\omega$ with $\mathsf{Fill}(\sum_{\rho:\Gamma} A\,\rho, \lambda_{(\rho,a)} B\,\rho\,a)$, we have

$$\mathsf{Fill}(\Gamma, \lambda_{\rho:\Gamma} \textstyle\prod_{a:A\,\rho} B\,\rho\,a).$$

- The Glue set former preserves filling structures with equivalences. By this, we mean the following. Let $A : \Gamma \to \mathcal{U}_\omega$ and $\varphi : \mathbb{F}$. For ρ in Γ and x in $[\varphi]$, let $B\,\rho\,x : \mathcal{U}_\omega$ with a map $e\,\rho\,x : A\,\rho \to B\,\rho\,x$. Assume $\mathsf{Fill}(\Gamma, A)$ and $\mathsf{Fill}(\sum_{\rho:\Gamma} [\varphi], \lambda_{(\rho,x)} B\,\rho\,\mathsf{tt})$ and that $e\,\rho\,\mathsf{tt}$ is an equivalence for $\rho : \Gamma$ on $[\varphi]$. Then we have

$$\mathsf{Fill}(\Gamma, \lambda_\rho \,\mathsf{Glue}\,(A\,\rho)) \,[\varphi\,\rho \mapsto (B\,\rho\,\mathsf{tt}, e\,\rho\,\mathsf{tt})]$$

  and the map $\mathsf{unglue}\,\rho$ from the glue set to $A\,\rho$ is an equivalence for ρ as above.

All of the above closure observations satisfy naturality under substitution.

Above, we have recorded closure of Fill under various set formers. From this, we use external reasoning to deduce the corresponding closure properties for C. Specifically, we have that C is closed under $\Pi, \Sigma, \mathsf{Path}, \mathsf{Glue}$ (adding equivalence data in the case of Glue), and that $\mathsf{C}(A)$ implies $\mathsf{C}(A^S)$ for $A, S : \mathcal{U}_\omega$.[9]

We explain how this works in the example case of Π.

Given (A, c_A) in $\mathcal{U}_\omega^{\mathsf{fib}}$ and $\langle B, c_B \rangle : A \to \mathcal{U}_\omega^{\mathsf{fib}}$, we wish to show $\mathsf{C}(\prod_A B)$. We set

$$\Delta = \textstyle\sum_{(A,c_A):\mathcal{U}_\omega^{\mathsf{fib}}} A \to \mathcal{U}_\omega^{\mathsf{fib}}$$

for the "generic context" of the closure statement. Then the goal is a global element of

$$\textstyle\prod_{((A,c_A),\langle B,c_B \rangle):\Delta} C(\textstyle\prod_A B).$$

By (2.1), this amounts to a global element of

$$\mathsf{Fill}(\Delta, \lambda_{((A,c_A),\langle B,c_B \rangle)} \textstyle\prod_A B).$$

Now we reason internally. Since Fill is closed under dependent products, the goal reduces to

$$\mathsf{Fill}(\Delta, \lambda_{((A,c_A),\langle B,c_B \rangle)} A),$$
$$\mathsf{Fill}(\textstyle\sum_{((A,c_A),\langle B,c_B \rangle):\Delta} A, \lambda_{(((A,c_A),\langle B,c_B \rangle),a)} B\,a).$$

Elements of these are given by Remark 2.3 since all families here are valued in fibrant sets (as witnessed by the components c_A and c_B).

Note that in the case of Glue with $(A, c) : \mathcal{U}_\omega^{\mathsf{fib}}$, $\varphi : \mathbb{F}$, $\langle B, d \rangle : [\varphi] \to \mathcal{U}_\omega^{\mathsf{fib}}$, and $e : [\varphi] \to \mathsf{Equiv}(B\,\mathsf{tt}, A)$, naturality of the forward map of (2.1) is needed to see that the element $c : \mathsf{C}(\mathsf{Glue}\,A\,[\varphi \mapsto (B, e)])$ constructed in the same fashion as above for dependent products equals $d\,\mathsf{tt} : C(B\,\mathsf{tt})$ on $[\varphi]$.

As in [CCHM18, OP16, LOPS18], glueing shows $\mathsf{Fill}(1, \mathcal{U}_n^{\mathsf{fib}})$ for $n \geq 0$. Using (2.1), we conclude $\mathsf{C}(\mathcal{U}_n^{\mathsf{fib}})$.

Let $\mathbb{N}$ denote the natural number object in presheaves over $\mathcal{C}$, the constant presheaf with value the natural numbers. From [CCHM18, OP16], we have $\mathsf{Fill}(1, \mathbb{N})$. Using (2.1), we conclude $\mathsf{C}(\mathbb{N})$.

We justify fibrant indexed inductive sets in Appendix B.

---

[9]Note that naturality in S of the latter operation is used in substitutional stability of universes in the sconing in Section 3.

2.3. **Standard model.** Making the same assumptions on $\mathcal{C}, \mathbb{I}, \mathbb{F}$ as in Subsection 2.2, we can now specify the standard model $\mathcal{S}$ of cubical type theory in the sense of the current article as a cubical cwf (with respect to parameters $\mathcal{C}, \mathbb{I}, \mathbb{F}$) purely using the internal language of the presheaf topos. The cwf is induced by the family over $\mathcal{U}_\omega^{\mathsf{fib}}$ given by the first projection as follows.

- The category of contexts is $\mathcal{U}_\omega$, with $\mathsf{Hom}(\Delta, \Gamma)$ the functions from Δ to Γ.
- The types over Γ are maps from Γ to $\mathcal{U}_\omega^{\mathsf{fib}}$; a type $\langle A, p \rangle$ is of level n if A is in $\Gamma \to \mathcal{U}_n$. This is clearly functorial in Γ.
- The elements of $\langle A, p \rangle : \Gamma \to \mathcal{U}_\omega^{\mathsf{fib}}$ are $\prod_{\rho:\Gamma} A \rho$. This is clearly functorial in Γ.
- The terminal context is given by 1.
- The context extension of Γ by $\langle A, p \rangle$ is given by $\sum_{\rho:\Gamma} A \rho$, with p, q given by projections and substitution extension given by pairing.

We briefly go through the necessary type formers and operations, omitting evident details. Whenever we mention an induced witness of fibrancy, this refers to the observations recorded in Subsubsection 2.2.2.

- The dependent product of $\langle A, c \rangle : \Gamma \to \mathcal{U}_\omega^{\mathsf{fib}}$ and $\langle B, d \rangle : \sum_{\rho:\Gamma} A \rho \to \mathcal{U}_\omega^{\mathsf{fib}}$ is

$$\langle \lambda_\rho \textstyle\prod_{a:A\rho} B(\rho, a), e \rangle$$

where $e\,\rho : \mathsf{C}(\prod_{a:A\rho} B(\rho, a))$ is induced by $c\,\rho : \mathsf{C}(A\,\rho)$ and $d\,\rho\,a : \mathsf{C}(B(\rho, a))$ for $a : A$.
- The dependent sum of $\langle A, c \rangle : \Gamma \to \mathcal{U}_\omega^{\mathsf{fib}}$ and $\langle B, d \rangle : \sum_{\rho:\Gamma} A \rho \to \mathcal{U}_\omega^{\mathsf{fib}}$ is

$$\langle \lambda_\rho \textstyle\sum_{a:A\rho} B(\rho, a), e \rangle$$

where e is induced by c and d.
- The universe $\mathtt{U}_n : \Gamma \to \mathcal{U}_{n+1}^{\mathsf{fib}}$ is constantly

$$(\mathcal{U}_n^{\mathsf{fib}}, c)$$

with $c : \mathsf{C}(\mathcal{U}_n^{\mathsf{fib}})$ as recorded before. According to our definition of the types in $\mathcal{S}$, this universe is actually Russell-style, i.e., the evident isomorphism $\mathsf{Type}_n(\Gamma) \cong \mathsf{Elem}(\Gamma, \mathtt{U}_n)$ is an identity.
- The natural number type $\mathtt{N} : \Gamma \to \mathcal{U}_0^{\mathsf{fib}}$ is constantly

$$(\mathbb{N}, c)$$

with $c : \mathsf{C}(\mathbb{N})$ as recorded before. The zero and successor constructors and the eliminator are given by the corresponding features of the natural number object $\mathbb{N}$.

We now turn to the cubical aspects.

- The filling operation

$$\mathtt{fill} : \mathsf{Fill}(\Gamma \to \mathcal{U}_\omega^{\mathsf{fib}}, \lambda_{\langle A, p \rangle} \textstyle\prod_{\rho:\Gamma} A \rho)$$

is derived from (2.2) by closure of filling structures under exponentiation.
- Given $\langle A, c \rangle : \mathbb{I} \to \Gamma \to \sum_{A:\mathcal{U}_\omega} \mathsf{C}(A)$ and $a_b : \prod_{\rho:\Gamma} A\,b\,\rho$ for $b \in \{0, 1\}$, we define $\mathsf{Path}(A, a_0, a_1) : \Gamma \to \sum_{A:\mathcal{U}_\omega} \mathsf{C}(A)$ as

$$\langle \textstyle\prod_{\rho:\Gamma} \mathsf{Path}_{\lambda_i A\,i\,\rho}\, c_0\, c_1), d \rangle$$

where $d\,\rho : \mathsf{C}(\mathsf{Path}_{\lambda_i A\,i\,\rho}\, c_0\, c_1))$ is induced by $\lambda_i\, c\,i\,\rho$. Path abstraction and application operations are defined from those of Path.

Before defining glue types, we note that the notions $\texttt{isContr}_c$ and $\texttt{isEquiv}_c$ in the cubical cwf we are defining correspond to the notions $\mathsf{isContr}$ and $\mathsf{isEquiv}$. For example, given a type $A : \Gamma \to \mathcal{U}_\omega^{\mathsf{fib}}$, then the elements of $\texttt{isContr}_c(A)$, given by $\prod_{\rho:\Gamma} \texttt{isContr}_c(A).1\,\rho$, are in bijection with $\prod_{\rho:\Gamma} \mathsf{isContr}(A.1\,\rho)$ naturally in Γ.

- Given $\langle A, c\rangle : \Gamma \to \mathcal{U}_\omega^{\mathsf{fib}}$, $\varphi : \mathbb{F}$, $\langle T, d\rangle : [\varphi] \to \Gamma \to \mathcal{U}_\omega^{\mathsf{fib}}$, and $e : [\varphi] \to \texttt{Equiv}_c(T\,\mathsf{tt}, A)$, we define $\texttt{Glue}_c(\langle A, c\rangle, \varphi, \langle T, d\rangle, e) : \Gamma \to \mathcal{U}_\omega^{\mathsf{fib}}$ as

$$\lambda_\rho\,(\mathsf{Glue}\,(A\,\rho)\,[\varphi \mapsto (T\,\mathsf{tt}\,\rho, (e'\,\mathsf{tt}\,\rho).1)], q\,\rho)$$

where $e'\,\mathsf{tt}\,\rho : \mathsf{Equiv}(T\,\mathsf{tt}\,\rho, A\,\rho)$ is induced by $e\,\mathsf{tt}\,\rho$ and $q\,\rho$ is induced by $c\,\rho$ and $\lambda_x\,d\,x\,\rho$ and $\lambda_x\,(e'\,\mathsf{tt}\,\rho).2$.

We have thus verified the following statement.

Theorem 2.4. *Assuming the parameters* $\mathcal{C}, \mathbb{I}, \mathbb{F}$ *satisfy the assumptions of Subsection 2.2, the standard model* $\mathcal{S}$ *forms a cubical cwf.*

3. Sconing

We make the same assumptions on our parameters $\mathcal{C}, \mathbb{I}, \mathbb{F}$ as in Subsection 2.2. In the global context, let $\mathcal{M}$ be a cubical cwf (with respect to these parameters) denoted $\mathsf{Con}, \mathsf{Hom}, \ldots$ as in Subsection 1.3. We assume that $\mathcal{M}$ is *size-compatible* with the standard model, by which we mean $\mathsf{Hom}(\Delta, \Gamma) : \mathcal{U}_\omega$ for all Γ, Δ and $\mathsf{Elem}(\Gamma, A) : \mathcal{U}_i$ for $i \in \{0, 1, \ldots, \omega\}$ and all Γ and $A : \mathsf{Type}_i(\Gamma)$. We will then define a new cubical cwf $\mathcal{M}^*$ denoted $\mathsf{Con}^*, \mathsf{Hom}^*, \ldots$, the *Artin glueing* of $\mathcal{M}$ with the standard model $\mathcal{S}$ along an (internal) global sections functor, i.e. the *sconing* of $\mathcal{M}$. (We refrain from referring it to as just glueing to avoid confusion with the glue types of cubical cwfs.)

Recall from Subsection 1.3 the operation $\texttt{fill}$ of $\mathcal{M}$. Instantiating it to the terminal context, we get $\Box\mathsf{Fill}(\mathsf{Type}(1), \lambda_A\,\mathsf{Elem}(1, A))$. Using the forward direction of (2.1), we thus have an internal operation $k : \prod_{A:\mathsf{Type}(1)} \mathsf{C}(\mathsf{Elem}(1, A))$.

From now on, we will work in the internal language of presheaves over $\mathcal{C}$. We start by defining a global sections operation $|-|$ mapping contexts, types, and elements of $\mathcal{M}$ to those of $\mathcal{S}$.

- Given $\Gamma : \mathsf{Con}$, we define $|\Gamma| : \mathcal{U}_\omega$ as the set of substitutions $\mathsf{Hom}(1, \Gamma)$. Given a substitution $\sigma : \mathsf{Hom}(\Delta, \Gamma)$, we define $|\sigma| : |\Delta| \to |\Gamma|$ as $|\sigma|\rho = \sigma\rho$. This evidently defines a functor.
- Given $A : \mathsf{Type}(\Gamma)$, we define $|A| : |\Gamma| \to \mathcal{U}_\omega^{\mathsf{fib}}$ as $|A|\,\rho = (\mathsf{Elem}(1, A\rho), k\,(A\,\rho))$. This evidently natural in Γ. If A is of level n, then $|A| : |\Gamma| \to \mathcal{U}_n^{\mathsf{fib}}$.
- Given $a : \mathsf{Elem}(\Gamma, A)$ we define $|a| : \prod_{\rho:\Gamma} (|A|\,\rho).1$ as $|a|\,\rho = a\rho$. This is evidently natural in Γ.

Note that $|-|$ preserves the terminal context and context extension up to canonical isomorphism in the category of contexts. One could thus call $|-|$ an (internal) *pseudomorphism* cwfs from $\mathcal{M}$ to $\mathcal{S}$. The sconing $\mathcal{M}^*$ will be defined as essentially the Artin glueing along this pseudomorphism, but we will be as explicit as possible and not define Artin glueing at the level of generality of an abstract pseudomorphism.

For convenience, we also just write $|A| : |\Gamma| \to \mathcal{U}_\omega$ instead of $\lambda_\rho\,(|A|\,\rho).1$, implicitly applying the first projection. We also write just $|A|$ for $|A|\,|()|$ if Γ is the terminal context.

3.1. **Contexts, substitutions, types, and elements.** We start by defining the cwf $\mathcal{M}^*$.

- A context $(\Gamma, \Gamma') : \mathsf{Con}^*$ consists of a context $\Gamma : \mathsf{Con}$ in $\mathcal{M}$ and a family Γ' over $|\Gamma|$ (which in the context of Artin glueing should be thought of as a substitution in $\mathcal{S}$ from some context to $|\Gamma|$). We think of Γ' as a *proof-relevant computability predicate*. A substitution $(\sigma, \sigma') : \mathsf{Hom}^*((\Delta, \Delta'), (\Gamma, \Gamma'))$ consists of a substitution $\sigma : \Delta \to \Gamma$ in $\mathcal{M}$ and a map

$$\sigma' : \prod_{\nu : |\Delta|} \Delta'(\nu) \to \Gamma'(\sigma\nu).$$

This evidently has the structure of a category.

- A type $(A, A') : \mathsf{Type}^*(\Gamma, \Gamma')$ consists of a type $A : \mathsf{Type}(\Gamma)$ in $\mathcal{M}$ and

$$A' : \prod_{\rho : |\Gamma|} \prod_{\rho' : \Gamma'\,\rho} |A|\,\rho \to \mathcal{U}_\omega^{\mathsf{fib}}.$$

We think of A' as a fibrant *proof-relevant computability family* on A. In the abstract context of Artin glueing for cwfs, we should think of it as

$$A' : \mathsf{Type}(\textstyle\sum_{\rho : |\Gamma|} \sum_{\rho' : \Gamma'\,\rho} |A|\,\rho)$$

in $\mathcal{S}$. However, we choose the former as the official definition so that the construction of $\mathcal{M}^*$ from $\mathcal{M}$ preserves Russell-style universes, as we shall see later. Recalling $\mathcal{U}_\omega^{\mathsf{fib}} = \sum_{X : \mathcal{U}_\omega} C(X)$, we also write $\langle A', \mathsf{fib}_{A'} \rangle$ instead of A' if we want to directly access the family and split off its proof of fibrancy.

The type (A, A') is of level n if A and A' are.

The action of a substitution $(\sigma, \sigma') : \mathsf{Hom}^*((\Delta, \Delta'), (\Gamma, \Gamma'))$ on (A, A') is given by

$$(A\sigma, \lambda_{\nu, \nu', a}\, A'\,(\sigma\nu)\,(\sigma'\,\nu\,\nu')\,a).$$

- An element $(a, a') : \mathsf{Elem}^*((\Gamma, \Gamma'), (A, \langle A', \mathsf{fib}_{A'} \rangle))$ consists of $a : \mathsf{Elem}(\Gamma, A)$ in $\mathcal{M}$ and

$$a' : \prod_{\rho : |\Gamma|} \prod_{\rho' : \Gamma'\,\rho} A'(\rho, \rho', a\rho).$$

In the context of Artin glueing of cwfs (with types in $\mathcal{M}^*$ presented correspondingly), this should be thought of as an element

$$a' : \mathsf{Elem}(\textstyle\sum_{\rho : |\Gamma|} \Gamma'\,\rho, \lambda_{(\rho, \rho')}\,A'(\rho, \rho', |a|\,\rho))$$

of $\mathcal{S}$.

The action of a substitution $(\sigma, \sigma') : \mathsf{Hom}^*((\Delta, \Delta'), (\Gamma, \Gamma'))$ on the element (a, a') is given by

$$(a\sigma, \lambda_{\nu, \nu'}\, a'\,\sigma\nu\,(\sigma'\,\nu\,\nu')).$$

- The terminal context is given by $(1, 1')$ defined by $1'\,() = 1$.
- The extension in $\mathcal{M}^*$ of a context (Γ, Γ') by a type (A, A') is given by $(\Gamma.A, (\Gamma.A)')$ where

$$(\Gamma.A')(\rho, a) = \textstyle\sum_{\rho' : \Gamma'\,\rho} (A'\,\rho\,\rho'\,a).1.$$

The projection $\mathsf{p}^* : \mathsf{Hom}^*((\Gamma, \Gamma').(A, A'), (\Gamma, \Gamma'))$ is $(\mathsf{p}, \mathsf{p}')$ where

$$\mathsf{p}'\,(\rho, a)\,(\rho', a') = \rho'$$

and the generic term $\mathsf{q}^* : \mathsf{Elem}((\Gamma, \Gamma').(A, A')\mathsf{p}^*)$ is $(\mathsf{q}, \mathsf{q}')$ where

$$\mathsf{q}'\,(\rho, a)\,(\rho', a') = a'.$$

The extension of $(\sigma, \sigma') : \mathsf{Hom}^*((\Delta, \Delta'), (\Gamma, \Gamma'))$ with $(a, a') : \mathsf{Elem}^*((\Delta, \Delta'), (A, A')(\sigma, \sigma'))$ is

$$((\sigma, a), \lambda_{\nu, \nu'}\,(\sigma'\,\nu\,\nu', a'\,\nu\,\nu')).$$

3.2. Type formers and operations.

3.2.1. *Dependent products.* Let

$$(A, \langle A', \mathsf{fib}_{A'} \rangle) : \mathsf{Type}^*(\Gamma, \Gamma'),$$

$$(B, \langle B', \mathsf{fib}_{B'} \rangle) : \mathsf{Type}^*((\Gamma, \Gamma').(A, \langle A', \mathsf{fib}_{A'} \rangle)).$$

We define the dependent product

$$\Pi^*((A, \langle A', \mathsf{fib}_{A'} \rangle), (B, \langle B', \mathsf{fib}_{B'} \rangle)) = (\Pi(A, B), \langle \Pi(A, B)', \mathsf{fib}_{\Pi(A,B)'} \rangle)$$

where

$$\Pi(A, B)'(\rho, \rho', f) = \prod\nolimits_{a:|A| \, \rho} \prod\nolimits_{a':A' \, \rho \, \rho' \, a} B'(\rho, a)(\rho', a')(\mathsf{app}(f, a))$$

and $\mathsf{fib}_{\Pi(A,B)'}(\rho, \rho', f)$ is given by closure of C under dependent product applied to $(|A| \, \rho).2$, $\mathsf{fib}_{A'} \, \rho \, \rho' \, a$ for $a : |A| \, \rho$, and $\mathsf{fib}_{B'}(\rho, a)(\rho', a')(\mathsf{app}(f, a))$ for additionally $a' : A' \, \rho \, \rho' \, a$.

Given an element (b, b') of $(B, \langle B', d \rangle))$ in $\mathcal{M}^*$, we define the abstraction $\mathtt{lam}^*(b, b') = (\mathtt{lam}(b), \mathtt{lam}(b)')$ where

$$\mathtt{lam}(b)' \, \rho \, \rho' \, a \, a' = b'(\rho, a)(\rho', a').$$

Given elements (f, f') of $\Pi^*((A, \langle A', c \rangle), (B, \langle B', d \rangle))$ and (a, a') of $(A, \langle A', \mathsf{fib}_{A'} \rangle))$ in $\mathcal{M}^*$, we define the application $\mathsf{app}^*((f, f'), (a, a')) = (\mathsf{app}(f, a), \mathsf{app}(f, a)')$ where

$$\mathsf{app}(f, a)' \, \rho \, \rho' = f' \, \rho \, \rho' \, a\rho \, (a' \, \rho \, \rho').$$

3.2.2. *Dependent sums.* Let

$$(A, \langle A', \mathsf{fib}_{A'} \rangle) : \mathsf{Type}^*(\Gamma, \Gamma'),$$

$$(B, \langle B', \mathsf{fib}_{B'} \rangle) : \mathsf{Type}^*((\Gamma, \Gamma').(A, \langle A', \mathsf{fib}_{A'} \rangle)).$$

We define the dependent sum

$$\Sigma^*((A, \langle A', \mathsf{fib}_{A'} \rangle), (B, \langle B', \mathsf{fib}_{B'} \rangle)) = (\Sigma(A, B), \langle \Sigma(A, B)', \mathsf{fib}_{\Sigma(A,B)'} \rangle)$$

where

$$\Sigma(A, B)' \, \rho \, \rho' \, (\mathtt{pair}(a, b)) = \sum\nolimits_{a':A' \, \rho \, \rho' \, a} B'(\rho, a)(\rho', a') \, b$$

and $\mathsf{fib}_{\Sigma(A,B)'} \, \rho \, \rho' \, (\mathtt{pair}(a, b))$ is given by closure of C under dependent sum applied to $\mathsf{fib}_{A'} \, \rho \, \rho' \, a$ and $\mathsf{fib}_{B'}(\rho, a)(\rho', a') \, b$.

Given elements (a, a') of $(A, \langle A', \mathsf{fib}_{A'} \rangle)$ and (b, b') of $(B, \langle B', \mathsf{fib}_{B'} \rangle)[(a, a')]$ in $\mathcal{M}^*$, we define the pairing $\mathtt{pair}^*((a, a'), (b, b')) = (\mathtt{pair}(a, b), \langle a', b' \rangle)$.

Given an element $(\mathtt{pair}(a, b), \langle a', b' \rangle)$ of $\Sigma^*((A, \langle A', \mathsf{fib}_{A'} \rangle), (B, \langle B', \mathsf{fib}_{B'} \rangle))$ in $\mathcal{M}^*$, we define the projections $\mathtt{fst}^*(\mathtt{pair}(a, b), \langle a', b' \rangle) = (a, a')$ and $\mathtt{snd}^*(\mathtt{pair}(a, b), \langle a', b' \rangle) = (b, b')$.

3.2.3. *Universes.* We define the universe $\mathsf{U}_n^* : \mathsf{Type}^*(\Gamma, \Gamma')$ as $\mathsf{U}_n^* = (\mathsf{U}_n, \langle \mathsf{U}_n', \mathsf{fib}_{\mathsf{U}_n'} \rangle)$ where

$$\mathsf{U}_n' \, \rho \, \rho' \, A = |A| \, \rho \rightarrow \mathcal{U}_n^{\mathsf{fib}}$$

and $\mathsf{fib}_{\mathsf{U}_n'} \, \rho \, \rho' \, A$ is given by $\mathsf{C}(\mathcal{U}_n^{\mathsf{fib}})$ and closure of C under exponentiation (note that fibrancy of $|A| \, \rho$ is not used). We have carefully chosen our definitions so that the evident natural isomorphism $\mathsf{Elem}^*((\Gamma, \Gamma'), \mathsf{U}_n^*) \cong \mathsf{Type}_n^*(\Gamma, \Gamma')$ is an identity if the corresponding isomorphism in $\mathsf{Type}_n(\Gamma) \cong \mathsf{Elem}(\Gamma, \mathsf{U}_n)$ in $\mathcal{M}$ is an identity. Thus, Russell-style universes are preserved by our presentation of the sconing model.

3.2.4. *Natural numbers.* As per Appendix B, we have a fibrant indexed inductive set $\mathtt{N}' : |\mathtt{N}| \to \mathcal{U}_0^{\mathsf{fib}}$ (where $\mathtt{N} : \mathsf{Type}_0(1)$, hence $|\mathtt{N}| : \mathcal{U}_0$) with constructors

$$0' : \mathtt{N}'\, 0,$$

$$\mathsf{S}' : \textstyle\prod_{n:|\mathtt{N}|\,\rho} \mathtt{N}'\, n \to \mathtt{N}'\, (\mathsf{S}\, n).$$

In context $(\Gamma, \Gamma') : \mathsf{Con}^*$, we then define $\mathtt{N}^* = (\mathtt{N}, \lambda_{\rho,\rho'}\, \mathtt{N}')$. We have $0^* = (0, \lambda_{\rho,\rho'}\, 0')$ and $\mathsf{S}^*(n, n') = (\mathsf{S}(n), \lambda_{\rho,\rho'}\, \mathsf{S}'\, n\rho\, n')$ for $(n, n') : \mathsf{Elem}^*((\Gamma, \Gamma'), \mathtt{N}^*)$.

Given $(P, P') : \mathsf{Type}((\Gamma, \Gamma').\mathtt{N}^*)$ with

$$(z, z') : \mathsf{Elem}^*((\Gamma, \Gamma')(P, P')[0^*]),$$

$$(s, s') : \mathsf{Elem}^*((\Gamma, \Gamma').\mathtt{N}^*.(P, P'), (P, P')(\mathsf{p}, S^*(\mathsf{q}))\mathsf{p})$$

and $(n, n') : \mathsf{Elem}^*((\Gamma, \Gamma'), \mathtt{N}^*)$, we define the elimination

$$\mathtt{natrec}^*((P, P'), (z, z'), (s, s'), (n, n')) = (\mathtt{natrec}(P, z, s, n), \lambda_{\rho,\rho'}\, h'\, n\rho\, (n'\, \rho\, \rho'))$$

where

$$h' : \textstyle\prod_{m:|\mathtt{N}|} \prod_{m':\mathtt{N}'\, m} P'\, (\rho, m)\, (\rho', m')\, (\mathtt{natrec}(P\rho^+, z\rho, s\rho^{+++}, m))$$

is given by induction on $\mathtt{N}'$ with defining equations

$$h'\, 0\, 0' = z'\, \rho\, \rho',$$

$$h'\, (\mathsf{S}(n))\, (\mathsf{S}'\, n\, n') = s'\, (\rho, n, \mathtt{natrec}(P, z, s, n))\, (\rho', n', h'\, n\, n').$$

3.2.5. *Dependent paths.* Let $\langle A, A' \rangle : \mathbb{I} \to \mathsf{Type}^*(\Gamma, \Gamma')$ and $(a_b, a_b') : \mathsf{Elem}^*((\Gamma, \Gamma'), (A\, b, A'\, b))$ for $b \in \{0, 1\}$. We then define

$$\mathtt{Path}^*(\langle A, A' \rangle, (a_0, a_0'), (a_1, a_1')) = (\mathtt{Path}(A, a_0, a_1), \langle \mathtt{Path}(A, a_0, a_1)', \mathsf{fib}_{\mathtt{Path}(A,a_0,a_1)'} \rangle)$$

where

$$\mathtt{Path}(A, a_0, a_1)'\, \rho\, \rho'\, (\langle\rangle(u)) = \mathtt{Path}_{\lambda_i\, (A'\, i\, \rho\, \rho'\, (u\, i)).1}(a_0'\, \rho\, \rho')(a_1'\, \rho\, \rho')$$

and $\mathsf{fib}_{\mathtt{Path}(A,a_0,a_1)'}\, \rho\, \rho'\, (\langle\rangle(u))$ is closure of C under Path applied to $(A'\, i\, \rho\, \rho'\, (u\, i)).2$ for $i : \mathbb{I}$.

Given $\langle u, u' \rangle : \prod_{i:\mathbb{I}} \mathsf{Elem}^*((\Gamma, \Gamma'), (A\, i, A'\, i))$, we define the path abstraction as

$$\langle\rangle^*(\langle u, u' \rangle) = (\langle\rangle(u), \lambda_{\rho,\rho',i}\, u'\, i\, \rho\, \rho').$$

Given $(p, p') : \mathsf{Elem}^*((\Gamma, \Gamma'), \mathtt{Path}^*(\langle A, A' \rangle, (a_0, a_0'), (a_1, a_1')))$ and $i : \mathbb{I}$, we define the path application

$$\mathtt{ap}^*(p, i) = (\mathtt{ap}(p, i), \lambda_{\rho,\rho'}\, u'\, \rho\, \rho'\, i).$$

3.2.6. *Filling operation.* Given $\langle A, A' \rangle : \mathbb{I} \to \mathsf{Type}^*(\Gamma, \Gamma')$, $\varphi : \mathbb{F}$, $b \in \{0, 1\}$, and $\langle u, u' \rangle : \prod_{i:\mathbb{I}} [\varphi] \vee (i = b) \to \mathsf{Elem}^*((\Gamma, \Gamma'), (A\, i, A'\, i))$, we have to extend u to

$$\mathtt{fill}^*(\langle A, A' \rangle, \varphi, b, \langle u, u' \rangle) : \textstyle\prod_{i:\mathbb{I}} \mathsf{Elem}^*((\Gamma, \Gamma'), (A\, i, A'\, i)).$$

We define $\mathtt{fill}^*(\langle A, A' \rangle, \varphi, b, \langle u, u' \rangle) = \langle \mathtt{fill}(A, \varphi, b, u), \mathtt{fill}(A, \varphi, b, u)' \rangle$ where

$$\mathtt{fill}(A, \varphi, b, u)'\, i\, \rho\, \rho' : A'\, i\, \rho\, \rho'\, (\mathtt{fill}(A, \varphi, b, u)\, i)\rho$$

is defined using fill from (2.2) as

$$\mathtt{fill}(A, \varphi, b, u)'\, i\, \rho\, \rho' = \mathsf{fill}(\lambda_i\, A'\, i\, \rho\, \rho'\, (\mathtt{fill}(A, \varphi, b, u)\, i)\rho, \varphi, b, \lambda_{i,x}\, u'\, i\, x\, \rho\, \rho').$$

3.2.7. *Glue types.* Before defining the glueing operation in $\mathcal{M}^*$, we will develop several lemmas relating notions such as contractibility and equivalences in $\mathcal{M}$ with the corresponding notions of Subsection 2.2. Given $f : \mathsf{Elem}(\Gamma, A \to B)$ in $\mathcal{M}$, we write $|f| : \prod_{\rho:|\Gamma|} |A| \, \rho \to |B| \, \rho$ for $|f| \, \rho \, a = \mathsf{app}(f\rho, a)$. This notation overlaps with the action of $|-|$ on elements, but we will not use that one here.

Just in this subsection, we will use the alternative definition via given left and right homotopy inverses instead of contractible homotopy fibers of both equivalences Equiv_c in the cubical cwf $\mathcal{M}$ and equivalences Equiv in the (current) internal language. In both settings, there are maps back and forth to the usual definition, which are furthermore natural in the context in the case of the cubical cwf $\mathcal{M}$. The statements we will prove are then also valid for the usual definition.

Lemma 3.1. *Given* $f : \mathsf{Elem}(\Gamma, A \to B)$ *in* $\mathcal{M}$ *with* $\mathsf{Elem}(\Gamma, \mathsf{isEquiv}_c(f))$, *we have* $\prod_{\rho:|\Gamma|} \mathsf{isEquiv}(|f| \, \rho)$. *This is natural in* Γ.

Proof. A (left or right) homotopy inverse $g : \mathsf{Elem}(\Gamma, B \to A)$ to f in $\mathcal{M}$ becomes a (left or right, respectively) homotopy inverse $|g| \, \rho$ to $|f| \, \rho$ for $\rho : |\Gamma|$. $\qquad\square$

Lemma 3.2. *Given* $(f, f') : \mathsf{Elem}((\Gamma, \Gamma'), (A, A') \to (B, B'))$ *in* $\mathcal{M}^*$, *the following statements are logically equivalent, naturally in* (Γ, Γ'):

$$\mathsf{Elem}((\Gamma, \Gamma'), \mathsf{isEquiv}_c^*(f, f')), \tag{3.1}$$

$$\mathsf{Elem}(\Gamma, \mathsf{isEquiv}_c(f)) \times \textstyle\prod_{\rho:|\Gamma|} \prod_{\rho':\Gamma' \rho} \mathsf{isEquiv}(\textstyle\sum_{|f| \, \rho} f' \, \rho \, \rho'), \tag{3.2}$$

$$\mathsf{Elem}(\Gamma, \mathsf{isEquiv}_c(f)) \times \textstyle\prod_{\rho:|\Gamma|} \prod_{\rho':\Gamma' \rho} \prod_{a:|A| \, \rho} \mathsf{isEquiv}(f' \, \rho \, \rho' \, a) \tag{3.3}$$

where $\sum_{|f| \, \rho} f' \, \rho \, \rho' : \sum_{a:|A| \, \rho} A' \, \rho \, \rho' \, a \to \sum_{b:|B| \, \rho} B' \, \rho \, \rho' \, b$.

Proof. Let us only look at homotopy left inverses.

For $(3.1) \to (3.2)$, a homotopy left inverse (g, g') to (f, f') in $\mathcal{M}^*$ gives a homotopy left inverse $\sum_{|g| \, \rho} g' \, \rho \, \rho'$ to $\sum_{|f| \, \rho} f' \, \rho \, \rho'$ for all ρ, ρ'.

For $(3.2) \to (3.3)$, we use Lemma 3.1 and note that a fiberwise map over an equivalence is a fiberwise equivalence exactly if it is an equivalence on total spaces (the corresponding statement for identity types instead of paths is [Uni13, Theorem 4.7.7]).

For $(3.3) \to (3.1)$, given a homotopy left inverse g to the equivalence f in $\mathcal{M}$ and a homotopy left inverse $\overline{g}' \, \rho \, \rho' \, a : B' \, \rho \, \rho' \, (|f| \, a) \to A' \, \rho \, \rho' \, a$ to $f' \, \rho \, \rho' \, a$ for all ρ, ρ', a, we use Lemma 3.1 to transpose $\overline{g}'$ to the second component $g' \, \rho \, \rho' \, b : B' \, \rho \, \rho' \, b \to A' \, \rho \, \rho' \, (|g| \, b)$ for all ρ, ρ', b of a homotopy left inverse (g, g') to (f, f') in $\mathcal{M}^*$. $\qquad\square$

We can now define glue types in $\mathcal{M}^*$. Let $(A, A') : \mathsf{Type}(\Gamma, \Gamma')$, $\varphi : \mathbb{F}$, $\langle T, T' \rangle : [\varphi] \to \mathsf{Type}(\Gamma, \Gamma')$, and

$$\langle e, e' \rangle : [\varphi] \to \mathsf{Elem}((\Gamma, \Gamma'), \mathsf{Equiv}_c^*((T\mathsf{tt}, T'\mathsf{tt}), (A, A'))).$$

We define

$$\mathsf{Glue}_c^*((A, A'), \varphi, \langle T, T' \rangle, \langle e, e' \rangle) = (\mathsf{Glue}_c(A, \varphi, , T, e), \langle G', \mathsf{fib}_{G'} \rangle)$$

where

$$G' \, \rho \, \rho' \, (\mathsf{glue}(a, t)) = \mathsf{Glue} \, (A' \, \rho \, \rho' \, a).1 \, [\varphi \mapsto (T' \, \mathsf{tt} \, \rho \, \rho' \, (t \, \mathsf{tt}), ((e' \, \mathsf{tt} \, \rho \, \rho').1 \, (t \, \mathsf{tt})))]$$

where $\mathsf{fib}_{G'} \, \rho \, \rho' \, (\mathsf{glue}(a, t))$ is given by closure of C under Glue applied to $(A' \, \rho \, \rho' \, a).2$ and $T' \, \mathsf{tt} \, \rho \, \rho' \, (t \, \mathsf{tt})$ on $[\varphi]$ and the witness that $(e' \, \mathsf{tt} \, \rho \, \rho').1 \, (t \, \mathsf{tt})$ is an equivalence provided by the

direction from (3.1) to (3.3) of Lemma 3.2. We define $\mathtt{unglue}^* = (\mathtt{unglue}, \mathtt{unglue}')$ where

$$\mathtt{unglue}' \, \rho \, \rho' \, (\mathtt{glue}(a, t)) = \mathtt{unglue}.$$

Given $(a, a') : \mathsf{Elem}((\Gamma.\Gamma'), (A, A'))$ and

$$(t, t') : [\varphi] \to \mathsf{Elem}((\Gamma, \Gamma'), (T \, \mathtt{tt}, T' \, \mathtt{tt}))$$

such that $\mathtt{app}^*(\mathtt{fst}^*(e, e') \, \mathtt{tt}, (t, t') \, \mathtt{tt}) = (a, a')$ on $[\varphi]$, we define $\mathtt{glue}^*((a, a'), (t, t'))$ as the pair $(\mathtt{glue}(a, t), \mathtt{glue}(a, t)')$ where

$$\mathtt{glue}(a, t)' \, \rho \, \rho' = \mathtt{glue} \, (a' \, \rho \, \rho') \, [\varphi \mapsto t' \, \mathtt{tt} \, \rho \, \rho'].$$

3.3. **Main result.** One checks in a mechanical fashion that the operations we have defined above satisfy the required laws, including stability under substitution in the context (Γ, Γ'). We thus obtain the following statement.

Theorem 3.3 (Sconing)**.** *Assume the parameters $\mathcal{C}, \mathbb{I}, \mathbb{F}$ satisfy the assumptions of Subsection 2.2. Then given any cubical cwf $\mathcal{M}$ that is size-compatible in the sense of the beginning of Section 3, the sconing $\mathcal{M}^*$ is a cubical cwf with operations defined as above. We further have a morphism $\mathcal{M}^* \to \mathcal{M}$ of cubical cwfs given by the first projection.*

4. Homotopy canonicity

We fix parameters $\mathcal{C}, \mathbb{I}, \mathbb{F}$ as before. To make our homotopy canonicity result independent of Conjecture 2.1 concerning initiality of the term model, we phrase it directly using the *initial model $\mathcal{I}$*, initial in the category of cubical cwfs with respect to the parameters $\mathcal{C}, \mathbb{I}, \mathbb{F}$. Its existence can be justified generically following [Ste19, PV07]. It is size-compatible in the sense of Section 3: internally, $\mathsf{Hom}_\mathcal{I}(\Delta, \Gamma)$ and $\mathsf{Elem}_\mathcal{I}(\Gamma, A)$ live in the lowest universe $\mathcal{U}_0$ for all Γ, Δ, A.

Theorem 4.1 (Homotopy canonicity)**.** *Assume the parameters $\mathcal{C}, \mathbb{I}, \mathbb{F}$ satisfy the assumptions of Subsection 2.2. In the internal language of presheaves over $\mathcal{C}$, given a closed natural $n : \mathsf{Elem}(1, \mathtt{N})$ in the initial model $\mathcal{I}$, we have a numeral $k : \mathbb{N}$ with $p : \mathsf{Elem}(1, \mathtt{Path}(\mathtt{N}, n, \mathtt{S}^k(\mathtt{0})))$.*

Proof. We start the arguing reasoning externally. Using Theorem 3.3, we build the sconing $\mathcal{I}^*$ of $\mathcal{I}$. Using initiality, we obtain a section F of the cubical cwf morphism $\mathcal{I}^* \to \mathcal{I}$.

Let us now proceed in the internal language. Recall the construction of Subsubsection 3.2.4 of natural numbers in $\mathcal{I}^*$. We observe that $\sum_{n:|\mathtt{N}|} \mathtt{N}' \, n$ forms a fibrant natural number set (in the sense of Appendix B). It is thus homotopy equivalent to $\mathbb{N}$. Under this equivalence, the first projection $\sum_{n:|\mathtt{N}|} \mathtt{N}' \, n \to |\mathtt{N}|$ implements the map sending $k : \mathbb{N}$ to $\mathtt{S}^k(\mathtt{0})$.

Inspecting the action of F on $n : \mathsf{Elem}(1, \mathtt{N})$, we obtain $n' : \mathtt{N}' \, n$. By the preceding paragraph, this corresponds to $k : \mathbb{N}$ with a path $p' : \mathbb{I} \to |\mathtt{N}|$ from n to $\mathtt{S}^k(\mathtt{0})$. Now $p = \langle\rangle(p')$ is the desired witness of homotopy canonicity. $\qquad\square$

5. Extensions

5.1. **Identity types.** Our treatment extends to the variation of cubical cwfs that includes identity types.

Identity types in a cubical cwf denoted as in Subsection 1.3 consist of the following operations and laws (omitting stability under substitution), internal to presheaves over $\mathcal{C}$. Fix A in $\mathsf{Type}(\Gamma)$. Given x, y in $\mathsf{Elem}(\Gamma, A)$, we have $\mathtt{Id}(A, x, y)$ in $\mathsf{Type}(\Gamma)$, of level n if A is. Given a in $\mathsf{Elem}(\Gamma, A)$, we have $\mathtt{refl}(a)$ in $\mathsf{Elem}(\Gamma, \mathtt{Id}(A, a, a))$. Given P in $\mathsf{Type}(\Gamma.A.\mathit{A}\mathsf{p}.\mathtt{Id}(\mathit{A}\mathsf{pp}, \mathsf{qp}, \mathsf{q}))$ and d in $\mathsf{Elem}(\Gamma.A, P[\mathsf{q}, \mathsf{q}, \mathtt{refl}(q)])$ and x, y in $\mathsf{Elem}(\Gamma, A)$ and p in $\mathsf{Elem}(\Gamma, \mathtt{Id}(A, x, y))$, we have $\mathtt{J}(P, d, x, y, p)$ in $\mathsf{Elem}(\Gamma, P[x, y, p])$. We have

$$\mathtt{J}(P, d, a, a, \mathtt{refl}(a)) = d[a].$$

We can interpret univalent type theory in any cubical cwf with identity types as per Remark 1.1.

The standard model of Subsection 2.3 has identity type $\mathtt{Id}(\langle A, \mathsf{fib}_A \rangle, x, y) : \mathsf{Type}(\Gamma)$ given by $\prod_{\rho : \Gamma} \mathsf{Id}_{A\,\rho}\,(x\,\rho)\,(y\,\rho)$ using Andrew Swan's construction of Id referenced in Appendix B. We omit the evident description of the remaining operations.

To obtain homotopy canonicity in this setting, it suffices to extend the sconing construction $\mathcal{M}^*$ of Section 3 to identity types. Given $A : \mathsf{Type}(1)$ and $A' : |A| \to \mathcal{U}_\omega^{\mathsf{fib}}$, we define $\mathsf{Id}'_{A,A'}$ as the fibrant indexed inductive set (as per Appendix B) over $x, y : |A|$, $p : |\mathtt{Id}(A, x, y)|$, $x' : A'\,x$, $y' : A'\,y$ with constructor

$$\mathsf{refl}' : \textstyle\prod_{a:|A|} \prod_{a':A'\,a} \mathsf{Id}'_{A,A'}\,a\,a\,(\mathtt{refl}(a))\,a'\,a'.$$

Now fix $(A, A') : \mathsf{Type}^*(\Gamma, \Gamma')$. Given $\rho : |\Gamma|$, $\rho' : \Gamma'\rho$, and elements $(x, x'), (y, y')$ of (A, A') in $\mathcal{M}^*$, we define

$$\mathtt{Id}^*((A, A'), (x, x'), (y, y')) = (\mathtt{Id}(A, x, y), \lambda_{\rho, \rho', p}\,\mathsf{Id}'_{A\rho, A'\,\rho\,\rho'}\,x\rho\,y\rho\,p\,(x'\,\rho\,\rho')\,(y'\,\rho\,\rho')).$$

Given an element (a, a') of (A, A') in $\mathcal{M}^*$, we define $\mathtt{refl}^*(a, a') = (\mathtt{refl}(a), \mathtt{refl}(a)')$ where

$$\mathtt{refl}(a)'\,\rho\,\rho' = \mathsf{refl}'\,a\rho\,(a'\,\rho\,\rho').$$

The eliminator $\mathtt{J}((C, C'), (d, d'), (x, x'), (y, y'), (p, p'))$ is defined as

$$(\mathtt{J}(C, d, x, y, p), \lambda_{\rho, \rho'}\,h'\,x\rho\,y\rho\,p\rho\,(x'\,\rho\,\rho')\,(y'\,\rho\,\rho')\,(p'\,\rho\,\rho'))$$

where

$$h' : \textstyle\prod_{x,y:|A|\,\rho} \prod_{p:|\mathtt{Id}(A,x,y)|\,\rho} \prod_{x':A'\,\rho'} \prod_{y':A'\,\rho'} \prod_{p':\mathsf{Id}'_{A\rho,A'\,\rho\,\rho'}\,x\,y\,p\,x'\,y'}$$
$$P'\,(\rho, x, y, p)\,(\rho', x', y, p')\,(\mathtt{J}(P\rho^{+++}, d\rho^+, x, y, p))$$

is given by induction on $\mathsf{Id}'_{A\rho, A'\,\rho\,\rho'}$ via the clause

$$h'\,a\,a\,(\mathtt{refl}(a))\,a'\,a'\,(\mathsf{refl}'\,a\,a') = d'\,(\rho, a)\,(\rho', a').$$

5.2. **Higher inductive types.** Our treatment extends to higher inductive types [Uni13], following the semantics presented in [CHM18]. Crucially, we have fibrant *indexed* higher inductive sets in presheaves over $\mathcal{C}$ as we have what we would call fibrant *uniformly indexed* higher inductive sets in the same fashion as in [CHM18] and fibrant identity sets [CCHM18, OP16], mirroring the derivation of fibrant indexed inductive sets from fibrant uniformly indexed inductive sets and fibrant identity sets recollected in Appendix B.[10]

Let us look at the case of the *suspension* operation in a cubical cwf, where $\mathsf{Susp}(A) : \mathsf{Type}(\Gamma)$ has constructors $\mathtt{north}, \mathtt{south}$ and $\mathtt{merid}(a, i)$ for $a : A$ and $i : \mathbb{I}$ with $\mathtt{merid}(a, 0) = \mathtt{north}$ and $\mathtt{merid}(a, 1) = \mathtt{south}$.

For the sconing model of Section 3, we define for $A : \mathsf{Type}(1)$ and $A' : |A| \to \mathcal{U}_\omega^{\mathsf{fib}}$ the indexed higher inductive set $\mathsf{Susp}'_{A,A'}$ over $|\mathsf{Susp}(A)|$ with constructors

$$\mathsf{north}' : \mathsf{Susp}'_{A,A'} \, \mathtt{north},$$
$$\mathsf{south}' : \mathsf{Susp}'_{A,A'} \, \mathtt{south},$$
$$\mathsf{merid}' \, a \, a' \, i : (\mathsf{Susp} \, A)'(\mathtt{merid}(a, i))[i = 0 \mapsto \mathsf{north}', i = 1 \mapsto \mathsf{south}']$$

for $a : |a|$ and $a' : A' \, a$ and $i : \mathbb{I}$ (using the notation of [CHM18]). In the above translation to a uniformly indexed higher inductive set, the constructor north' will for example be replaced by

$$\mathsf{north}'' : \mathsf{Id}_{|\mathsf{Susp}(A)|} \, u \, \mathtt{north} \to \mathsf{Susp}'_{A,A'} \, u.$$

Given $(A, A') : \mathsf{Type}^*(\Gamma, \Gamma')$, we then define

$$\mathtt{Susp}^*(A, A') = (\mathtt{Susp}(A), \lambda_{\rho,\rho'} \, \mathsf{Susp}'_{A\rho, A' \, \rho \, \rho'}),$$

with constructors and eliminator treated as in Subsection 5.1.

6. Canonicity

The goal of this section is to show *canonicity* for cubical type theory, stating that any closed term of type $\mathtt{N}$ is (strictly) equal to a numeral. This is a priori a stronger result than merely *homotopy canonicity*. However, it requires us to add further computation rules for the filling operation to the theory. For this purpose, we have defined in Subsection 1.4 the notion of computational cubical cwf, modelling a modified version of cubical type theory where the filling operation is replaced by the composition operation (filling is then a *derived* operation). This is our notion of model in this section.

The main point is how to define the right notion of computability structure. Once this is done, we can essentially construct the sconing model as in [Coq19]. If we apply this to the initial computational cubical cwf, we get the canonicity result: any closed natural number term is convertible to a numeral. This result was already proved in [Hub19], but like for the proof in [Coq19], our new argument completely avoids the need to define a reduction relation, which is quite subtle for cubical type theory in [Hub19] since it is not closed under name substitution.

As in Subsection 1.4, we have not just connection structure on the interval, but also a compatible reversal structure. Other than that, we make the same assumptions on the $\mathbb{I}$ and $\mathbb{F}$ as in Subsection 2.2. Starting from an arbitrary computational cubical cwf $\mathcal{M}$ in the

---

[10]We stress that the use of "set" in this context refers to the types of the language of presheaves over $\mathcal{C}$, not homotopy sets.

global context satisfying size-compatibility as in Section 3, we build a new computational cwf $\mathcal{M}^*$, the sconing of $\mathcal{M}$.

6.1. **Sconing model: cwf structure.** The underlying category of $\mathcal{M}^*$ is defined as in Section 3. It is the Artin glueing of $\mathcal{M}$ along the global sections functor $|-| : \mathcal{M} \to \mathcal{U}_\omega$. (Recall that $|\Gamma|$ is $\mathsf{Hom}(1,\Gamma)$ for Γ in Con. In particular, a context in $\mathcal{M}^*$ is a pair (Γ, Γ') where Γ is a context in $\mathcal{M}$ and $\Gamma' : |\Gamma| \to \mathcal{U}_\omega$.)

In contrast to Section 3, we cannot view the rest of the structure $\mathcal{M}^*$ as being obtained by glueing along the pseudomorphism $|-| : \mathcal{M} \to \mathcal{S}$ of (computational) cubical cwfs. In particular, we will not make use of the standard model $\mathcal{S}$. Rather, $\mathcal{M}^*$ can be seen as the total space of a fibration that presents a fibred version of the standard model over $\mathcal{M}$. In the definition of types, we need to track computability of the composition operation, expressed using the right adjoint to exponentiation with $\mathbb{I}$ to get a fiberwise notion.

For A in $\mathsf{Type}(1)$, we write $|A|$ for $\mathsf{Elem}(1, A)$. Set $\mathsf{Pred}_n = \sum_{A:\mathsf{Type}(1)} |A| \to \mathcal{U}_n$. For $\lambda_i\,(A_i, A_i')$ in $\mathsf{Pred}_\omega^\mathbb{I}$, we define $\mathsf{Red}(\lambda_i\,(A_i, A_i'))$ to be the type of operations c taking as argument ψ in $\mathbb{F}$ and a family u_i, u_i' for $[\psi] \vee (i = 0)$ with u_i in A_i and u_i' in $A_i'\,u_i$ and producing an element $c(\psi, u, u')$ in $A_1'(\mathsf{comp}(A, \psi, u))$ which is equal to u_1' for $\psi = 1$.

Proposition 6.1. *We have $R : \mathsf{Pred}_\omega \to \mathcal{U}_\omega$ with, naturally in global A and $B : A \to \mathsf{Pred}_\omega$, a bijection between $\square(\prod_A R \circ B)$ and $\square(\prod_{A^\mathbb{I}} \mathsf{Red} \circ B^\mathbb{I})$. For $n \geq 0$, we have that R descends to an operation $R : \mathsf{Pred}_n \to \mathcal{U}_n$.*

Proof. This follows from Lemma 2.2. $\qquad\square$

Proposition 6.2. *Given e in $\prod_{i:\mathbb{I}} R(A_i, A_i')$, there is an operation $c(e)$ which, given ψ in $\mathbb{F}$ and u_i' in $A_i'\,u_i$ for $[\psi] \vee (i = 0)$, produces an element $c(e)(\psi, u, u')$ in $A_1'(\mathsf{comp}(\lambda_i\,A_i, \psi, u))$ which is equal to u_1' for $[\psi]$.*

Proof. This follows from Proposition 6.1 by setting $T = \sum_{\mathsf{Pred}_\omega} R$. $\qquad\square$

Remark 6.3. Define $C(X)$ for $X : \mathcal{U}_\omega$ as in Subsubsection 2.2.1, but using composition instead of filling. As in Section 3, we have a map $k : \prod_{A:\mathsf{Type}(1)} C(|A|)$. In the same fashion as above for R, one may construct C' over $\sum_{A:\mathcal{U}_\omega} \sum_{c:C(A)} A' : A \to \mathcal{U}_\omega$ encoding that the family A' has "composition over c". Then R can be defined as the restriction of C' along the map induced by k. Under the equivalence between families $A' : A \to \mathcal{U}_\omega$ over A and $\overline{A} : \mathcal{U}_\omega$ with a map $p : \overline{A} \to A$, this corresponds to an element of $C(\overline{A})$ such that p forms a composition-preserving morphism of fibrant types, a notion defined (like C and C') using the adjunction recorded in Lemma 2.2.

The closure properties of R under type formers proved in Subsection 6.2 and Lemma 6.6 below can be proved at the level of C'. This has the advantage of eliminating the dependency on the computational cubical cwf $\mathcal{M}$ (in the case of natural numbers, the input is instead a natural number algebra in fibrant types). Then the actual sconing construction can proceed without external reasoning as in Section 3.

The sconing of the standard model for computational cwfs (defined as $\mathcal{S}$ in Subsection 2.3, but using composition instead of filling) will coincide, externally, with the standard model constructed internally in presheaves over $[1] \times \mathcal{C}$ (where $[1]$ is the poset with elements $0 < 1$) with interval object and cofibration classifier defined by projection to $\mathcal{C}$.

We define $\mathsf{Type}^*(\Gamma, \Gamma')$ to be the set of triples (A, A', e_A) where A is in $\mathsf{Type}(\Gamma)$ and $A' \, \rho \, \rho'$ is in $|A \, \rho| \to \mathcal{U}_\omega$ for ρ in $|\Gamma|$ and ρ' in $\Gamma' \, \rho$ and $e_A \, \rho \, \rho'$ is in $R(A\rho, A \, \rho \, \rho')$ for ρ and ρ' as before.

We define $\mathsf{Elem}^*((\Gamma, \Gamma'), (A, A', e_A))$ to be the set of pairs (a, a') where a is in $\mathsf{Elem}(\Gamma, A)$ and $a' \, \rho \, \rho'$ is in $A' \, \rho \, \rho' \, (a\rho)$ for ρ in $|\Gamma|$ and ρ' in $\Gamma' \, \rho$.

Since the elements do not make use of the last component of the triple defining a type, the operations involving context extension are defined as in Subsection 3.1. For example, we define $(\Gamma, \Gamma').(A, A', e_A)$ to be $(\Gamma.A, (\Gamma.A'))$ where $(\Gamma.A)'(\rho, u)$ is $\sum_{\rho':\Gamma' \, \rho} A' \, \rho \, \rho' \, u$.

6.2. Example of dependent sum types.
Before explaining the example of the dependent sum type, we need the following preliminary lemma. It intuitively says that the filling operation is computable if the composition operation is computable.

Lemma 6.4. *Given e in $\prod_{i:\mathbb{I}} R(A_i, A_i')$, the filling operation on A is "computable": for ψ in $\mathbb{F}$ and a partial family u_i of elements in $|A_i|$ together with u_i' in $A_i' \, u_i$ defined for $[\psi] \vee (i = 0)$, then for any r in $\mathbb{I}$ we have*

$$\mathtt{fill}'(A, \psi, u, u') \, r : A_r' \, (\mathtt{fill}(A, \psi, u) \, r)$$

equal to u_r' for $[\psi] \vee (r = 0)$.

Proof. Given r in $\mathbb{I}$, we define e_r in $\prod_{i:\mathbb{I}} R(A_{i\wedge r}, A_{i\wedge r}')$ by $e_r \, i = e_{r\wedge i}$. Using Proposition 6.2, we can define

$$\mathtt{fill}'(A, \psi, u, u') \, r = c(e_r)(\psi, u_r, u_r')$$

with $u_r \, i \, x = u \, (i \wedge r) \, x$ and $u_r' \, i \, x = u' \, (i \wedge r) \, x$. $\qquad \square$

Given (A, A') in Pred_ω and $B : \mathsf{Type}(1.A)$ and $B' : \prod_{u:|A|} A' \, u \to |B[u]| \to \mathcal{U}_\omega$ we define $\Sigma(A, B)'$ by

$$\Sigma(A, B)' \, w = \sum_{u':A' \, (\mathtt{fst}(w))} B' \, (\mathtt{fst}(w)) \, u' \, (\mathtt{snd}(w)).$$

We then want to define an operation

$$R(A, A') \to (\textstyle\prod_{u:|A|, u':A' \, u} R(B[u], B' \, u \, u')) \to R(\Sigma(A, B), \Sigma(A, B)').$$

In order to do this, we consider the iterated dependent sum Δ corresponding to the context

$$\begin{aligned}
(A, A') &: \mathsf{Pred}_n, \\
B &: \mathsf{Type}_n(A), \\
B' &: \textstyle\prod_{u:|A|} A' \, u \to |B[u]| \to \mathcal{U}_n, \\
e_A &: R(A, A'), \\
e_B &: \textstyle\prod_{u:|A|} \prod_{u':A' \, u} R(B[u], B' \, u \, u').
\end{aligned}$$

We want to build a global element of

$$\textstyle\prod_{(A, A', B, B', e_A, e_B):\Delta} R(\Sigma(A, B), \Sigma(A, B)').$$

Using Proposition 6.1, we are reduced to show the following statement.

Proposition 6.5. *There is an element of*

$$\textstyle\prod_{\lambda_i \, (A_i, A_i', B_i, B_i', e_A^i, e_B^i):\Delta^{\mathbb{I}}} \mathsf{Red}(\lambda_i \, (\Sigma(A_i, B_i), \Sigma(A_i, B_i)')).$$

Proof. We assume a family A_i, A'_i in Pred_n and e^i_A in $R(A_i, A'_i)$ for $i : \mathbb{I}$. We also have $B_i : \mathsf{Type}_n(A_i)$ and B'_i in $\prod_{u:|A_i|} A'_i u \to |B_i[u]| \to \mathcal{U}_n$ and $e^i_B u u'$ in $R(B_i[u], B'_i u u')$. We also have ψ in $\mathbb{F}$ with $\mathtt{pair}(a_i, b_i) : \Sigma(A_i, B_i)$ and a'_i in $A'_i a_i$ and b'_i in $B'_i a_i a'_i b_i$ defined for $[\psi] \vee (i = 0)$.

Given all this, we want to build

$$(a'_1, b'_1) : \Sigma(A_1, B_1)'(\mathtt{comp}(\Sigma(A, B), \psi, (a, b))).$$

By the computation rule for Σ, we have

$$\mathtt{comp}(\Sigma(A, B), \psi, (a, b)) = (\mathtt{comp}(A, \psi, a), \mathtt{comp}(\lambda_i B[u\,i], \psi, b))$$

where $u = \mathtt{fill}(A, \psi, a)$.

Using Lemma 6.4, we have $u' = \mathtt{fill}'(A, \psi, a, a')$ and we define $a'_1 = u'\, 1$. We next define $e_B\, i = e^i_B (u\,i)(u'\,i)$, and using Proposition 6.2, we take $b'_1 = c(e_B)(\psi, b, b')$. $\qquad\square$

We can use these results to interpret dependent sum types in the model $\mathcal{M}^*$, following essentially the interpretation in [Coq19].

Given (A, A', e_A) in $\mathsf{Type}^*(\Gamma, \Gamma')$ and (B, B', e_B) in $\mathsf{Type}^*((\Gamma, \Gamma').(A, A', e_A))$ we define T, T', e_T where $T = \Sigma(A, B)$ in $\mathsf{Type}^*(\Gamma, \Gamma')$.

Given ρ and ρ' we have $A_1 = A\rho$ in $\mathsf{Type}(1)$ and $A'_1 = A' \rho \rho'$ in $|A_1| \to \mathcal{U}_\omega$. We also have $B_1 = B\rho^+$ in $\mathsf{Type}(A_1)$ and $B_1 u u' = B'(\rho, \mathtt{fst}(w))(\rho', u')$ in $\prod_{u:|A_1|} A'_1 u \to |B_1[u]| \to \mathcal{U}_\omega$ since $B_1[u] = B\rho^+[u] = B(\rho, u)$. We can then define $T' \rho \rho'$ to be $\Sigma(A_1, B_1)'$ and use (1) to define $e_T \rho \rho'$.

6.3. Natural numbers.
Let $\mathbb{N}$ be the (internal) set of natural numbers (given by the constant presheaf of natural numbers). We have a canonical map $\mathtt{quote} : \mathbb{N} \to |\mathbb{N}|$ sending k to $\mathbf{S}^k(\mathbf{0})$. We define a (non-fibrant) family $\mathbb{N}'$ over $|\mathbb{N}|$ by $\mathbb{N}'(t) = \sum_{k:\mathbb{N}} t =_{|\mathbb{N}|} \mathtt{quote}(k)$ (using the *strict* equality on the set $|\mathbb{N}|$).[11]

Lemma 6.6. *We have an element of $R(\mathbb{N}, \mathbb{N}')$.*

Proof. Using the adjoint definition of R in Proposition 6.1, we must build $c : \mathsf{Red}(\lambda_i (\mathbb{N}, \mathbb{N}'))$. Exponentiation with $\mathbb{I}$ preserves external coproducts since $\mathbb{I}$ is tiny. Since $\mathbb{N}$ is a countable coproduct of 1, it follows that any function $\mathbb{I} \to \mathbb{N}$ is constant (formally, factors uniquely through $\mathbb{I} \to 1$).

Let $u_i : |\mathbb{N}|$ and $u'_i : \mathbb{N}' u_i$ for $[\psi] \vee (i = 0)$. The latter means $k_i : \mathbb{N}$ such that $u_i = \mathtt{quote}(k_i)$ for $[\psi] \vee (i = 0)$. Using the observation from the previous paragraph, there is unique $k : \mathbb{N}$ such that $k_i = k$ for $[\psi] \vee (i = 0)$. From the equations for the composition operation on $\mathbb{N}$ in a computational cubical cwf and induction on k, we get that

$$\mathtt{comp}(\lambda_i \mathbb{N}, \psi, \lambda_{i,x} u_i) = \mathtt{comp}(\lambda_i \mathbb{N}, \psi, \lambda_{i,x} \mathtt{quote}(k)) = \mathtt{quote}(k).$$

We are forced to set $c(\psi, u, u') = (k, \mathtt{tt})$. $\qquad\square$

This provides the interpretation of the type of natural numbers in the model $\mathcal{M}^*$.

We see here a key difference compared to the sconing model used for proving homotopy canonicity: the computability predicate used in this case is not valued in fibrant sets. Note

---

[11]An isomorphic alternative is to define $\mathbb{N}'$ as a (*non-fibrant*) indexed inductive set in the presheaf model, with constructors of type $\mathbb{N}'\, \mathbf{0}$ and $\mathbb{N}'\, n \to \mathbb{N}'\, (\mathbf{S}\, n)$ for $n : |\mathbb{N}|$. Indeed, it is this approach that generalizes to the interpretation of inductive types with parameters.

that the family $\sum_{k:\mathbb{N}} |\mathsf{Path}(\mathbb{N}, t, \mathsf{quote}(k))|$ for $t : |\mathbb{N}|$ is fibrant, but would not work for showing canonicity since it does not support an interpretation of `natrec`.

6.4. **Proof of canonicity.** Starting from any computational cubical cwf $\mathcal{M}$, we have built a new model $\mathcal{M}^*$, the associated *computability model*, with a (strict) projection map $\mathcal{M}^* \to \mathcal{M}$. Like in [Coq19], if we apply this to the initial model, we get that any closed term of type $\mathbb{N}$ is "computable", i.e., is *strictly* equal to a numeral.

Conclusion

We have given proofs of two forms of canonicity for cubical type theory. The first one is homotopy canonicity (every closed term of type $\mathbb{N}$ is *path equal* to a numeral) in a cubical type theory without structural computation rules for the composition operation. The second one is canonicity (every closed term of type $\mathbb{N}$ is *strictly equal* to a numeral) in a cubical type theory with these computation rules. While our arguments rely on an interplay between internal and external reasoning, the main part of the first argument can be seen as happening internally in the model of fibrant sets. The second argument can hopefully be refined to a constructive proof of normalisation.

References

[ABC+21] C. Angiuli, G. Brunerie, T. Coquand, R. Harper, K.-B. Hou (Favonia), and D. R. Licata. Syntax and models of cartesian cubical type theory. *Mathematical Structures in Computer Science*, 31:424–468, 2021. `doi:10.1017/S0960129521000347`.

[Acz99] P. Aczel. On relating type theories and set theories. In T. Altenkirch, B. Reus, and W. Naraschewski, editors, *Types for Proofs and Programs*, volume 1657 of *Lecture Notes in Computer Science*, pages 1–18. Springer Verlag, Berlin, Heidelberg, New York, 1999. `doi:10.1007/3-540-48167-2_1`.

[BCH14] M. Bezem, T. Coquand, and S. Huber. A model of type theory in cubical sets. In R. Matthes and A. Schubert, editors, *19th International Conference on Types for Proofs and Programs (TYPES 2013)*, volume 26 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 107–128. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2014. `doi:10.4230/LIPIcs.TYPES.2013.107`.

[CCHM18] C. Cohen, T. Coquand, S. Huber, and A. Mörtberg. Cubical type theory: A constructive interpretation of the univalence axiom. In T. Uustalu, editor, *21st International Conference on Types for Proofs and Programs (TYPES 2015)*, volume 69 of *Leibniz International Proceedings in Informatics (LIPIcs)*. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2018. `doi:10.4230/LIPIcs.TYPES.2015.5`.

[CHM18] T. Coquand, S. Huber, and A. Mörtberg. On higher inductive types in cubical type theory. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '18, pages 255–264, New York, NY, USA, 2018. ACM. `doi:10.1145/3209108.3209197`.

[Coq18] T. Coquand. A survey of constructive presheaf models of univalence. *ACM SIGLOG News*, 5(3):54–65, 2018. `doi:10.1145/3242953.3242962`.

[Coq19] T. Coquand. Canonicity and normalization for dependent type theory. *Theoretical Computer Science*, 777:184–191, 2019. `doi:10.1016/j.tcs.2019.01.015`.

[Dyb96] P. Dybjer. Internal Type Theory. In *Lecture Notes in Computer Science*, pages 120–134. Springer Verlag, Berlin, Heidelberg, New York, 1996. `doi:10.1007/3-540-61780-9_66`.

[Ehr88] T. Ehrhard. *Une sémantique catégorique des types dépendants: Application au Calcul des Constructions*. Phd thesis, University Paris VII, 1988.

[Eil39] S. Eilenberg. On the relation between the fundamental group of a space and the higher homotopy groups. *Fundamenta Mathematicae*, 32(1):169–175, 1939. `doi:10.4064/fm-32-1-167-175`.

[GH03]   N. Gambino and M. Hyland. Wellfounded trees and dependent polynomial functors. In *International Workshop on Types for Proofs and Programs*, pages 210–225. Springer, 2003. `doi:10.1007/978-3-540-24849-1_14`.

[GK13]   N. Gambino and J. Kock. Polynomial functors and polynomial monads. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 154, pages 153–192. Cambridge University Press, 2013. `doi:10.1017/S0305004112000394`.

[Göd58]   K. Gödel. Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes. *Dialectica*, 12:280–287, 1958. `doi:10.1111/j.1746-8361.1958.tb01464.x`.

[GS17]   N. Gambino and C. Sattler. The Frobenius condition, right properness, and uniform fibrations. *Journal of Pure and Applied Algebra*, 221(12):3027–3068, 2017. `doi:10.1016/j.jpaa.2017.02.013`.

[GZ67]   P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*, volume 35 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 1967. `doi:10.1007/978-3-642-85844-4`.

[Hof97]   M. Hofmann. Syntax and semantics of dependent types. In A.M. Pitts and P. Dybjer, editors, *Semantics and logics of computation*, volume 14 of *Publ. Newton Inst.*, pages 79–130. Cambridge University Press, Cambridge, 1997. `doi:10.1007/978-1-4471-0963-1_2`.

[Hub19]   S. Huber. Canonicity for cubical type theory. *Journal of Automated Reasoning*, 63:172–210, 2019. `doi:10.1007/s10817-018-9469-1`.

[Joy17]   A. Joyal. Notes on clans and tribes, 2017.

[Kel80]   G.M. Kelly. A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on. *Bulletin of the Australian Mathematical Society*, 22(1):1–83, 1980. `doi:10.1017/S0004972700006353`.

[KL12]   K. Kapulkin and P. LeFanu Lumsdaine. The simplicial model of univalent foundations (after Voevodsky). *Journal of the European Mathematical Society*, 23:2071–2126, 2012. `doi:10.4171/JEMS/1050`.

[KV20]   K. Kapulkin and V. Voevodsky. A cubical approach to straightening. *Journal of Topology*, 13:1682–1700, 12 2020. `doi:10.1112/topo.12173`.

[LOPS18]   D. R. Licata, I. Orton, A. M. Pitts, and B. Spitters. Internal universes in models of homotopy type theory. In Hélène Kirchner, editor, *3rd International Conference on Formal Structures for Computation and Deduction (FSCD 2018)*, volume 108 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 22:1–22:17. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2018. `doi:10.4230/LIPIcs.FSCD.2018.22`.

[LS20]   P. LeFanu Lumsdaine and M. Shulman. Semantics of higher inductive types. *Mathematical Proceedings of the Cambridge Philosophical Society*, 169(1):159–208, 2020. `doi:10.1017/S030500411900015X`.

[OP16]   I. Orton and A. M. Pitts. Axioms for modelling cubical type theory in a topos. In *25th EACSL Annual Conference on Computer Science Logic (CSL 2016)*, volume 62 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 24:1–24:19, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum für Informatik. `doi:10.4230/LIPIcs.CSL.2016.24`.

[PV07]   E. Palmgren and S. J. Vickers. Partial Horn logic and cartesian categories. *Annals of Pure and Applied Logic*, 145(3):314–353, 2007. `doi:10.1016/j.apal.2006.10.001`.

[RS18]   E. Riehl and M. Shulman. A type theory for synthetic ∞-categories. *Higher Structures*, 1(1), 2018.

[Shu15]   M. Shulman. Univalence for inverse diagrams and homotopy canonicity. *Mathematical Structures in Computer Science*, 25(5):1203–1277, 2015. `doi:10.1017/S0960129514000565`.

[Ste19]   J. Sterling. Algebraic type theory and universe hierarchies. *arXiv preprint arXiv:1902.08848*, 2019.

[Str91]   T. Streicher. *Semantics of type theory: correctness, completeness, and independence results*. Progress in Theoretical Computer Science. Birkhäuser Basel, 1991. `doi:10.2307/2275776`.

[Swa16]   A. Swan. An algebraic weak factorisation system on 01-substitution sets: A constructive proof. *Journal of Logic & Analysis*, 8(1):1–35, 2016. `doi:10.4115/jla.2016.8.1`.

[Swa17]   A. Swan. Semantics of higher inductive types, 2017. On the HoTT mailing list.

[Tai67]   W. W. Tait. Intensional interpretations of functionals of finite type I. *Journal of Symbolic Logic*, 32(2):198–212, 1967. `doi:10.2307/2271658`.

[Uni13]   The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. http://homotopytypetheory.org/book, Institute for Advanced Study, 2013.

[Voe14]   V. Voevodsky. The equivalence axiom and univalent models of type theory. (Talk at CMU on February 4, 2010). Preprint arXiv:1402.5556 [math.LO], 2014. URL: https://arxiv.org/pdf/1402.5556.pdf.

Appendix A. Rules of the term model

We denote the objects of our base category $\mathcal{C}$ by X, Y, Z and its morphisms by f, g, h. In the term model $\mathcal{T}$ morphisms $f \colon Y \to X$ act on judgments at stage X via an implicit substitution, while for substitutions on object variables we will use explicit substitutions. For this to make sense we first define the raw expressions as a presheaf: at stage X this is given by

$$
\begin{aligned}
\Gamma, \Delta \quad &::= \quad \varepsilon \mid \Gamma.A \\
A, B, t, u, v \quad &::= \quad \mathsf{q} \mid t\sigma \mid \mathsf{U}_n \mid \Pi(A, B) \mid \mathtt{lam}(u) \mid \mathtt{app}(u, v) \\
&\quad\mid \quad \Sigma(A, B) \mid \mathtt{pair}(u, v) \mid \mathtt{fst}(u) \mid \mathtt{snd}(u) \\
&\quad\mid \quad \mathtt{Path}(\bar{A}, u, v) \mid \langle\rangle\bar{u} \mid \mathtt{ap}(u, r) \\
&\quad\mid \quad \mathtt{Glue}_c(A, \varphi, \bar{B}, \bar{u}) \mid \mathtt{glue}(v, \bar{u}) \mid \mathtt{unglue}(u) \\
&\quad\mid \quad \mathtt{fill}(\bar{A}, \varphi, b, \bar{u}, r) \mid \ldots \\
\bar{A}, \bar{B}, \bar{u}, \bar{v} \quad &::= \quad (A_{f,r})_{f,r} \mid (A_f)_{f \in [\varphi]} \\
\sigma, \tau, \delta \quad &::= \quad \mathsf{p} \mid \mathsf{id} \mid \sigma\tau \mid (\sigma, u) \mid ()
\end{aligned}
$$

where $b \in \{0, 1\}$, $\varphi \in \mathbb{F}(X)$, and we skipped the constants for natural numbers. Above, we have families of expressions, say $\bar{A} = (A_{f,r})_{f,r}$, whose index set ranges over certain Y, $f \colon Y \to X$, and $r \in \mathbb{I}(Y)$, and $A_{f,r}$ is a raw expression at stage Y; likewise $(A_f)_{f \in [\varphi](X)}$ consists of raw expressions A_f at stage Y for $f \colon Y \to X$ in the sieve $[\varphi]$ on X. (The exact index sets will be clear from the typing rules below.) All other occurrences of r above have $r \in \mathbb{I}(X)$. The restrictions along $f \colon Y \to X$ on the raw syntax then leave all the usual cwf structure untouched, so we have $\mathsf{q}f = \mathsf{q}$ and $(\Pi(A, B))f = \Pi(Af, Bf)$, and uses the restrictions in $\mathbb{I}$ and $\mathbb{F}$ accordingly, e.g., $(\mathtt{ap}(u, r))f = \mathtt{ap}(uf, rf)$, and we will re-index families according to $\bar{A}f = (A_{gf,rf})$ for $\bar{A} = (A_{g,r})_{g,r}$.

To get the *initial* cubical cwf we in fact need more annotations to the syntax in order to be able to define a partial interpretation (cf. [Str91, Hof97]) on the raw syntax. But to enhance readability we suppress these annotations.

We will now describe a type system indexed by stages X. The forms of judgment are:

$$
\Gamma \vdash_X \qquad \Gamma \vdash_X A \qquad \Gamma \vdash_X A = B \qquad \Gamma \vdash_X t : A \qquad \Gamma \vdash_X t = u : A \qquad \sigma \colon \Delta \to_X \Gamma
$$

where the involved expressions are at stage X.

Remark A.1. In cubical type theory as described in [CCHM18] we did not index judgments by objects X but allowed extending context by interval variables instead. Loosely speaking, a judgment $\Gamma \vdash_{\{i_1, \ldots, i_n\}} \mathcal{J}$ corresponds to $i_1 : \mathbb{I}, \ldots, i_n : \mathbb{I}, \Gamma \vdash \mathcal{J}$ given the setting of [CCHM18].

As mentioned above we have the rule:

$$
\frac{\Gamma \vdash_X \mathcal{J} \qquad f \colon Y \to X}{\Gamma f \vdash_Y \mathcal{J}f}
$$

At each stage we have all the usual rules valid in a cwf with Π-types, Σ-types, universes, and natural numbers. We will present some of the rules, but skip all congruence rules.

$$\frac{}{\varepsilon \vdash_X} \qquad \frac{\Gamma \vdash_X \qquad \Gamma \vdash_X A}{\Gamma.A \vdash_X} \qquad \frac{\Gamma \vdash_X A \qquad \sigma: \Delta \to_X \Gamma}{\Delta \vdash_X A\sigma} \qquad \frac{\Gamma \vdash_X t: A \qquad \sigma: \Delta \to_X \Gamma}{\Delta \vdash_X t\sigma: A\sigma}$$

$$\frac{\Gamma \vdash_X A}{\Gamma.A \vdash_X \mathsf{q}: A\mathsf{p}} \qquad \frac{\Gamma \vdash_X t: A \qquad \Gamma \vdash_X A = B}{\Gamma \vdash_X t: B} \qquad \frac{\Gamma \vdash_X}{\mathsf{id}: \Gamma \to_X \Gamma} \qquad \frac{\Gamma \vdash_X}{(): \Gamma \to \varepsilon}$$

$$\frac{\Gamma \vdash_X A}{\mathsf{p}: \Gamma.A \to_X \Gamma} \qquad \frac{\sigma: \Delta \to_X \Gamma \qquad \tau: \Theta \to_X \Delta}{\sigma\tau: \Theta \to_X \Gamma}$$

$$\frac{\sigma: \Delta \to_X \Gamma \qquad \Gamma \vdash_X A \qquad \Delta \vdash_X u: A\sigma}{(\sigma, u): \Delta \to_X \Gamma.A}$$

$$\frac{\Gamma.A \vdash_X B}{\Gamma \vdash_X \Pi(A, B)} \qquad \frac{\Gamma.A \vdash_X B \qquad \Gamma.A \vdash_X b: B}{\Gamma \vdash_X \mathtt{lam}(b): \Pi(A, B)} \qquad \frac{\Gamma \vdash_X w: \Pi(A, B) \qquad \Gamma \vdash_X u: A}{\Gamma \vdash_X \mathtt{app}(w, u): B[u]}$$

where we write $[u]$ for (id, u) and σ^+ for $(\sigma\mathsf{p}, \mathsf{q})$. The judgmental equalities (skipping suitable premises, types, and contexts) are:

$$\mathsf{id}\,\sigma = \sigma\,\mathsf{id} = \sigma \qquad (\sigma\tau)\delta = \sigma(\tau\delta) \qquad ()\sigma = () \qquad (\sigma, u)\delta = (\sigma\delta, u\delta) \qquad \mathsf{p}(\sigma, u) = \sigma$$

$$\mathsf{q}(\sigma, u) = u \qquad (\mathsf{p}, \mathsf{q}) = \mathsf{id} \qquad A\,\mathsf{id} = A \qquad (A\sigma)\delta = A(\sigma\delta) \qquad u\,\mathsf{id} = u \qquad (u\sigma)\delta = u(\sigma\delta)$$

$$(\Pi(A, B))\sigma = \Pi(A\sigma, B\sigma^+) \qquad (\mathtt{lam}(b))\sigma = \mathtt{lam}(b\sigma^+) \qquad \mathtt{app}(w, u)\delta = \mathtt{app}(w\delta, u\delta)$$

$$\mathtt{app}(\mathtt{lam}(b), u) = b[u] \qquad\qquad w = \mathtt{lam}(\mathtt{app}(w\mathsf{p}, \mathsf{q}))$$

We skip the rules for Σ-types and natural numbers as they are standard, but simply indexed with an object X as we did for Π-types. The rules for universes are:

$$\frac{\Gamma \vdash_X}{\Gamma \vdash_X \mathsf{U}_n} \qquad \frac{\Gamma \vdash_X}{\Gamma \vdash_X \mathsf{U}_n: \mathsf{U}_{n+1}} \qquad \frac{\Gamma \vdash_X A: \mathsf{U}_n}{\Gamma \vdash_X A: \mathsf{U}_{n+1}} \qquad \frac{\Gamma \vdash_X A: \mathsf{U}_n}{\Gamma \vdash_X A}$$

and we skip the rules for equality and closure under the type formers Π, Σ, natural numbers, $\mathtt{Path}$, and $\mathtt{Glue}_c$.

To state the rules for dependent path-types we introduce the following abbreviations. We write $\Gamma.\mathbb{I} \vdash_X \bar{A}$ if $\bar{A} = (A_{f,r})$ is a family indexed by Y, $f: Y \to X$, and $r \in \mathbb{I}(Y)$ such that

$$\Gamma f \vdash_Y A_{f,r} \quad \text{and} \quad \Gamma fg \vdash_Z (A_{f,r})g = A_{fg,r}.$$

Given $\Gamma.\mathbb{I} \vdash_X \bar{A}$ we write $\Gamma.\mathbb{I} \vdash_X \bar{u}: \bar{A}$ whenever $\bar{u} = (u_{f,r})$ is a family indexed by Y, $f: Y \to X$, and $r \in \mathbb{I}(Y)$ such that

$$\Gamma f \vdash_Y u_{f,r}: A_{f,r} \quad \text{and} \quad \Gamma fg \vdash_Z (u_{f,r})g = u_{fg,rg}: A_{fg,rg}.$$

The rules for the dependent path type are:

$$\frac{\Gamma.\mathbb{I} \vdash_X \bar{A} \qquad \Gamma \vdash_X u : A_{\mathsf{id}_X,0} \qquad \Gamma \vdash_X u : A_{\mathsf{id}_X,1}}{\Gamma \vdash_X \mathtt{Path}(\bar{A}, u, v)} \qquad\qquad \frac{\Gamma.\mathbb{I} \vdash_X \bar{A} \qquad \Gamma.\mathbb{I} \vdash_X \bar{u} : \bar{A}}{\Gamma \vdash_X \mathtt{lam}(\bar{u}) : \mathtt{Path}(\bar{A}, u_{\mathsf{id}_X,0}, u_{\mathsf{id}_X,1})}$$

$$\frac{\Gamma \vdash_X t : \mathtt{Path}(\bar{A}, u, v) \qquad r \in \mathbb{I}(X)}{\Gamma \vdash_X \mathtt{ap}(t, r) : A_{\mathsf{id}_X,r}}$$

$$\mathtt{ap}(\mathtt{lam}(\bar{u}), r) = u_{\mathsf{id},r} \qquad t = \mathtt{lam}(\mathtt{ap}(tf, r)_{f,r}) \qquad \mathtt{Path}(\bar{A}, u, v)\sigma = \mathtt{Path}((A_{f,r}\sigma f)_{f,r}, u\sigma, v\sigma)$$

$$(\mathtt{lam}(\bar{u}))\sigma = \mathtt{lam}((u_{f,r}\sigma f)_{f,r}) \qquad\qquad (\mathtt{ap}(t, r))\sigma = \mathtt{ap}(t\sigma, r)$$

Note that in general these rules might have infinitely many premises. We get the non-dependent path type for $\Gamma \vdash_X A$ by using the family $A_{f,r} := Af$.

Given $\Gamma.\mathbb{I} \vdash_X \bar{A}$ and $b \in \{0,1\}$ we write $\Gamma.\mathbb{I} \vdash_X^{\varphi,b} \bar{u} : \bar{A}$ for $\bar{u} = (u_{f,r})$ a family indexed over all Y, $f : Y \to X$, and $r \in \mathbb{I}(Y)$ such that either f is in the sieve $[\varphi]$ or $r = b$ and we have

$$\Gamma f \vdash_Y u_{f,r} : A_{f,r} \text{ and } \Gamma fg \vdash_Z (u_{f,r})g = u_{fg,rg} : A_{fg,rg}$$

for all $g : Z \to Y$. The rule for the filling operation is given by:

$$\frac{\Gamma.\mathbb{I} \vdash_X \bar{A} \qquad \varphi \in \mathbb{F}(X) \qquad b \in \{0,1\} \qquad \Gamma.\mathbb{I} \vdash_X^{\varphi,b} \bar{u} : \bar{A} \qquad r \in \mathbb{I}(X)}{\Gamma \vdash_Y \mathtt{fill}(\bar{A}, \varphi, b, \bar{u}, r) : A_{\mathsf{id},r}}$$

with judgmental equality

$$\mathtt{fill}(\bar{A}, \varphi, b, \bar{u}, r) = u_{\mathsf{id},r} \text{ whenever } [\varphi] \text{ is the maximal sieve or } r = b.$$

For the glueing operation we only present the formation rule; the other rules are similar as in [CCHM18] but adapted to our setting. We write $\Gamma \vdash_X^\varphi \bar{B}$ if $\bar{B}$ is a family of B_f for $f : Y \to X$ in $[\varphi]$ with $\Gamma f \vdash_Y B_f$ which is compatible, i.e. $\Gamma fg \vdash_Z B_f g = B_{fg}$. In this case, we write likewise $\Gamma \vdash_X \bar{u} : \bar{B}$ if $\bar{u}$ is a compatible family of terms $\Gamma f \vdash_Y u_f : B_f$.

$$\frac{\Gamma \vdash_X A \qquad \varphi \in \mathbb{F}(X) \qquad \Gamma \vdash_X^\varphi \bar{B} \qquad \Gamma \vdash_X^\varphi \bar{u} : \mathtt{isEquiv}_c(\bar{B}, A)}{\Gamma \vdash_X \mathtt{Glue}_c(A, \varphi, \bar{B}, \bar{u})}$$

and the judgmental equality $\mathtt{Glue}_c(A, \varphi, \bar{B}, \bar{u}) = B_{\mathsf{id}}$ in case $[\varphi]$ is the maximal sieve, and an equation for substitution.

This formal system gives rise to a cubical cwf $\mathcal{T}$ as follows. First, define judgmental equality for contexts and substitutions as usual (we could also have those as primitive judgments). Next, we define presheaves Con and Hom on $\mathcal{C}$ by taking, say, $\mathsf{Con}(X)$ equivalence classes $[\Gamma]_\sim$ of Γ with $\Gamma \vdash_X$ modulo judgmental equality; restrictions are induced by the (implicit) substitution: $[\Gamma]_\sim f = [\Gamma f]_\sim$. Types $\mathsf{Type}(X, [\Gamma]_\sim)$ are equivalence classes of A with $\Gamma \vdash_X A$ modulo judgmental equality, and elements are defined similarly as equivalence classes.

For type formers in $\mathcal{T}$ let us look at path types: we have to give an element of $\mathsf{Type}(\Gamma)$ in a context (w.r.t. the internal language) $\Gamma : \mathsf{Con}, A : \mathbb{I} \to \mathsf{Type}(\Gamma), u : \mathsf{Elem}(\Gamma, A\,0), v : \mathsf{Elem}(\Gamma, A\,1)$. Unfolding the use of internal language, given $[\Gamma]_\sim \in \mathsf{Con}(X)$, a compatible family $[A_{f,r}]_\sim \in \mathsf{Type}(Y, [\Gamma]_\sim f)$ (for $f : Y \to X$ and $r \in \mathbb{I}(Y)$) and elements $[u]_\sim \in \mathsf{Elem}([\Gamma]_\sim, [A_{\mathsf{id},0}]_\sim)$ and $[v]_\sim \in ([\Gamma]_\sim, [A_{\mathsf{id},1}]_\sim)$, we have to give an element of $\mathsf{Type}(X, [\Gamma]_\sim)$, which we do by the formation rule for $\mathtt{Path}$.

The remainder of the cubical cwf structure for $\mathcal{T}$ is defined in a similar manner, in fact the rules are designed to reflect the laws of cubical cwfs. We conjecture that we can follow a similar argument as in [Str91] to show that $\mathcal{T}$ is the *initial* cubical cwf. Given a cubical cwf $\mathcal{M}$ over $\mathcal{C}, \mathbb{I}, \mathbb{F}$ we first have to define *partial* interpretations of the raw syntax and then show that each derivable judgment has a defined interpretation in $\mathcal{M}$, and for equality judgments both sides of the equation have a defined interpretation in $\mathcal{M}$ and are equal. In an intuitionistic framework, this partial interpretation should be described as an inductively defined relation, which is shown to be functional. The partial interpretation $[\![-]\!]$ assigns meanings to raw judgments with the following signature:

$$\begin{array}{lcl}
[\![\Gamma \vdash_X]\!] & \in & \mathsf{Con}_{\mathcal{M}}(X) \\
[\![\sigma : \Delta \to \Gamma]\!] & \in & \mathsf{Hom}_{\mathcal{M}}(X, [\![\Delta \vdash_X]\!], [\![\Gamma \vdash_X]\!]) \\
[\![\Gamma \vdash_X A]\!] & \in & \mathsf{Type}_{\mathcal{M}}(X, [\![\Gamma \vdash_X]\!]) \\
[\![\Gamma \vdash_X u : A]\!] & \in & \mathsf{Elem}_{\mathcal{M}}(X, [\![\Gamma \vdash_X]\!], [\![\Gamma \vdash_X A]\!])
\end{array}$$

where among the conditions for the interpretation on the left-hand side to be defined is that all references to the interpretation on the right-hand side are defined. This proceeds by structural induction on the raw syntax and for $[\![\Gamma \vdash_X \mathcal{J}]\!]$ to be defined we assume all the ingredients needed are already defined. E.g. for the path type $[\![\Gamma \vdash_X \mathtt{Path}(\bar{A}, u, v)]\!]$ we in particular have to assume that the assignment $f, r \mapsto [\![\Gamma f \vdash_Y A_{f,r}]\!]$ is defined and gives rise to a suitable input of $\mathtt{Path}_{\mathcal{M}}$.

Appendix B. Indexed inductive sets in presheaves over $\mathcal{C}$

We work in the setting of Subsection 2.2 given by presheaves over $\mathcal{C}$.

Given a set I, a family A over I, a family B over $i : I$, an element $a : A\,i$, and a map

$$s : \prod_{i:I} \prod_{a:A\,i} B\,i\,a \to I,$$

the *indexed inductive set* $\mathsf{W}_{I,A,B,s}$ is the initial algebra of the polynomial endofunctor [GK13] on the (internal) category of families over I sending a family X to the family

$$[\![I, A, B, s]\!]\,i = \sum_{a:A\,i} \prod_{b:B\,i\,a} X(s\,i\,a\,b).$$

Its constructive justification as an operation in the internal language of the presheaf topos using inductive constructions of the metatheory is folklore (in a classical setting, one can use transfinite colimits [Kel80]).[12] If I, A, B are small with respect to a universe $\mathcal{U}_i$ with $i \in \{0, 1, \dots, \omega\}$, then $\mathsf{W}_{I,A,B,s} : I \to \mathcal{U}_i$.

Let I, A, B now be small with respect to $\mathcal{U}_\omega$. Given $\mathsf{Fill}(I, A)$ and $\mathsf{Fill}(\sum_{i:I} A\,i, \lambda_{(i,a)}\,B\,i\,a)$, we may use induction (i.e. the universal property of $\mathsf{W}_{I,A,B,s}$) to derive an element of $\mathsf{Fill}(I, \mathsf{W}_{I,A,B,s})$. As in Subsection 2.2 for dependent products, this implies (using external reasoning) the internal statement $\mathsf{C}(\mathsf{W}_{I,A,B,s}\,i)$ for $i : I$ given $\mathsf{C}(A\,i)$ for all i and $\mathsf{C}(B\,i\,a)$ for all i, a. We then call $\mathsf{W}_{I,A,B,s}$ a *fibrant uniformly indexed inductive set*. The qualifier

---

[12]An indexed inductive set in presheaves unfolds externally to an indexed inductive-recursive definition where one defines the values at every level simultaneously with the restriction operations between levels. In turn, this indexed inductive-recursive definition can be encoded as an indexed inductive-inductive definition, which in turn reduces to an indexed inductive definition (which one may further reduce to an inductive definition [GH03]). Both steps use the idea of encoding functions via their graphs. Alternatively, one can directly transform the inductive-recursive definition to an indexed inductive definition by first omitting about the naturality condition that mentions restriction, then define restriction recursively, and finally carve out the elements that recursively satisfy the naturality condition.

uniformly indexed indicates that A is a fibrant family over I rather than a fibrant set with a "target" map to I that indicates the target sort of the constructor sup.

Given $A : \mathcal{U}_\omega$ with $\mathsf{Fill}(1, A)$, we may use the technique of Andew Swan [Swa16, OP16] to construct a (level preserving) identity set $\mathsf{Id}_A\, a_0\, a_1$ for $a_0, a_1 : A$ (different from the equality set $a_0 = a_1$) with $\mathsf{Fill}(A \times A, \lambda_{(a_0,a_1)}\, \mathsf{Id}_A\, a_0\, a_1))$ and constructor $\mathsf{refl}_a : \mathsf{Id}_A\, a\, a$ for $a : A$ that has the usual elimination with respect to families $P : \prod_{a_0\, a_1 : A} \mathsf{Id}_A\, a_0\, a_1 \to \mathcal{U}_\omega$ that satisfy $\mathsf{Fill}(\sum_{a_0\, a_1 : A} \mathsf{Id}_A\, a_0\, a_1, P)$. Using external reasoning as before, one has $\mathsf{C}(\mathsf{Id}_A\, a_0\, a_1)$ given $\mathsf{C}(A)$, justifying calling $\mathsf{Id}_A\, a_0\, a_1$ a *fibrant identity set*; using (2.2) one has elimination with respect to families P of the previous signature with $\mathsf{C}(C\, a_0\, a_1\, p)$ for all a_0, a_1, p.

Using a folklore technique, we may use fibrant identity sets to derive *fibrant indexed inductive sets* from fibrant uniformly indexed inductive sets, by which we mean the following. Given $(I, \mathsf{fib}_I) : \mathcal{U}_\omega^{\mathsf{fib}}$, $(A, \mathsf{fib}_A) : \mathcal{U}_\omega^{\mathsf{fib}}$, $\langle B, \mathsf{fib}_B \rangle : A \to \mathcal{U}_\omega^{\mathsf{fib}}$ with maps $t : A \to I$ and $s : \prod_{a:A} B\, a \to I$, we have $\langle W_{I,A,B,s,t}, \mathsf{fib}_W \rangle : I \to \mathcal{U}_\omega^{\mathsf{fib}}$ (we omit the subscripts to W for readability), W living in $\mathcal{U}_i$ if I, A, B do, with

$$\mathrm{sup} : \textstyle\prod_{a:A} \prod_{f:\, \prod_{b:B\, a} W\, (s\, a\, b)} W\, (t\, a).$$

Given $\langle P, \mathsf{fib}_P \rangle : \prod_{i:I} W \to \mathcal{U}_\omega^{\mathsf{fib}}$ with

$$h : \textstyle\prod_{a:A} \prod_{f:\, \prod_{b:B\, a} W\, (s\, a\, b)} \left(\textstyle\prod_{b:B\, a} P\, (s\, a\, b)\, (f\, b) \right) \to P\, (t\, a)\, (\mathrm{sup}\, a\, f),$$

we have $v : \prod_{i:I} \prod_{w:W\, i} P\, i\, w$ such that

$$v\, (t\, a)\, (\mathrm{sup}\, a\, f) = h\, a\, f\, (\lambda_b\, v\, (s\, a\, b)\, (f\, b)).$$

Fibrant indexed inductive sets are used for the interpretation in the sconing model of natural numbers in Section 3, higher inductive types in Subsection 5.2, and identity types in Subsection 5.1. In practise, we will usually not bother to bring the fibrant indexed inductive set needed into the above form and instead work explicitly with the more usual specification in terms of a list of constructors, each taking a certain number non-recursive and recursive arguments.[13]

As an example, we construct the fibrant indexed inductive set N' needed in Section 3. There, we have a fibrant set $|\mathsf{N}| : \mathcal{U}_0$ (satisfying $\mathsf{C}(|\mathsf{N}|)$) with an element $\mathsf{0} : |\mathsf{N}|$ and an endofunction $\mathsf{S} : |\mathsf{N}| \to |\mathsf{N}|$. We wish to define the fibrant indexed inductive set $\mathsf{N}' : |\mathsf{N}| \to \mathcal{U}_0$ with constructors $\mathsf{0}' : \mathsf{N}'\, \mathsf{0}$ and $\mathsf{S}' : \prod_{n:|\mathsf{N}|_\rho} \mathsf{N}'\, n \to \mathsf{N}'\, (\mathsf{S}\, n)$. We let N' be the uniformly indexed inductive set over $m : |\mathsf{N}|$ with constructors

$$\mathsf{0}'' : \mathsf{Id}_{|\mathsf{N}|}\, m\, \mathsf{0} \to \mathsf{N}'\, m,$$
$$\mathsf{S}'' : \textstyle\prod_{n:|\mathsf{N}|} \mathsf{Id}_{|\mathsf{N}|}\, m\, (\mathsf{S}\, n) \to \mathsf{N}'\, n \to \mathsf{N}'\, m.$$

and define $\mathsf{0}' = \mathsf{0}''\, \mathsf{refl}_\mathsf{0}$ and $\mathsf{S}'\, n\, n' = \mathsf{S}''\, n\, \mathsf{refl}_{\mathsf{S}(n)}\, n'$. Fibrancy of Id ensures fibrancy of N' (i.e. $\mathsf{C}(\mathsf{N}'\, n)$ for $n : |\mathsf{N}|$). For elimination, we are given a fibrant family $P\, n\, n'$ for $n : |\mathsf{N}|$ and $n' : \mathsf{N}'\, n$ with $z' : P\, \mathsf{0}\, \mathsf{0}'$ and $s'\, n\, n'\, x : P\, (\mathsf{S}\, n)\, (\mathsf{S}'\, n\, n')$ for all n, n' and $x : P\, n\, n'$. We have to define $h'\, n\, n' : P\, n\, n'$ for all n, n' such that $h'\, \mathsf{0}\, \mathsf{0}' = z'$ and $h'\, (\mathsf{S}\, n)\, (\mathsf{S}'\, n\, n') = s'\, n\, n'\, (h'\, n\, n')$. We define h' by induction on the uniformly indexed inductive set N' and fibrant identity sets

---

[13]Note that the latter is really an instance of the former since our dependent sums, dependent products, and finite coproducts are extensional (satisfy universal properties). Conversely, the former is an instance of the latter with a single constructor taking a non-recursive and a recursive argument.

(using fibrancy of P) via defining equations

$$h'\, 0''\, \mathsf{refl}_0 = z,$$
$$h'\, (\mathsf{S}n)\, (\mathsf{S}''\, n\, \mathsf{refl}_{\mathsf{S}\, n}\, n') = s'\, n\, n'\, (h'\, n\, n').$$

Appendix C. Variations

C.1. **Univalence as an axiom.** Our treatment extends to the case where the glue types in a cubical cwf as in Subsection 1.3 are replaced by an operation $\mathsf{Elem}(\Gamma, \mathtt{iUnivalence}_n)$ for $\Gamma : \mathsf{Con}$ and $n \geq 0$, with $\mathtt{iUnivalence}_n$ defined in Remark 1.1.

To define this operation in the sconing model of Section 3, one first shows analogously to Lemmas 3.1 and 3.2 that $|-|$ preserves contractible types and that $(A, A') : \mathsf{Type}^*(\Gamma, \Gamma')$ is contractible exactly if A is contractible and $A'\, \rho\, \rho'\, a$ for $\rho : |\Gamma|$ and $\rho' : \Gamma'\, \rho$ where $a : |A|$ is the induced center of contraction. We have analogous statements for types of homotopy level $n \geq 0$i in $\mathcal{M}$, in which case we instead have to quantify over all $a : |A|$.

Given $(A, A') : \mathsf{Type}^*_n(\Gamma, \Gamma')$, we have show that the type

$$(S, S') = \Sigma^*(\mathsf{U}_n, \mathsf{Equiv}^*(\mathsf{q}, A))$$

over (Γ, Γ') is contractible in $\mathcal{M}^*$. Without loss of generality, we may assume the center of contraction of univalence in $\mathcal{M}$ is given by the identity equivalence. Using the observations of the preceding paragraph, it suffices to show that

$$V' = S'\, \rho\, \rho'\, (\mathtt{pair}(A\rho, (\mathtt{lam}(\mathsf{q}), w)))$$

is contractible for $\rho : |\Gamma|$ and $\rho' : \Gamma'\, \rho$ where w denotes the canonical witness that the identity map $\mathtt{lam}(\mathsf{q})$ on $A\rho$ is an equivalence in $\mathcal{M}$. Inhabitation is evident, and so it remains to show propositionality. By the case of the preceding paragraph for propositions, the second component of V' is a proposition, and thus we can ignore it for the current goal, which then becomes

$$\mathsf{isProp}\big(\textstyle\sum_{T':|A|\to\mathcal{U}_n} \prod_{a:|A|} \mathsf{Equiv}(T'\, a, A'\, \rho\, \rho'\, a)\big)$$

and follows from univalence in the standard model, justified by glueing.

Appendix D. Simplicial set model

Choosing for $\mathcal{C}$ the simplex category Δ, for $\mathbb{I}$ the usual interval Δ^1 in simplicial sets, and for $\mathbb{F}$ a small copy $\Omega_{0,\mathsf{dec}}$ of the sublattice of Ω_0 of decidable sieves, we obtain a notion of cubical cwf with a simplicial notion of shape.

Assume now the law of excluded middle. The above choice of $\mathcal{C}, \mathbb{I}, \mathbb{F}$ satisfies all of the assumptions of Subsection 2.2 but one: the existence of a right adjoint to exponentiation with $\mathbb{I}$. However, except for Section 6, the only place our development makes use of this assumption is in establishing (2.1). We will instead give a different definition of C that still satisfies (2.1). Then the rest of our development, except for Section 6, still applies to simplicial sets.

A *Kan fibration structure* on a family $Y\colon X \to \mathcal{U}_\omega$ in simplicial sets consists of a choice of diagonal fillers in all commuting squares of the form

$$
\begin{array}{ccc}
\Lambda_k^m & \longrightarrow & \sum_{x:X} Y\,x \\
\downarrow & \nearrow & \downarrow \\
\Delta^m & \longrightarrow & X
\end{array}
$$

with left map a *horn inclusion* and right map the evident projection. Note that the codomains of horn inclusions are representable. It follows that the presheaf of Kan fibration structures indexed over the slice of simplicial sets over $\mathcal{U}_\omega$ is representable. Given $[n] \in \Delta$ and $A \in (\mathcal{U}_\omega)_n$ (i.e. an ω-small presheaf on $\Delta/[n]$), we define $\mathsf{C}([n], A)$ as the set of Kan fibration structures on $A\colon \Delta^n \to \mathcal{U}_\omega$. This defines a level preserving map $\mathsf{C}\colon \mathcal{U}_\omega \to \mathcal{U}_\omega$. Then the representing object of the above presheaf is given by the first projection $\mathcal{U}_\omega^{\mathsf{fib}} \to \mathcal{U}_\omega$ where $\mathcal{U}_\omega^{\mathsf{fib}} = \sum_{X:\mathcal{U}_\omega} \mathsf{C}(X)$ is defined as before.

Let us now verify (2.1). Given a simplicial set X with $Y\colon X \to \mathcal{U}_\omega$, a global element of $\mathsf{Fill}(X, Y)$ corresponds to a *uniform Kan fibration* structure on $\sum_{x:X}(Y\,x) \to X$ in the sense of [GS17]. A uniform Kan fibration structure induces a Kan fibration structure naturally in X, giving the forward direction of (2.1). For the reverse direction, it suffices to give a uniform Kan fibration structure in the generic case, i.e. a global element of $\mathsf{Fill}(\mathcal{U}_\omega^{\mathsf{fib}}, \lambda_{(A,c)}\, A)$. This is [GS17, Theorem 8.9, part (ii)] together with the fact proved in [GZ67, Chapter IV] that Kan fibrations lift against pushout products of interval endpoint inclusions with (levelwise decidable) monomorphisms.[14]

Having verified (2.1), the rest of our development applies just as well to the case of simplicial sets. In particular, we obtain in the standard model $\mathcal{S}$ of Subsection 2.3 a version of the simplicial set model [KL12] of univalent type theory (using Subsection 5.1 for identity types).[15] As per Subsection 5.2, we furthermore obtain higher inductive types in the simplicial set model in a way that avoids (as suggested by Andrew Swan [Swa17]) the pitfall of fibrant replacement failing to preserve size encountered in [LS20].

Seeing simplicial sets as a full subtopos of distributive lattice cubical sets as observed in [KV20], there is a functor from cubical cwfs with $(\mathcal{C}, \mathbb{I}, \mathbb{F}) = (\Delta, \Delta^1, \Omega_{0,\mathsf{dec}})$ to cubical cwfs where $\mathcal{C}$ is the Lawvere theory of distributive lattices, $\mathbb{I}$ is represented by the generic object, and $\mathbb{F}$ is the (small) sublattice of Ω_0 generated by distributive lattice equations. The cubical cwfs in the image of this functor satisfy a *sheaf condition*, which can be represented syntactically as an operation allowing one to e.g. uniquely glue together to a type $\Gamma \vdash_{\{i,j\}} A$ coherent families of types $\Gamma f \vdash_X A_f$ for f a map to X from the free distributive lattice on symbols $\{i, j\}$ such that $f\,i \le f\,j$ or $f\,j \le f\,i$ (compare also the tope logic of [RS18]).

Applying this functor to the simplicial set model $\mathcal{S}$ discussed above, we obtain an interpretation of distributive lattice cubical type theory (with $\mathbb{I}$ and $\mathbb{F}$ as above) in the sense of the current article (crucially, without computation rules for filling at type formers) in simplicial sets. Thus, this cubical type theory is homotopically sound: can only derive statements which hold for standard homotopy types.

---

[14]This is the only place where excluded middle is used, to produce a cellular decomposition in terms of simplex boundary inclusions of such a monomorphism.

[15]Instead of Kan fibration structures, we can also work with the property of being a Kan fibration. Then C is valued in propositions and we would obtain in $\mathcal{S}$ a version of the simplicial set model in which being a type is truly just a property. However, choice would be needed to obtain (2.1).