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Localization of eigenfunctions in a thin cylinder with a locally periodic oscillating boundary

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A B S T R A C T

We study a Dirichlet spectral problem for a second-order elliptic operator with locally periodic coefficients in a thin cylinder. The lateral boundary of the cylinder is assumed to be locally periodic. When the thickness of the cylinder ε tends to zero, the eigenvalues are of order ε^{-2} and described in terms of the first eigenvalue $\mu(x_1)$ of an auxiliary spectral cell problem parametrized by x_1, while the eigenfunctions localize with rate $\sqrt{\varepsilon}$.

1. Introduction

This paper deals with the leading terms in the asymptotics of the eigenvalues and the eigenfunctions to the following Dirichlet spectral problem in a thin cylinder with a periodically oscillating boundary:

$$-\mathrm{div}\big(A\big(x_1, \frac{x}{\varepsilon}\big)\nabla u^\varepsilon\big) = \lambda^\varepsilon u^\varepsilon \quad \text{in } \Omega_\varepsilon, \tag{1}$$

with the Dirichlet condition on the boundary $\partial\Omega_\varepsilon$. The asymptotics is established in the case of locally periodically oscillating coefficients, under structural assumptions given in terms of an eigenvalue problem on the associated periodicity cell.

In [8], Friedlander and Solomyak studied the spectrum of the Dirichlet Laplacian in a narrow strip

$$\Omega_\varepsilon = \{(x,y) \in \mathbf{R}^2 : -a < x < a,\ 0 < y < \varepsilon h(x)\},$$

where ε is a small positive parameter. The following structural assumptions were made: $x = 0$ is the unique global maximum of positive function $h(x)$, such that

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$$h(x) > 0 \text{ continuous on } I = [-a, a],\ a > 0$$

$$h(x) \text{ is } C^1 \text{ on } I \setminus \{0\}, \text{ and}$$

$$h(x) = \begin{cases} M - c_+ x^m + O(x^{m+1}), & x > 0, \\ M - c_- |x|^m + O(x^{m+1}), & x < 0, \end{cases} \qquad M, c_-, c_+ > 0, \quad m \geq 1.$$

In particular, the authors have proved the following asymptotics result.

Theorem 1.1 *(Friedlander, Solomyak, 2009). Let λ_j^ε be the eigenvalues of $-\Delta$ on Ω^ε with the Dirichlet condition on $\partial\Omega^\varepsilon$. Then*

$$\mu_j = \lim_{\varepsilon \to 0} \varepsilon^{4(m+2)^{-1}} \Big(\lambda_j^\varepsilon - \frac{\pi^2}{M^2 \varepsilon^2} \Big),$$

where μ_j are the eigenvalues of the operator on $L^2(\mathbf{R})$ given by

$$-\frac{d^2}{dx^2} + q(x), \quad q(x) = \begin{cases} 2\pi M^{-3} c_+ x^m, & x > 0, \\ 2\pi M^{-3} c_- x^m, & x < 0. \end{cases}$$

One notes that Theorem 1.1 tells that the decay of the profile (or diameter) $h(x_1)$ in the vicinity of its maximum dictates the growth of the eigenvalues λ_j^ε, and that it indicates the rate of localization of the eigenfunctions, as ε tends to zero.

The present paper aims at describing the effect of oscillating coefficients and oscillating boundary on the spectral asymptotics. In Theorem 1.1, the authors imposed structural assumptions on the strip profile $h(x_1)$, requiring for $h(x_1)$ a unique maximum point. In singularly perturbed problems, when the leading term of the asymptotics contains an oscillating function, which is the first eigenfunction of an auxiliary spectral cell problem, a standard assumption is often made on the corresponding first eigenvalue (see for example [2]). In our case, we use also the factorization technique, impose an assumption on the first eigenvalue $\mu_1(x_1)$ in the periodicity cell eigenvalue problem (see (8)), and require that it has a unique minimum point. Even if such assumptions are standard, it is sometimes not apparent how they are related to the geometry of the domain or the coefficients of the equation. In the present paper we show that the assumption in Theorem 1.1 is a specific case of (8) (see Example 5.1). We show that the global minimizer of μ_1 determines the point where the eigenfunctions localize. In addition, the growth of μ in the vicinity of its minimum dictates the rate of localization as well as the behavior of the eigenvalues.

The method we use to obtain the leading terms of the asymptotics of the eigenvalue and the eigenfunctions to (1) is homogenization via two-scale (see [16,1]). In particular, we use the singular measure approach, and the two-scale compactness theorem essentially contained in [24,25] (see also [7]), which is expressed in the current setting in [19].

The fundamentals of spectral asymptotics for elliptic operators has been described in [23], which has been successfully applied in homogenization problems (see for example [17]). There are many works in which localization of eigenfunctions has been described in the context of elliptic operators with oscillating coefficients. The works most closely related to our problem seem to be the following. In [20,2], spectral asymptotics and localization of eigenvalues in bulk domains with large potentials were considered. In [18], spectral asymptotics and localization of eigenvalues in thin domains with large potentials were considered. Homogenization in domains with low amplitude oscillating boundaries were considered in [13,3,19]. Spectral asymptotics of the Laplace operator in thin domains with slowly varying thickness were considered in [8,6,14], where under the Dirichlet boundary conditions the localization of eigenfunctions occur.

In order to capture the oscillations of the eigenfunctions, in [20,2] the factorization by an eigenfunction of an auxiliary spectral cell problem was used. In the present paper, we also use this kind of factorization. Due to the homogeneous Dirichlet boundary conditions in our problem, the first eigenfunction $\psi_1(x_1, y)$ of the auxiliary spectral problem (6) vanishes on the boundary of the cell, and the new unknown function satisfies a problem with degenerate coefficients posed in a weighted Sobolev space. Significant contributions to the development of the factorization principle were made by Vanninathan in [22], and by Kozlov in [11]. The case of the Dirichlet condition treated in [22] is closely related to the spectral asymptotics in the problem considered in this paper.

The rest of this paper is organized as follows. In Section 2, we describe the domain with oscillating boundary and state the problem with the hypotheses. In Section 3, we establish a priori estimates for the eigenvalues and state the main result, and in Section 4 we prove the spectral asymptotics result of this paper. In Section 5, we describe a scheme to compute the leading terms in the expansions of the eigenvalues and the eigenfunctions to (1).

2. Problem statement

The problem we consider is to describe the leading terms in the asymptotics of the first eigenvalue λ_1^ε and eigenfunction u_1^ε to the problem (2), as ε tends to zero. The domain under consideration is a thin cylinder with a locally periodic oscillating boundary, and the coefficients are assumed to be smooth, periodically oscillating, and to satisfy the strong ellipticity condition. The Dirichlet condition is imposed on the boundary. Here we specify the assumptions on the domain, and the coefficients and boundary conditions separately.

2.1. A thin cylinder with a locally periodic oscillating boundary

Let $\varepsilon > 0$ be a small parameter; the points in $\mathbf{R}^d$ are denoted by $x = (x_1, x')$, and $I = (-1/2, 1/2)$. We are going to work in a thin cylinder

$$\Omega_\varepsilon = \Big\{ x = (x_1, x') : x_1 \in I,\, x' \in \varepsilon Q(x_1, \frac{x_1}{\varepsilon}) \Big\}.$$

Here $Q(x_1, x_1/\varepsilon)$ describes the locally periodic varying cross section of the cylinder introduced in the following way.

We denote

$$Q(x_1, y_1) = \{ y' \in \mathbf{R}^{d-1} : F(x_1, y_1, y') > 0 \},$$

where $F(x_1, y_1, y')$ is such that

(H1) $F(x_1, y_1, y') \in C^2(\overline{I} \times \mathbf{T}^1 \times \mathbf{R}^{d-1})$, where $\mathbf{T}^1$ is the one-dimensional torus.
(H2) $Q(x_1, y_1)$ is non-empty, bounded, and simply connected.

The conditions (H1), (H2) are fulfilled, for instance, if F has the following properties:

(F1) For each x_1 and y_1, $F(x_1, y_1, 0) > 0$ and $F(x_1, y_1, y') < 0$ for $|y'| \geq R$, for some $R > 0$.
(F2) $F(x_1, y_1, \cdot)$ does not have a nonpositive local maximum/minimum.

The condition (F1) guarantees that $Q(x_1, y_1)$ is not empty and bounded, and the condition (F2) guarantees that $Q(x_1, y_1)$ is simply connected. When $F = F(y_1, y')$ we have a periodic oscillating boundary, when $F = F(x_1, y')$ we are in the case of slowly varying thickness, and finally, when $F = F(y')$ the cylinder has uniform cross-section, constant along the cylinder.

The boundary of Ω_ε consists of the lateral boundary of the cylinder

$$\Sigma_\varepsilon = \Big\{x = (x_1, x') : x_1 \in I,\ F\big(x_1, \frac{x_1}{\varepsilon}, \frac{x'}{\varepsilon}\big) = 0\Big\},$$

and the bases

$$\Gamma_\varepsilon^\pm = \Big\{\pm\frac{1}{2}\Big\} \times \varepsilon Q\Big(\pm\frac{1}{2}, \pm\frac{1}{\varepsilon}\Big).$$

The periodicity cell depends on x_1 and is defined as follows

$$\begin{aligned}\square(x_1) &= \{y = (y_1, y') \in \mathbf{T}^1 \times \mathbf{R}^{d-1} : y_1 \in \mathbf{T}^1,\ y' \in Q(x_1, y_1)\}\\ &= \{y \in \mathbf{T}^1 \times \mathbf{R}^{d-1} : F(x_1, y) > 0\},\end{aligned}$$

where $\mathbf{T}^1$ is the one-dimensional torus, realized with unit measure. Since $F(x_1, y_1, y')$ is periodic in y_1, and F is regular, the lateral boundary $\partial\square(x_1) = \{y : F(x_1, y) = 0\}$ of the periodicity cell $\square(x_1)$ is Lipschitz.

2.2. Dirichlet spectral problem in a thin oscillating domain

In the thin domains with oscillating boundary Ω_ε of Section 2.1, we consider the following Dirichlet spectral problem:

$$\begin{aligned} -\mathrm{div}\big(A\big(x_1, \frac{x}{\varepsilon}\big)\nabla u^\varepsilon\big) &= \lambda^\varepsilon u^\varepsilon \qquad \text{in } \Omega_\varepsilon,\\ u^\varepsilon &= 0 \qquad \text{on } \partial\Omega_\varepsilon. \end{aligned} \tag{2}$$

The matrix $A(x_1, y)$ in (2) is assumed to be symmetric with entries a_{ij} in $C^2(\overline{I} \times \mathbf{T}^1 \times \mathbf{R}^{d-1})$, and satisfy the ellipticity condition

$$A(x_1, y)\xi \cdot \xi \geq \alpha|\xi|^2, \quad x_1 \in I, \quad y \in \square(x_1), \quad \xi \in \mathbf{R}^d.$$

By the Hilbert-Schmidt theorem, for each $\varepsilon > 0$, the spectrum of (2) is discrete and may be arranged as follows:

$$0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon \leq \lambda_3^\varepsilon \leq \cdots, \qquad \lim_{j\to\infty} \lambda_j^\varepsilon = \infty,$$

where the eigenvalues are counted as many times as their finite multiplicity. Orthonormalized by

$$\int_{\Omega_\varepsilon} u_i^\varepsilon u_j^\varepsilon\, dx = \varepsilon^{1/2}\varepsilon^{d-1}\delta_{ij}, \tag{3}$$

where δ_{ij} is the Kronecker delta, the corresponding eigenfunctions form a Hilbert basis in $L^2(\Omega_\varepsilon)$, up to scaling. The scaling in (3) is natural for the statement of the result of this paper.

We study the asymptotic behavior of the eigenpairs $(\lambda^\varepsilon, u^\varepsilon)$, as ε tends to zero. The homogenization result for the eigenpairs $(\lambda^\varepsilon, u^\varepsilon)$, as $\varepsilon \to 0$, is given in Theorem 3.3. In order to present the limit problem, we need an auxiliary spectral cell problem, which is introduced in Section 3.1 (see (6)).

3. Spectral asymptotics

In this section, we identify the first two terms in the asymptotic expansion of the first eigenvalue λ_1^ε to (2):

$$\lambda_1^\varepsilon = \frac{\mu_0(0)}{\varepsilon^2} + \frac{\lambda_1^0}{\varepsilon} + o\Big(\frac{1}{\varepsilon}\Big), \tag{4}$$

as ε tends to zero. For comparison with the result of Theorem 1.1, for smooth h with quadratic growth at its maximum ($m = 2$), one has for the first eigenvalue λ_1^ε of the Dirichlet Laplacian in $\mathbf{R}^2$,

$$\lambda_1^\varepsilon = \frac{\pi^2/h(0)^2}{\varepsilon^2} + \frac{\mu_1}{\varepsilon} + o\Big(\frac{1}{\varepsilon}\Big), \tag{5}$$

as ε tends to zero, where μ_1 is the first eigenvalue of a harmonic oscillator. When comparing the asymptotics in (4) and (5) one notes that the oscillations in the coefficients and the domain do not change the structure of the asymptotics, while the factors in both the leading terms $1/\varepsilon^2$ and $1/\varepsilon$ will change. In particular, the eigenvalue λ_1^ε is still shifted a bit to the right as the factor λ_1^0 is positive. One notes also that the factors $\mu_1(0)$ and λ_1^0 in (4) agree with $\pi^2/h(0)^2$ and μ_1 in (5), for the Dirichlet Laplacian in $\mathbf{R}^2$ as explained in Section 5.

To obtain the asymptotics (4) we proceed as follows. First we derive a priori estimates for the first eigenvalue λ_1^ε to (2). Then we make a change of variables with the aid of hypothesis (8) such that the eigenvalues of the rescaled problem are bounded. Finally we show the convergence of the corresponding eigenvalues and their eigenfunction in L^2 using the method of two-scale convergence in domains with measure. The result, which includes (4), is stated as Theorem 3.3.

3.1. A priori estimates for the first eigenvalue λ_1^ε

The radius of the cylinder Ω_ε is of order ε, so the Dirichlet condition on the lateral boundary suggests that in view of the variational principle, the first eigenvalue λ_1^ε is expected to be of order $1/\varepsilon^2$, as ε tends to zero. Consider a smooth function of the form

$$v^\varepsilon(x) = v(x_1)w\Big(\frac{x'}{\varepsilon}\Big),$$

$v(x_1) \in C_0^\infty(\mathbf{R})$ with support in I, $w(x') \in C_0^\infty(B_0(\rho))$, where $B_0(\rho)$ is the ball of a small fixed radius ρ centered at the origin in $\mathbf{R}^{d-1}$ such that $I \times \varepsilon B_0(\rho) \subset \Omega_\varepsilon$ (cf. (F1)). Using $v^\varepsilon(x)$ as a test function in the variational principle for λ_1^ε, we obtain

$$\lambda_1^\varepsilon = \min_{v \in H_0^1(\Omega_\varepsilon)\setminus\{0\}} \frac{\int_{\Omega_\varepsilon} A\big(x_1, \frac{x}{\varepsilon}\big)\nabla v \cdot \nabla v \, dx}{\int_{\Omega_\varepsilon} v^2 \, dx} \le C \frac{\int_{\Omega_\varepsilon} |\nabla v^\varepsilon|^2 \, dx}{\int_{\Omega_\varepsilon} (v^\varepsilon)^2 \, dx} \le \frac{C}{\varepsilon^2},$$

for all small enough ε such that $v^\varepsilon \in H_0^1(\Omega_\varepsilon)$, and for some constant C independent of ε. To identify the factor in $1/\varepsilon^2$, and to obtain the corresponding estimate from below, one needs to choose the test functions more carefully. In particular, for $O(1/\varepsilon^2)$ to be sharp an optimal oscillating ε-periodic profile ψ along x_1 has to be selected, because $|\nabla\psi(x_1/\varepsilon)|^2 = O(1/\varepsilon^2)$.

Let $(\mu_1, \psi_1(x_1, y))$, for each $x_1 \in \overline{I}$, be the first eigenpair to the following cell eigenvalue problem:

$$\begin{aligned} -\mathrm{div}_y(A(x_1, y)\nabla_y \psi) &= \mu(x_1)\psi && \text{in } \square(x_1), \\ \psi &= 0 && \text{on } \partial\square(x_1), \end{aligned} \tag{6}$$

normalized by

$$\int_{\square(x_1)} \psi_1(x_1,y)^2\,dy = 1. \tag{7}$$

By the Krein-Rutman theorem (cf. [5]), the first eigenvalue $\mu_1(x_1) > 0$ is simple, and $\psi_1(x_1,y)$ does not change sign in $\square(x_1)$, and may for example be chosen positive. By the regularity of the coefficients A and the domain $\square(x_1)$ in x_1, and the simplicity of the first eigenvalue $\mu_1(x_1)$, one has $\mu_1(x_1) \in C^2(\overline{I})$ (see [9,10]).

We use ψ_1 to construct a test function in the variational formulation for λ_1^ε. Namely, we are going to take a test function as a product of ψ_1, taking care of the transverse oscillations, and a function of x_1 only. In the proof of Lemma 3.1 we see that to minimize the ratio in the variational principle, the latter function should localize in the vicinity of the minimum point of μ_1. This motivates the following structural assumption:

$$\mu_1(x_1) \text{ has a unique global minimum at } x_1 = 0, \text{ and } \mu_1''(0) > 0. \tag{8}$$

We make further remarks on the assumption (8) in Section 5, and there illustrate how it is related to the assumptions on the geometry of the cylinder in Theorem 1.1 in [8].

The following a priori estimate holds for the first eigenvalue to (2).

Lemma 3.1. *Let λ_1^ε be the first eigenvalue to (2), and suppose that the first eigenvalue $\mu_1(x_1)$ to (6) satisfies (8). Then*

$$\lambda_1^\varepsilon = \frac{\mu_1(0)}{\varepsilon^2} + O\Big(\frac{1}{\varepsilon}\Big),$$

as ε tends to zero.

Proof. In the Rayleigh quotient,

$$\lambda_1^\varepsilon = \min_{v\in H_0^1(\Omega_\varepsilon)\setminus\{0\}} \frac{\int_{\Omega_\varepsilon} A(x_1,\frac{x}{\varepsilon})\nabla v\cdot\nabla v\,dx}{\int_{\Omega_\varepsilon} v^2\,dx},$$

we use test functions of the form

$$v^\varepsilon(x) = \psi_1\Big(x_1,\frac{x}{\varepsilon}\Big)v\Big(\frac{x_1}{\sqrt{\varepsilon}}\Big),$$

where $\psi_1 \geq 0$ is the first eigenfunction to (6), and nonzero $v \in C_0^\infty(\mathbf{R})$ with support contained in I. Then $v^\varepsilon \in H_0^1(\Omega_\varepsilon)$. The Taylor theorem and hypothesis (8) give

$$\mu_1(\sqrt{\varepsilon}x_1) = \mu_1(0) + \varepsilon\mu_1''(0)x_1^2 + o(\varepsilon x_1^2),$$

as ε tends to zero. Using the boundedness of a_{ij}, the regularity properties of ψ_1, re-scaling and integrating by parts, we obtain

$$\begin{aligned}
\lambda_1^\varepsilon &\leq \frac{\int_{\Omega_\varepsilon} A\big(x_1,\frac{x}{\varepsilon}\big)\nabla v^\varepsilon\cdot\nabla v^\varepsilon\,dx}{\int_{\Omega_\varepsilon}(v^\varepsilon)^2\,dx} \\
&= \frac{\varepsilon^{-1}\int_{\mathbf{R}^d}(\psi_1^2 A)\big(\sqrt{\varepsilon}x_1,\frac{x}{\sqrt{\varepsilon}}\big)\nabla v\cdot\nabla v\,dx}{\int_{\mathbf{R}^d}\psi_1^2\big(\sqrt{\varepsilon}x_1,\frac{x}{\sqrt{\varepsilon}}\big)v^2\,dx} + \frac{\int_{\mathbf{R}^d}(\psi_1 b + \varepsilon^{-1}\psi_1 c + \varepsilon^{-2}\mu_1\psi_1^2)\big(\sqrt{\varepsilon}x_1,\frac{x}{\sqrt{\varepsilon}}\big)v^2\,dx}{\int_{\mathbf{R}^d}\psi_1^2\big(\sqrt{\varepsilon}x_1,\frac{x}{\sqrt{\varepsilon}}\big)v^2\,dx} \\
&\leq \frac{\mu_1(0)}{\varepsilon^2} + \frac{C}{\varepsilon},
\end{aligned}$$

for some absolute constant C which is independent of ε, where

$$b(x_1, y) = -\mathrm{div}_x(A(x_1, y)\nabla_x \psi_1(x_1, y)), \tag{9}$$

$$c(x_1, y) = -\mathrm{div}_y(A(x_1, y)\nabla_x \psi_1(x_1, y)) - \mathrm{div}_x(A(x_1, y)\nabla_y \psi_1(x_1, y)). \tag{10}$$

We proceed with the estimate from below. Let u_1^ε be the first eigenfunction to (2), normalized by $\int_{\Omega_\varepsilon}(u_1^\varepsilon)^2\,dx = 1$, and let v_1^ε be such that

$$u_1^\varepsilon(x) = \psi_1\big(x_1, \frac{x}{\varepsilon}\big)v_1^\varepsilon(x). \tag{11}$$

Note that v_1^ε may blow up at the boundary due to the Dirichlet condition on ψ_1, and this will be addressed with weighted Sobolev spaces in the further analysis. By the hypothesis (8) that $\mu_1(0)$ is the minimal value of $\mu_1(x_1)$, we get under the chosen normalization,

$$\begin{aligned}\lambda_1^\varepsilon &= \int_{\Omega_\varepsilon} (\psi_1^2 A)\big(x_1, \frac{x}{\varepsilon}\big)\nabla v_1^\varepsilon \cdot \nabla v_1^\varepsilon\,dx + \int_{\Omega_\varepsilon} (\psi_1 b + \varepsilon^{-1}\psi_1 c)\big(x_1, \frac{x}{\varepsilon}\big)(v_1^\varepsilon)^2\,dx \\ &\quad + \int_{\Omega_\varepsilon} \frac{\mu_1(x_1)}{\varepsilon^2}\psi_1^2\big(x_1, \frac{x}{\varepsilon}\big)(v_1^\varepsilon)^2\,dx \\ &\geq \frac{\mu_1(0)}{\varepsilon^2} + \int_{\Omega_\varepsilon} (\psi_1^2 A)\big(x_1, \frac{x}{\varepsilon}\big)\nabla v_1^\varepsilon \cdot \nabla v_1^\varepsilon\,dx + \int_{\Omega_\varepsilon} (\psi_1 b + \varepsilon^{-1}\psi_1 c)\big(x_1, \frac{x}{\varepsilon}\big)(v_1^\varepsilon)^2\,dx. \end{aligned} \tag{12}$$

We note that v_1^ε defined in (11) belongs to the weighted space

$$H^1\big(\Omega_\varepsilon, \psi_1^2\big(x_1, \frac{x}{\varepsilon}\big)\big) = \big\{v : \psi_1\big(x_1, \frac{x}{\varepsilon}\big)v \in L^2(\Omega_\varepsilon),\ \psi_1\big(x_1, \frac{x}{\varepsilon}\big)\nabla v \in L^2(\Omega_\varepsilon)\big\},$$

equipped with the natural norm. The space $H^1\big(\Omega_\varepsilon, (\psi_1)^2\big(x_1, \frac{x}{\varepsilon}\big)\big)$ is continuously embedded into $L^2(\Omega_\varepsilon)$: More precisely, there exists a constant C independent of ε such that

$$\int_{\Omega_\varepsilon} v^2\,dx \leq C\Big(\int_{\Omega_\varepsilon} v^2\psi_1^2\big(x_1, \frac{x}{\varepsilon}\big)\,dx + \varepsilon^2\int_{\Omega_\varepsilon} |\nabla_{x'}v|^2\psi_1^2\big(x_1, \frac{x}{\varepsilon}\big)\,dx\Big), \quad v \in H^1\big(\Omega_\varepsilon, \psi_1^2\big(x_1, \frac{x}{\varepsilon}\big)\big). \tag{13}$$

Recall that x' are the transverse coordinates, and the radius of the cylinder Ω_ε is of order ε. To verify (13) one may use the following one-dimensional inequality:

$$\int_0^\infty \Big(\int_x^\infty |v(\xi)|\,d\xi\Big)^2 dx \leq 4\int_0^\infty v^2x^2\,dx, \quad \int_0^\infty v^2x^2\,dx < \infty,$$

arguing in local charts as in the standard proof of the Sobolev embedding theorems (cf. [15]), together with the estimates (30) on the growth of ψ_1 in the vicinity of the lateral boundary of $\square(x_1)$ (see Lemma 4.1 below). It then follows from the estimate (13), the uniform boundedness of ψ_1, $b(x_1, y)$, and $c(x_1, y)$, and the ellipticity condition for A, that

$$\begin{aligned}\Big|\int_{\Omega_\varepsilon} (\psi_1 b + \varepsilon^{-1}\psi_1 c)\big(x_1, \frac{x}{\varepsilon}\big)(v_1^\varepsilon)^2\,dx\Big| &\leq \frac{C}{\varepsilon}\Big(1 + \varepsilon^2\int_{\Omega_\varepsilon} |\nabla_{x'}v_1^\varepsilon|^2\psi_1^2\big(x_1, \frac{x}{\varepsilon}\big)\,dx\Big) \\ &\leq \frac{C}{\varepsilon} + C_1\varepsilon\int_{\Omega_\varepsilon} (\psi_1^2 A)\big(x_1, \frac{x}{\varepsilon}\big)\nabla v_1^\varepsilon \cdot \nabla v_1^\varepsilon\,dx,\end{aligned}$$

which together with (12) give

$$\lambda_1^\varepsilon \geq \frac{\mu_1(0)}{\varepsilon^2} - \frac{C}{\varepsilon},$$

for small enough $\varepsilon > 0$, and the desired estimate from below for the first eigenvalue λ_1^ε. □

It follows from the estimate in Lemma 3.1 that the first eigenfunction u_1^ε concentrates (or localizes) in the vicinity of the minimum point of $\mu_1(x_1)$ (here at $x_1 = 0$). Namely, for any $\gamma > 0$,

$$\int\limits_{\Omega_\varepsilon \setminus B_\gamma(0)} (u_1^\varepsilon)^2 \, dx < \gamma,$$

for all small enough $\varepsilon > 0$, where $B_\gamma(0)$ is the ball of radius γ centered at the origin (cf. Lemma 3.3 in [21]).

3.2. Spectral asymptotics

We introduce the homogenized problem for eigenpairs (ν, v) on $\mathbf{R}$:

$$-\frac{d}{dz_1}\Big(a^{\text{eff}} \frac{dv}{dz_1}\Big) + \Big(c^{\text{eff}} + \frac{1}{2}\mu_1''(0) z_1^2\Big) v = \nu v, \quad z_1 \in \mathbf{R}, \tag{14}$$

where the constant coefficients a^{eff}, c^{eff} are defined as follows. Let μ_1, ψ_1 be the first eigenpair to (6) normalized by (7). Let

$$c^{\text{eff}} = -\int\limits_{\square(0)} \psi_1(0,y)\Big(\text{div}_y(A(x_1,y)\nabla_x \psi_1(x_1,y)) + \text{div}_x(A(x_1,y)\nabla_y \psi_1(x_1,y))\Big)\Big|_{x_1=0} dy.$$

The effective coefficient a^{eff} is such that $a^{\text{eff}} > 0$, and it is given by

$$a^{\text{eff}} = \int\limits_{\square(0)} \sum_{j=1}^d \psi_1^2(0,y) a_{1j}(0,y)(\delta_{1j} + \partial_{y_j} N(y)) \, dy, \tag{15}$$

with $N \in H^1(\square(0), \psi_1^2(0,y))$ the unique solution such that

$$\int\limits_{\square(0)} N(y)\psi_1^2(0,y) \, dy = 0,$$

to the following auxiliary cell problem:

$$-\text{div}_y((\psi_1^2 A)(0,y)\nabla_y N) = \sum_{j=1}^d \partial_{y_j}(\psi_1^2 a_{j1})(0,y), \quad y \in \square(0). \tag{16}$$

In the cell problem (16), the classical non-homogeneous Neumann condition is not present and this because the weight $\psi_1^2(0,y)$ effectively removes the lateral boundary $\partial\square(0)$. This is seen in the variational form of the problem, presented as (18)–(19) below.

In the definition of the effective coefficient a^{eff}, the weighted Sobolev space $H^1(\square(0), \psi_1^2(0,y))$ is used. Denote

$$L^2(\square(0), \psi_1^2(0,y)) = \Big\{ v : \int_{\square(0)} v^2(y)\psi_1^2(0,y)\, dy < \infty \Big\},$$

the weighted Lebesgue space. Then $H^1(\square(0), \psi_1^2(0,y))$ is defined as

$$H^1(\square(0), \psi_1^2(0,y)) = \big\{ v \in L^2(\square(0), \psi_1^2(0,y)) : \nabla v \in L^2(\square(0), \psi_1^2(0,y)) \big\},$$

with inner product

$$(u,v)_{H^1(\square(0),\psi_1^2(0,y))} = \int_{\square(0)} uv\, \psi_1^2(0,y)\, dy + \int_{\square(0)} (\nabla_y u \cdot \nabla_y v)\psi_1^2(0,y)\, dy.$$

One notes that $\psi_1^2(0,y) > 0$ for $y \in \square(0)$, and $\psi_1^2(0,y) = 0$ for y on $\partial\square(0)$. Moreover,

$$\psi_1^2(0,y) \in L^1_{\mathrm{loc}}(\square(0)), \qquad \frac{1}{\psi_1^2(0,y)} \in L^1_{\mathrm{loc}}(\square(0)). \tag{17}$$

The first property in (17) ensures that $C^\infty(\square(0))$ belongs to $H^1(\square(0), \psi_1^2(0,y))$, and the second property in (17) ensures that $H^1(\square(0), \psi_1^2(0,y))$ is a separable Hilbert space (cf. [15,12]).

The cell problem (16) is well-posed in $H^1(\square(0), \psi_1^2(0,y))$, and its variational form is

$$N \in H^1(\square(0), \psi_1^2(0,y)), \quad \int_{\square(0)} N(y)\psi_1^2(0,y)\, dy = 0, \tag{18}$$

$$\int_{\square(0)} (\psi_1^2 A)(0,y)\nabla_y N \cdot \nabla_y \varphi\, dy = -\int_{\square(0)} \sum_{j=1}^{d} (\psi_1^2 a_{1j})(0,y) \frac{\partial \varphi}{\partial y_j}\, dy, \quad \varphi \in H^1(\square(0), \psi_1^2(0,y)). \tag{19}$$

Lemma 3.2. *The spectrum of the harmonic oscillator problem (14) is bounded from below and discrete:*

$$\nu_1 < \nu_2 < \cdots < \nu_j < \cdots, \quad \lim_{j\to\infty} v_j = \infty.$$

The corresponding eigenfunctions $v_j \in L^2(\mathbf{R})$ may be normalized by

$$\int_{\mathbf{R}} v_i v_j\, dz_1 = \delta_{ij}. \tag{20}$$

The following theorem is the result of this paper.

Theorem 3.3. *Suppose that (8) holds. Let λ_i^ε, u_i^ε be the ith eigenpair to (2), u_i^ε normalized by (3), and let μ_1, ψ_1 be the first eigenpair to (6), ψ_1 normalized by (58). Then*

$$(i) \quad \lambda_i^\varepsilon = \frac{\mu_1(0)}{\varepsilon^2} + \frac{\nu_i}{\varepsilon} + o\Big(\frac{1}{\varepsilon}\Big),$$

$$(ii) \quad \frac{1}{\varepsilon^{d-1}\sqrt{\varepsilon}} \int_{\Omega_\varepsilon} \Big| u_i^\varepsilon(x) - \psi_1\Big(0, \frac{x}{\varepsilon}\Big) v_i^0\Big(\frac{x_1}{\sqrt{\varepsilon}}\Big) \Big|^2 dx = o(1),$$

as ε tends to zero, where (ν_i, v_i) is the ith eigenpair to (14), v_i normalized by (20).

The Dirichlet condition for u_1^ε on the lateral boundary Σ_ε of the cylinder Ω_ε is captured in the limit by the radial profile $\psi_1(0, y)$ solving the cell eigenvalue problem (6), while the Dirichlet condition on the ends/bases $\Gamma_\varepsilon^\pm$ of the cylinder Ω_ε is translated into exponential decay of the longitudinal profile v solving the homogenized eigenvalue problem (14) as $|z_1|$ tends to infinity.

4. Proof of Theorem 3.3

This section is devoted to the convergence of the spectrum, Theorem 3.3.

4.1. Rescaled and shifted problem

The estimate for the first eigenvalue λ_1^ε to (2) in Lemma 3.1 suggests studying the asymptotics of

$$\varepsilon\lambda_i^\varepsilon - \frac{\mu_1(0)}{\varepsilon}, \tag{21}$$

as ε tends to zero, as for $i = 1$ this is $O(1)$. Moreover, the first eigenfunction tends to localize in the vicinity of $x_1 = 0$, which is the minimum point of $\mu_1(x_1)$. Let us subtract $\mu_1(0)/\varepsilon^2$ from both sides of the equation (2), shifting the spectrum to the left, and make the following change of variables:

$$z = \frac{x}{\sqrt{\varepsilon}}, \qquad u^\varepsilon(\sqrt{\varepsilon}z) = \psi_1\Big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\Big)v^\varepsilon(z) \equiv \psi_1^\varepsilon(z)v^\varepsilon(z). \tag{22}$$

The corresponding problem in the up-scaled domain and bases

$$\widetilde{\Omega_\varepsilon} = \frac{1}{\sqrt{\varepsilon}}\Omega_\varepsilon, \qquad \widetilde{\Gamma_\varepsilon^\pm} = \frac{1}{\sqrt{\varepsilon}}\Gamma_\varepsilon^\pm,$$

takes the form

$$-\mathrm{div}_z\big(\widetilde{A^\varepsilon}\nabla_z v^\varepsilon\big) + \Big(C^\varepsilon + \frac{\mu_1(\sqrt{\varepsilon}z_1) - \mu_1(0)}{\varepsilon}(\psi_1^\varepsilon)^2\Big)v^\varepsilon = \nu^\varepsilon(\psi_1^\varepsilon)^2 v^\varepsilon \qquad \text{in } \widetilde{\Omega_\varepsilon}, \tag{23}$$

$$v^\varepsilon = 0 \qquad \text{on } \widetilde{\Gamma_\varepsilon^\pm}. \tag{24}$$

Here, the coefficients and the potential are given by

$$\widetilde{A^\varepsilon}(z) \equiv \widetilde{A}\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big) = \psi_1^2\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)A\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big), \tag{25}$$

and

$$\begin{aligned} C^\varepsilon(z) \equiv C\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big) = \Big[&- \psi_1(x_1, y)\mathrm{div}_y(A(x_1, y)\nabla_x\psi_1(x_1, y)) \\ &- \psi_1(x_1, y)\mathrm{div}_x(A(x_1, y)\nabla_y\psi_1(x_1, y)) \\ &- \varepsilon\psi_1(x_1, y)\mathrm{div}_x(A(x_1, y)\nabla_x\psi_1(x_1, y))\Big]\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big), \end{aligned} \tag{26}$$

and by choice (21),

$$\nu^\varepsilon = \varepsilon\lambda^\varepsilon - \frac{\mu_1(0)}{\varepsilon}. \tag{27}$$

The functions v_i^ε and the number ν_i^ε are well-defined by (22) and (27), in terms of λ_i^ε, u_i^ε and ψ_1, and so $\nu_i^\varepsilon, v_i^\varepsilon$ is an eigenpair to the problem (23)–(24). The problem for v^ε (23)–(24) is well-posed in the weighted Sobolev space $H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2, \widetilde{\Gamma_\varepsilon^\pm})$ defined as follows. Denote

$$L^2(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2) = \Big\{ v : \int_{\widetilde{\Omega_\varepsilon}} v^2(z)(\psi_1^\varepsilon)^2 \, dz < \infty \Big\},$$

the weighted Lebesgue space. Then $H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2, \widetilde{\Gamma_\varepsilon^\pm})$ is defined as

$$H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2, \widetilde{\Gamma_\varepsilon^\pm}) = \{ v \in L^2(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2) : \nabla v \in L^2(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2),\ v = 0 \text{ on } \widetilde{\Gamma_\varepsilon^\pm} \}, \tag{28}$$

with inner product

$$(u, v)_{H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2, \widetilde{\Gamma_\varepsilon^\pm})} = \int_{\widetilde{\Omega_\varepsilon}} u\, v\, (\psi_1^\varepsilon)^2 \, dz + \int_{\widetilde{\Omega_\varepsilon}} \nabla_z u \cdot \nabla_z v \, (\psi_1^\varepsilon)^2 \, dz,$$

where the meaning of the trace on $\widetilde{\Gamma_\varepsilon^\pm}$ in (28) is described below. One notes that $(\psi_1^\varepsilon)^2 > 0$ for $z \in \widetilde{\Omega_\varepsilon}$, and $(\psi_1^\varepsilon)^2 = 0$ on the lateral boundary $\widetilde{\Sigma_\varepsilon} = \partial\widetilde{\Omega_\varepsilon} \setminus \widetilde{\Gamma_\varepsilon^\pm}$, and $(\psi_1^\varepsilon)^2 > 0$ almost everywhere on the ends $\widetilde{\Gamma_\varepsilon^\pm}$. Furthermore,

$$(\psi_1^\varepsilon)^2 \in L^1_{\text{loc}}(\widetilde{\Omega_\varepsilon}), \qquad \frac{1}{(\psi_1^\varepsilon)^2} \in L^1_{\text{loc}}(\widetilde{\Omega_\varepsilon}). \tag{29}$$

The first property in (29) ensures that $C^\infty(\widetilde{\Omega_\varepsilon})$ belongs to $H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2, \widetilde{\Gamma_\varepsilon^\pm})$, and the second property in (29) ensures that $H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2, \widetilde{\Gamma_\varepsilon^\pm})$ is a separable Hilbert space (cf. [15,12]).

The usual trace can be used in the definition (28) of $H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2, \widetilde{\Gamma_\varepsilon^\pm})$, say by arguing separately on disjoint parts of the boundary. Let $C^\infty(\widetilde{\overline{\Omega^\varepsilon}}, \widetilde{\Gamma^\varepsilon_\pm})$ denote the space of functions in $C^\infty(\widetilde{\overline{\Omega^\varepsilon}})$ with support not intersecting $\widetilde{\Gamma^\varepsilon_\pm}$. As $(\psi_1^\varepsilon)^2 \in L^1_{\text{loc}}(\widetilde{\overline{\Omega^\varepsilon}})$, all functions in $C^\infty(\widetilde{\overline{\Omega^\varepsilon}}, \widetilde{\Gamma^\varepsilon_\pm})$ belong to $H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2) = \{ v \in L^2(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2) : \nabla v \in L^2(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2)\}$, equipped with the considered norm. It is therefore meaningful to let $H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2, \widetilde{\Gamma_\varepsilon^\pm})$ be the closure of $C^\infty(\widetilde{\overline{\Omega^\varepsilon}}, \widetilde{\Gamma^\varepsilon_\pm})$ with respect to the considered norm. The space of traces of functions $H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2)$ on $\widetilde{\Gamma^\varepsilon_\pm}$ is the factor space $H^{1/2}(\widetilde{\Gamma^\varepsilon_\pm}) = H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2) / H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2, \widetilde{\Gamma_\varepsilon^\pm})$ equipped with the factor norm (cf. [12]).

Note that C^ε is uniformly bounded by the regularity properties of ψ_1, and consequently adding a potential Cv^ε to (23) shifts the spectrum by C and makes the potential positive.

Although the spectrum of (2) is characterized, it is useful to remark that it is at most shifted after the change of variables. To this end we make use of the following estimates on the growth of the first eigenfunction to (6) in the vicinity of the Dirichlet boundary (see [4] for a general analysis).

Lemma 4.1. *There exist absolute constants c_1, c_2, δ such that in local charts $(y'_r, y_{rd}), a(y'_r)$ of $\square(x_1)$,*

$$c_1\psi_1(x_1, y) \le y'_{rd} - a(y'_r) \le c_2\psi_1(x_1, y), \tag{30}$$

for $y \in V_\delta$, where $V_\delta = \{ y \in \overline{\square(x_1)} : y'_{rd} - a(y'_r) < \delta \}$.

Proof. By the regularity of the coefficient matrix A and the boundary $\partial\square(x_1)$, Ψ_1 is $C^2(\overline{\square(x_1)})$ for all $x_1 \in \overline{I}$. Because the boundary $\partial\square(x_1)$ is C^2, the distance function $d(y, \partial\square(x_1))$ is in $C^2(V_\delta)$ for some $\delta > 0$, where V_δ is a neighborhood of $\partial\square(x_1)$:

$$V_\delta = \{y \in \overline{\square(x_1)} : d(y, \partial\square(x_1)) < \delta\}.$$

Let φ be a smooth cutoff such that $\varphi = 1$ in $V_{\delta/2}$, and $\varphi = 0$ outside V_δ. Then for some constant $C(x_1)$,

$$|\mathrm{div}_y(A(x_1, y)\nabla_y(\varphi d(y, \partial\square(x_1))))| \le C(x_1), \quad y \in \overline{\square(x_1)}.$$

It follows that for $\alpha > 0$,

$$-\mathrm{div}_y(A(x_1, y)\nabla_y(\Psi_1 - \alpha\varphi d(y, \partial\square(x_1)))) \ge \mu_1(x_1)\Psi_1(x_1, y) - \alpha C(x_1),$$

for $y \in \square(x_1)$. By the Taylor theorem, for $y_0 \in \partial\square(x_1)$,

$$\Psi_1(x_1, y) = \frac{\partial\Psi_1}{\partial\nu}(x_1, y_0)d(y, \partial\square(x_1)) + O(d(y, \partial\square(x_1))^2), \tag{31}$$

and by the strong maximum principle, the inward normal derivative is positive, if Ψ_1 is chosen positive in $\square(x_1)$:

$$\frac{\partial\Psi_1}{\partial\nu}(x_1, y) > 0, \quad x_1 \in \overline{I},\ y \in \overline{\square(x_1)}.$$

One notes that in the vicinity of the boundary, the quadratic term in (31) is small, while in the interior a positive Ψ_1 is globally bounded away from zero by a positive constant. Thus by choosing Ψ_1 to be positive in $\square(x_1)$, one has for all small enough α, δ, that $\Psi_1 - \alpha\varphi d(y, \partial\square(x_1))$ is a subsolution that satisfies the Dirichlet condition on the boundary:

$$\begin{aligned} -\mathrm{div}_y(A(x_1, y)\nabla_y(\Psi_1 - \alpha\varphi d(y, \partial\square(x_1)))) &\ge 0, \quad y \in \square(x_1), \\ \Psi_1 - \alpha\varphi d(y, \partial\square(x_1)) &= 0, \quad y \in \partial\square(x_1). \end{aligned}$$

It follows from the maximum principle that

$$\psi_1(x_1, y) \ge C(x_1)d(y, \partial\square(x_1)), \quad y \in V_{\delta/2},$$

for some constant $C(x_1)$. The second inequality in (30) follows, by the compactness of $\partial\square(x_1)$. The first inequality in (30) follows from the Taylor theorem,

$$\Psi_1(x_1, y) = \nabla_y\Psi_1(x_1, y_0) \cdot (y - y_0) + O(|y - y_0|^2),$$

for $y_0 \in \partial\square(x_1)$, and the quadratic term is dominated by the linear term in a neighborhood of the boundary. □

Lemma 4.2. *The spectrum of problem (23) is discrete, bounded from below, and consists of a countably infinite number of points:*

$$C \le \nu_1^\varepsilon < \nu_2^\varepsilon \le \nu_3^\varepsilon \le \cdots \qquad \lim_{i\to\infty} v_i^\varepsilon = \infty,$$

counted as many times as their finite multiplicity, and there exists a sequence of corresponding eigenfunctions v_i^ε *that may be normalized by*

$$\varepsilon^{-(d-1)/2} \int\limits_{\widetilde{\Omega}_\varepsilon} |\square(\sqrt{\varepsilon}z)|^{-1} v_i^\varepsilon v_j^\varepsilon\, (\psi_1^\varepsilon)^2\, dz = \delta_{ij}. \tag{32}$$

Proof. Let $G^\varepsilon : L^2(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2) \to H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2, \widetilde{\Gamma_\varepsilon^\pm})$ be the Green operator that maps $f \in L^2(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2)$ to the solution to the equation

$$-\mathrm{div}_z\big(\widetilde{A^\varepsilon}\nabla_z v\big) + \big(D^\varepsilon + C_1(\psi_1^\varepsilon)^2\big)v = (\psi_1^\varepsilon)^2 f \quad \text{in } \widetilde{\Omega_\varepsilon}, \tag{33}$$

$$v = 0 \qquad \text{on } \widetilde{\Gamma_\varepsilon^\pm}, \tag{34}$$

where

$$D^\varepsilon = C^\varepsilon + \frac{\mu_1(\sqrt{\varepsilon}z_1) - \mu_1(0)}{\varepsilon}(\psi_1^\varepsilon)^2,$$

where C^ε is given by (26), and C_1 is a constant. Because D^ε is uniformly bounded, the solution $v \in H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2, \widetilde{\Gamma_\varepsilon^\pm})$ is uniquely defined for all ε, by the Riesz representation theorem, for large enough C_1. By Lemma 4.1, there exist absolute constants c_1, c_2, c_3 such that in local charts $(x_r', x_{rd}), a(x_r')$,

$$\sqrt{\varepsilon}c_1\psi_1^\varepsilon(x) \le x_{rd}' - a(x_r') + \kappa(x_r') \le \sqrt{\varepsilon}c_2\psi_1^\varepsilon(x), \tag{35}$$

for some κ such that $0 \le \kappa(x_r') \le c_3$, where κ is necessary to compensate for Ψ_1^ε being nonzero on the ends $\widetilde{\Gamma_\varepsilon^\pm}$. Because the boundary of $\widetilde{\Omega_\varepsilon}$ is continuous, the condition (35) guaranties that $H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2, \widetilde{\Gamma_\varepsilon^\pm})$ is compactly embedded into $L^2(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2)$ (see e.g. [15]). Let f_n be a bounded sequence in $H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2, \widetilde{\Gamma_\varepsilon^\pm})$. By weak compactness, there along a subsequence, f_n converges weakly to f in $H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2, \widetilde{\Gamma_\varepsilon^\pm})$, and strongly in $L^2(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2)$. It follows that $G^\varepsilon f_n$ converges to $G^\varepsilon f$ in $H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2, \widetilde{\Gamma_\varepsilon^\pm})$, along some subsequence. This shows that G^ε is compact from $H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2, \widetilde{\Gamma_\varepsilon^\pm})$ to $H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2, \widetilde{\Gamma_\varepsilon^\pm})$. In particular, G^ε is self-adjoint on $H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2, \widetilde{\Gamma_\varepsilon^\pm})$, and positive for large enough C_1. The statement of the lemma then follows from the spectral theorem for such operators. □

We may obtain information about the asymptotics of the eigenvalues and eigenfunctions to (2) by studying the spectral asymptotics of the problem (23)–(24), or view it as a change of variables in studying (2). In any point of view the following a priori estimate for the eigenfunction v_1^ε to the rescaled problem is a starting point of the analysis.

4.2. A priori estimates for the rescaled problem

For the dimension reduction in the problem, we introduce the natural measure on $\mathbf{R}^d$ that charges the thin up-scaled cylinder $\widetilde{\Omega_\varepsilon}$. Namely, let

$$d\mu_\varepsilon(z) = \varepsilon^{-(d-1)/2}|\square(0)|^{-1}\chi_{\widetilde{\Omega_\varepsilon}}(z)\,dz. \tag{36}$$

In this way, we divide by the measure of the cross-section such that the rescaled measure converges to the one-dimensional Lebesgue measure charging the real line.

Lemma 4.3. *The sequence of measures μ_ε defined by (36) converges weakly to μ defined by*

$$d\mu_\varepsilon \rightharpoonup d\mu = dz_1 \times \delta(z'), \quad z = (z_1, z') \in \mathbf{R}^d,$$

as ε tends to zero (weak in the space of Radon measures on $\mathbf{R}^d$).*

Proof. The proof relies on the mean value property and one only has to check that the rescalings are appropriate (cf. Lemma 2.1 in [19]).

Let $\varphi \in C_0(\mathbf{R}^d)$. Then

$$\int_{\mathbf{R}^d} \varphi(z)\, d\mu_\varepsilon(z) = \int_{\varepsilon^{-1/2} I} \frac{\varepsilon^{-\frac{(d-1)}{2}}}{|\square(0)|} \int_{\sqrt{\varepsilon} Q(\sqrt{\varepsilon} z_1, \frac{z_1}{\sqrt{\varepsilon}})} \varphi(z) dz' dz_1.$$

Rescaling $y' = z'/\sqrt{\varepsilon}$ gives

$$\frac{1}{|\square(0)|} \int_{\mathbf{R}^d} \varphi(z)\, d\mu_\varepsilon(z) = \frac{1}{|\square(0)|} \int_{\varepsilon^{-1/2} I} \frac{1}{|\square(0)|} \int_{Q(\sqrt{\varepsilon} z_1, \frac{z_1}{\sqrt{\varepsilon}})} \varphi(z_1, \sqrt{\varepsilon} y') dy' dz_1.$$

Let us divide the interval $\varepsilon^{-1/2} I$ into subintervals (translated scaled periods) $I_j^\varepsilon = \sqrt{\varepsilon}[0,1) + \sqrt{\varepsilon} j$, $j \in \mathbf{Z}$.

On each interval we use the mean-value theorem choosing a point $\sqrt{\varepsilon}\xi_j$ and get

$$\frac{1}{|\square(0)|} \sum_j \int_{I_j^\varepsilon} \int_{Q(\sqrt{\varepsilon} z_1, \frac{z_1}{\sqrt{\varepsilon}})} \varphi(z_1, \sqrt{\varepsilon} y') dy' dz_1 = \frac{1}{|\square(0)|} \sum_j \int_{I_j^\varepsilon} \int_{Q(\sqrt{\varepsilon} \xi_j z_1, \frac{z_1}{\sqrt{\varepsilon}})} \varphi(\xi_j, \sqrt{\varepsilon} y') dy' dz_1.$$

Since $Q(x_1, y_1)$ is periodic with respect to y_1, rescaling $z_1 = y_1\sqrt{\varepsilon}$ yields

$$\frac{1}{|\square(0)|} \sum_j \sqrt{\varepsilon} \int_{\mathbf{T}^1} \int_{Q(\sqrt{\varepsilon}\xi_j, y_1)} \varphi(\xi_j, \varepsilon y') dy' dy_1 = \frac{1}{|\square(0)|} \sum_j \sqrt{\varepsilon} \int_{\square(\sqrt{\varepsilon}\xi_j)} \varphi(\xi_j, \varepsilon y') dy.$$

The last sum is a Riemann sum converging to the following integral

$$\frac{1}{|\square(0)|} \sum_j \sqrt{\varepsilon} \int_{\square(\sqrt{\varepsilon}\xi_j)} \varphi(\xi_j, \varepsilon y') dy \to \frac{1}{|\square(0)|} \int_{\mathbf{R}} \int_{\square(0)} \varphi(z_1, 0)\, dy dz_1 = \int_I \varphi(z_1, 0)\, dz_1 = \int_{\mathbf{R}^d} \varphi(z)\, d\mu,$$

as ε tends to zero. □

Lemma 4.4. *Suppose that $\mu_1(x_1)$ has a unique global minimum point at $x_1 = 0$, i.e. (8) holds. Let v_1^ε be the first eigenfunction to (23), normalized by (32). Then the following estimate holds:*

$$\|\psi_1^\varepsilon \nabla v_1^\varepsilon\|_{L^2(\mathbf{R}^d, d\mu_\varepsilon)} + \|\psi_1^\varepsilon v_1^\varepsilon\|_{L^2(\mathbf{R}^d, d\mu_\varepsilon)} + \|\psi_1^\varepsilon z_1 v_1^\varepsilon\|_{L^2(\mathbf{R}^d, d\mu_\varepsilon)} \le C.$$

Proof. The weak form of the equation for $\nu^\varepsilon, v^\varepsilon$ is

$$\int_{\widetilde{\Omega_\varepsilon}} \widetilde{A^\varepsilon} \nabla v^\varepsilon \cdot \nabla \varphi\, dz + \int_{\widetilde{\Omega_\varepsilon}} \Big(C^\varepsilon + \frac{\mu_1(\sqrt{\varepsilon} z_1) - \mu_1(0)}{\varepsilon} (\psi_1^\varepsilon)^2 \Big) v^\varepsilon \varphi\, dz = \nu^\varepsilon \int_{\widetilde{\Omega_\varepsilon}} (\psi_1^\varepsilon)^2 v^\varepsilon \varphi\, dz, \tag{37}$$

for all $\varphi \in H^1(\widetilde{\Omega_\varepsilon}, (\psi_1^\varepsilon)^2, \widetilde{\Gamma_\varepsilon^\pm})$, where C^ε is given by (26).

We turn to the a priori estimates for the first eigenfunction v_1^ε, under the following normalization:

$$\int_{\mathbf{R}^d} (\psi_1^\varepsilon)^2 (v_1^\varepsilon)^2\, d\mu_\varepsilon = 1.$$

By Lemma 3.1,

$$\nu_1^\varepsilon = \varepsilon\lambda_1^\varepsilon - \frac{\mu_1(0)}{\varepsilon} = O(1),$$

as ε tends to zero. Taking the first eigenfunction as a test function in (37), we get

$$\begin{aligned}
&\int_{\mathbf{R}^d} \psi_1^2\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)|\nabla_z v_1^\varepsilon|^2\, d\mu_\varepsilon(z) \\
&\quad\le C\int_{\mathbf{R}^d} (\psi_1^2 A)\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)\nabla_z v_1^\varepsilon \cdot \nabla_z v_1^\varepsilon\, d\mu_\varepsilon(z) \\
&\quad= C\Big(\nu_1^\varepsilon \int_{\mathbf{R}^d} \psi_1^2\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)(v_1^\varepsilon)^2\, d\mu_\varepsilon(z) - \int_{\mathbf{R}^d} (\psi_1 b + \varepsilon\psi_1 c)\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)(v_1^\varepsilon)^2\, d\mu_\varepsilon(z) \\
&\qquad\quad - \int_{\mathbf{R}^d} \Big(\frac{\mu_1(\sqrt{\varepsilon}z_1) - \mu_1(0)}{\varepsilon}\psi_1^2\Big)\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)(v_1^\varepsilon)^2\, d\mu_\varepsilon(z)\Big) \\
&\quad\le C,
\end{aligned}$$

for some constant C which is independent of ε, where one has used the boundedness of ψ_1, b, and c.

By hypothesis (8), there exists an absolute constant C that is independent of ε and such that

$$\frac{\mu_1(\sqrt{\varepsilon}z_1) - \mu_1(0)}{\varepsilon} \ge Cz_1^2, \quad z_1 \in \frac{1}{\sqrt{\varepsilon}}\overline{I}.$$

Indeed, suppose that there exists a sequence ζ_j such that

$$\frac{\mu_1(\sqrt{\varepsilon}\zeta_j) - \mu_1(0)}{|\sqrt{\varepsilon}\zeta_j|^2} \to 0, \quad j \to \infty.$$

Since $z_1 \in (\sqrt{\varepsilon})^{-1}\overline{I}$, then for each fixed ε, $|\sqrt{\varepsilon}\zeta_j|^2$ is bounded, which yields

$$\mu_1(\sqrt{\varepsilon}\zeta_j) \to \mu_1(0), \quad j \to \infty.$$

Then $\zeta_j \to 0$, by uniqueness of the minimum point 0. On the other hand, since $\mu_1''(0)$ is strictly positive by assumption,

$$\frac{\mu_1(\sqrt{\varepsilon}\zeta_j) - \mu_1(0)}{|\sqrt{\varepsilon}\zeta_j|^2} = \frac{1}{2}\mu_1''(0) + o(1) > \alpha\mu_1''(0) > 0, \quad \zeta_j \to 0$$

for some $\alpha > 0$, and we arrive at a contradiction.

Therefore, again by the integral identity for the eigenpair $(\nu_1^\varepsilon, v_1^\varepsilon)$,

$$\begin{aligned}
&\int_{\mathbf{R}^d} \psi_1^2\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)z_1^2(v_1^\varepsilon)^2\, d\mu_\varepsilon(z) \\
&\quad\le C_1\int_{\mathbf{R}^d} \psi_1^2\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)\frac{\mu_1(\sqrt{\varepsilon}z_1) - \mu_1(0)}{\varepsilon}(v_1^\varepsilon)^2\, d\mu_\varepsilon(z) \\
&\quad= C_2\Big(\nu_1^\varepsilon \int_{\mathbf{R}^d} \psi_1^2\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)(v_1^\varepsilon)^2\, d\mu_\varepsilon(z) - \int_{\mathbf{R}^d} (\psi_1^2 A)\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)\nabla_z v_1^\varepsilon \cdot \nabla_z v_1^\varepsilon\, d\mu_\varepsilon(z)
\end{aligned}$$

$$- \int_{\mathbf{R}^d} (\psi_1 b + \varepsilon \psi_1 c)\big(\sqrt{\varepsilon} z_1, \frac{z}{\sqrt{\varepsilon}}\big)(v_1^\varepsilon)^2 \, d\mu_\varepsilon(z)\Big)$$
$$\leq C,$$

for some constant C which is independent of ε, by the boundedness of A, ψ_1, b, and c, and the above estimate for $\psi_1\big(\sqrt{\varepsilon} z_1, \frac{z}{\sqrt{\varepsilon}}\big)\nabla_z v_1^\varepsilon$. This proves Lemma 4.4. □

4.3. Two-scale convergence

The definition of two-scale convergence for our particular setting is the following. For the specific weakly convergent measures in (36),

$$d\mu_\varepsilon(z) = \frac{\varepsilon^{-(d-1)/2}}{|\Box(\sqrt{\varepsilon} z_1)|} \chi_{\widetilde{\Omega_\varepsilon}}(z) dz \rightharpoonup d\mu(z) = dz_1 \times d\delta(z'),$$

as ε tends to zero. A bounded sequence v_ε in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$, that is

$$\limsup_{\varepsilon\to\infty} \int_{\mathbf{R}^d} v_\varepsilon^2(z) \, d\mu_\varepsilon(z) < \infty, \tag{38}$$

is said to be weakly two-scale convergent at rate $\sqrt{\varepsilon}$ to a function $v = v(z, y) \in L^2(\mathbf{R}^d \times \Box(0), d\mu \times dy)$, $v_\varepsilon \overset{2}{\rightharpoonup} v$, if

$$\lim_{\varepsilon\to 0} \int_{\mathbf{R}^d} v_\varepsilon(z)\varphi(z)\phi\Big(\frac{z}{\sqrt{\varepsilon}}\Big) \, d\mu_\varepsilon(z) = \frac{1}{|\Box(0)|} \int_{\mathbf{R}^d} \int_{\Box(0)} v(z, y)\varphi(z)\phi(y) \, dy \, d\mu(z),$$

for any $\phi \in C^\infty(\mathbf{T}^1 \times \mathbf{R}^{d-1})$ and any $\varphi \in C_0^\infty(\mathbf{R}^d)$.

A bounded sequence v_ε in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$ is said to be strongly two-scale converging to a function $v \in L^2(\mathbf{R}^d \times \Box(0), d\mu \times dy)$ if

$$\lim_{\varepsilon\to 0} \int_{\mathbf{R}^d} v_\varepsilon(z) w_\varepsilon(z) \, d\mu_\varepsilon(z) = \frac{1}{|\Box(0)|} \int_{\mathbf{R}^d} \int_{\Box(0)} v(z, y) w(z, y) \, dy \, d\mu(z),$$

for any weakly two-scale converging sequence $w_\varepsilon \overset{2}{\rightharpoonup} w$ in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$.

The following two-scale compactness principle holds.

Lemma 4.5. *Let v_ε be a bounded sequence in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$, that is it satisfies (38). Then, along a subsequence, v_ε converges weakly two-scale in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$ to some $v = v(z, y) \in L^2(\mathbf{R}^d \times \Box(0), d\mu \times dy)$.*

A proof Lemma 4.5 goes as the proof of the weak compactness principle in $L^2(\mathbf{R}^d)$ (cf. Lemma 2.2 in [25]) utilizing the property of the mean value.

4.4. Passage to the limit

Lemma 4.6. *Let $(\nu_1^\varepsilon, v_1^\varepsilon)$ be the first eigenpair to (23), normalized by (32). Then, along some subsequence, $(\nu_1^\varepsilon, \psi_1^\varepsilon v_1^\varepsilon)$ converge in $\mathbf{R}$ and weakly two-scale in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$, respectively, to a pair $(\nu, \psi_1(0, y)v)$, where (ν, v) is an eigenpair to the effective problem (14).*

Note that at this point we do not claim that (ν, v) is the first eigenpair to the limit problem.

Proof of Lemma 4.6. By the two-scale compactness principle, using the a priori estimates in Lemma 4.4 for $\psi_1^\varepsilon v_1^\varepsilon$, there exist $v, w \in L^2(\mathbf{R}^d \times \square(0), dz_1 \times d\delta(z') \times dy)$ such that along some subsequence, still denoted by ε, the following weak two-scale convergences hold in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$:

$$\psi_1\big(\sqrt{\varepsilon} z_1, \frac{z}{\sqrt{\varepsilon}}\big) v_1^\varepsilon \overset{2}{\rightharpoonup} w(z_1, y), \tag{39}$$

$$\psi_1\big(\sqrt{\varepsilon} z_1, \frac{z}{\sqrt{\varepsilon}}\big) \nabla_z v_1^\varepsilon \overset{2}{\rightharpoonup} p(z_1, y), \tag{40}$$

as ε tends to zero, and in particular $w, p \in L^2(\mathbf{R} \times \square(0))$. Let ν be such that ν^ε converges to ν in $\mathbf{R}$, restricting to a further subsequence if necessary.

By the boundedness of the gradient of v_1^ε, Lemma 4.4 or (39)–(40), one notes that necessarily $w = \psi_1(0, y) v(z_1)$ for some function $v = v(z_1) \in L^2(\mathbf{R})$. To see this one uses the two-scale convergence in (39) and oscillating test functions of the form

$$\Phi^\varepsilon(z) = \sqrt{\varepsilon} \varphi(z) \phi\big(\frac{z}{\sqrt{\varepsilon}}\big),$$

where $\varphi \in C_0^\infty(\mathbf{R}^d)$ and $\phi \in C_0^\infty(\mathbf{T}^1 \times \mathbf{R}^{d-1})$. On the one hand,

$$\begin{aligned}\int_{\mathbf{R}^d} (\psi_1^2)^\varepsilon v_1^\varepsilon \partial_{z_i} \Phi^\varepsilon \, d\mu_\varepsilon &= \int_{\mathbf{R}^d} (\psi_1^2)^\varepsilon v_1^\varepsilon \big(\sqrt{\varepsilon}(\partial_{z_i}\varphi)\phi(y) + \varphi \partial_{y_i}\phi\big)\big(\sqrt{\varepsilon} z_1, \frac{z}{\sqrt{\varepsilon}}\big) \, d\mu_\varepsilon \\ &\to \int_{\mathbf{R}} \frac{1}{|\square(0)|} \int_{\square(0)} \psi_1(0, y) w(z_1, y) \varphi \partial_{y_i}\phi \, dy \, dz_1, \end{aligned} \tag{41}$$

as ε tends to zero, where one has used that $\psi_1^\varepsilon \partial_{z_i} \Phi^\varepsilon$ is strongly two-scale convergent to $\psi_1(0, y)\varphi(z)\partial_{y_i}\phi(y)$ in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$. On the other hand,

$$\int_{\mathbf{R}^d} (\psi_1^2)^\varepsilon v_1^\varepsilon \partial_{z_i} \Phi^\varepsilon \, d\mu_\varepsilon = -\int_{\mathbf{R}^d} \partial_{z_i}((\psi_1^2)^\varepsilon v_1^\varepsilon) \Phi^\varepsilon \, d\mu_\varepsilon. \tag{42}$$

Because

$$\partial_{z_i}\big(\psi_1^2\big(\sqrt{\varepsilon} z_1, \frac{z}{\sqrt{\varepsilon}}\big) v_1^\varepsilon\big) = (\psi_1^2)^\varepsilon \partial_{z_i} v_1^\varepsilon + 2\sqrt{\varepsilon}(\partial_{z_i}\psi_1)^\varepsilon \psi_1^\varepsilon v_1^\varepsilon + \frac{1}{\sqrt{\varepsilon}}(\partial_{y_i}\psi_1)^\varepsilon \psi_1^\varepsilon v_1^\varepsilon,$$

one has, by the boundedness of $\psi_1^\varepsilon \nabla_z v_1^\varepsilon$,

$$\int_{\mathbf{R}^d} \partial_{z_i}((\psi_1^2)^\varepsilon v_1^\varepsilon) \Phi^\varepsilon \, d\mu_\varepsilon \to \int_{\mathbf{R}} \frac{1}{|\square(0)|} \int_{\square(0)} 2w \varphi(z) \phi(y) \partial_{y_i}\psi_1(0, y) \, dy \, dz_1, \tag{43}$$

as ε tends to zero. It follows from (41)–(43) that

$$\psi_1(0, y) \nabla_y w = 2w \nabla \psi_1(0, y),$$

almost everywhere in $\mathbf{R} \times \square(0)$. One concludes that,

$$w(z_1, y) = \psi_1(0, y) v(z_1),$$

for some $v = v(z_1) \in L^2(\mathbf{R})$.

We proceed to the two-scale limit p of $\psi_1^\varepsilon \nabla_z v_1^\varepsilon$. Let $\Phi = \Phi(z_1, y)$ be such that

$$\operatorname{div}_y(\psi_1^2(0,y)\Phi(z_1,y)) = 0 \quad \text{in } \mathbf{T} \times \mathbf{R}^{d-1}. \tag{44}$$

Then one the one hand by (40),

$$\int_{\mathbf{R}^d} (\psi_1^2)^\varepsilon \nabla_z v_1^\varepsilon \cdot \Phi^\varepsilon \, d\mu_\varepsilon \to \int_{\mathbf{R}} \frac{1}{|\square(0)|} \int_{\square(0)} \psi_1(0,y) p(z_1,y) \cdot \Phi(0,y) \, dy \, dz_1,$$

as ε tends to zero. On the other hand by the choice of test functions, compactly supported Φ,

$$\begin{aligned}
\int_{\mathbf{R}^d} (\psi_1^2)^\varepsilon \nabla_z v_1^\varepsilon \cdot \Phi^\varepsilon \, d\mu_\varepsilon &= \int_{\mathbf{R}^d} (\psi_1^2)(0, \frac{z}{\sqrt{\varepsilon}}) \nabla_z v_1^\varepsilon \cdot \Phi^\varepsilon \, d\mu_\varepsilon + o(1) \\
&= - \int_{\mathbf{R}^d} v_1^\varepsilon (\psi_1^2)^\varepsilon (\operatorname{div}_z \Phi)^\varepsilon \, d\mu_\varepsilon + o(1) \\
&\to - \int_{\mathbf{R}} \frac{1}{|\square(0)|} \int_{\square(0)} \psi_1^2(0,y) v(z_1) (\operatorname{div}_z \Phi)(0,y) \, dy \, dz_1 \\
&= \int_{\mathbf{R}} \frac{1}{|\square(0)|} \int_{\square(0)} \nabla_z v(z_1) \cdot \psi_1^2(0,y) \Phi(0,y) \, dy \, dz_1,
\end{aligned}$$

as ε tends to zero, for any $dz_1 \times d\delta(z')$ gradient $\nabla_z v$ of v. Therefore,

$$\int_{\mathbf{R}} \int_{\square(0)} \Big(\frac{p}{\psi_1(0,y)} + \nabla_z v \Big) \cdot \psi_1^2(0,y) \Phi(0,y) \, dy \, dz_1 = 0,$$

for any solution Φ to (44). By the solenoidal nature of $\psi_1^2(0,y)\Phi(z_1,y)$,

$$\frac{p}{\psi_1(0,y)} + \nabla_z v = \nabla_y q,$$

for some $q \in L^2(\mathbf{R}, H^1(\square(0)))$. It follows that

$$p = \psi_1(0,y)(\nabla_z v + \nabla_y q),$$

almost everywhere in $\mathbf{R} \times \square(0)$, for some $q \in L^2(\mathbf{R}, H^1(\square(0)))$.

To sum up, by the two-scale compactness principle for sequences with bounded gradient, there exists $v \in L^2(\mathbf{R})$ and $q \in L^2(\mathbf{R}, H^1(\square(0)))$ such that two-scale weakly in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$,

$$\begin{aligned}
\psi_1^\varepsilon v_1^\varepsilon &\overset{2}{\rightharpoonup} \psi_1(0,y) v(z_1), \\
\psi_1^\varepsilon \nabla_z v_1^\varepsilon &\overset{2}{\rightharpoonup} \psi_1(0,y)(\nabla^{dz_1 \times d\delta(z')} v(z_1) + \nabla_y q(z_1,y)),
\end{aligned}$$

as ε tends to zero. One notes that

$$\nabla^{dz_1 \times d\delta(z')} v = (\partial_{z_1} v, r),$$

where $r \in L^2(\mathbf{R})$ is some transverse gradient of v with respect to the measure $dz_1 \times d\delta(z')$. Moreover, by the a priori estimate $\|z_1\psi_1^\varepsilon v_1^\varepsilon\|_{L^2(\mathbf{R}^d,d\mu_\varepsilon)} \le C$, the convergence of $\psi_1^\varepsilon v_1^\varepsilon$ is strong in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$.

We are now in a position to pass to the limit in the equation for $\nu^\varepsilon, v^\varepsilon$. The variational form of the equation for $\nu^\varepsilon, v^\varepsilon$, in terms of the measure $d\mu_\varepsilon$, is

$$\int_{\mathbf{R}^d} (\psi_1^2 A)\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)\nabla_z v^\varepsilon \cdot \nabla_z \varphi \, d\mu_\varepsilon + \int_{\mathbf{R}^d} (\psi_1(b+\varepsilon c) + \frac{\mu_1(x_1)-\mu_1(0)}{\varepsilon}\psi_1^2)\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big) v^\varepsilon \varphi \, d\mu_\varepsilon$$
$$= \nu^\varepsilon \int_{\mathbf{R}^d} \psi_1^2\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big) v^\varepsilon \varphi \, d\mu_\varepsilon, \tag{45}$$

for any $\varphi \in H^1(\mathbf{R}^d)$.

We pass to the limit as ε tends to zero in (45) using the two-scale converge. Let $\varphi \in C_0^\infty(\mathbf{R}^d)$. Then

$$\begin{aligned}\int_{\mathbf{R}^d} (\psi_1^2 A)\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)\nabla_z v^\varepsilon \cdot \nabla_z \varphi \, d\mu_\varepsilon &= \int_{\mathbf{R}^d} \psi_1\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)\nabla_z v^\varepsilon \cdot (\psi_1 A)\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)\nabla_z \varphi \, d\mu_\varepsilon \\ &\to \int_{\mathbf{R}} \int_{\square(0)} p(z_1,y) \cdot (\psi_1 A)(0,y)\nabla_z \varphi(z_1,0)\, dy\, dz_1 \\ &= \int_{\mathbf{R}} \int_{\square(0)} (\psi_1 A)(0,y) p(z_1,y)\, dy \cdot \nabla_z \varphi(z_1,0)\, dz_1,\end{aligned}$$

as ε tends to zero, because $(\psi_1 A)\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)\nabla_z\varphi$ converges strongly two-scale in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$ to $(\psi_1 A)(0,y)\nabla_z\varphi(z_1,0) \in L^2(\mathbf{R}\times\square(0))$. Because $((b+\varepsilon c)\psi_1)\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)\varphi$ converges strongly two-scale in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$ to $b(0,y)\psi_1(0,y)\varphi(z_1,0) \in L^2(\mathbf{R}\times\square(0))$,

$$\int_{\mathbf{R}^d} (\psi_1(b+\varepsilon c))\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big) v^\varepsilon \varphi \, d\mu_\varepsilon \to \int_{\mathbf{R}} \int_{\square(0)} b(0,y) w(z_1,y)\, dy\, \varphi(z_1,0)\, dz_1,$$

as ε tends to zero. Because $\nu^\varepsilon\psi_1\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)\varphi$ converges strongly two-scale in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$ to $\nu\psi_1(0,y)\varphi(z_1,0) \in L^2(\mathbf{R}\times\square(0))$,

$$\nu^\varepsilon \int_{\mathbf{R}^d} \psi_1^2\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big) v^\varepsilon \varphi \, d\mu_\varepsilon \to \nu \int_{\mathbf{R}} \int_{\square(0)} \psi_1(0,y) w(z_1,y)\, dy\, \varphi(z_1,0)\, dz_1,$$

as ε tends to zero. By the Taylor theorem, and hypothesis (8),

$$\frac{\mu_1(\sqrt{\varepsilon}z_1)-\mu_1(0)}{\varepsilon} = \frac{1}{2}\mu_1''(0)z_1^2 + o(z_1^2),$$

as ε tends to zero, so by the compact support of φ,

$$\begin{aligned}\int_{\mathbf{R}^d} \frac{\mu_1(\sqrt{\varepsilon}z_1)-\mu_1(0)}{\varepsilon}\psi_1^2\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big) v^\varepsilon \varphi \, d\mu_\varepsilon &= \int_{\mathbf{R}^d} \frac{1}{2}\mu_1''(0) z_1^2 \psi_1^2\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big) v^\varepsilon \varphi \, d\mu_\varepsilon + o(1) \\ &\to \int_{\mathbf{R}} \frac{1}{2}\mu_1''(0) z_1^2 \int_{\square(0)} \psi_1(0,y) w(z_1,y)\, dy\, \varphi(z_1,0)\, dz_1,\end{aligned}$$

as ε tends to zero, because $\frac{1}{2}\mu_1''(0)z_1^2\psi_1\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)\varphi$ converges strongly two-scale in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$ to $\frac{1}{2}\mu_1''(0)z_1^2\psi_1(0,y)\varphi \in L^2(\mathbf{R}\times\square(0))$. In conclusion,

$$\begin{aligned}
&\int_{\mathbf{R}}\int_{\square(0)} (\psi_1 A)(0,y)p(z_1,y)\,dy\cdot\nabla_z\varphi(z_1,0)\,dz_1 + \int_{\mathbf{R}}\int_{\square(0)} b(0,y)w(z_1,y)\,dy\,\varphi(z_1,0)\,dz_1 \\
&\quad + \int_{\mathbf{R}} \frac{1}{2}\mu_1''(0)z_1^2 \int_{\square(0)} \psi_1(0,y)w(z_1,y)\,dy\,\varphi(z_1,0)\,dz_1 \\
&= \nu \int_{\mathbf{R}}\int_{\square(0)} \psi_1(0,y)w(z_1,y)\,dy\,\varphi(z_1,0)\,dz_1,
\end{aligned} \tag{46}$$

for any $\varphi \in C_0^\infty(\mathbf{R}^d)$.

Using

$$\begin{aligned}
w(z_1,y) &= \psi_1(0,y)v(z_1),\\
p(z_1,y) &= \psi_1(0,y)\big((\partial_{z_1}v(z_1), r) + \nabla_y q(z_1,y)\big),
\end{aligned}$$

and the normalization of ψ_1, transforms (46) into

$$\begin{aligned}
&\int_{\mathbf{R}}\int_{\square(0)} (\psi_1^2 A)(0,y)\big((\partial_{z_1}v(z_1), r) + \nabla_y q(z_1,y)\big)\,dy\cdot\nabla_z\varphi(z_1,0)\,dz_1 \\
&\quad + \int_{\mathbf{R}}\int_{\square(0)} (\psi_1 b)(0,y)\,dy\,v(z_1)\varphi(z_1,0)\,dz_1 + \int_{\mathbf{R}} \frac{1}{2}\mu_1''(0)z_1^2 v(z_1)\varphi(z_1,0)\,dz_1 \\
&= \nu\int_{\mathbf{R}} v(z_1)\varphi(z_1,0)\,dz_1,
\end{aligned} \tag{47}$$

for any $\varphi \in C_0^\infty(\mathbf{R}^d)$.

To compute q one uses oscillating test functions of the form

$$\Phi^\varepsilon(z) = \sqrt{\varepsilon}\phi(z)\varphi\big(\frac{z}{\sqrt{\varepsilon}}\big),$$

with $\phi \in C_0^\infty(\mathbf{R}^d)$ and $\varphi \in C^\infty(\overline{\square(0)})$. One has

$$\nabla_z\Phi^\varepsilon(z) = \phi(z)\nabla_y\varphi\big(\frac{z}{\sqrt{\varepsilon}}\big) + \sqrt{\varepsilon}\varphi\big(\frac{z}{\sqrt{\varepsilon}}\big)\nabla_z\phi(z).$$

By passing to the limit as ε tends to zero in (45) with test functions Φ^ε, using their fixed compact support, one obtains from the weak two-scale convergence of $\psi_1\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)v_1^\varepsilon$,

$$\int_{\mathbf{R}}\int_{\square(0)} (\psi_1^2 A)(0,y)((\partial_{z_1}v, r) + \nabla_y q)\cdot\nabla_y\varphi\,dy\,\phi(z_1,0)\,dz_1 = 0, \tag{48}$$

for any $\phi \in C_0^\infty(\mathbf{R}^d)$ and any $\varphi \in C^\infty(\overline{\square(0)})$. If

$$q(z_1,y) = N(y)\cdot(\partial_{z_1}v(z_1), r), \tag{49}$$

equation (48) requires for $N(y)$ to satisfies

$$\int_{\mathbf{R}}\int_{\square(0)}\sum_{r,j=1}^{d}(\psi_1^2 a_{rj})(0,y)\partial_{y_j}N_1\partial_{y_r}\varphi\,dy\,\partial_{z_1}v\phi(z_1,0)\,dz_1$$
$$+\int_{\mathbf{R}}\int_{\square(0)}\sum_{k=2}^{d}\sum_{r,j=1}^{d}(\psi_1^2 a_{rj})(0,y)\partial_{y_j}N_k\partial_{y_r}\varphi\,dy\,r_k\phi(z_1,0)\,dz_1$$
$$=-\int_{\mathbf{R}}\int_{\square(0)}\sum_{r=1}^{d}(\psi_1^2 a_{r1})(0,y)\partial_{y_r}\varphi\,dy\,\partial_{z_1}v\,\phi(z_1,0)\,dz_1$$
$$-\int_{\mathbf{R}}\int_{\square(0)}\sum_{r=1}^{d}\sum_{j=2}^{d}(\psi_1^2 a_{rj})(0,y)\partial_{y_r}\varphi\,dy\,r_j\,\phi(z_1,0)\,dz_1,$$

for any $\phi \in C_0^\infty(\mathbf{R}^d)$ and any $\varphi \in C^\infty(\overline{\square(0)})$. Let $N_k \in H^1(\square(0), \psi_1^2(0,y))$ be such that

$$\int_{\square(0)} N_k(y)\psi_1^2(0,y)\,dy = 0,$$

and

$$-\mathrm{div}_y((\psi_1^2 A)(0,y)\nabla_y N_k) = \sum_{j=1}^{d}\partial_{y_j}(\psi_1^2 a_{kj})(0,y), \quad y \in \square(0).$$

That is,

$$\int_{\square(0)}(\psi_1^2 A)(0,y)\nabla_y N_k\cdot\nabla_y\varphi\,dy = -\int_{\square(0)}\sum_{j=1}^{d}(\psi_1^2 a_{kj})(0,y)\frac{\partial\varphi}{\partial y_j}\,dy, \tag{50}$$

for any $\varphi \in H^1(\square(0), \psi_1^2(0,y))$. Then N_k are well defined because the bilinear form on the left hand side in (50) is coercive on $H^1(\square(0), \psi_1^2(0,y))/\mathbf{R}$, by the ellipticity condition on A and positivity of $\psi_1^2(0,y)$, and the compatibility condition is satisfied:

$$\int_{\square(0)}\mathrm{div}((\psi_1^2 a_k)(0,y))\,dy = \int_{\square(0)}(\psi_1^2 a_k)(0,y)\cdot\nu\,dy = 0,$$

where a_k is the kth row of A, using that $\psi_1(0,y)$ is zero on $\partial\square(0)$. Therefore, the vector N is such that (49) holds, and in particular,

$$\psi_1\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)\nabla_z v_1^\varepsilon \overset{2}{\rightharpoonup} \psi_1(0,y)((\partial_{z_1}v, r) + \nabla_y N\cdot(\partial_{z_1}v, r)), \tag{51}$$

weakly in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$ as ε tends to zero.

Let the effective $d\times d$ matrix A^{eff} with entries a_{ij}^{eff} be defined by

$$a_{ij}^{\mathrm{eff}} = \int_{\square(0)}\sum_{k=1}^{d}(\psi_1^2 a_{ik})(0,y)(\delta_{kj} + \partial_{y_k}N_j(y))\,dy. \tag{52}$$

We compute the effective flux $A^{\mathrm{eff}}(\partial_{z_1} v, r)$. Let $\varphi(z) = z \cdot \phi(z_1)$, where $\phi(z_1)$ is a vector with components $\phi_j \in C_0^\infty(\mathbf{R})$ and $\phi_1(z_1) = 0$. Then

$$\varphi \to 0, \qquad \nabla_z \varphi \to \phi(z_1),$$

strongly in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$, as ε tends to zero. By passing to the limit as ε tends to zero in the variational form (45) of the equation for v_1^ε (or equivalently setting $\varphi = z \cdot \phi$ in (47)), and using the definition of A^{eff} and the characterization of the limit of $\nabla_z v_1^\varepsilon$, (49), (51), one obtains

$$\int_{\mathbf{R}} A^{\mathrm{eff}}(\partial_{z_1} v, r) \cdot \phi(z_1)\, dz_1 = 0,$$

for any $\phi(z_1) \in C_0^\infty(\mathbf{R})$. It follows that the transverse component of the effective flux is zero:

$$A^{\mathrm{eff}}(\partial_{z_1} v, r) = \Big(\sum_j a_{1j}^{\mathrm{eff}} \partial_{z_1} v, 0\Big) = A^{\mathrm{eff}}(\partial_{z_1} v, 0), \tag{53}$$

recalling that $\phi_1 = 0$. More precisely, setting $\varphi = y_i$, $i = 2, \ldots, d$, respectively, as test functions in the variational form (50) of the equations for N_j gives

$$\int_{\square(0)} \sum_{k=1}^{d} (\psi_1^2 a_{ik})(0, y)(\delta_{kj} + \partial_{y_k} N_j(y))\, dy = 0,$$

for $j = 1, \ldots, d$. In view of the definition of A^{eff} (52), this means that

$$a_{ij}^{\mathrm{eff}} = 0, \tag{54}$$

for all $(i, j) \neq (1, 1)$. That is, all transverse components of the effective matrix are zero. It follows that the effective flux in (53) reduces to

$$A^{\mathrm{eff}}(\partial_{z_1} v, r) = (a_{11}^{\mathrm{eff}} \partial_{z_1} v, 0).$$

One may verify that $a_{11}^{\mathrm{eff}} > 0$ as follows. Use N_i as a test function in the variational form (50) of the equation for N_j to obtain,

$$a_{ij}^{\mathrm{eff}} = \int_{\square(0)} (\psi_1^2 A)(0, y) \nabla_y (y_i + N_i(y)) \cdot \nabla_y (y_j + N_j(y))\, dy. \tag{55}$$

It follows from (54) and (55) that A^{eff} is symmetric and positive semidefinite by the same properties of A. In particular, by (55),

$$a_{11}^{\mathrm{eff}} = \int_{\square(0)} (\psi_1^2 A)(0, y) \nabla_y (y_1 + N_1(y)) \cdot \nabla_y (y_1 + N_1(y))\, dy \geq \int_{\square(0)} \psi_1^2(0, y) |\nabla_y (y_1 + N_1(y))|^2\, dy.$$

Suppose that $a_{11}^{\mathrm{eff}} = 0$. Then by the last inequality, $\psi_1^2(0, y) \nabla_y (y_1 + N_1(y)) = 0$ a.e. in $\square(0)$, which by the connectedness of $\square(0)$ and the positivity of ψ_1^2 implies that $y_1 + N_1(y)$ is constant, which contradicts the periodicity of N_1 in y_1. Therefore,

$$a_{11}^{\text{eff}} > 0.$$

We show that ν, v is an eigenpair to the effective equation (14). By passing to the limit in the variational form (45) of the equation for v_1^ε (or equivalently reading off from (47)), using a test function $\varphi \in C_0^\infty(\mathbf{R}^d)$, one obtains

$$\int_{\mathbf{R}} a^{\text{eff}} \partial_{z_1} v(z_1) \partial_{z_1} \varphi(z_1, 0)\, dz_1 + \int_{\mathbf{R}} \Big(c^{\text{eff}} + \frac{1}{2}\mu_1''(0) z_1^2\Big) v(z_1)\varphi(z_1, 0)\, dz_1 = \nu \int_{\mathbf{R}} v(z_1)\varphi(z_1, 0)\, dz_1,$$

for any $\varphi \in C_0^\infty(\mathbf{R}^d)$.

By the strong convergence of $\psi_1^\varepsilon v_1^\varepsilon$ in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$, with the normalization (32), the limit v is nonzero. This shows that ν, v is an eigenpair to the problem (14). □

Lemma 4.7. *The whole sequence $(\nu_1^\varepsilon, \psi_1^\varepsilon v_1^\varepsilon)$ converges to $(\nu_1, \psi_1(0, y) v_1)$, in $\mathbf{R}$ and weakly two-scale in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$, respectively, where ν_1, v_1 is the first eigenpair of the limit problem (14).*

Proof. Let ϕ^ε be a smooth cutoff for the interval I such that

$$\begin{aligned} &\phi^\varepsilon \in C^\infty(\overline{I}), \qquad \phi^\varepsilon = 0 \text{ on } \partial I, \qquad 0 \le \phi^\varepsilon \le 1 \text{ in } I,\\ &\phi^\varepsilon(x_1) = 1 \text{ for } \operatorname{dist}(x_1, \partial I) > \varepsilon, \qquad \varepsilon|\partial_{x_1}\phi^\varepsilon| \le 1. \end{aligned}$$

Say,

$$\phi^\varepsilon(x_1) = \begin{cases} \operatorname{dist}(\frac{x_1}{\varepsilon}, \partial I), & 0 \le \operatorname{dist}(x_1, \partial I) \le \varepsilon,\\ 1, & \text{otherwise.} \end{cases}$$

Then the function $\phi^\varepsilon(\sqrt{\varepsilon} z_1)$ is smooth and cuts off the growing interval $\frac{1}{\sqrt{\varepsilon}} I$ in a $\sqrt{\varepsilon}$ neighborhood of its boundary, with gradient satisfying the estimate

$$\big|\partial_{z_1}\big(\phi^\varepsilon(\sqrt{\varepsilon} z_1)\big)\big| \le \frac{1}{\sqrt{\varepsilon}}.$$

Let ν_1, v_1 be the first eigenpair to the limit problem (14). Now we consider the following test function in the variational principle for the eigenvalue ν^ε:

$$w_\varepsilon(z) = \phi^\varepsilon(\sqrt{\varepsilon} z_1)\Big(v_1(z_1) + \sqrt{\varepsilon} N_1\big(\frac{z}{\sqrt{\varepsilon}}\big)\partial_{z_1} v_1(z_1)\Big).$$

By the definitions of ϕ^ε, v_1, N_1, one has for any $\varepsilon > 0$, $w^\varepsilon \in H^1(\mathbf{R}^d) \setminus \{0\}$, with vanishing trace on the bases

$$\widetilde{\Gamma_\varepsilon^\pm} = \Big\{z = (z_1, z') \in \mathbf{R}^d : z_1 \in \partial \frac{1}{\sqrt{\varepsilon}} I\Big\}.$$

Then for $\nu_1^\varepsilon = \varepsilon\lambda_1^\varepsilon - \mu_1(0)/\varepsilon$ one has from the variational principle, using the test function w^ε,

$$\begin{aligned} \nu_1^\varepsilon \le\ & \frac{\int_{\mathbf{R}^d} (\psi_1^2 A)\big(\sqrt{\varepsilon} z_1, \frac{z}{\sqrt{\varepsilon}}\big)\nabla_z w_\varepsilon(z)\cdot\nabla_z w_\varepsilon(z)\, d\mu_\varepsilon(z)}{\int_{\mathbf{R}^d} \psi_1^2\big(\sqrt{\varepsilon} z_1, \frac{z}{\sqrt{\varepsilon}}\big) w_\varepsilon^2(z)\, d\mu_\varepsilon(z)}\\ &+ \frac{\int_{\mathbf{R}^d} (\varepsilon\psi_1 b + \psi_1 c + \varepsilon^{-1}(\mu_1 - \mu_1(0))\psi_1^2)\big(\sqrt{\varepsilon} z_1, \frac{z}{\sqrt{\varepsilon}}\big) w_\varepsilon^2(z)\, d\mu_\varepsilon(z)}{\int_{\mathbf{R}^d} \psi_1^2\big(\sqrt{\varepsilon} z_1, \frac{z}{\sqrt{\varepsilon}}\big) w_\varepsilon^2(z)\, d\mu_\varepsilon(z)}. \end{aligned}$$

One has with $z = (z_1, z')$,

$$\partial_{z_1} w_\varepsilon(z) = (\partial_{z_1}(\phi^\varepsilon(\varepsilon z_1)))(v_1(z_1) + \sqrt{\varepsilon} N_1\big(\sqrt{\varepsilon} z_1, \frac{z}{\sqrt{\varepsilon}}\big)\partial_{z_1} v_1(z_1)) \\ + \phi^\varepsilon(\sqrt{\varepsilon} z_1)\big(\big(1 + \partial_{y_1} N\big(\sqrt{\varepsilon} z_1, \frac{z}{\sqrt{\varepsilon}}\big)\big)\partial_{z_1} v_1(z_1)\big),$$

$$\nabla_{z'} w_\varepsilon(z) = \partial_{z_1} v_1(z_1) \nabla_{y'} N_1\big(\frac{z}{\sqrt{\varepsilon}}\big).$$

The following estimates follow:

$$\int\limits_{\mathbf{R}^d} \psi_1^2\big(\sqrt{\varepsilon} z_1, \frac{z}{\sqrt{\varepsilon}}\big) w_\varepsilon^2(z)\, d\mu_\varepsilon(z) = \int\limits_{\mathbf{R}} v_1^2(z_1)\, dz_1 + o(1),$$

$$\frac{\int_{\mathbf{R}^d} (\psi_1^2 A)\big(\sqrt{\varepsilon} z_1, \frac{z}{\sqrt{\varepsilon}}\big) \nabla_z w_\varepsilon(z) \cdot \nabla_z w_\varepsilon(z)\, d\mu_\varepsilon(z)}{\int_{\mathbf{R}^d} \psi_1^2\big(\sqrt{\varepsilon} z_1, \frac{z}{\sqrt{\varepsilon}}\big) w_\varepsilon^2(z)\, d\mu_\varepsilon(z)} \le \int\limits_{\mathbf{R}} a^{\text{eff}} (\partial_{z_1} v_1(z_1))^2\, dz_1 + o(1),$$

$$\frac{\int_{\mathbf{R}^d} (\varepsilon \psi_1 b + \psi_1 c + \varepsilon^{-1}(\mu_1 - \mu_1(0))\psi_1^2)\big(\sqrt{\varepsilon} z_1, \frac{z}{\sqrt{\varepsilon}}\big) w_\varepsilon^2(z)\, d\mu_\varepsilon(z)}{\int_{\mathbf{R}^d} \psi_1^2\big(\sqrt{\varepsilon} z_1, \frac{z}{\sqrt{\varepsilon}}\big) w_\varepsilon^2(z)\, d\mu_\varepsilon(z)} = \int\limits_{\mathbf{R}} \big(c^{\text{eff}} + \frac{1}{2} z_1^2 \mu_1''(0)\big) v_1^2(z_1)\, dz_1 + o(1),$$

as ε tends to zero. By the variational principle (or directly from the variational form of the equation with v_1 as test function) for the first eigenvalue ν_1 for the limit problem (14),

$$\nu_1 = \frac{\int_{\mathbf{R}} a^{\text{eff}} (\partial_{z_1} v_1(z_1))^2\, dz_1 + \int_{\mathbf{R}} \big(c^{\text{eff}} + \frac{1}{2} z_1^2 \mu_1''(0)\big) v_1^2(z_1)\, dz_1}{\int_{\mathbf{R}} v_1^2(z_1)\, dz_1}.$$

One gets the estimate, along a subsequence,

$$\nu_1^\varepsilon \le \nu_1 + o(1),$$

as ε tends to zero. One concludes that for the whole ε sequence,

$$\lim_{\varepsilon \to 0} \nu_1^\varepsilon = \nu = \nu_1.$$

In terms of λ_1^ε this estimate reads,

$$\lambda_1^\varepsilon = \frac{\mu_1(0)}{\varepsilon^2} + \frac{\nu_1}{\varepsilon} + o\Big(\frac{1}{\varepsilon}\Big),$$

as ε tends to zero.

By the simplicity of the first eigenvalue ν_1 to the limit problem (14), the whole sequence v_1^ε converges to $v(z_1) = v_1(z_1)$, the first eigenfunction to the limit problem (14). □

Using the fact that the limit of $\nu_1^\varepsilon, v_1^\varepsilon$ is the first eigenpair to the effective equation (14), we derive the following a priori estimate for the second eigenvalue ν_2^ε to the problem (23).

Lemma 4.8. *Let ν_2^ε be the second eigenvalue to the problem (23). Then*

$$\nu_2^\varepsilon \le \nu_2 + o(1),$$

as ε tends to zero.

Proof. One notes that v_1^ε is almost orthogonal to $v_2(z_1)$ as ε tends to zero because v_1^ε converges to $v_1(z_1)$ which is orthogonal to $v_2(z_1)$. Let

$$w_2^\varepsilon = \phi^\varepsilon(\sqrt{\varepsilon}z_1)\Big(v_2(z_1) + \sqrt{\varepsilon}N_1\big(\frac{z}{\sqrt{\varepsilon}}\big)\partial_{z_1}v_2(z_1)\Big).$$

Then v_1^ε is almost orthogonal to w_2^ε. One may therefore use

$$w_2^\varepsilon(z) - \frac{(w_2^\varepsilon(z), v_1^\varepsilon)}{(v_1^\varepsilon, v_1^\varepsilon)}v_1^\varepsilon$$

as a test function in the variational principle for ν_2^ε. □

Lemma 4.9. *Suppose that for all $i \le k$,*

$$\lim_{\varepsilon\to 0}\psi_1\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)v_i^\varepsilon = v_i \qquad \textit{strongly in } L^2(\mathbf{R}^d, d\mu_\varepsilon).$$

Let $m > k$. Then v_m^ε is asymptotically orthogonal to v_i for all $i \le k$:

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^d}\psi_1^2\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)v_m^\varepsilon(z)v_i(z_1)\,d\mu_\varepsilon(z) = 0,$$

and v_m is asymptotically orthogonal to v_i^ε for all $i \le k$:

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^d}\psi_1^2\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)v_m(z_1)v_i^\varepsilon(z)\,d\mu_\varepsilon(z) = 0.$$

Proof. Let $m > k$. Then by the orthonormalization (32) of the eigenfunctions v_j^ε,

$$\begin{aligned}
&\int_{\mathbf{R}^d}\psi_1^2\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)v_m^\varepsilon(z)v_i(z_1)\,d\mu_\varepsilon(z)\\
&= \sum_{i=1}^k\int_{\mathbf{R}^d}\psi_1^2\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)v_m^\varepsilon(z)v_i^\varepsilon(z)\,d\mu_\varepsilon(z) + \sum_{i=1}^k\int_{\mathbf{R}^d}\psi_1^2\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)v_m^\varepsilon(z)(v_i(z_1) - v_i^\varepsilon(z))\,d\mu_\varepsilon(z)\\
&= \sum_{i=1}^k\int_{\mathbf{R}^d}\psi_1^2\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)v_m^\varepsilon(z)(v_i(z_1) - v_i^\varepsilon(z))\,d\mu_\varepsilon(z).
\end{aligned}$$

By the normalization of v_m^ε, $\psi_1\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)v_m^\varepsilon$ is bounded in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$. The first asymptotic orthogonality follows from the convergence of $\psi_1\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)v_i^\varepsilon$ to v_i in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$ because

$$\int_{\mathbf{R}^d}\psi_1^2\big(\sqrt{\varepsilon}z_1, \frac{z}{\sqrt{\varepsilon}}\big)v_m^\varepsilon(z)(v_i(z_1) - v_i^\varepsilon(z))\,d\mu_\varepsilon(z) = o(1),$$

as ε tends to zero, for any $i \le k$.

The second asserted asymptotic orthogonality follows from the strong convergence and the orthogonality of v_m to v_i for $i \ne m$. □

We approach the convergence of spectrum by considering the second eigenvalue for illustration.

Lemma 4.10. *The second eigenvalue converges to the second eigenvalue: $\nu_2^\varepsilon \to \nu_2$, as ε tends to* 0*.*

Proof. The second eigenfunction v_2^ε is almost orthogonal to v_1^ε when ε small. Then $\nu_2^\varepsilon, v_2^\varepsilon$ converges along a subsequence (using the estimate in previous lemma for a priori estimates for two-scale convergence) to an eigenpair ν, v such that v is almost orthogonal to v_1. By a previous lemma we must have $\nu = \nu_2$, and thus $v = v_2$, using that all eigenvalues of the limit problem are simple. □

Lemma 4.11. *The eigenpairs $(\nu_i^\varepsilon, v_i^\varepsilon)$ converge to (ν_i, v_i), as ε tends to zero.*

Proof. We know that $(\nu_1^\varepsilon, v_1^\varepsilon) \to (\nu_1, v_1)$. Suppose that $(\nu_i^\varepsilon, v_i^\varepsilon) \to (\nu_i, v_i)$ for all $i \leq k$. Let

$$w_{k+1}^\varepsilon = \phi^\varepsilon(\sqrt{\varepsilon} z_1)\Big(v_{k+1}(z_1) + \sqrt{\varepsilon} N_1(\frac{z}{\sqrt{\varepsilon}})\partial_{z_1} v_{k+1}(z_1)\Big).$$

One verifies that w_{k+1}^ε is asymptotically orthogonal to v_i for all $i \leq k$, using the above lemma. Then one verifies that

$$v_{k+1}^\varepsilon(z) - \sum_{i=1}^{k} \frac{(w_{k+1}^\varepsilon(z), v_i^\varepsilon(z))}{(v_i^\varepsilon(z), v_i^\varepsilon(z))} v_i^\varepsilon(z)$$

are nonzero test functions for all ε. Using these test functions one obtains the estimate

$$\nu_{k+1}^\varepsilon \leq \nu_{k+1} + o(1),$$

as ε tends to zero. It follows that $\nu_{k+1}^\varepsilon, v_{k+1}^\varepsilon$ converges to some eigenpair. Using the simplicity and the upper estimate for ν_{k+1}^ε one concludes that for the full ε sequence

$$\lim_{\varepsilon\to 0} \nu_{k+1}^\varepsilon = \nu_{k+1}.$$

This shows by induction that

$$\lim_{\varepsilon\to 0} \nu_i^\varepsilon = \nu_i,$$

for any i, and that the corresponding eigenfunctions converge. □

Putting the above sequence of lemmas together, one concludes Theorem 3.3.

Remark that in general for $d\mu_\varepsilon \rightharpoonup d\mu$, and a sequence v^ε weakly converging in $L^2(\mathbf{R}^d, d\mu_\varepsilon)$ to $v \in L^2(\mathbf{R}^d, d\mu)$, it is not always the case that

$$\lim_{\varepsilon\to 0} \int_{\mathbf{R}^d} (v^\varepsilon - v)^2 \, d\mu_\varepsilon = 0. \tag{56}$$

To the positive, (56) holds for instance if v is bounded and the measure of $\mathbf{R}^d$ is finite in the limit. In our case the measure of $\mathbf{R}^d$ is not finite, while the limit function is bounded and exponentially decaying, which compensates.

5. Computing the leading terms

In this section we connect hypothesis (8) with the hypothesis in Theorem 1.1. We also describe a scheme to compute the effective coefficients and the leading terms in the expansions of the eigenpairs in Theorem 3.3. The procedure goes as follows:

1. Locating the global minimum of the principle eigenvalue μ_1.
2. Computing the effective coefficients a^{eff}, c^{eff}.
3. Computing the eigenpair λ_1^0, v_1^0.

Because we do not have an effective characterization of the minimum of principal eigenvalue μ_1, some iterative procedure could be useful, say the Newton method. For this purpose we compute the derivative μ_1' with respect to x_1. The Hessian μ_1'' can be obtained by the similar procedure to that of Lemma 5.1 below, or by a finite difference after computing $\mu_1(x_1)$ at points close to the minimum.

Lemma 5.1. *Let μ_1, ψ_1 be the principle eigenpair to the problem*

$$\begin{aligned} -\operatorname{div}_y(A(x_1,y)\nabla_y\psi) &= \mu(x_1)\psi && \text{in } \square(x_1), \\ \psi &= 0 && \text{on } \partial\square(x_1), \end{aligned} \tag{57}$$

normalized by

$$\int\limits_{\square(x_1)} \psi_1^2 \, dy = 1. \tag{58}$$

Let V_n be the outward normal velocity of the boundary $\partial\square(x_1)$, and let V be a globally defined velocity field for $\square(x_1)$, with respect to x_1, namely $V_n = V \cdot \nu = -\frac{\partial_{x_1}F}{|\nabla_y F|}$ on $\partial\square(x_1)$. Then

$$\begin{aligned} \mu_1' = &\int\limits_{\square(x_1)} \frac{\partial A}{\partial x_1}\nabla\psi_1 \cdot \nabla\psi_1 \, dy - \int\limits_{\partial\square(x_1)} (A\nabla_y\psi_1 \cdot \nabla_y\psi_1)(V\cdot\nu)\, d\sigma \\ &+ 2\int\limits_{\square(x_1)} (A\nabla_y\psi_1 \cdot \nabla_y(\nabla_y\psi_1 \cdot V) - \mu_1\psi_1(\nabla_y\psi_1 \cdot V))\, dy. \end{aligned}$$

To prove Lemma 5.1 we consider separately the contributions to the linearizations from the dependence on the coefficients $A(x_1,y)$ and the dependence on the change in shape of $\square(x_1)$. Lemma 5.1 follows directly from Lemma 5.3 and 5.4 below.

The following example relates hypothesis (8) to the hypothesis used in [23] for a smooth profile h. We make use of the following boundary point property for the first eigenfunction ψ_1 to the problem (6).

Lemma 5.2. *Let μ_1, ψ_1 be the first eigenpair to the problem (6), with sign chosen such that $\psi_1(x_1,y) > 0$ everywhere in $\square(x_1)$. Then*

$$\nabla_y\psi_1(x_1,y)\cdot\nu < 0 \qquad a.e.\ y \in \partial\square(x_1),$$

for any $x_1 \in \overline{I}$.

Proof. One notes that for any $x \in \overline{I}$, $\psi_1(x_1,y)$ is continuous up to the boundary in y, and satisfies $\psi_1(x_1,y_0) = 0$ for any $y_0 \in \partial\square(x_1)$, and in particular $\psi_1(x_1,y) > \psi_1(x_1,y_0) = 0$ for every $y \in \square(x_1)$. Moreover, ψ_1 is a subsolution to the equation

$$-\operatorname{div}_y(A(x_1,y)\nabla_y\psi_1) - \mu_1\psi_1 = 0, \quad y \in \square(0).$$

By the Lipschitz continuity of $\partial\square(x_1)$, at almost every $y_0 \in \partial\square(x_1)$, the outward unit normal ν exists, the outward normal derivative exists $\nabla_y\psi_1 \cdot \nu$, and there is a ball $B_R(y) \subset \square(x_1)$ with $y_0 \in \partial B_R(y)$. The assertion then follows from the classical argument using the weak maximum principle (cf. Lemma 3.4 in [9]). □

Example 5.1. Consider the case of the Dirichlet Laplacian $-\Delta$ in a finite thin strip in $\mathbf{R}^2$, with profile given by a smooth positive $h(x_1)$, $x_1 \in [-1,1]$:

$$\Omega_\varepsilon = \{x : -1 < x_1 < 1,\, 0 < x_2 < \varepsilon h(x_1)\}.$$

A function $F : [-1,1] \times \mathbf{T}^1 \times \mathbf{R} \to \mathbf{R}$ such that

$$\Omega_\varepsilon = \{x : x_1 \in (-1,1),\, F(x_1, x/\varepsilon) > 0\}$$

is

$$F(x_1, y) = \frac{y_2}{h(x_1)}\Big(1 - \frac{y_2}{h(x_1)}\Big).$$

The corresponding up-scaled cell is

$$\square(x_1) = \{y : F(x_1, y) > 0\} = \{y : y_1 \in \mathbf{T}^1 : 0 < y_2 < h(x_1)\}.$$

The normal velocity of $\partial\square(x_1)$ is

$$V \cdot \nu = -\frac{\partial_{x_1} F}{|\nabla_y F|} = -\frac{h' y_2}{h}, \tag{59}$$

and a globally defined smooth domain velocity field on $\overline{\square(x_1)}$ is

$$V = \Big(0, -\frac{h' y_2}{h}\Big).$$

By Lemma 5.1,

$$\begin{aligned}
\mu_1' &= -\int_{\partial\square(x_1)} |\nabla_y \psi_1|^2 (V\cdot\nu)\, d\sigma + 2\int_{\square(x_1)} ((\nabla_y\psi_1\cdot\nu)\nabla_y(\nabla_y\psi_1\cdot V) - \mu_1\psi_1(\nabla_y\psi_1\cdot V))\, dy \\
&= -\int_{\partial\square(x_1)} |\nabla_y \psi_1|^2 (V\cdot\nu)\, d\sigma + 2\int_{\partial\square(x_1)} (\nabla_y\psi_1\cdot\nu)(\nabla_y\psi_1\cdot V)\, d\sigma \\
&= \int_{\partial\square(x_1)} |\nabla_y \psi_1|^2 (V\cdot\nu)\, d\sigma,
\end{aligned}$$

where one in the second step has used the divergence theorem and the equation (57), and in the third step has used that here $(\xi\cdot\nu)(\xi\cdot V) = |\xi|^2(V\cdot\nu)$, $\xi \in \mathbf{T}^1 \times \mathbf{R}$. One has $\int_{\partial\square(x_1)} |\nabla_y\psi_1|^2\, d\sigma > 0$, by the boundary point property, Lemma 5.2, irregardless of the sign chosen for ψ_1. (In other words, because otherwise the critical set $\{y \in \overline{\square(x_1)} : \psi_1 = 0,\, \nabla_y\psi_1 = 0\}$ would be of positive $(d-1)$-dimensional measure.) It follows that $\mu_1'(x_1) = 0$ if and only if $h'(x_1) = 0$. Therefore by (59), the hypothesis of unique minimum of $\mu_1(x_1)$ is equivalent to the existence of a unique maximum of $h(x_1)$ in this example. One might remark that in this example, one also has access to both the domain monotonicity of the eigenvalues, as well as the exact eigenpair, none of which is available if $A(x_1, y)$ is not constant in the fast variable y.

We conclude this section by computing the linearization given in Lemma 5.1. We first compute μ_1' for a constant cell $\square$, that is $\square(x_1)$ independent of x_1.

Lemma 5.3. *Suppose that $\square(x_1) = \square$ is independent of x_1. Let $A(x_1, y)$ and $\partial\square$ be sufficiently smooth. Let μ_1, ψ_1 be the principle eigenpair to*

$$-\mathrm{div}_y(A(x_1, y)\nabla_y \psi) = \mu(x_1)\psi \qquad \textit{in } \square,$$

with the homogeneous Dirichlet condition on $\partial\square$, and normalized by

$$\int_\square \psi_1^2\, dy = 1.$$

Then by the Fréchet differentiability of the eigenpair and the bilinear forms associated to the problem, remarking that μ_1 is simple,

$$\mu_1' = \int_\square \frac{\partial A}{\partial x_1} \nabla_y \psi_1 \cdot \nabla_y \psi_1\, dy.$$

Proof. For any test function $\varphi \in H_0^1(\square)$,

$$\int_\square A\nabla_y \psi_1 \cdot \nabla_y \varphi\, dy = \mu_1 \int_\square \psi_1 \varphi\, dy. \tag{60}$$

Differentiate both sides with respect to x_1, to obtain

$$\int_\square \frac{\partial A}{\partial x_1} \nabla_y \psi_1 \cdot \nabla_y \varphi\, dy + \int_\square A\nabla_y \frac{\partial \psi_1}{\partial x_1} \cdot \nabla_y \varphi\, dy = \mu_1' \int_\square \psi_1 \varphi\, dy + \mu_1 \int_\square \frac{\partial \psi_1}{\partial x_1} \varphi\, dy. \tag{61}$$

Noting that $\partial_{x_1}\psi_1 \in H_0^1(\square)$, and using that μ_1, ψ_1 is an eigenpair with test function $\partial_{x_1}\psi_1$ in (60), and the normalizing condition $\int_\square \psi_1^2\, dy = 1$, yields after using ψ_1 as test function in (61),

$$\mu_1' = \int_\square \frac{\partial A}{\partial x_1} \nabla \psi_1 \cdot \nabla \psi_1\, dy. \quad \square$$

Now we suppose that $A = A(x_1, y)$ is constant in x_1, and let $\square(x_1)$ vary. To compute $\mu_1'(x_1)$ we need to compute how points on the boundary of $\square(x_1)$ move in the normal direction when x_1 is varied, i.e. the normal velocity of the boundary, which we compute it terms of F. By definition,

$$\square(x_1) = \{y : F(x_1, y) > 0\}.$$

The boundary of $\square(x_1)$ is given by

$$\partial\square(x_1) = \{y : F(x_1, y) = 0\}.$$

Let V_n denote the outward normal velocity, that is

$$V_n = -\frac{\partial_{x_1} F}{|\nabla_y F|}.$$

Let $\phi \in H^1(\square(x_1))$ be such that $\int_{\square(x_1)} \phi\, dy = 0$ and

$$\begin{aligned} -\Delta_y \phi &= 0 && \text{in } \square(x_1) \\ \nabla_y \phi \cdot \nu &= -\frac{\partial_{x_1} F}{|\nabla_y F|} && \text{on } \partial\square(x_1) \end{aligned}$$

If the compatibility condition is not satisfied, we set $V = 0$ in some interior ball, or curve of positive measure. Then $V = \nabla_y \phi$ is a globally defined velocity field such that $V_n = V \cdot \nu = -\frac{\partial_{x_1} F}{|\nabla_y F|}$ on $\partial\square(x_1)$, and where $\nu = -\frac{\nabla_y F}{|\nabla_y F|}$ is the outward unit normal to $\square(x_1)$.

Lemma 5.4. *Let $A(x_1, y)$ and $\partial\square(x_1)$ be sufficiently smooth. Suppose that $A(x_1, y)$ is constant in x_1. Let μ_1, ψ_1 be the principle eigenpair to*

$$-\mathrm{div}_y(A(y))\nabla_y \psi) = \mu(x_1)\psi \quad \text{in } \square(x_1),$$

with homogeneous Dirichlet condition on $\partial\square(x_1)$, normalized by

$$\int_{\square(x_1)} \psi_1^2\, dy = 1.$$

Then

$$\mu_1' = 2 \int_{\square(x_1)} (A(y)\nabla_y \psi_1 \cdot \nabla_y(\nabla_y \psi_1 \cdot V) - \mu_1 \psi_1 (\nabla_y \psi_1 \cdot V))\, dy - \int_{\partial\square(x_1)} (A(y)\nabla_y \psi_1 \cdot \nabla_y \psi_1) V_n\, d\sigma. \quad (62)$$

Proof. Let $\dot{v} = \partial_{x_1} v - \nabla_y v \cdot V$ denote the material derivative. For any test function $\varphi \in H_0^1(\square)$,

$$\int_{\square(x_1)} A(y)\nabla_y \psi_1 \cdot \nabla_y \varphi\, dy = \mu_1 \int_{\square(x_1)} \psi_1 \varphi\, dy. \quad (63)$$

Differentiate both sides with respect to x_1, to obtain

$$\begin{aligned} &\int_{\square(x_1)} \Big(A(y)\nabla_y \frac{\partial \psi_1}{\partial x_1} \cdot \nabla_y \varphi + A(y)\nabla_y \psi_1 \cdot \nabla_y \frac{\partial \varphi}{\partial x_1} \Big)\, dy \\ &\quad - \int_{\partial\square(x_1)} (A(y)\nabla_y \psi_1 \cdot \nabla_y \varphi) V_n\, d\sigma + \int_{\square(x_1)} A(y)\nabla_y \dot{\psi}_1 \cdot \nabla_y \varphi\, dy \\ &= \mu_1' \int_{\square(x_1)} \psi_1 \varphi\, dy + \mu_1 \int_{\square(x_1)} \Big(\frac{\partial \psi_1}{\partial x_1} \varphi + \psi_1 \frac{\partial \varphi}{\partial x_1} \Big)\, dy - \mu_1 \int_{\partial\square(x_1)} \psi_1 \varphi V_n\, d\sigma + \mu_1 \int_{\square(x_1)} \dot{\psi}_1 \varphi\, dy. \quad (64) \end{aligned}$$

Use ψ_1 as a test function in (64), substitute $\partial_{x_1} \psi_1 = \dot{\psi}_1 - \nabla_y \psi_1 \cdot V$, and note that $\dot{\psi}_1$ can be used as a test function in (63). Using that $\psi_1 = 0$ on $\partial\square(x_1)$, and the normalization $\int_{\square(x_1)} \psi_1^2\, dy = 1$, yield (62). $\square$

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