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Higher Order Deformed Elliptic Ruijsenaars Operators

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Abstract: We present four infinite families of mutually commuting difference operators which include the deformed elliptic Ruijsenaars operators. The trigonometric limit of this kind of operators was previously introduced by Feigin and Silantyev. They provide a quantum mechanical description of two kinds of relativistic quantum mechanical particles which can be identified with particles and anti-particles in an underlying quantum field theory. We give direct proofs of the commutativity of our operators and of some other fundamental properties such as kernel function identities. In particular, we give a rigorous proof of the quantum integrability of the deformed Ruijsenaars model.

1. Introduction

The quantum Calogero–Moser–Sutherland systems form an important class of integrable systems in quantum mechanics. Chalykh, Feigin and Veselov [6] discovered that certain deformations of such systems maintain integrability. For instance, the Schrödinger operator

$$
\begin{aligned}
H = &-\sum_{i=1}^{m} \frac{1}{2}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{r} \frac{g}{2}\frac{\partial^2}{\partial y_i^2} \\
&+ \sum_{1\le i<j\le m} \frac{g(g+1)}{(x_i-x_j)^2} + \sum_{1\le i<j\le r} \frac{1/g+1}{(y_i-y_j)^2} + \sum_{i=1}^{m}\sum_{j=1}^{r} \frac{g+1}{(x_i-y_j)^2}
\end{aligned}
$$

is integrable for arbitrary variable numbers m and r and coupling parameter g [6,24], where $r = 0$ is the non-deformed case first studied by Calogero [3]. Such deformed models turned out to be intimately connected to Lie superalgebras and related analogues of symmetric functions such as super-Jack polynomials [24,25]. From a physics point of

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view, the deformed model describes a system of arbitrary numbers of two different kinds of identical particles. The Schrödinger operator above corresponds to the rational case; the most general elliptic case is obtained by replacing the interaction potential $1/x^2$ by the Weierstrass $\wp$-function $\wp(x|\omega_1, \omega_2)$.

Ruijsenaars [21] introduced relativistic generalizations of quantum Calogero–Moser–Sutherland systems, defined by difference operators rather than differential operators. Deformed versions of such systems were first considered by Chalykh [4,5]. In greater generality, they were introduced and studied by Sergeev and Veselov [25,26] in the trigonometric case and by Atai together with two of us [1] in the elliptic case. They describe systems of two kinds of identical particles which can be interpreted as particles and anti-particles in an underlying relativistic quantum field theory [2]. Feigin and Silantyev [9] constructed higher order operators that commute with the first order operators of Sergeev and Veselov. They also showed that a sufficiently large subset of these operators is algebraically independent, concluding that the deformed models remain integrable in the relativistic setting.

In the present paper, we introduce and study elliptic extensions of the operators of Feigin and Silantyev. To be more precise, for fixed non-negative integers m and r, we introduce a family of operators [see (2.7) for the explicit expression]

$$H_{m,r}^{(k)}(x_1, \dots, x_m; y_1, \dots, y_r; \delta, \kappa), \qquad k \in \mathbb{Z}_{\geq 0}. \tag{1.1a}$$

They are linear combinations of shift operators acting on functions in the x- and y-variables as

$$f(x_1, \dots, x_m; y_1, \dots, y_r) \mapsto f(x_1 + \mu_1\delta, \dots, x_m + \mu_m\delta; y_1 - I_1\kappa, \cdots, y_r - I_r\kappa),$$

where $\mu_j \in \mathbb{Z}_{\geq 0}$, $I_j \in \{0, 1\}$ and the total degree $\sum_j \mu_j + \sum_j I_j$ is fixed to k. The Ruijsenaars operators correspond to the case $m = 0$ and the case $r = 0$ give the operators of Noumi and Sano [19].

In the original Ruijsenaars model, the Hamiltonian and momentum operator are (up to a similary transformation) linear combinations of the operator $H_{0,r}^{(1)}$ and the same operator with (δ, κ) replaced by $(-\delta, -\kappa)$. These linear combinations satisfy Poincaré algebra relations (see (A.1)) and thus describe a system of relativistic quantum particles. As explained in Appendix A, the Poincaré relations extend to the deformed case. For this purpose, it is essential to consider commutation relations between $H_{m,r}^{(k)}$ and modifications of these operators with shifts acting in the opposite direction. It turns out that there are four mutually commuting infinite families, given by (1.1a) together with

$$H_{r,m}^{(k)}(y_1, \dots, y_r; x_1, \dots, x_m; -\kappa, -\delta), \qquad k \in \mathbb{Z}_{\geq 0}, \tag{1.1b}$$

$$H_{m,r}^{(k)}(x_1 - \delta, \dots, x_m - \delta; y_1 + \kappa, \dots, y_r + \kappa; -\delta, -\kappa), \qquad k \in \mathbb{Z}_{\geq 0}, \tag{1.1c}$$

$$H_{r,m}^{(k)}(y_1 + \kappa, \dots, y_r + \kappa; x_1 - \delta, \dots, x_m - \delta; \kappa, \delta), \qquad k \in \mathbb{Z}_{\geq 0}. \tag{1.1d}$$

Roughly speaking, in (1.1b) we have interchanged the roles of the two types of particles, in (1.1c) we have reversed the direction of the shift operators and in (1.1d) we have made both these changes. The shifts in the x- and y-variables present in (1.1c) and (1.1d) could be eliminated by an overall translation (see (A.2)), but we avoid that since it would make most of our formulas slightly more complicated.

The parameters δ and κ are related to the standard parameters of Macdonald polynomial theory by $q = e^{2i\pi\delta}$, $t = e^{2i\pi\kappa}$. In the trigonometric limit case, the operators (1.1a) have super-Macdonald polynomials as joint eigenfunctions. The fact that there are four types of commuting operators (1.1) is reflected in symmetries of super-Macdonald polynomials under the transformations $(q, t) \mapsto (t^{-1}, q^{-1})$, (q^{-1}, t^{-1}) and (t, q), see [11, Prop. 3.3]. Very recently, several significant new results on joint eigenfunctions for the original Ruijsenaars operators have been obtained, including elliptic generalisations of results on Macdonald polynomials [7,16,18] and new connections to the fusion ring of conformal field theories [8]; see references therein for earlier important work on the subject. Generalising any of these results to the deformed operators (1.1a) remains an intriguing open problem.

The main result of the present paper is that the four infinite families of operators (1.1) mutually commute. Moreover, we prove that for generic δ and κ, the operators $H_{m,r}^{(k)}$ are algebraically independent for $1 \leq k \leq m+r$. This gives a rigorous proof that the deformed elliptic Ruijsenaars model is quantum integrable, which has until now been an unsolved problem. We also prove that the operators (1.1b) are in the algebraic closure of the operators (1.1a) (and vice versa). This generalizes the result of [19] that the Noumi–Sano operators are in the algebraic closure of the Ruijsenaars operators. The other two families are clearly outside this closure, since they act by shifts in the opposite direction. Finally, we show that our operators satisfy kernel function identities with respect to the same kernel function that was obtained in [1] in the first order case.

Our proofs are direct and based on non-trivial identities for theta functions that we refer to as source identities. They are also closely related to transformation formulas for multiple elliptic hypergeometric series found in [13,15,20].

In the main text, we present and prove the results in an additive notation close to the one used by Ruijsenaars [21]. For the convenience of the reader, in Appendix B we give the key formulas in the multiplicative notation generalizing the one used in the theory of Macdonald polynomials [17].

2. Main Results

We fix a non-zero odd entire function $x \mapsto [x]$, which satisfies the identity

$$[x+y][x-y][u+v][u-v]+[x+v][x-v][y+u][y-u]=[x+u][x-u][y+v][y-v]. \tag{2.1}$$

A generic such function can be written

$$[x]=Ce^{cx^2}\sigma(x|\omega_1,\omega_2), \tag{2.2}$$

where σ is the Weierstrass sigma function [29]. For our purposes, the prefactor Ce^{cx^2} can be viewed as a normalization and plays no essential role. Degenerate cases include the trigonometric solutions $[x]=\sin(\pi x/\omega)$, the hyperbolic solutions $[x]=\sinh(\pi x/\omega)$ and the rational solution $[x]=x$.

Throughout, δ and κ are fixed parameters. For simplicity, we will assume that

$$[n\delta]\neq 0, \quad [n\kappa]\neq 0, \qquad n\in\mathbb{Z}_{>0}; \tag{2.3}$$

see the end of Appendix A for a discussion of this condition.

For $k\in\mathbb{Z}_{\geq 0}$ we will write

$$[x]_k=[x][x+\delta]\cdots[x+(k-1)\delta] \tag{2.4}$$

and, for negative subscripts,

$$[x]_{-k} = \frac{1}{[x-k\delta]_k} = \frac{1}{[x-k\delta][x-(k-1)\delta]\cdots[x-\delta]}.$$

Occasionally, we indicate the dependence on δ as $[x;\delta]_k$.

We write T_x^δ for the difference operator

$$T_x^\delta f(x) = f(x+\delta)$$

and, more generally,

$$T_x^{\delta\mu} f(x_1,\cdots,x_n) = f(x_1+\mu_1\delta,\dots,x_n+\mu_n\delta),$$

when x and μ are vectors. We will write $\langle n\rangle = \{1,\dots,n\}$ (the notation $[n]$ is more common, but we wish to avoid confusion with the function satisfying (2.1)). Subsets $I\subseteq\langle n\rangle$ will be identified with vectors $(I_1,\dots,I_n)\in\{0,1\}^n$, where $I_j=1$ for $j\in I$ and $I_j=0$ otherwise. With this identification, we can write $T_x^{\delta I}=\prod_{j\in I}T_{x_j}^\delta$. The higher order Ruijsenaars operators are defined by

$$D_n^{(k)} = \sum_{I\subseteq\langle n\rangle,\,|I|=k}\ \prod_{i\in I,\,j\in I^c}\frac{[x_i-x_j+\kappa]}{[x_i-x_j]}\cdot T_x^{\delta I}, \tag{2.5}$$

where I^c denotes the complement of I in $\langle n\rangle$. It is a non-trivial fact that these operators commute for $0\le k\le n$ [21].

Noumi and Sano [19] introduced another family of elliptic difference operators, which we denote

$$H_n^{(k)} = \sum_{\mu\in\mathbb{Z}_{\geq 0}^n,\,|\mu|=k}\ \prod_{1\le i<j\le n}\frac{[x_i-x_j+(\mu_i-\mu_j)\delta]}{[x_i-x_j]}\prod_{i,j=1}^n\frac{[x_i-x_j+\kappa]_{\mu_i}}{[x_i-x_j+\delta]_{\mu_i}}\cdot T_x^{\delta\mu}.$$

Here, $|\mu|=\mu_1+\cdots+\mu_n$. They proved that they are related to the Ruijsenaars operators through the so called Wronski relation

$$\sum_{k+l=K}(-1)^k[k\kappa+l\delta]D_n^{(k)}H_n^{(l)}=0,\qquad K=1,2,3,\dots. \tag{2.6}$$

Since $D_n^{(0)}=\mathrm{id}$, this can be used to recursively write $H_n^{(l)}$ as a polynomial in the operators $D_n^{(k)}$ (and vice versa). As a consequence, all these operators commute.

In the present work we introduce and study a family of difference operators $H_{m,r}^{(k)}$ in $m+r$ variables

$$(x_1,\dots,x_m,y_1,\dots,y_r),$$

which generalize both $D_m^{(k)}$ and $H_r^{(k)}$. They are defined by

$$H_{m,r}^{(k)} = \sum_{\substack{\mu\in\mathbb{Z}_{\geq 0}^m,\,I\subseteq\langle r\rangle,\\ |\mu|+|I|=k}} C_{\mu,I}(x;y)\,T_x^{\delta\mu}T_y^{-\kappa I}, \tag{2.7a}$$

where

$$C_{\mu,I}(x;y)=(-1)^{|I|}\prod_{1\le i<j\le m}\frac{[x_i-x_j+(\mu_i-\mu_j)\delta]}{[x_i-x_j]}\prod_{i\in I,\,j\in I^c}\frac{[y_i-y_j-\delta]}{[y_i-y_j]}$$
$$\times\prod_{i,j=1}^{m}\frac{[x_i-x_j+\kappa]_{\mu_i}}{[x_i-x_j+\delta]_{\mu_i}}\prod_{i=1}^{m}\left(\prod_{j\in I}\frac{[x_i-y_j-\kappa]}{[x_i-y_j+\mu_i\delta]}\prod_{j\in I^c}\frac{[x_i-y_j-\delta]}{[x_i-y_j+(\mu_i-1)\delta]}\right). \quad (2.7b)$$

If $m>0$, $H_{m,r}^{(k)}$ is well-defined only if $[j\delta]\neq 0$ for $1\le j\le k$, since otherwise the factor $[x_i-x_j+\delta]_{\mu_i}$ vanishes for $j=i$ and $\mu_i=k$. This is guaranteed by our assumption (2.3).

Several special cases of the operators (2.7a) are known in the literature. Clearly, $H_{m,0}^{(k)}$ is equal to the Noumi–Sano operator $H_m^{(k)}$. The operator $H_{0,r}^{(k)}$ is equal to the Ruijsenaars operator $(-1)^k D_r^{(k)}$, with δ and $-\kappa$ interchanged. The operators $H_{m,r}^{(1)}$ are the deformed Ruijsenaars operators introduced in [1]. Finally, the trigonometric limit of the general operators $H_{m,r}^{(k)}$ were introduced by Sergeev and Veselov [25] for $k=1$ and by Feigin and Silantyev [9] in general; see also [11]. To make the connection to the operators in [9, Eq. (4.19)] one should rewrite (2.7b) using the elementary identity

$$\prod_{1\le i<j\le m}\frac{[x_i-x_j+(\mu_i-\mu_j)\delta]}{[x_i-x_j]}\prod_{i,j=1}^{m}\frac{1}{[x_i-x_j+\delta]_{\mu_i}}=(-1)^{|\mu|}\prod_{i,j=1}^{m}\frac{1}{[x_i-x_j-\delta\mu_j]_{\mu_i}}.$$

Our first main result is that these operators commute.

Theorem 2.1. *The operators* (2.7a) *satisfy* $[H_{m,r}^{(k)},H_{m,r}^{(l)}]=0$ *for all* $k,l\in\mathbb{Z}_{\ge 0}$.

Next, we prove that $m+r$ of the operators $H_{m,r}^{(k)}$ are algebraically independent. We interpret this as a rigorous formulation of quantum integrability for the deformed elliptic Ruijsenaars model.

Theorem 2.2. *For generic κ and δ, the operators $H_{m,r}^{(k)}$, $k=1,\dots,m+r$, are algebraically independent.*

As mentioned in the introduction, one can construct further commuting operators by making appropriate modification to $H_{m,r}^{(k)}$. We will first consider the family (1.1c). Writing the coefficients (2.7b) as $C_{\mu,I}(x;y;\delta,\kappa)$, we denote the corresponding operators

$$\hat{H}_{m,r}^{(k)}=\sum_{\substack{\mu\in\mathbb{Z}_{\ge 0}^m,\ I\subseteq\langle r\rangle,\\ |\mu|+|I|=k}}C_{\mu,I}(x_1-\delta,\dots,x_m-\delta;y_1+\kappa,\dots,y_r+\kappa;-\delta,-\kappa)\,T_x^{-\delta\mu}T_y^{\kappa I}. \quad (2.8)$$

Since they are obtained from $H_{m,r}^{(k)}$ by a change of variables, these operators commute among themselves. Our second main result states that the two families are mutually commuting.

Theorem 2.3. *We have* $[H_{m,r}^{(k)},\hat{H}_{m,r}^{(l)}]=0$ *for all* $k,l\in\mathbb{Z}_{\ge 0}$.

In the special case $m = 0$, Theorem 2.3 follows from Theorem 2.1 and the observation that [21]

$$\hat{H}_{0,r}^{(k)} = H_{0,r}^{(r-k)} \left(H_{0,r}^{(r)}\right)^{-1}$$

(note that $H_{0,r}^{(r)} = (-1)^r T_{y_1}^{-\kappa} \cdots T_{y_r}^{-\kappa}$ is invertible). We stress that when $m > 0$ this simple argument does not work, and Theorem 2.1 requires a separate proof.

Next, we consider the family (1.1b), which we denote

$$D_{m,r}^{(k)} = \sum_{\substack{\mu \in \mathbb{Z}_{\geq 0}^r,\, I \subseteq \langle m \rangle, \\ |\mu|+|I|=k}} C_{\mu,I}(y; x; -\kappa, -\delta)\, T_x^{\delta I} T_y^{-\kappa\mu}.$$

The Ruijsenaars operator (2.5) can be written $D_m^{(k)} = (-1)^k D_{m,0}^{(k)}$. Our third main result states that the Wronski relation (2.6) extends to the deformed case $r > 0$.

Theorem 2.4. *The operators $D_{m,r}^{(k)}$ and $H_{m,r}^{(l)}$ are related by*

$$\sum_{k+l=K} [k\kappa + l\delta]\, D_{m,r}^{(k)} H_{m,r}^{(l)} = 0, \qquad K \in \mathbb{Z}_{>0}. \tag{2.9}$$

Since $D_{m,r}^{(0)} = \mathrm{id}$, we can alternatively write

$$H_{m,r}^{(K)} = -\frac{1}{[K\delta]} \sum_{k=1}^{K} [k\kappa + (K-k)\delta]\, D_{m,r}^{(k)} H_{m,r}^{(K-k)}. \tag{2.10}$$

This gives a recursion for computing $H_{m,r}^{(l)}$ as a polynomial in the operators $D_{m,r}^{(k)}$. As a consequence, we have the following result.

Corollary 2.1. *The operator $H_{m,r}^{(l)}$ is in the algebra generated by $D_{m,r}^{(k)}$ for $1 \leq k \leq l$. In particular, $[D_{m,r}^{(k)}, H_{m,r}^{(l)}] = 0$ for $k, l \in \mathbb{Z}_{\geq 0}$.*

In [19], the recursion (2.10) for $r = 0$ is solved explicitly in terms of determinants. This solution extends immediately to general r.

Corollary 2.2. *The operator $H_{m,r}^{(l)}$ can be expressed in terms of the operators $D_{m,r}^{(k)}$ as*

$$H_{m,r}^{(l)} = (-1)^l \det_{1 \leq i,j \leq l} \left(\frac{[(i-j+1)\kappa + (j-1)\delta]}{[i\delta]} D_{m,r}^{(i-j+1)} \right),$$

where matrix elements with $i - j + 1 < 0$ are interpreted as zero.

Interchanging δ and $-\kappa$ gives the inverse relation

$$D_{m,r}^{(l)} = (-1)^l \det_{1 \leq i,j \leq l} \left(\frac{[(i-j+1)\delta + (j-1)\kappa]}{[i\kappa]} H_{m,r}^{(i-j+1)} \right).$$

The identities in [19, Prop. 1.4] also extend immediately to our more general operators; we will not reproduce them here.

The fourth family of operators is

$$\hat{D}_{m,r}^{(k)} = \sum_{\substack{\mu\in\mathbb{Z}_{\geq 0}^r,\, I\subseteq\langle m\rangle,\\ |\mu|+|I|=k}} C_{\mu,I}(y_1+\kappa,\dots,y_r+\kappa;x_1-\delta,\dots,x_m-\delta;\kappa,\delta)\, T_x^{-\delta I} T_y^{\kappa\mu}.$$

It follows from Corollary 2.1 that $\hat{D}_{m,r}^{(k)}$ is a polynomial in $\hat{H}_{m,r}^{(l)}$ for $l \leq k$. We can now conclude that all these operators commute.

Corollary 2.3. *For fixed m and r, and arbitrary $k_j \in \mathbb{Z}_{\geq 0}$, the operators $H_{m,r}^{(k_1)}$, $\hat{H}_{m,r}^{(k_2)}$, $D_{m,r}^{(k_3)}$ and $\hat{D}_{m,r}^{(k_4)}$ commute.*

Finally, we consider the so called kernel function. To this end, we fix a meromorphic solution G_δ to the functional equation

$$G_\delta(x+\delta) = [x]G_\delta(x). \tag{2.11}$$

In the generic case, G_δ can be constructed from Ruijsenaars' elliptic gamma function, see (B.3) below.

Theorem 2.5. *Assuming that*

$$(m-n)\kappa = (r-s)\delta, \tag{2.12}$$

the function

$$\begin{aligned}\Phi^{(m,r,n,s)}&(x_1,\dots,x_m;y_1,\dots,y_r;X_1,\dots,X_n;Y_1,\dots,Y_s)\\ &= \prod_{\substack{1\leq i\leq m,\\ 1\leq j\leq n}} \frac{G_\delta(x_i+X_j-\kappa)}{G_\delta(x_i+X_j)} \prod_{\substack{1\leq i\leq r,\\ 1\leq j\leq s}} \frac{G_{-\kappa}(y_i+Y_j+\delta)}{G_{-\kappa}(y_i+Y_j)}\\ &\quad\times \prod_{\substack{1\leq i\leq m,\\ 1\leq j\leq s}} [x_i+Y_j] \prod_{\substack{1\leq i\leq r,\\ 1\leq j\leq n}} [y_i+X_j]\end{aligned} \tag{2.13}$$

satisfies the kernel function identity

$$H_{m,r}^{(k)}(x;y)\Phi^{(m,r,n,s)}(x;y;X;Y) = H_{n,s}^{(k)}(X;Y)\Phi^{(m,r,n,s)}(x;y;X;Y), \tag{2.14}$$

where we indicate on which variables the difference operators act.

The so called balancing condition (2.12) stems from the fact that the sum of the zeroes of an elliptic function (modulo periods) equals the sum of the poles. This condition seems to be unavoidable in the elliptic case, but in the trigonometric case there is a modified version of (2.14) without this condition [11].

Clearly, (2.12) holds for generic parameters when $m = n$ and $r = s$. However, also exceptional cases when $\kappa/\delta \in \mathbb{Q}$ may be of interest. In that situation, the undeformed Ruijsenaars operators (2.5) act on a finite-dimensional space of theta functions spanned by affine Lie algebra characters [14]. Another type of finite-dimensional reduction was recently studied in [7,8] and leads to relations to the fusion ring of conformal field theories. (To be precise, the latter papers assume a relation of the form $m\kappa = r\delta + 2\omega_1$, where ω_1 is a real quasi-period, such that $[x+\omega_1] = -[x]$. It is easy to see that Theorem 2.5 remains valid under this type of shifts.) One could hope that Theorem 2.5 is useful for studying deformed versions of these two types of finite-dimensional reductions.

3. Source Identities

Clearly, the commutation relation in Theorem 2.1 can be translated to an identity involving the coefficients (2.7b). This is also the case for Theorems 2.3, 2.4 and 2.5. We refer to these scalar equations for the coefficients as *source identities* for the corresponding facts about operators. It turns out that the operator identities can be obtained from the same source identities as in the non-deformed case ($r = 0$), but with the variables specialized in a non-obvious way.

Theorems 2.1 and 2.3 will both be derived from the source identity [21, Thm. A.2]

$$\sum_{I\subseteq\langle n\rangle,\,|I|=k}\ \prod_{i\in I,\,j\in I^c}\frac{[z_i-z_j-a][z_i-z_j-b]}{[z_i-z_j][z_i-z_j-a-b]} = \sum_{I\subseteq\langle n\rangle,\,|I|=n-k}\ \prod_{i\in I,\,j\in I^c}\frac{[z_i-z_j-a][z_i-z_j-b]}{[z_i-z_j][z_i-z_j-a-b]}. \tag{3.1a}$$

Ruijsenaars used this identity to prove commutativity of the operators (2.5). In the case of Theorem 2.3 we need to combine (3.1a) with an argument of analytic continuation. Incidentally, this leads to a new proof of an elliptic hypergeometric transformation formula due to Langer et al. [15].

For the Wronski relation (2.9), the source identity is the same as the one used by Noumi and Sano [19] in the case $r = 0$, that is,

$$\sum_{I\subseteq\langle n\rangle}(-1)^{|I|}\frac{[|z|-|w|+|I|a]}{[|z|-|w|]}\prod_{i\in I,\,j\in I^c}\frac{[z_i-z_j+a]}{[z_i-z_j]}\prod_{i\in I,\,j\in\langle n\rangle}\frac{[z_i-w_j]}{[z_i-w_j+a]}=0. \tag{3.1b}$$

Here, the notation $|z| = \sum_j z_j$ is used also for complex vectors.

Finally, the kernel function identity (2.14) will be obtained from the Kajihara–Noumi identity [13]

$$\sum_{I\subseteq\langle n\rangle,\,|I|=k}\ \prod_{i\in I,\,j\in I^c}\frac{[z_i-z_j-a]}{[z_i-z_j]}\prod_{i\in I,\,j\in\langle n\rangle}\frac{[z_i+w_j+a]}{[z_i+w_j]} = \sum_{I\subseteq\langle n\rangle,\,|I|=k}\ \prod_{i\in I,\,j\in I^c}\frac{[w_i-w_j-a]}{[w_i-w_j]}\prod_{i\in I,\,j\in\langle n\rangle}\frac{[w_i+z_j+a]}{[w_i+z_j]}. \tag{3.1c}$$

The same identity was used by Ruijsenaars [23] to prove the non-deformed case ($m = n = 0$) of (2.14). Just as for Theorem 2.3, it must in the general case be combined with an analytic continuation argument.

Although the three source identities (3.1) may look similar at first glance, none of them seem to follow easily from the others. Both (3.1b) and (3.1c) can be derived as consequences of the Frobenius determinant evaluation [10]

$$\det_{1\leq i,j\leq n}\left(\frac{[\lambda+z_i+w_j]}{[\lambda][z_i+w_j]}\right)=\frac{[\lambda+|z|+|w|]\prod_{1\leq i<j\leq n}[z_i-z_j][w_i-w_j]}{[\lambda]\prod_{1\leq i,j\leq n}[z_i+w_j]}.$$

However, we are not aware of an analogous proof of (3.1a).

4. Commutativity

In this section we prove Theorem 2.1. Consider the product

$$H_{m,r}^{(k)}H_{m,r}^{(l)}=\sum_{\substack{\mu,\,\nu\in\mathbb{Z}_{\geq 0}^m,\,I,\,J\subseteq\langle r\rangle,\\ |\mu|+|I|=k,\,|\nu|+|J|=l}} C_{\mu,I}(x;y)C_{\nu,J}(x+\delta\mu;y-\kappa I)\,T_x^{\delta(\mu+\nu)}T_y^{-\kappa(I+J)}. \quad (4.1)$$

Here, $I+J$ denotes the sum of the corresponding sequences, that is, $I+J\in\{0,1,2\}^r$. It will be convenient to introduce the sets

$$K=I\cap J,\quad L=I\triangle J,\quad M=\langle r\rangle\setminus(I\cup J),\quad P=I\setminus J,\quad Q=J\setminus I,$$

where $\triangle$ denotes symmetric difference. We then have the disjoint unions

$$\langle r\rangle=K\sqcup L\sqcup M,\qquad L=P\sqcup Q.$$

Substituting $\nu\mapsto\lambda-\mu$, (4.1) takes the form

$$H_{m,r}^{(k)}H_{m,r}^{(l)}=\sum_{\substack{\lambda\in\mathbb{Z}_{\geq 0}^m,\,K,\,L\subseteq\langle r\rangle,\\ K\cap L=\emptyset,\,|\lambda|+2|K|+|L|=k+l}} S_k(x;y)\,T_x^{\delta\lambda}T_y^{-\kappa(2K+L)},$$

where

$$S_k(x;y)=\sum_{\substack{0\leq\mu_j\leq\lambda_j,\,1\leq j\leq m,\\ P\sqcup Q=L,\,|\mu|+|P|=k-|K|}} C_{\mu,K\cup P}(x;y)C_{\lambda-\mu,K\cup Q}(x+\delta\mu;y-\kappa(K\cup P)). \quad (4.2)$$

Hence, the commutativity is equivalent to the symmetry

$$S_k(x;y)=S_{|\lambda|+2|K|+|L|-k}(x;y), \quad (4.3)$$

for fixed λ, K and L.

We now insert the expression (2.7b) into (4.2). Consider first the factors involving only y-variables. They have the form

$$\prod_{i\in K\sqcup P,\,j\in M\sqcup Q}\frac{[y_i-y_j-\delta]}{[y_i-y_j]}\prod_{t\in K\sqcup Q,\,u\in M\sqcup P}\frac{[y_t-y_u+\varepsilon_{t,u}\kappa-\delta]}{[y_t-y_u+\varepsilon_{t,u}\kappa]},$$

where

$$\varepsilon_{t,u}=(K\sqcup P)_u-(K\sqcup P)_t=\begin{cases}1, & t\in Q,\ u\in P,\\ -1, & t\in K,u\in M,\\ 0, & \text{else.}\end{cases}$$

The factors with $(i,j)\in P\times Q$ and $(t,u)\in Q\times P$ can be combined as

$$\prod_{i\in P,\,j\in Q}\frac{[y_i-y_j-\delta][y_i-y_j-\kappa+\delta]}{[y_i-y_j][y_i-y_j-\kappa]} \quad (4.4)$$

and the remaining factors can be written

$$\prod_{i\in K,\, j\in M}\frac{[y_i-y_j-\delta][y_i-y_j-\kappa-\delta]}{[y_i-y_j][y_i-y_j-\kappa]}\\ \times\prod_{i\in L,\, j\in M}\frac{[y_i-y_j-\delta]}{[y_i-y_j]}\prod_{i\in K,\, j\in L}\frac{[y_i-y_j-\delta]}{[y_i-y_j]}, \tag{4.5}$$

For our purpose, the only relevant factors are (4.4), since (4.5) can be cancelled from (4.3).

The factors in (4.2) involving only x-variables can be expressed as

$$\prod_{1\le i<j\le m}\frac{[x_i-x_j+(\mu_i-\mu_j)\delta]}{[x_i-x_j]}\prod_{i,j=1}^{m}\frac{[x_i-x_j+\kappa]_{\mu_i}}{[x_i-x_j+\delta]_{\mu_i}}\\ \times\prod_{1\le i<j\le m}\frac{[x_i-x_j+(\lambda_i-\lambda_j)\delta]}{[x_i-x_j+(\mu_i-\mu_j)\delta]}\prod_{i,j=1}^{m}\frac{[x_i-x_j+(\mu_i-\mu_j)\delta+\kappa]_{\lambda_i-\mu_i}}{[x_i-x_j+(\mu_i-\mu_j+1)\delta]_{\lambda_i-\mu_i}}\\ =\prod_{1\le i<j\le m}\frac{[x_i-x_j+(\lambda_i-\lambda_j)\delta]}{[x_i-x_j]}\prod_{i,j=1}^{m}\frac{[x_i-x_j+\kappa]_{\lambda_i}}{[x_i-x_j+\delta]_{\lambda_i}}\\ \times\prod_{i,j=1}^{m}\frac{[x_i-x_j+\delta]_{\mu_i-\mu_j}[x_i-x_j+\kappa]_{\mu_i}[x_i-x_j-\lambda_j\delta]_{\mu_i}}{[x_i-x_j+\kappa]_{\mu_i-\mu_j}[x_i-x_j+\delta]_{\mu_i}[x_i-x_j-(\lambda_j-1)\delta-\kappa]_{\mu_i}},$$

where the first two double products can be cancelled from (4.3).

Finally, the factors that mix x-variables and y-variables are

$$\prod_{i=1}^{m}\left(\prod_{j\in K\sqcup P}\frac{[x_i-y_j-\kappa]}{[x_i-y_j+\mu_i\delta]}\prod_{j\in M\sqcup Q}\frac{[x_i-y_j-\delta]}{[x_i-y_j+(\mu_i-1)\delta]}\right)\\ \times\prod_{i=1}^{m}\left(\prod_{j\in K\sqcup Q}\frac{[x_i-y_j+\mu_i\delta-Q_j\kappa]}{[x_i-y_j+\lambda_i\delta+K_j\kappa]}\prod_{j\in M\sqcup P}\frac{[x_i-y_j+(\mu_i-1)\delta+P_j\kappa]}{[x_i-y_j+(\lambda_i-1)\delta+P_j\kappa]}\right).$$

Here, all factors with $j\in K\sqcup M$ can be cancelled from (4.3). The remaining factors can be written

$$\prod_{i=1}^{m}\left(\prod_{j\in P}\frac{[x_i-y_j-\kappa][x_i-y_j+(\mu_i-1)\delta+\kappa]}{[x_i-y_j+\mu_i\delta][x_i-y_j+(\lambda_i-1)\delta+\kappa]}\right.\\ \left.\times\prod_{j\in Q}\frac{[x_i-y_j-\delta][x_i-y_j+\mu_i\delta-\kappa]}{[x_i-y_j+(\mu_i-1)\delta][x_i-y_j+\lambda_i\delta]}\right).$$

From this expression, we factor out

$$\prod_{i=1}^{m}\prod_{j\in P\sqcup Q}\frac{[x_i-y_j-\kappa]}{[x_i-y_j+\lambda_i\delta]},$$

which can again be cancelled from (4.3), and are left with

$$\prod_{i=1}^{m}\left(\prod_{j\in P}\frac{[x_i-y_j+\lambda_i\delta][x_i-y_j+(\mu_i-1)\delta+\kappa]}{[x_i-y_j+\mu_i\delta][x_i-y_j+(\lambda_i-1)\delta+\kappa]}\right.$$
$$\left.\times\prod_{j\in Q}\frac{[x_i-y_j-\delta][x_i-y_j+\mu_i\delta-\kappa]}{[x_i-y_j+(\mu_i-1)\delta][x_i-y_j-\kappa]}\right).$$

To summarize, to prove Theorem 2.1 it is enough to verify that (4.3) holds with

$$S_k=\sum_{\substack{0\leq\mu_j\leq\lambda_j,\,1\leq j\leq m,\\ P\sqcup Q=L,\,|\mu|+|P|=k-|K|}}\prod_{i\in P,\,j\in Q}\frac{[y_i-y_j-\delta][y_i-y_j-\kappa+\delta]}{[y_i-y_j][y_i-y_j-\kappa]}$$
$$\times\prod_{i,j=1}^{m}\frac{[x_i-x_j+\delta]_{\mu_i-\mu_j}[x_i-x_j+\kappa]_{\mu_i}[x_i-x_j-\lambda_j\delta]_{\mu_i}}{[x_i-x_j+\kappa]_{\mu_i-\mu_j}[x_i-x_j+\delta]_{\mu_i}[x_i-x_j-(\lambda_j-1)\delta-\kappa]_{\mu_i}}$$
$$\times\prod_{i=1}^{m}\left(\prod_{j\in P}\frac{[x_i-y_j+\lambda_i\delta][x_i-y_j+(\mu_i-1)\delta+\kappa]}{[x_i-y_j+\mu_i\delta][x_i-y_j+(\lambda_i-1)\delta+\kappa]}\right.$$
$$\left.\times\prod_{j\in Q}\frac{[x_i-y_j-\delta][x_i-y_j+\mu_i\delta-\kappa]}{[x_i-y_j+(\mu_i-1)\delta][x_i-y_j-\kappa]}\right)$$

(which differs from (4.2) by a factor independent of k). It is enough to do this for $L=\langle r\rangle$, since the general case then follows by changing the variables $\{y_1,\dots,y_r\}$ to $\{y_j\}_{j\in L}$. We have thus reduced Theorem 2.1 to the following identity.

Proposition 4.1. *For $\lambda\in\mathbb{Z}^m_{\geq 0}$, let*

$$S_k=\sum_{\substack{0\leq\mu_j\leq\lambda_j,\,1\leq j\leq m\\ P\subseteq\langle r\rangle,\,|\mu|+|P|=k}}\prod_{i\in P,\,j\in P^c}\frac{[y_i-y_j-\delta][y_i-y_j+\delta-\kappa]}{[y_i-y_j][y_i-y_j-\kappa]}$$
$$\times\prod_{i,j=1}^{m}\left(\frac{[x_i-x_j+\delta]_{\mu_i-\mu_j}}{[x_i-x_j+\kappa]_{\mu_i-\mu_j}}\frac{[x_i-x_j+\kappa]_{\mu_i}[x_i-x_j-\lambda_j\delta]_{\mu_i}}{[x_i-x_j+\delta]_{\mu_i}[x_i-x_j-(\lambda_j-1)\delta-\kappa]_{\mu_i}}\right)$$
$$\times\prod_{i=1}^{m}\left(\prod_{j\in P}\frac{[x_i-y_j+\lambda_i\delta][x_i-y_j+(\mu_i-1)\delta+\kappa]}{[x_i-y_j+\mu_i\delta][x_i-y_j+(\lambda_i-1)\delta+\kappa]}\right.$$
$$\left.\times\prod_{j\in P^c}\frac{[x_i-y_j-\delta][x_i-y_j+\mu_i\delta-\kappa]}{[x_i-y_j-\kappa][x_i-y_j+(\mu_i-1)\delta]}\right).$$

Then, $S_k=S_{|\lambda|+r-k}$.

As we explain in § 6, Proposition 4.1 is closely related to an elliptic hypergeometric transformation formula due to Langer, Schlosser and Warnaar [15].

Proof. Consider (3.1a) with $a = \delta$, $b = \kappa - \delta$ and

$$(z_1, \dots, z_n) = \big(x_1, x_1 + \delta, \dots, x_1 + (\lambda_1 - 1)\delta, \dots, \\ x_m, x_m + \delta, \dots, x_m + (\lambda_m - 1)\delta, y_1, \dots, y_r\big), \tag{4.6}$$

where $n = |\lambda| + r$. As usual, we identify sets $I \subseteq \langle n \rangle$ with sequences in $\{0, 1\}^n$. We claim that, to give a non-zero contribution to the sums in (3.1a), I has to be of the form

$$I = (\underbrace{1, \dots, 1}_{\mu_1}, \underbrace{0, \dots, 0}_{\lambda_1 - \mu_1}, \dots, \underbrace{1, \dots, 1}_{\mu_m}, \underbrace{0, \dots, 0}_{\lambda_m - \mu_m}, P),$$

where $0 \le \mu_k \le \lambda_k$ for each k and $P \subseteq \langle r \rangle$. Otherwise, there is an index k such that $z_{k+1} = z_k + \delta$, $k \notin I$ and $k + 1 \in I$. Then, the corresponding term in (3.1a) contains the factor $[z_{k+1} - z_k - \delta] = 0$.

The general term in (3.1a) can be written $F(\delta)/F(\kappa)$, where

$$F(c) = \prod_{i \in I,\, j \in I^c} \frac{[z_i - z_j - c]}{[z_i - z_j - c + \delta]}.$$

Specializing z as in (4.6), $F(c)$ splits naturally into four parts, depending on whether z_i and z_j are specialized to shifted x-variables or to y-variables. The first part is

$$\begin{aligned} F_1(c) &= \prod_{i,j=1}^{m} \prod_{\substack{1 \le k \le \mu_i, \\ \mu_j + 1 \le l \le \lambda_j}} \frac{[x_i - x_j + (k - l)\delta - c]}{[x_i - x_j + (k - l + 1)\delta - c]} \\ &= \prod_{i,j=1}^{m} \prod_{k=1}^{\mu_i} \frac{[x_i - x_j + (k - \lambda_j)\delta - c]}{[x_i - x_j + (k - \mu_j)\delta - c]} = \prod_{i,j=1}^{m} \frac{[x_i - x_j + (1 - \lambda_j)\delta - c]_{\mu_i}}{[x_i - x_j + (1 - \mu_j)\delta - c]_{\mu_i}}, \end{aligned}$$

where we used that the product in l telescopes. Using that

$$[x_i - x_j + (1 - \mu_j)\delta - c]_{\mu_i} = (-1)^{\mu_i} \frac{[x_j - x_i + c]_{\mu_j}}{[x_j - x_i + c]_{\mu_j - \mu_i}},$$

we obtain

$$F_1(c) = (-1)^{m|\mu|} \prod_{i,j=1}^{m} \frac{[x_i - x_j + c]_{\mu_i - \mu_j} [x_i - x_j + (1 - \lambda_j)\delta - c]_{\mu_i}}{[x_i - x_j + c]_{\mu_i}}. \tag{4.7a}$$

The second part of the product, when z_i is specialized to a shifted x-variable and z_j to a y-variable, can be written

$$F_2(c) = \prod_{i=1}^{m} \prod_{j \in P^c} \prod_{k=1}^{\mu_i} \frac{[x_i + (k - 1)\delta - y_j - c]}{[x_i + k\delta - y_j - c]} = \prod_{i=1}^{m} \prod_{j \in P^c} \frac{[x_i - y_j - c]}{[x_i - y_j + \mu_i\delta - c]} \tag{4.7b}$$

and similarly the third part is

$$F_3(c) = \prod_{i=1}^{m} \prod_{j \in P} \prod_{k=\mu_i + 1}^{\lambda_i} \frac{[x_i + (k - 1)\delta - y_j + c]}{[x_i + (k - 2)\delta - y_j + c]} = \prod_{i=1}^{m} \prod_{j \in P} \frac{[x_i - y_j + (\lambda_i - 1)\delta + c]}{[x_i - y_j + (\mu_i - 1)\delta + c]}. \tag{4.7c}$$

Finally, the last part is simply

$$F_4(c) = \prod_{i\in P,\ j\in P^c} \frac{[y_i - y_j - c]}{[y_i - y_j + \delta - c]}. \tag{4.7d}$$

The general term of the sums in (3.1a) is

$$\frac{F_1(\delta)F_2(\delta)F_3(\delta)F_4(\delta)}{F_1(\kappa)F_2(\kappa)F_3(\kappa)F_4(\kappa)}.$$

Inserting the explicit expressions (4.7) yields the desired result. □

5. Algebraic Independence

We will now prove Theorem 2.2. We will first give a proof of algebraic independence in the special case $\kappa = \delta$, and then deduce the general case.

Lemma 5.1. *For $\kappa = \delta$, the operators $H_{m,r}^{(k)}$, $k = 1, \ldots, m + r$, are algebraically independent.*

Proof. It is easy to check that

$$C_{\mu,I}(x; y)\Big|_{\kappa=\delta} = (-1)^{|I|} F(x; y)^{-1} T_x^{\delta\mu} T_y^{-\delta I} F(x; y),$$

where

$$F(x; y) = \frac{\prod_{1\leq i<j\leq m}[x_i - x_j]\prod_{1\leq i<j\leq r}[y_i - y_j]}{\prod_{1\leq i\leq m,\ 1\leq j\leq r}[x_i - y_j - \delta]}.$$

It follows that

$$H_{m,r}^{(k)}(x; y)\Big|_{\kappa=\delta} = F(x; y)^{-1} \sum_{\substack{\mu\in\mathbb{Z}^m_{\geq 0},\ I\subseteq\langle r\rangle,\\ |\mu|+|I|=k}} (-1)^{|I|} T_x^{\delta\mu} T_y^{-\delta I} F(x; y).$$

Thus, it is enough to prove algebraic independence of the operators

$$\sum_{\substack{\mu\in\mathbb{Z}^m_{\geq 0},\ I\subseteq\langle r\rangle,\\ |\mu|+|I|=k}} (-1)^{|I|} T_x^{\delta\mu} T_y^{-\delta I} = \sum_{\substack{i\geq 0,\ 0\leq j\leq r,\\ i+j=k}} (-1)^j h_i(T_x^{\delta}) e_j(T_y^{-\delta}), \qquad 1 \leq k \leq m + r,$$

where h_i and e_j denote complete homogeneous and elementary symmetric polynomials, respectively. This is in turn equivalent to algebraic independence of the polynomials

$$h_{m,r}^{(k)}(\xi; \eta) = \sum_{\substack{i\geq 0,\ 0\leq j\leq r,\\ i+j=k}} (-1)^j h_i(\xi) e_j(\eta), \qquad 1 \leq k \leq m + r,$$

which are invariant under the natural action of the product $\mathrm{S}_m \times \mathrm{S}_r$ of symmetric groups. We note that

$$\mathbb{C}[\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_r]^{\mathrm{S}_m\times\mathrm{S}_r} = \mathbb{C}[e_1(\xi), \ldots, e_m(\xi), e_1(\eta), \ldots, e_r(\eta)]$$

and introduce the total order on the monomials

$$e_r(\eta)^{l_r}\cdots e_1(\eta)^{l_1}e_m(\xi)^{k_m}\cdots e_1(\xi)^{k_1}$$

corresponding to lexicographic order of the multi-indices $(l_r,\dots,l_1,k_m,\dots,k_1)$.

It is well-known that

$$h_i(\xi)=(-1)^{i-1}e_i(\xi)+\text{lower order terms},\qquad 1\le i\le m.$$

Hence, for $1\le k\le r$ we have

$$h_{m,r}^{(k)}(\xi;\eta)=(-1)^k e_k(\eta)+\text{lower order terms},$$

whereas for $r+1\le k\le m+r$,

$$\begin{aligned}h_{m,r}^{(k)}(\xi;\eta)&=(-1)^r h_{k-r}(\xi)e_r(\eta)+\text{lower order terms}\\&=(-1)^{k+1}e_{k-r}(\xi)e_r(\eta)+\text{lower order terms}.\end{aligned}$$

It follows that the polynomial

$$h_{m,r}^{(1)}(\xi;\eta)^{\lambda_1}\cdots h_{m,r}^{(m+r)}(\xi;\eta)^{\lambda_{m+r}},\qquad \lambda_1,\dots,\lambda_{m+r}\in\mathbb{Z}_{\ge 0}\tag{5.1}$$

has leading term

$$e_1(\eta)^{\lambda_1}\cdots e_{r-1}(\eta)^{\lambda_{r-1}}e_r(\eta)^{\lambda_r+\lambda_{r+1}+\cdots+\lambda_{m+r}}e_1(\xi)^{\lambda_{r+1}}\cdots e_m(\xi)^{\lambda_{m+r}},$$

up to an irrelevant sign factor. Since these terms are all distinct, the polynomials (5.1) are linearly independent. Equivalently, $h_{m,r}^{(k)}$ are algebraically independent for $1\le k\le m+r$. □

We will now prove Theorem 2.2. Algebraic independence is equivalent to linear independence of the operators

$$H_{m,r}^{\lambda}=(H_{m,r}^{(1)})^{\lambda_1}\cdots(H_{m,r}^{(m+r)})^{\lambda_{m+r}},\qquad \lambda\in\mathbb{Z}_{\ge 0}^{m+r}.\tag{5.2}$$

These operators have the form

$$H_{m,r}^{\lambda}=\sum_{|\mu|+|\nu|=\|\lambda\|}C_{\mu,\nu}^{\lambda}(x;y)T_x^{\delta\mu}T_y^{-\kappa\nu},$$

where the coefficients $C_{\mu,\nu}^{\lambda}$ are meromorphic and

$$\|\lambda\|=\lambda_1+2\lambda_2+\cdots+(m+r)\lambda_{m+r}.$$

A linear relation

$$\sum_{\lambda}c_\lambda H_{m,r}^{\lambda}=0$$

between the operators (5.2) is equivalent to the corresponding relations

$$\sum_{\|\lambda\|=|\mu|+|\nu|}c_\lambda C_{\mu,\nu}^{\lambda}(x;y)=0,\qquad \mu\in\mathbb{Z}_{\ge 0}^m,\quad \nu\in\mathbb{Z}_{\ge 0}^r$$

between the coefficients. In particular, the operators are linearly independent if and only if the matrices

$$(C_{\mu,\nu}^{\lambda}(x;y))_{|\mu|+|\nu|=N,\,\|\lambda\|=N}$$

of meromorphic functions have maximal rank for each $N\in\mathbb{Z}_{\ge 0}$. Since we know from Lemma 5.1 that this is the case when $\kappa=\delta$, it must be true for generic values of κ and δ.

6. Second Commutation Relation

The proof of Theorem 2.3 is similar to that of Theorem 2.1. We write (2.8) as

$$\hat{H}_{m,r}^{(k)} = \sum_{\substack{\mu\in\mathbb{Z}_{\geq 0}^m,\ I\subseteq\langle r\rangle,\\ |\mu|+|I|=k}} D_{\mu,I}(x;y)\,T_x^{-\delta\mu}T_y^{\kappa I},$$

where

$$\begin{aligned} &D_{\mu,I}(x;y)\\ &\quad=(-1)^{|I|}\prod_{1\leq i<j\leq m}\frac{[x_i-x_j-(\mu_i-\mu_j)\delta]}{[x_i-x_j]}\prod_{i,j=1}^{m}\frac{[x_j-x_i+\kappa]_{\mu_i}}{[x_j-x_i+\delta]_{\mu_i}}\prod_{i\in I,\ j\in I^c}\frac{[y_i-y_j+\delta]}{[y_i-y_j]}\\ &\qquad\times\prod_{i=1}^{m}\left(\prod_{j\in I}\frac{[x_i-y_j-\delta]}{[x_i-y_j-(\mu_i+1)\delta-\kappa]}\prod_{j\in I^c}\frac{[x_i-y_j-\kappa]}{[x_i-y_j-\mu_i\delta-\kappa]}\right). \end{aligned} \tag{6.1}$$

This gives

$$\begin{aligned} &H_{m,r}^{(k)}\hat{H}_{m,r}^{(l)}\\ &\quad=\sum_{\substack{\lambda\in\mathbb{Z}^m,\ K\in\{-1,0,1\}^r,\\ |\lambda|+|K|=k-l}}\ \sum_{\substack{\mu\in\mathbb{Z}_{\geq 0}^m,\ I\subseteq\langle r\rangle,\\ |\mu|+|I|=k}} C_{\mu,I}(x;y)D_{\mu-\lambda,I-K}(x+\delta\mu;y-\kappa I)\,T_x^{\delta\lambda}T_y^{-\kappa K}. \end{aligned}$$

Here, we should interpret $D_{\mu-\lambda,I-K}$ as zero unless $\mu_j\geq\lambda_j$ and $I_j-K_j\in\{0,1\}$ for all j. In the notation

$$K^i=\{j\in\langle r\rangle;\ K_j=i\},\qquad i=-1,\,0,\,1,$$

the latter condition is equivalent to $K^1\subseteq I\subseteq K^0\cup K^1$. Then, the vector difference $I-K$ coincides with $K^{-1}\cup(I\cap K^0)$. Writing $\hat{H}_{m,r}^{(l)}H_{m,r}^{(k)}$ in the same way we find that Theorem 2.3 is equivalent to the scalar equations

$$\begin{aligned} &\sum_{\substack{\mu_j\geq\max(0,\lambda_j),\ 1\leq j\leq m,\\ K^1\subseteq I\subseteq K^0\cup K^1,\\ |\mu|+|I|=k}} C_{\mu,I}(x;y)D_{\mu-\lambda,I-K}(x+\delta\mu;y-\kappa I)\\ &\quad=\sum_{\substack{\mu_j\geq\max(0,\lambda_j),\ 1\leq j\leq m,\\ K^1\subseteq I\subseteq K^0\cup K^1,\\ |\mu|+|I|=k}} C_{\mu,I}(x-\delta(\mu-\lambda);y+\kappa(I-K))D_{\mu-\lambda,I-K}(x;y). \end{aligned} \tag{6.2}$$

We want to factor

$$C_{\mu,I}(x;y)D_{\mu-\lambda,I-K}(x+\delta\mu;y-\kappa I)=F(x;y)G(x;y),$$

where F is independent of μ and I, and G is normalized to take the value 1 if $\mu=0$ and $I=K^1$. Inserting the explicit expressions (2.7b) and (6.1), we find after a tedious computation that

$$\begin{aligned}
F(x;y) &= (-1)^{|K|}\prod_{1\le i<j\le m}\frac{[x_i-x_j+(\lambda_i-\lambda_j)\delta]}{[x_i-x_j]}\prod_{i\in K^0,\,j\in K^{-1}}\frac{[y_i-y_j-\delta]}{[y_i-y_j]}\\
&\quad\times\prod_{i\in K^1,\,j\in K^0}\frac{[y_i-y_j-\delta]}{[y_i-y_j]}\prod_{i\in K^1,\,j\in K^{-1}}\frac{[y_i-y_j-\delta][y_i-y_j-\delta-\kappa]}{[y_i-y_j][y_i-y_j-\kappa]}\\
&\quad\times\prod_{i=1}^m\left(\prod_{j\in K^{-1}}\frac{[x_i-y_j-\delta]}{[x_i-y_j+(\lambda_i-1)\delta-\kappa]}\prod_{j\in K^0}\frac{[x_i-y_j-\kappa]}{[x_i-y_j+\lambda_i\delta-\kappa]}\right.\\
&\quad\left.\times\prod_{j\in K^1}\frac{[x_i-y_j-\kappa]}{[x_i-y_j+\lambda_i\delta]}\right),\\
G(x;y) &= \prod_{i\in(K^0\cap I),\,j\in(K^0\setminus I)}\frac{[y_i-y_j-\delta][y_i-y_j+\delta-\kappa]}{[y_i-y_j][y_i-y_j-\kappa]}\\
&\quad\times\prod_{i,j=1}^m\left(\frac{[x_i-x_j+\delta]_{\mu_i-\mu_j}}{[x_i-x_j+\kappa]_{\mu_i-\mu_j}}\frac{[x_i-x_j+\kappa]_{\mu_i}[x_i-x_j+\kappa]_{\mu_i-\lambda_j}}{[x_i-x_j+\delta]_{\mu_i}[x_i-x_j+\delta]_{\mu_i-\lambda_j}}\right)\\
&\quad\times\prod_{i=1}^m\left(\prod_{j\in K^0\cap I}\frac{[x_i-y_j+\lambda_i\delta-\kappa][x_i-y_j+(\mu_i-1)\delta+\kappa]}{[x_i-y_j+(\lambda_i-1)\delta][x_i-y_j+\mu_i\delta]}\right.\\
&\quad\left.\times\prod_{j\in K^0\setminus I}\frac{[x_i-y_j-\delta][x_i-y_j+\mu_i\delta-\kappa]}{[x_i-y_j-\kappa][x_i-y_j+(\mu_i-1)\delta]}\right).
\end{aligned}$$

In the same way, one gets

$$\begin{aligned}
&C_{\mu,I}(x-\delta(\mu-\lambda);y+\kappa(I-K))D_{\mu-\lambda,I-K}(x;y)\\
&\quad=\prod_{i,j=1}^m\frac{[x_i-x_j+\kappa]_{\lambda_i-\lambda_j}}{[x_i-x_j+\delta]_{\lambda_i-\lambda_j}}F(x;y)G(\hat{x};\hat{y}),
\end{aligned}$$

where we have introduced the variables

$$\hat{x}_j=-x_j-\lambda_j\delta+\kappa,\qquad \hat{y}_j=-y_j-\delta. \tag{6.3}$$

We now observe that the variables y_j with $j\in K^{-1}\cup K^1$ only appear in the prefactor F that can be cancelled from (6.2). Thus, it suffices to prove the case $K^0=\langle r\rangle$, that is, the identity

$$\begin{aligned}
&\sum_{\substack{\mu_j\ge\max(0,\lambda_j),\,1\le j\le m,\\ I\subseteq\langle r\rangle,\,|\mu|+|I|=k}}G(x;y)\\
&\quad=\prod_{i,j=1}^m\frac{[x_i-x_j+\kappa]_{\lambda_i-\lambda_j}}{[x_i-x_j+\delta]_{\lambda_i-\lambda_j}}\sum_{\substack{\mu_j\ge\max(0,\lambda_j),\,1\le j\le m,\\ I\subseteq\langle r\rangle,\,|\mu|+|I|=k}}G(\hat{x};\hat{y}).
\end{aligned}\tag{6.4}$$

We will identify (6.4) with a version of Proposition 4.1 where the conditions $0 \leq \mu_j \leq \lambda_j$ are replaced by $\mu_j \geq \max(0, \lambda_j)$. To this end, we first rewrite the identity $S_k = S_{|\lambda|+r-k}$. On the left-hand side, we note that the factor $[x_i - x_j - \lambda_j\delta]_{\mu_i}$ vanishes if $j = i$ and $\mu_i > \lambda_i$. Hence, we may ignore the restrictions $\mu_i \leq \lambda_i$ on the summation indices. It will be convenient to introduce the variables $z_j = -x_j - \lambda_j\delta$. We can then write

$$S_k = T_k(x; y; -x - \delta\lambda),$$

where

$$\begin{aligned}
&T_k(x_1, \dots, x_m; y_1, \dots, y_r; z_1, \dots, z_m)\\
&= \sum_{\substack{\mu\in\mathbb{Z}_{\geq 0}^m,\ P\subseteq\langle r\rangle,\\ |\mu|+|P|=k}} \prod_{i\in P,\ j\in P^c} \frac{[y_i - y_j - \delta][y_i - y_j + \delta - \kappa]}{[y_i - y_j][y_i - y_j - \kappa]}\\
&\quad\times \prod_{i,j=1}^m \left(\frac{[x_i - x_j + \delta]_{\mu_i-\mu_j}}{[x_i - x_j + \kappa]_{\mu_i-\mu_j}} \frac{[x_i - x_j + \kappa]_{\mu_i}[x_i + z_j]_{\mu_i}}{[x_i - x_j + \delta]_{\mu_i}[x_i + z_j + \delta - \kappa]_{\mu_i}} \right)\\
&\quad\times \prod_{i=1}^m \left(\prod_{j\in P} \frac{[z_i + y_j][x_i - y_j + (\mu_i - 1)\delta + \kappa]}{[z_i + y_j + \delta - \kappa][x_i - y_j + \mu_i\delta]}\right.\\
&\quad\times \left.\prod_{j\in P^c} \frac{[x_i - y_j - \delta][x_i - y_j + \mu_i\delta - \kappa]}{[x_i - y_j - \kappa][x_i - y_j + (\mu_i - 1)\delta]} \right).
\end{aligned}$$

In the sum $S_{|\lambda|+r-k}$, we replace $\mu_i \mapsto \lambda_i - \mu_i$ for all i and $P \mapsto P^c$. By a straightforward computation, we obtain

$$S_{|\lambda|+r-k} = T_k(-x - \delta\lambda; \hat{y}; x),$$

where $\hat{y}$ is as in (6.3). Thus, Proposition 4.1 can be formulated as

$$T_k(x; y; z) = T_k(z; \hat{y}; x), \tag{6.5}$$

where

$$x_j + z_j = -\lambda_j\delta, \qquad \lambda_j \in \mathbb{Z}_{\geq 0}, \quad j = 1, \dots, m. \tag{6.6}$$

Proposition 6.1. *The identity* (6.5) *holds also without the condition* (6.6).

Proof. We apply a standard analytic continuation argument, see e.g. [28]. It is straightforward to check that each term in (6.5) has the form

$$C \prod_{j=1}^N \frac{[x_1 + a_j]}{[x_1 + b_j]},$$

where C is independent of x_1 and

$$\sum_{j=1}^N (a_j - b_j) = k(\kappa - \delta).$$

Assume that we are in a generic situation, when $[x]$ is given by (2.2). Then, all these terms have the same quasi-periodicity with respect to the lattice $\Gamma = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. It follows that (6.5) holds if

$$x_1 \in -z_1 - \mathbb{Z}_{\geq 0}\delta + \Gamma.$$

By our assumption (2.3), these values are all distinct mod Γ, so by analytic continuation (6.5) holds for generic x_1. By symmetry, the same argument applies to the other variables x_j. □

In the special case $r = 0$, Proposition 6.1 reduces to the elliptic hypergeometric transformation formula

$$\sum_{\mu\in\mathbb{Z}^m_{\geq 0},\,|\mu|=k} \prod_{i,j=1}^{m} \left(\frac{[x_i - x_j + \delta]_{\mu_i-\mu_j}}{[x_i - x_j + \kappa]_{\mu_i-\mu_j}} \frac{[x_i - x_j + \kappa]_{\mu_i}[x_i + z_j]_{\mu_i}}{[x_i - x_j + \delta]_{\mu_i}[x_i + z_j + \delta - \kappa]_{\mu_i}} \right)$$
$$= \sum_{\mu\in\mathbb{Z}^m_{\geq 0},\,|\mu|=k} \prod_{i,j=1}^{m} \left(\frac{[z_i - z_j + \delta]_{\mu_i-\mu_j}}{[z_i - z_j + \kappa]_{\mu_i-\mu_j}} \frac{[z_i - z_j + \kappa]_{\mu_i}[z_i + x_j]_{\mu_i}}{[z_i - z_j + \delta]_{\mu_i}[z_i + x_j + \delta - \kappa]_{\mu_i}} \right). \tag{6.7}$$

Replacing m by $m+1$ and eliminating the summation index μ_{m+1}, it is straight-forward to check that this is equivalent to [15, Cor. 4.3]. Conversely, one can recover Proposition 6.1 from (6.7) by replacing m by $m + r$ and then specializing $x_j + z_j = -\delta$ for $m + 1 \leq j \leq m + r$. The proof of (6.7) given here is very similar to that in [15]. However, the observation that (6.7) can be derived from Ruijsenaars' identity (3.1a) is new. In [15], it is derived from a more complicated source identity.

We will now consider (6.5) when $z_j = \hat{x}_j$ is given by (6.3). More precisely, to avoid division by zero we first multiply both sides with

$$\prod_{i,j=1}^{n} \frac{[x_i + z_j + \delta - \kappa]_{\lambda_j}}{[x_i + z_j]_{\lambda_j}}$$

and then use that

$$\lim_{z_j\to -x_j-\lambda_j\delta+\kappa} \frac{[x_i + z_j + \delta - \kappa]_{\lambda_j}[x_i + z_j]_{\mu_i}}{[x_i + z_j]_{\lambda_j}[x_i + z_j + \delta - \kappa]_{\mu_i}} = \frac{[x_i - x_j + \kappa]_{\mu_i-\lambda_j}}{[x_i - x_j + \delta]_{\mu_i-\lambda_j}},$$

which by definition vanishes if $j = i$ and $\mu_i < \lambda_i$. On the right-hand side, we use

$$\lim_{x_j\to -z_j-\lambda_j\delta+\kappa} \frac{[z_i + x_j + \delta - \kappa]_{\lambda_i}[z_i + x_j]_{\mu_i}}{[z_i + x_j]_{\lambda_i}[z_i + x_j + \delta - \kappa]_{\mu_i}} = \frac{[x_j - x_i + \kappa]_{\lambda_j-\lambda_i}[z_i - z_j + \kappa]_{\mu_i-\lambda_j}}{[x_j - x_i + \delta]_{\lambda_j-\lambda_i}[z_i - z_j + \delta]_{\mu_i-\lambda_j}}.$$

It is now clear that the resulting limit case of (6.5) is (6.4) (with $I{=}P$). This completes the proof of Theorem 2.3.

7. Wronski Relations

We now turn to the proof of Theorem 2.4. Indicating the parameter-dependence in (2.7b) as $C_{\mu,I}(x;y;\delta,\kappa)$, the left-hand side of (2.9) may be expressed as

$$\sum_{\substack{\mu\in\mathbb{Z}_{\geq 0}^r,\, I\subseteq\langle m\rangle,\\ |\nu|\in\mathbb{Z}_{\geq 0}^m,\, J\subseteq\langle r\rangle,\\ |\mu|+|I|+|\nu|+|J|=K}} [(|\mu|+|I|)\kappa+(|\nu|+|J|)\delta] \\ \times C_{\mu,I}(y;x;-\kappa,-\delta)C_{\nu,J}(x+\delta I;y-\kappa\mu;\delta,\kappa)\,T_x^{\delta(\nu+I)}T_y^{-\kappa(J+\mu)}.$$

We make the change of variables $\nu\mapsto\nu-I$ and $\mu\mapsto\mu-J$. We then have $\nu_j\geq I_j$ and $\mu_j\geq J_j$ for all j, that is,

$$I\subseteq\operatorname{supp}(\nu)=\{j\in\langle m\rangle;\ \nu_j>0\}$$

and $J\subseteq\operatorname{supp}(\mu)$. This gives the expression

$$\sum_{\substack{\mu\in\mathbb{Z}_{\geq 0}^r,\,\nu\in\mathbb{Z}_{\geq 0}^m,\\ |\mu|+|\nu|=K}}\Bigg(\sum_{\substack{I\subseteq\operatorname{supp}(\nu),\\ J\subseteq\operatorname{supp}(\mu)}}[(|\mu|+|I|-|J|)\kappa+(|\nu|+|J|-|I|)\delta] \\ \times C_{\mu-J,I}(y;x;-\kappa,-\delta)C_{\nu-I,J}(x+\delta I;y+\kappa(J-\mu);\delta,\kappa)\Bigg)T_x^{\delta\nu}T_y^{-\kappa\mu}.$$

We introduce the notation $M=\operatorname{supp}(\nu)$ and $N=\operatorname{supp}(\mu)$, and normalize the inner sum so that the term with $I=\emptyset$ and $J=N$ is 1. That is, we define

$$D_{\mu,\nu}(x;y)=\sum_{I\subseteq M,\,J\subseteq N}\frac{[(\mu+|I|-|J|)\kappa+(|\nu|+|J|-|I|)\delta]}{[(|\mu|-|N|)\kappa+(|\nu|+|N|)\delta]} \\ \times\frac{C_{\mu-J,I}(y;x;-\kappa,-\delta)C_{\nu-I,J}(x+\delta I;y+\kappa(J-\mu);\delta,\kappa)}{C_{\mu-N,\emptyset}(y;x;-\kappa,-\delta)C_{\nu,N}(x;y+\kappa(N-\mu);\delta,\kappa)}. \tag{7.1}$$

Then, Theorem 2.6 is equivalent to the identity

$$D_{\mu,\nu}(x;y)=0,\qquad |\mu|+|\nu|>0. \tag{7.2}$$

We now insert (2.7b) into (7.1). To distinguish the shifted factorials (2.4) with base δ from shifted factorials with base $-\kappa$, we use the notation

$$[x;-\kappa]_k=[x][x-\kappa]\cdots[x-(k-1)\kappa].$$

By a straight-forward computation, the factors involving only y-variables can be simplified to

$$\prod_{1\le i<j\le r}\frac{[y_i-y_j+(\mu_j-J_j-\mu_i+J_i)\kappa]}{[y_i-y_j+(\mu_j-N_j-\mu_i+N_i)\kappa]}$$

$$\times\prod_{i,j=1}^{r}\frac{[y_i-y_j-\delta;-\kappa]_{\mu_i-J_i}[y_i-y_j-\kappa;-\kappa]_{\mu_i-N_i}}{[y_i-y_j-\kappa;-\kappa]_{\mu_i-J_i}[y_i-y_j-\delta;-\kappa]_{\mu_i-N_i}}$$

$$\times\prod_{i\in J,\, j\in\langle r\rangle\setminus J}\frac{[y_i-y_j+(\mu_j-\mu_i+1)\kappa-\delta]}{[y_i-y_j+(\mu_j-\mu_i+1)\kappa]}$$

$$\times\prod_{i\in N,\, j\in\langle r\rangle\setminus N}\frac{[y_i-y_j+(\mu_j-\mu_i+1)\kappa]}{[y_i-y_j+(\mu_j-\mu_i+1)\kappa-\delta]}$$

$$=\prod_{i\in N\setminus J}\left(\prod_{j\in J}\frac{[y_i-y_j+(\mu_j-\mu_i-1)\kappa+\delta]}{[y_i-y_j+(\mu_j-\mu_i)\kappa]}\prod_{j\in N}\frac{[y_i-y_j-(\mu_i-1)\kappa-\delta]}{[y_i-y_j-\mu_i\kappa]}\right).$$

The factors involving both x- and y-variables are

$$\prod_{i=1}^{r}\left(\prod_{j\in I}\frac{[y_i-x_j+\delta]}{[y_i-x_j-(\mu_i-J_i)\kappa]}\prod_{j\in\langle m\rangle\setminus I}\frac{[y_i-x_j+\kappa]}{[y_i-x_j-(\mu_i-J_i-1)\kappa]}\right.$$

$$\left.\times\prod_{j=1}^{m}\frac{[y_i-x_j-(\mu_i-N_i-1)\kappa]}{[y_i-x_j+\kappa]}\right)$$

$$\times\prod_{i=1}^{m}\left(\prod_{j\in J}\frac{[x_i-y_j+I_i\delta+(\mu_j-2)\kappa]}{[x_i-y_j+\nu_i\delta+(\mu_j-1)\kappa]}\prod_{j\in\langle r\rangle\setminus J}\frac{[x_i-y_j+(I_i-1)\delta+\mu_j\kappa]}{[x_i-y_j+(\nu_i-1)\delta+\mu_j\kappa]}\right.$$

$$\left.\times\prod_{j\in N}\frac{[x_i-y_j+\nu_i\delta+(\mu_j-1)\kappa]}{[x_i-y_j+(\mu_j-2)\kappa]}\prod_{j\in\langle r\rangle\setminus N}\frac{[x_i-y_j+(\nu_i-1)\delta]}{[x_i-y_j-\delta]}\right)$$

$$=\prod_{i\in I}\left(\prod_{j\in J}\frac{[x_i-y_j+\delta+(\mu_j-2)\kappa]}{[x_i-y_j+(\mu_j-1)\kappa]}\prod_{j\in N}\frac{[x_i-y_j-\delta]}{[x_i-y_j-\kappa]}\right)$$

$$\times\prod_{i\in N\setminus J}\left(\prod_{j\in M\setminus I}\frac{[y_i-x_j+\delta-\mu_i\kappa]}{[y_i-x_j-(\mu_i-1)\kappa]}\prod_{j\in M}\frac{[y_i-x_j-\nu_j\delta-(\mu_i-1)\kappa]}{[y_i-x_j-(\nu_j-1)\delta-\mu_i\kappa]}\right).$$

Finally, the factors involving only x-variables are

$$\prod_{i\in I,\, j\in\langle m\rangle\setminus I}\frac{[x_i-x_j+\kappa]}{[x_i-x_j]}\prod_{1\leq i<j\leq m}\frac{[x_i-x_j]}{[x_i-x_j+(I_i-I_j)\delta]}$$
$$\times\prod_{i,j=1}^{m}\frac{[x_i-x_j+\delta;\delta]_{\nu_i}[x_i-x_j+(I_i-I_j)\delta+\kappa;\delta]_{\nu_i-I_i}}{[x_i-x_j+\kappa;\delta]_{\nu_i}[x_i-x_j+(I_i-I_j+1)\delta;\delta]_{\nu_i-I_i}}$$
$$=\prod_{i\in I}\left(\prod_{j\in M\setminus I}\frac{[x_i-x_j+\delta-\kappa]}{[x_i-x_j]}\prod_{j\in M}\frac{[x_i-x_j-\nu_j\delta]}{[x_i-x_j-(\nu_j-1)\delta-\kappa]}\right).$$

We conclude that

$$\begin{aligned}D_{\mu,\nu}(x;y)=\sum_{I\subseteq M,\,J\subseteq N}&(-1)^{|I|+|J|+|N|}\frac{[(\mu+|I|-|J|)\kappa+(|\nu|+|J|-|I|)\delta]}{[(|\mu|-|N|)\kappa+(|\nu|+|N|)\delta]}\\
&\times\prod_{i\in I}\left(\prod_{j\in M\setminus I}\frac{[x_i-x_j+\delta-\kappa]}{[x_i-x_j]}\prod_{j\in J}\frac{[x_i-y_j+\delta+(\mu_j-2)\kappa]}{[x_i-y_j+(\mu_j-1)\kappa]}\right.\\
&\left.\times\prod_{j\in M}\frac{[x_i-x_j-\nu_j\delta]}{[x_i-x_j-(\nu_j-1)\delta-\kappa]}\prod_{j\in N}\frac{[x_i-y_j-\delta]}{[x_i-y_j-\kappa]}\right)\\
&\times\prod_{i\in N\setminus J}\left(\prod_{j\in M\setminus I}\frac{[y_i-x_j+\delta-\mu_i\kappa]}{[y_i-x_j-(\mu_i-1)\kappa]}\prod_{j\in J}\frac{[y_i-y_j+(\mu_j-\mu_i-1)\kappa+\delta]}{[y_i-y_j+(\mu_j-\mu_i)\kappa]}\right.\\
&\left.\times\prod_{j\in M}\frac{[y_i-x_j-\nu_j\delta-(\mu_i-1)\kappa]}{[y_i-x_j-(\nu_j-1)\delta-\mu_i\kappa]}\prod_{j\in N}\frac{[y_i-y_j-(\mu_i-1)\kappa-\delta]}{[y_i-y_j-\mu_i\kappa]}\right).\end{aligned}\tag{7.3}$$

We now explain how to identify (7.2) with a special case of the source identity (3.1b). As a first step, we write the index set in (3.1b) as a disjoint union $\langle n\rangle=M\sqcup N$. We make a corresponding change of variables $z_i\mapsto x_i$ for $i\in M$, $z_i\mapsto y_i$ for $i\in N$, $w_i\mapsto u_i$ for $i\in M$ and $w_i\mapsto v_i$ for $i\in N$. Finally, we make the substitutions $I\cap M\mapsto I$, $I^c\cap N\mapsto J$. The left-hand side of (3.1b) then takes the form

$$\begin{aligned}\sum_{I\subseteq M,\,J\subseteq N}&(-1)^{|I|+|J|+|N|}\frac{[|x|+|y|-|u|-|v|+(|I|+|N|-|J|)a]}{[|x|+|y|-|u|-|v|]}\\
&\times\prod_{i\in I}\left(\prod_{j\in M\setminus I}\frac{[x_i-x_j+a]}{[x_i-x_j]}\prod_{j\in J}\frac{[x_i-y_j+a]}{[x_i-y_j]}\prod_{j\in M}\frac{[x_i-u_j]}{[x_i-u_j+a]}\prod_{j\in N}\frac{[x_i-v_j]}{[x_i-v_j+a]}\right)\\
&\times\prod_{i\in N\setminus J}\left(\prod_{j\in M\setminus I}\frac{[y_i-x_j+a]}{[y_i-x_j]}\prod_{j\in J}\frac{[y_i-y_j+a]}{[y_i-y_j]}\right.\\
&\left.\times\prod_{j\in M}\frac{[y_i-u_j]}{[y_i-u_j+a]}\prod_{j\in N}\frac{[y_i-v_j]}{[y_i-v_j+a]}\right).\end{aligned}$$

Substituting $a\mapsto\delta-\kappa$ and, for all i, $x_i\mapsto x_i$, $y_i\mapsto y_i-(\mu_i-1)\kappa$, $u_i\mapsto x_i+\nu_i\delta$, $v_i\mapsto y_i+\delta$ in this expression gives (7.3). This completes the proof of Theorem 2.4.

8. Kernel Function Identities

To prove Theorem 2.5 we will be need the following elliptic hypergeometric transformation formula.

Proposition 8.1. *Assume that the parameters* $x_1, \ldots, x_m$*,* $y_1, \ldots, y_r$*,* $X_1, \ldots, X_n$*,* $Y_1, \ldots, Y_s$ *and* $a_1, \ldots, a_{m+n}$ *satisfy the balancing condition*

$$|x| + |a| + s\delta = |X| + r\delta. \tag{8.1}$$

Then,

$$\begin{aligned}
&\sum_{\substack{\mu\in\mathbb{Z}^m_{\geq 0},\, I\subseteq\langle r\rangle,\\ |\mu|+|I|=k}} (-1)^{|I|} \prod_{1\leq i<j\leq m} \frac{[x_i - x_j + (\mu_i - \mu_j)\delta]}{[x_i - x_j]} \\
&\quad\times \prod_{i\in I,\, j\in I^c} \frac{[y_i - y_j - \delta]}{[y_i - y_j]} \prod_{1\leq i\leq m,\, j\in I^c} \frac{[x_i - y_j - \delta]}{[x_i - y_j + (\mu_i - 1)\delta]} \\
&\quad\times \prod_{i=1}^{m} \left(\frac{\prod_{j=1}^{m+n}[x_i + a_j]_{\mu_i}}{\prod_{j=1}^{m}[x_i - x_j + \delta]_{\mu_i} \prod_{j=1}^{n}[x_i + X_j]_{\mu_i}} \prod_{j=1}^{s} \frac{[x_i + Y_j + \mu_i\delta]}{[x_i + Y_j]} \right) \\
&\quad\times \prod_{i\in I} \left(\frac{\prod_{j=1}^{m+n}[y_i + a_j]}{\prod_{j=1}^{m}[y_i - x_j - \mu_j\delta] \prod_{j=1}^{n}[y_i + X_j]} \prod_{j=1}^{s} \frac{[y_i + Y_j + \delta]}{[y_i + Y_j]} \right) \\
&= \sum_{\substack{\mu\in\mathbb{Z}^n_{\geq 0},\, I\subseteq\langle s\rangle,\\ |\mu|+|I|=k}} (-1)^{|I|} \prod_{1\leq i<j\leq n} \frac{[X_i - X_j + (\mu_i - \mu_j)\delta]}{[x_i - x_j]} \\
&\quad\times \prod_{i\in I,\, j\in I^c} \frac{[Y_i - Y_j - \delta]}{[Y_i - Y_j]} \prod_{1\leq i\leq n,\, j\in I^c} \frac{[X_i - Y_j - \delta]}{[X_i - Y_j + (\mu_i - 1)\delta]} \\
&\quad\times \prod_{i=1}^{n} \left(\frac{\prod_{j=1}^{m+n}[X_i - a_j]_{\mu_i}}{\prod_{j=1}^{n}[X_i - X_j + \delta]_{\mu_i} \prod_{j=1}^{m}[X_i + x_j]_{\mu_i}} \prod_{j=1}^{r} \frac{[X_i + y_j + \mu_i\delta]}{[X_i + y_j]} \right) \\
&\quad\times \prod_{i\in I} \left(\frac{\prod_{j=1}^{m+n}[Y_i - a_j]}{\prod_{j=1}^{n}[Y_i - X_j - \mu_j\delta] \prod_{j=1}^{m}[Y_i + x_j]} \prod_{j=1}^{r} \frac{[Y_i + y_j + \delta]}{[Y_i + y_j]} \right).
\end{aligned} \tag{8.2}$$

Proposition 8.1 is a slight variation of a transformation formula found by Kajihara [12] in the trigonometric case and, in general, in [13] and [20]. To be precise, that transformation appears as the special case $r = s = 0$. On the other hand, given that special case, the general case follows by substituting $x \mapsto (x, y)$, $X \mapsto (X, Y)$ and

$$(a_1, \ldots, a_{m+n}) \mapsto (a_1, \ldots, a_{m+n}, -y_1 - \delta, \ldots, -y_r - \delta, Y_1 + \delta, \ldots, Y_s + \delta).$$

Alternatively, one can follow the approach of [13] and derive Proposition 8.1 from the source identity (3.1c). We find it instructive to sketch this proof. We start from (3.1c), with $a = \delta$ and n replaced by N. We first specialize the z-variables as in (4.6) and make a similar specialization

$$\begin{aligned}&(w_1, \dots, w_N)\\&\quad = \big(X_1, X_1+\delta, \dots, X_1+(\nu_1-1)\delta, \dots, X_n, X_n+\delta, \dots, X_n+(\nu_n-1)\delta, Y_1, \dots, Y_s\big).\end{aligned}$$

Here, we must have

$$N = |\lambda| + r = |\nu| + s. \tag{8.3}$$

Just as in the proof of Proposition 4.1, the left-hand side of (3.1c) reduces to a sum over $(\mu_1, \dots, \mu_m, P)$, where $0 \le \mu_j \le \lambda_j$ for each j, $P \subseteq \langle r\rangle$ and $|\mu| + |P| = k$. The resulting expression contains the product

$$F(\delta) = \prod_{i\in I,\, j\in I^c} \frac{[z_i - z_j - \delta]}{[z_i - z_j]},$$

which is computed in (4.7). The remaining factors are easily computed in a similar way. Apart from a sign factor $(-1)^k$, the left-hand side of (3.1c) takes the form

$$\begin{aligned}&\sum_{\substack{\mu\in\mathbb{Z}^m_{\ge 0},\, P\subseteq\langle r\rangle,\\ 0\le\mu_j\le\lambda_j,\, 1\le j\le m,\\ |\mu|+|P|=k}} (-1)^{|P|} \prod_{1\le i<j\le m} \frac{[x_i - x_j + (\mu_i - \mu_j)\delta]}{[x_i - x_j]} \prod_{i,j=1}^{m} \frac{[x_i - x_j - \lambda_j\delta]_{\mu_i}}{[x_i - x_j + \delta]_{\mu_i}}\\
&\times \prod_{i=1}^{m} \left(\prod_{j\in P} \frac{[x_i - y_j + \lambda_i\delta]}{[x_i - y_j + \mu_i\delta]} \prod_{j\in P^c} \frac{[x_i - y_j - \delta]}{[x_i - y_j + (\mu_i - 1)\delta]} \right) \prod_{i\in P,\, j\in P^c} \frac{[y_i - y_j - \delta]}{[y_i - y_j]}\\
&\times \prod_{\substack{1\le i\le m,\\ 1\le j\le n}} \frac{[x_i + X_j + \nu_j\delta]_{\mu_i}}{[x_i + X_j]_{\mu_i}} \prod_{\substack{1\le i\le m,\\ 1\le j\le s}} \frac{[x_i + Y_j + \mu_i\delta]}{[x_i + Y_j]}\\
&\times \prod_{i\in P,\, 1\le j\le n} \frac{[y_i + X_j + \nu_j\delta]}{[y_i + X_j]} \prod_{i\in P,\, 1\le j\le s} \frac{[y_i + Y_j + \delta]}{[y_i + Y_j]}.\end{aligned}$$

Here, the restrictions $\mu_j \le \lambda_j$ may be ignored, since $[x_i - x_j - \lambda_j\delta]_{\mu_i}$ vanishes if $i = j$ and $\mu_j > \lambda_j$. We then obtain the left-hand side of (8.2), in the special case when

$$(a_1, \dots, a_{m+n}) = (-x_1 - \lambda_1\delta, \dots, -x_m - \lambda_m\delta, X_1 + \nu_1\delta, \dots, X_n + \nu_n\delta). \tag{8.4}$$

By (8.3), this is consistent with the balancing condition (8.1). It is clear from symmetry considerations that the right-hand side of (3.1c) reduces to the corresponding right-hand side of (8.2). We conclude that (8.2) holds in the infinitely many special cases (8.4), with λ_j and ν_j non-negative integers subject to (8.3). Finally, by the same type of analytic continuation argument that was used in the proof of Proposition 6.1, (8.2) holds for general values of a_j, as long as (8.1) is satisfied. This proves Proposition 8.1.

We now turn to the proof of Theorem 2.5. We write the kernel function identity as

$$\sum_{\substack{\mu\in\mathbb{Z}_{\geq 0}^m,\, I\subseteq\langle r\rangle,\\ |\mu|+|I|=k}} C_{\mu,I}(x;y)\frac{\Phi(x+\delta\mu;y-\kappa I;X;Y)}{\Phi(x;y;X;Y)} = \sum_{\substack{\mu\in\mathbb{Z}_{\geq 0}^n,\, I\subseteq\langle s\rangle,\\ |\mu|+|I|=k}} C_{\mu,I}(X;Y)\frac{\Phi(x;y;X+\delta\mu;Y-\kappa I)}{\Phi(x;y;X;Y)}. \tag{8.5}$$

It is straight-forward to compute

$$\frac{\Phi(x+\delta\mu;y-\kappa I;X;Y)}{\Phi(x;y;X;Y)} = \prod_{i=1}^m\left(\prod_{j=1}^n\frac{[x_i+X_j-\kappa]_{\mu_i}}{[x_i+X_j]_{\mu_i}}\prod_{j=1}^s\frac{[x_i+Y_j+\delta\mu_i]}{[x_i+Y_j]}\right) \times\prod_{i\in I}\left(\prod_{j=1}^n\frac{[y_i+X_j-\kappa]}{[y_i+X_j]}\prod_{j=1}^s\frac{[y_i+Y_j+\delta]}{[y_i+Y_j]}\right).$$

Inserting (2.7b), the left-hand side of (8.5) is

$$\sum_{\substack{\mu\in\mathbb{Z}_{\geq 0}^m,\, I\subseteq\langle r\rangle,\\ |\mu|+|I|=k}}(-1)^{|I|}\prod_{1\leq i<j\leq m}\frac{[x_i-x_j+(\mu_i-\mu_j)\delta]}{[x_i-x_j]}\prod_{i\in I,\,j\in I^c}\frac{[y_i-y_j-\delta]}{[y_i-y_j]}$$
$$\times\prod_{i,j=1}^m\frac{[x_i-x_j+\kappa]_{\mu_i}}{[x_i-x_j+\delta]_{\mu_i}}\prod_{i=1}^m\left(\prod_{j\in I}\frac{[x_i-y_j-\kappa]}{[x_i-y_j+\mu_i\delta]}\prod_{j\in I^c}\frac{[x_i-y_j-\delta]}{[x_i-y_j+(\mu_i-1)\delta]}\right)$$
$$\times\prod_{i=1}^m\left(\prod_{j=1}^n\frac{[x_i+X_j-\kappa]_{\mu_i}}{[x_i+X_j]_{\mu_i}}\prod_{j=1}^s\frac{[x_i+Y_j+\delta\mu_i]}{[x_i+Y_j]}\right)$$
$$\times\prod_{i\in I}\left(\prod_{j=1}^n\frac{[y_i+X_j-\kappa]}{[y_i+X_j]}\prod_{j=1}^s\frac{[y_i+Y_j+\delta]}{[y_i+Y_j]}\right).$$

This agrees with the left-hand side of (8.2), under the specialization

$$(a_1,\dots,a_{m+n})=(\kappa-x_1,\dots,\kappa-x_m,X_1-\kappa,\dots,X_m-\kappa).$$

Note that the balancing condition (8.1) reduces to (2.12) in this case. By symmetry, the right-hand side of (8.5) reduces to the corresponding right-hand side of (8.2). This proves Theorem 2.5.

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Declarations

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A Relation to deformed Ruijsenaars model

The conventions used in this paper differ from the ones that are more common in the physics literature, going back to the work of Ruijsenaars [21]. For the convenience of the reader, we explain the relation between these conventions. In particular, we give the precise relation between the operators $H^{(1)}_{m,r}$ and the deformed Ruijsenaars model introduced in [1].

The Ruijsenaars systems are defined by two difference operators, S^+ and S^-, such that the Hamiltonian $H = S^+ + S^-$, the momentum operator $P = S^+ - S^-$ and a boost operator B represent the Poincaré algebra in 1 + 1 spacetime dimensions. That is, the commutation relations

$$[H, P] = 0, \quad [H, B] = \mathrm{i}P, \quad [P, B] = \mathrm{i}H. \tag{A.1}$$

are satisfied [21]. In particular, for the deformed elliptic Ruijsenaars model, the corresponding operators are given by (we rename $(\beta, \beta g)$ in [1, Eq. (16)] to $(\mathrm{i}\delta, \mathrm{i}\kappa)$ and drop an irrelevant overall constant)

$$S^{\pm} = \sum_{i=1}^{m} \frac{[\kappa]}{[\delta]} A_i^{\mp} e^{\pm\delta \frac{\partial}{\partial x_i}} A_i^{\pm} - \sum_{i=1}^{r} B_i^{\mp} e^{\mp\kappa \frac{\partial}{\partial y_i}} B_i^{\pm},$$

where

$$A_i^{\pm} = \prod_{\substack{1\le j\le m\\ j\ne i}} \left(\frac{[x_i - x_j \mp \kappa]}{[x_i - x_j]}\right)^{1/2} \prod_{j=1}^{r} \left(\frac{[x_i - y_j \mp \kappa/2 \pm \delta/2]}{[x_i - y_j \mp \kappa/2 \mp \delta/2]}\right)^{1/2},$$

$$B_i^{\pm} = \prod_{\substack{1\le j\le r\\ j\ne i}} \left(\frac{[y_i - y_j \pm \delta]}{[y_i - y_j]}\right)^{1/2} \prod_{j=1}^{m} \left(\frac{[y_i - x_j \pm \delta/2 \mp \kappa/2]}{[y_i - x_j \pm \delta/2 \pm \kappa/2]}\right)^{1/2}.$$

One can check that $H = S^+ + S^-$, $P = S^+ - S^-$ and

$$B = \frac{\mathrm{i}}{\delta} \sum_{i=1}^{m} x_i - \frac{\mathrm{i}}{\kappa} \sum_{i=1}^{r} y_i,$$

indeed satisfy (A.1).

We will now show that, up to a similarity transformation and shifts of the variables, the operators S^+ and S^- are equal to, respectively, our operators $H^{(1)}_{m,r}$ and $\hat{H}^{(1)}_{m,r}$. To this end, we introduce the function

$$\Delta = \prod_{\substack{1\leq i,j\leq m\\ i\neq j}} \frac{G_\delta(x_i - x_j + \kappa)}{G_\delta(x_i - x_j)} \prod_{\substack{1\leq i,j\leq r\\ i\neq j}} \frac{G_{-\kappa}(y_i - y_j - \delta)}{G_{-\kappa}(y_i - y_j)} \times \prod_{i=1}^{m}\prod_{j=1}^{r} \frac{1}{[x_i - y_j + \kappa/2 - \delta/2][y_i - x_i + \kappa/2 - \delta/2]}.$$

A straight-forward computation gives

$$\Delta^{-1/2} S^{\pm} \Delta^{1/2} = \frac{[\kappa]}{[\delta]} \sum_{i=1}^{m} \prod_{\substack{1\leq j\leq m\\ j\neq i}} \frac{[x_i - x_j \pm \kappa]}{[x_i - x_j]} \prod_{j=1}^{r} \frac{[x_i - y_j \pm \kappa/2 \mp \delta/2]}{[x_i - y_j \pm \kappa/2 \pm \delta/2]} e^{\pm\delta \frac{\partial}{\partial x_i}} - \sum_{i=1}^{r} \prod_{\substack{1\leq j\leq r\\ j\neq i}} \frac{[y_i - y_j \mp \delta]}{[y_i - y_j]} \prod_{j=1}^{m} \frac{[y_i - x_j \mp \delta/2 \pm \kappa/2]}{[y_i - x_j \mp \delta/2 \mp \kappa/2]} e^{\mp\kappa \frac{\partial}{\partial y_i}}.$$

Moreover, the case $k = 1$ of (2.7) and (2.8) can be written

$$H^{(1)}_{m,r} = \frac{[\kappa]}{[\delta]} \sum_{i=1}^{m} \prod_{\substack{1\leq j\leq m\\ j\neq i}} \frac{[x_i - x_j + \kappa]}{[x_i - x_j]} \prod_{j=1}^{r} \frac{[x_i - y_j - \delta]}{[x_i - y_j]} e^{\delta \frac{\partial}{\partial x_i}} - \sum_{i=1}^{r} \prod_{\substack{1\leq j\leq r\\ j\neq i}} \frac{[y_i - y_j - \delta]}{[y_i - y_j]} \prod_{j=1}^{m} \frac{[y_i - x_j + \kappa]}{[y_i - x_j]} e^{-\kappa \frac{\partial}{\partial y_i}},$$

$$\hat{H}^{(1)}_{m,r} = \frac{[\kappa]}{[\delta]} \sum_{i=1}^{m} \prod_{\substack{1\leq j\leq m\\ j\neq i}} \frac{[x_i - x_j - \kappa]}{[x_i - x_j]} \prod_{j=1}^{r} \frac{[x_i - y_j - \kappa]}{[x_i - y_j - \delta - \kappa]} e^{-\delta \frac{\partial}{\partial x_i}} - \sum_{i=1}^{r} \prod_{\substack{1\leq j\leq r\\ j\neq i}} \frac{[y_i - y_j + \delta]}{[y_i - y_j]} \prod_{j=1}^{m} \frac{[y_i - x_j + \delta]}{[y_i - x_j + \kappa + \delta]} e^{\kappa \frac{\partial}{\partial y_i}}.$$

This makes manifest that, after shifting the variables in $\Delta^{-1/2} S^{\pm} \Delta^{1/2}$ as

$$x_i \to x_i \pm \delta/2, \quad y_j \to y_j \mp \kappa/2, \quad i = 1, \ldots, m, \ j = 1, \ldots, r, \tag{A.2}$$

one obtains the operators $H^{(1)}_{m,r}$ and $\hat{H}^{(1)}_{m,r}$, respectively.

It is interesting to note that Δ is the weight function in a natural scalar product on the space of common eigenfunctions of the operators $S^{\pm}$ proposed recently in [2].

Finally, we comment on the role of the condition (2.3). In the elliptic case, we can normalize the function $[x]$ so that its zero set is $\mathbb{Z}+\tau\mathbb{Z}$ for some τ in the upper half-plane. Then, (2.3) means that

$$\delta, \ \kappa \notin \mathbb{Q} + \tau\mathbb{Q}.$$

The physically most natural case is when $\tau \in \mathrm{i}\mathbb{R}_{>0}$ and the parameters $\beta = \mathrm{i}\delta$ and $g = \kappa/\delta$ are real. This gives the conditions

$$\beta,\ g\beta \notin \mathrm{i}\tau\mathbb{Q}.$$

We need these conditions to make the operators $H_{m,r}^{(k)}$ and $D_{m,r}^{(k)}$ well-defined for all k. If they are violated, the operators still make sense for a finite range of k, which could conceivably be extended by appropriate renormalization.

B Multiplicative notation

We have considered our operators as acting by additive shifts. In the trigonometric and elliptic cases, they can also be realized by multiplicative shifts. We will restate our main results in this form, as it is very common in the literature.

Excluding the rational case, the function $x \mapsto [x]$ can be chosen as periodic. After rescaling the variable, we can assume that the primitive period is 2. We then normalize the function as

$$[x] = e^{-\mathrm{i}\pi x}\theta(e^{2\mathrm{i}\pi x}; p), \tag{B.1}$$

where

$$\theta(z; p) = \prod_{j=0}^{\infty}(1 - p^j z)\left(1 - \frac{p^{j+1}}{z}\right)$$

and the elliptic nome p satisfies $|p| < 1$. The trigonometric case is included as

$$[x]_{p=0} = e^{-\mathrm{i}\pi x}(1 - e^{2\mathrm{i}\pi x}) = -2\mathrm{i}\sin(\pi x).$$

If $z = e^{2\mathrm{i}\pi x}$, the additive shifts $x \mapsto x + \delta$ and $x \mapsto x - \kappa$ correspond to $z \mapsto qz$, $z \mapsto t^{-1}z$, where

$$q = e^{2\mathrm{i}\pi\delta}, \qquad t = e^{2\mathrm{i}\pi\kappa}.$$

The assumption (2.3) means that $q,\ t \notin e^{\mathrm{i}\pi\mathbb{Q}}p^{\mathbb{Q}}$.

Consider the operators $H_{m,r}^{(k)}$ as acting on functions that are 1-periodic in the variables x_j and y_j, and hence can be expressed in terms of $z_j = e^{2\mathrm{i}\pi x_j}$ and $w_j = e^{2\mathrm{i}\pi y_j}$. We will normalize the resulting multiplicative difference operator as

$$\mathbf{H}_{m,r}^{(k)} = \mathbf{H}_{m,r}^{(k)}(z_1, \dots, z_m; w_1, \dots, w_r; q, t) = e^{\mathrm{i}\pi k((r-1)\delta - m\kappa)} H_{m,r}^{(k)}. \tag{B.2a}$$

We also introduce the modified operators

$$\mathbf{D}_{m,r}^{(k)} = \mathbf{H}_{r,m}^{(k)}(w_1, \dots, w_r; z_1, \dots, z_m; t^{-1}, q^{-1}), \tag{B.2b}$$

$$\hat{\mathbf{H}}_{m,r}^{(k)} = \mathbf{H}_{m,r}^{(k)}(q^{-1}z_1, \dots, q^{-1}z_m; tw_1, \dots, tw_r; q^{-1}, t^{-1}), \tag{B.2c}$$

$$\hat{\mathbf{D}}_{m,r}^{(k)} = \mathbf{H}_{r,m}^{(k)}(tw_1, \dots, tw_r; q^{-1}z_1, \dots, q^{-1}z_m; t, q), \tag{B.2d}$$

which are related to the additive operators used in the main text by

$$\begin{aligned}
\mathbf{D}_{m,r}^{(k)} &= e^{\mathrm{i}\pi k(r\delta - (m-1)\kappa)} D_{m,r}^{(k)}, \\
\hat{\mathbf{H}}_{m,r}^{(k)} &= e^{\mathrm{i}\pi k(m\kappa - (r-1)\delta)} \hat{H}_{m,r}^{(k)}, \\
\hat{\mathbf{D}}_{m,r}^{(k)} &= e^{\mathrm{i}\pi k((m-1)\kappa - r\delta)} \hat{D}_{m,r}^{(k)}.
\end{aligned}$$

It is straight-forward to verify that, in the notation

$$\begin{aligned}(a;q,p)_k &= \theta(a;p)\theta(aq;p)\cdots\theta(aq^{k-1};p),\\ T_{q,z}^{\mu} f(z_1,\dots,z_m) &= f(q^{\mu_1}z_1,\dots,q^{\mu_m}z_m),\end{aligned}$$

we have

$$\mathbf{H}_{m,r}^{(k)} = \sum_{\substack{\mu\in\mathbb{Z}_{\geq 0}^m,\, I\subseteq\langle r\rangle,\\ |\mu|+|I|=k}} C_{\mu,I}(z;w)\, T_{q,z}^{\mu} T_{t^{-1},w}^{I},$$

where

$$\begin{aligned}C_{\mu,I}(z;w) &= (-1)^{|I|}(t^{-m}q^r)^{|\mu|} q^{\binom{|I|}{2}} \prod_{i\in I,\, j\in I^c} \frac{\theta(qw_j/w_i;p)}{\theta(w_j/w_i;p)} \prod_{i,j=1}^{m} \frac{(tz_i/z_j;q,p)_{\mu_i}}{(qz_i/z_j;q,p)_{\mu_i}}\\ &\quad\times \prod_{1\leq i<j\leq m} \frac{q^{\mu_j}\theta(q^{\mu_i-\mu_j}z_i/z_j;p)}{\theta(z_i/z_j;p)}\\ &\quad\times \prod_{i=1}^{m}\left(\prod_{j\in I}\frac{\theta(z_i/tw_j;p)}{\theta(q^{\mu_i}z_i/w_j;p)} \prod_{j\in I^c}\frac{\theta(z_i/qw_j;p)}{\theta(q^{\mu_i-1}z_i/w_j;p)}\right).\end{aligned}$$

In multiplicative notation, Theorems 2.2, 2.4, Corollaries 2.1, 2.2 and 2.3 can be summarized as follows.

Theorem B.1. *For fixed m and r, the four infinite families of operators (B.2) mutually commute. If q and t are generic, the operators (B.2a) are algebraically independent for $1\leq k\leq m+r$. The operators (B.2a) and (B.2b) are related by*

$$\sum_{k+l=N} t^k\theta(q^kt^l)\mathbf{H}_{m,r}^{(k)}\mathbf{D}_{m,r}^{(l)} = 0, \qquad N\geq 1,$$

and by

$$\mathbf{H}_{m,r}^{(l)} = (-t)^{-l} \det_{1\leq i,j\leq l}\left(\frac{\theta(t^{i-j+1}q^{j-1})}{\theta(q^i)}\,\mathbf{D}_{m,r}^{(i-j+1)}\right),$$

where one should interpret matrix elements with $i-j+1<0$ as zero.

To write Theorem 2.5 in multiplicative notation takes some more work. We will express the kernel function in terms of the elliptic gamma function [22]

$$\Gamma(z;p,q) = \prod_{j,k=0}^{\infty}\frac{1-p^{j+1}q^{k+1}/z}{1-p^jq^kz}, \qquad |p|,\ |q|<1,$$

which satisfies the q-difference equation

$$\frac{\Gamma(qz;p,q)}{\Gamma(z;p,q)} = \theta(z;p).$$

Equivalently, the function

$$G_\delta(x) = e^{\frac{\mathrm{i}\pi x(\delta-x)}{2\delta}} \Gamma(e^{2\mathrm{i}\pi x}; p, q) \tag{B.3a}$$

satisfies (2.11). This solution is valid for $|q| < 1$, that is, $\mathrm{Im}(\delta) > 0$. If $\mathrm{Im}(\delta) < 0$, one can instead take

$$G_\delta(x) = \frac{e^{\frac{\mathrm{i}\pi x(\delta-x)}{2\delta}}}{\Gamma(q^{-1}e^{2\mathrm{i}\pi x}; p, q^{-1})}. \tag{B.3b}$$

In either case, the general solution of (2.11) is G_δ times an arbitrary δ-periodic meromorphic function. The construction of solutions to (2.11) with real δ (that is, $|q| = 1$) is more complicated [27], so we will assume for simplicity that $|q|,\ |t| \neq 1$. In the case $|q| < 1 < |t|$, we introduce the multiplicative kernel function

$$\begin{aligned}
&\mathbf{\Phi}^{(m,r,n,s)}(z_1, \dots, z_m; w_1, \dots, w_r; Z_1, \dots, Z_n; W_1, \dots, W_s) \\
&\quad = \prod_{\substack{1\le i\le m,\\ 1\le j\le n}} \frac{\Gamma(t^{-1}z_iZ_j; p, q)}{\Gamma(z_iZ_j; p, q)} \prod_{\substack{1\le i\le r,\\ 1\le j\le s}} \frac{\Gamma(qw_iW_j; p, t^{-1})}{\Gamma(w_iW_j; p, t^{-1})} \\
&\qquad \times \prod_{\substack{1\le i\le m,\\ 1\le j\le s}} \theta(z_iW_j; p) \prod_{\substack{1\le i\le r,\\ 1\le j\le n}} \theta(w_iZ_j; p).
\end{aligned} \tag{B.4}$$

If one or both the parameters $|q|$ and $|t^{-1}|$ is larger than 1, we define $\mathbf{\Phi}$ by the expression obtained from (B.4) after making the formal replacement

$$\Gamma(x; p, s) \mapsto \frac{1}{\Gamma(sx; p, 1/s)}, \qquad |s| > 1.$$

Theorem B.2. *Assuming $t^{m-n} = q^{r-s}$, the kernel function identity*

$$\mathbf{H}_{m,r}^{(k)}(z; w)\mathbf{\Phi}^{(m,r,n,s)}(z; w; Z; W) = \mathbf{H}_{n,s}^{(k)}(Z; W)\mathbf{\Phi}^{(m,r,n,s)}(z; w; Z; W) \tag{B.5}$$

holds.

In particular, (B.5) holds if $m = n$ and $r = s$.

To prove Theorem B.2, we insert (B.1) and (B.3) into (2.13). In terms of the multiplicative variables $z_j = e^{2\mathrm{i}\pi x_j}$, $w_j = e^{2\mathrm{i}\pi y_j}$, $Z_j = e^{2\mathrm{i}\pi X_j}$ and $W_j = e^{2\mathrm{i}\pi Y_j}$, the additive and multiplicative kernel functions are related by

$$\begin{aligned}
&\frac{\Phi^{(m,r,n,s)}(x; y; X; Y)}{\mathbf{\Phi}^{(m,r,n,s)}(z; w; Z; W)} \\
&\quad = \prod_{\substack{1\le i\le m,\\ 1\le j\le n}} \frac{e^{\mathrm{i}\pi(x_i+X_j-\kappa)(\delta-x_i-X_j+\kappa)/2\delta}}{e^{\mathrm{i}\pi(x_i+X_j)(\delta-x_i-X_j)/2\delta}} \prod_{\substack{1\le i\le r,\\ 1\le j\le s}} \frac{e^{-\mathrm{i}\pi(y_i+Y_j+\delta)(-\kappa-y_i-Y_j-\delta)/2\kappa}}{e^{-\mathrm{i}\pi(y_i+Y_j)(-\kappa-x_i-X_j)/2\kappa}} \\
&\qquad \times \prod_{\substack{1\le i\le m,\\ 1\le j\le s}} e^{-\mathrm{i}\pi(x_i+Y_j)} \prod_{\substack{1\le i\le r,\\ 1\le j\le n}} e^{-\mathrm{i}\pi(y_i+X_j)} = Ce^{\mathrm{i}\pi(A(x;y)+B(X;Y))},
\end{aligned}$$

where C is an irrelevant constant and

$$A(x;y) = \frac{\kappa}{\delta} n|x| + \frac{\delta}{\kappa} s|y| - s|x| - n|y|,$$
$$B(X;Y) = \frac{\kappa}{\delta} m|X| + \frac{\delta}{\kappa} r|Y| - r|X| - m|Y|.$$

We can now write the kernel function identity (2.14) as

$$\begin{aligned} &e^{-\mathrm{i}\pi A(x;y)} H_{m,r}^{(k)}(x;y) e^{\mathrm{i}\pi A(x;y)} \boldsymbol{\Phi}^{m,r,n,s}(z;w;Z;W) \\ &\quad = e^{-\mathrm{i}\pi B(X;Y)} H_{n,s}^{(k)}(X;Y) e^{\mathrm{i}\pi B(X;Y)} \boldsymbol{\Phi}^{(m,r,n,s)}(z;w;Z;W). \end{aligned} \tag{B.6}$$

The operator on the left is a sum of terms of the form

$$e^{-\mathrm{i}\pi A(x;y)} T_x^{\delta\mu} T_y^{-\kappa I} e^{\mathrm{i}\pi A(x;y)} = e^{\mathrm{i}\pi(A(x+\delta\mu;y-\kappa I)-A(x;y))} T_x^{\delta\mu} T_y^{-\kappa I} = e^{\mathrm{i}\pi k(n\kappa - s\delta)} T_x^{\delta\mu} T_y^{-\kappa I}.$$

Hence,

$$e^{-\mathrm{i}\pi A(x;y)} H_{m,r}^{(k)} e^{\mathrm{i}\pi A(x;y)} = e^{\mathrm{i}\pi k(n\kappa - s\delta)} H_{m,r}^{(k)} = e^{\mathrm{i}\pi k((m+n)\kappa - (r+s-1)\delta)} \mathbf{H}_{m,r}^{(k)}.$$

On the right-hand side of (B.6), the same exponential prefactor appears and can be canceled. This proves Theorem B.2.

References

1. Atai, F., Hallnäs, M., Langmann, E.: Source identities and kernel functions for deformed (quantum) Ruijsenaars models. Lett. Math. Phys. **104**, 811–835 (2014)
2. Atai, F., Hallnäs, M., Langmann, E.: Super-Macdonald polynomials: orthogonality and Hilbert space interpretation. Commun. Math. Phys. **388**, 435–468 (2021)
3. Calogero, F.: Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials. J. Math. Phys. **12**, 419–436 (1971)
4. Chalykh, O.A.: Duality of the generalized Calogero and Ruijsenaars problems. Russ. Math. Surv. **52**, 1289–1291 (1997)
5. Chalykh, O.A.: Macdonald polynomials and algebraic integrability. Adv. Math. **166**, 193–259 (2002)
6. Chalykh, O., Feigin, M., Veselov, A.: New integrable generalizations of Calogero–Moser quantum problem. J. Math. Phys. **39**, 695–703 (1998)
7. van Diejen, J.F., Görbe, T.: Elliptic Ruijsenaars difference operators on bounded partitions. Internat. Math. Res. Notices (to appear)
8. van Diejen, J.F., Görbe, T.: Elliptic Ruijsenaars difference operators, symmetric polynomials, and Wess–Zumino–Witten fusion rings, arXiv:2106.14919
9. Feigin, M., Silantyev, A.: Generalized Macdonald–Ruijsenaars systems. Adv. Math. **250**, 144–192 (2014)
10. Frobenius, F.G.: Über die elliptischen Functionen zweiter Art. J. Reine Angew. Math. **93**, 53–68 (1882)
11. Hallnäs, M., Langmann, E., Noumi, M., Rosengren, H.: From Kajihara's transformation formula to deformed Macdonald–Ruijsenaars and Noumi–Sano operators. Selecta Math. (N.S.) **28**, 24 (2022)
12. Kajihara, Y.: Euler transformation formula for multiple basic hypergeometric series of type A and some applications. Adv. Math. **187**, 53–97 (2004)
13. Kajihara, Y., Noumi, M.: Multiple elliptic hypergeometric series. An approach from the Cauchy determinant. Indag. Math. **14**, 395–421 (2003)
14. Komori, Y.: Ruijsenaars' commuting difference operators and invariant subspace spanned by theta functions. J. Math. Phys. **42**, 4503–4522 (2001)
15. Langer, R., Schlosser, M.J., Warnaar, S.O.: Theta functions, elliptic hypergeometric series, and Kawanaka's Macdonald polynomial conjecture. SIGMA **5**, 055 (2009)
16. Langmann, E., Noumi, M., Shiraishi, J.: Construction of eigenfunctions for the elliptic Ruijsenaars difference operators. Commun. Math. Phys. (to appear)
17. Macdonald, I.G.: Symmetric Functions and Hall Polynomials, 2nd edn. Oxford University Press, New York (1995)

18. Mironov, A., Morozov, A., Zenkevich, Y.: Duality in elliptic Ruijsenaars system and elliptic symmetric functions. Eur. Phys. J. C **81**, 461 (2021)
19. Noumi, M., Sano, A.: An infinite family of higher-order difference operators that commute with Ruijsenaars operators of type A. Lett. Math. Phys. **111**, 91 (2021)
20. Rosengren, H.: New transformations for elliptic hypergeometric series on the root system A_n. Ramanujan J. **12**, 155–166 (2006)
21. Ruijsenaars, S.N.M.: Complete integrability of relativistic Calogero–Moser systems and elliptic function identities. Commun. Math. Phys. **110**, 191–213 (1987)
22. Ruijsenaars, S.N.M.: First order analytic difference equations and integrable quantum systems. J. Math. Phys. **38**, 1069–1146 (1997)
23. Ruijsenaars, S.N.M.: Eigenfunctions with a zero eigenvalue for differences of elliptic relativistic Calogero–Moser Hamiltonians. Theor. Math. Phys. **146**, 25–33 (2006)
24. Sergeev, A.: Superanalogs of the Calogero operators and Jack polynomials. J. Nonlinear Math. Phys. **8**(1), 59–64 (2001)
25. Sergeev, A.N., Veselov, A.P.: Deformed quantum Calogero–Moser systems and Lie superalgebras. Commun. Math. Phys. **245**, 249–278 (2004)
26. Sergeev, A.N., Veselov, A.P.: Deformed Macdonald–Ruijsenaars operators and super Macdonald polynomials. Commun. Math. Phys. **288**, 653–675 (2009)
27. Spiridonov, V.P.: Theta hypergeometric integrals. St. Petersb. Math. J. **15**, 929–967 (2004)
28. Warnaar, S.O.: Summation and transformation formulas for elliptic hypergeometric series. Constr. Approx. **18**, 479–502 (2002)
29. Whittaker, E.T., Watson, G.N.: A Course of Modern Analysis. Cambridge University Press, Cambridge (1927)

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