Principal series of Hermitian Lie groups induced from Heisenberg parabolic subgroups

Journal of Functional Analysis 282 (2022) 109399

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

Principal series of Hermitian Lie groups induced from Heisenberg parabolic subgroups ☆

Genkai Zhang

Mathematical Sciences, Chalmers University of Technology and Mathematical Sciences, Göteborg University, SE-412 96 Göteborg, Sweden

ARTICLE INFO

Article history:
Received 24 May 2021
Accepted 11 January 2022
Available online 20 January 2022
Communicated by P. Delorme

MSC:
17B15
17B60
22D30
43A80
43A85

Keywords:
Hermitian Lie groups
Heisenberg parabolic subgroups
Induced representations
Composition and Complementary series

ABSTRACT

Let G be an irreducible Hermitian Lie group and $D = G/K$ its bounded symmetric domain in $\mathbb{C}^d$ of rank r. Each γ of the Harish-Chandra strongly orthogonal roots $\{\gamma_1, \cdots, \gamma_r\}$ defines a Heisenberg parabolic subgroup $P = MAN$ of G. We study the principal series representations $\mathrm{Ind}_P^G(1 \otimes e^{\nu} \otimes 1)$ of G induced from P. These representations can be realized as the L^2-space on the minimal K-orbit $S = Ke = K/L$ of a root vector e of γ in $\mathbb{C}^d$, and S is a circle bundle over a compact Hermitian symmetric space K/L_0 of K of rank one or two. We find the complementary series, reduction points, and unitary sub-quotients in this family of representations.

0. Introduction

In the present paper we shall study composition series and complementary series for degenerate principal series representations for an irreducible Hermitian Lie group G induced from a Heisenberg parabolic subgroup. Let $D = G/K \subset \mathbb{C}^d$ be the bounded

☆ Research supported partially by the Swedish Science Council (VR), Grant 2018-03402.

E-mail address: genkai@chalmers.se.

https://doi.org/10.1016/j.jfa.2022.109399

symmetric domain of G in its Harish-Chandra realization in $\mathbb{C}^d = \mathfrak{p}^+$. Any choice of a Harish-Chandra strongly orthogonal root determines a one-dimensional split subgroup $A = \mathbb{R}^+$ of G and a Heisenberg parabolic subgroup $P = MAN$ of G. We study the induced representation $I(\nu) = \mathrm{Ind}_P^G(1 \otimes e^\nu \otimes 1)$ of G from P. We shall find its complementary series, the reduction points, explicit realization of certain finite dimensional representations, and unitarizable subrepresentations.

The representation $I(\nu)$ can be realized on the L^2-space on a homogeneous space K/L of K, $L = M \cap K$, and $G/P = K/L = K \cdot e$ is an orbit of K in $\mathbb{C}^d$, with the Harish-Chandra root vector e as a base point. The vector e is also realized as a boundary point of the domain G/K in $\mathfrak{p}^+ = \mathbb{C}^d$ where G is acting as rational maps, and the G-orbit $G \cdot e$ is a bundle over $K/L = K \cdot e$ with fiber being a bounded symmetric domain of rank $r-1$; see [19]. The orbit K/L is a circle bundle $K/L \to K/L_0$ over its projectivization K/L_0. The space K/L_0 itself is a *compact Hermitian symmetric space*. We can find the irreducible decomposition of $L^2(K/L)$ by using the Cartan-Helgason theorem for line bundles over K/L_0. We then study the Lie algebra action of $\mathfrak{g}$ on $L^2(K/L)$.

The induced representations $I(\nu) = Ind_P^G(\nu)$ from Heisenberg parabolic subgroup P can be viewed as the counterpart of the representations $Ind_Q^G(\nu)$ from the Siegel parabolic subgroup Q, both P and Q being maximal parabolic subgroups. The nilpotent part in P is a Heisenberg nilpotent and non-abelian group whereas it is abelian in Q for tube domains D. The representations $Ind_Q^G(\nu)$ and their analogues are well studied and are of considerable interests as they are closely related to the holomorphic discrete series [23]. The general case of maximal parabolic subgroups with abelian nilradical can be put in the setup of Koecher's construction of Lie algebras and the corresponding induced representations have also been intensively studied; see e.g. [10,17,25,30]. The analysis is done in [25] by using eigenvalues of intertwining differential operators and the tensor product structure of induced representations, and in [10,30] by computing differentiations and recursions of spherical polynomials along a torus in the symmetric space K/L; in [17] the results for the multiplicity free $Sp(p,p)$-case are applied to the $Sp(p,q)$-case. In our case the nilpotent group N is non-abelian, K/L is not symmetric and the decomposition of $L^2(K/L)$ under K is not immediate, so the methods in [25,17] seem not possible, and we shall adapt the method in [30]. Now the manifold K/L_0 is a complex manifold and the differentiation involves vector fields in $T^{(1,0)}(K/L_0)$, we have to develop further the technique in [30]. We consider the differentiation along products of copies of the projective sphere $SU(2)/U(1)$ in K/L_0 and find the differentiation formulas by considering classical spherical polynomials of $SU(2)$. By using Weyl group symmetry of spherical functions we find then the required formulas for the Lie algebra actions. This then determines the complementary series and the composition series.

The study of induced representations from Heisenberg parabolic subgroups can be put in a rather general context. We give a very brief account of some background and related works. It is known that in most semisimple Lie groups G there are Heisenberg parabolic subgroups P, and these groups have been all classified. Howe [7] studied the induced representations from P from the point of view of minimal representations and his notion of ranks of representations. Recently Frahm [3] has studied the intertwining differential operators for the induced representations in some non-Hermitian Lie groups. When $G = SU(p, q)$ the induced representations can be realized on a homogeneous cone in $\mathbb{C}^{p+q}$ and they have been studied by Howe-Tan [8] in greater details. In [2] the authors developed a different method to study principal series representations and it might be interesting to adapt that method in our setup. The classification of spherical duals of classical groups has been studied extensively, and composition series of degenerate principal series are closely related to theta correspondence; see further [1,13,14,16–18,21]. In a forthcoming paper [4] we shall give a different proof for the results in this paper by using the non-compact picture of the induced representations and study related intertwining differential operators.

We mention also briefly some geometric perspectives of the spaces involved. The compact space K/L_0 has an interesting geometry, it is the variety of minimal rational tangents in a fixed tangent space $V = T_0^{(1,0)}(D)$ of the symmetric space D; more precisely it is the projectivization $\mathbb{P}(V)$ of all tangent vectors $v \in V$ with maximal holomorphic sectional curvature. It is also the projectivization of the space K/L of minimal tripotents and plays important role in complex differential geometry; see [9,20]. The G-orbit $G \cdot e$ above in the boundary of $D = G/K$ is a bundle

$$G \cdot e = G/LAN \to G/P = G/MAN = G \cdot (e + M/L) = K/L$$

with typical fiber M/L being the bounded symmetric space of M of rank $r-1$ in V_0 realized as a holomorphic bundary component $e + M/L \subset e + V_0 \subset V$ of $D = G/K \subset V$, where $V = V_2 + V_1 + V_0$ is the Peirce decomposition of V. The minimal nilpotent orbit of G in $\mathfrak{g}$ is the homogeneous space G/MN so that we have K-equivariant fiberations

$$G/MN \to G/MAN = K/L \to K/L_0$$

with fibers being $\mathbb{R}^+$ and the circle S^1 respectively, and it is tempting to put the representations studied here into some general context of nilpotent orbits.

The paper is organized as follows. In Section 1 we introduce the parabolic subalgebra $\mathfrak{m}+\mathfrak{a}+\mathfrak{n} = \mathfrak{m}+\mathbb{R}\xi+\mathfrak{n}$ and the principal series $(I(\nu), \pi_\nu)$. In Section 2 we find the irreducible decomposition for $I(\nu)|_K = L^2(K/L)$ under K. The action of $\pi_\nu(\xi)$ on $I(\nu)$ is done in Section 3 and it is one of our main results. As consequences we find in Section 4 the complementary series, certain unitarizable subquotients, and also realization of certain finite dimensional representations of $\mathfrak{g}^{\mathbb{C}}$ in the induced representations. In Section 5 we treat the case when $\mathfrak{g} = \mathfrak{su}(d,1), \mathfrak{sp}(r,\mathbb{R})$ where K/L is the sphere and K/L_0 is the complex projective space. We give a simpler proof of the classical result of Johnson-Wallach [11]. In Appendix A we compute certain recurrence formulas for spherical polynomials over the complex projective line $\mathbb{P}^1$; we need only the leading coefficients in the formulas which can be easily proved by other methods, but we present the complete formulas as they might be of independent interests in special functions. The complete lists of the spaces G/K and K/L_0 are given in Appendix B.

Notation. Hermitian symmetric spaces have rather rich structure and so is the notation. For the convenience of the reader we list the main symbols used in our paper.

(1) Real Lie algebras will be denoted by $\mathfrak{g}, \mathfrak{h}, \ldots$, and their complexifications $\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}, \ldots$. The adjoint action in $\mathfrak{g}^{\mathbb{C}}$ will be denoted by $\operatorname{ad} X(Y) = [X, Y], X, Y \in \mathfrak{g}^{\mathbb{C}}$, and the adjoint action in G as well as its induced action on $\mathfrak{g}$ by $\operatorname{Ad} g(h) = ghg^{-1}$, $g, h \in G$, $\operatorname{Ad} g(X) = (\operatorname{Ad} g)_*(X), X \in \mathfrak{g}$.

(2) $D = G/K$, bounded symmetric domain of D of rank r realized in the Jordan triple system $V = \mathbb{C}^d$ with Jordan triple product $\{x, \bar{y}, z\} = D(x, y)z$ and Jordan characteristic (a, b) (or root multiplicities); $d = r + \frac{a}{2}r(r-1) + rb$, the dimension.

(3) $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, Cartan decomposition, $\mathfrak{p} = \{\xi_v = v + Q(z)\bar{v}; v \in V\}$ as holomorphic vector fields on D, $2Q(z)\bar{v} = D(z, v)z$.

(4) $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^- + \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^+$, Harish-Chandra decomposition with respect to the center element Z of $\mathfrak{k}$, $\operatorname{ad}(Z)|_{\mathfrak{p}^\pm} = \pm i$; $\mathfrak{p}^+ = V$.

(5) $e = e_1$, a fixed minimal tripotent, $V = V_1 + V_1 + V_0 = V_1(e) + V_1(e) + V_0(e)$, the Peirce decomposition with respect to e, and $\{e, v_1, w, v_2\}$ a Jordan quadrangle.

(6) $\{e, \bar{e}, D(e, e)\}$, standard $\mathfrak{sl}(2)$-triple; $H_0 = iD(e, e)$.

(7) $\xi = \xi_e = e + \bar{e} \in \mathfrak{p}$, $\mathfrak{a} = \mathbb{R}\xi$, $\mathfrak{g} = \mathfrak{n}_{-2} + \mathfrak{n}_{-1} + \mathfrak{m} + \mathfrak{a} + \mathfrak{n}_1 + \mathfrak{n}_2$, the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$, $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2$, a Heisenberg Lie algebra; $\rho_{\mathfrak{g}} = 1 + (r-1)a + b$, half sum of positive roots.

(8) M, A, N, the corresponding subgroups with Lie algebra $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{n}$, $L = M \cap K = \{k \in K; ke = e\} \subseteq K$ with Lie algebra $\mathfrak{l}$.

(9) $\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}]$, the semisimple part of $\mathfrak{k}$, and $K_1 \subset K$ the corresponding Lie group.

(10) $S = K/L = G/P$, the manifold of rank one tripotents.

(11) $S_1 = \mathbb{P}(S) = K/L_0 = K_1/L_1$, the projective space of S, also called the variety of minimal rational tangents; $L_0 = \{k \in K; ke = \chi(k)e, \chi(k) \in U(1)\}$ the subgroup of elements of K fixing the line $\mathbb{C}e$ with Lie algebra $\mathfrak{l}_0 = \{X \in \mathfrak{k}; Xe = \chi(X)e, c \in i\mathbb{R}\}$. $S_1 = K/L_0 = K_1/L_1$, compact Hermitian symmetric space of rank two and of dimension $d_1 = \dim_{\mathbb{C}} V_1 = (r-1)a_1 + b_1$, (a_1, b_1), the Jordan characteristic of S_1;

(12) $\chi_l(k) = \chi(k)^l$, character on L_0, $l \in \mathbb{Z}$.

(13) $\mathfrak{k} = \mathfrak{q} + \mathfrak{l}_0$, $\mathfrak{k}_1 = \mathfrak{q} + \mathfrak{l}_1$ Cartan decomposition for the symmetric space $S_1 = K/L_0 = K_1/L_1$; $\mathfrak{q} = \{D(v, e) - D(e, v), v \in V_1\}$; $\mathfrak{l}_1 = \mathfrak{l}_0 \cap \mathfrak{k}_1$, the semisimple component of $\mathfrak{k}_0$.

(14) $\mathfrak{k}^{\mathbb{C}} = \mathfrak{q}^- + \mathfrak{l}_0^{\mathbb{C}} + \mathfrak{q}^+$, $\mathfrak{k}_1^{\mathbb{C}} = \mathfrak{q}^- + \mathfrak{l}_1^{\mathbb{C}} + \mathfrak{q}^+$, the Harish-Chandra decomposition of $\mathfrak{k}^{\mathbb{C}}, \mathfrak{k}_1^{\mathbb{C}}$ for the Hermitian symmetric space $K/L_0 = K_1/L_1$; $\mathfrak{q}^+ = \{D(v, e); v \in V_1\}$; $\mathfrak{q}^- = \{D(e, v); v \in V_1\}$; the Jordan triple product on $\mathfrak{q}^+ = \{D(v, e); v \in V_1\}$ is via the Lie bracket,

$$[[D(v, e), D(e, w)], D(u, e)] = D(D(v, w)u, e),$$

and is isomorphic to the Jordan triple system $V_1 \subset V$; $\mathfrak{q}^+$ is the holomorphic tangent space of $S_1 = K/L_0 = K_1/L_1$ at $L_0 \in K/L_0$.

(15) $\mathfrak{k}_1^* = \mathfrak{l}_1 + i\mathfrak{q}$, the non-compact dual of $\mathfrak{k}_1 = \mathfrak{l}_1 + \mathfrak{q}$; $\mathfrak{h}_{i\mathfrak{q}} \subset i\mathfrak{q}$, Cartan subspace.

(16) $\rho := \rho_{\mathfrak{k}_1^*} := \rho_1\alpha_1 + \rho_2\alpha_2 = (\rho_1, \rho_2)$, $\rho_1 = 1 + a_1 + b_1$, $\rho_2 = 1 + b_1$, half sum of positive restricted roots of $\mathfrak{k}_1^*$ with respect to $\mathfrak{h}_{i\mathfrak{q}}$.
(17) $\Phi_{\lambda,l}$, Harish-Chandra spherical function for the symmetric pair $(\mathfrak{k}_1^*, \mathfrak{l}_1)$ with one-dimensional character χ_l, $c(\lambda, l) = c(\lambda, -l)$, Harish-Chandra c-function for the symmetric space $S_1 = K/L_0 = K_1/L_1$ with character χ_l; $\phi_{\mu,l}$ spherical polynomial.

1. Preliminaries

We shall use the Jordan triple description of Hermitian symmetric spaces; see [19,28].

1.1. Hermitian symmetric space $D = G/K$

Let D be an irreducible bounded symmetric domain of rank r in $V = \mathbb{C}^d$. Let G be the group of bi-holomorphic automorphisms of D, and $K = \{k \in G; k0 = 0\}$ the maximal compact subgroup of G, so that $D = G/K$. The space V has the structure of an irreducible Jordan triple system with triple product $\{x, \overline{y}, z\} = D(x, y)z$ with the corresponding $\mathrm{End}(\bar{V}, V)$-valued quadratic form $Q(x)$, $Q(x)\overline{y} = \frac{1}{2}D(x, y)x$, where $\bar{V}$ is the space V with the conjugated complex structure. Note that in [19] $D(x, y)$ is written as $D(x, \overline{y})$, and to ease notation we write it just as $D(x, y)$ so it is *conjugate linear* in y. Let (a, b) be the Jordan characteristic of V, and $b = 0$ when D is a tube domain. The dimension $d = r + \frac{a}{2}r(r-1) + rb$.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. Realized as holomorphic vector fields on D, i.e., as V-valued functions on D, the space $\mathfrak{p}$ is

$$\mathfrak{p} = \{\xi_v = v - Q(z)\overline{v}; v \in V\}. \tag{1.1}$$

The adjoint action $v \mapsto \mathrm{Ad}(k)v$ of $k \in K$ as well as $\mathfrak{k}$ on $\mathfrak{p}$ coincides with its defining action on D and will be written just as $kv = \mathrm{Ad}(k)v$, $Xv = \mathrm{ad}(X)v$, $k \in K$, $X \in \mathfrak{k}$ when no confusion would arise.

Denote $Z \in \mathfrak{k}$ the central element defining the complex structure of $\mathfrak{p}$ and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^+ + \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^-$ be the Harish-Chandra decomposition, $Z|_{\mathfrak{p}^\pm} = \pm i$. The space $\mathfrak{p}^+$ is identified with V via the identification $V \ni v = \frac{1}{2}(\xi_v - i\xi_{iv}) \in \mathfrak{p}^+$ and $\overline{V} = \{\overline{v}; v \in V\}$ with $\{-Q(z)v\} = \mathfrak{p}^-$. The Lie algebra $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathbb{R}Z$, where $\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}]$ is the semisimple part of $\mathfrak{k}$ with trivial center. Let $K_1 \subset K$ be the corresponding semisimple subgroup of K with Lie algebra $\mathfrak{k}_1$.

We fix the Euclidean inner product on V so that a minimal tripotent has norm 1, and fix the corresponding normalization of the Killing form on $\mathfrak{g}^{\mathbb{C}}$. All orthogonality in the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ below is with respect to the Killing form unless otherwise specified.

1.2. Maximal parabolic subgroup $P = MAN$ of G and induced representation $Ind_P^G(\nu)$

We fix in the rest of the paper a minimal tripotent $e = e_1$ and denote

$$\xi = \xi_e, \quad H_0 = iD(e, e), \quad \mathfrak{a} = \mathbb{R}\xi \subset \mathfrak{p}. \tag{1.2}$$

A Harish-Chandra strongly orthogonal root γ_1 for $\mathfrak{g}^{\mathbb{C}}$ can be chosen so that its co-root is $D(e,e)$, $\gamma_1(D(e,e)) = 2$. We shall only need γ_1 below.

The Peirce decomposition of $V = \mathbb{C}^d$ with respect to the tripotent e is

$$V = V_2 + V_1 + V_0,\ V_j = V_j(e) := \{v \in V; D(e,e)v = jv\},\ j = 0,1,2. \tag{1.3}$$

Furthermore $V_2 = \mathbb{C}e$ is one-dimensional, V_1 is of dimension $d_1 = \dim_{\mathbb{C}} V_1 = (r-1)a + b$, and V_0 is a Jordan triple system of rank $r-1$ and dimension $1 + \frac{1}{2}a(r-1)(r-2) + (r-1)b$. The Jordan rank of V_1 is

$$\operatorname{rank} V_1 = \begin{cases} 2, & \mathfrak{g} \neq \mathfrak{su}(d,1), \mathfrak{sp}(r,\mathbb{R}) \\ 1, & \mathfrak{g} = \mathfrak{su}(d,1), \mathfrak{sp}(r,\mathbb{R}). \end{cases} \tag{1.4}$$

Certain computations have to be done depending on the different cases.

We shall need the description for the root spaces of $\mathfrak{g}$ under $\operatorname{ad}(\xi) : X \to [\xi, X]$. A linear functional $\nu \in (\mathfrak{a}^{\mathbb{C}})^*$ will be identified as $\nu \in \mathbb{C}, \nu = \nu(\xi)$.

Lemma 1.1. *The root space decomposition of $\mathfrak{g}$ under $\mathfrak{a} = \mathbb{R}\xi$ is*

$$\mathfrak{g} = \mathfrak{n}_- + (\mathfrak{m} + \mathfrak{a}) + \mathfrak{n}, \quad \mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2, \quad \mathfrak{n}_- = \mathfrak{n}_{-1} + \mathfrak{n}_{-2}, \tag{1.5}$$

where $\mathfrak{n}_{\pm 2}$, $\mathfrak{n}_{\pm 1}$, and $\mathfrak{m} + \mathfrak{a}$ are the root spaces of ξ with roots $\pm 2, \pm 1, 0$, respectively. The subspaces are given by

$$\mathfrak{m} = \mathfrak{l} \oplus \{\xi_v; v \in V_0\}, \quad \mathfrak{l} = \mathfrak{m} \cap \mathfrak{k} = \{X \in \mathfrak{k}; Xe = 0\}, \tag{1.6}$$

$$\mathfrak{n}_1 = \{\xi_v + (D(e,v) - D(v,e)); v \in V_1\}, \tag{1.7}$$

and

$$\mathfrak{n}_2 = \mathbb{R}(\xi_{ie} - H_0). \tag{1.8}$$

The half sum of positive roots is

$$\rho_{\mathfrak{g}} := \rho_{\mathfrak{g}}(\xi) = 1 + (r-1)a + b = 1 + \dim_{\mathbb{C}} V_1. \tag{1.9}$$

Proof. The root spaces are described in [19, Lemma 9.14]. The root space $\mathfrak{n}_1$ has real dimension $2\dim_{\mathbb{C}} V_1$ and thus $\rho_{\mathfrak{g}} = 1 + \dim_{\mathbb{C}} V_1 = 1 + (r-1)a + b$. □

The nilpotent algebra $\mathfrak{n}$ is a Heisenberg Lie algebra. Their appearance in general semisimple Lie algebras has been classified; see e.g. [3,7].

Let M, A, N, $L = M \cap K = \{k \in K; ke = e\}$, be the corresponding Lie subgroups and $P = MAN$ the parabolic group of G.

The main object of this paper is the induced representation

$$I(\nu) = Ind_P^G(\nu) := Ind_P^G(1 \otimes e^{\nu} \otimes 1) \tag{1.10}$$

defined as the space of measurable functions on G such that

$$f(gme^{t\xi}n) = e^{-t\nu} f(g), m \in M, n \in N, t \in \mathbb{R},$$

and $f|_K \in L^2(K)$. Any $f \in I(\nu)$ as function on K is right L-invariant, and thus $f|_K \in L^2(K/L)$. The corresponding $(\mathfrak{g}^{\mathbb{C}}, K)$-representation will also be denoted by $I(\nu)$. If $\nu = \rho_{\mathfrak{g}} + i\lambda, \lambda \in \mathbb{R}, \lambda \neq 0$, then $I(\nu)$ is a unitary irreducible representation of G on $L^2(K/L)$; see e.g. [12].

2. Decomposition of $(L^2(K/L), K)$ and spherical polynomials

We assume throughout Sections 2, 3 and 4 that the Hermitian symmetric domain $D = G/K$ is irreducible of rank $r \geq 2$ and is not the Siegel domain $Sp(r, \mathbb{R})/U(n)$; this case and the rank one domain $SU(d,1)/U(d)$ will be treated in Section 5.

2.1. The homogeneous space $S = G/P = K/L$ as circle bundle over compact Hermitian symmetric space $S_1 = K/L_0 = K_1/L_1$

The homogeneous space $S := G/P = K/L$ of K can be realized as the manifold of rank one tripotents in V, $S = Ke \subset V$. We shall fix this realization in the rest of the paper and use the global coordinates on V for S when needed.

To find the decomposition of $L^2(K/L)$ under K we consider the projectivization

$$S \to S_1 = \mathbb{P}(S) = \{[v] := \mathbb{C}v \in \mathbb{P}(V); v \in S\}, \quad v \mapsto [v]$$

of S in the projective space $\mathbb{P}(V)$ of V. Then $S_1 = K/L_0$, where $L_0 = \{k \in K; ke \in \mathbb{C}e\}$, and $L \subset L_0$ is a normal subgroup with L_0/L being the circle group $U(1)$. The natural map $S = K/L \to S_1 = \mathbb{P}(S) = K/L_0$ defines a fibration

$$S = K/L \to S_1 = K/L_0 \tag{2.1}$$

of $\mathbb{P}(S)$ with fiber the circle $U(1) = L_0/L$.

The space $S_1 = \mathbb{P}(S)$ is a compact Hermitian symmetric space of (non-simple) K. The complete list of (G, K, L_0), $D = G/K$, $S = G/P = K/L \to S_1 = K/L_0$ is given in Tables 1-2. The space $S_1 = K/L_0$ is also the variety of minimal rational tangents (VMRT) [9] of the symmetric space D.

The action of L_0 on e defines a character χ of L_0, namely

$$\chi(k) = c, \quad \text{if} \quad k \in L_0, ke = ce. \tag{2.2}$$

Let $L_1 = \{k \in K_1; ke = \chi(k)e\} \subset L_0$, then $S_1 = K/L_0 = K_1/L_1$, $S = K_1/L \cap L_1$. The fibration (2.1) above becomes a circle bundle $S = K/L \to S_1 = K/L_0 = K_1/L_1$ for K_1-homogeneous spaces with the fiber $U(1) = L_1/L \cap L_1$.

The element $\exp(\pi H_0), H_0 = iD(e,e) \in \mathfrak{l}_0$, defines a Cartan involution $\exp(\pi \operatorname{ad} H_0)$ on $\mathfrak{k}$ with the corresponding Cartan decomposition

$$\mathfrak{k} = \mathfrak{l}_0 + \mathfrak{q}$$

with

$$\mathfrak{q} = \{D(v,e) - D(e,v); v \in V_1\}. \tag{2.3}$$

We fix now a complex structure on $\mathfrak{q}$ and the corresponding Harish-Chandra decomposition of $\mathfrak{k}^{\mathbb{C}}$. As a convention the complex structure for $(\mathfrak{k}, \mathfrak{l}_0)$ or $(\mathfrak{k}_1, \mathfrak{l}_1)$ is defined using an element in $\mathfrak{l}_0$ respectively $\mathfrak{l}_1$.

Lemma 2.1. *Define the K-invariant complex structure on $S_1 = K/L_0$ by the element $-\frac{1}{2}H_0 = -\frac{1}{2}iD(e,e) \in \mathfrak{l}_0$, i.e. by $X \to -\frac{1}{2}\operatorname{ad}(H_0)(X)$, $X \in \mathfrak{q} = T_{[e]}(S_1)$, at the base point $[e] = \mathbb{C}e \in S_1$. The corresponding complex structure for the pair $(\mathfrak{k}_1, \mathfrak{l}_1)$ is defined by $-\frac{1}{2}H_0'$,*

$$-\frac{1}{2}\operatorname{ad}(H_0)|_{\mathfrak{q}} = -\frac{1}{2}\operatorname{ad}(H_0')|_{\mathfrak{q}}, \quad H_0' = H_0 - i\frac{p}{d}Z \in \mathfrak{l}_1 \subset \mathfrak{k}_1. \tag{2.4}$$

The Harish-Chandra decompositions of $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{k}_1^{\mathbb{C}}$ are

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{q}^- + \mathfrak{l}_0^{\mathbb{C}} + \mathfrak{q}^+, \quad \mathfrak{k}_1^{\mathbb{C}} = \mathfrak{q}^- + \mathfrak{l}_1^{\mathbb{C}} + \mathfrak{q}^+ \tag{2.5}$$

with

$$T_{[e]}^{(1,0)}(S_1) = \mathfrak{q}^+ = \{D(v,e); v \in V_1\}, T_{[e]}^{(0,1)} = \mathfrak{q}^- = \{-D(e,v); v \in V_1\}. \tag{2.6}$$

Proof. For any $c \in \mathbb{R}$, $\operatorname{ad}(X) = \operatorname{ad}(cZ + X)$ on $\mathfrak{k}$ for any $X \in \mathfrak{k}$, since Z is in the center of $\mathfrak{k}$. The semisimple component of X in $\mathfrak{k}_1$ is obtained as $X - \frac{1}{d}\operatorname{tr}\operatorname{ad}(X)|_{\mathfrak{p}^+}$, $d = \dim V$. Thus we have (2.4). The Harish-Chandra decomposition is obtained by the commutator formula in Jordan triples [19],

$$[D(u,v), D(x,y)] = D(D(u,v)x, y) - D(x, D(v,u)y). \quad \square$$

Here the complex structure on $\mathfrak{q}$ is chosen so that

$$v \in V_1 \subset V = \mathfrak{p}^+ \to D(v,e) \in \mathfrak{q}^+$$

is complex linear so that the complex structures in $\mathfrak{p}$ and $\mathfrak{q}$ match in this sense. The Lie algebra structure in $\mathfrak{k}^{\mathbb{C}}$ defines $\mathfrak{q}^{\pm}$ as a Jordan triple system, and it is isomorphic to V_1. To avoid confusion we shall keep the notation $\mathfrak{q}^+$.

2.2. Cartan subalgebra $\mathfrak{h}_{i\mathfrak{q}} \subset i\mathfrak{q}$ and the restricted root system for the non-compact symmetric pair $(\mathfrak{k}^*, \mathfrak{l}_0) = (\mathfrak{l}_0 + i\mathfrak{q}, \mathfrak{l}_0)$

We construct now a split Cartan subalgebra in $i\mathfrak{q}$ for the symmetric pair $(\mathfrak{k}^*, \mathfrak{l}_0) = (\mathfrak{l}_0 + i\mathfrak{q}, \mathfrak{l}_0)$ and its semisimple part $(\mathfrak{k}_1^*, \mathfrak{l}_1) = (\mathfrak{l}_1 + i\mathfrak{q}, \mathfrak{l}_1)$, and find the corresponding root system. They will be used in the decomposition of $L^2(K/L)$ and computation of Harish-Chandra c-function. We need the notion of a Jordan *quadrangle* [22, p. 12, p. 16].

An ordered quadruple (u_0, u_1, u_2, u_3) of minimal tripotents is called Jordan quadrangle if the following three conditions are satisfied, for all i modulo 4,

(1) u_i and u_{i+1} are in each other's Peirce V_1-space, $u_i \in V_1(u_{i+1})$, $u_{i+1} \in V_1(u_i)$;
(2) u_i and u_{i+2} are orthogonal as tripotents;
(3) $D(u_i, u_{i+1})u_{i+2} = u_{i+3}$.

Recall that we have assumed in this section that the domain $D \neq III_n = Sp(n, \mathbb{R})/U(n)$, $D \neq I_{n,1} = SU(n,1)/U(n)$ (the Type IV domain $IV_3 = III_2$ is also excluded). Then starting with the fixed minimal tripotent e there are minimal tripotents v_1, w, v_2 such that $(u_0, u_1, u_2, u_3) = (e, v_1, w, v_2)$ is a Jordan quadrangle. This is implicitly in [22] where orthogonal bases (called grids) are constructed for Jordan triple systems, and we provide brief arguments. The Jordan triple system V is of rank $r \geq 2$ so there exists a Jordan algebra V' as a sub-triple of V of rank two with $e_1 + e_2$ as identity element, where $e_1 = e, e_2$ are the Harish-Chandra strongly orthogonal root vectors; for

$$D = I_{r,r+b},\ II_{2r},\ II_{2r+1},\ IV_n(n > 4),\ V,\ VI$$

the corresponding D' is

$$D' = I_{2,2},\ II_4,\ II_4,\ IV_4,\ IV_8,\ IV_{10}.$$

In all cases the Jordan algebra $M_{2,2}$ of square 2×2-matrices, $D = I_{2,2} = IV_4$, forms a Jordan sub-triple system, since

$$I_{2,2} = IV_4 \subset II_4 = IV_6 \subset IV_8 \subset IV_{10}$$

in the sense of Jordan sub-triple systems. The following standard matrices

$$E_{11}, E_{12}, E_{22}, E_{21}$$

form a Jordan quadrangle in $M_{2,2}$ and in V.

Fix in the rest of the paper the Jordan quadrangle $\{e, v_1, w, v_2\}$. We have

$$H_0 v_1 = iD(e,e)v_1 = iv_1,\ H_0 v_2 = iD(e,e)v_2 = iv_2,\ D(v_1,v_1)e = D(v_2,v_2)e = e, \tag{2.7}$$

and

$$D(e,w)=D(v_1,v_2)=0,\ D(v_1,e)v_1=D(v_2,e)v_2=0, D(e,v_1)w=v_2,\ D(v_1,e)v_2=w, \tag{2.8}$$

which we shall use below. See [22, p. 12, p.16] for further details.

The above construction results in the following two commuting copies of $\mathfrak{sl}(2,\mathbb{C})$-triples in $\mathfrak{k}_1^{\mathbb{C}}$,

$$E_j^+=D(v_j,e)\in\mathfrak{q}^+,\ E_j^-=D(e,v_j)\in\mathfrak{q}^-,\ H_j=D(v_j,v_j)-D(e,e)\in\mathfrak{k}^{\mathbb{C}},\ \ E_j:=E_j^+-E_j^-\in\mathfrak{q}, \tag{2.9}$$

with the canonical relation

$$[H_j,E_j^{\pm}]=2E_j^{\pm},\ [E_j^+,E_j^-]=H_j,\ j=1,2.$$

Moreover $\mathbb{R}(iE_1)+\mathbb{R}(iE_2)\subset i\mathfrak{q}$ is maximal abelian.

Definition 2.2. Let

$$\mathfrak{h}_{\mathfrak{q}}=\mathbb{R}E_1+\mathbb{R}E_2\subset\mathfrak{q},\quad \mathfrak{h}_{i\mathfrak{q}}=\mathbb{R}(iE_1)+\mathbb{R}(iE_2)\subset i\mathfrak{q},\quad \mathfrak{h}_{\mathfrak{q}}^{\mathbb{C}}=\mathbb{C}E_1+\mathbb{C}E_2. \tag{2.10}$$

Extend the abelian subalgebra $\mathbb{C}E_1+\mathbb{C}E_2$ of $\mathfrak{k}_1^{\mathbb{C}}$ to a Cartan subalgebra

$$\mathfrak{h}_1^{\mathbb{C}}:=(\mathbb{C}E_1+\mathbb{C}E_2)\oplus\mathfrak{h}_+,\quad \mathfrak{h}_+\subset\mathfrak{l}_1^{\mathfrak{c}}\subset\mathfrak{k}_1^{\mathbb{C}},$$

of $\mathfrak{k}_1^{\mathbb{C}}$, so that

$$\mathfrak{h}^{\mathbb{C}}:=(\mathbb{C}Z+\mathbb{C}E_1+\mathbb{C}E_2)\oplus\mathfrak{h}_+$$

is a Cartan subalgebra of $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}}$. Define $\{\alpha_0,\alpha_1,\alpha_2\}$ to be the dual basis vectors of $\{iZ,iE_1,iE_2\}$ that are vanishing on $\mathfrak{h}_+$.

Now

$$(\mathbb{C}Z+\mathbb{C}H_1+\mathbb{C}H_2)\oplus\mathfrak{h}_+\subset\mathfrak{l}_0^{\mathbb{C}}\subset\mathfrak{k}^{\mathbb{C}}\subset\mathfrak{g}^{\mathbb{C}} \tag{2.11}$$

is a Cartan subalgebra of three algebras $\mathfrak{l}_0^{\mathbb{C}},\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}}$.

We shall need the Cartan-Helgason theorem in [26] for line bundles over K/L_0 defined by the characters $\chi_l=\chi^l$; the character in [26] is defined using Cayley transform and Cartan subalgebras instead of the geometric definition here. The relevant Cayley transform in our setup is

$$c=c_{\mathfrak{k}}=\exp\Big(-\frac{\pi i}{4}\operatorname{ad}(E_1^++E_2^++E_1^-+E_2^-)\Big).$$

Lemma 2.3.

(1) *The subspace* $\mathfrak{h}_{i\mathfrak{q}}$ *is maximal abelian in* $i\mathfrak{q}$. *If* $\mathfrak{g} \neq \mathfrak{su}(r+b, r)$, $r > 1$ *then* K/L_0 *is an irreducible symmetric space and the restricted root system for the non-compact dual* $\mathfrak{k}^* = \mathfrak{l}_0 + i\mathfrak{q}$ *with respect to* $\mathfrak{h}_{i\mathfrak{q}}$ *is*

$$R(\mathfrak{k}^*, \mathfrak{h}_{i\mathfrak{q}}) = R(\mathfrak{k}_1^*, \mathfrak{h}_{i\mathfrak{q}}) := \{\pm 2\alpha_1, \pm 2\alpha_2\} \cup \{\pm\alpha_1 \pm \alpha_2\} \cup \{\pm\alpha_1, \pm\alpha_2\} \tag{2.12}$$

with root multiplicities $(1, a_1, 2b_1)$ *for the three subsets of roots,* a_1, b_1 *being given in Tables 1-2. The half-sum of the positive roots with respect to the ordering* $\alpha_1 > \alpha_2 > 0$ *is*

$$\rho := \rho_{\mathfrak{k}_1^*} = \rho_1\alpha_1 + \rho_2\alpha_2 = (\rho_1, \rho_2), \quad \rho_1 = 1 + a_1 + b_1,\ \rho_2 = 1 + b_1. \tag{2.13}$$

The two linear functionals $\rho_{\mathfrak{k}_1^*}$ *and* $\rho_{\mathfrak{g}}$ *are related by*

$$\rho_{\mathfrak{g}} = 1 + \rho_1 + \rho_2.$$

If $\mathfrak{g} = \mathfrak{su}(r+b, r)$, $r > 1$ *then* $K/L_0 = K_1/L_1$ *is reducible,* $\mathfrak{k}_1 = \mathfrak{su}(r+b) + \mathfrak{su}(r)$, $\mathfrak{k}_1^* = \mathfrak{su}(1, r+b-1) + \mathfrak{su}(1, r-1)$, *and the restricted root system is*

$$R(\mathfrak{k}^*, \mathfrak{h}_{i\mathfrak{q}}) = R(\mathfrak{k}_1^*, \mathfrak{h}_{i\mathfrak{q}}) := \{\pm 2\alpha_1, \pm\alpha_1\} \cup \{\pm 2\alpha_2, \pm\alpha_2\} \tag{2.14}$$

with root multiplicities $(1, r+b-1), (1, r-1)$ *respectively. The corresponding* ρ *is*

$$\rho_{\mathfrak{k}_1^*} = \rho_1\alpha_1 + \rho_2\alpha_2 = (\rho_1, \rho_2), \quad \rho_1 = r+b-1,\ \rho_2 = r-1 \tag{2.15}$$

The relation $\rho_{\mathfrak{g}} = 1 + \rho_1 + \rho_2$ *holds also in this case.*

(2) *The Cayley transform* c *exchanges two Cartan subalgebras*

$$c : (\mathbb{C}Z + \mathbb{C}H_1 + \mathbb{C}H_2) \oplus \mathfrak{h}_+ \to \mathfrak{h}^{\mathbb{C}},$$

$$c : H_j \to iE_j$$

and the pullbacks

$$c^*(2\alpha_j), j = 1, 2,$$

are the Harish-Chandra orthogonal roots for $(\mathfrak{k}_1^*, \mathfrak{l}_1)$.

Proof. We have assumed that D is not $SU(d,1)/U(d)$ nor $Sp(n,\mathbb{R})/U(n)$, so that $K/L_0 = K_1/L_1$ is of rank two; see the Tables 1-2. The rest is a consequence of general results on root systems algebra applied to the non-compact Helgason dual $\mathfrak{k}^* = \mathfrak{l}_0 + i\mathfrak{q}$ of

$\mathfrak{k} = \mathfrak{l}_0 + i\mathfrak{q}$. It follows also from [19, Lemma 9.14], the Lie algebra $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ there being replaced by our $\mathfrak{k}_1^* = i\mathfrak{q} + \mathfrak{l}_1$. The abelian subspace in $i\mathfrak{q}$ is obtained from the Harish-Chandra root vectors corresponding to a frame of minimal tripotents in $\mathfrak{q}^+$, and in the present case the frame in $\mathfrak{q}^+$ is $\{D(v_1, e), D(v_2, e)\}$. Finally the dimension of a general Jordan triple system of characteristics (r, a, b) is $r + \frac{1}{2}r(r-1)a + rb$. We have by Lemma 1.1, $\rho_{\mathfrak{g}} = 1 + \dim_{\mathbb{C}} V_1$ and $\dim_{\mathbb{C}} V_1 = 2 + a_1 + 2b_1 = \rho_1 + \rho_2$ since V_1 is a Jordan triple system with characteristic $(2, a_1, b_1)$, and thus $\rho_{\mathfrak{g}} = 1 + \rho_1 + \rho_2$. The fact about the Cayley transform is well-known; see e.g. [26], [19, Proposition 10.6(3)]. □

Note that the evaluation of the character χ on H_0, H_1, H_2 is given by

$$\chi(H_0) = 2\chi(Z) = 2i,\ \chi(H_1) = \chi(H_2) = -i, \tag{2.16}$$

since $H_0 e = 2Ze = 2ie$, $H_j e = i(D(v_j, v_j) - D(e, e))e = -ie$. Observe also that the geometrically natural choice of the complex structure of $S_1 = K/L_0 = K_1/L_1$ results in some discrepancy: For $H_0 = iD(e, e), \mathrm{ad}(-iH_0) = \mathrm{ad}(D(e, e))$, has *non-negative* eigenvalues $2, 1, 0$ on $\mathfrak{p}^+ = V = V_2 + V_1 + V_0$, whereas it has *negative* eigenvalue -1 on $\mathfrak{q}^+$,

$$[D(e, e), D(v, e)] = -D(v, e), \quad D(v, e) \in \mathfrak{q}^+. \tag{2.17}$$

2.3. Cartan-Helgason theorem for K/L_0

Let $L^2(K, L_0, \chi_l)$ be the L^2-space of sections of the homogeneous line bundle $K \times_{(L_0, \chi_l^{-1})} \mathbb{C}$ defined by χ_l^{-1} of L_0. The space $L^2(K, L_0, \chi_l)$ consists of $f \in L^2(K)$ such that

$$f(gh) = \chi^l(h) f(g), \quad g \in K, h \in L_0,\ he = \chi(h)e. \tag{2.18}$$

It follows immediately from the definitions of L_0 and χ that

$$L^2(K/L) = \sum_{l=-\infty}^{\infty} L^2(K, L_0, \chi_l), \tag{2.19}$$

under the left regular action of K. This is the Fourier series expansion along the fiber of $K/L \to K/L_0$.

We shall treat extensively functions on K that are transforming under L_0 as in (2.18) and it is convenient to give the following

Definition 2.4. An element $f \in L^2(K)$ is called (l_1, l_2)-spherical if

$$f(h_1 k h_2) = \chi_{l_1}(h_1)\chi_{l_2}(h_2) f(k), \quad h_1, h_2 \in L_0.$$

Lemma 2.5.

(1) *Let $\mathfrak{g} \neq \mathfrak{su}(r+b,r)$, $r>1$. The space $L^2(K, L_0, \chi_l)$ is decomposed as a sum of irreducible representations of K,*

$$L^2(K, L_0, \chi_l) = \sum_{\mu} W_{\mu,l},$$

where each $W_{\mu,l}$ has highest weight given by

$$l\alpha_0 + \mu,\ \mu = (\mu_1, \mu_2) = \mu_1\alpha_1 + \mu_2\alpha_2, \quad \mu_1 \geq \mu_2 \geq |l|, \quad \mu_1 = \mu_2 = l, \operatorname{mod} 2. \tag{2.20}$$

Moreover each space $W_{\mu,l}$ contains a unique vector (l,l)-spherical element $\phi_{\mu,l}$ up to nonzero scalars.

(2) *Let $\mathfrak{g} = \mathfrak{su}(r+b,r)$, $r>1$. The space $L^2(K, L_0, \chi_l)$ is decomposed as above with*

$$\mu_1, \mu_2 \geq |l| \quad \mu_1 = \mu_2 = l, \operatorname{mod} 2.$$

The highest weight vector in $W_{\mu,l}$ can be chosen as

$$f(x) = z_1^p \overline{z_{r+b}}^q w_1^{p'} \overline{w_r}^{q'}, \quad x = zw^* \in S \subset M_{r+b,r}(\mathbb{C}),$$

where we have written a rank one projection $x \in S \subset M_{r+b,r}(\mathbb{C})$ as $x = zw^*, z \in \mathbb{C}^{r+b}, w \in \mathbb{C}^r$, $\|z\| = \|w\| = 1$, and where (p, q, p', q') are subject to the condition

$$\mu_1 = p + q, \mu_2 = p' + q', l = p - q = p' - q'.$$

The spherical polynomial $\phi_{\mu,l}$ in this case is $\phi_{\mu,l}(zw^*) = \phi_{\mu_1,-l}(z)\phi_{\mu_2,l}(w)$ where $\phi_{m,l}$ is (l,l)-spherical polynomial on $\mathbb{P}(\mathbb{C}^n)$.

Proof. The statement for the decomposition of $L^2(K, L_0, \chi_l)$ as representation of the semi-simple group K_1 is in [26, Theorem 7.2]; our $(\mathfrak{k}_1^*, \mathfrak{l}_1)$ corresponds to $(\mathfrak{g}, \mathfrak{k})$ there. More precisely our character χ_l is precisely the same as χ_{-l} in [26]. The character χ_{-l} on K in [26] for the Hermitian symmetric space $D = G/K \subset \mathbb{C}^d = \mathfrak{p}^+$ is defined by

$$X \in \mathfrak{k}^{\mathbb{C}} \mapsto \frac{l}{\operatorname{tr}\operatorname{ad}(D(e,e))|_{\mathfrak{p}^+}} \operatorname{tr}\operatorname{ad} X|_{\mathfrak{p}^+};$$

equivalently it is determined [26, (5.1)] by $\chi_{-l} : D(e,e) \mapsto l$. For our symmetric pair $(\mathfrak{k}_1^*, \mathfrak{l}_1)$ the corresponding $D(e,e)$ is $H_1 = D(v_1, v_1) - D(e,e)$ described in Lemma 2.3 and $H_1 e = -le$ and thus $\chi_l(H_1) = -l$ by the definition. (Alternatively we can also prove this by using the duality relation in Appendix B.) The results in [26] then determine the highest weights of $W_{\mu,l}$ on $\mathfrak{h}_{i\mathfrak{q}} = \mathbb{R}(iE_1) + \mathbb{R}(iE_2)$ as $\mu_1\alpha_1 + \mu_2\alpha_2$ in our statement.

Finally it is trivial to find the weight of $W_{\mu,l}$ on the central element Z. The right action of $\exp(sZ)$ on $\phi \in W_{\mu,l}$ is, using $Ze = ie$,

$$\pi_\nu(\exp(sZ))\phi(h) = \phi(\exp(-sZ)h) = \phi(h\exp(-sZ)) = \exp(-isl)\phi(h), \quad h \in K. \tag{2.21}$$

Thus $\pi_\nu(Z)\phi = -il\phi$, and $\pi_\nu(iZ)\phi = l\phi$. Thus the highest weight of $W_{\mu,l}$ as representation of $\mathfrak{k}^{\mathbb{C}}$ is $l\alpha_0 + \mu_1\alpha_1 + \mu_2\alpha_2$.

The second part (2) is well-known; see e.g. [8,11]. □

Altogether we have now $I(\nu)$ is

$$I(\nu) = L^2(K/L) = \sum_{l=-\infty}^{\infty} \sum_{\mu} W_{\mu,l}$$

with μ being specified above.

Remark 2.6. The exact formulas for the highest weight vectors above in the case $\mathfrak{g} = \mathfrak{su}(r+b,r)$ are not needed in our paper. However it is possible to prove our Theorem 3.1 below for $\mathfrak{g} = \mathfrak{su}(r+b,r)$ by using the weight vectors instead of spherical vectors; see [8]. Note also that the parametrization of the spherical polynomials on $\mathbb{P}(\mathbb{C}^n)$ generated by the (p,q)-spherical harmonic polynomial $z_1^p\overline{z_n}^q$ as $\phi_{\mu_1,-l}$, $\mu_1 = p+q, l = p-q$, is due to our geometric definition of character χ. Recall that due to (2.16) $\phi_{\mu,l}$ satisfies

$$\phi_{\mu,l}(ke^{itH_j}) = e^{-ilt}\phi_{\mu,l}(k), \quad j = 1,2. \tag{2.22}$$

The same parametrization is used in Appendix A.

2.4. Harish-Chandra c-function and expansion of the (l,l)-spherical polynomials

A major technical step in the proof of Theorem 3.1 below is to use the Harish-Chandra c-function to compute certain expansions and differentiations involving the spherical polynomials $\phi_{\mu,l}$. We recall that the spherical polynomial $\phi_{\mu,l}(h)$ on $K/L = K_1/L_1$ is a special case of the Harish-Chandra spherical function $\Phi_{\lambda,l}(h)$ for the non-compact pair $(\mathfrak{k}_1^*, \mathfrak{l}_1)$ corresponding to the character χ_l of $L_1 \subset L_0$; see [26,27].

The precise relation between $\phi_{\mu,l}$ and $\Phi_{\lambda,l}$ is

$$\phi_{\mu,l}(h) = \Phi_{-i(\mu+\rho),l}(h), h \in K_1, \tag{2.23}$$

where $\rho = \rho_{\mathfrak{k}_1^*}$; see [6,26,27]. The spherical function $\Phi_{\lambda,l}$ is invariant with respect to the Weyl group $W(\mathfrak{k}_1^*, \mathfrak{h}_{i\mathfrak{q}})$ of the root system $R(\mathfrak{k}_1^*, \mathfrak{h}_{i\mathfrak{q}})$ in (2.12), acting on the parameter λ. Eventually we shall replace λ by $-i(\mu+\rho)$ and use *Weyl group symmetry in* $\mu+\rho$. An important property is that the leading term of $\Phi_{\lambda,l}(h)$ is given by the limit formula

$$\lim_{H\to\infty} e^{-(i\lambda-\rho)(H)}\Phi_{\lambda,l}(\exp(H)) = c(\lambda,l), \tag{2.24}$$

for H in the positive Weyl Chamber of the root system (2.12), i.e., for $H = x_1(iE_1) + x_2(iE_2) \in \mathfrak{h}_{i\mathfrak{q}}, x_1 > x_2 > 0$, and for $\mathrm{Re}(i\lambda) = y_1\alpha_1 + y_2\alpha_2, y_1 > y_2 > 0$; see [6, Ch. IV, Theorem 6.14; Ch. V, Section 4], [27, Theorem 3.6]. In particular

$$\phi_{\mu,l}(\exp(H)) = \Phi_{\lambda,l}(\exp(H)) = c(\lambda,l)e^{(i\lambda-\rho)(H)} + L.O.T., \, H \in \mathfrak{h}_{\mathfrak{q}}^{\mathbb{C}} \tag{2.25}$$

as an expansion of trigonometric polynomial on the complexification $\exp(\mathfrak{h}_{\mathfrak{q}}^{\mathbb{C}})[e]$ of the real torus $\exp(\mathfrak{h}_{\mathfrak{q}})[e] \subset K/L_0 = \mathbb{P}(K/L), [e] = \mathbb{C}e$, with lower order terms (L.O.T.) being trigonometric polynomials of lower order in the sense defined by the Weyl Chamber. Here $c(\lambda,l) = c(\lambda,-l)$ is the Harish-Chandra c-function, and in our case it is given by

$$c(\lambda,l) = c_0 \prod_{\epsilon=\pm1} \frac{\Gamma(\frac{i}{2}(\lambda_1+\epsilon\lambda_2))}{\Gamma(\frac{1}{2}a_1 + i(\lambda_1+\epsilon\lambda_2))} \prod_{j=1,2} \frac{2^{-i\lambda_j}\Gamma(i\lambda_j)}{\Gamma(\frac{1}{2}(b_1+1+i\lambda_j+l))\Gamma(\frac{1}{2}(b_1+1+i\lambda_j-l))}$$

for $\lambda = \lambda_1\alpha_1 + \lambda_2\alpha_2$, where c_0 is normalized so that $c(-i\rho,0) = 1$ for the Harish-Chandra c-function $c(-i\rho,0)$ with trivial line bundle, $l = 0$; see [27]. We observe also that the c-function $c(\lambda,l)$ is positive for $\lambda = -i(\mu+\rho)$.

We shall need the spherical polynomials $\phi_{(1,1),\pm1}(k)$ for $\mu = (1,1)$. The corresponding representations space of K is $\mathfrak{p}^+ = V$ or $\mathfrak{p}^- = \bar{V}$, and the spherical polynomial is the matrix coefficient

$$\phi_{(1,1),1}(k) = \langle ke, e\rangle, \quad \phi_{(1,1),-1}(k) = \langle e, ke\rangle. \tag{2.26}$$

Indeed the space $\mathfrak{p}^+ = V$ is a representation of $\mathfrak{k}_1^{\mathbb{C}}$ of highest weight $\alpha_1+\alpha_2$ and representation of $\mathfrak{k}^{\mathbb{C}}$ of highest weight $\alpha_0+\alpha_1+\alpha_2$ since Z acts as i, the corresponding highest weight vector is

$$v_0 = \frac{1}{2}\left((v_1 - ie) + (v_2 + iw)\right). \tag{2.27}$$

In other words, recalling that e is the root vector of the Harish-Chandra strongly orthogonal root γ_1, we see that $\alpha_1+\alpha_2$ is conjugated to γ_1.

3. Lie algebra $\mathfrak{g}$-action on $I(\nu) = Ind_P^G(\nu)$

We compute the Lie algebra action of $\mathfrak{g}^{\mathbb{C}}$ on $I(\nu)$. For that purpose we denote the right differentiation of Lie algebra elements $X \in \mathfrak{g}^{\mathbb{C}}$ on functions f on G by Xf,

$$Xf(g) = \frac{d}{ds}f(g\exp(sX))|_{s=0}. \tag{3.1}$$

Then X commutes with the left regular action

$$X(f)(hx) = X\Big(f(h\cdot)\Big)(x), \tag{3.2}$$

and intertwines the right action $f(x) \to f(xh) = f_h(x)$ as

$$(Xf)_h(x) = (Xf)(xh) = \Big((\mathrm{Ad}\, h(X)) f_h\Big)(x). \tag{3.3}$$

First it follows from (1.2) and (2.2) that $H_0 e = 2ie$, $\chi_l(\exp(tH_0)) = e^{2ilt}$, $\chi_l(H_0) = 2il$. Thus any element $f \in L^2(K, L_0, \chi_l)$ is an eigenfunction of the differentiation by H_0,

$$H_0 f = 2il f. \tag{3.4}$$

Theorem 3.1. *Let $\mathfrak{g}$ be a simple Hermitian Lie algebra of rank $r \geq 2$ and $\mathfrak{g} \not\simeq \mathfrak{sp}(r, \mathbb{R})$. The action of $\pi_\nu(\xi)$ on $\phi_{\mu,l}$ is given by*

$$\begin{aligned} &2^3\, \pi_\nu(\xi)\phi_{\mu,l} \\ &= \sum_{\sigma=(\sigma_1,\sigma_2)=(\pm1,\pm1)} \Big(\nu + \sigma_1(\mu_1+\rho_1) + \sigma_2(\mu_2+\rho_2) - (\rho_1+\rho_2)\Big) \\ &\times \Big(c_{\mu,l}(\mu+\sigma, l+1)\phi_{\mu+\sigma,l+1} + c_{\mu,l}(\mu+\sigma, l-1)\phi_{\mu+\sigma,l-1}\Big), \end{aligned}$$

where the coefficients $c_{\mu,l}(\mu+\sigma, l\pm1)$ are given by

$$c_{\mu,l}(\mu+\sigma, l\pm1) = \frac{c(-i(\sigma(\mu+\rho)), l)}{c(-i(\alpha_1+\alpha_2+\sigma(\mu+\rho)), l\pm1)}, \tag{3.5}$$

and $\sigma = (\sigma_1, \sigma_2)$ is viewed as element in the Weyl group W of the root system (2.12) such that $\sigma(\alpha_1+\alpha_2) = \sigma_1\alpha_1 + \sigma_2\alpha_2$. Moreover all the coefficients are positive.

It is understood here that the term $\phi_{\mu+\sigma,l\pm1} = \phi_{(\mu_1+\sigma_1,\mu_2+\sigma_2),l\pm1}$ will not appear in the RHS if $(\mu_1+\sigma_1)\alpha_1 + (\mu_2+\sigma_2)\alpha_2$ is not one of the highest weights specified in Lemma 2.5.

Remark 3.2. It is remarkable that all the coefficients of $\phi_{\mu+\sigma,l\pm1}$ have a rather uniform formula. Actually it is relatively easy to find the coefficient of the leading term $\phi_{\mu+(1,1),l\pm1}$ and the other coefficients can be obtained from the Weyl group symmetry and by unitarity of π_ν for $\nu = \rho_{\mathfrak{g}} + ix, x \in \mathbb{R}$. We shall find all the coefficients independent of the unitarity by proving some recursion and differentiation formulas for spherical polynomials, which might be of independent interests [31].

Proof. We claim first that for any $X \in \mathfrak{p}^{\mathbb{C}}$,

$$\pi_\nu(X)W_{\mu,l} \subseteq \sum_{\sigma=(\sigma_1,\sigma_2)=(\pm1,\pm1)} W_{\mu+\sigma,l\pm1}.$$

This follows by considering the tensor product $\mathfrak{p}^{\mathbb{C}} \otimes W_{\mu,l}$ as representation of K. Indeed consider the case $\mathfrak{g} \neq \mathfrak{su}(r+b, r)$. The adjoint action of the central element $Z \in \mathfrak{k}$ on $\mathfrak{p}^{\pm}$ is $\pm i$, and its right action on $W_{\mu,l}$ is il. Let $X \in \mathfrak{p}^+$, then $\pi_\nu(X)W_{\mu,l}$ is of weight $i(l+1)$

under Z for any $X \in \mathfrak{p}^+$. It is also a classical fact that the highest weights in the tensor product decomposition of $W_{\mu,l} \otimes \mathfrak{p}^\pm$ under $\mathfrak{k}^{\mathbb{C}}$ are of the form $\mu + l\alpha_0 + \nu'$ where ν' is a weight appearing in $\mathfrak{p}^+$. The space $\mathfrak{p}^+$ is of highest weight $(1,1) = \alpha_1 + \alpha_2$ under the Cartan subalgebra $(\mathfrak{h}_{i\mathfrak{q}})^{\mathbb{C}} = \mathbb{C}E_1 + \mathbb{C}iE_2$ of $\mathfrak{h}^{\mathbb{C}}$ and the only non-zero weights in $\mathfrak{p}^\pm$ of the form $c_1\alpha_1 + c_2\alpha_2$ are $c_1, c_2 = 0, \pm 1$, namely they are of the form

$$\sigma_1\alpha_1 + \sigma_2\alpha_2, \sigma_1\alpha_1,\ \sigma_2\alpha_2,\ \sigma_1 = \pm 1, \sigma_2 = \pm 1.$$

However by the Cartan-Helgason theorem, Lemma 2.5, we see that $\sigma_1\alpha_1, \sigma_2\alpha_2$ are not eligible since the center Z-action is $i(l+1)$. Thus $\pi_\nu(X)W_{\mu,l}$ is of the claimed form.

When $\mathfrak{g} = \mathfrak{su}(r+b, r)$ the Weyl group for the root system of $(\mathfrak{k}^*, \mathfrak{h}_{i\mathfrak{q}})$ is $(\mathbb{Z}_2)^2$ consisting of only sign changes instead of all signed permutations $(\mathbb{Z}_2)^2 \rtimes S_2$, but all the relevant weights $\sigma_1\alpha_1 + \sigma_2\alpha_2$ are still in the orbit of the Weyl group $(\mathbb{Z}_2)^2$ so the arguments are valid for $\mathfrak{g} = \mathfrak{su}(r+b, r)$ as well.

Next, the element $\xi = \xi_e$ is invariant under $L \subset K$, thus $\pi_\nu(\xi)\phi_{\mu,l}$ is a sum of the L-invariant vectors in $\sum_{\sigma_1,\sigma_2=\pm 1} W_{\mu+\sigma,l\pm 1}$, and is further by Lemma 2.5 a linear combination of $\phi_{\mu+\sigma,l\pm 1}$. The rest of the proof is to determine the coefficients. Notice also that each function in the linear combination is determined by its restriction on the complex torus $\exp(\mathfrak{h}_{\mathfrak{q}}^{\mathbb{C}})e \subset K/L$ once the line parameter l is given, so it is enough to find the expansion restricted on the complex torus (after the differentiations) as the line bundle parameters of each term in the expansion are already fixed.

We have

$$\begin{aligned}\pi_\nu(\xi)\phi_{\mu,l}(k) &= \frac{d}{ds}\phi_{\mu,l}(\exp(-s\xi)k)|_{s=0} = \frac{d}{ds}\phi_{\mu,l}(kk^{-1}\exp(-s\xi)k)|_{s=0} \\ &= \frac{d}{ds}\phi_{\mu,l}(k\exp(-s\,\mathrm{Ad}(k^{-1})\xi))|_{s=0} = -\Big((\mathrm{Ad}(k^{-1})\xi)\phi_{\mu,l}\Big)(k),\ k \in K,\end{aligned} \tag{3.6}$$

where $\Big((\mathrm{Ad}(k^{-1})\xi)\phi_{\mu,l}\Big)(k)$ is right differentiation of the Lie algebra valued *vector field* $-\mathrm{Ad}(k^{-1})\xi$ on $\phi_{\mu,l}$ evaluated at $k \in K$. The element $\phi_{\mu,l}$ is in the induced representation, any differentiation of $\phi_{\mu,l}$ along the Lie algebra $\mathfrak{m} + \mathfrak{n}$ is zero, and we need formulas for $\mathrm{Ad}(k^{-1})\xi = \mathrm{Ad}(k^{-1})\xi_e = \xi_{k^{-1}e} \mod \mathfrak{m} + \mathfrak{n}$.

Lemma 3.3. *Let $V = V_2 + V_1 + V_0 = \mathbb{C}e + V_1 + V_0$ be the Peirce decomposition with respect to the minimal tripotent e and P_2, P_1, P_0 the corresponding projections. Any element $\xi_u \in \mathfrak{p}$ has the following decomposition according to (1.5),* $\mathrm{mod}\,\mathfrak{m} + \mathfrak{n}$,

$$\xi_u = \mathrm{Re}\langle u, e\rangle\xi + \mathrm{Im}\langle u, e\rangle H_0 + D(P_1u, e) - D(e, P_1u). \tag{3.7}$$

Proof. Write $u = P_2u + P_1u + P_0u = \langle u, e\rangle e + u_1 + u_0 = \mathrm{Re}\langle u, e\rangle e + i\,\mathrm{Im}\langle u, e\rangle e + u_1 + u_0$. Then

$$\xi_{\langle u,e\rangle e} = \mathrm{Re}\langle u, e\rangle\xi_e + \mathrm{Im}\langle u, e\rangle\xi_{ie},$$

with

$$\xi_{ie} = (\xi_{ie} - H_0) + H_0 = H_0, \bmod \mathfrak{n}$$

by (1.8). In view of (1.7) we have

$$\xi_{u_1} = \Big(\xi_{u_1} + D(e, u_1) - D(u_1, e)\Big) + \Big(D(u_1, e) - D(e, u_1)\Big) = D(u_1, e) - D(e, u_1), \bmod \mathfrak{n},$$

and $\xi_{u_0} \in \mathfrak{m}$. This proves (3.7). □

Using Lemma 3.3 and the formula (2.26) we see that $\mathrm{Ad}(k^{-1})\xi$, mod $\mathfrak{m} + \mathfrak{n}$, is

$$\begin{aligned} \mathrm{Ad}(k^{-1})\xi &= \mathrm{Re}\langle k^{-1}e, e\rangle\xi + \mathrm{Im}\langle k^{-1}e, e\rangle H_0 + D(P_1(k^{-1}e), e) - D(e, P_1(k^{-1}e)) \\ &= \mathrm{Re}\langle ke, e\rangle\xi - \mathrm{Im}\langle ke, e\rangle H_0 + D(P_1(k^{-1}e), e) - D(e, P_1(k^{-1}e)). \end{aligned} \tag{3.8}$$

Hence

$$\pi_\nu(\xi)\phi_{\mu,l}(k) = I + II + III \tag{3.9}$$

with

$$I = -\mathrm{Re}\langle k^{-1}e, e\rangle\Big(\xi\phi_{\mu,l}\Big)(k),$$
$$II = \mathrm{Im}\langle ke, e\rangle\Big(H_0\phi_{\mu,l}\Big)(k)$$

and

$$III = \Big(\Big[D(e, P_1(k^{-1}e)) - D(P_1(k^{-1}e), e)\Big]\phi_{\mu,l}\Big)(k).$$

Using the definition (1.10) of the induced representation we have that the right differentiation of ξ on any element $f \in L^2(K/L) = Ind_P^G(\nu)$ has eigenvalue $-\nu$, $\xi\phi_{\mu,l} = -\nu\phi_{\mu,l}$, and the first term is

$$\begin{aligned} I &= -\mathrm{Re}\langle k^{-1}e, e\rangle\Big(\xi\phi_{\mu,l}\Big)(k) \\ &= \nu\,\mathrm{Re}\langle k^{-1}e, e\rangle\phi_{\mu,l}(k) \\ &= \frac{\nu}{2}\langle ke, e\rangle\phi_{\mu,l}(k) + \frac{\nu}{2}\langle e, ke\rangle\phi_{\mu,l}(k) \\ &= I^+ + I^- \end{aligned}$$

with

$$I^+ = \frac{\nu}{2}\langle ke, e\rangle\phi_{\mu,l}(k) = \frac{\nu}{2}\phi_{(1,1),1}(k)\phi_{\mu,l}(k)$$

and

$$I^- = \frac{\nu}{2}\phi_{(1,1),-1}(k)\phi_{\mu,l}(k).$$

The second term II, in view of (3.4), is

$$\begin{aligned} II &= \mathrm{Im}\langle ke, e\rangle(2il\phi_{\mu,l})(k) \\ &= l\langle ke, e\rangle\phi_{\mu,l}(k) - l\langle e, ke\rangle\phi_{\mu,l}(k) \\ &= l\phi_{(1,1),1}(k)\phi_{\mu,l}(k) - l\phi_{(1,1),-1}(k)\phi_{\mu,l}(k) = II^{+} + II^{-}. \end{aligned}$$

We proceed to find recursion formulas for $\phi_{(1,1),1}\phi_{\mu,l}$. For that purpose we find explicit coordinates for the complex torus $\exp(\mathfrak{h}_{\mathfrak{q}}^{\mathbb{C}})e$, $\exp(\mathfrak{h}_{\mathfrak{q}})e \subset S = K/L \subset V$. We shall treat all relevant functions as trigonometric functions on the compact homogeneous space K/L and on the complex torus $\exp(\mathfrak{h}_{\mathfrak{q}}^{\mathbb{C}})[e]$.

Lemma 3.4. *Recall the Jordan quadrangle $\{e, v_1, w, v_2\}$ and E_1, E_2 in (2.9). If $k = \exp(x_1E_1 + x_2E_2), x_1, x_2 \in \mathbb{C}$, then*

$$k^{-1}e = \cos x_1 \cos x_2 e - (\sin x_1 \cos x_2 v_1 + \sin x_2 \cos x_1 v_2) + \sin x_1 \sin x_2 w.$$

Proof. Using (2.8) we find

$$E_1e = (D(v_1, e) - D(e, v_1))e = v_1, E_1^2e = E_1v_1 = (D(v_1, e) - D(e, v_1))v_1 = -e,$$

and generally

$$E_1^{2m}e = (-1)^m e,\ E_1^{2m+1}e = (-1)^m v_1.$$

Therefore

$$e^{-x_1E_1}e = \cos x_1 e - \sin x_1 v_1,$$

and also $e^{-x_2E_2}e = \cos x_2 e - \sin x_2 v_2$. We compute further

$$E_2v_1 = (D(v_2, e) - D(e, v_2))v_1 = w,\ E_2^2v_1 = (D(v_2, e) - D(e, v_2))w = -v_1,$$

and in general

$$E_2^{2m}v_1 = (-1)^m v_1, E_2^{2m+1}v_1 = (-1)^m w.$$

This implies that

$$e^{-x_2E_2}v_1 = \cos x_2 v_1 - \sin x_2 w.$$

We have then

$$\begin{aligned} k^{-1}e &= e^{-x_1E_1-x_2E_2}e = e^{-x_2E_2}(\cos x_1 e - \sin x_1 v_1) \\ &= (\cos x_1(\cos x_2 e - \sin x_2 v_2) + \sin x_1(\cos x_2 v_1 - \sin x_2 w) \\ &= \cos x_1 \cos x_2 e - (\sin x_1 \cos x_2 v_1 + \sin x_2 \cos x_1 v_2) + \sin x_1 \sin x_2 w. \quad \square \end{aligned} \tag{3.10}$$

Lemma 3.5. *The following recursion formulas hold,*

$$\phi_{(1,1),1}\phi_{\mu,l} = \frac{1}{4}\sum_{\sigma_1,\sigma_2=\pm 1} c_{\mu,l}(\mu+\sigma, l+1)\phi_{\mu+\sigma,l+1}, \tag{3.11}$$

$$\phi_{(1,1),-1}\phi_{\mu,l} = \frac{1}{4}\sum_{\sigma_1,\sigma_2=\pm 1} c_{\mu,l}(\mu+\sigma, l-1)\phi_{\mu+\sigma,l-1}, \tag{3.12}$$

where $c_{\mu,l}(\mu+\sigma, l\pm 1)$ *are given in Theorem 3.1.*

Proof. Observe again that by general tensor product arguments the product $\phi_{(1,1),\pm 1}\phi_{\mu,l}$ is a sum of $\phi_{\mu+\sigma,l\pm 1}$, $\sigma = (\pm 1, \pm 1)$. We use the idea in [29] by considering the leading term of $\phi_{(1,1),1}\Phi_{\lambda,l}$ and the Harish-Chandra limit formula (2.24); see also [30]. We recall (2.23) and consider the expansion

$$\phi_{(1,1),1}\Phi_{\lambda,l} = \sum_{\sigma_1,\sigma_2=\pm 1} A_{\lambda-i\sigma,l+1}\Phi_{\lambda-i\sigma,l+1} + L.O.T. \tag{3.13}$$

(Presumably L.O.T. will not appear for *general* λ but it will not concern us here.) We let $h = \exp(H), H = x_1(iE_1) + x_2(iE_2)$. First it is clear from (2.26) and (3.10) that

$$\phi_{(1,1),1}(h) = \langle he, e\rangle = \cosh x_1 \cosh x_2 = \frac{1}{4}(e^{x_1}+e^{-x_1})(e^{x_2}+e^{-x_2}) = \frac{1}{4}e^{x_1+x_2} + L.O.T. \tag{3.14}$$

The coefficient $\frac{1}{4}$ can also be obtained using the general formula (2.24); indeed the evaluation of Harish-Chandra c-function is

$$c((1,1),1) = c(\alpha_1+\alpha_2, 1) = \frac{1}{4}.$$

Using the limit formulas (2.24) again we see that the coefficient $A_{\lambda-i(1,1),l+1}$ of the leading term $\Phi_{\lambda-i(1,1),l+1}$ is

$$A_{\lambda-i(1,1),l+1} = \frac{1}{4}\frac{c(\lambda, l)}{c(\lambda - i(\alpha_1+\alpha_2), l+1)}.$$

Next we use Weyl group symmetry to find $A_{\lambda-i\sigma,l+1}$. With some abuse of notation we view $\sigma = (\sigma_1, \sigma_2) = \sigma(1,1)$ as an element in the Weyl group. The term $A_{\lambda-i\sigma,l+1}\Phi_{\lambda-i\sigma,l+1}$ in the above expansion is also

$$A_{\lambda-i\sigma,l+1}\Phi_{\lambda-i\sigma,l+1} = A_{\lambda-i\sigma,l+1}\Phi_{\sigma\lambda-i(1,1),l+1},$$

since $\Phi_{\lambda,l}$ is W-invariant in λ and $\sigma^2 = 1$. But the coefficients are unique in the above expansion, thus $A_{\lambda-i\sigma,l+1}$ is precisely $A_{\sigma\lambda-i(1,1),l+1}$,

$$A_{\lambda-i\sigma,l+1} = A_{\sigma\lambda-i(1,1),l+1} = \frac{1}{4}\frac{c(\sigma\lambda,l)}{c(\sigma\lambda-i(\alpha_1+\alpha_2),l+1)}.$$

Specifying the result to the case $\lambda = -i(\mu+\rho)$ we prove our claim; (3.12) is proved by the same method. □

We can now apply the lemma to both terms I and II,

$$I^+ = \frac{\nu}{2}\phi_{(1,1),1}(k)\phi_{\mu,l}(k) = \frac{\nu}{2^3}\sum_{\sigma_1,\sigma_2=\pm1} c_{\mu,l}(\mu+\sigma,l+1)\phi_{\mu+\sigma,l+1}(k), \tag{3.15}$$

$$I^- = \frac{\nu}{2}\phi_{(1,1),-1}\phi_{\mu,l}(k) = \frac{\nu}{2^3}\sum_{\sigma_1,\sigma_2=\pm1} c_{\mu,l}(\mu+\sigma,l-1)\phi_{\mu+\sigma,l-1}(k), \tag{3.16}$$

$$II^+ = l\phi_{(1,1),1}(k)\phi_{\mu,l}(k) = \frac{l}{4}\sum_{\sigma_1,\sigma_2=\pm1} c_{\mu,l}(\mu+\sigma,l+1)\phi_{\mu+\sigma,l+1}, \tag{3.17}$$

$$II^- = -l\phi_{(1,1),-1}(k)\phi_{\mu,l}(k) = -\frac{l}{4}\sum_{\sigma_1,\sigma_2=\pm1} c_{\mu,l}(\mu+\sigma,l-1)\phi_{\mu+\sigma,l-1}. \tag{3.18}$$

The third term III is

$$III = \Big(D(e,P_1(k^{-1}e))\phi_{\mu,l}\Big)(k) - \Big(D(P_1(k^{-1}e),e)\phi_{\mu,l}\Big)(k) =: III^+ + III^-.$$

Lemma 3.6. *We have the following recurrence formula for the right differentiations of the vector fields $D(e,P_1(k^{-1}e))$ and $-D(P_1(k^{-1}e),e)$ on $\phi_{\mu,l}$,*

$$III^+ = \Big(D(e,P_1(k^{-1}e))\phi_{\mu,l}\Big)(k) = \sum_{\sigma_1,\sigma_2=\pm1} b_{\mu+\sigma,l+1}\phi_{\mu+\sigma,l+1}(k), \tag{3.19}$$

$$III^- = -\Big(D(P_1(k^{-1}e),e)\phi_{\mu,l}\Big)(k) = \sum_{\sigma_1,\sigma_2=\pm1} b_{\mu+\sigma,l-1}\phi_{\mu+\sigma,l-1}(k), \tag{3.20}$$

where the coefficients $b_{\mu+\sigma,l\pm1}$ are given by

$$b_{\mu+\sigma,l+1} = \frac{1}{2^3}\Big(\sigma_1(\mu_1+\rho_1)+\sigma_2(\mu_2+\rho_2)-(\rho_1+\rho_2)-2l\Big)c_{\mu,l}(\mu+\sigma,l+1),$$
$$b_{\mu+\sigma,l-1} = \frac{1}{2^3}\Big(\sigma_1(\mu_1+\rho_1)+\sigma_2(\mu_2+\rho_2)-(\rho_1+\rho_2)+2l\Big)c_{\mu,l}(\mu+\sigma,l-1).$$

Proof. Denote X the vector field

$$X(k) = D(e,P_1(k^{-1}e)) \tag{3.21}$$

acting on functions on K by right differentiation, $f \to (X(k)f)(k)$. With some abuse of notation we abbreviate it sometimes as $(Xf)(k)$, $III^+ = \Big(X(k)\phi_{\mu,l}\Big)(k) = (X\phi_{\mu,l})(k)$.

We prove first that $X\phi_{\mu,l}$ is $(l+1,l+1)$-spherical. Some care has to be taken as X is vector field taking values in the Lie algebra of $\mathfrak{k}^{\mathbb{C}}$; the transformation rule of $X\phi_{\mu,l}$ under the center of K in L_0 is easily checked but we have to prove it for all L_0. The space V_1 is invariant under the subgroup $L_0 \subset K$, and

$$P_1((hk)^{-1}e) = P_1(k^{-1}h^{-1}e) = \chi(h)^{-1}P_1(k^{-1}e),\, hP_1((kh)^{-1}e) = P_1(k^{-1}e), h \in L_0.$$

Also elements $h \in K$ act on $D(x,y)$ as Jordan triple automorphisms $\mathrm{Ad}(h)D(u,v) = D(hu,hv)$, $D(u,v)$ is conjugate linear in v, and $\overline{\chi(h)} = \chi^{-1}(h), h \in L_0$; elements $k \in L$ act as Jordan triple isomorphism as $Le = e$ and L_0 as isomorphism up to the character χ. Thus the vector field $X(k) = D(e, P_1(k^{-1}e))$ satisfies

$$X(hk) = \chi(h)X(k), \quad \mathrm{Ad}(h)\Big(X(kh)\Big) = \chi(h)X(k), \quad h \in L_0.$$

It follows by the chain rules (3.2) and (3.3) that

$$\begin{aligned}\Big(X(hk)\phi_{\mu,l}\Big)(hk) &= \left(X(hk)\phi_{\mu,l}(h\cdot)\right)(k)\\ &= \left(\chi(h)X(k)\chi_l(h)\phi_{\mu,l}(\cdot)\right)(k)\\ &= \chi_{l+1}(h)\Big(X(k)\phi_{\mu,l}\Big)(k)\end{aligned}$$

and

$$\begin{aligned}\Big(X(kh)\phi_{\mu,l}\Big)(kh) &= \Big(\,\mathrm{Ad}(h)X(kh)\Big)\Big(\phi_{\mu,l}(\cdot h)\Big)(k) = \chi(h)\chi_l(h)\Big(X(k)\phi_{\mu,l}\Big)(k)\\ &= \chi_{l+1}(h)\Big(X(k)\phi_{\mu,l}\Big)(k).\end{aligned}$$

Thus $\Big(X(k)\phi_{\mu,l}\Big)(k)$ must be of the form (3.19). To find the coefficients we consider the subgroup $SL(2,\mathbb{C})^2 = SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ with the Lie algebra $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ generated by (2.9) and the restriction of $\phi_{\mu,l}$ on $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ and its expansion in terms of spherical polynomials of $SL(2,\mathbb{C})$, namely we consider the branching of the representation $(K^{\mathbb{C}}, W_{\mu,l})$ under $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. (The connected subgroup in $G^{\mathbb{C}}$ with Lie algebra $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ can be a finite quotient of $SL(2,\mathbb{C})^2$ by a finite normal subgroup. But this would not change the arguments below.) The highest weight of the representation $W_{\mu,l}$ restricted to $SL(2) \times SL(2)$ is $\mu = (\mu_1,\mu_2) = \mu_1\alpha_1 + \mu_2\alpha_2$, and thus the representation $\odot^{\mu_1}\mathbb{C}^2 \otimes \odot^{\mu_2}\mathbb{C}^2$ of $SL(2) \times SL(2)$ appears in $(K^{\mathbb{C}}, \mu)$, and all other representations are of the form (μ_1', μ_2') with $\mu_1' < \mu_1$, or $\mu_2' < \mu_2$. Let $\psi_{m,l}(g)$ be the (l,l)-spherical polynomial for the group $SL(2,\mathbb{C})$ in the Appendix A, (A.3). Comparing the leading term of $\phi_{\mu,l}(g_1,g_2)$ and $\psi_{\mu_1,l}(g_1)\psi_{\mu_2,l}(g_2)$ we have then

$$\phi_{\mu,l}(g_1,g_2) = c(-i(\mu+\rho),l)\psi_{\mu_1,l}(g_1)\psi_{\mu_2,l}(g_2) + L.O.T. \quad (g_1,g_2) \in SL(2,\mathbb{C})^2.$$

The vector field $X(k) = D(e, P_1(k^{-1}e))$ restricted to

$$k = (k_1, k_2) = \Big(\exp(x_1 E_1), \exp(x_2 E_2) \Big) \in SU(2) \times SU(2)$$

is, by Lemma 3.3, of the form

$$\begin{aligned} X(k) &= -\sin x_1 \cos x_2 D(e, v_1) - \sin x_2 \cos x_1 D(e, v_2) \\ &= -\sin x_1 \cos x_2 E_1^- - \sin x_2 \cos x_1 E_2^- . \end{aligned}$$

The vector field $X(k)$ takes values also in the complexification of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})+\mathfrak{sl}(2, \mathbb{C})$. Thus the restriction

$$\Big(X\phi_{\mu,l}\Big)|_{SU(2)\times SU(2)} = X\left(\phi_{\mu,l}|_{SU(2)\times SU(2)}\right),$$

i.e., the restriction of the differentiation is the same as the differentiation of the restriction. Moreover the Lie algebra differentiations by $D(e, v_1), D(e, v_2)$ clearly preserve the degree. Consequently

$$\begin{aligned} \Big(X\phi_{\mu,l}\Big)(k_1, k_2) &= c(-i(\mu+\rho), l)\Big(-\sin x_1 (E_1^- \psi_{\mu_1,l})(k_1)\Big)\Big(\cos x_2 \psi_{\mu_2,l}(k_2)\Big) \\ &\quad + c(-i(\mu+\rho), l)\Big(-\sin x_2 (E_2^- \psi_{\mu_2,l})(k_2)\Big)\Big(\cos x_1 \psi_{\mu_1,l}(k_1)\Big) + L.O.T. \end{aligned}$$

We use now Lemma A.1, (A.6), and obtain

$$-\sin x_j (E_j^- \psi_{\mu_j,l})(k_j) = \frac{1}{4}(\mu_j - l)\psi_{\mu_j+1,l+1}(k_j) + L.O.T., \quad j = 1, 2. \tag{3.22}$$

The leading term of $\cos x_2 \psi_{\mu_2,l}(k_2)$, $k_2 = \exp(x_2 E_2)$, is clearly the same as $\frac{1}{2}\psi_{\mu_2+1,l}(k_2)$. Thus

$$\Big(-\sin x_1 (E_1^- \psi_{\mu_1,l})(k_1)\Big)\Big(\cos x_2 \psi_{\mu_2,l}(k_2)\Big) = \frac{1}{8}(\mu_1 - l)\psi_{\mu_j+1,l+1}(k_j)\psi_{\mu_2+1,l}(k_2);$$

similarly for $\Big(-\sin x_2 (E_2^- \psi_{\mu_2,l})(k_2)\Big)\Big(\cos x_1 \psi_{\mu_1,l}(k_1)\Big)$. We have then

$$\begin{aligned} \Big(X\phi_{\mu,l}\Big)(k_1, k_2) &= \frac{1}{8}(\mu_1 - l + \mu_2 - l)c(-i(\mu+\rho), l)\psi_{\mu_1+1,l+1}(k_1)\psi_{\mu_2+1,l+1}(k_2) + L.O.T. \\ &= \frac{1}{8}(\mu_1 + \mu_2 - 2l)c(-i(\mu+\rho), l)\psi_{\mu_1+1,l+1}(k_1)\psi_{\mu_2+1,l+1}(k_2) + L.O.T. \end{aligned}$$

On the other hand the leading term in RHS of (3.19) is

$$\begin{aligned} &b_{\mu+(1,1),l+1}\phi_{\mu+(1,1),l+1}(k_1, k_2) \\ &= b_{\mu+(1,1),l+1}c(-i(\mu+(1,1)+\rho), l+1)\psi_{\mu_1+1,l+1}(k_1)\psi_{\mu_2+1,l+1}(k_2) + L.O.T. \end{aligned}$$

It follows then that

$$b_{\mu+(1,1),l+1} = \frac{1}{8}(\mu_1 + \mu_2 - 2l)c_{\mu,l}(\mu + (1,1), l+1)$$

where $c_{\mu,l}(\mu + (1,1), l+1)$ is given in (3.5). This proves the formula for the leading coefficient.

To find the other coefficients we write

$$\mu_1 + \mu_2 - 2l = (\mu_1 + \rho_1) + (\mu_2 + \rho_2) - (\rho_1 + \rho_2) - 2l$$

and use the Weyl group symmetry as in the proof of Lemma above to get

$$b_{\mu+\sigma,l+1} = \frac{1}{8}\Big(\sigma_1(\mu_1 + \rho_1) + \sigma_2(\mu_2 + \rho_2) - (\rho_1 + \rho_2) - 2l)\Big)c_{\mu,l}(\mu + \sigma, l+1),$$

for $\sigma_1, \sigma_2 = \pm 1$.

To prove (3.20) we consider the vector field $Y(k) = -\Big(D(P_1(k^{-1}e), e)\phi_{\mu,l}\Big)$ and its restriction to $SL(2,\mathbb{C})^2$. We have

$$Y = Y(k) = \sin x_1 \cos x_2 E_1^+ + \sin x_2 \cos x_1 E_2^+.$$

We use then (A.5) to find the leading term of the expansion $Y\phi_{\mu,l}$ and obtain all the coefficients by Weyl group symmetry. $\square$

Hence

$$\begin{aligned} III^+ &= \Big(D(e, P_1(k^{-1}e))\phi_{\mu,l}\Big)(k) \\ &= \sum_{\sigma_1,\sigma_2=\pm 1} \frac{1}{2^3}\Big(\sigma_1(\mu_1 + \rho_1) + \sigma_2(\mu_2 + \rho_2) - (\rho_1 + \rho_2) - 2l\Big)c_{\mu,l}(\mu + \sigma, l+1)\phi_{\mu+\sigma,l+1}(k), \\ III^- &= -\Big(D(P_1(k^{-1}e), e)\phi_{\mu,l}\Big)(k) \\ &= \sum_{\sigma_1,\sigma_2=\pm 1} \frac{1}{2^3}\Big(\sigma_1(\mu_1 + \rho_1) + \sigma_2(\mu_2 + \rho_2) - (\rho_1 + \rho_2) + 2l\Big)c_{\mu,l}(\mu + \sigma, l-1)\phi_{\mu+\sigma,l-1}(k). \end{aligned}$$

Altogether we find $\pi_\nu(\xi)\phi_{\mu,l} = (I^+ + II^+ + III^+) + (I^- + II^- + III^-)$,

$$(I^+ + II^+ + III^+) = \frac{1}{2^3}\sum_{\sigma_1,\sigma_2=\pm 1}\Big(\nu + \sigma_1(\mu_1 + \rho_1) + \sigma_2(\mu_2 + \rho_2) - (\rho_1 + \rho_2)\Big)\phi_{\mu+\sigma,l+1},$$

$$(I^- + II^- + III^-) = \frac{1}{2^3}\sum_{\sigma_1,\sigma_2=\pm 1}\Big(\nu + (\sigma_1(\mu_1 + \rho_1) + \sigma_2(\mu_2 + \rho_2) - (\rho_1 + \rho_2)\Big)\phi_{\mu+\sigma,l-1}.$$

This finished the proof. $\square$

Remark 3.7. We have used restrictions to subgroups in our expanding the differentiation $\pi_\nu(\xi)\phi_{\mu,l}$ of spherical polynomials. It might be important to note that generally differentiations and restrictions are not commuting and restrictions are not injective. Here we have proved in apriori that there is an expansion of $\pi_\nu(\xi)\phi_{\mu,l}$ in terms of spherical polynomials $\phi_{\mu+\sigma,l\pm 1}$ each of them being uniquely determined by their restriction on $\exp(\eta_{\mathfrak{q}}^{\mathbb{C}})$, and eventually we used the fact that right differentiations by Lie algebra elements commute with left multiplications by Lie group elements.

Remark 3.8. It is remarkable that in the formula the line bundle parameter l disappears due to the cancellation of $2l$ in the sum $II + III$. When $\mathfrak{g} = \mathfrak{su}(r+b, r)$, $r > 1$, $\mathfrak{k} = \mathfrak{s}(u(r+b) + \mathfrak{u}(r))$, the action of π on $I(\nu)$ has been studied in details in [8]. In this case the parameter l indeed does not appear in affine term $\nu+\sigma_1(\mu_1+\rho_1)+\sigma_2(\mu_2+\rho_2)-\rho_1-\rho_2)$ for the action; also the coefficients $-A^{\pm,\pm}$ in [8, Lemma 4.1] can be formulated, writing $\mu_1 = (\mu_1 + \rho_1) - \rho_1, \mu_2 = (\mu_2 + \rho_2) - \rho_2$, as

$$\nu \pm (\mu_1 + \rho_1) \pm (\mu_2 + \rho_2) - \rho_1 - \rho_2,$$

with our ν being their $-a = -(\alpha+\beta)$, $\mu_1 = m_1 + m_2$, $\mu_2 = n_1 + n_2$, $\rho_1 = p - 1$, $\rho_2 = q - 1$, and our l their $m_1 - m_2 = n_2 - n_1$; see further [8, (4.10)-(4.11)]. The Weyl group symmetry is again manifest here.

4. Reductions points, complementary and composition series

4.1. Reduction points and finite dimensional subrepresentations

We study now the existence of intertwining operators between representations $I(\nu)$ and $I(\nu')$, and we find certain finite dimensional representations at the reduction point of $I(\nu)$.

Theorem 4.1. *Let $\mathfrak{g} \neq \mathfrak{sp}(r, \mathbb{R})$ be a simple Hermitian Lie algebra of rank $r \geq 2$.*

(1) *There exists an intertwining operator between the induced irreducible representations $I(\nu)$ and $I(\nu')$ if and only if $\nu = \nu'$ or $\nu + \nu' = 2\rho_{\mathfrak{g}}$.*
(2) *$I(\nu)$ is reducible if and only if ν is an even integer, $\nu \geq 2\rho_1 + 2$ or $\nu \leq 2\rho_2 - 2$. Moreover at the point $\nu = -2k$, $k \geq 1$, the irreducible submodule in symmetric tensor product $S^k(\mathfrak{g}^{\mathbb{C}})$ generated by $\otimes^k E_0$ is realized as finite-dimensional subrepresentation of $I(\nu)$ via*

$$T : S^k(\mathfrak{g}^{\mathbb{C}}) \to I(\nu), X \mapsto f(g) = (\otimes^k \operatorname{Ad}(g)(E_0), X), g \in G, \tag{4.1}$$

where $E_0 = \xi_{ie} - iH_0 \in \mathfrak{n}_2$ is the basis vector of the center $\mathfrak{n}_2$ of the nilpotent algebra $\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2$ and $(\cdot,\cdot)$ is the Killing form in $\mathfrak{g}^{\mathbb{C}}$ extended to $S^k(\mathfrak{g}^{\mathbb{C}})$.

Proof. The first part and the second part on reductions points are done similarly as in [8,25,30]. Now let $\nu = -2k$ be an negative even integer. We prove that T above is an intertwining operator from $S^k(\mathfrak{g}^{\mathbb{C}})$ into $I(\nu), \nu = -2k$. The functions $f = f_X$ transform under $P = MAN$ as

$$\begin{aligned} f(gm) &= (\otimes^k \operatorname{Ad}(gm)(E_0), X) = (\otimes^k \operatorname{Ad}(g)\operatorname{Ad}(m)(E_0), X) \\ &= (\otimes^k \operatorname{Ad}(g), X) = f(g), \quad m \in M, \end{aligned}$$

since M centralizes E_0, $f(gn) = f(g)$, $n \in N$, as E_0 is in the center of N, and

$$\begin{aligned} f(ge^{t\xi}) &= (\otimes^k \operatorname{Ad}(g)\operatorname{Ad}(e^{t\xi})(E_0), X) = (\otimes^k (e^{2t}\operatorname{Ad}(g)(E_0), X) \\ &= e^{2kt} f(g) = e^{-\nu t} f(g), \quad m \in N, \end{aligned}$$

since $\operatorname{ad}(\xi)E_0 = 2E_0$, $\operatorname{Ad}(e^{t\xi})(E_0) = e^{2t}E_0$.

Thus $f \in I(\nu)$. The intertwining property of T is obvious by its definition. This completes the proof. □

4.2. Complementary series

We determine the complementary series i.e., that case when ν is real and the whole module $(\mathfrak{g}^{\mathbb{C}}, K)$-module $I(\nu)$ is unitary and irreducible.

Theorem 4.2. *The complementary series $I(\nu)$ appears precisely in the range $\nu = \rho_{\mathfrak{g}} + \delta$, $|\delta| < \delta_0$,*

$$\delta_0 = \begin{cases} 1+b, & \mathfrak{g} = \mathfrak{su}(r+b, r), \\ 3, & \mathfrak{g} = \mathfrak{so}^*(2r), \\ n-3, & \mathfrak{g} = \mathfrak{so}(2,n),\ n \geq 4, \\ 3, & \mathfrak{g} = \mathfrak{e}_{6(-14)}, \\ 5, & \mathfrak{g} = \mathfrak{e}_{7(-25)}. \end{cases}$$

Proof. The abstract arguments in determining the complementary series here are the same as in [8,25,30], so we will only present some brief computations. Let ν be real. Suppose the $(\mathfrak{g}^{\mathbb{C}}, K)$-module $I(\nu)$ is irreducible with invariant Hermitian inner product $\langle \cdot, \cdot \rangle_\nu$. By Schur's Lemma we have $\langle f, f \rangle_\nu = S(\mu, l)\|f\|^2$ for all $f \in W_{\mu,l}$, where $\|f\|^2$ is the norm square in $L^2(K/L)$ and $S(\mu, l) = S(\nu, \mu, l)$ is the Schur proportionality constant. Then $\pi_\nu(\xi)$ is skew symmetric and in particular

$$\langle \pi_\nu(\xi)\phi_{\mu,l}, \phi_{\mu+\sigma,l+1} \rangle_\nu = -\langle \phi_{\mu,l}, \pi_\nu(\xi)\phi_{\mu+\sigma,l+1} \rangle_\nu.$$

Write the expansion of $\pi_\nu(\xi)\phi_{\mu,l}$ in Theorem 3.1 as

$$\pi_\nu(\xi)\phi_{\mu,l} = \sum_{\sigma_1,\sigma_2=\pm 1} A(\nu, \mu, l; \mu+\sigma, l+1)\phi_{\mu+\sigma,l+1} + \sum_{\sigma_1,\sigma_2=\pm 1} A(\nu, \mu, l; \mu+\sigma, l-1)\phi_{\mu+\sigma,l-1}.$$

Thus the invariance of the Hermitian form above becomes

$$A(\nu,\mu,l;\mu+\sigma,l+1)S(\mu+\sigma,l+1)=-A(\nu,\mu+\sigma,l+1;\mu,l)S(\mu+\sigma,l),$$
$$A(\nu,\mu,l;\mu+\sigma,l-1)S(\mu+\sigma,l-1)=-A(\nu,\mu+\sigma,l-1;\mu,l)S(\mu,l).$$

That $I(\nu)$ is unitary and irreducible is equivalent to all Schur proportionality constants $S(\mu,l)$ being positive. It implies that $A(\nu,\mu,l;\mu+\sigma,l+1)$ and $A(\nu,\mu+\sigma,l+1;\mu,l)$ have opposite signs, as well as $A(\nu,\mu,l;\mu+\sigma,l-1)$ and $A(\nu,\mu+\sigma,l-1;\mu,l)$. This determines the range of ν, given by the condition

$$|\delta|<\min\{\rho,\rho-2\rho_2,2\rho_1-\rho+2\}.$$

By case computations we get the range as claimed. □

Remark 4.3. The complementary series for $SU(p,q)$ has been found before in [13, (ii)(b), p. 49; 5.2, p. 69], [8, 4.4].

4.3. Composition series and unitarizable subrepresentations

The composition series for $I(\nu)$ at reducible points ν is a bit involved. We shall only determine the unitary subrepresentations at the reduction points $\nu=-2k$ in Theorem 4.1 (2) for $\mathfrak{g}\neq\mathfrak{su}(p,q)$; the case of $\mathfrak{g}=\mathfrak{su}(p,q)$ is studied in [8]. Note that the formulas for the K-type (μ_1,μ_2,l) in the composition series for $\mathfrak{g}\neq\mathfrak{su}(p,q)$ are somewhat simpler than $\mathfrak{su}(p,q)$. This is because there is a constraint $\mu_1\geq\mu_2$ if $\mathfrak{g}\neq\mathfrak{su}(p,q)$ whereas it disappears for $\mathfrak{g}=\mathfrak{su}(p,q)$.

The proof of the following result is done by examining the signs of the coefficients $\nu+\sigma_1(\mu_1+\rho_1)+\sigma_2(\mu_2+\rho_2)-\rho_1-\rho_2$ in Theorem 3.1.

Theorem 4.4. *Suppose $\nu=-2k$, $k\geq 1$, is an even integer. Then there is a composition series of $I(-2k)$ with K-types,*

$$0\subset\mathcal{M}_1=\{(\mu,l);\mu_1+\mu_2\leq 2k\}\subset\mathcal{M}_2=\{(\mu,l);\mu_1-\mu_2\leq 2k+2\rho_2\}\subset I(-2k).$$

The quotient $\mathcal{I}(-2k)/\mathcal{M}_2$ is unitarizable, sub-representation $\mathcal{M}_1$ and the sub-quotient $\mathcal{M}_2/\mathcal{M}_1$ are not unitarizable.

Remark 4.5. The anonymous referee raised the question of whether some sub-quotients of $I(\nu)$ are unitarizable highest weight representations. Indeed for the rank-one group $SU(p,1)$ some unitarizable highest weight representations can appear as sub-quotients of $I(\nu)$; the same is true for Siegel parabolic $Q=MAN$ with non-trivial representations of Levi component M [23]. However it seems that for higher rank groups and for Heisenberg parabolic $P=MAN$ one has to take infinite-dimensional unitary highest weight representations of M and study the induced representations from $P=MAN$ to G in order to realize the unitarizable highest weight representations; indeed M is a Hermitian Lie group and has unitary highest weight representations. See e.g. [13,14] for the case of $SU(2,p)$.

5. The cases of $\mathfrak{g} = \mathfrak{su}(d, 1)$ and $\mathfrak{g} = \mathfrak{sp}(r, \mathbb{R})$

We treat the remaining cases when $\mathfrak{g} = \mathfrak{sp}(r, \mathbb{R}), \mathfrak{su}(d, 1)$ where the space K/L_0 is the complex projective space $\mathbb{P}^{r-1}, \mathbb{P}^{d-1}$, respectively.

5.1. $\mathfrak{g} = \mathfrak{su}(d, 1)$

This case is already treated in [11] by using rather explicit differentiation of hypergeometric functions. We shall give somewhat easier proof of their results by using our method above; this avoids explicit computations involving special functions and gives conceptual expression for the action of $\pi_\nu(\xi)$ in terms of Harish-Chandra c-functions.

The Cartan decomposition is $\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \mathfrak{u}(d) + \mathbb{C}^d$, with $\mathfrak{k}^{\mathbb{C}} = \mathfrak{gl}(d)$. The Jordan triple system $V = \mathbb{C}^d$ with $\{x, y, z\} = D(x, y)z = \langle x, y\rangle z + \langle z, y\rangle x$. We fix the tripotent $e = e_1$, the standard basis vector of $\mathbb{C}^d$ and $\xi := \xi_e$. The half-sum $\rho_{\mathfrak{g}}$ is $\rho_{\mathfrak{g}} = d$. The space $S = K/L$ is the sphere S in $\mathbb{C}^d$ with L the isotropic subgroup of $e \in S$, and $S_1 = K/L_0$ the projective space $\mathbb{P}(\mathbb{C}^d)$ with $S \to S_1$ as a circle bundle over $\mathbb{P}(\mathbb{C}^d)$ by the defining map $z \mapsto [z]$. The tangent space of S_1 is realized as $T^{(1,0)}_{[e]} S_1 = \{D(v, e); v \in V_1 = \mathbb{C}^{d-1}\}$, $T^{(0,1)}_{[e]} S_1 = \{D(e, v); v \in V_1\}$. We fix an $\mathfrak{sl}(2)$-subalgebra in $\mathfrak{k}$ as

$$E^+ = D(e_2, e), \; E^- = D(e, e_2), E = E^+ - E^- \in T_{[e]}(K/L_0).$$

Put $H = [E^+, E^-] = D(e_2, e_2) - D(e, e)$. The element iE generates a Cartan subalgebra for the non-compact dual $(\mathfrak{k}_1^*, \mathfrak{l}_1)$, and positive roots are $\{2\alpha_1, \alpha_1\}$ with α_1 the dual element of iE, and the half-sum is $\rho_{\mathfrak{k}_1^*} = d - 1$.

The decomposition of $L^2(K/L)$ is well-known,

$$L^2(K/L) = \sum_{m \geq |l|, m = l \bmod 2} W_{m,l}.$$

Each space $W_{m,l}$ is generated by $\overline{z}_1{}^p z_2{}^q$, with

$$p = \frac{m + l}{2}, q = \frac{m - l}{2}, \quad m \geq |l|, m = l \mod 2.$$

Now the coefficients in the expansion of $\pi_\nu(\xi)\phi_{m,l}$ can be written as $\Big(\nu \pm (m + d - 1) + c\Big)$ for some constants c as in Theorem 3.1. We write them explicitly.

Theorem 5.1.

(1) *The action of* $\pi_\nu(\xi)$ *on* $\phi_{m,l}$ *is given by*

$$2^2 \pi_\nu(\xi)\phi_{m,l} = \Big(\nu + m + l\Big) c_{m,l}(m + 1, l + 1)\phi_{m+1,l+1}$$

$$+\Big(\nu - m - 2d + 2 + l\Big)c_{m,l}(m+1, l+1)\phi_{m-1,l+1}$$
$$+\Big(\nu + m - l\Big)c_{m,l}(m+1, l-1)\phi_{m+1,l-1}$$
$$+\Big(\nu - m - l - 2d + 2\Big)c_{m,l}(m-1, l-1)\phi_{m-1,l-1}.$$

(2) *The complementary series is in the range* $\nu = \rho_{\mathfrak{g}} + \delta$, $|\delta| < d = \rho_{\mathfrak{g}}$.

Proof. We follow the computations of $\pi_\nu(\xi)\phi_{m,l}$ in the proof of Theorem 3.1 and indicate the necessary changes. We find first the coefficient of $\phi_{m,l}$ and the other coefficients will be found by general arguments. We have $\pi_\nu(\xi)\phi_{m,l} = I + II + III$, $I = I^+ + I^-$,

$$I^+ = \frac{\nu}{2}\phi_{1,1}(k)\phi_{m,l}(k)$$

and the spherical polynomial $\phi_{1,1}(k)$ is

$$\phi_{1,1}(k) = \phi_{1,1}(z) = \langle ke, e\rangle = z_1, \quad z = ke \in S,$$

as a function on the sphere $S = K/L$. For $k = \exp(tE)$ we have $\phi_{1,1}(k) = \cos x$ and its complexification is $\cosh x = \frac{1}{2}(e^x + e^{-x})$. The expansion (3.14) now has coefficient $\frac{1}{2}$. Thus the leading term in I^+ is

$$\frac{\nu}{4}c_{m,l}(m+1, l+1)\phi_{m+1,l+1}$$

where $c_{m,l}(m+1, l+1)$ is a quotient of two Harish-Chandra c-functions.

The term II^+ in (3.17) in the present case becomes $II^+ = l\phi_{1,1}\phi_{m,l}$ and has the leading term

$$\frac{l}{2}c_{m,l}(m+1, l+1)\phi_{m+1,l+1}.$$

The third term III^+ is treated similarly as in Lemma 3.6, and (3.22) gives that the leading term of III^+ is $\frac{m-l}{4}c_{m,l}(m+1, l+1)\phi_{m+1,l+1}$. Altogether terms involving $\phi_{m+1,l+1}$ in $\pi_\nu(\xi)\phi_{m,l}$ are

$$\frac{\nu + 2l + m - l}{4}c_{m,l}(m+1, l+1)\phi_{m+1,l+1} = \frac{\nu + m + l}{4}c_{m,l}(m+1, l+1)\phi_{m+1,l+1}$$

with

$$\frac{\nu + m + l}{4} = \frac{\nu + (m + \rho_{\mathfrak{k}_1^*}) - \rho_{\mathfrak{k}_1^*} + l}{4}$$

in terms of the Weyl group invariant parameter $m + \rho_{\mathfrak{k}_1^*}$. Observe again that the highest weight with respect to E is $m\alpha_1$ so the Weyl group symmetry is with respect to $(m + \rho_{\mathfrak{k}_1^*})\alpha_1$. The coefficient of $\phi_{m+1,l+1}$ involving ν is

$$\nu + m + l = \nu + (m + \rho_{\mathfrak{k}_1^*}) - \rho_{\mathfrak{k}_1^*} + l.$$

Thus $\phi_{m-1,l+1}$ has coefficient

$$\nu - (m + \rho_{\mathfrak{k}_1^*}) - \rho_{\mathfrak{k}_1^*} + l = \nu - m - 2\rho_{\mathfrak{k}_1^*} + l = \nu - m - 2d + 2 + l.$$

The other two coefficient $\phi_{m\pm1,l-1}$ is found by the unitarity of $\pi_\nu(\xi)$ at $\nu = \rho_{\mathfrak{g}} + ix, x \in \mathbb{R}$ and by the Weyl group symmetry.

The rest on the complementary series is obtained as in [11]. □

Remark 5.2. The statement of [11, Theorem 4.1] for the group $SU(n,1)$ is (their $d = 2$)

$$\begin{aligned}\pi_\nu(H)e_{m,l} &= a_{m+1,l+1}(\nu + m + l)e_{m+1,l+1} + a_{m-1,l+1}(\nu - m + l - 2n + 2)e_{m-1,l+1}\\ &\quad + a_{m+1,l-1}(\nu + m - l)e_{m+1,l-1} + a_{m-1,l+1}(\nu - m - l - 2n + 2)e_{m-1,l-1}\end{aligned}$$

where $a_{m\pm1,l\pm1}$ are certain positive constants independent of ν. The important coefficients are $\nu+m+l$ for $e_{m+1,l+1}$ and $\nu+m-l$ for $e_{m+1,l-1}$, which have rather simple form, and the rest is obtained by Weyl group symmetry. This coincides with our formula. As mentioned above in Remark 3.2 it is enough to determine the leading coefficient $a_{m+1,l+1}(\nu+m+l)$.

5.2. $\mathfrak{g} = \mathfrak{sp}(r, \mathbb{R})$, $r \geq 3$

The Jordan triple system here is $V = M_r^s = \{v \in M_r(\mathbb{C}); v = v^t\}$ of complex matrices with the triple product $D(u, v)w = uv^*w + wv^*u$. This case is rather special and we provide all details. The normalization of the Euclidean norm in $\mathfrak{p}^+$ is as before with minimal tripotents having norm 1. The group $K = U(r)$ acts on V by $A \in K : Z \to AZA^t$. To avoid confusion with various realizations we recall that all Lie algebra elements are realized via Jordan triple products as in Section 1, in particular Lie algebra elements of $\mathfrak{k}^{\mathbb{C}} = \mathfrak{gl}(r, \mathbb{C})$ appear as $D(u, v) : w \to D(u, v)w$; $D(u, v)$ are identified with usual matrices uv^* if we still want matrix realizations.

We fix the minimal tripotent the diagonal matrix $e = \mathrm{diag}(1, 0, \cdots, 0) \in V$ and $\xi = \xi_e \in \mathfrak{p}$ in (1.2). The functional $\rho_{\mathfrak{g}}$ is now $\rho_{\mathfrak{g}} = r$. A subtle point here is that the group $L = M \cap K \subset K$ is

$$L = \mathbb{Z}_2 \times U(r - 1) = \{h = \mathrm{diag}(h_0, h_1); h_0 = \pm1, h \in U(r - 1)\}.$$

Let

$$L_0 = U(1) \times U(r - 1) = \{h = \mathrm{diag}(h_0, h_1); h_0 \in U(1), h \in U(r - 1)\}.$$

Thus $S = K/L \subset V$ is the real projective space $\mathbb{P}(\mathbb{R}^{2r}) \subset V$ and $S_1 = K/L_0$ is again the complex projective space $\mathbb{P}(\mathbb{C}^r) \subset \mathbb{P}(V)$.

The Peirce decomposition $V = V_2 + V_1 + V_0$ with respect to e is the block $(1 + r - 1) \times (1 \times r - 1)$-partition of V. We fix

$$v = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in V_1, \; w = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in V_0,$$

v is a tripotent of *rank two* and w of *rank one*. The Cartan decomposition of $\mathfrak{k}$ is $\mathfrak{k} = \mathfrak{u}(r-1) + \mathfrak{q}$ where elements in $\mathfrak{q} = \{D(u,e) - D(e,u); u \in V_1\} = \mathbb{C}^{r-1}$ as real spaces.

We fix also the following $\mathfrak{sl}(2)$ elements,

$$E = D(v,e) - D(e,v) = E^+ - E^- \in \mathfrak{q}, \quad H = [E^+, E^-] = D(v,v) - 2D(e,e) \tag{5.1}$$

with $[H, E^+] = 2E^+$. As matrices $D(e,v) = E_{12} \in \mathfrak{gl}(r, \mathbb{C})$. (Note the difference between this case and (2.9), and this has been studied in greater details in [22].) The symmetric pairs $(\mathfrak{k}^*, \mathfrak{l}_0) = \mathfrak{u}(r-1,1), \mathfrak{u}(r-1) + \mathfrak{u}(1))$, $(\mathfrak{k}_1^*, \mathfrak{l}_1) = (\mathfrak{su}(r-1,1), \mathfrak{s}(\mathfrak{u}(r-1) + \mathfrak{u}(1)))$, and the roots of $(\mathfrak{k}_1^*, iE)$ are 2 and 1 with $\rho_{\mathfrak{k}_1 *} = r - 1$, and $\rho_{\mathfrak{g}} = 1 + \rho_{\mathfrak{k}_1^*}$.

Lemma 2.5 in the present case is

$$L^2(K/L) = \sum_{m \geq |l|} W_{2m,2l}.$$

Here each space $W_{2m,2l}$ is generated by $\overline{z}_1{}^{(m+l)} z_2{}^{(m-l)}$ and contains the spherical polynomial $\phi_{2m,2l}$; it is the space of (p,q)-harmonic polynomials of even degree $2m$, $p = m + l$, $q = m - l$.

Theorem 5.3.

(1) *The action of* $\pi_\nu(\xi)$ *on* $\phi_{2m,2l}$ *is given by*

$$\begin{aligned} 2^3 \pi_\nu(\xi) \phi_{2m,2l} = &\Big(\nu + 2m\Big) c_{m,l}(m+1, l+1) \phi_{2m+2,2l+2} \\ &+ \Big(\nu - 2m - 2r + 2\Big) c_{m,l}(m-1, l+1) \phi_{2m-2,2l+2} \\ &+ \Big(\nu + 2m\Big) c_{m,l}(m+1, l-1) \phi_{2m+2,2l-2} \\ &+ \Big(\nu - 2m - 2r + 2\Big) c_{m,l}(m-1, l-1) \phi_{2m-2,2l-2} \\ &+ \Big(\nu - r\Big) c_{m,l}(m, l+1) \phi_{2m,2l+2} + \Big(\nu - r\Big) c_{m,l}(m, l-1) \phi_{2m,2l-2} \end{aligned}$$

(2) *There is no complementary series in the family* $I(\nu)$.

Proof. First we find the irreducible components in the action $X \otimes \phi \mapsto \pi_\nu(X)\phi$, $X \in \mathfrak{p}^\pm$, $\phi \in W_{2m,2l}$ as representations of $U(r)$. In the realization above $W_{2m,2l}$ is the space of $(p,q) = (m+1, m-l)$-spherical harmonics on the sphere in $\mathbb{C}^n$ of highest weight $p\epsilon_1 - q\epsilon_r$, with $\mathfrak{p}^+$ of highest weight $2\epsilon_1$. The tensor product $\mathfrak{p}^+ \otimes W_{2m,2l}$, written in terms of highest weights, is generically

$$2\epsilon_1 \otimes (p\epsilon_1 - q\epsilon_r) = (p+2)\epsilon_1 \oplus \Big((p+1)\epsilon_1 - (q-1)\epsilon_r\Big) \oplus \Big(p\epsilon_1 - (q-2)\epsilon_r\Big) \oplus \mathrm{REST}$$

with the term REST containing those summands which are not of the form $W_{2m,2l}$ and irrelevant to us. The linear term in ν in $\pi_\nu(X)\phi$ involves the pointwise product $\mathfrak{p}^+ \cdot W_{2m,2l} = W_{2,2} \cdot W_{2m,2l}$ of polynomials in $W_{2,2}$ and in $W_{2m,2l}$. It is a general fact [24, Theorem 12.4.4] that for $r \geq 3$,

$$W_{2,2} \cdot W_{2m,2l} = W_{2m+2,2l+2} + W_{2m-2,2l+2} + W_{2m,2l+2}$$

whenever each term makes sense. Thus $\pi_\nu(\xi)\phi_{2m,2l}$ is a sum of six terms

$$\begin{aligned} &a_{1,1}\phi_{2m+2,2l+2} + a_{-1,1}\phi_{2m-2,2l+2} + a_{1,-1}\phi_{2m+2,2l-2} + a_{-1,-1}\phi_{2m-2,2l-21} \\ &\quad + a_{0,1}\phi_{2m,2l+2} + a_{0,-1}\phi_{2m,2l-2}. \end{aligned}$$

Next we follow the earlier computations in Section 4 and consider the linear terms $I^\pm$ in ν, with

$$I^+ = \frac{\nu}{2}\langle ke, e\rangle \phi_{2m,2l}(k).$$

The matrix coefficient $\langle ke, e\rangle = \phi_{2,2}(k)$, and its restriction on the torus $\exp(\mathbb{R}E)e$ is

$$\exp(xE)e = \begin{bmatrix} \cos^2 x & \cos x \sin x \\ \cos x \sin x & \sin^2 x \end{bmatrix} = \cos^2 xe + \cos x \sin xv + \sin^2 xw. \tag{5.2}$$

Thus $\phi_{2,2}(k) = (\cos x)^2$ and it has an expansion

$$\phi_{2,2}(k) = (\cos x)^2 = \frac{1}{4}(e^{2ix} + 2 + e^{-2ix}).$$

The leading term in I^+ is

$$\frac{\nu}{8} c_{m,l}(m+1, l+1)\phi_{2m+2,2l+2}$$

where $c_{m,l}(m+1, l+1)$ is the quotient Harish-Chandra c-functions for $\phi_{2m+2,2l+2}$ and $\phi_{2m,2l}$.

In the second term II we have $II^+ = l\phi_{2,2}\phi_{2m,2l}$, which has the leading term

$$\frac{l}{4} c_{m,l}(m+1, l+1)\phi_{2m+2,2l+2}.$$

The vector field $X(k) = D(e, P_1(k^{-1}e))$ in (3.21) has restriction

$$X(\exp(xE)) = D(e, P_1(\exp(-xE)e)) = -\cos x \sin x D(e, v),$$

by (5.2). The leading term of the expansion $-\cos x \sin x E^{-}\phi_{2m,2l}$ is obtained, using the proof of Lemma A.1 in Appendix A, as

$$-\cos x \sin x E^{-}\phi_{2m,2l}(\exp(xE)) = \frac{2m-2l}{8}e^{i2mx} + L.O.T.$$

Altogether we find

$$\begin{aligned} I^{+} + II^{+} + III^{+} &= \frac{\nu + 2l + 2m - 2l}{8}c_{m,l}(m+1, l+1)\phi_{2m+2,2l+2} + L.O.T. \\ &= \frac{\nu + 2m}{8}c_{m,l}(m+1, l+1)\phi_{2m+2,2l+2} + L.O.T. \end{aligned}$$

By the same considerations using Weyl group symmetry we can find the first 4 coefficients of a's.

Finally to find the coefficients $a_{0,1}$ and $a_{0,-1}$ we use the results in [15]. It is proved there that $(I(\nu), \pi_\nu)$ for $\nu = r = \rho_{\mathfrak{g}}$ (in our parametrization) is reducible with two irreducible components consisting of $W_{2m,2l}$, $m + l = 0$ or $m + l = 1$ (mod 2), respectively. The coefficient $a_{0,1}$ is affine linear in ν, hence $a_{0,1} = 0$ for $\nu = r$ since the parity of $m+l$ changes from $W_{2m,2l}$ to $W_{2m,2l+2}$. Consequently $a_{0,1} = (\nu - r)c$ for some scalar constant c. This constant c is the coefficient of ν in the expansion of the product $\frac{\nu}{2}\langle ke, e\rangle\phi_{2m,2l}(k)$ of two spherical polynomials on the projective sphere K/L. The polynomial $\langle ke, e\rangle = z_1^2$, with $kL \in K/L$ being represented by z on the sphere. Thus $\frac{\nu}{2}\langle ke, e\rangle\phi_{2m,2l}(k) = \frac{\nu}{2}z_1^2\phi_{2m,2l}(z)$ can be found by repeatedly using the elementary expansion of $z_1\phi_{m',l'}(z)$, and we find the coefficient c of $\phi_{2m,2l+2}$ is positive whenever $m \geq |l+1|$. The same argument applies also to $a_{0,-1}$.

The rest is done as in the proof of Theorem 3.1 above. □

Remark 5.4. Even integers $\nu = -2m$ are reduction points for $I(\nu)$ in both cases above, $\mathfrak{g} = \mathfrak{su}(d,1), \mathfrak{sp}(r,\mathbb{R})$. The map (4.1) realizes the leading component in $S^m(\mathfrak{g}^{\mathbb{C}})$ as a subrepresentations of $I(\nu)$. The coeffcient $(\nu + 2m)$ in the above theorem can also be obtained using this result.

As mentioned above it might be interesting to study induced representations $Ind_{MAN}^{G}(\tau \otimes e^{\nu} \otimes 1)$ with τ being a unitary highest weight representation of $M = \mathbb{Z}_2 \times Sp(r-1,\mathbb{R})$.

Acknowledgments

I would like to thank Jan Frahm and Clemens Weiske for reading versions of this paper and for discussions on the relevance of the results in [15] with my study in Section 5 which led me to find two terms being missed in Theorem 5.3 in an earlier version. I'm also grateful to Dan Barbarsch for explaining his work on unitary spherical duals of classical groups and Harald Upmeier for explanation of Jordan quadrangles. I thank the anonymous referee for a very thorough reading of an earlier version of this paper and for offering many expert suggestions and comments.

Appendix A. Recursion formulas for differentiations of spherical polynomials

Let

$$SU(2) = \left\{ g = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}; |a|^2 + |b|^2 = 1 \right\}$$

and fix the following Lie algebra $sl(2, \mathbb{C})$ elements,

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, E^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E = E^+ + E^- \in i\mathfrak{su}(2).$$

Then the $\mathfrak{sl}(2)$-algebras $\mathfrak{sl}(2) = \mathbb{C}H_j + \mathbb{C}E_j^+ + \mathbb{C}E_j^-, j = 1, 2$, defined in (2.9) are isomorphic to the present $\mathfrak{sl}(2)$ via the identification

$$H_j \longleftrightarrow H, E_j^\pm \longleftrightarrow E^\pm;$$

as well as the identification of the compact $\mathfrak{su}(2)$-real forms

$$\mathfrak{su}(2) = \mathbb{R}iH_j + \mathbb{R}E_j + \mathbb{R}(i(E_j^+ + E_j^-)) \longleftrightarrow \mathbb{R}iH + \mathbb{R}E + \mathbb{R}(i(E^+ + E^-)).$$

Let

$$U(1) = \left\{ u_\theta = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} = e^{i\theta H} \right\}.$$

Recall $\chi_l(H_j) = -l$ and the transformation rule (2.22) of $\phi_{\mu,l}$ under $\exp(itH_j)$. Accordingly we let $\chi_l(H) = -l$, and the spherical polynomials $\phi_{m,l}$ in the present case satisfy $\phi_{m,l}(e^{i\theta H}g) = \phi_{m,l}(ge^{i\theta H}) = e^{-il\theta}\phi_{m,l}(g)$. Consider the representation of $SL(2, \mathbb{C})$ on the symmetric tensor $S^m := \odot^m \mathbb{C}^2$ of the defining representation $\mathbb{C}^2$, and write the action simply as $g \to gv$, $v \in S^m\mathbb{C}^2$, as well as the Lie algebra action. Let $-m \le l \le m$, $m = l$ mod 2. The l-spherical polynomial is given by the matrix coefficient

$$\phi_{m,l}(g) = \binom{m}{k} \langle g(e_1^k e_2^{m-k}), e_1^k e_2^{m-k} \rangle, 0 \le k \le m,\ l = m - 2k, \tag{A.1}$$

where the tensor $e_1^k e_2^{m-k}$ is as usual viewed as polynomial on the dual space of $\mathbb{C}^2$, namely the polynomial $z_1^k z_2^{m-}$ on the dual space. This is verified by

$$\begin{aligned} \phi_{m,l}(gu_\theta) = \phi_{m,l}(ge^{i\theta H}) &= \langle ge^{i\theta H}(e_1^k e_2^{m-k}), e_1^k e_2^{m-k} \rangle = e^{i(2k-m)\theta}\phi_{m,l}(g) \\ &= e^{-il\theta}\phi_{m,l}(g) = \chi_l(u_\theta)\phi_{m,l}(g), \end{aligned} \tag{A.2}$$

also $\phi_{m,l}(u_\theta g) = \chi_l(u_\theta)\phi_{m,l}(g)$ along with the normalization $\phi_{m,l}(I) = \binom{m}{k}\|e_1^k e_2^{m-k}\|^2 = 1$. For our purpose in Section 3 it is more convenient to consider the spherical polynomial

$$\psi_{m,l}(g) = \frac{2^m}{\binom{m}{k}}\phi_{m,l}(g) = 2^m\langle g(e_1^k e_2^{m-k}), e_1^k e_2^{m-k}\rangle,\ l = m - 2k, \tag{A.3}$$

which has the normalization the leading term of $\psi_{m,l}(\exp tE)$ being e^{imt}, i.e.

$$\psi_{m,l}(\exp tE) = e^{imt} + L.O.T.,$$

where L.O.T. is a trigonometric polynomial of lower order. The following lemma is used in the proof of Lemma 3.6. Actually we need only to find the leading term of the trigonometric polynomials $\langle ge_1, e_2\rangle(E^-\phi_{m,l})(g)$ and $\langle ge_2, e_1\rangle(E^+\phi_{m,l})(g)$ for $g = \exp(tE)$, which is elementary.

Lemma A.1. *The following recurrence formulas hold,*

$$\langle ge_1, e_2\rangle(E^-\phi_{m,l})(g) = \frac{1}{4}\frac{(m-l)(m+l+2)}{m+1}\left(\phi_{m+1,l+1} - \phi_{m-1,l+1}\right), \tag{A.4}$$

$$\langle ge_2, e_1\rangle(E^+\phi_{m,l})(g) = \frac{1}{4}\frac{(m+l)(m-l+2)}{m+1}\left(\phi_{m+1,l-1} - \phi_{m-1,l-1}\right). \tag{A.5}$$

When restricted to $g = \exp(xE) = \begin{bmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{bmatrix}$ *and written in terms of* $\psi_{m,l}$ *they are*

$$-\sin x(E^-\psi_{m,l})(g) = \frac{1}{4}(m-l)\psi_{m+1,l+1} - \frac{1}{4}\frac{(m+l+2)(m-l)^2}{m(m+1)}\psi_{m-1,l+1}, \tag{A.6}$$

$$\sin x(E^+\psi_{m,l})(g) = \frac{1}{4}(m+l)\psi_{m+1,l-1}(g) - \frac{1}{4}\frac{(m-l+2)(m+l)}{m(m+1}\psi_{m-1,l-1}(g). \tag{A.7}$$

Proof. We prove (A.5) and (A.4) is proved by similar computations. First we find the weight of $f(g) = \langle ge_2, e_1\rangle(E^+\psi_{m,l})(g)$ under the regular left and right actions of $\exp(i\theta H)$. We have E^+ is of weight 2 under $\mathrm{ad}(H)$. Also the matrix coefficient $\langle ge_2, e_1\rangle$ transforms as $\langle ge^{i\theta H}e_2, e_1\rangle = e^{-i\theta}\langle ge_2, e_1\rangle$, thus it is of weight -1 under the right regular action of H. The character χ_l is defined by $\chi_l(H) = -l$ thus $f(g) = \langle ge_2, e_1\rangle(E^+\psi_{m,l})(g)$ is of weight $-l+2-1 = -(l-1) = \chi_{l-1}(H)$ in the sense $f(g\exp(i\theta H)) = \chi_{l-1}(\exp(i\theta H))f(g)$. Similarly $\langle\exp(i\theta H)ge_2, e_1\rangle = e^{i\theta}\langle ge_2, e_1\rangle$ and the right differentiation by E^+ commutes with the left action. Thus $f(\exp(i\theta H)g) = e^{-i(l-1)\theta}f(g)$, and $f(g)$ is a linear combination of $\phi_{m+1,l+1}$ and $\phi_{m-1,l+1}$ as it is the matrix coefficient of $\mathbb{C}^2\otimes S^m = S^{m+1}\oplus S^{m-1}\mathbb{C}^2$,

$$f(g) = A\phi_{m+1,l-1} + B\phi_{m-1,l-1}, \tag{A.8}$$

with some unknown constants A, B.

We have $\phi_{m,l}(g) = \binom{m}{k}\langle g(e_1^k e_2^{m-k}), e_1^k e_2^{m-k}\rangle$, and

$$(E^+\phi_{m,l})(g) = \binom{m}{k}\langle g(E^+(e_1^k e_2^{m-k})), e_1^k e_2^{m-k}\rangle = \binom{m}{k}(m-k)\langle g(e_1^{k+1}e_2^{m-k-1}), e_1^k e_2^{m-k}\rangle,$$

since $E^+ e_1^k e_2^{m-k} = (m-k) e_1^{k+1} e_2^{m-k-1}$. Its inner product with $\langle ge_2, e_1\rangle$ is

$$\langle ge_2, e_1\rangle \langle g(e_1^{k+1} e_2^{m-k-1}), e_1^k e_2^{m-k}\rangle = \langle g(e_2 \otimes e_1^{k+1} e_2^{m-k-1}), e_1 \otimes e_1^k e_2^{m-k}\rangle.$$

Both $e_1 \otimes e_1^k e_2^{m-k}$ and $e_2 \otimes e_1^{k+1} e_2^{m-k-1}$ are of weight $i(1+2k-m)$ under H, the corresponding weight vector in the space S^{m+1} respectively S^{m-1} is $e_1^{k+1} e_2^{m-k}$ resp. $e_1^k e_2^{m-k-1}$ with matrix coefficient $\langle g(e_1^{k+1} e_2^{m-k}), e_1^{k+1} e_2^{m-k}\rangle$, $\langle g(e_1^k e_2^{m-k-1}), e_1^k e_2^{m-k-1}\rangle$. In view of (A.1) the formula (A.8) becomes

$$\begin{aligned} &\binom{m}{k}(m-k)\langle g(e_2 \otimes e_1^{k+1} e_2^{m-k-1}), e_1 \otimes e_1^k e_2^{m-k}\rangle \\ &= A\binom{m+1}{k+1}\langle g(e_1^{k+1} e_2^{m-k}), e_1^{k+1} e_2^{m-k}\rangle + B\binom{m-1}{k}\langle g(e_1^k e_2^{m-k-1}), e_1^k e_2^{m-k-1}\rangle. \end{aligned} \tag{A.9}$$

Evaluating at $g = I$ we get $B = -A$. Next we specify the equality to the self adjoint element

$$g = \begin{bmatrix} \operatorname{ch} x & \operatorname{sh} x \\ \operatorname{sh} x & \operatorname{ch} x \end{bmatrix} = \begin{bmatrix} \operatorname{ch} \frac{x}{2} & \operatorname{sh} \frac{x}{2} \\ \operatorname{sh} \frac{x}{2} & \operatorname{ch} \frac{x}{2} \end{bmatrix}^2 =: h^2$$

and look for the coefficients of $e^{(m+1)x}$. We have

$$\langle g(e_1 \otimes e_1^k e_2^{m-k}), e_2 \otimes e_1^{k+1} e_2^{m-k-1}\rangle = \langle h(e_1 \otimes e_1^k e_2^{m-k}), h(e_2 \otimes e_1^{k+1} e_2^{m-k-1})\rangle,$$

and its leading term is

$$\frac{e^{(m+1)x}}{2^{2(m+1)}}\langle (e_1+e_2)^{m+1}, (e_1+e_2)^{m+1}\rangle = \frac{e^{(m+1)x}}{2^{m+1}},$$

and the LHS has leading term $\binom{m}{k}(m-k)\frac{e^{(m+1)x}}{2^{m+1}}$. The term $e^{(m+1)x}$ appears only in the first summand in the RHS which has leading term $A\binom{m+1}{k+1}\frac{e^{(m+1)x}}{2^{m+1}}$. Thus

$$A = \frac{\binom{m}{k}(m-k)}{\binom{m+1}{k+1}} = \frac{(m-k)(k+1)}{m+1} = \frac{(m+l)(m-l+2)}{4(m+1)}.$$

This proves (A.5). $\square$

Remark A.2. In terms of $\psi_{m,l}$ they become

$$\langle ge_1, e_2\rangle (E^- \psi_{m,l})(g) = \frac{1}{4}(m-l)\psi_{m+1,l+1} - \frac{1}{4}\frac{(m+l+2)(m+l)^2}{m(m+1)}\psi_{m-1,l+1}, \tag{A.10}$$

$$\langle ge_2, e_1\rangle (E^+ \psi_{m,l})(g) = \frac{1}{4}(m+l)\psi_{m+1,l-1} - \frac{1}{4}\frac{(m+2-l)(m+l)^2}{m(m+1)}\psi_{m+1,l-1}. \tag{A.11}$$

Table 1
Non-compact Hermitian symmetric spaces $D = G/K$.

$D = G/K$	G	K	(a, b)
$I_{r+b,r}$	$SU(r+b,r))$	$S(U(r+b) \times U(r))$	$(2,b)$
II_{2r}	$SO^*(4r)$	$U(2r)$	$(4,0)$
II_{2r+1}	$SO^*(4r+2)$	$U(2r+1)$	$(4,2)$
III_r	$Sp(r,\mathbb{R})$	$U(r)$	$(1,0)$
$IV_n, n > 4.(r=2)$	$SO(n,2)$	$SO(n) \times SO(2)$	$(n-2,0)$
$V(r=2)$	$E_{6(-14)}$	$Spin(10) \times SO(2)$	$(6,4)$
$VI(r=3)$	$E_{7(-25)}$	$E_6 \times SO(2)$	$(8,0)$

Table 2
The compact Hermitian symmetric spaces $\mathbb{P}(S) = K/L_0 = K'/L'$. For type I domain $I_{r,r+b}$, $r \geq 2$, $\mathbb{P}(S)$ is a product $\mathbb{P}^{r-1} \times \mathbb{P}^{r+b-1}$ of projective spaces with the corresponding (a_1, b_1) being $(0, r+b-2), (0, r-2)$ for each factor.

$D = G/K$	$\mathbb{P}(S) = K/L_0 = K_1/L_1$	(a_1, b_1)
$I_{r+b,r}$	$I^*_{r+b-1} \times I^*_{r-1}$	$(0, r+b-2), (0, r-2)$
II_{2r}	$I^*_{2,2r-2}$	$(2, 2r-4)$
II_{2r+1}	$I^*_{2,2r-1}$	$(2, 2r-3)$
III_r	I^*_{r-1}	$(0, r-2)$
$IV_n, n > 4$	IV^*_{n-2}	$(n-4, 0)$
V	II^*_5	$(4, 2)$
VI	V^*	$(6, 4)$

The spherical polynomial $\phi_{m,l}$ is a special case of the spherical function $\Phi_{\lambda,l}$ with $\lambda = -i(m+\rho) = -i(m+1)$ in our case, and $\Phi_{\lambda,l}$ is invariant with respect to the Weyl group action $\lambda \to -\lambda$. Namely, the pair of the coefficients

$$\pm\frac{1}{4}\frac{(m-l))(m+l+2)}{m+1} = \pm\frac{1}{4}\frac{((m+1)-(l+1))((m+1)+(l+1))}{m+1}$$

in the lemma is invariant by the change $m+1 \to -(m+1)$ and this symmetry is indeed obvious here. These formulas are all classical trigonometric identities and can be obtained by other methods.

Appendix B. Table of Hermitian symmetric spaces G/K and their varieties of minimal rational tangents K/L_0. Duality relation for $(\dim(X), \mathrm{genus}(X))$ for $X = G/K, K/L_0$

B.1. Tables

We give a list of G/K and the corresponding projective space $S_1 = \mathbb{P}(S) = K/L_0 = K_1/L_1$ as compact Hermitian symmetric space; see [5,19,9]. The compact dual of a non-compact Hermitian symmetric space D is denoted by D^*.

B.2. Duality between (d, p) and (d_1, p_1) for G/K and K/L

Let $d = \dim_{\mathbb{C}} D = r + \frac{1}{2}ar(r-1) + rb$, $p = 2 + a(r-1) + b$, the dimension and the genus of D. In terms of Lie algebra actions they are

$$(d, p) = (\operatorname{tr}\operatorname{ad}(-iZ)|_{\mathfrak{p}^+}, \operatorname{tr}\operatorname{ad}(D(e,e))|_{\mathfrak{p}^+})$$

where $D(e,e)$ is the Harish-Chandra co-root of γ_1, $\gamma_1(D(e,e)) = 2$. Similarly let

$$(d_1, p_1) = (\dim(K_1^*/L), \operatorname{genus}(K_1^*/L))$$

if $D \neq SU(r, r+b)/S(U(r) \times U(r+b))$. Put

$$d' = \dim(\mathbb{P}^{r-1}) = r - 1, p' = r; d'' = \dim(\mathbb{P}^{r+b-1}) = r + b - 1, p'' = r + b$$

when $D = SU(r, r+b)/S(U(r) \times U(r+b))$.

The following duality between the pairs $(\dim(D), \operatorname{genus}(D))$ and $(\dim(K/L_0), \operatorname{genus}(K/L_0))$ is mentioned in Lemma 2.5 and might be of independent interest. It can be proved by trace computations or by case-by-case computations of the tables above.

Lemma B.1.

(1) *Let D be of rank $r \geq 2$ and is one of the domains II, IV, V, VI. Then*

$$\frac{p}{d} + \frac{d_1}{p_1} = 2.$$

(2) *Let D be of Type I with $r \geq 2$. Then*

$$\frac{p}{d} + \frac{d'}{p'} + \frac{d''}{p''} = 2.$$

(3) *Let D be the Siegel domain II. Then*

$$\frac{p}{d} + 2\frac{d_1}{p_1} = 2.$$

References

[1] D. Barbasch, The unitary spherical spectrum for split classical groups, J. Inst. Math. Jussieu 9 (2) (2010) 265–356.
[2] T. Branson, G. Ólafsson, B. Ørsted, Spectrum generating operators and intertwining operators for representations induced from a maximal parabolic subgroup, J. Funct. Anal. 135 (1996) 163–205.
[3] J. Frahm, Conformally invariant differential operators on Heisenberg groups and minimal representations, preprint, arXiv:2012.05952v1.
[4] J. Frahm, C. Weiske, G. Zhang, Principal series for Hermitian Lie groups induced from Heisenberg parabolic subgroups and intertwining operators, in preparation.

[5] S. Helgason, Groups and Geometric Analysis, Academic Press, New York, London, 1984.
[6] S. Helgason, Geometric Analysis on Symmetric Spaces, Mathematical Surveys and Monographs, vol. 39, American Mathematical Society, Providence, RI, 1994.
[7] R. Howe, Some recent applications of induced representations, in: Group Representations, Ergodic Theory, and Mathematical Physics: a Tribute to George W. Mackey, in: Contemp. Math., vol. 449, 2008, pp. 173–191.
[8] R. Howe, E. Tan, Homogeneous functions on light cones: the infinitesimal structure of some degenerate principal series representations, Bull. Am. Math. Soc. 28 (1) (1993) 1–74.
[9] J.-M. Hwang, N. Mok, Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation, Invent. Math. 131 (1998) 393–418.
[10] K.D. Johnson, Degenerate principal series and compact groups, Math. Ann. 287 (1990) 703–718.
[11] K.D. Johnson, N.R. Wallach, Composition series and intertwining operators for the spherical principal series. I, Trans. Am. Math. Soc. 229 (1977) 137–173.
[12] A. Knapp, Representation Theory of Semisimple Groups. An Overview Based on Examples, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 2001.
[13] A. Knapp, B. Speh, Irreducible unitary representations of $SU(2,2)$, J. Funct. Anal. 45 (1982) 41–73.
[14] A. Knapp, B. Speh, The role of basic cases in classification: theorems about unitary representations applicable to SU(N,2), in: Noncommutative Harmonic Analysis and Lie Groups, Marseille, 1982, in: Lecture Notes in Math., vol. 1020, Springer, 1983, pp. 119–160.
[15] T. Kobayashi, B. Ørsted, M. Pevzner, Geometric analysis on small unitary representations of $GL(N,\mathbb{R})$, J. Funct. Anal. 260 (6) (2011) 1682–1720.
[16] S.T. Lee, Covariants of $Sp_n(\mathbb{C})$ and degenerate principal series of $GL_n(\mathbb{H})$, J. Funct. Anal. 253 (1) (2007) 18–42.
[17] S.T. Lee, H.Y. Loke, Degenerate principal series representations of Sp(p,q), Isr. J. Math. 137 (2003) 355–379.
[18] S.T. Lee, C.-B. Zhu, Degenerate principal series and local theta correspondence, Trans. Am. Math. Soc. 350 (1998) 5017–5046.
[19] O. Loos, Bounded Symmetric Domains and Jordan Pairs, University of California, Irvine, 1977.
[20] N. Mok, Geometric structures on uniruled projective manifolds defined by their varieties of minimal rational tangents, Astérisque 322 (2008) 151–205, vol. II.
[21] J. Möllers, B. Schwarz, Structure of the degenerate principal series on symmetric R-spaces and small representations, J. Funct. Anal. 266 (2014) 3508–3542.
[22] E. Neher, Jordan Triple Systems by the Grid Approach, Lect. Notes in Math., vol. 1280, Springer, 1987.
[23] B. Ørsted, G. Zhang, Generalized principal series representations and tube domains, Duke Math. J. 78 (1995) 335–357.
[24] W. Rudin, Function Theory in the unit ball of $\mathbb{C}^n$, Springer-Verlag, Berlin, Heidelberg, 2008.
[25] S. Sahi, Jordan algebras and degenerate principal series, J. Reine Angew. Math. 462 (1995) 1–18.
[26] H. Schlichtkrull, One-dimensional K-types in finite dimensional representations of semisimple Lie groups: a generalization of Helgason's theorem, Math. Scand. 54 (1984) 279–294.
[27] N. Shimeno, The Plancherel formula for spherical functions with a one-dimensional K-type on a simply connected simple Lie group of Hermitian type, J. Funct. Anal. 121 (1994) 330–388.
[28] H. Upmeier, Harald Jordan algebras and harmonic analysis on symmetric spaces, Am. J. Math. 108 (1) (1986) 1–25.
[29] L. Vretare, Elementary spherical functions on symmetric spaces, Math. Scand. 39 (1976) 343–358.
[30] G. Zhang, Jordan algebras and generalized principal series representations, Math. Ann. 302 (1995) 773–786.
[31] G. Zhang, Some recurrence formulas for spherical polynomials on tube domains, Trans. Am. Math. Soc. 347 (1995) 1725–1734.