CHALMERS
UNIVERSITY OF TECHNOLOGY

On the maximal operator of a general Ornstein–Uhlenbeck semigroup

Downloaded from: https://research.chalmers.se, 2025-12-09 23:30 UTC

Citation for the original published paper (version of record):
CASARINO, V., CIATTI, P., Sjögren, P. (2022). On the maximal operator of a general Ornstein–Uhlenbeck semigroup. Mathematische Zeitschrift, 301(3): 2393-2413.
http://dx.doi.org/10.1007/s00209-022-02986-w

N.B. When citing this work, cite the original published paper.

research.chalmers.se offers the possibility of retrieving research publications produced at Chalmers University of Technology. It covers all kind of research output: articles, dissertations, conference papers, reports etc. since 2004. research.chalmers.se is administrated and maintained by Chalmers Library

(article starts on next page)

Mathematische Zeitschrift
https://doi.org/10.1007/s00209-022-02986-w

Mathematische Zeitschrift

On the maximal operator of a general Ornstein–Uhlenbeck semigroup

Valentina Casarino[1] · Paolo Ciatti[2] · Peter Sjögren[3]

Received: 28 February 2020 / Accepted: 11 January 2022
© The Author(s) 2022

Abstract
If Q is a real, symmetric and positive definite $n \times n$ matrix, and B a real $n \times n$ matrix whose eigenvalues have negative real parts, we consider the Ornstein–Uhlenbeck semigroup on $\mathbb{R}^n$ with covariance Q and drift matrix B. Our main result says that the associated maximal operator is of weak type (1, 1) with respect to the invariant measure. The proof has a geometric gist and hinges on the "forbidden zones method" previously introduced by the third author.

Keywords Ornstein–Uhlenbeck semigroup · Maximal operator · Gaussian measure · Mehler kernel · Weak type (1,1)

Mathematics Subject Classification 47D03 · 42B25

1 Introduction

In this paper we prove a weak type (1, 1) theorem for the maximal operator associated to a general Ornstein–Uhlenbeck semigroup. We extend the proof given by the third author in 1983 in a symmetric context. Our setting is the following.

V. Casarino and P. Ciatti were partially supported by GNAMPA (Project 2018 "Operatori e disuguaglianze integrali in spazi con simmetrie") and MIUR (PRIN 2015 "Real and Complex Manifolds: Geometry, Topology and Harmonic Analysis"). This research was carried out while the third author was a visiting scientist at the University of Padova, Italy, and he is grateful for its hospitality.

✉ Peter Sjögren
peters@chalmers.se

Valentina Casarino
valentina.casarino@unipd.it

Paolo Ciatti
paolo.ciatti@unipd.it

1 Università degli Studi di Padova, Stradella san Nicola 3, 36100 Vicenza, Italy

2 Università degli Studi di Padova, Via Marzolo 9, 35100 Padua, Italy

3 Mathematical Sciences, University of Gothenburg and Mathematical Sciences, Chalmers University of Technology, 412 96 Göteborg, Sweden

In $\mathbb{R}^n$ we will consider the semigroup generated by the elliptic operator

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{n} q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i,j=1}^{n} b_{ij} x_i \frac{\partial}{\partial x_j},$$

or, equivalently,

$$\mathcal{L} = \frac{1}{2} \operatorname{tr}\big(Q\nabla^2\big) + \langle Bx, \nabla\rangle,$$

where ∇ is the gradient and ∇^2 the Hessian. Here $Q = (q_{ij})$ is a real, symmetric and positive definite $n \times n$ matrix, indicating the covariance of $\mathcal{L}$. The real $n \times n$ matrix $B = (b_{ij})$ is negative in the sense that all its eigenvalues have negative real parts, and it gives the drift of $\mathcal{L}$.

The semigroup is formally $\mathcal{H}_t = e^{t\mathcal{L}}$, $t > 0$, but to write it more explicitly we first introduce the positive definite, symmetric matrices

$$Q_t = \int_0^t e^{sB} Q e^{sB^*} \, ds, \qquad 0 < t \le +\infty, \tag{1.1}$$

and the normalized Gaussian measures γ_t in $\mathbb{R}^n$, with $t \in (0, +\infty]$, having density

$$y \mapsto (2\pi)^{-\frac{n}{2}} (\det Q_t)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\langle Q_t^{-1} y, y\rangle\right)$$

with respect to Lebesgue measure. Then for functions f in the space of bounded continuous functions in $\mathbb{R}^n$ one has

$$\mathcal{H}_t f(x) = \int f(e^{tB}x - y) \, d\gamma_t(y)\,, \quad x \in \mathbb{R}^n\,, \tag{1.2}$$

a formula due to Kolmogorov. The measure γ_∞ is invariant under the action of $\mathcal{H}_t$; it will be our basic measure, replacing Lebesgue measure.

We remark that $\big(\mathcal{H}_t\big)_{t>0}$ is the transition semigroup of the stochastic process

$$\chi(x,t) = e^{tB} + \int_0^t e^{(t-s)B} \, dW(s),$$

where W is a Brownian motion in $\mathbb{R}^n$ with covariance Q.

We are interested in the maximal operator defined as

$$\mathcal{H}_* f(x) = \sup_{t>0} \big|\mathcal{H}_t f(x)\big|.$$

Under the above assumptions on Q and B, our main result is the following.

Theorem 1.1 *The Ornstein–Uhlenbeck maximal operator $\mathcal{H}_*$ is of weak type $(1,1)$ with respect to the invariant measure γ_∞, with an operator quasinorm that depends only on the dimension and the matrices Q and B.*

In other words, the inequality

$$\gamma_\infty\{x \in \mathbb{R}^n : \mathcal{H}_* f(x) > \alpha\} \le \frac{C}{\alpha} \, \|f\|_{L^1(\gamma_\infty)}, \qquad \alpha > 0, \tag{1.3}$$

holds for all functions $f \in L^1(\gamma_\infty)$, with $C = C(n, Q, B)$.

For large values of the time parameter, we also obtain a refinement of this result. Indeed, we prove in Proposition 6.1 that

$$\gamma_\infty \left\{ x \in \mathbb{R}^n : \sup_{t>1} |\mathcal{H}_t f(x)| > \alpha \right\} \le \frac{C}{\alpha \sqrt{\log \alpha}} \tag{1.4}$$

for large $\alpha > 0$ and all normalized functions $f \in L^1(\gamma_\infty)$. Here $C = C(n, Q, B)$, and this estimate is shown to be sharp. It cannot be extended to $\mathcal{H}_*$, since the maximal operator corresponding to small values of t only satisfies the ordinary weak type inequality. This sharpening is not surprising, in the light of some recent results for the standard case $Q = I$ and $B = -I$ by Lehec [8]. He proved the following conjecture, proposed by Ball, Barthe, Bednorz, Oleszkiewicz and Wolff [2]:

For each fixed $t > 0$, there exists a function $\psi_t = \psi_t(\alpha)$, with $\lim_{\alpha \to +\infty} \psi_t(\alpha) = 0$, satisfying

$$\gamma_\infty \{x \in \mathbb{R}^n : |\mathcal{H}_t f(x)| > \alpha\} \le \frac{\psi_t(\alpha)}{\alpha}$$

for all large $\alpha > 0$ and all $f \in L^1(\gamma_\infty)$ such that $\|f\|_{L^1(\gamma_\infty)} = 1$. Lehec proved this conjecture with $\psi_t(\alpha) = C(t)/\sqrt{\log \alpha}$ independent of the dimension, and this ψ_t is sharp. Our estimates depend strongly on the dimension n, but on the other hand we estimate the supremum over large t.

The history of $\mathcal{H}_*$ is quite long and started with the first attempts to prove L^p estimates. When $\left(\mathcal{H}_t\right)_{t>0}$ is symmetric, i.e., when each operator $\mathcal{H}_t$ is self-adjoint on $L^2(\gamma_\infty)$, then $\mathcal{H}_*$ is bounded on $L^p(\gamma_\infty)$ for $1 < p \le \infty$, as a consequence of the general Littlewood–Paley–Stein theory for symmetric semigroups of contractions on L^p spaces [16, Ch. III].

It is easy to see that the maximal operator is unbounded on $L^1(\gamma_\infty)$. This led, about fifty years ago, to the study of the weak type (1, 1) of $\mathcal{H}_*$ with respect to γ_∞. The first positive result is due to B. Muckenhoupt [13], who proved the estimate (1.3) in the one-dimensional case with $Q = I$ and $B = -I$. The analogous question in the higher-dimensional case was an open problem until 1983, when the third author [15] proved the weak type (1, 1) in any finite dimension. Other proofs are due to Menárguez, Pérez and Soria [11] (see also [10, 14]) and to Garcìa-Cuerva, Mauceri, Meda, Sjögren and Torrea [7]. Moreover, a different proof of the weak type (1, 1) of $\mathcal{H}_*$, based on a covering lemma halfway between covering results by Besicovitch and Wiener, was given by Aimar, Forzani and Scotto [1]. A nice overview of the literature may be found in [17, Ch.4].

In [4] the present authors recently considered a normal Ornstein–Uhlenbeck semigroup in $\mathbb{R}^n$, that is, we assumed that $\mathcal{H}_t$ is for each $t > 0$ a normal operator on $L^2(\gamma_\infty)$. Under this extra assumption, we proved that the associated maximal operator is of weak type (1, 1) with respect to the invariant measure γ_∞. This extends earlier work in the non-symmetric framework by Mauceri and Noselli [9], who proved that if $Q = I$ and $B = \lambda(R - I)$ for some positive λ and a real skew-symmetric matrix R generating a periodic group, then the maximal operator $\mathcal{H}_*$ is of weak type (1, 1).

In Theorem 1.1 we go beyond the hypothesis of normality. The proof has a geometric core and relies on the *ad hoc* technique developed by the third author in [15]. It is worth noticing that, while the proof in [4] required an analysis of the special case when $Q = I$ and $B = (-\lambda_1, \dots, -\lambda_n)$, with $\lambda_j > 0$ for $j = 1, \dots, n$, and then the application of factorization results, we apply here directly, avoiding many intermediate steps, the "forbidden zones" technique introduced in [15].

Since the maximal operator $\mathcal{H}_*$ is trivially bounded from L^∞ to L^∞, we obtain by interpolation the following corollary.

Corollary 1.2 *The Ornstein–Uhlenbeck maximal operator $\mathcal{H}_*$ is bounded on $L^p(\gamma_\infty)$ for all $p > 1$.*

This result improves Theorem 4.2 in [9], where the L^p boundedness of $\mathcal{H}_*$ is proved for all $p > 1$ in the normal framework, under the additional assumption that the infinitesimal generator of $\left(\mathcal{H}_t\right)_{t>0}$ is a sectorial operator of angle less than $\pi/2$.

In this paper we focus our attention on the Ornstein–Uhlenbeck semigroup in $\mathbb{R}^n$. In view of possible applications to stochastic analysis and to SPDE's, it would be very interesting to investigate the case of the infinite-dimensional Ornstein-Uhlenbeck maximal operator as well (see [3, 6, 18] for an introduction to the infinite-dimensional setting). The Riesz transforms associated to a general Ornstein–Uhlenbeck semigroup in $\mathbb{R}^n$ have been studied in the authors' paper [5].

The scheme of the paper is as follows. In Sect. 2 we introduce the Mehler kernel $K_t(x, u)$, that is, the integral kernel of $\mathcal{H}_t$. Some estimates for the norm and the determinant of Q_t and related matrices are provided in Sect. 3. As a consequence, we obtain bounds for the Mehler kernel. In Sect. 4 we consider the relevant geometric features of the problem, and introduce in Sect. 4.1 a system of polar-like coordinates. We also express Lebesgue measure in terms of these coordinates. Sections 5, 6, 7 and 8 are devoted to the proof of Theorem 1.1. First, Sect. 5 introduces some preliminary simplifications of the proof; in particular, we restrict the variable x to an ellipsoidal annulus. In Sect. 6 we consider the supremum in the definition of the maximal operator taken only over $t > 1$ and prove the sharp estimate (1.4). Section 7 is devoted to the case of small t under an additional local condition. Finally, in Sect. 8 we treat the remaining case and conclude the proof of Theorem 1.1, by proving the estimate (1.3) for small t under a global assumption.

In the following, we use the "variable constant convention", according to which the symbols $c > 0$ and $C < \infty$ will denote constants which are not necessarily equal at different occurrences. They all depend only on the dimension and on Q and B. For any two nonnegative quantities a and b we write $a \lesssim b$ instead of $a \le Cb$ and $a \gtrsim b$ instead of $a \ge cb$. The symbol $a \simeq b$ means that both $a \lesssim b$ and $a \gtrsim b$ hold.

By $\mathbb{N}$ we mean the set of all nonnegative integers. If A is an $n \times n$ matrix, we write $\|A\|$ for its operator norm on $\mathbb{R}^n$ with the Euclidean norm $|\cdot|$.

2 The Mehler kernel

For $t > 0$, the difference

$$Q_\infty - Q_t = \int_t^\infty e^{sB} Q e^{sB^*} ds \tag{2.1}$$

is a symmetric and strictly positive definite matrix. So is the matrix

$$Q_t^{-1} - Q_\infty^{-1} = Q_t^{-1}(Q_\infty - Q_t)Q_\infty^{-1}, \tag{2.2}$$

and we can define

$$D_t = (Q_t^{-1} - Q_\infty^{-1})^{-1} Q_t^{-1} e^{tB}, \qquad t > 0. \tag{2.3}$$

Then formula (1.2), the definition of the Gaussian measure and some elementary computations yield

$$\begin{aligned}\mathcal{H}_t f(x) &= (2\pi)^{-\frac{n}{2}} (\det Q_t)^{-\frac{1}{2}} \int f(e^{tB}x - y) \exp\left[-\frac{1}{2}\langle Q_t^{-1} y, y\rangle\right] dy \\ &= \left(\frac{\det Q_\infty}{\det Q_t}\right)^{1/2} \exp\left[\frac{1}{2}\langle Q_t^{-1} e^{tB}x, D_t\, x - e^{tB}x\rangle\right] \\ &\quad\times \int f(u) \exp\left[\frac{1}{2}\langle (Q_\infty^{-1} - Q_t^{-1})(u - D_t\, x)\,,\ u - D_t\, x\rangle\right] d\gamma_\infty(u)\,, \qquad (2.4)\end{aligned}$$

where we repeatedly used the fact that $Q_\infty^{-1} - Q_t^{-1}$ is symmetric. We now express the matrix D_t in various ways.

Lemma 2.1 *For all $x \in \mathbb{R}^n$ and $t > 0$ we have*

(i) $D_t = Q_\infty e^{-tB^*} Q_\infty^{-1}$;
(ii) $D_t = e^{tB} + Q_t e^{-tB^*} Q_\infty^{-1}$.

Proof (i) The formulae (2.1) and (1.1) imply

$$Q_\infty - Q_t = e^{tB} Q_\infty e^{tB^*} \qquad (2.5)$$

(see also [12, formula (2.1)]). From (2.3) and (2.2) it follows that

$$D_t = Q_\infty (Q_\infty - Q_t)^{-1}\, e^{tB},$$

and combining this with (2.5) we arrive at (i).
(ii) Multiplying (2.5) by $e^{-tB^*} Q_\infty^{-1}$ from the right, we obtain

$$Q_\infty e^{-tB^*} Q_\infty^{-1} - Q_t e^{-tB^*} Q_\infty^{-1} = e^{tB},$$

and (ii) now follows from (i).

□

By means of (i) in this lemma, we can define D_t for all $t \in \mathbb{R}$, and they will form a one-parameter group of matrices.
Now (ii) in Lemma 2.1 yields

$$\langle Q_t^{-1} e^{tB}x, D_t\, x - e^{tB}x\rangle = \langle Q_t^{-1} e^{tB}x, Q_t e^{-tB^*} Q_\infty^{-1} x\rangle = \langle Q_\infty^{-1} x, x\rangle.$$

Thus (2.4) may be rewritten as

$$\mathcal{H}_t f(x) = \int K_t(x, u)\, f(u)\, d\gamma_\infty(u)\,,$$

where K_t denotes the Mehler kernel, given by

$$\begin{aligned}&K_t(x, u) \\ &= \left(\frac{\det Q_\infty}{\det Q_t}\right)^{1/2} \exp\big(R(x)\big) \exp\left[-\frac{1}{2}\big\langle (Q_t^{-1} - Q_\infty^{-1})(u - D_t\, x)\,,\ u - D_t\, x\big\rangle\right] \qquad (2.6)\end{aligned}$$

for $x, u \in \mathbb{R}^n$. Here we introduced the quadratic form

$$R(x) = \frac{1}{2}\big\langle Q_\infty^{-1} x, x\big\rangle, \qquad x \in \mathbb{R}^n.$$

3 Some auxiliary results

In this section we collect some preliminary bounds, which will be essential for the sequel.

Lemma 3.1 *For $s > 0$ and for all $x \in \mathbb{R}^n$ the matrices D_s and $D_{-s} = D_s^{-1}$ satisfy*

$$e^{cs}|x| \lesssim |D_s\, x| \lesssim e^{Cs}|x|,$$

and

$$e^{-Cs}|x| \lesssim |D_{-s}\, x| \lesssim e^{-cs}|x|.$$

This also holds with D_s replaced by e^{-sB} and e^{-sB^*}.

Proof We make a Jordan decomposition of B^*, thus writing it as the sum of a complex diagonal matrix and a triangular, nilpotent matrix, which commute with each other. This leads to expressions for e^{-sB^*} and e^{sB^*}, and since B^* like B has only eigenvalues with negative real parts, we see that

$$\|e^{-sB^*}\| \lesssim e^{Cs} \qquad \text{and} \qquad \|e^{sB^*}\| \lesssim e^{-cs}. \tag{3.1}$$

From (i) in Lemma 2.1, we now get the claimed upper estimates for $D_{\pm s}$. To prove the lower estimate for D_s, we write

$$|x| = |D_{-s}\, D_s\, x| \lesssim e^{-cs}|D_s\, x|.$$

The other parts of the lemma are completely analogous. □

In the following lemma, we collect estimates of some basic quantities related to the matrices Q_t.

Lemma 3.2 *For all $t > 0$ we have*

(i) $\det\, Q_t \simeq (\min(1, t))^n$;
(ii) $\|Q_t^{-1}\| \simeq (\min(1, t))^{-1}$;
(iii) $\|Q_\infty - Q_t\| \lesssim e^{-ct}$;
(iv) $\|Q_t^{-1} - Q_\infty^{-1}\| \lesssim t^{-1}\, e^{-ct}$;
(v) $\left\| \left(Q_t^{-1} - Q_\infty^{-1}\right)^{-1/2} \right\| \lesssim t^{1/2}\, e^{Ct}$.

Proof (i) and (ii) Using (3.1), we see that for each $t > 0$ and for all $v \in \mathbb{R}^n$

$$\begin{aligned}\langle Q_t v, v\rangle &= \left\langle \int_0^t e^{sB} Q e^{sB^*} v\, ds,\, v \right\rangle = \int_0^t \langle Q^{1/2} e^{sB^*} v,\, Q^{1/2} e^{sB^*} v\rangle\, ds \\ &= \int_0^t \left|Q^{1/2} e^{sB^*} v\right|^2 ds \simeq \int_0^t \left|e^{sB^*} v\right|^2 ds \\ &\lesssim \int_0^t e^{-cs}\, ds\, |v|^2 \simeq \min(1, t)\, |v|^2.\end{aligned}$$

Since $\left\| \left(e^{sB^*}\right)^{-1} \right\| = \|e^{-sB^*}\| \lesssim e^{Cs}$, there is also a lower estimate

$$\int_0^t \left|e^{sB^*} v\right|^2 ds \gtrsim \int_0^t e^{-Cs}\, ds\, |v|^2 \simeq \min(1, t)|v|^2.$$

Thus any eigenvalue of Q_t has order of magnitude $\min(1, t)$, and (i) and (ii) follow.

(iii) From the definition of Q_t and (3.1), we get

$$\|Q_\infty - Q_t\| = \left\| \int_t^\infty e^{sB} Q e^{sB^*} ds \right\| \lesssim e^{-ct}.$$

(iv) Using now (ii) and (iii), we have

$$\begin{aligned}\|Q_t^{-1} - Q_\infty^{-1}\| = \|Q_t^{-1}(Q_\infty - Q_t)Q_\infty^{-1}\| &\lesssim \|Q_t^{-1}\| \, \|Q_\infty - Q_t\| \\ &\lesssim (\min(1,t))^{-1} e^{-ct} \lesssim t^{-1} e^{-ct}.\end{aligned}$$

(v) Since $\|A^{1/2}\| = \|A\|^{1/2}$ for any symmetric positive definite matrix A, we consider $(Q_t^{-1} - Q_\infty^{-1})^{-1}$, which can be rewritten as

$$(Q_t^{-1} - Q_\infty^{-1})^{-1} = (Q_\infty^{-1}(Q_\infty - Q_t)Q_t^{-1})^{-1} = Q_t(Q_\infty - Q_t)^{-1}Q_\infty. \tag{3.2}$$

It follows from (2.5) that $(Q_\infty - Q_t)^{-1} = e^{-tB^*} Q_\infty^{-1} e^{-tB}$,
so that

$$\|(Q_\infty - Q_t)^{-1}\| \lesssim e^{Ct}$$

as a consequence of (3.2). Inserting this and the simple estimate $\|Q_t\| \lesssim t$ in (3.2), we obtain $\|(Q_t^{-1} - Q_\infty^{-1})^{-1}\| \lesssim te^{Ct}$, and (v) follows. □

Proposition 3.3 *For $t \geq 1$ and $w \in \mathbb{R}^n$, we have*

$$\langle (Q_t^{-1} - Q_\infty^{-1}) D_t\, w,\ D_t\, w\rangle \simeq |w|^2.$$

Proof By (2.3) and Lemma 2.1 (i) we have

$$\begin{aligned}\langle (Q_t^{-1} - Q_\infty^{-1}) D_t\, w,\ D_t\, w\rangle &= \langle Q_t^{-1} e^{tB} w\,,\ Q_\infty e^{-tB^*} Q_\infty^{-1}\, w\rangle \\ &= \langle Q_\infty Q_t^{-1} e^{tB} w\,,\ e^{-tB^*} Q_\infty^{-1}\, w\rangle.\end{aligned}$$

Since $Q_\infty Q_t^{-1} = I + (Q_\infty - Q_t)Q_t^{-1}$, this leads to

$$\begin{aligned}&\langle (Q_t^{-1} - Q_\infty^{-1}) D_t\, w,\ D_t\, w\rangle \\ &= \langle e^{tB} w\,,\ e^{-tB^*} Q_\infty^{-1}\, w\rangle + \langle (Q_\infty - Q_t) Q_t^{-1} e^{tB} w\,,\ e^{-tB^*} Q_\infty^{-1}\, w\rangle \\ &= \langle Q_\infty^{-1} w, w\rangle + \langle e^{-tB}(Q_\infty - Q_t) Q_t^{-1} e^{tB} w\,,\ Q_\infty^{-1}\, w\rangle.\end{aligned}$$

Here $\langle Q_\infty^{-1} w, w\rangle \simeq |w|^2$. Using (2.1) and then the definition of Q_∞, we observe that the last term can be written as

$$\begin{aligned}&\left\langle \int_t^\infty e^{(s-t)B} Q e^{(s-t)B^*} ds\, e^{tB^*}\, Q_t^{-1} e^{tB} w\,,\ Q_\infty^{-1}\, w \right\rangle \\ &= \left\langle Q_\infty\, e^{tB^*}\, Q_t^{-1} e^{tB} w\ ,\ Q_\infty^{-1}\, w\right\rangle \\ &= \langle\, e^{tB^*}\, Q_t^{-1} e^{tB} w\ ,\ w\rangle \\ &= \left| Q_t^{-1/2} e^{tB} w \right|^2.\end{aligned} \tag{3.3}$$

Since $\left|Q_t^{-1/2} e^{tB} w\right|^2 \lesssim |w|^2$ for $t \geq 1$ by Lemmata 3.1 and 3.2 (ii), the proposition follows. □

We finally give estimates of the kernel K_t, for small and large values of t. When $t \leq 1$, one has $\|(Q_t^{-1} - Q_\infty^{-1})^{1/2}\| \simeq t^{-1/2}$ and $\|(Q_t^{-1} - Q_\infty^{-1})^{-1/2}\| \simeq t^{1/2}$, by (iv) and (v) in Lemma 3.2. Combined with (2.6), this implies

$$\frac{e^{R(x)}}{t^{n/2}} \exp\left(-C\,\frac{|u-D_t\,x|^2}{t}\right) \lesssim K_t(x,u) \lesssim \frac{e^{R(x)}}{t^{n/2}} \exp\left(-c\,\frac{|u-D_t\,x|^2}{t}\right), \quad 0<t\le 1. \tag{3.4}$$

Lemma 3.4 *For $t \ge 1$ and $x, u \in \mathbb{R}^n$, we have*

$$e^{R(x)} \exp\Big[-C\big|D_{-t}\,u-x\big|^2\Big] \lesssim K_t(x,u) \lesssim e^{R(x)} \exp\Big[-c\big|D_{-t}\,u-x\big|^2\Big]. \tag{3.5}$$

Proof This follows from (2.6), if we write $u - D_t\,x = D_t(D_{-t}\,u - x)$ and apply Proposition 3.3 with $w = D_{-t}\,u - x$. □

4 Geometric aspects of the problem

4.1 A system of adapted polar coordinates

We first need a technical lemma.

Lemma 4.1 *For all x in $\mathbb{R}^n$ and $s \in \mathbb{R}$, we have*

$$\langle B^* Q_\infty^{-1} x, x\rangle = -\frac{1}{2}\,|Q^{1/2}\,Q_\infty^{-1} x|^2; \tag{4.1}$$

$$\frac{\partial}{\partial s} D_s\,x = -Q_\infty B^*\,Q_\infty^{-1} D_s\,x = -Q_\infty e^{-sB^*} B^* Q_\infty^{-1} x; \tag{4.2}$$

$$\frac{\partial}{\partial s} R\big(D_s\,x\big) = \frac{1}{2}\,\big|Q^{1/2} Q_\infty^{-1} D_s\,x\big|^2 \simeq \big|D_s\,x\big|^2. \tag{4.3}$$

Proof To prove (4.1), we use the definition of Q_∞ to write for any $z \in \mathbb{R}^n$

$$\begin{aligned}
\langle B^* z, Q_\infty z\rangle &= \int_0^\infty \langle B^* z, e^{sB}\,Q\,e^{sB^*} z\rangle\,ds \\
&= \int_0^\infty \langle e^{sB^*}\,B^* z,\; Q\,e^{sB^*} z\rangle\,ds \\
&= \frac{1}{2}\int_0^\infty \frac{d}{ds}\langle e^{sB^*}\,z,\; Q\,e^{sB^*} z\rangle\,ds \\
&= -\frac{1}{2}\,|Q^{1/2}\,z|^2.
\end{aligned}$$

Setting $z = Q_\infty^{-1} x$, we get (4.1).

Further, (4.2) easily follows if we observe that

$$\frac{\partial}{\partial s} D_s\,x = \frac{\partial}{\partial s}\left(Q_\infty e^{-sB^*} Q_\infty^{-1} x\right) = -Q_\infty B^*\,Q_\infty^{-1} Q_\infty e^{-sB^*} Q_\infty^{-1} x = -Q_\infty B^*\,Q_\infty^{-1} D_s\,x.$$

Finally, we get by means of (4.2) and (4.1)

$$\begin{aligned}
\frac{\partial}{\partial s} R\,(D_s\,x) &= \frac{1}{2}\frac{\partial}{\partial s}\langle Q_\infty^{-1/2} D_s\,x,\, Q_\infty^{-1/2} D_s\,x\rangle \\
&= -\langle Q_\infty^{-1/2} Q_\infty B^* Q_\infty^{-1} D_s\,x,\, Q_\infty^{-1/2} D_s\,x\rangle \\
&= \frac{1}{2}\big|Q^{1/2} Q_\infty^{-1} D_s\,x\big|^2,
\end{aligned}$$

and (4.3) is verified. □

We observe here that an integration of (4.2) leads to

$$|x - D_t\, x| \lesssim t\, |x|, \qquad 0 \le t \le 1. \tag{4.4}$$

Fix now $\beta > 0$ and consider the ellipsoid

$$E_\beta = \{x \in \mathbb{R}^n :\ R(x) = \beta\}\,.$$

As a consequence of (4.3), the map $s \mapsto R(D_s z)$ is strictly increasing for each $0 \ne z \in \mathbb{R}^n$. Hence any $x \in \mathbb{R}^n,\ x \ne 0$, can be written uniquely as

$$x = D_s\, \tilde{x}\,, \tag{4.5}$$

for some $\tilde{x} \in E_\beta$ and $s \in \mathbb{R}$. We consider s and $\tilde{x}$ as the polar coordinates of x. Our estimates in what follows will be uniform in β.

Next, we shall write Lebesgue measure in terms of these polar coordinates. A normal vector to the surface E_β at the point $\tilde{x} \in E_\beta$ is $\mathbf{N}(\tilde{x}) = Q_\infty^{-1}\tilde{x}$, and the tangent hyperplane at $\tilde{x}$ is $\mathbf{N}(\tilde{x})^\perp$. For $s > 0$ the tangent hyperplane of the surface $D_s E_\beta = \{D_s\, \tilde{x} : \tilde{x} \in E_\beta\}$ at the point $D_s\, \tilde{x}$ is $D_s(\mathbf{N}(\tilde{x})^\perp)$, and a normal to $D_s E_\beta$ at the same point is $w = (D_s^{-1})^*(\mathbf{N}(\tilde{x})) = D_{-s}^* Q_\infty^{-1}\tilde{x} = Q_\infty^{-1} e^{sB}\tilde{x}$.

The scalar product of w and the tangent of the curve $s \mapsto D_s\, \tilde{x}$ at the point $D_s\, \tilde{x}$ is, because of (4.2) and (4.1),

$$\begin{aligned}&\left\langle \frac{\partial}{\partial s} D_s\, \tilde{x},\, w \right\rangle \\ &\quad = -\langle Q_\infty e^{-sB^*} B^* Q_\infty^{-1}\tilde{x},\ Q_\infty^{-1} e^{sB}\tilde{x}\rangle = -\langle B^* Q_\infty^{-1}\tilde{x}, \tilde{x}\rangle = \frac{1}{2}\, |Q^{1/2}\, Q_\infty^{-1}\tilde{x}|^2 > 0.\end{aligned} \tag{4.6}$$

Thus the curve $s \mapsto D_s\, \tilde{x}$ is transversal to each surface $D_s E_\beta$. Let dS_s denote the area measure of $D_s E_\beta$. Then Lebesgue measure is given in terms of our polar coordinates by

$$dx = H(s, \tilde{x})\, dS_s(D_s\, \tilde{x})\, ds, \tag{4.7}$$

where

$$H(s, \tilde{x}) = \left\langle \frac{\partial}{\partial s} D_s\, \tilde{x},\, \frac{w}{|w|} \right\rangle = \frac{|Q^{1/2}\, Q_\infty^{-1}\tilde{x}|^2}{2\, |Q_\infty^{-1} e^{sB}\tilde{x}|}.$$

To see how dS_s varies with s, we take a continuous function $\varphi = \varphi(\tilde{x})$ on E_β and extend it to $\mathbb{R}^n \setminus \{0\}$ by writing $\varphi(D_s\, \tilde{x}) = \varphi(\tilde{x})$. For any $t > 0$ and small $\varepsilon > 0$, we define the shell

$$\Omega_{t,\varepsilon} = \{D_s\, \tilde{x} : t < s < t + \varepsilon,\ \ \tilde{x} \in E_\beta\}.$$

Then $\Omega_{t,\varepsilon}$ is the image under D_t of $\Omega_{0,\varepsilon}$, and the Jacobian of this map is $\det D_t = e^{-t \operatorname{tr} B}$. Thus

$$\int_{\Omega_{t,\varepsilon}} \varphi(x)\, dx = e^{-t \operatorname{tr} B} \int_{\Omega_{0,\varepsilon}} \varphi(D_t\, x)\, dx,$$

which we can rewrite as

$$\begin{aligned}&\int_{t<s<t+\varepsilon} \int_{\tilde{x}\in E_\beta} \varphi(\tilde{x})\, H(s, \tilde{x})\, dS_s(D_s\, \tilde{x})\, ds \\ &\quad = e^{-t \operatorname{tr} B} \int_{0<s<\varepsilon} \int_{\tilde{x}\in E_\beta} \varphi(\tilde{x})\, H(s, \tilde{x})\, dS_s(D_s\, \tilde{x})\, ds.\end{aligned}$$

Now we divide by ε and let $\varepsilon \to 0$, getting

$$\int_{E_\beta} \varphi(\tilde{x})\, H(t,\tilde{x})\, dS_t(D_t\,\tilde{x}) = e^{-t\,\mathrm{tr}\,B} \int_{E_\beta} \varphi(\tilde{x})\, H(0,\tilde{x})\, dS_0(\tilde{x}).$$

Since this holds for any φ, it follows that

$$dS_t(D_t\,\tilde{x}) = e^{-t\,\mathrm{tr}\,B}\, \frac{H(0,\tilde{x})}{H(t,\tilde{x})}\, dS_0(\tilde{x}).$$

Together with (4.7), this implies the following result.

Proposition 4.2 *The Lebesgue measure in $\mathbb{R}^n$ is given in terms of polar coordinates $(t,\tilde{x})$ by*

$$dx = e^{-t\,\mathrm{tr}\,B}\, \frac{|Q^{1/2}\,Q_\infty^{-1}\tilde{x}|^2}{2\,|Q_\infty^{-1}\tilde{x}|}\, dS_0(\tilde{x})\, dt\,.$$

We also need estimates of the distance between two points in terms of the polar coordinates. The following result is a generalization of Lemma 4.2 in [4], and its proof is analogous.

Lemma 4.3 *Fix $\beta > 0$. Let $x^{(0)},\ x^{(1)} \in \mathbb{R}^n \setminus \{0\}$ and assume $R(x^{(0)}) > \beta/2$. Write*

$$x^{(0)} = D_{s^{(0)}}(\tilde{x}^{(0)}) \qquad \text{and} \qquad x^{(1)} = D_{s^{(1)}}(\tilde{x}^{(1)})$$

with $s^{(0)},\ s^{(1)} \in \mathbb{R}$ and $\tilde{x}^{(0)},\ \tilde{x}^{(1)} \in E_\beta$.

(i) *Then*

$$\left|x^{(0)} - x^{(1)}\right| \gtrsim c\,\left|\tilde{x}^{(0)} - \tilde{x}^{(1)}\right|. \tag{4.8}$$

(ii) *If also $s^{(1)} \geq 0$, then*

$$\left|x^{(0)} - x^{(1)}\right| \gtrsim c\,\sqrt{\beta}\,|s^{(0)} - s^{(1)}|. \tag{4.9}$$

Proof Let $\Gamma : [0,1] \to \mathbb{R}^n \setminus \{0\}$ be a differentiable curve with $\Gamma(0) = x^{(0)}$ and $\Gamma(1) = x^{(1)}$. It suffices to bound the length of any such curve from below by the right-hand sides of (4.8) and (4.9).

For each $\tau \in [0,1]$, we write

$$\Gamma(\tau) = D_{s(\tau)}\,\tilde{x}(\tau),$$

with $\tilde{x}(\tau) \in E_\beta$ and $\tilde{x}(i) = \tilde{x}^{(i)}$, $s(i) = s^{(i)}$ for $i = 0, 1$. Thus

$$\Gamma'(\tau) = -s'(\tau)\, \tfrac{\partial}{\partial s}\, D_s\big|_{s=s(\tau)}\, \tilde{x}(\tau) + D_{s(\tau)}\, \tilde{x}'(\tau).$$

The group property of D_s implies that

$$\frac{\partial}{\partial s} D_s\big|_{s=s(\tau)} = D_{s(\tau)} \frac{\partial}{\partial s} D_s\big|_{s=0},$$

and so

$$\Gamma'(\tau) = D_{s(\tau)}\, v,$$

with

$$v = -s'(\tau)\, \tfrac{\partial}{\partial s}\, D_s\big|_{s=0}\, \tilde{x}(\tau) + \tilde{x}'(\tau).$$

The vector $\tilde{x}'(\tau)$ is tangent to E_β and thus orthogonal to $\mathbf{N}(\tilde{x})$. Then (4.6) (with $s = 0$) implies that the angle between $\frac{\partial}{\partial s} D_s\big|_{s=0} \tilde{x}(\tau)$ and $\tilde{x}'(\tau)$ is larger than some positive constant. It follows that

$$|v|^2 \gtrsim |s'(\tau)|^2 \left|\frac{\partial}{\partial s} D_s\big|_{s=0} \tilde{x}(\tau)\right|^2 + \left|\tilde{x}'(\tau)\right|^2 \gtrsim |s'(\tau)|^2 \beta + \left|\tilde{x}'(\tau)\right|^2, \tag{4.10}$$

where we also used the fact that, by (4.2),

$$\left|\frac{\partial}{\partial s} D_s\big|_{s=0} \tilde{x}(\tau)\right| \simeq |\tilde{x}(\tau)| \simeq \sqrt{\beta}.$$

Since

$$|v| = \left|D_{-s(\tau)} \Gamma'(\tau)\right| \leq \left\|D_{-s(\tau)}\right\| \left|\Gamma'(\tau)\right| \lesssim e^{-C \min(s(\tau),0)} \left|\Gamma'(\tau)\right|$$

because of Lemma 3.1, we obtain from (4.10)

$$\left|\Gamma'(\tau)\right| \gtrsim e^{C \min(s(\tau),0)} \left(\sqrt{\beta}\, |s'(\tau)| + \left|\tilde{x}'(\tau)\right|\right). \tag{4.11}$$

Next, we derive a lower bound for $s(0)$; assume first that $s(0) < 0$. The assumption $R(x^{(0)}) > \beta/2$ implies, together with Lemma 3.1,

$$\beta/2 \leq R(D_{s(0)}\, \tilde{x}^{(0)}) \lesssim \left|D_{s(0)}\, \tilde{x}^{(0)}\right|^2 \lesssim e^{c\, s(0)} \left|\tilde{x}^{(0)}\right|^2 \simeq e^{c\, s(0)} \beta.$$

It follows that

$$s(0) > -\tilde{s},$$

for some $\tilde{s}$ with $0 < \tilde{s} < C$, and this obviously holds also without the assumption $s(0) < 0$.

Assume now that $s(\tau) > -\tilde{s} - 1$ for all $\tau \in [0, 1]$. Then (4.11) implies

$$\left|\Gamma'(\tau)\right| \gtrsim \sqrt{\beta}\, |s'(\tau)|$$

and

$$\left|\Gamma'(\tau)\right| \gtrsim |\tilde{x}'(\tau)|.$$

Integrating these estimates with respect to τ in $[0, 1]$, we immediately see that one can control the length of Γ from below by the right-hand sides of (4.8) and (4.9).

If instead $s(\tau) \leq -\tilde{s} - 1$ for some $\tau \in [0, 1]$, we can proceed as in the proof of Lemma 4.2 in [4]. More precisely, since the image $s([0, 1])$ contains the interval $[-\tilde{s} - 1, \max(s(0), s(1))]$, we can find a closed subinterval I of $[0, 1]$ whose image $s(I)$ is exactly the interval $[-\tilde{s} - 1, \max(s(0), s(1))]$. Thus we may use (4.11) to control the length of Γ by

$$\int_0^1 \left|\Gamma'(\tau)\right| d\tau \geq \int_I \left|\Gamma'(\tau)\right| d\tau \gtrsim \sqrt{\beta} \int_I |s'(\tau)|\, d\tau \geq \sqrt{\beta} \left(\max\left(s(0), s(1)\right) \,+\, \tilde{s} + 1\right).$$

Here

$$\sqrt{\beta} \left(\max\left(s(0), s(1)\right) \,+\, \tilde{s} + 1\right) \gtrsim \sqrt{\beta} \gtrsim \operatorname{diam} E_\beta \geq \left|\tilde{x}^{(0)} - \tilde{x}^{(1)}\right|,$$

and (4.8) follows. Under the additional hypothesis $s(1) \geq 0$ of (ii), we have

$$\tilde{s} \geq \max\left(-s(0), -s(1)\right) = -\min\left(s(0), s(1)\right).$$

Then

$$\sqrt{\beta}\,\big(\max\,(s(0),s(1))\;+\;\tilde{s}+1\big)\gtrsim\sqrt{\beta}\,\big(\max\,(s(0),s(1))-\min\,(s(0),s(1))\big)$$
$$=\sqrt{\beta}\,|s(0)-s(1)|,$$

and (4.9) follows. □

4.2 The Gaussian measure of a tube

We fix a large $\beta>0$. Define for $x^{(1)}\in E_\beta$ and $a>0$ the set

$$\Omega=\left\{x\in E_\beta:\left|x-x^{(1)}\right|<a\right\}.$$

This is a spherical cap of the ellipsoid E_β, centered at $x^{(1)}$. Observe that $|x|\simeq\sqrt{\beta}$ for $x\in\Omega$, and that the area of Ω is $|\Omega|\simeq\min\,(a^{n-1},\beta^{(n-1)/2})$. Then consider the tube

$$Z=\{D_s\,\tilde{x}:s\geq 0,\ \ \tilde{x}\in\Omega\}. \tag{4.12}$$

Lemma 4.4 *There exists a constant C such that $\beta>C$ implies that the Gaussian measure of the tube Z fulfills*

$$\gamma_\infty(Z)\lesssim\frac{a^{n-1}}{\sqrt{\beta}}\,e^{-\beta}.$$

Proof Proposition 4.2 yields, since $H(0,\tilde{x})\simeq|\tilde{x}|\simeq\sqrt{\beta}$,

$$\gamma_\infty(Z)\simeq\int_0^\infty e^{-s\,\mathrm{tr}\,B}\,e^{-R(D_s\,\tilde{x})}\int_\Omega H(0,\tilde{x})\,dS(\tilde{x})\,ds\lesssim\sqrt{\beta}\,a^{n-1}\int_0^\infty e^{-s\,\mathrm{tr}\,B}e^{-R(D_s\,\tilde{x})}\,ds.$$

By (4.3) we have

$$R(D_s\,\tilde{x})-R(\tilde{x})\simeq\int_0^s\left|D_{s'}\,\tilde{x}\right|^2ds'\gtrsim s|\tilde{x}|^2\simeq s\beta,$$

which implies

$$\gamma_\infty(Z)\lesssim\sqrt{\beta}\,a^{n-1}\,e^{-\beta}\int_0^\infty e^{-s\,\mathrm{tr}\,B}\,e^{-cs\beta}\;ds.$$

Assuming β large enough, one has $c\beta>-2\,\mathrm{tr}\,B$, and then the last integral is finite and no larger than C/β. The lemma follows. □

5 Simplifications

In this section, we introduce some preliminary simplifications and reductions for the proof of (1.3), i.e., of Theorem 1.1.

(1) We may assume that f is nonnegative and normalized in the sense that

$$\|f\|_{L^1(\gamma_\infty)}=1,$$

since this involves no loss of generality.

(2) We may assume that α is large, $\alpha>C$, since otherwise (1.3) and (1.4) are trivial.

(3) In many cases, we may restrict x in (1.3) and (1.4) to the ellipsoidal annulus

$$\mathcal{E}_\alpha = \left\{x \in \mathbb{R}^n : \frac{1}{2}\log\alpha \le R(x) \le 2\log\alpha\right\}.$$

To begin with, we can always forget the unbounded component of the complement of $\mathcal{E}_\alpha$, since

$$\begin{aligned}&\gamma_\infty\{x \in \mathbb{R}^n : R(x) > 2\log\alpha\}\\ &\lesssim \int_{R(x)>2\log\alpha} \exp(-R(x))\,dx \lesssim (\log\alpha)^{(n-2)/2}\exp(-2\log\alpha) \lesssim \frac{1}{\alpha}.\end{aligned} \tag{5.1}$$

(4) When $t > 1$, we may forget also the inner region where $R(x) < \frac{1}{2}\log\alpha$. Indeed, from (3.5) we get, if $(x, u) \in \mathbb{R}^n \times \mathbb{R}^n$ with $R(x) < \frac{1}{2}\log\alpha$,

$$K_t(x, u) \lesssim e^{R(x)} < \sqrt{\alpha} < \alpha,$$

since α is large. In other words, for any $(x, u) \in \mathbb{R}^n \times \mathbb{R}^n$

$$R(x) < \frac{1}{2}\log\alpha \qquad \Rightarrow \qquad K_t(x, u) \lesssim \alpha, \tag{5.2}$$

for all $t > 1$.
Replacing α by $C\alpha$ for some C, we see from (3) and (4) that we can assume $x \in \mathcal{E}_\alpha$ in the proof of (1.3) and (1.4), when the supremum in the maximal operator is taken only over $t > 1$.

Before introducing the last simplification, we need to define a global region

$$G = \left\{(x, u) \in \mathbb{R}^n \times \mathbb{R}^n : |x - u| > \frac{1}{1 + |x|}\right\}$$

and a local region

$$L = \left\{(x, u) \in \mathbb{R}^n \times \mathbb{R}^n : |x - u| \le \frac{1}{1 + |x|}\right\}.$$

Notice that the definition of G and L does not depend on Q and B.

(5) When $t \le 1$ and $(x, u) \in G$, we shall see that (5.2) is still valid, and it is again enough to consider $x \in \mathcal{E}_\alpha$.

To prove this, we need a lemma which will also be useful later.

Lemma 5.1 *If* $(x, u) \in G$ *and* $0 < t \le 1$, *then*

$$\frac{1}{(1 + |x|)^2} \lesssim t^2|x|^2 + |u - D_t\, x|^2.$$

Proof From the definition of G and (4.4) we get

$$\frac{1}{1 + |x|} \le |x - u| \le |x - D_t\, x| + |D_t\, x - u| \lesssim t|x| + |u - D_t\; x|.$$

The lemma follows. □

To verify now (5.2) in the global region with $t \le 1$, we recall from (3.4) that

$$K_t(x,u) \lesssim \frac{e^{R(x)}}{t^{n/2}} \exp\left(-c\,\frac{|u - D_t\,x|^2}{t}\right).$$

It follows from Lemma 5.1 that

$$t^2 \gtrsim \frac{1}{(1+|x|)^4} \qquad \text{or} \qquad \frac{|u - D_t\,x|^2}{t} \gtrsim \frac{1}{(1+|x|)^2 t}. \tag{5.3}$$

The first inequality here implies that

$$K_t(x,u) \lesssim e^{R(x)}\,(1+|x|)^n \lesssim e^{2R(x)},$$

and (5.2) follows. If the second inequality of (5.3) holds, we have

$$K_t(x,u) \lesssim \frac{e^{R(x)}}{t^{n/2}} \exp\left(-\frac{c}{(1+|x|)^2 t}\right) \lesssim e^{R(x)}\,(1+|x|)^n,$$

and we get the same estimate. Thus (5.2) is verified.

Finally, let

$$\mathcal{H}_*^G f(x) = \sup_{0<t\le 1} \left| \int K_t(x,u)\,\chi_G(x,u)\,f(u)\,d\gamma_\infty(u) \right|,$$

and

$$\mathcal{H}_*^L f(x) = \sup_{0<t\le 1} \left| \int K_t(x,u)\,\chi_L(x,u)\,f(u)\,d\gamma_\infty(u) \right|.$$

6 The case of large *t*

In this section, we consider the supremum in the definition of the maximal operator taken only over $t > 1$, and we prove (1.4).

Proposition 6.1 *For all functions $f \in L^1(\gamma_\infty)$ such that $\|f\|_{L^1(\gamma_\infty)} = 1$,*

$$\gamma_\infty \left\{ x : \sup_{t>1} |\mathcal{H}_t f(x)| > \alpha \right\} \lesssim \frac{1}{\alpha\sqrt{\log\alpha}}, \qquad \alpha > 2. \tag{6.1}$$

In particular, the maximal operator

$$\sup_{t>1} |\mathcal{H}_t f(x)|$$

is of weak type $(1, 1)$ *with respect to the invariant measure* γ_∞.

Proof We can assume that $f \ge 0$. Looking at the arguments in Sect. 5, items (3) and (4), we see that it suffices to consider points $x \in \mathcal{E}_\alpha$. For both x and u we use the coordinates introduced in (4.5) with $\beta = \log\alpha$, that is,

$$x = D_s\,\tilde{x}, \qquad u = D_{s'}\,\tilde{u},$$

where $\tilde{x}, \tilde{u} \in E_{\log\alpha}$ and $s, s' \in \mathbb{R}$.

From (3.5) we have

$$K_t(x,u) \lesssim \exp(R(x)) \exp\left(-c\left|D_{-t}\,u - x\right|^2\right)$$

for $t > 1$ and $x, u \in \mathbb{R}^n$. Since $x \in \mathcal{E}_\alpha$ and $D_{-t}\, u = D_{s'-t}\, \tilde{u}$, we can apply Lemma 4.3 (i), getting

$$\left|D_{-t}\, u - x\right| \gtrsim \left|\tilde{x} - \tilde{u}\right|,$$

so that

$$\int K_t(x,u) f(u)\, d\gamma_\infty(u) \lesssim \exp\left(R(D_s\, \tilde{x})\right) \int \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) f(u)\, d\gamma_\infty(u).$$

In view of (4.3), the right-hand side here is strictly increasing in s, and therefore the inequality

$$\exp\left(R(D_s\, \tilde{x})\right) \int \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) f(u)\, d\gamma_\infty(u) > \alpha \tag{6.2}$$

holds if and only if $s > s_\alpha(\tilde{x})$ for some function $\tilde{x} \mapsto s_\alpha(\tilde{x})$, with equality for $s = s_\alpha(\tilde{x})$. Since $\alpha > 2$ and $\|f\|_{L^1(\gamma_\infty)} = 1$, it follows that $s_\alpha(\tilde{x}) > 0$.

For some C, the set of points $x \in \mathcal{E}_\alpha$ where the supremum in (6.1) is larger than $C\alpha$ is contained in the set $\mathcal{A}(\alpha)$ of points $D_s\, \tilde{x} \in \mathcal{E}_\alpha$ fulfilling (6.2). We use Proposition 4.2 to estimate the γ_∞ measure of $\mathcal{A}(\alpha)$. Observe that $H(0, \tilde{x}) \simeq |\tilde{x}| \simeq \sqrt{\log \alpha}$ and that $D_s\, \tilde{x} \in \mathcal{E}_\alpha$ implies $s \lesssim 1$, so that also $e^{-s \operatorname{tr} B} \lesssim 1$. We get

$$\begin{aligned}
\gamma_\infty(\mathcal{A}(\alpha)) &= \int_{\mathcal{A}(\alpha)\cap\mathcal{E}_\alpha} e^{-R(x)}\, dx \\
&\lesssim \sqrt{\log \alpha} \int_{E_{\log \alpha}} \int_{s_\alpha(\tilde{x})}^{C} e^{-R(D_s\, \tilde{x})}\, ds\, dS(\tilde{x}) \\
&\lesssim \sqrt{\log \alpha} \int_{E_{\log \alpha}} \int_{s_\alpha(\tilde{x})}^{+\infty} \exp\left(-R(D_{s_\alpha(\tilde{x})}\, \tilde{x}) - c \log \alpha\, (s - s_\alpha(\tilde{x}))\right) ds\, dS(\tilde{x}),
\end{aligned}$$

where the last inequality follows from (4.3), since $|D_s\, \tilde{x}|^2 \gtrsim |\tilde{x}|^2 \simeq \log \alpha$. Integrating in s, we obtain

$$\gamma_\infty(\mathcal{A}(\alpha)) \lesssim \frac{1}{\sqrt{\log \alpha}} \int_{E_{\log \alpha}} \exp\left(-R(D_{s_\alpha(\tilde{x})}\, \tilde{x})\right) dS(\tilde{x}).$$

Now combine this estimate with the case of equality in (6.2) and change the order of integration, to get

$$\begin{aligned}
\gamma_\infty(\mathcal{A}(\alpha)) &\lesssim \frac{1}{\alpha\sqrt{\log \alpha}} \int \int_{E_{\log \alpha}} \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right) dS(\tilde{x})\, f(u)\, d\gamma_\infty(u) \\
&\lesssim \frac{1}{\alpha\sqrt{\log \alpha}} \int f(u)\, d\gamma_\infty(u)\,,
\end{aligned}$$

which proves Proposition 6.1. □

Finally, we show that the factor $1/\sqrt{\log \alpha}$ in (6.1) is sharp.

Proposition 6.2 *For any $t > 1$ and any large α, there exists a function f normalized in $L^1(\gamma_\infty)$ and such that*

$$\gamma_\infty\, \{x : |\mathcal{H}_t\, f(x)| > \alpha\} \simeq \frac{1}{\alpha\sqrt{\log \alpha}}\,.$$

Proof Take a point z with $R(z) = \log\alpha$, and let f be (an approximation of) a Dirac measure at the point $u = D_t z$. Then, as a consequence of (3.5), $K_t(x,u) \simeq \exp(R(x))$ when x is in the ball $B(D_{-t}u, 1) = B(z, 1)$. We then have $\mathcal{H}_t f(x) = K_t(x,u) \gtrsim \alpha$ in the set $\mathcal{B} = \{x \in B(z,1) \colon R(x) > R(z)\}$, whose measure is

$$\gamma_\infty(\mathcal{B}) \simeq e^{-R(z)} \frac{1}{\sqrt{R(z)}} = \frac{1}{\alpha\sqrt{\log\alpha}}.$$

□

7 The local case for small t

Proposition 7.1 *If* $(x,u) \in L$ *and* $0 < t \le 1$, *then*

$$\left|K_t(x,u)\right| \lesssim \frac{\exp\left(R(x)\right)}{t^{n/2}} \exp\left(-c\,\frac{|u-x|^2}{t}\right).$$

Proof In view of (3.4), it is enough to show that

$$\frac{|u - D_t x|^2}{t} \ge \frac{|u-x|^2}{t} - C. \tag{7.1}$$

We write

$$|u - D_t x|^2 = |u - x + x - D_t x|^2 = |u-x|^2 + 2\langle u-x, x - D_t x\rangle + |x - D_t x|^2$$
$$\ge |u-x|^2 - 2|u-x|\,|x - D_t x|.$$

By (4.4),

$$|u-x|\,|x - D_t x| \lesssim |u-x|\,t\,|x| \le t$$

since $(x,u) \in L$, and (7.1) follows. □

Proposition 7.2 *The maximal operator* $\mathcal{H}_*^L$ *is of weak type* $(1,1)$ *with respect to the invariant measure* γ_∞.

Proof The proof is standard, since Proposition 7.1 implies

$$\mathcal{H}_*^L f(x) \lesssim \sup_{0<t\le 1} \frac{\exp\left(R(x)\right)}{t^{n/2}} \int \exp\left(-c\,\frac{|x-u|^2}{t}\right) \chi_L(x,u)\, f(u)\, d\gamma_\infty(u).$$

The supremum here defines an operator of weak type $(1,1)$ with respect to Lebesgue measure in $\mathbb{R}^n$. From this the proposition follows, cf. [7, Section 3]. □

8 The global case for small t

In this section, we conclude the proof of Theorem 1.1.

Proposition 8.1 *The maximal operator* $\mathcal{H}_*^G$ *is of weak type* $(1,1)$ *with respect to the invariant measure* γ_∞.

Proof We take f and α as in items (1) and (2) of Sect. 5. Then item (5) tells us that we need only consider $\mathcal{H}_*^G f(x)$ for $x \in \mathcal{E}_\alpha$.

For $m \in \mathbb{N}$ and $0 < t \leq 1$, we introduce regions $\mathcal{S}_t^m$. If $m > 0$, we let

$$\mathcal{S}_t^m = \left\{(x,u) \in G : 2^{m-1}\sqrt{t} < |u - D_t\, x| \leq 2^m \sqrt{t}\right\}.$$

If $m = 0$, we replace the condition $2^{m-1}\sqrt{t} < |u - D_t\, x| \leq 2^m\sqrt{t}$ by $|u - D_t\, x| \leq \sqrt{t}$. Note that for any fixed $t \in (0,1]$ these sets form a partition of G.

In the set $\mathcal{S}_t^m$ we have, because of (3.4),

$$K_t(x,u) \lesssim \frac{\exp(R(x))}{t^{n/2}} \exp\left(-c 2^{2m}\right).$$

Then setting

$$\mathcal{K}_t^m(x,u) = \frac{\exp(R(x))}{t^{n/2}} \chi_{\mathcal{S}_t^m}(x,u),$$

one has, for all $(x,u) \in G$ and $0 < t < 1$,

$$K_t(x,u) \lesssim \sum_{m=0}^{\infty} \exp\left(-c2^{2m}\right) \mathcal{K}_t^m(x,u).$$

Hence, it suffices to prove that for $m = 0, 1, \ldots$

$$\gamma_\infty \left\{x \in \mathcal{E}_\alpha : \sup_{0<t\leq 1} \int \mathcal{K}_t^m(x,u)\, f(u)\, d\gamma_\infty(u) > \alpha\right\} \lesssim \frac{2^{Cm}}{\alpha}, \tag{8.1}$$

for large α and some C, since this will allow summing in m in the space $L^{1,\infty}(\gamma_\infty)$.

Fix $m \in \mathbb{N}$ and assume that $(x,u) \in S_t^m$ for some $t \in (0,1]$, so that $|u - D_t\, x| \leq 2^m\sqrt{t}$. Then Lemma 5.1 leads to

$$1 \lesssim (1+|x|)^4 t^2 + (1+|x|)^2\, 2^{2m}\, t \leq ((1+|x|)^2\, 2^{2m}\, t)^2 + (1+|x|)^2\, 2^{2m}\, t.$$

Consequently, a point $x \in \mathcal{E}_\alpha$ satisfies

$$(1+|x|)^2\, 2^{2m}\, t \gtrsim 1 \tag{8.2}$$

as soon as there exists a point u with $\mathcal{K}_t^m(x,u) \neq 0$, and then $t \geq \varepsilon > 0$ for some $\varepsilon = \varepsilon(\alpha, m) > 0$. Hence the supremum in (8.1) will be the same if taken only over $\varepsilon \leq t \leq 1$, and it follows that this supremum is a continuous function of $x \in \mathcal{E}_\alpha$.

To prove (8.1), the idea, which goes back to [15], is to construct a finite sequence of pairwise disjoint balls $\left(\mathcal{B}^{(\ell)}\right)_{\ell=1}^{\ell_0}$ in $\mathbb{R}^n$ and a finite sequence of sets $\left(\mathcal{Z}^{(\ell)}\right)_{\ell=1}^{\ell_0}$ in $\mathbb{R}^n$, called forbidden zones. These zones will together cover the level set in (8.1). We will then verify that

$$\left\{x \in \mathcal{E}_\alpha : \sup_{\varepsilon\leq t\leq 1} \int \mathcal{K}_t^m(x,u)\, f(u)\, d\gamma_\infty(u) \geq \alpha\right\} \subset \bigcup_{\ell=1}^{\ell_0} \mathcal{Z}^{(\ell)}, \tag{8.3}$$

that for each ℓ

$$\gamma_\infty(\mathcal{Z}^{(\ell)}) \lesssim \frac{2^{Cm}}{\alpha} \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty(u), \tag{8.4}$$

and that the $\mathcal{B}^{(\ell)}$ are pairwise disjoint. This would imply

$$\gamma_\infty\Big(\bigcup_{\ell=1}^{\ell_0} \mathcal{Z}^{(\ell)}\Big) \lesssim \frac{2^{Cm}}{\alpha} \sum_{\ell=1}^{\ell_0} \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty(u) \lesssim \frac{2^{Cm}}{\alpha},$$

and thus also (8.1) and Proposition 8.1.

The sets $\mathcal{B}^{(\ell)}$ and $\mathcal{Z}^{(\ell)}$ will be introduced by means of a sequence of points $x^{(\ell)}$, $\ell = 1, \dots, \ell_0$, which we define by recursion. To start, we choose as $x^{(1)}$ a point where the quadratic form $R(x)$ takes its minimal value in the compact set

$$\mathcal{A}_1(\alpha) = \left\{ x \in \mathcal{E}_\alpha : \sup_{\varepsilon \le t \le 1} \int \mathcal{K}_t^m(x, u)\, f(u)\, d\gamma_\infty \ge \alpha \right\}.$$

However, should this set be empty, (8.1) is immediate.

We now describe the recursion to construct $x^{(\ell)}$ for $\ell \ge 2$. Like $x^{(1)}$, these points will satisfy

$$\sup_{\varepsilon \le t \le 1} \int \mathcal{K}_t^m(x^{(\ell)}, u)\, f(u)\, d\gamma_\infty \ge \alpha.$$

Once an $x^{(\ell)}$, $\ell \ge 1$, is defined, we can thus by continuity choose $t_\ell \in [\varepsilon, 1]$ such that

$$\int \mathcal{K}_{t_\ell}^m(x^{(\ell)}, u)\, f(u)\, d\gamma_\infty \ge \alpha. \tag{8.5}$$

Using this t_ℓ, we associate with $x^{(\ell)}$ the tube

$$\mathcal{Z}^{(\ell)} = \left\{ D_s\, \eta \in \mathbb{R}^n : s \ge 0,\ R(\eta) = R(x^{(\ell)}),\ |\eta - x^{(\ell)}| < A\, 2^{3m} \sqrt{t_\ell} \right\},$$

Here the constant $A > 0$ is to be determined, depending only on n, Q and B.

All the $x^{(\ell)}$ will be minimizing points of $R(x)$. To avoid having them too close to one another, we will not allow $x^{(\ell)}$ to be in any $\mathcal{Z}^{(\ell')}$ with $\ell' < \ell$. More precisely, assuming $x^{(1)}, \dots, x^{(\ell)}$ already defined, we will choose $x^{(\ell+1)}$ as a minimizing point of $R(x)$ in the set

$$\mathcal{A}_{\ell+1}(\alpha) = \left\{ x \in \mathcal{E}_\alpha \setminus \bigcup_{\ell'=1}^{\ell} \mathcal{Z}^{(\ell')} : \sup_{\varepsilon \le t \le 1} \int \mathcal{K}_t^m(x, u)\, f(u)\, d\gamma_\infty(u) \ge \alpha \right\}, \tag{8.6}$$

provided this set is nonempty. But if $\mathcal{A}_{\ell+1}(\alpha)$ is empty, the process stops with $\ell_0 = \ell$ and (8.3) follows. We will see that this actually occurs for some finite ℓ.

Now assume that $\mathcal{A}_{\ell+1}(\alpha) \ne \emptyset$. In order to assure that a minimizing point exists, we must verify that $\mathcal{A}_{\ell+1}(\alpha)$ is closed and thus compact, although the $\mathcal{Z}^{(\ell')}$ are not open. To do so, observe that for $1 \le \ell' \le \ell$, the minimizing property of $x^{(\ell')}$ means that there is no point x in $\mathcal{A}_{\ell'}(\alpha)$ with $R(x) < R(x^{(\ell')})$. Thus we have the inclusions

$$\mathcal{A}_{\ell+1}(\alpha) \subset \mathcal{A}_{\ell'}(\alpha) \subset \left\{ x : R(x) \ge R(x^{(\ell')}) \right\}, \qquad 1 \le \ell' \le \ell.$$

It follows that

$$\begin{aligned} \mathcal{A}_{\ell+1}(\alpha) &= \mathcal{A}_{\ell+1}(\alpha) \cap \bigcap_{1 \le \ell' \le \ell} \{x : R(x) \ge R(x^{(\ell')})\} \\ &= \bigcap_{\ell'=1}^{\ell} \left\{ x \in \mathcal{E}_\alpha \setminus \mathcal{Z}^{(\ell')} : R(x) \ge R(x^{(\ell')}),\ \sup_{\varepsilon \le t \le 1} \int \mathcal{K}_t^m(x, u)\, f(u)\, d\gamma_\infty(u) \ge \alpha \right\}. \end{aligned}$$

For each $\ell' = 1, \dots, \ell$ we have

$$\begin{aligned}&\{x \in \mathcal{E}_\alpha \setminus \mathcal{Z}^{(\ell')} : R(x) \geq R(x^{(\ell')})\}\\&= \left\{D_s\, \eta \in \mathcal{E}_\alpha : s \geq 0,\ R(\eta) = R(x^{(\ell')}),\ |\eta - x^{(\ell')}| \geq A 2^{3m} \sqrt{t_{\ell'}}\right\},\end{aligned}$$

and this set is closed. It follows that $\mathcal{A}_{\ell+1}(\alpha)$ is compact, and a minimizing point $x^{(\ell+1)}$ can be chosen. Thus the recursion is well defined.

We observe that (8.2) applies to t_ℓ and $x^{(\ell)}$, and $|x^{(\ell)}|$ is large, so

$$|x^{(\ell)}|^2\, 2^{2m}\, t_\ell \gtrsim 1. \tag{8.7}$$

Further, we define balls

$$\mathcal{B}^{(\ell)} = \{u \in \mathbb{R}^n : |u - D_{t_\ell}\, x^{(\ell)}| \leq 2^m \sqrt{t_\ell}\,\}.$$

Because of the definitions of $\mathcal{K}_t^m$ and $\mathcal{S}_t^m$, the inequality (8.5) implies

$$\alpha \leq \frac{\exp\left(R(x^{(\ell)})\right)}{t_\ell^{n/2}} \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty(u). \tag{8.8}$$

It remains to verify the claimed properties of $\mathcal{B}^{(\ell)}$ and $\mathcal{Z}^{(\ell)}$. The arguments below follow the lines of the proof of Lemma 6.2 in [4], with only slight modifications.

Lemma 8.2 *The balls $\mathcal{B}^{(\ell)}$ are pairwise disjoint.*

Proof Two balls $\mathcal{B}^{(\ell)}$ and $\mathcal{B}^{(\ell')}$ with $\ell < \ell'$ will be disjoint if

$$\left|D_{t_\ell}\, x^{(\ell)} - D_{t_{\ell'}}\, x^{(\ell')}\right| > 2^m (\sqrt{t_\ell} + \sqrt{t_{\ell'}}). \tag{8.9}$$

By means of our polar coordinates with $\beta = R(x^{(\ell)})$, we write

$$x^{(\ell')} = D_s\, \tilde{x}^{(\ell')}$$

for some $\tilde{x}^{(\ell')}$ with $R(\tilde{x}^{(\ell')}) = R(x^{(\ell)})$ and some $s \in \mathbb{R}$. Note that $s \geq 0$, because $R(x^{(\ell')}) \geq R(x^{(\ell)})$. Since $x^{(\ell')}$ does not belong to the forbidden zone $\mathcal{Z}^{(\ell)}$, we must have

$$|\tilde{x}^{(\ell')} - x^{(\ell)}| \geq A 2^{3m} \sqrt{t_\ell}. \tag{8.10}$$

We first assume that $t_{\ell'} \geq M\, 2^{4m}\, t_\ell$, for some $M = M(n, Q, B) \geq 2$ to be chosen. Lemma 4.3 (ii) implies

$$\left|D_{t_\ell}\, x^{(\ell)} - D_{t_{\ell'}}\, x^{(\ell')}\right| = \left|D_{t_\ell}\, x^{(\ell)} - D_{t_{\ell'}+s}\, \tilde{x}^{(\ell')}\right| \gtrsim |x^{(\ell)}|\, (t_{\ell'} + s - t_\ell) \gtrsim |x^{(\ell)}|\, t_{\ell'},$$

the last step by our assumption. Using again the assumption and then (8.7), we get

$$|x^{(\ell)}|\, t_{\ell'} \gtrsim |x^{(\ell)}|\, \sqrt{M}\, 2^{2m} \sqrt{t_\ell}\, \sqrt{t_{\ell'}} \gtrsim \sqrt{M}\, 2^m \sqrt{t_{\ell'}} \simeq \sqrt{M}\, 2^m\, (\sqrt{t_{\ell'}} + \sqrt{t_\ell}).$$

Fixing M suitably large, we obtain (8.9) from the last two formulae.

It remains to consider the case when $t_{\ell'} < M\, 2^{4m}\, t_\ell$. Then

$$\sqrt{t_\ell} > \frac{2^{-2m-1}}{\sqrt{M}} (\sqrt{t_{\ell'}} + \sqrt{t_\ell}).$$

Applying this to (8.10), we obtain (8.9) by choosing A so that $A/\sqrt{M}$ is large enough. □

We next verify that the sequence $(x^{(\ell)})$ is finite. For $\ell < \ell'$, we have (8.10), and Lemma 4.3 (i) implies

$$\left|x^{(\ell')} - x^{(\ell)}\right| \gtrsim A\, 2^{3m} \sqrt{t_\ell}.$$

Since $t_\ell \geq \varepsilon$, we see that the distance $\left|x^{(\ell')} - x^{(\ell)}\right|$ is bounded below by a positive constant. But all the $x^{(\ell)}$ are contained in the bounded set $\mathcal{E}_\alpha$, so they are finite in number. Thus the set considered in (8.6) must be empty for some ℓ, and the recursion stops. This implies (8.3).

We finally prove (8.4) . Observe that the forbidden zone $\mathcal{Z}^{(\ell)}$ is a tube as defined in (4.12), with $a = A\, 2^{3m}\sqrt{t_\ell}$ and $\beta = R(x^{(\ell)})$. This value of β is large since $x^{(\ell)} \in \mathcal{E}_\alpha$, and thus we can apply Lemma 4.4 to obtain

$$\gamma_\infty(\mathcal{Z}^{(\ell)}) \lesssim \frac{\left(A 2^{3m}\sqrt{t_\ell}\right)^{n-1}}{\sqrt{R(x^{(\ell)})}} \exp\left(-R(x^{(\ell)})\right).$$

We bound the exponential here by means of (8.8) and observe that $R(x^{(\ell)}) \sim |x^{(\ell)}|^2$, getting

$$\gamma_\infty(\mathcal{Z}^{(\ell)}) \lesssim \frac{1}{\alpha |x^{(\ell)}| \sqrt{t_\ell}} (A 2^{3m})^{n-1} \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty(u).$$

As a consequence of (8.7), we obtain

$$\gamma_\infty(\mathcal{Z}^{(\ell)}) \lesssim \frac{2^m}{\alpha} \left(A 2^{3m}\right)^{n-1} \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty(u) \lesssim \frac{2^{Cm}}{\alpha} \int_{\mathcal{B}^{(\ell)}} f(u)\, d\gamma_\infty(u),$$

proving (8.4). This concludes the proof of Proposition 8.1. □

Funding Open access funding provided by University of Gothenburg.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Aimar, H., Forzani, L., Scotto, R.: On Riesz transforms and maximal functions in the context of Gaussian harmonic analysis. Trans. Am. Math. Soc. **359**, 2137–2154 (2005)
2. Ball, K., Barthe, F., Bednorz, W., Oleszkiewicz, K., Wolff, P.: L^1-smoothing for the Ornstein–Uhlenbeck semigroup. Mathematika **59**, 160–168 (2013)
3. Carbonaro, A., Dragičević, O.: Bounded holomorphic functional calculus for nonsymmetric Ornstein–Uhlenbeck operators. Ann. Sc. Norm. Super. Pisa Cl. Sci. **XIX** (5), 1497–1533 (2019)
4. Casarino, V., Ciatti, P., Sjögren, P.: The maximal operator of a normal Ornstein–Uhlenbeck semigroup is of weak type (1,1). Ann. Sc. Norm. Sup. Pisa Cl. Sci. **XXI**(5), 385–410 (2020)
5. Casarino, V., Ciatti, P., Sjögren, P.: Riesz transforms of a general Ornstein–Uhlenbeck semigroup. Calculus Var. Partial Differ. Equ. **60**, 135 (2021)
6. Chojnowska-Michalik, A., Goldys, B.: Nonsymmetric Ornstein–Uhlenbeck semigroup as second quantized operator. J. Math. Kyoto Univ. **36**, 481–498 (1996)
7. Garcìa-Cuerva, J., Mauceri, G., Meda, S., Sjögren, P., Torrea, J.L.: Maximal operators for the holomorphic Ornstein–Uhlenbeck semigroup. J. Lond. Math. Soc. **67**, 219–234 (2003)

8. Lehec, J.: Regularization in L^1 for the Ornstein–Uhlenbeck semigroup. Ann. Faculté des Sci. Toulouse Math. **25**, 191–204 (2016)
9. Mauceri, G., Noselli, L.: The maximal operator associated to a non symmetric Ornstein–Uhlenbeck semigroup. J. Fourier Anal. Appl. **15**, 179–200 (2009)
10. Menárguez, T., Pérez, S., Soria, F.: Pointwise and norm estimates for operators associated with the Ornstein–Uhlenbeck semigroup. C. R. Acad. Sci. Paris Sér. I **326**, 25–30 (1998)
11. Menárguez, T., Pérez, S., Soria, F.: The Mehler maximal function: a geometric proof of the weak type (1,1). J. Lond. Math. Soc. **62**, 846–856 (2000)
12. Metafune, G., Prüss, J., Rhandi, A., Schnaubelt, R.: The domain of the Ornstein–Uhlenbeck operator on a L^p-space with invariant measure. Ann. Sc. Norm. Super. Pisa Cl. Sci. **1**, 471–487 (2002)
13. Muckenhoupt, B.: Poisson integrals for Hermite and Laguerre expansions. Trans. Am. Math. Soc. **139**, 231–242 (1969)
14. Pérez, S., Soria, F.: Operators associated with the Ornstein–Uhlenbeck semigroup. J. Lond. Math. Soc. **61**, 857–871 (2000)
15. Sjögren, P.: On the maximal function for the Mehler kernel. In: Mauceri G., Weiss G. (eds.) Harmonic Analysis, Cortona 1982. Springer Lecture Notes in Mathematics, vol. 992, pp. 73–82 (1983)
16. Stein, E.M.: Topics in Harmonic Analysis Related to the Littlewood–Paley Theory, Annals in Mathematics Studies. Princeton University Press, Princeton (1970)
17. Urbina-Romero, W.: Gaussian Harmonic Analysis. Springer Monographs in Mathematics (2019)
18. van Neerven, J.M.A.M.: Nonsymmetric Ornstein–Uhlenbeck semigroups in Banach spaces. J. Funct. Anal. **155**, 495–535 (1998)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.