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# On the maximal operator of a general Ornstein–Uhlenbeck semigroup

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**Abstract**
If $Q$ is a real, symmetric and positive definite $n \times n$ matrix, and $B$ a real $n \times n$ matrix whose eigenvalues have negative real parts, we consider the Ornstein–Uhlenbeck semigroup on $\mathbb{R}^n$ with covariance $Q$ and drift matrix $B$. Our main result says that the associated maximal operator is of weak type $(1, 1)$ with respect to the invariant measure. The proof has a geometric gist and hinges on the "forbidden zones method" previously introduced by the third author.

**Keywords** Ornstein–Uhlenbeck semigroup · Maximal operator · Gaussian measure · Mehler kernel · Weak type (1,1)

**Mathematics Subject Classification** 47D03 · 42B25

## 1 Introduction

In this paper we prove a weak type $(1, 1)$ theorem for the maximal operator associated to a general Ornstein–Uhlenbeck semigroup. We extend the proof given by the third author in 1983 in a symmetric context. Our setting is the following.

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In $\mathbb{R}^n$ we will consider the semigroup generated by the elliptic operator

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{n} q_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j=1}^{n} b_{ij}x_i\frac{\partial}{\partial x_j},$$

or, equivalently,

$$\mathcal{L} = \frac{1}{2}\operatorname{tr}\big(Q\nabla^2\big) + \langle Bx, \nabla\rangle,$$

where $\nabla$ is the gradient and $\nabla^2$ the Hessian. Here $Q = (q_{ij})$ is a real, symmetric and positive definite $n \times n$ matrix, indicating the covariance of $\mathcal{L}$. The real $n \times n$ matrix $B = (b_{ij})$ is negative in the sense that all its eigenvalues have negative real parts, and it gives the drift of $\mathcal{L}$.

The semigroup is formally $\mathcal{H}_t = e^{t\mathcal{L}}$, $t > 0$, but to write it more explicitly we first introduce the positive definite, symmetric matrices

$$Q_t = \int_0^t e^{sB}Qe^{sB^*}\,ds, \qquad 0 < t \leq +\infty, \tag{1.1}$$

and the normalized Gaussian measures $\gamma_t$ in $\mathbb{R}^n$, with $t \in (0, +\infty]$, having density

$$y \mapsto (2\pi)^{-\frac{n}{2}}(\det Q_t)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\langle Q_t^{-1}y, y\rangle\right)$$

with respect to Lebesgue measure. Then for functions $f$ in the space of bounded continuous functions in $\mathbb{R}^n$ one has

$$\mathcal{H}_t f(x) = \int f(e^{tB}x - y)\,d\gamma_t(y)\,, \quad x \in \mathbb{R}^n\,, \tag{1.2}$$

a formula due to Kolmogorov. The measure $\gamma_\infty$ is invariant under the action of $\mathcal{H}_t$; it will be our basic measure, replacing Lebesgue measure.

We remark that $\big(\mathcal{H}_t\big)_{t>0}$ is the transition semigroup of the stochastic process

$$\chi(x,t) = e^{tB} + \int_0^t e^{(t-s)B}\,dW(s),$$

where $W$ is a Brownian motion in $\mathbb{R}^n$ with covariance $Q$.

We are interested in the maximal operator defined as

$$\mathcal{H}_* f(x) = \sup_{t>0}\big|\mathcal{H}_t f(x)\big|.$$

Under the above assumptions on $Q$ and $B$, our main result is the following.

**Theorem 1.1** *The Ornstein–Uhlenbeck maximal operator $\mathcal{H}_*$ is of weak type $(1,1)$ with respect to the invariant measure $\gamma_\infty$, with an operator quasinorm that depends only on the dimension and the matrices $Q$ and $B$.*

In other words, the inequality

$$\gamma_\infty\{x \in \mathbb{R}^n : \mathcal{H}_* f(x) > \alpha\} \leq \frac{C}{\alpha}\,\|f\|_{L^1(\gamma_\infty)}, \qquad \alpha > 0, \tag{1.3}$$

holds for all functions $f \in L^1(\gamma_\infty)$, with $C = C(n, Q, B)$.

For large values of the time parameter, we also obtain a refinement of this result. Indeed, we prove in Proposition 6.1 that

$$\gamma_\infty \left\{ x \in \mathbb{R}^n : \sup_{t>1} |\mathcal{H}_t f(x)| > \alpha \right\} \leq \frac{C}{\alpha \sqrt{\log \alpha}} \tag{1.4}$$

for large $\alpha > 0$ and all normalized functions $f \in L^1(\gamma_\infty)$. Here $C = C(n, Q, B)$, and this estimate is shown to be sharp. It cannot be extended to $\mathcal{H}_*$, since the maximal operator corresponding to small values of $t$ only satisfies the ordinary weak type inequality. This sharpening is not surprising, in the light of some recent results for the standard case $Q = I$ and $B = -I$ by Lehec [8]. He proved the following conjecture, proposed by Ball, Barthe, Bednorz, Oleszkiewicz and Wolff [2]:

For each fixed $t > 0$, there exists a function $\psi_t = \psi_t(\alpha)$, with $\lim_{\alpha \to +\infty} \psi_t(\alpha) = 0$, satisfying

$$\gamma_\infty\{x \in \mathbb{R}^n : |\mathcal{H}_t f(x)| > \alpha\} \leq \frac{\psi_t(\alpha)}{\alpha}$$

for all large $\alpha > 0$ and all $f \in L^1(\gamma_\infty)$ such that $\|f\|_{L^1(\gamma_\infty)} = 1$. Lehec proved this conjecture with $\psi_t(\alpha) = C(t)/\sqrt{\log \alpha}$ independent of the dimension, and this $\psi_t$ is sharp. Our estimates depend strongly on the dimension $n$, but on the other hand we estimate the supremum over large $t$.

The history of $\mathcal{H}_*$ is quite long and started with the first attempts to prove $L^p$ estimates. When $\left(\mathcal{H}_t\right)_{t>0}$ is symmetric, i.e., when each operator $\mathcal{H}_t$ is self-adjoint on $L^2(\gamma_\infty)$, then $\mathcal{H}_*$ is bounded on $L^p(\gamma_\infty)$ for $1 < p \leq \infty$, as a consequence of the general Littlewood–Paley–Stein theory for symmetric semigroups of contractions on $L^p$ spaces [16, Ch. III].

It is easy to see that the maximal operator is unbounded on $L^1(\gamma_\infty)$. This led, about fifty years ago, to the study of the weak type (1, 1) of $\mathcal{H}_*$ with respect to $\gamma_\infty$. The first positive result is due to B. Muckenhoupt [13], who proved the estimate (1.3) in the one-dimensional case with $Q = I$ and $B = -I$. The analogous question in the higher-dimensional case was an open problem until 1983, when the third author [15] proved the weak type (1, 1) in any finite dimension. Other proofs are due to Menárguez, Pérez and Soria [11] (see also [10, 14]) and to Garcìa-Cuerva, Mauceri, Meda, Sjögren and Torrea [7]. Moreover, a different proof of the weak type (1, 1) of $\mathcal{H}_*$, based on a covering lemma halfway between covering results by Besicovitch and Wiener, was given by Aimar, Forzani and Scotto [1]. A nice overview of the literature may be found in [17, Ch.4].

In [4] the present authors recently considered a normal Ornstein–Uhlenbeck semigroup in $\mathbb{R}^n$, that is, we assumed that $\mathcal{H}_t$ is for each $t > 0$ a normal operator on $L^2(\gamma_\infty)$. Under this extra assumption, we proved that the associated maximal operator is of weak type (1, 1) with respect to the invariant measure $\gamma_\infty$. This extends earlier work in the non-symmetric framework by Mauceri and Noselli [9], who proved that if $Q = I$ and $B = \lambda(R - I)$ for some positive $\lambda$ and a real skew-symmetric matrix $R$ generating a periodic group, then the maximal operator $\mathcal{H}_*$ is of weak type (1, 1).

In Theorem 1.1 we go beyond the hypothesis of normality. The proof has a geometric core and relies on the *ad hoc* technique developed by the third author in [15]. It is worth noticing that, while the proof in [4] required an analysis of the special case when $Q = I$ and $B = (-\lambda_1, \dots, -\lambda_n)$, with $\lambda_j > 0$ for $j = 1, \dots, n$, and then the application of factorization results, we apply here directly, avoiding many intermediate steps, the "forbidden zones" technique introduced in [15].

Since the maximal operator $\mathcal{H}_*$ is trivially bounded from $L^\infty$ to $L^\infty$, we obtain by interpolation the following corollary.

**Corollary 1.2** *The Ornstein–Uhlenbeck maximal operator $\mathcal{H}_*$ is bounded on $L^p(\gamma_\infty)$ for all $p > 1$.*

This result improves Theorem 4.2 in [9], where the $L^p$ boundedness of $\mathcal{H}_*$ is proved for all $p > 1$ in the normal framework, under the additional assumption that the infinitesimal generator of $\left(\mathcal{H}_t\right)_{t>0}$ is a sectorial operator of angle less than $\pi/2$.

In this paper we focus our attention on the Ornstein–Uhlenbeck semigroup in $\mathbb{R}^n$. In view of possible applications to stochastic analysis and to SPDE's, it would be very interesting to investigate the case of the infinite-dimensional Ornstein-Uhlenbeck maximal operator as well (see [3, 6, 18] for an introduction to the infinite-dimensional setting). The Riesz transforms associated to a general Ornstein–Uhlenbeck semigroup in $\mathbb{R}^n$ have been studied in the authors' paper [5].

The scheme of the paper is as follows. In Sect. 2 we introduce the Mehler kernel $K_t(x, u)$, that is, the integral kernel of $\mathcal{H}_t$. Some estimates for the norm and the determinant of $Q_t$ and related matrices are provided in Sect. 3. As a consequence, we obtain bounds for the Mehler kernel. In Sect. 4 we consider the relevant geometric features of the problem, and introduce in Sect. 4.1 a system of polar-like coordinates. We also express Lebesgue measure in terms of these coordinates. Sections 5, 6, 7 and 8 are devoted to the proof of Theorem 1.1. First, Sect. 5 introduces some preliminary simplifications of the proof; in particular, we restrict the variable $x$ to an ellipsoidal annulus. In Sect. 6 we consider the supremum in the definition of the maximal operator taken only over $t > 1$ and prove the sharp estimate (1.4). Section 7 is devoted to the case of small $t$ under an additional local condition. Finally, in Sect. 8 we treat the remaining case and conclude the proof of Theorem 1.1, by proving the estimate (1.3) for small $t$ under a global assumption.

In the following, we use the "variable constant convention", according to which the symbols $c > 0$ and $C < \infty$ will denote constants which are not necessarily equal at different occurrences. They all depend only on the dimension and on $Q$ and $B$. For any two nonnegative quantities $a$ and $b$ we write $a \lesssim b$ instead of $a \le Cb$ and $a \gtrsim b$ instead of $a \ge cb$. The symbol $a \simeq b$ means that both $a \lesssim b$ and $a \gtrsim b$ hold.

By $\mathbb{N}$ we mean the set of all nonnegative integers. If $A$ is an $n \times n$ matrix, we write $\|A\|$ for its operator norm on $\mathbb{R}^n$ with the Euclidean norm $|\cdot|$.

## 2 The Mehler kernel

For $t > 0$, the difference

$$Q_\infty - Q_t = \int_t^\infty e^{sB} Q e^{sB^*} ds \tag{2.1}$$

is a symmetric and strictly positive definite matrix. So is the matrix

$$Q_t^{-1} - Q_\infty^{-1} = Q_t^{-1}(Q_\infty - Q_t)Q_\infty^{-1}, \tag{2.2}$$

and we can define

$$D_t = (Q_t^{-1} - Q_\infty^{-1})^{-1} Q_t^{-1} e^{tB}, \qquad t > 0. \tag{2.3}$$

Then formula (1.2), the definition of the Gaussian measure and some elementary computations yield

$$
\begin{aligned}
\mathcal{H}_t f(x) &= (2\pi)^{-\frac{n}{2}} (\det Q_t)^{-\frac{1}{2}} \int f(e^{tB}x - y) \exp\left[-\frac{1}{2}\langle Q_t^{-1} y, y\rangle\right] dy \\
&= \left(\frac{\det Q_\infty}{\det Q_t}\right)^{1/2} \exp\left[\frac{1}{2}\langle Q_t^{-1} e^{tB} x, D_t\, x - e^{tB} x\rangle\right] \\
&\quad \times \int f(u) \exp\left[\frac{1}{2}\langle (Q_\infty^{-1} - Q_t^{-1})(u - D_t\, x)\,,\; u - D_t\, x\rangle\right] d\gamma_\infty(u)\,, \qquad (2.4)
\end{aligned}
$$

where we repeatedly used the fact that $Q_\infty^{-1} - Q_t^{-1}$ is symmetric. We now express the matrix $D_t$ in various ways.

**Lemma 2.1** *For all $x \in \mathbb{R}^n$ and $t > 0$ we have*

(i) $D_t = Q_\infty e^{-tB^*} Q_\infty^{-1}$;
(ii) $D_t = e^{tB} + Q_t e^{-tB^*} Q_\infty^{-1}$.

***Proof*** (i) The formulae (2.1) and (1.1) imply

$$
Q_\infty - Q_t = e^{tB} Q_\infty e^{tB^*} \qquad (2.5)
$$

(see also [12, formula (2.1)]). From (2.3) and (2.2) it follows that

$$
D_t = Q_\infty (Q_\infty - Q_t)^{-1}\, e^{tB},
$$

and combining this with (2.5) we arrive at (i).

(ii) Multiplying (2.5) by $e^{-tB^*} Q_\infty^{-1}$ from the right, we obtain

$$
Q_\infty e^{-tB^*} Q_\infty^{-1} - Q_t e^{-tB^*} Q_\infty^{-1} = e^{tB},
$$

and (ii) now follows from (i).

□

By means of (i) in this lemma, we can define $D_t$ for all $t \in \mathbb{R}$, and they will form a one-parameter group of matrices.

Now (ii) in Lemma 2.1 yields

$$
\langle Q_t^{-1} e^{tB} x, D_t\, x - e^{tB} x\rangle = \langle Q_t^{-1} e^{tB} x, Q_t e^{-tB^*} Q_\infty^{-1} x\rangle = \langle Q_\infty^{-1} x, x\rangle.
$$

Thus (2.4) may be rewritten as

$$
\mathcal{H}_t f(x) = \int K_t(x, u)\, f(u)\, d\gamma_\infty(u)\,,
$$

where $K_t$ denotes the Mehler kernel, given by

$$
\begin{aligned}
&K_t(x, u) \\
&= \left(\frac{\det Q_\infty}{\det Q_t}\right)^{1/2} \exp\big(R(x)\big) \exp\left[-\frac{1}{2}\big\langle (Q_t^{-1} - Q_\infty^{-1})(u - D_t\, x)\,,\; u - D_t\, x\big\rangle\right] \qquad (2.6)
\end{aligned}
$$

for $x, u \in \mathbb{R}^n$. Here we introduced the quadratic form

$$
R(x) = \frac{1}{2}\big\langle Q_\infty^{-1} x, x\big\rangle, \qquad x \in \mathbb{R}^n.
$$

## 3 Some auxiliary results

In this section we collect some preliminary bounds, which will be essential for the sequel.

**Lemma 3.1** *For $s > 0$ and for all $x \in \mathbb{R}^n$ the matrices $D_s$ and $D_{-s} = D_s^{-1}$ satisfy*

$$e^{cs}|x| \lesssim |D_s\, x| \lesssim e^{Cs}|x|,$$

*and*

$$e^{-Cs}|x| \lesssim |D_{-s}\, x| \lesssim e^{-cs}|x|.$$

*This also holds with $D_s$ replaced by $e^{-sB}$ and $e^{-sB^*}$.*

***Proof*** We make a Jordan decomposition of $B^*$, thus writing it as the sum of a complex diagonal matrix and a triangular, nilpotent matrix, which commute with each other. This leads to expressions for $e^{-sB^*}$ and $e^{sB^*}$, and since $B^*$ like $B$ has only eigenvalues with negative real parts, we see that

$$\|e^{-sB^*}\| \lesssim e^{Cs} \qquad \text{and} \qquad \|e^{sB^*}\| \lesssim e^{-cs}. \tag{3.1}$$

From (i) in Lemma 2.1, we now get the claimed upper estimates for $D_{\pm s}$. To prove the lower estimate for $D_s$, we write

$$|x| = |D_{-s}\, D_s\, x| \lesssim e^{-cs}|D_s\, x|.$$

The other parts of the lemma are completely analogous. □

In the following lemma, we collect estimates of some basic quantities related to the matrices $Q_t$.

**Lemma 3.2** *For all $t > 0$ we have*

(i) $\det\, Q_t \simeq (\min(1, t))^n$;
(ii) $\|Q_t^{-1}\| \simeq (\min(1, t))^{-1}$;
(iii) $\|Q_\infty - Q_t\| \lesssim e^{-ct}$;
(iv) $\|Q_t^{-1} - Q_\infty^{-1}\| \lesssim t^{-1}\, e^{-ct}$;
(v) $\left\| \left(Q_t^{-1} - Q_\infty^{-1}\right)^{-1/2} \right\| \lesssim t^{1/2}\, e^{Ct}$.

***Proof*** (i) and (ii) Using (3.1), we see that for each $t > 0$ and for all $v \in \mathbb{R}^n$

$$\begin{aligned}\langle Q_t v, v\rangle &= \left\langle \int_0^t e^{sB} Q e^{sB^*} v\, ds,\, v \right\rangle = \int_0^t \langle Q^{1/2} e^{sB^*} v,\, Q^{1/2} e^{sB^*} v\rangle\, ds \\ &= \int_0^t \left|Q^{1/2} e^{sB^*} v\right|^2 ds \simeq \int_0^t \left|e^{sB^*} v\right|^2 ds \\ &\lesssim \int_0^t e^{-cs}\, ds\, |v|^2 \simeq \min(1, t)\, |v|^2.\end{aligned}$$

Since $\left\| \left(e^{sB^*}\right)^{-1} \right\| = \|e^{-sB^*}\| \lesssim e^{Cs}$, there is also a lower estimate

$$\int_0^t \left|e^{sB^*} v\right|^2 ds \gtrsim \int_0^t e^{-Cs}\, ds\, |v|^2 \simeq \min(1, t)|v|^2.$$

Thus any eigenvalue of $Q_t$ has order of magnitude $\min(1, t)$, and (i) and (ii) follow.

(iii) From the definition of $Q_t$ and (3.1), we get

$$\|Q_\infty - Q_t\| = \left\| \int_t^\infty e^{sB} Q e^{sB^*} ds \right\| \lesssim e^{-ct}.$$

(iv) Using now (ii) and (iii), we have

$$\begin{aligned}\|Q_t^{-1} - Q_\infty^{-1}\| &= \|Q_t^{-1}(Q_\infty - Q_t)Q_\infty^{-1}\| \lesssim \|Q_t^{-1}\| \, \|Q_\infty - Q_t\| \\ &\lesssim (\min(1,t))^{-1} \, e^{-ct} \lesssim t^{-1} \, e^{-ct}.\end{aligned}$$

(v) Since $\|A^{1/2}\| = \|A\|^{1/2}$ for any symmetric positive definite matrix $A$, we consider $(Q_t^{-1} - Q_\infty^{-1})^{-1}$, which can be rewritten as

$$(Q_t^{-1} - Q_\infty^{-1})^{-1} = (Q_\infty^{-1}(Q_\infty - Q_t)Q_t^{-1})^{-1} = Q_t(Q_\infty - Q_t)^{-1}Q_\infty. \tag{3.2}$$

It follows from (2.5) that $(Q_\infty - Q_t)^{-1} = e^{-tB^*} Q_\infty^{-1} e^{-tB}$,
so that

$$\|(Q_\infty - Q_t)^{-1}\| \lesssim e^{Ct}$$

as a consequence of (3.2). Inserting this and the simple estimate $\|Q_t\| \lesssim t$ in (3.2), we obtain $\|(Q_t^{-1} - Q_\infty^{-1})^{-1}\| \lesssim te^{Ct}$, and (v) follows. □

**Proposition 3.3** *For $t \geq 1$ and $w \in \mathbb{R}^n$, we have*

$$\langle (Q_t^{-1} - Q_\infty^{-1})D_t\, w,\ D_t\, w\rangle \simeq |w|^2.$$

***Proof*** By (2.3) and Lemma 2.1 (i) we have

$$\begin{aligned}\langle (Q_t^{-1} - Q_\infty^{-1})D_t\, w,\ D_t\, w\rangle &= \langle Q_t^{-1}e^{tB}w\,,\ Q_\infty e^{-tB^*} Q_\infty^{-1}\, w\rangle \\ &= \langle Q_\infty Q_t^{-1} e^{tB} w\,,\ e^{-tB^*} Q_\infty^{-1}\, w\rangle.\end{aligned}$$

Since $Q_\infty Q_t^{-1} = I + (Q_\infty - Q_t)Q_t^{-1}$, this leads to

$$\begin{aligned}&\langle (Q_t^{-1} - Q_\infty^{-1})D_t\, w,\ D_t\, w\rangle \\ &= \langle e^{tB} w\,,\ e^{-tB^*} Q_\infty^{-1}\, w\rangle + \langle (Q_\infty - Q_t)Q_t^{-1}e^{tB}w\,,\ e^{-tB^*} Q_\infty^{-1}\, w\rangle \\ &= \langle Q_\infty^{-1} w, w\rangle + \langle e^{-tB}(Q_\infty - Q_t)Q_t^{-1}e^{tB}w\,,\ Q_\infty^{-1}\, w\rangle.\end{aligned}$$

Here $\langle Q_\infty^{-1} w, w\rangle \simeq |w|^2$. Using (2.1) and then the definition of $Q_\infty$, we observe that the last term can be written as

$$\begin{aligned}&\left\langle \int_t^\infty e^{(s-t)B} Q e^{(s-t)B^*}\, ds\; e^{tB^*}\; Q_t^{-1} e^{tB} w\,,\ Q_\infty^{-1}\, w \right\rangle \\ &= \left\langle Q_\infty\, e^{tB^*}\; Q_t^{-1} e^{tB} w\ ,\ Q_\infty^{-1}\, w \right\rangle \\ &= \langle\, e^{tB^*}\; Q_t^{-1} e^{tB} w\ ,\ w\rangle \\ &= \left| Q_t^{-1/2} e^{tB} w \right|^2.\end{aligned} \tag{3.3}$$

Since $\left|Q_t^{-1/2} e^{tB} w\right|^2 \lesssim |w|^2$ for $t \geq 1$ by Lemmata 3.1 and 3.2 (ii), the proposition follows. □

We finally give estimates of the kernel $K_t$, for small and large values of $t$. When $t \leq 1$, one has $\|(Q_t^{-1} - Q_\infty^{-1})^{1/2}\| \simeq t^{-1/2}$ and $\|(Q_t^{-1} - Q_\infty^{-1})^{-1/2}\| \simeq t^{1/2}$, by (iv) and (v) in Lemma 3.2. Combined with (2.6), this implies

$$\frac{e^{R(x)}}{t^{n/2}} \exp\left(-C\,\frac{|u-D_t\,x|^2}{t}\right) \lesssim K_t(x,u) \lesssim \frac{e^{R(x)}}{t^{n/2}} \exp\left(-c\,\frac{|u-D_t\,x|^2}{t}\right), \qquad 0<t\le 1. \tag{3.4}$$

**Lemma 3.4** *For $t \ge 1$ and $x, u \in \mathbb{R}^n$, we have*

$$e^{R(x)} \exp\Big[-C\big|D_{-t}\,u-x\big|^2\Big] \lesssim K_t(x,u) \lesssim e^{R(x)} \exp\Big[-c\big|D_{-t}\,u-x\big|^2\Big]. \tag{3.5}$$

***Proof*** This follows from (2.6), if we write $u - D_t\,x = D_t(D_{-t}\,u - x)$ and apply Proposition 3.3 with $w = D_{-t}\,u - x$. □

## 4 Geometric aspects of the problem

### 4.1 A system of adapted polar coordinates

We first need a technical lemma.

**Lemma 4.1** *For all $x$ in $\mathbb{R}^n$ and $s \in \mathbb{R}$, we have*

$$\langle B^* Q_\infty^{-1} x, x\rangle = -\frac{1}{2}\,|Q^{1/2}\,Q_\infty^{-1} x|^2; \tag{4.1}$$

$$\frac{\partial}{\partial s} D_s\,x = -Q_\infty B^*\,Q_\infty^{-1} D_s\,x = -Q_\infty e^{-sB^*} B^* Q_\infty^{-1} x; \tag{4.2}$$

$$\frac{\partial}{\partial s} R\big(D_s\,x\big) = \frac{1}{2}\,\big|Q^{1/2} Q_\infty^{-1} D_s\,x\big|^2 \simeq \big|D_s\,x\big|^2. \tag{4.3}$$

***Proof*** To prove (4.1), we use the definition of $Q_\infty$ to write for any $z \in \mathbb{R}^n$

$$\begin{aligned}
\langle B^* z, Q_\infty z\rangle &= \int_0^\infty \langle B^* z, e^{sB}\,Q\,e^{sB^*} z\rangle\,ds \\
&= \int_0^\infty \langle e^{sB^*}\,B^* z,\; Q\,e^{sB^*} z\rangle\,ds \\
&= \frac{1}{2}\int_0^\infty \frac{d}{ds}\langle e^{sB^*}\,z,\; Q\,e^{sB^*} z\rangle\,ds \\
&= -\frac{1}{2}\,|Q^{1/2}\,z|^2.
\end{aligned}$$

Setting $z = Q_\infty^{-1} x$, we get (4.1).

Further, (4.2) easily follows if we observe that

$$\frac{\partial}{\partial s} D_s\,x = \frac{\partial}{\partial s}\left(Q_\infty e^{-sB^*}\,Q_\infty^{-1} x\right) = -Q_\infty B^*\,Q_\infty^{-1}\,Q_\infty e^{-sB^*}\,Q_\infty^{-1} x = -Q_\infty B^*\,Q_\infty^{-1} D_s\,x.$$

Finally, we get by means of (4.2) and (4.1)

$$\begin{aligned}
\frac{\partial}{\partial s} R\,(D_s\,x) &= \frac{1}{2}\frac{\partial}{\partial s}\langle Q_\infty^{-1/2} D_s\,x,\, Q_\infty^{-1/2} D_s\,x\rangle \\
&= -\langle Q_\infty^{-1/2} Q_\infty B^* Q_\infty^{-1} D_s\,x,\, Q_\infty^{-1/2} D_s\,x\rangle \\
&= \frac{1}{2}\big|Q^{1/2} Q_\infty^{-1} D_s\,x\big|^2,
\end{aligned}$$

and (4.3) is verified. □

We observe here that an integration of (4.2) leads to

$$|x - D_t\, x| \lesssim t\, |x|, \qquad 0 \le t \le 1. \tag{4.4}$$

Fix now $\beta > 0$ and consider the ellipsoid

$$E_\beta = \{x \in \mathbb{R}^n : \ R(x) = \beta\}\,.$$

As a consequence of (4.3), the map $s \mapsto R(D_s z)$ is strictly increasing for each $0 \neq z \in \mathbb{R}^n$. Hence any $x \in \mathbb{R}^n,\ x \neq 0$, can be written uniquely as

$$x = D_s\, \tilde{x}\,, \tag{4.5}$$

for some $\tilde{x} \in E_\beta$ and $s \in \mathbb{R}$. We consider $s$ and $\tilde{x}$ as the polar coordinates of $x$. Our estimates in what follows will be uniform in $\beta$.

Next, we shall write Lebesgue measure in terms of these polar coordinates. A normal vector to the surface $E_\beta$ at the point $\tilde{x} \in E_\beta$ is $\mathbf{N}(\tilde{x}) = Q_\infty^{-1}\tilde{x}$, and the tangent hyperplane at $\tilde{x}$ is $\mathbf{N}(\tilde{x})^\perp$. For $s > 0$ the tangent hyperplane of the surface $D_s E_\beta = \{D_s\, \tilde{x} : \tilde{x} \in E_\beta\}$ at the point $D_s\, \tilde{x}$ is $D_s(\mathbf{N}(\tilde{x})^\perp)$, and a normal to $D_s E_\beta$ at the same point is $w = (D_s^{-1})^*(\mathbf{N}(\tilde{x})) = D_{-s}^* Q_\infty^{-1}\tilde{x} = Q_\infty^{-1} e^{sB}\tilde{x}$.

The scalar product of $w$ and the tangent of the curve $s \mapsto D_s\, \tilde{x}$ at the point $D_s\, \tilde{x}$ is, because of (4.2) and (4.1),

$$\begin{aligned}&\left\langle \frac{\partial}{\partial s} D_s\, \tilde{x}, w \right\rangle \\ &\quad = -\langle Q_\infty e^{-sB^*} B^* Q_\infty^{-1}\tilde{x},\ Q_\infty^{-1} e^{sB}\tilde{x}\rangle = -\langle B^* Q_\infty^{-1}\tilde{x}, \tilde{x}\rangle = \frac{1}{2}\, |Q^{1/2}\, Q_\infty^{-1}\tilde{x}|^2 > 0.\end{aligned} \tag{4.6}$$

Thus the curve $s \mapsto D_s\, \tilde{x}$ is transversal to each surface $D_s E_\beta$. Let $dS_s$ denote the area measure of $D_s E_\beta$. Then Lebesgue measure is given in terms of our polar coordinates by

$$dx = H(s, \tilde{x})\, dS_s(D_s\, \tilde{x})\, ds, \tag{4.7}$$

where

$$H(s, \tilde{x}) = \left\langle \frac{\partial}{\partial s} D_s\, \tilde{x}, \frac{w}{|w|} \right\rangle = \frac{|Q^{1/2}\, Q_\infty^{-1}\tilde{x}|^2}{2\, |Q_\infty^{-1} e^{sB}\tilde{x}|}.$$

To see how $dS_s$ varies with $s$, we take a continuous function $\varphi = \varphi(\tilde{x})$ on $E_\beta$ and extend it to $\mathbb{R}^n \setminus \{0\}$ by writing $\varphi(D_s\, \tilde{x}) = \varphi(\tilde{x})$. For any $t > 0$ and small $\varepsilon > 0$, we define the shell

$$\Omega_{t,\varepsilon} = \{D_s\, \tilde{x} : t < s < t + \varepsilon,\ \ \tilde{x} \in E_\beta\}.$$

Then $\Omega_{t,\varepsilon}$ is the image under $D_t$ of $\Omega_{0,\varepsilon}$, and the Jacobian of this map is $\det D_t = e^{-t \operatorname{tr} B}$. Thus

$$\int_{\Omega_{t,\varepsilon}} \varphi(x)\, dx = e^{-t \operatorname{tr} B} \int_{\Omega_{0,\varepsilon}} \varphi(D_t\, x)\, dx,$$

which we can rewrite as

$$\begin{aligned}&\int_{t<s<t+\varepsilon} \int_{\tilde{x}\in E_\beta} \varphi(\tilde{x})\, H(s, \tilde{x})\, dS_s(D_s\, \tilde{x})\, ds \\ &\qquad = e^{-t \operatorname{tr} B} \int_{0<s<\varepsilon} \int_{\tilde{x}\in E_\beta} \varphi(\tilde{x})\, H(s, \tilde{x})\, dS_s(D_s\, \tilde{x})\, ds.\end{aligned}$$

Now we divide by $\varepsilon$ and let $\varepsilon \to 0$, getting

$$\int_{E_\beta} \varphi(\tilde{x})\, H(t,\tilde{x})\, dS_t(D_t\,\tilde{x}) = e^{-t\,\mathrm{tr}\,B} \int_{E_\beta} \varphi(\tilde{x})\, H(0,\tilde{x})\, dS_0(\tilde{x}).$$

Since this holds for any $\varphi$, it follows that

$$dS_t(D_t\,\tilde{x}) = e^{-t\,\mathrm{tr}\,B}\, \frac{H(0,\tilde{x})}{H(t,\tilde{x})}\, dS_0(\tilde{x}).$$

Together with (4.7), this implies the following result.

**Proposition 4.2** *The Lebesgue measure in $\mathbb{R}^n$ is given in terms of polar coordinates $(t, \tilde{x})$ by*

$$dx = e^{-t\,\mathrm{tr}\,B}\, \frac{|Q^{1/2}\, Q_\infty^{-1}\tilde{x}|^2}{2\,|Q_\infty^{-1}\tilde{x}|}\, dS_0(\tilde{x})\, dt\,.$$

We also need estimates of the distance between two points in terms of the polar coordinates. The following result is a generalization of Lemma 4.2 in [4], and its proof is analogous.

**Lemma 4.3** *Fix $\beta > 0$. Let $x^{(0)},\ x^{(1)} \in \mathbb{R}^n \setminus \{0\}$ and assume $R(x^{(0)}) > \beta/2$. Write*

$$x^{(0)} = D_{s^{(0)}}(\tilde{x}^{(0)}) \qquad \text{and} \qquad x^{(1)} = D_{s^{(1)}}(\tilde{x}^{(1)})$$

*with $s^{(0)},\ s^{(1)} \in \mathbb{R}$ and $\tilde{x}^{(0)},\ \tilde{x}^{(1)} \in E_\beta$.*

(i) *Then*

$$\left|x^{(0)} - x^{(1)}\right| \gtrsim c\,\left|\tilde{x}^{(0)} - \tilde{x}^{(1)}\right|. \tag{4.8}$$

(ii) *If also $s^{(1)} \ge 0$, then*

$$\left|x^{(0)} - x^{(1)}\right| \gtrsim c\,\sqrt{\beta}\,|s^{(0)} - s^{(1)}|. \tag{4.9}$$

***Proof*** Let $\Gamma : [0, 1] \to \mathbb{R}^n \setminus \{0\}$ be a differentiable curve with $\Gamma(0) = x^{(0)}$ and $\Gamma(1) = x^{(1)}$. It suffices to bound the length of any such curve from below by the right-hand sides of (4.8) and (4.9).

For each $\tau \in [0, 1]$, we write

$$\Gamma(\tau) = D_{s(\tau)}\,\tilde{x}(\tau),$$

with $\tilde{x}(\tau) \in E_\beta$ and $\tilde{x}(i) = \tilde{x}^{(i)}$, $s(i) = s^{(i)}$ for $i = 0, 1$. Thus

$$\Gamma'(\tau) = -s'(\tau)\, \tfrac{\partial}{\partial s}\, D_s\big|_{s=s(\tau)}\, \tilde{x}(\tau) + D_{s(\tau)}\, \tilde{x}'(\tau).$$

The group property of $D_s$ implies that

$$\frac{\partial}{\partial s} D_s\Big|_{s=s(\tau)} = D_{s(\tau)} \frac{\partial}{\partial s} D_s\Big|_{s=0},$$

and so

$$\Gamma'(\tau) = D_{s(\tau)}\, v,$$

with

$$v = -s'(\tau)\, \tfrac{\partial}{\partial s}\, D_s\big|_{s=0}\, \tilde{x}(\tau) + \tilde{x}'(\tau).$$

The vector $\tilde{x}'(\tau)$ is tangent to $E_\beta$ and thus orthogonal to $\mathbf{N}(\tilde{x})$. Then (4.6) (with $s = 0$) implies that the angle between $\frac{\partial}{\partial s} D_s{}_{|s=0}\, \tilde{x}(\tau)$ and $\tilde{x}'(\tau)$ is larger than some positive constant. It follows that

$$|v|^2 \gtrsim |s'(\tau)|^2 \left| \frac{\partial}{\partial s} D_s{}_{|s=0}\, \tilde{x}(\tau)\right|^2 + \left|\tilde{x}'(\tau)\right|^2 \gtrsim |s'(\tau)|^2\, \beta + \left|\tilde{x}'(\tau)\right|^2, \tag{4.10}$$

where we also used the fact that, by (4.2),

$$\left| \frac{\partial}{\partial s} D_s{}_{|s=0}\, \tilde{x}(\tau)\right| \simeq |\tilde{x}(\tau)| \simeq \sqrt{\beta}.$$

Since

$$|v| = \left|D_{-s(\tau)}\, \Gamma'(\tau)\right| \le \left\|D_{-s(\tau)}\right\| \left|\Gamma'(\tau)\right| \lesssim e^{-C \min(s(\tau),0)} \left|\Gamma'(\tau)\right|$$

because of Lemma 3.1, we obtain from (4.10)

$$\left|\Gamma'(\tau)\right| \gtrsim e^{C \min(s(\tau),0)} \left(\sqrt{\beta}\, |s'(\tau)| + \left|\tilde{x}'(\tau)\right|\right). \tag{4.11}$$

Next, we derive a lower bound for $s(0)$; assume first that $s(0) < 0$. The assumption $R(x^{(0)}) > \beta/2$ implies, together with Lemma 3.1,

$$\beta/2 \le R(D_{s(0)}\, \tilde{x}^{(0)}) \lesssim \left|D_{s(0)}\, \tilde{x}^{(0)}\right|^2 \lesssim e^{c\, s(0)} \left|\tilde{x}^{(0)}\right|^2 \simeq e^{c\, s(0)} \beta.$$

It follows that

$$s(0) > -\tilde{s},$$

for some $\tilde{s}$ with $0 < \tilde{s} < C$, and this obviously holds also without the assumption $s(0) < 0$.

Assume now that $s(\tau) > -\tilde{s} - 1$ for all $\tau \in [0, 1]$. Then (4.11) implies

$$\left|\Gamma'(\tau)\right| \gtrsim \sqrt{\beta}\, |s'(\tau)|$$

and

$$\left|\Gamma'(\tau)\right| \gtrsim |\tilde{x}'(\tau)|.$$

Integrating these estimates with respect to $\tau$ in $[0, 1]$, we immediately see that one can control the length of $\Gamma$ from below by the right-hand sides of (4.8) and (4.9).

If instead $s(\tau) \le -\tilde{s} - 1$ for some $\tau \in [0, 1]$, we can proceed as in the proof of Lemma 4.2 in [4]. More precisely, since the image $s([0, 1])$ contains the interval $[-\tilde{s} - 1, \max(s(0), s(1))]$, we can find a closed subinterval $I$ of $[0, 1]$ whose image $s(I)$ is exactly the interval $[-\tilde{s} - 1, \max(s(0), s(1))]$. Thus we may use (4.11) to control the length of $\Gamma$ by

$$\int_0^1 \left|\Gamma'(\tau)\right| d\tau \ge \int_I \left|\Gamma'(\tau)\right| d\tau \gtrsim \sqrt{\beta} \int_I |s'(\tau)|\, d\tau \ge \sqrt{\beta}\left(\max\left(s(0), s(1)\right) + \tilde{s} + 1\right).$$

Here

$$\sqrt{\beta}\left(\max\left(s(0), s(1)\right) + \tilde{s} + 1\right) \gtrsim \sqrt{\beta} \gtrsim \operatorname{diam} E_\beta \ge \left|\tilde{x}^{(0)} - \tilde{x}^{(1)}\right|,$$

and (4.8) follows. Under the additional hypothesis $s(1) \ge 0$ of (ii), we have

$$\tilde{s} \ge \max\left(-s(0), -s(1)\right) = -\min\left(s(0), s(1)\right).$$

Then

$$\sqrt{\beta}\,\big(\max\,(s(0),s(1))\;+\;\tilde{s}+1\big)\gtrsim\sqrt{\beta}\,\big(\max\,(s(0),s(1))-\min\,(s(0),s(1))\big)$$
$$=\sqrt{\beta}\,|s(0)-s(1)|,$$

and (4.9) follows. □

### 4.2 The Gaussian measure of a tube

We fix a large $\beta>0$. Define for $x^{(1)}\in E_\beta$ and $a>0$ the set

$$\Omega=\left\{x\in E_\beta:\left|x-x^{(1)}\right|<a\right\}.$$

This is a spherical cap of the ellipsoid $E_\beta$, centered at $x^{(1)}$. Observe that $|x|\simeq\sqrt{\beta}$ for $x\in\Omega$, and that the area of $\Omega$ is $|\Omega|\simeq\min\,(a^{n-1},\beta^{(n-1)/2})$. Then consider the tube

$$Z=\{D_s\,\tilde{x}:s\ge 0,\;\;\tilde{x}\in\Omega\}. \tag{4.12}$$

**Lemma 4.4** *There exists a constant $C$ such that $\beta>C$ implies that the Gaussian measure of the tube $Z$ fulfills*

$$\gamma_\infty(Z)\lesssim\frac{a^{n-1}}{\sqrt{\beta}}\,e^{-\beta}.$$

***Proof*** Proposition 4.2 yields, since $H(0,\tilde{x})\simeq|\tilde{x}|\simeq\sqrt{\beta}$,

$$\gamma_\infty(Z)\simeq\int_0^\infty e^{-s\operatorname{tr}B}\;e^{-R(D_s\,\tilde{x})}\int_\Omega H(0,\tilde{x})\,dS(\tilde{x})\,ds\lesssim\sqrt{\beta}\,a^{n-1}\int_0^\infty e^{-s\operatorname{tr}B}e^{-R(D_s\,\tilde{x})}\,ds.$$

By (4.3) we have

$$R(D_s\,\tilde{x})-R(\tilde{x})\simeq\int_0^s\left|D_{s'}\,\tilde{x}\right|^2 ds'\gtrsim s|\tilde{x}|^2\simeq s\beta,$$

which implies

$$\gamma_\infty(Z)\lesssim\sqrt{\beta}\,a^{n-1}\,e^{-\beta}\int_0^\infty e^{-s\operatorname{tr}B}\,e^{-cs\beta}\;ds.$$

Assuming $\beta$ large enough, one has $c\beta>-2\operatorname{tr}B$, and then the last integral is finite and no larger than $C/\beta$. The lemma follows. □

## 5 Simplifications

In this section, we introduce some preliminary simplifications and reductions for the proof of (1.3), i.e., of Theorem 1.1.

(1) We may assume that $f$ is nonnegative and normalized in the sense that

$$\|f\|_{L^1(\gamma_\infty)}=1,$$

since this involves no loss of generality.

(2) We may assume that $\alpha$ is large, $\alpha>C$, since otherwise (1.3) and (1.4) are trivial.

(3) In many cases, we may restrict $x$ in (1.3) and (1.4) to the ellipsoidal annulus

$$\mathcal{E}_\alpha = \left\{ x \in \mathbb{R}^n : \frac{1}{2}\log\alpha \le R(x) \le 2\log\alpha \right\}.$$

To begin with, we can always forget the unbounded component of the complement of $\mathcal{E}_\alpha$, since

$$\begin{aligned}&\gamma_\infty\{x \in \mathbb{R}^n : R(x) > 2\log\alpha\}\\ &\lesssim \int_{R(x)>2\log\alpha} \exp(-R(x))\,dx \lesssim (\log\alpha)^{(n-2)/2}\exp(-2\log\alpha) \lesssim \frac{1}{\alpha}.\end{aligned} \tag{5.1}$$

(4) When $t > 1$, we may forget also the inner region where $R(x) < \frac{1}{2}\log\alpha$. Indeed, from (3.5) we get, if $(x,u) \in \mathbb{R}^n \times \mathbb{R}^n$ with $R(x) < \frac{1}{2}\log\alpha$,

$$K_t(x,u) \lesssim e^{R(x)} < \sqrt{\alpha} < \alpha,$$

since $\alpha$ is large. In other words, for any $(x,u) \in \mathbb{R}^n \times \mathbb{R}^n$

$$R(x) < \frac{1}{2}\log\alpha \qquad \Rightarrow \qquad K_t(x,u) \lesssim \alpha, \tag{5.2}$$

for all $t > 1$.
Replacing $\alpha$ by $C\alpha$ for some $C$, we see from (3) and (4) that we can assume $x \in \mathcal{E}_\alpha$ in the proof of (1.3) and (1.4), when the supremum in the maximal operator is taken only over $t > 1$.

Before introducing the last simplification, we need to define a global region

$$G = \left\{ (x,u) \in \mathbb{R}^n \times \mathbb{R}^n : |x-u| > \frac{1}{1+|x|} \right\}$$

and a local region

$$L = \left\{ (x,u) \in \mathbb{R}^n \times \mathbb{R}^n : |x-u| \le \frac{1}{1+|x|} \right\}.$$

Notice that the definition of $G$ and $L$ does not depend on $Q$ and $B$.

(5) When $t \le 1$ and $(x,u) \in G$, we shall see that (5.2) is still valid, and it is again enough to consider $x \in \mathcal{E}_\alpha$.

To prove this, we need a lemma which will also be useful later.

**Lemma 5.1** *If $(x,u) \in G$ and $0 < t \le 1$, then*

$$\frac{1}{(1+|x|)^2} \lesssim t^2|x|^2 + |u - D_t\,x|^2.$$

***Proof*** From the definition of $G$ and (4.4) we get

$$\frac{1}{1+|x|} \le |x-u| \le |x - D_t\,x| + |D_t\,x - u| \lesssim t|x| + |u - D_t\;x|.$$

The lemma follows. □

To verify now (5.2) in the global region with $t \le 1$, we recall from (3.4) that

$$K_t(x,u) \lesssim \frac{e^{R(x)}}{t^{n/2}} \exp\left(-c\,\frac{|u - D_t\,x|^2}{t}\right).$$

It follows from Lemma 5.1 that

$$t^2 \gtrsim \frac{1}{(1+|x|)^4} \qquad \text{or} \qquad \frac{|u - D_t\,x|^2}{t} \gtrsim \frac{1}{(1+|x|)^2 t}. \tag{5.3}$$

The first inequality here implies that

$$K_t(x,u) \lesssim e^{R(x)}\,(1+|x|)^n \lesssim e^{2R(x)},$$

and (5.2) follows. If the second inequality of (5.3) holds, we have

$$K_t(x,u) \lesssim \frac{e^{R(x)}}{t^{n/2}} \exp\left(-\frac{c}{(1+|x|)^2 t}\right) \lesssim e^{R(x)}\,(1+|x|)^n,$$

and we get the same estimate. Thus (5.2) is verified.

Finally, let

$$\mathcal{H}_*^G f(x) = \sup_{0<t\le 1} \left| \int K_t(x,u)\,\chi_G(x,u)\,f(u)\,d\gamma_\infty(u) \right|,$$

and

$$\mathcal{H}_*^L f(x) = \sup_{0<t\le 1} \left| \int K_t(x,u)\,\chi_L(x,u)\,f(u)\,d\gamma_\infty(u) \right|.$$

## 6 The case of large *t*

In this section, we consider the supremum in the definition of the maximal operator taken only over $t > 1$, and we prove (1.4).

**Proposition 6.1** *For all functions $f \in L^1(\gamma_\infty)$ such that $\|f\|_{L^1(\gamma_\infty)} = 1$,*

$$\gamma_\infty \left\{ x : \sup_{t>1} |\mathcal{H}_t f(x)| > \alpha \right\} \lesssim \frac{1}{\alpha\sqrt{\log \alpha}}, \qquad \alpha > 2. \tag{6.1}$$

*In particular, the maximal operator*

$$\sup_{t>1} |\mathcal{H}_t f(x)|$$

*is of weak type* $(1,1)$ *with respect to the invariant measure* $\gamma_\infty$.

***Proof*** We can assume that $f \ge 0$. Looking at the arguments in Sect. 5, items (3) and (4), we see that it suffices to consider points $x \in \mathcal{E}_\alpha$. For both $x$ and $u$ we use the coordinates introduced in (4.5) with $\beta = \log \alpha$, that is,

$$x = D_s\,\tilde{x}, \qquad u = D_{s'}\,\tilde{u},$$

where $\tilde{x}, \tilde{u} \in E_{\log \alpha}$ and $s, s' \in \mathbb{R}$.

From (3.5) we have

$$K_t(x,u) \lesssim \exp(R(x)) \exp\left(-c\left|D_{-t}\,u - x\right|^2\right)$$

for $t > 1$ and $x, u \in \mathbb{R}^n$. Since $x \in \mathcal{E}_\alpha$ and $D_{-t}\, u = D_{s'-t}\, \tilde{u}$, we can apply Lemma 4.3 (i), getting

$$\left|D_{-t}\, u - x\right| \gtrsim \left|\tilde{x} - \tilde{u}\right|,$$

so that

$$\int K_t(x,u) f(u)\, d\gamma_\infty(u) \lesssim \exp\left(R(D_s\, \tilde{x})\right) \int \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) f(u)\, d\gamma_\infty(u).$$

In view of (4.3), the right-hand side here is strictly increasing in $s$, and therefore the inequality

$$\exp\left(R(D_s\, \tilde{x})\right) \int \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) f(u)\, d\gamma_\infty(u) > \alpha \tag{6.2}$$

holds if and only if $s > s_\alpha(\tilde{x})$ for some function $\tilde{x} \mapsto s_\alpha(\tilde{x})$, with equality for $s = s_\alpha(\tilde{x})$. Since $\alpha > 2$ and $\|f\|_{L^1(\gamma_\infty)} = 1$, it follows that $s_\alpha(\tilde{x}) > 0$.

For some $C$, the set of points $x \in \mathcal{E}_\alpha$ where the supremum in (6.1) is larger than $C\alpha$ is contained in the set $\mathcal{A}(\alpha)$ of points $D_s\, \tilde{x} \in \mathcal{E}_\alpha$ fulfilling (6.2). We use Proposition 4.2 to estimate the $\gamma_\infty$ measure of $\mathcal{A}(\alpha)$. Observe that $H(0, \tilde{x}) \simeq |\tilde{x}| \simeq \sqrt{\log \alpha}$ and that $D_s\, \tilde{x} \in \mathcal{E}_\alpha$ implies $s \lesssim 1$, so that also $e^{-s \operatorname{tr} B} \lesssim 1$. We get

$$\begin{aligned}
\gamma_\infty(\mathcal{A}(\alpha)) &= \int_{\mathcal{A}(\alpha)\cap\mathcal{E}_\alpha} e^{-R(x)} dx \\
&\lesssim \sqrt{\log \alpha} \int_{E_{\log \alpha}} \int_{s_\alpha(\tilde{x})}^{C} e^{-R(D_s\, \tilde{x})}\, ds\, dS(\tilde{x}) \\
&\lesssim \sqrt{\log \alpha} \int_{E_{\log \alpha}} \int_{s_\alpha(\tilde{x})}^{+\infty} \exp\left(-R(D_{s_\alpha(\tilde{x})}\, \tilde{x}) - c \log \alpha\, (s - s_\alpha(\tilde{x}))\right) ds\, dS(\tilde{x}),
\end{aligned}$$

where the last inequality follows from (4.3), since $|D_s\, \tilde{x}|^2 \gtrsim |\tilde{x}|^2 \simeq \log \alpha$. Integrating in $s$, we obtain

$$\gamma_\infty(\mathcal{A}(\alpha)) \lesssim \frac{1}{\sqrt{\log \alpha}} \int_{E_{\log \alpha}} \exp\left(-R(D_{s_\alpha(\tilde{x})}\, \tilde{x})\right) dS(\tilde{x}).$$

Now combine this estimate with the case of equality in (6.2) and change the order of integration, to get

$$\begin{aligned}
\gamma_\infty(\mathcal{A}(\alpha)) &\lesssim \frac{1}{\alpha\sqrt{\log \alpha}} \int \int_{E_{\log \alpha}} \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) dS(\tilde{x})\, f(u)\, d\gamma_\infty(u) \\
&\lesssim \frac{1}{\alpha\sqrt{\log \alpha}} \int f(u)\, d\gamma_\infty(u),
\end{aligned}$$

which proves Proposition 6.1. □

Finally, we show that the factor $1/\sqrt{\log \alpha}$ in (6.1) is sharp.

**Proposition 6.2** *For any $t > 1$ and any large $\alpha$, there exists a function $f$ normalized in $L^1(\gamma_\infty)$ and such that*

$$\gamma_\infty\{x : |\mathcal{H}_t f(x)| > \alpha\} \simeq \frac{1}{\alpha\sqrt{\log \alpha}}.$$

***Proof*** Take a point $z$ with $R(z) = \log \alpha$, and let $f$ be (an approximation of) a Dirac measure at the point $u = D_t z$. Then, as a consequence of (3.5), $K_t(x, u) \simeq \exp(R(x))$ when $x$ is in the ball $B(D_{-t}\, u, 1) = B(z, 1)$. We then have $\mathcal{H}_t f(x) = K_t(x, u) \gtrsim \alpha$ in the set $\mathcal{B} = \{x \in B(z, 1)\colon R(x) > R(z)\}$, whose measure is

$$\gamma_\infty(\mathcal{B}) \simeq e^{-R(z)} \frac{1}{\sqrt{R(z)}} = \frac{1}{\alpha \sqrt{\log \alpha}}.$$

□

## 7 The local case for small *t*

**Proposition 7.1** *If* $(x, u) \in L$ *and* $0 < t \le 1$, *then*

$$\left|K_t(x, u)\right| \lesssim \frac{\exp\left(R(x)\right)}{t^{n/2}} \exp\left(-c\,\frac{|u - x|^2}{t}\right).$$

***Proof*** In view of (3.4), it is enough to show that

$$\frac{|u - D_t\, x|^2}{t} \ge \frac{|u - x|^2}{t} - C. \tag{7.1}$$

We write

$$\begin{aligned} |u - D_t\, x|^2 &= |u - x + x - D_t\, x|^2 = |u - x|^2 + 2\langle u - x, x - D_t\, x\rangle + |x - D_t\, x|^2 \\ &\ge |u - x|^2 - 2|u - x|\,|x - D_t\, x|. \end{aligned}$$

By (4.4),

$$|u - x|\,|x - D_t\, x| \lesssim |u - x|\, t\, |x| \le t$$

since $(x, u) \in L$, and (7.1) follows. □

**Proposition 7.2** *The maximal operator* $\mathcal{H}_*^L$ *is of weak type* $(1, 1)$ *with respect to the invariant measure* $\gamma_\infty$.

***Proof*** The proof is standard, since Proposition 7.1 implies

$$\mathcal{H}_*^L f(x) \lesssim \sup_{0<t\le 1} \frac{\exp\left(R(x)\right)}{t^{n/2}} \int \exp\left(-c\,\frac{|x - u|^2}{t}\right) \chi_L(x, u)\, f(u)\, d\gamma_\infty(u).$$

The supremum here defines an operator of weak type $(1, 1)$ with respect to Lebesgue measure in $\mathbb{R}^n$. From this the proposition follows, cf. [7, Section 3]. □

## 8 The global case for small *t*

In this section, we conclude the proof of Theorem 1.1.

**Proposition 8.1** *The maximal operator* $\mathcal{H}_*^G$ *is of weak type* $(1, 1)$ *with respect to the invariant measure* $\gamma_\infty$.

***Proof*** We take $f$ and $\alpha$ as in items (1) and (2) of Sect. 5. Then item (5) tells us that we need only consider $\mathcal{H}_*^G f(x)$ for $x \in \mathcal{E}_\alpha$.

For $m \in \mathbb{N}$ and $0 < t \leq 1$, we introduce regions $\mathcal{S}_t^m$. If $m > 0$, we let

$$\mathcal{S}_t^m = \left\{ (x,u) \in G : 2^{m-1}\sqrt{t} < |u - D_t\, x| \leq 2^m \sqrt{t} \right\}.$$

If $m = 0$, we replace the condition $2^{m-1}\sqrt{t} < |u - D_t\, x| \leq 2^m\sqrt{t}$ by $|u - D_t\, x| \leq \sqrt{t}$. Note that for any fixed $t \in (0, 1]$ these sets form a partition of $G$.

In the set $\mathcal{S}_t^m$ we have, because of (3.4),

$$K_t(x,u) \lesssim \frac{\exp(R(x))}{t^{n/2}} \exp\left(-c2^{2m}\right).$$

Then setting

$$\mathcal{K}_t^m(x,u) = \frac{\exp(R(x))}{t^{n/2}} \chi_{\mathcal{S}_t^m}(x,u),$$

one has, for all $(x,u) \in G$ and $0 < t < 1$,

$$K_t(x,u) \lesssim \sum_{m=0}^{\infty} \exp\left(-c2^{2m}\right) \mathcal{K}_t^m(x,u).$$

Hence, it suffices to prove that for $m = 0, 1, \ldots$

$$\gamma_\infty \left\{ x \in \mathcal{E}_\alpha : \sup_{0<t\leq 1} \int \mathcal{K}_t^m(x,u)\, f(u)\, d\gamma_\infty(u) > \alpha \right\} \lesssim \frac{2^{Cm}}{\alpha}, \tag{8.1}$$

for large $\alpha$ and some $C$, since this will allow summing in $m$ in the space $L^{1,\infty}(\gamma_\infty)$.

Fix $m \in \mathbb{N}$ and assume that $(x,u) \in S_t^m$ for some $t \in (0, 1]$, so that $|u - D_t\, x| \leq 2^m\sqrt{t}$. Then Lemma 5.1 leads to

$$1 \lesssim (1+|x|)^4 t^2 + (1+|x|)^2\, 2^{2m}\, t \leq ((1+|x|)^2\, 2^{2m}\, t)^2 + (1+|x|)^2\, 2^{2m}\, t.$$

Consequently, a point $x \in \mathcal{E}_\alpha$ satisfies

$$(1+|x|)^2\, 2^{2m}\, t \gtrsim 1 \tag{8.2}$$

as soon as there exists a point $u$ with $\mathcal{K}_t^m(x,u) \neq 0$, and then $t \geq \varepsilon > 0$ for some $\varepsilon = \varepsilon(\alpha, m) > 0$. Hence the supremum in (8.1) will be the same if taken only over $\varepsilon \leq t \leq 1$, and it follows that this supremum is a continuous function of $x \in \mathcal{E}_\alpha$.

To prove (8.1), the idea, which goes back to [15], is to construct a finite sequence of pairwise disjoint balls $\left(\mathcal{B}^{(\ell)}\right)_{\ell=1}^{\ell_0}$ in $\mathbb{R}^n$ and a finite sequence of sets $\left(\mathcal{Z}^{(\ell)}\right)_{\ell=1}^{\ell_0}$ in $\mathbb{R}^n$, called forbidden zones. These zones will together cover the level set in (8.1). We will then verify that

$$\left\{ x \in \mathcal{E}_\alpha : \sup_{\varepsilon\leq t\leq 1} \int \mathcal{K}_t^m(x,u)\, f(u)\, d\gamma_\infty(u) \geq \alpha \right\} \subset \bigcup_{\ell=1}^{\ell_0} \mathcal{Z}^{(\ell)}, \tag{8.3}$$

that for each $\ell$

$$\gamma_\infty(\mathcal{Z}^{(\ell)}) \lesssim \frac{2^{Cm}}{\alpha} \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty(u), \tag{8.4}$$

and that the $\mathcal{B}^{(\ell)}$ are pairwise disjoint. This would imply

$$\gamma_\infty\Big(\bigcup_{\ell=1}^{\ell_0} \mathcal{Z}^{(\ell)}\Big) \lesssim \frac{2^{Cm}}{\alpha} \sum_{\ell=1}^{\ell_0} \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty(u) \lesssim \frac{2^{Cm}}{\alpha},$$

and thus also (8.1) and Proposition 8.1.

The sets $\mathcal{B}^{(\ell)}$ and $\mathcal{Z}^{(\ell)}$ will be introduced by means of a sequence of points $x^{(\ell)}$, $\ell = 1, \dots, \ell_0$, which we define by recursion. To start, we choose as $x^{(1)}$ a point where the quadratic form $R(x)$ takes its minimal value in the compact set

$$\mathcal{A}_1(\alpha) = \left\{ x \in \mathcal{E}_\alpha : \sup_{\varepsilon \le t \le 1} \int \mathcal{K}_t^m(x,u)\, f(u)\, d\gamma_\infty \ge \alpha \right\}.$$

However, should this set be empty, (8.1) is immediate.

We now describe the recursion to construct $x^{(\ell)}$ for $\ell \ge 2$. Like $x^{(1)}$, these points will satisfy

$$\sup_{\varepsilon \le t \le 1} \int \mathcal{K}_t^m(x^{(\ell)},u)\, f(u)\, d\gamma_\infty \ge \alpha.$$

Once an $x^{(\ell)}$, $\ell \ge 1$, is defined, we can thus by continuity choose $t_\ell \in [\varepsilon, 1]$ such that

$$\int \mathcal{K}_{t_\ell}^m(x^{(\ell)},u)\, f(u)\, d\gamma_\infty \ge \alpha. \tag{8.5}$$

Using this $t_\ell$, we associate with $x^{(\ell)}$ the tube

$$\mathcal{Z}^{(\ell)} = \left\{ D_s\, \eta \in \mathbb{R}^n : s \ge 0,\ R(\eta) = R(x^{(\ell)}),\ |\eta - x^{(\ell)}| < A\, 2^{3m} \sqrt{t_\ell} \right\},$$

Here the constant $A > 0$ is to be determined, depending only on $n$, $Q$ and $B$.

All the $x^{(\ell)}$ will be minimizing points of $R(x)$. To avoid having them too close to one another, we will not allow $x^{(\ell)}$ to be in any $\mathcal{Z}^{(\ell')}$ with $\ell' < \ell$. More precisely, assuming $x^{(1)}, \dots, x^{(\ell)}$ already defined, we will choose $x^{(\ell+1)}$ as a minimizing point of $R(x)$ in the set

$$\mathcal{A}_{\ell+1}(\alpha) = \left\{ x \in \mathcal{E}_\alpha \setminus \bigcup_{\ell'=1}^{\ell} \mathcal{Z}^{(\ell')} : \sup_{\varepsilon \le t \le 1} \int \mathcal{K}_t^m(x,u)\, f(u)\, d\gamma_\infty(u) \ge \alpha \right\}, \tag{8.6}$$

provided this set is nonempty. But if $\mathcal{A}_{\ell+1}(\alpha)$ is empty, the process stops with $\ell_0 = \ell$ and (8.3) follows. We will see that this actually occurs for some finite $\ell$.

Now assume that $\mathcal{A}_{\ell+1}(\alpha) \neq \emptyset$. In order to assure that a minimizing point exists, we must verify that $\mathcal{A}_{\ell+1}(\alpha)$ is closed and thus compact, although the $\mathcal{Z}^{(\ell')}$ are not open. To do so, observe that for $1 \le \ell' \le \ell$, the minimizing property of $x^{(\ell')}$ means that there is no point $x$ in $\mathcal{A}_{\ell'}(\alpha)$ with $R(x) < R(x^{(\ell')})$. Thus we have the inclusions

$$\mathcal{A}_{\ell+1}(\alpha) \subset \mathcal{A}_{\ell'}(\alpha) \subset \left\{ x : R(x) \ge R(x^{(\ell')}) \right\}, \qquad 1 \le \ell' \le \ell.$$

It follows that

$$\begin{aligned}
\mathcal{A}_{\ell+1}(\alpha) &= \mathcal{A}_{\ell+1}(\alpha) \cap \bigcap_{1 \le \ell' \le \ell} \{x : R(x) \ge R(x^{(\ell')})\} \\
&= \bigcap_{\ell'=1}^{\ell} \left\{ x \in \mathcal{E}_\alpha \setminus \mathcal{Z}^{(\ell')} : R(x) \ge R(x^{(\ell')}),\ \sup_{\varepsilon \le t \le 1} \int \mathcal{K}_t^m(x,u)\, f(u)\, d\gamma_\infty(u) \ge \alpha \right\}.
\end{aligned}$$

For each $\ell' = 1, \dots, \ell$ we have

$$\{x \in \mathcal{E}_\alpha \setminus \mathcal{Z}^{(\ell')} : R(x) \geq R(x^{(\ell')})\}$$
$$= \left\{ D_s\, \eta \in \mathcal{E}_\alpha : s \geq 0,\ R(\eta) = R(x^{(\ell')}),\ |\eta - x^{(\ell')}| \geq A 2^{3m} \sqrt{t_{\ell'}} \right\},$$

and this set is closed. It follows that $\mathcal{A}_{\ell+1}(\alpha)$ is compact, and a minimizing point $x^{(\ell+1)}$ can be chosen. Thus the recursion is well defined.

We observe that (8.2) applies to $t_\ell$ and $x^{(\ell)}$, and $|x^{(\ell)}|$ is large, so

$$|x^{(\ell)}|^2\, 2^{2m}\, t_\ell \gtrsim 1. \tag{8.7}$$

Further, we define balls

$$\mathcal{B}^{(\ell)} = \{u \in \mathbb{R}^n : |u - D_{t_\ell}\, x^{(\ell)}| \leq 2^m \sqrt{t_\ell}\,\}.$$

Because of the definitions of $\mathcal{K}_t^m$ and $\mathcal{S}_t^m$, the inequality (8.5) implies

$$\alpha \leq \frac{\exp\left(R(x^{(\ell)})\right)}{t_\ell^{n/2}} \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty(u). \tag{8.8}$$

It remains to verify the claimed properties of $\mathcal{B}^{(\ell)}$ and $\mathcal{Z}^{(\ell)}$. The arguments below follow the lines of the proof of Lemma 6.2 in [4], with only slight modifications.

**Lemma 8.2** *The balls $\mathcal{B}^{(\ell)}$ are pairwise disjoint.*

***Proof*** Two balls $\mathcal{B}^{(\ell)}$ and $\mathcal{B}^{(\ell')}$ with $\ell < \ell'$ will be disjoint if

$$\left|D_{t_\ell}\, x^{(\ell)} - D_{t_{\ell'}}\, x^{(\ell')}\right| > 2^m (\sqrt{t_\ell} + \sqrt{t_{\ell'}}). \tag{8.9}$$

By means of our polar coordinates with $\beta = R(x^{(\ell)})$, we write

$$x^{(\ell')} = D_s\, \tilde{x}^{(\ell')}$$

for some $\tilde{x}^{(\ell')}$ with $R(\tilde{x}^{(\ell')}) = R(x^{(\ell)})$ and some $s \in \mathbb{R}$. Note that $s \geq 0$, because $R(x^{(\ell')}) \geq R(x^{(\ell)})$. Since $x^{(\ell')}$ does not belong to the forbidden zone $\mathcal{Z}^{(\ell)}$, we must have

$$|\tilde{x}^{(\ell')} - x^{(\ell)}| \geq A 2^{3m} \sqrt{t_\ell}. \tag{8.10}$$

We first assume that $t_{\ell'} \geq M\, 2^{4m}\, t_\ell$, for some $M = M(n, Q, B) \geq 2$ to be chosen. Lemma 4.3 (ii) implies

$$\left|D_{t_\ell}\, x^{(\ell)} - D_{t_{\ell'}}\, x^{(\ell')}\right| = \left|D_{t_\ell}\, x^{(\ell)} - D_{t_{\ell'}+s}\, \tilde{x}^{(\ell')}\right| \gtrsim |x^{(\ell)}|\, (t_{\ell'} + s - t_\ell) \gtrsim |x^{(\ell)}|\, t_{\ell'},$$

the last step by our assumption. Using again the assumption and then (8.7), we get

$$|x^{(\ell)}|\, t_{\ell'} \gtrsim |x^{(\ell)}|\, \sqrt{M}\, 2^{2m} \sqrt{t_\ell}\, \sqrt{t_{\ell'}} \gtrsim \sqrt{M}\, 2^m \sqrt{t_{\ell'}} \simeq \sqrt{M}\, 2^m\, (\sqrt{t_{\ell'}} + \sqrt{t_\ell}).$$

Fixing $M$ suitably large, we obtain (8.9) from the last two formulae.

It remains to consider the case when $t_{\ell'} < M\, 2^{4m}\, t_\ell$. Then

$$\sqrt{t_\ell} > \frac{2^{-2m-1}}{\sqrt{M}} (\sqrt{t_{\ell'}} + \sqrt{t_\ell}).$$

Applying this to (8.10), we obtain (8.9) by choosing $A$ so that $A/\sqrt{M}$ is large enough. □

We next verify that the sequence $(x^{(\ell)})$ is finite. For $\ell < \ell'$, we have (8.10), and Lemma 4.3 (i) implies

$$\left|x^{(\ell')} - x^{(\ell)}\right| \gtrsim A\, 2^{3m} \sqrt{t_\ell}.$$

Since $t_\ell \geq \varepsilon$, we see that the distance $\left|x^{(\ell')} - x^{(\ell)}\right|$ is bounded below by a positive constant. But all the $x^{(\ell)}$ are contained in the bounded set $\mathcal{E}_\alpha$, so they are finite in number. Thus the set considered in (8.6) must be empty for some $\ell$, and the recursion stops. This implies (8.3).

We finally prove (8.4) . Observe that the forbidden zone $\mathcal{Z}^{(\ell)}$ is a tube as defined in (4.12), with $a = A\, 2^{3m} \sqrt{t_\ell}$ and $\beta = R(x^{(\ell)})$. This value of $\beta$ is large since $x^{(\ell)} \in \mathcal{E}_\alpha$, and thus we can apply Lemma 4.4 to obtain

$$\gamma_\infty(\mathcal{Z}^{(\ell)}) \lesssim \frac{\left(A 2^{3m} \sqrt{t_\ell}\right)^{n-1}}{\sqrt{R(x^{(\ell)})}} \exp\left(-R(x^{(\ell)})\right).$$

We bound the exponential here by means of (8.8) and observe that $R(x^{(\ell)}) \sim |x^{(\ell)}|^2$, getting

$$\gamma_\infty(\mathcal{Z}^{(\ell)}) \lesssim \frac{1}{\alpha |x^{(\ell)}| \sqrt{t_\ell}} (A 2^{3m})^{n-1} \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty(u).$$

As a consequence of (8.7), we obtain

$$\gamma_\infty(\mathcal{Z}^{(\ell)}) \lesssim \frac{2^m}{\alpha} \left(A 2^{3m}\right)^{n-1} \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty(u) \lesssim \frac{2^{Cm}}{\alpha} \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty(u),$$

proving (8.4). This concludes the proof of Proposition 8.1. □

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## References

1. Aimar, H., Forzani, L., Scotto, R.: On Riesz transforms and maximal functions in the context of Gaussian harmonic analysis. Trans. Am. Math. Soc. **359**, 2137–2154 (2005)
2. Ball, K., Barthe, F., Bednorz, W., Oleszkiewicz, K., Wolff, P.: $L^1$-smoothing for the Ornstein–Uhlenbeck semigroup. Mathematika **59**, 160–168 (2013)
3. Carbonaro, A., Dragičević, O.: Bounded holomorphic functional calculus for nonsymmetric Ornstein–Uhlenbeck operators. Ann. Sc. Norm. Super. Pisa Cl. Sci. **XIX** (5), 1497–1533 (2019)
4. Casarino, V., Ciatti, P., Sjögren, P.: The maximal operator of a normal Ornstein–Uhlenbeck semigroup is of weak type (1,1). Ann. Sc. Norm. Sup. Pisa Cl. Sci. **XXI**(5), 385–410 (2020)
5. Casarino, V., Ciatti, P., Sjögren, P.: Riesz transforms of a general Ornstein–Uhlenbeck semigroup. Calculus Var. Partial Differ. Equ. **60**, 135 (2021)
6. Chojnowska-Michalik, A., Goldys, B.: Nonsymmetric Ornstein–Uhlenbeck semigroup as second quantized operator. J. Math. Kyoto Univ. **36**, 481–498 (1996)
7. Garcìa-Cuerva, J., Mauceri, G., Meda, S., Sjögren, P., Torrea, J.L.: Maximal operators for the holomorphic Ornstein–Uhlenbeck semigroup. J. Lond. Math. Soc. **67**, 219–234 (2003)

8. Lehec, J.: Regularization in $L^1$ for the Ornstein–Uhlenbeck semigroup. Ann. Faculté des Sci. Toulouse Math. **25**, 191–204 (2016)
9. Mauceri, G., Noselli, L.: The maximal operator associated to a non symmetric Ornstein–Uhlenbeck semigroup. J. Fourier Anal. Appl. **15**, 179–200 (2009)
10. Menárguez, T., Pérez, S., Soria, F.: Pointwise and norm estimates for operators associated with the Ornstein–Uhlenbeck semigroup. C. R. Acad. Sci. Paris Sér. I **326**, 25–30 (1998)
11. Menárguez, T., Pérez, S., Soria, F.: The Mehler maximal function: a geometric proof of the weak type (1,1). J. Lond. Math. Soc. **62**, 846–856 (2000)
12. Metafune, G., Prüss, J., Rhandi, A., Schnaubelt, R.: The domain of the Ornstein–Uhlenbeck operator on a $L^p$-space with invariant measure. Ann. Sc. Norm. Super. Pisa Cl. Sci. **1**, 471–487 (2002)
13. Muckenhoupt, B.: Poisson integrals for Hermite and Laguerre expansions. Trans. Am. Math. Soc. **139**, 231–242 (1969)
14. Pérez, S., Soria, F.: Operators associated with the Ornstein–Uhlenbeck semigroup. J. Lond. Math. Soc. **61**, 857–871 (2000)
15. Sjögren, P.: On the maximal function for the Mehler kernel. In: Mauceri G., Weiss G. (eds.) Harmonic Analysis, Cortona 1982. Springer Lecture Notes in Mathematics, vol. 992, pp. 73–82 (1983)
16. Stein, E.M.: Topics in Harmonic Analysis Related to the Littlewood–Paley Theory, Annals in Mathematics Studies. Princeton University Press, Princeton (1970)
17. Urbina-Romero, W.: Gaussian Harmonic Analysis. Springer Monographs in Mathematics (2019)
18. van Neerven, J.M.A.M.: Nonsymmetric Ornstein–Uhlenbeck semigroups in Banach spaces. J. Funct. Anal. **155**, 495–535 (1998)