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The flatness of the $\mathcal{O}$-module of smooth functions and integral representation (*)

Mats Andersson (1)

Abstract. — We give a proof of the well-known fact that the $\mathcal{O}$-module $\mathcal{E}$ of smooth functions is flat by means of residue theory and integral formulas. A variant of the proof gives a similar result for classes of functions of lower regularity. We also prove a Briançon–Skoda type theorem for ideals of the form $\mathcal{E}a$, where a is an ideal in $\mathcal{O}$.

Résumé. — Nous donnons une preuve du fait bien connu que le $\mathcal{O}$-module $\mathcal{E}$ des fonctions lisses est plat au moyen de la théorie des résidus et des formules intègrales. Une variante de la preuve donne une résultat similaire pour classes de fonctions de moindre régularité. Nous prouvons également un théorème de type Briançon-Skoda pour des idéaux de la forme $\mathcal{E}a$, où a est un idéal dans $\mathcal{O}$.

1. Introduction

Let X be a complex manifold of dimension n with structure sheaf $\mathcal{O}$ of holomorphic functions, and let $\mathcal{E}$ be the analytic sheaf of smooth functions. It is well-known, and first proved by Malgrange already in the '60s, [14], that $\mathcal{E}$ is a flat $\mathcal{O}$-module; that is, an exact sequence of $\mathcal{O}$-modules remains exact when tensored by $\mathcal{E}$. It is enough to prove, see Section 2.1, noting that $\mathcal{E}^{\oplus m} = \mathcal{E} \otimes_{\mathcal{O}} \mathcal{O}^{\oplus m}$: If

$$\mathcal{O}^{\oplus m_2} \xrightarrow{f_2} \mathcal{O}^{\oplus m_1} \xrightarrow{f_1} \mathcal{O}^{\oplus m_0} \tag{1.1}$$

is exact, then the induced sequence

$$\mathcal{E}^{\oplus m_2} \xrightarrow{f_2} \mathcal{E}^{\oplus m_1} \xrightarrow{f_1} \mathcal{E}^{\oplus m_0} \tag{1.2}$$

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is exact, that is, locally there is a smooth solution to $f_2\psi = \phi$ for each smooth ϕ such that $f_1\phi = 0$. It is in fact enough to check the case when $m_0 = 1$.

Our first aim of this note is to give a proof based on residue theory and an integral formula that provides the desired smooth solution ψ. The idea to use integral formulas to find holomorphic solutions to this kind of equations, often referred to as division problems, was introduced by Berndtsson in [11]. It was further developed and adapted for a variety of situtions, see, e.g., [2, 4, 5, 6, 7, 8, 9, 10, 12, 15, 16, 17, 18].

One should notice that for instance the $\mathcal{O}$-module C of continuous functions is not flat. Let us consider the simple exact sequence

$$\mathcal{O} \xrightarrow{f_2} \mathcal{O}^{\oplus 2} \xrightarrow{f_1} \mathcal{O}, \tag{1.3}$$

in a neighborhood of the origin in $\mathbb{C}^2_{x_1,x_2}$, where $f_1 = (x_1, x_2)$ and $f_2 = (-x_2, x_1)^t$. Observe that $C^{\oplus m} = C \otimes_{\mathcal{O}} \mathcal{O}^{\oplus m}$. The induced sequence $C \to C^{\oplus 2} \to C$ is not exact. For instance, $\phi = (-x_2, x_1)^t/|x|^{1/3}$ is continuous and $f_1\phi = 0$ but there is no continuous function ψ such that $f_2\psi = \phi$; in fact, since f_2 is pointwise injective outside the origin the only possible solution is $\psi = 1/|x|^{1/3}$. However, if ϕ is in C^1 and $f_1\phi = 0$, then there is indeed a continuous solution to $f_2\psi = \phi$, cf. Example 3.4 below.

Following [8] one can associate a certain current U with (1.1), see Section 2.3.

THEOREM 1.1. — *Assume that* (1.1) *is exact in a neighborhood of* $0 \in \mathbb{C}^n$ *and let* M *be the order of the associated current* U*.*

(i) *If* ϕ *is in* $\mathcal{E}^{\oplus m_1}$ *and* $f_1\phi = 0$*, then there is* ψ *in* $\mathcal{E}^{\oplus m_2}$ *such that* $f_2\psi = \phi$*.*
(ii) *If* ϕ *in* $C^{k+2M+c_n} \otimes \mathcal{O}^{\oplus m_1}$ *and* $f_1\phi = 0$*, then there is* ψ *in* $C^k \otimes \mathcal{O}^{\oplus m_2}$ *such that* $f_2\psi = \phi$*. Here* c_n *is a constant that only depends on the dimension* n*.*

In view of the discussion above, (i) immediately implies that $\mathcal{E}$ is a flat $\mathcal{O}$-module.

The proof of Theorem 1.1 relies on of the construction of weighted integral representation formulas in [1, 4], the residue currents associated with free resolutions in [8, Section 5], and the special choice of weight in [2].

The classical Briançon–Skoda theorem, first proved in [19], states that if $a \subset \mathcal{O}$ is an ideal and $\phi \in \mathcal{O}$ is a function such that

$$|\phi| \leqslant C|a|^{\mu+r-1}, \tag{1.4}$$

where

$$\mu := \min(m, n) \tag{1.5}$$

and m is the minimal number of generators for a, then ϕ is in a^r. Here $|a| = \sum |a_j|$ for a (finite) set a_j of generators. Any other set of generators gives rise to a comparable quantity. By the Nullstellensatz ϕ is in the ideal a^r if it vanishes to a large enough order at the zero set of a. The important point is that the condition (1.4) is uniform in both a and r (if we replace μ by n).

There is a purely algebraic formulation in terms of integral closures, see, e.g., [19], and there are general versions for Noetherian, even non-regular, rings, [13]. We do not know if there is some kind of algebraic analogue for a non-Noetherian ring like $\mathcal{E}$. However, we have the following analytic variant for ideals $\mathcal{E}a \subset \mathcal{E}$, where a are ideals in $\mathcal{O}$. Notice that $(\mathcal{E}a)^r = \mathcal{E}a^r$.

THEOREM 1.2. — *Assume that ϕ is a germ of a smooth function at 0, $a \subset \mathcal{O}$ is an ideal and r is a positive integer. If there are constants C_α such that*

$$|\partial_{\bar{z}}^\alpha \phi| \leqslant C_\alpha |a|^{\mu+r-1} \tag{1.6}$$

for all multi-indices $\alpha \geqslant 0$, then ϕ is in $\mathcal{E}a^r$.

Notice that if $a_1, \dots, a_m$ generate a and ϕ is in $\mathcal{E}a^r$, then

$$\phi = \sum_{|I|=r} \xi_I a_1^{I_1} \dots a_m^{I_m},$$

where ξ_I are in $\mathcal{E}$. Noting that $|a^r| \sim |a|^r$, thus

$$\bar{\partial}_{\bar{z}}^\alpha \phi = \sum_{|I|=r} (\bar{\partial}_{\bar{z}}^\alpha \xi_I) a_1^{I_1} \dots a_m^{I_m}.$$

Therefore, (1.6) holds for all α with $\mu = 1$, and thus a condition like (1.6) is necessary.

The plan of this note is as follows. In Section 2 we recall some results about residue theory and integral representation of solutions to division problems. In the last sections we provide the proofs of Theorems 1.1 and 1.2.

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2. Preliminaries

We first recall some material that is basically known, but presented in a way that is adapted for the proofs in the last two sections.

2.1. Flatness

Let R be a commutative ring and M an R-module. There are many equivalent statements with the meaning that M is flat. One is the following:

M is flat if and only if $J\otimes_R M \to R\otimes_R M = M$ is injective for each finitely generated ideal $J \subset R$.

It is quite easy to see that this holds if and only if for any finite relation $\sum_1^d r_j\phi_j = 0$, where $r_j \in R$ and $\phi_j \in M$, letting $\phi = (\phi_1, \dots, \phi_d)^t$, there are a finite R-matrix A and a tuple b of elements in M such that $\phi = Ab$ and $(r_1 \dots r_d)A = 0$.

In our case R is the local Noetherian ring $\mathcal{O}$ at some point, and so all ideals are finitely generated. By Oka's lemma there is a holomorphic matrix f_2 such that

$$R^{m_2} \xrightarrow{f_2} R^d \xrightarrow{(r_1\dots r_d)} R \tag{2.1}$$

is exact (in particular, $(r_1 \dots r_d)f_2 = 0$) in a neighborhood of 0. Thus we are precisely in the situation in the introduction of this note, with $m_0 = 1$ and $m_1 = d$, and so the flatness of the $\mathcal{O}$-module $\mathcal{E}$ follows from the exactness of (1.2).

2.2. Integral representation

We first describe the idea in [1] to construct representation formulas for holomorphic functions in an open set $\Omega \subset \mathbb{C}^n$. Let z be a fixed point in Ω, let $\delta_{z-\zeta}$ be interior multiplication by the vector field

$$2\pi i \sum_1^n (\zeta_j - z_j)\frac{\partial}{\partial\bar\zeta_j},$$

and let $\nabla_{\zeta-z} = \delta_{\zeta-z} - \bar\partial$. Notice that $\nabla_{\zeta-z}$ satisfies Leibniz' rule

$$\nabla_{\zeta-z}(\alpha\wedge\beta) = \nabla_{\zeta-z}\alpha\wedge\beta + (-1)^{\deg\alpha}\alpha\wedge\nabla_{\zeta-z}\beta. \tag{2.2}$$

We say that a current $g = g_{0,0} + g_{1,1} + \dots + g_{n,n}$, where lower indices denote bidegree, is a *weight* with respect to z if $\nabla_{\zeta-z}g = 0$, g is smooth in

a neighborhood of z and $g_{0,0}(z) = 1$. If g^1 and g^2 are weights, and one of them is smooth, then by (2.2) also $g^1 \wedge g^2$ is a weight.

Let $b = \partial_\zeta |\zeta - z|^2/2\pi i|\zeta - z|^2$ and for $\zeta \neq z$ consider the form

$$v^z = \frac{b}{\nabla_{\zeta - z} b} = b + b \wedge \bar\partial_\zeta b + \cdots + b \wedge (\bar\partial_\zeta b)^{n-1}; \tag{2.3}$$

in [1] it is called the full Bochner–Martinelli form because the component $v^z_{n,n-1}$ of bidegree $(n, n-1)$ is precisely the classical Bochner–Martinelli kernel with pole at the point z. It is easy to see that $\nabla_{\zeta - z} v^z = 1$. Since v^z is locally integrable it has a natural current extension across z and

$$\nabla_{\zeta - z} v^z = 1 - [z], \tag{2.4}$$

where $[z]$ denotes the Dirac measure at z considered as an (n, n)-current, cf. [1, Section 2].

Proposition 2.1. — *Assume that g is a weight with respect to $z \in \Omega$ with compact support in Ω. If $\Phi = \Phi_{0,0} + \cdots + \Phi_{n,n}$ is a smooth form in Ω and $\nabla_{\zeta - z}\Phi = 0$, then*

$$\Phi_{0,0}(z) = \int g \wedge \Phi = \int (g \wedge \Phi)_{n,n}. \tag{2.5}$$

In particular the formula holds for $\Phi = \phi$ if ϕ is holomorphic in Ω.

Proof. — Notice that $v^z \wedge g \wedge \Phi$ is a well-defined current. By (2.2) and (2.4),

$$\nabla_{\zeta - z}(v^z \wedge g \wedge \Phi) = \nabla_{\zeta - z} v \wedge g \wedge \Phi = (1 - [z]) \wedge g \wedge \Phi = g \wedge \Phi - \Phi_{0,0}[z].$$

For degree reasons it follows that

$$d(v^z \wedge g \wedge \Phi)_{n,n-1} = (g \wedge \Phi)_{n,n} - \Phi_{0,0}[z]$$

and so (2.5) follows from Stokes' theorem. □

Assume that $E \to \Omega$ is a holomorphic vector bundle, Φ takes values in E, g takes values in $\mathrm{Hom}(E_\zeta, E_z)$, g is smooth in a neighborhood of z and $g_{0,0}(z) = I_{E_z}$. Then the same proof gives (2.5) for these g and Φ.

Example 2.2. — Assume that Ω is the unit ball. If

$$s = \frac{1}{2\pi i} \frac{\partial |\zeta|^2}{|\zeta|^2 - z \cdot \bar\zeta},$$

then $\delta_{\zeta - z} s = 1$ when $\zeta \neq z$. If

$$u = \frac{s}{\nabla_{\zeta - z} s} = s \wedge \big(1 + \bar\partial s + \cdots + (\bar\partial s)^{n-1}\big),$$

then $\nabla_{\zeta-z}u = 1$ when $\zeta \neq z$. If χ is a cutoff function in Ω that is 1 in a neighborhood of the closure of a smaller ball Ω', then

$$g = \chi - \bar\partial\chi \wedge u$$

is a weight with respect to z for each $z \in \Omega'$, depends holomorphically on z, and has compact support in Ω.

2.3. Residues associated with generically exact Hermitian complexes.

Assume that we have a complex

$$0 \longrightarrow E_N \xrightarrow{f_N} E_{N-1} \xrightarrow{f_{N-1}} \cdots \xrightarrow{f_3} E_2 \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 \tag{2.6}$$

of Hermitian vector bundles over a complex manifold X. Let us also assume that (2.6) is pointwise exact in $X \setminus Z$, where Z is an analytic set of positive codimension. Let $E = \bigoplus E_j$ and $f = \bigoplus f_j$, and let $\nabla_f = f - \bar\partial$. In [8] were defined currents U and R with components U_k^ℓ and R_k^ℓ with values in $\mathrm{Hom}(E_\ell, E_k)$ and of bidegree $(0, k-\ell-1)$ and $(0, k-\ell)$, respectively, such that

$$\nabla_f \circ U + U \circ \nabla_f = I_E - R \tag{2.7}$$

and R has support on Z. To be precise, one introduces a superstructure on $E \oplus T^*\Omega$ so that U, f and ∇_f are odd mappings whereas R is even; for details see [8, Section 2]. However, for the purpose of this paper, the precise signs are not essential.

It is proved that if ϕ is a holomorphic section of E_ℓ such that $f_\ell\phi = 0$ and, in addition, $R^\ell\phi = 0$, then locally there is a holomorphic solution to $f_{\ell+1}\psi = \phi$.

Example 2.3. — Let $A \to X$ be a Hermitian vector bundle of rank m and let a be a holomorphic section of its dual. If $E_k = \wedge^k A$ we get the so-called Koszul complex by letting f_k be interior multiplication δ_a by a. If Z is the set where a vanishes, then the complex is exact in $\Omega \setminus Z$. If σ is the section of E over $X \setminus Z$ with pointwise minimal norm such that $\delta_a\sigma = 1$, then

$$U_k^\ell = \sigma \wedge (\bar\partial\sigma)^{k-\ell}$$

there. Assume now that $a = a_0 a'$, where a_0 is a section of a line bundle $L^* \to X$ and a' is a non-vanishing section of $A^* \otimes L$. Then $\sigma = \sigma'/a_0$, where σ' is smooth. Since $\sigma' \wedge \sigma' = 0$, thus

$$U_k^\ell = \frac{1}{a_0^{k-\ell}}\sigma' \wedge (\bar\partial\sigma')^{k-\ell-1}.$$

It turns out that U_k^ℓ have extensions across Z as principal value currents. Since $R_k^0 = \delta_a U_{k+1}^0 - \bar\partial U_k^0$ one can check that

$$R_k^0 = \bar\partial \frac{1}{a_0^k} \wedge \sigma' \wedge (\bar\partial \sigma')^{k-1}.$$

For degree reasons this current must vanish when $k > n$ and since $\operatorname{rank} A = m$ it must vanish if $k > m$. Using that $a_0 \bar\partial (1/a_0^{k+1}) = \bar\partial (1/a_0^k)$ we conclude that

$$R^0 = \bar\partial \frac{1}{a_0^\mu} \wedge \omega, \tag{2.8}$$

where ω is a smooth form.

Example 2.4. — Let $A \to X$ and a be as in the previous example. If $r \geqslant 2$ we can construct a similar complex such that $E_0 = \mathbb{C}$, $E_1 = \otimes_1^r A$ and f_1 is $\delta_a \otimes \cdots \otimes \delta_a$. For a description of the whole complex, see, e.g., [3, Proof of Theorem 1.3]. If now $a = a_0 a'$, it follows by considerations as in the previous example that there is a smooth form ω such that

$$R^0 = \bar\partial \frac{1}{a_0^{\mu+r-1}} \wedge \omega. \tag{2.9}$$

2.4. Free resolutions

Assume that we have a sequence (1.1) in a neighborhood of a point $0 \in \mathcal{U} \subset \mathbb{C}^n$. Let $\mathcal{J}$ be the ideal generated by the entries in f_1, or equivalently, the image of the mapping f_1 in $\mathcal{O}$. After possibly shrinking $\mathcal{U}$ we can extend to an exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}^{\oplus m_N} \xrightarrow{f_n} \mathcal{O}^{\oplus m_{n-1}} \xrightarrow{f_{n-1}} \cdots$$
$$\cdots \xrightarrow{f_2} \mathcal{O}^{\oplus m_1} \xrightarrow{f_1} \mathcal{O}^{\oplus m_0} \longrightarrow \mathcal{O}^{\oplus m_0}/\mathcal{J} \longrightarrow 0, \tag{2.10}$$

that is, a free resolution of the $\mathcal{O}$-module $\mathcal{O}^{m_0}/\mathcal{J}$ in $\mathcal{U}$. By Hilbert's zyzygy theorem we can assume that $N \leqslant n$. We have an induced complex of (trivial) vector bundles (2.6), where $\operatorname{rank} E_k = m_k$, which is pointwise exact outside the zero set Z of f_1 (if $m_0 \geqslant 2$, then Z is the set where f_1 does not have optimal rank.) Let us equip these vector bundles with any Hermitian metrics, for instance trivial metrics with respect to global frames, and let R and U be the associated currents. A main result in [8] is that the exactness of (2.10) implies that $R^\ell = 0$ for $\ell \geqslant 1$. This in turn implies that if $\phi \in \mathcal{O}(E_0)$ (and ϕ is pointwise generically in the image of f_1), then $f_1 \psi = \phi$ has locally a holomorphic solution if and only if $R\phi = 0$.

Definition 2.5. — *Given an exact sequence* (1.1) *we let M be the minimal order of such an associated current U.*

2.5. Hefer mappings

Let us recall the idea from [4] to construct division-interpolation formulas. Given a generically exact complex like (2.6) in $\Omega \subset \mathbb{C}^n$ and $z \in \Omega$ one can locally find a tuple $H = (H_k^\ell)$, where H_k^ℓ is a smooth form-valued section of $\mathrm{Hom}(E_{k,\zeta}, E_{\ell,z})$, such that $H_k^\ell = 0$ for $k < \ell$, $H_\ell^\ell = I_{E_\ell}$ when $\zeta = z$, and in general

$$\nabla_{\zeta - z} H_k^\ell = H_{k-1}^\ell f_k - f_{\ell+1}(z) H_k^{\ell+1}. \tag{2.11}$$

In fact, if Ω is pseudoconvex, then one can find such an H that depends holomorphically on both $\zeta \in \Omega$ and $z \in \Omega' \subset\subset \Omega$. In that case H_k^ℓ has bidegree $(k-\ell, 0)$. It was proved in [4], see also the proof of Proposition 4.2 in [6], that if U and R are the currents above associated with the Hermitian complex (2.6), then

$$f(z)HU + HUf + HR \tag{2.12}$$

is $\nabla_{\zeta - z}$-closed. Let us fix a non-negative integer ℓ and let g' be the component of (2.12) with values in E_ℓ, that is,

$$\begin{aligned} g' &= f_{\ell+1}(z) H^{\ell+1} U^\ell + H^\ell U^{\ell-1} f_\ell + H^\ell R^\ell \\ &= \sum_k f_{\ell+1}(z) H_k^{\ell+1} U_k^\ell + \sum_k H_k^\ell U^{\ell-1} f_\ell + \sum_k H_k^\ell R_k^\ell. \end{aligned}$$

Then $g'_{0,0}(z) = I_{E_{\ell,z}}$. If $z \notin Z$, then $g'(\zeta)$ is smooth in a neighborhood of z, and hence it is a weight. If g is a weight with compact support and Φ is as in Proposition 2.1, but taking values in E_ℓ, then, since $g' \wedge g$ is a weight as well,

$$\Phi_{0,0}(z) = \int (g' \wedge g \wedge \Phi)_{n,n} = \int (g' \Phi \wedge g)_{n,n}. \tag{2.13}$$

Since (2.13) holds for $z \in \Omega' \setminus Z$, and both sides are smooth in z, we conclude that (2.13) holds for all $z \in \Omega'$. We can write (2.13) as

$$\Phi_{0,0}(z) = f_{\ell+1}(z) \int_\zeta H^{\ell+1} U^\ell \Phi \wedge g + \int_\zeta H^\ell U^{\ell-1} f_\ell \Phi \wedge g + \int_\zeta H^\ell R^\ell \Phi \wedge g. \tag{2.14}$$

If the sheaf complex (2.10) is exact and $\ell \geqslant 1$, as mentioned above, then $R^\ell = 0$, and hence

$$\Phi_{0,0}(z) = f_{\ell+1}(z) \int_\zeta H^{\ell+1} U^\ell \Phi \wedge g + \int_\zeta H^\ell U^{\ell-1} f_\ell \Phi \wedge g. \tag{2.15}$$

In particular, if $\Phi = \phi$ is a holomorphic section of E_ℓ and $f_\ell \phi = 0$, then the middle term is a holomorphic solution to $f_{\ell+1}\psi = \phi$.

3. Proof of Theorem 1.1

As we have seen the flatness of the $\mathcal{O}$-module $\mathcal{E}$ follows from Theorem 1.1 (i).

In [14] it is proved that $\mathcal{E}$ is a flat C^ω-module, where C^ω is the sheaf of real-analytic functions in $\mathbb{R}^N$. This result is obtained by a sophisticated study of local properties of real-analytic functions and goes via formal power series. The same proof can be applied to $\mathcal{O}$ instead of C^ω and then gives the flatness of $\mathcal{E}$ as an $\mathcal{O}$-module. One can also derive this statement quite easily directly from the flatness of $\mathcal{E}$ as a C^ω-module without reference to the proof in [14]. The proof of the flatness in this paper is independent of [14] but relies on the possibility to resolve singularities, i.e., Hironaka's theorem, which we need to define the currents we use.

3.1. Proof of Theorem 1.1 (i)

Let $\Omega \subset \mathbb{C}^n$ be a neighborhood of 0 where we have the exact complex (1.1). In a possibly smaller neighborhood we can extend to an exact complex (2.10) that we equip with some Hermitian metrics. Let (2.6) be the associated generically exact Hermitian complex and let U and R be the associated currents. Let us identify Ω with the set $\{(\zeta,\bar\zeta) \in \mathbb{C}^{2n};\ \zeta \in \Omega\}$ and let $\widetilde\Omega$ be an open neighborhood of Ω in $\mathbb{C}^{2n}_{\zeta,\omega}$.

Notice that (2.6) induces a generically exact complex in $\widetilde\Omega$ if we let $\widetilde f_\ell(\zeta,\eta) := f_\ell(\zeta)$. The associated currents $\widetilde U$ and $\widetilde R$ are the tensor products $U \otimes 1$ and $R \otimes 1$. In particular, $\widetilde R^\ell = 0$ for $\ell \geqslant 1$, cf. Section 2.4. We will use formula (2.15) in $\widetilde\Omega$.

If ϕ is a smooth section of E_ℓ in Ω, then let us consider the formal sum

$$\widetilde\phi(\zeta,\omega) = \sum_{\alpha\geqslant 0} (\partial^\alpha_{\bar\zeta}\phi)(\zeta)\frac{(\omega-\bar\zeta)^\alpha}{\alpha!}\chi(\lambda_{|\alpha|}(\omega-\bar\zeta)), \tag{3.1}$$

where χ is a cutoff function in $\mathbb{C}^n$ which is 1 in a neighborhood of 0, and λ_k are positive numbers. If $\lambda_k \to \infty$ fast enough, then, possibly after shrinking Ω, the series (3.1) converges to a smooth section of E_ℓ in $\widetilde\Omega$ such that

$$\widetilde\phi(\zeta,\bar\zeta) = \phi(\zeta), \tag{3.2}$$

and

$$\bar\partial\widetilde\phi(\zeta,\omega) = \mathcal{O}(|\omega-\bar\zeta|^\infty). \tag{3.3}$$

Such a $\widetilde\phi(\zeta,\omega)$, satisfying (3.2) and (3.3), is called an almost holomorphic extension of ϕ from Ω to $\widetilde\Omega$.

If ϕ is real-analytic one can take $\lambda_k = 1$ for all k; then $\widetilde{\phi}$ is the holomorphic extension of ϕ. The requirement on λ_k is related to which ultra-differentiable class ϕ belongs to, that is, how fast its Fourier transform decays. If ϕ is in a certain such class and h is holomorphic, then $h\phi$ is in the same class.

LEMMA 3.1. — *Let ϕ be a smooth section of E_ℓ in Ω, let v^z denote the Bochner–Martinelli form in $\widetilde{\Omega}$ with respect to the point $(z,\bar z)$, and let*

$$\Phi^z(\zeta,\omega) = \widetilde{\phi}(\zeta,\omega) - \bar\partial\widetilde{\phi}\wedge v^z.$$

Then Φ^z is smooth in ζ,ω,z and $\nabla_{(\zeta,\omega)-(z,\bar z)}\Phi^z = 0$. Moreover, if $f_\ell\phi = 0$, then $f_\ell\Phi^z = 0$.

Proof. — Since

$$v^z = \frac{b}{\nabla_{(\zeta,\omega)-(z,\bar z)}b},$$

where

$$b = \frac{1}{2\pi i}\left(\sum_1^n(\zeta_j - z_j)\,\mathrm{d}\zeta_j + \sum_1^n(\omega_j - \bar z_j)\,\mathrm{d}\omega_j\right),$$

cf. (2.3), we have that

$$\Phi^z(\zeta,\omega) = \widetilde{\phi}(\zeta,\omega) + \sum_{\ell=1}^{2n}\frac{\mathcal{O}(|\omega-\bar\zeta|^\infty)}{(|\zeta-z|^2+|\omega-\bar z|^2)^{\ell-1/2}},$$

and thus Φ^z is smooth. Since $\nabla_{(\zeta,\omega)-(z,\bar z)}v^z = 1$ outside the point $(z,\bar z)$ it follows that $\nabla_{(\zeta,\omega)-(z,\bar z)}\Phi^z = 0$. If $f_\ell\phi(\zeta) = 0$, then $f_\ell(\partial^\alpha_{\bar\zeta}\phi)(\zeta) = 0$ for all α and therefore $f_\ell\widetilde{\phi}(\zeta,\omega) = 0$. Thus also $f_\ell(\bar\partial\widetilde{\phi})(\zeta,\omega) = 0$, and so the lemma follows. □

Let $\Omega' \subset\subset \Omega$ be an open subset and for each $z\in\Omega'$, let g be a smooth weight with respect to $(z,\bar z)\in\widetilde{\Omega}'$ with compact support in $\widetilde{\Omega}$, cf. Example 2.2; of course we may assume that $\widetilde{\Omega}$ is a ball.

Assume that ϕ is a smooth section of E_ℓ in Ω and choose λ_k such that (3.1) defines an almost holomorphic extension. Then also $f_\ell\phi$ admits such an extension, in fact $\widetilde{f_\ell\phi} = \widetilde{f}_\ell\widetilde{\phi}$. We can then define the operator

$$T_\ell\colon \mathcal{E}(\Omega,R_\ell)\longrightarrow\mathcal{E}(\Omega',E_{\ell+1});\quad T_\ell\phi(z) = \int_{\zeta,\omega}\widetilde{H}\widetilde{U}\Phi^z\wedge g,\quad z\in\Omega'.$$

It follows from Lemma 3.1 that $T_\ell\phi$ is smooth in Ω'. Moreover, if $\ell\geqslant 1$ we have from (2.15) that

$$\phi = f_{\ell+1}T_\ell\phi + T_{\ell-1}(f_\ell\phi) \tag{3.4}$$

in Ω'. Now Theorem 1.1 (i) follows if $\ell = 1$.

Remark 3.2. — If we instead use (2.14) for $\ell = 0$ we can obtain a proof of the following residue condition for membership in an ideal of germs of smooth functions in $\mathcal{E}a$, where $a \subset \mathcal{O}$.

THEOREM 3.3. — *Assume that $a \subset \mathcal{O}$ is an ideal. If R is the residue current associated with the resolution* (2.10) *of $\mathcal{O}/\mathcal{J}$, then a germ of a smooth function ϕ is in $\mathcal{E}a$ if and only if*

$$(\partial_{\bar z}^{\alpha}\phi)R = 0 \tag{3.5}$$

for each multiindex $\alpha \geqslant 0$.

This result was proved in [2] in case a is a complete intersection, and for a general ideal, in fact for any submodule $a \subset \mathcal{O}^{\oplus m}$, in [8].

3.2. Proof of Theorem 1.1 (ii)

Let us first consider a simple example with lower regularity.

Example 3.4. — Consider the sequence (1.3) and assume that $\phi = (\phi_1, \phi_2)$ is in C^1 and that $f_1\phi = 0$. This means that

$$x_1\phi_1 + x_2\phi_2 = 0 \tag{3.6}$$

and thus $\phi_2 = 0$ when $x_1 = 0$. Since ϕ_1 is C^1 therefore $\phi_2 = x_1\psi$, where ψ is continuous. In the same way $\phi_1 = -x_2\widetilde{\psi}$. From (3.6) it follows that $\widetilde{\psi} = \psi$ and hence ψ is a continuous solution to $f_2\psi = \phi$.

We will use the same set-up and notation as in the proof above of part (i) except for the extension $\widetilde{\phi}$ of ϕ. Let c_n be a positive integer. If ϕ is in $C^{c_n+2M+k}(\Omega, E_\ell)$, then

$$\widetilde{\phi}(\zeta,\omega) := \sum_{|\alpha|\leqslant c_n+M+k} (\partial_{\bar\zeta}^{\alpha}\phi)(\zeta)\frac{(\omega-\bar\zeta)^{\alpha}}{\alpha!} \tag{3.7}$$

is in $C^M(\widetilde{\Omega}, E_\ell)$, $\widetilde{\phi}(\zeta,\bar\zeta) = \phi(\zeta)$, and

$$\bar\partial\widetilde{\phi}(\zeta,\omega) = \mathcal{O}\big(|\omega-\bar\zeta|^{c_n+M+k}\big). \tag{3.8}$$

We have the following analogue to Lemma 3.1.

LEMMA 3.5. — *There is a constant c_n, only depending on the dimension n, such that if ϕ is in $C^{c_n+2M+k}(\Omega, E_\ell)$ and*

$$\Phi^z = \widetilde{\phi}(\zeta,\omega) - \bar\partial\widetilde{\phi}\wedge v^z,$$

then Φ^z is in $C^M(\widetilde{\Omega}, E_\ell)$ even after taking up to k derivatives with respect to z, and $\nabla_{(\zeta,\omega)-(z,\bar z)}\Phi^z = 0$. Moreover, if $f_\ell\phi = 0$, then $f_\ell\Phi^z = 0$.

Proof. — Notice that

$$\Phi^z(\zeta,\omega) = \widetilde{\phi}(\zeta,\omega) + \sum_{\ell=1}^{2n} \frac{\mathcal{O}(|\omega-\bar\zeta|^{c_n+M+k})}{(|\zeta-z|^2+|\omega-\bar z|^2)^{\ell-1/2}}.$$

If c_n is suitably chosen, then we can take k derivatives with respect to z and still remain in $C^M(\widetilde{\Omega}, E_\ell)$. As before it follows that $\nabla_{(\zeta,\omega)-(z,\bar z)}\Phi^z = 0$. If $f_\ell\phi(\zeta) = 0$, then $f_\ell(\partial^\alpha_{\bar\zeta}\phi)(\zeta) = 0$ for all α such that $|\alpha| \leqslant M + c_n + k$, and therefore $f_\ell\widetilde{\phi}(\zeta,\omega) = 0$. It follows that also $f_\ell(\bar\partial\widetilde{\phi})(\zeta,\omega) = 0$. □

We can now conclude the proof of Theorem 1.1 (ii). Notice that Proposition 2.1 holds if g has order M (and is smooth at z) and Φ is at least in C^M. Our weight g leading to (2.14) contains the current R, that may have order $M+1$, and to avoid keeping track of relevant components, although it is not necessary, it is convenient to replace c_n by $c_n + 2$. We can then proceed precisely as in the smooth case and obtain integral operators

$$T_\ell \colon C^{c_n+2M+k}(\Omega, E_\ell) \longrightarrow C^k(\Omega', E_{\ell-1})$$

such that (3.4) holds in Ω' if $\ell \geqslant 1$. Now part (ii) of Theorem 1.1 follows if $\ell = 1$.

4. Proof of Theorem 1.2

Assume that we have a generically exact Hermitian complex (2.6) in X with associated currents U and R. If $\pi\colon X' \to X$ is a modification, then the pullback of (2.6) is a generically exact Hermitian complex in X' and thus we have associated currents U' and R'. It follows from the definition, cf. [8], that $\pi_*U' = U$ and $\pi_*R' = R$. In particular, if we have a Koszul complex on X generated by the section a of A^* as in Example 2.3, then the pull-back is a Koszul complex in X' generated by the section π^*a of π^*A^*. It follows from the example that if $\pi^*a = a_0a'$, where a' is non-vanishing, then

$$R^0 = \pi_*\big(\frac{1}{a_0^\mu}\wedge\omega\big),$$

where ω is a smooth form in X'. In the same way if $a \subset \mathcal{O}$ is an ideal in X and we take the Koszul-type complex in Example 2.4 associated with a^r, then there is a smooth form on X' such that the associated current has the form

$$R^0 = \pi_*\big(\frac{1}{a_0^{\mu+r-1}}\wedge\omega\big). \tag{4.1}$$

Let us now assume that $X = \Omega$ is a neighborhood of $0 \in \mathbb{C}^n$ and let ϕ be a smooth function in Ω. If we proceed precisely as in the proof of

Theorem 1.1 (i) above but with the currents U and R from Example 2.4, we get from (2.15) for $\ell = 0$ the formula

$$\phi = f_1 T_1 \phi + S\phi, \tag{4.2}$$

where

$$S\phi(z) = \int \widetilde{H}\widetilde{R}^0\Phi^z \wedge g. \tag{4.3}$$

In view of (3.1) we get from (4.2) and (4.3), cf. [2, Theorem 1.1]:

LEMMA 4.1. — *If*

$$(\partial_{\bar{z}}^{\alpha}\phi)R^0 = 0 \tag{4.4}$$

for all $\alpha \geqslant 0$, *then* $\widetilde{R}^0\Phi^z = 0$ *and* ϕ *is in* $\mathcal{E}a^r$.

Proof of Theorem 1.2. — In view of Lemma 4.1 we have to prove that (1.6) implies (4.4). Let as choose a modification $\pi\colon \Omega' \to \Omega$ as above. For simplicity we skip the upper index 0 and write (4.1) as $R = \pi_* R'$, where

$$R' = \bar{\partial}\frac{1}{a_0^{\mu+r-1}} \wedge \omega, \tag{4.5}$$

ω is a smooth form with values in $L^{\mu+r-1}$, and L is the line bundle defined by a_0.

Let $\xi = \partial_{\bar{z}}^{\alpha}\phi$ for some α. We must prove that $\xi R = 0$. Since ξ is smooth,

$$\xi R = \pi_*(\pi^*\xi \cdot R').$$

It is thus enough to verify that $\pi^*\xi \cdot R' = 0$.

Since $|a'| > 0$, the assumption (1.6) implies that

$$|\pi^*\xi| \leqslant C|a_0|^{\mu+r-1} \tag{4.6}$$

for some constant C.

In a neighborhood $\mathcal{V}$ of a regular point on the zero set of a_0 we can assume that we have local coordinates $s_1, \dots, s_n$ such that $s_1^t = a_0$ for some positive integer t, or more correctly, s_1^t is the representative of a_0 with respect to a local frame for L. In any case, in view of (4.6), a Taylor expansion gives that $\pi^*\xi$ is a finite sum of terms $s_1^{t(\mu+r-1)-j}\bar{s}_1^j\nu$, where ν is smooth. It is well-known that each such term annihilates the residue current $\bar{\partial}(1/s_1^{t(\mu+r-1)})$. In fact, already one single factor $\bar{s}_1$ is enough. The "worst" case is thus $s_1^{t(\mu+r-1)}\nu$, and this factor precisely annihilates $\bar{\partial}(1/s_1^{t(\mu+r-1)})$. It follows that $\mu = \pi^*\xi\bar{\partial}(1/a_0^{\mu+r-1})$ has support on a set with codimension $\geqslant 2$. Since μ is pseudomeromorphic and has bidegree $(0,1)$ it must therefore vanish identically in view of the dimension principle, see, e.g., [7].

Alternatively we can assume that the modification is chosen so that a_0 locally is a monomial (normal crossings), say $a_0 = s_1^{t_1}\dots s_k^{t_k}$. Then (4.6)

and a Taylor expansion reveals that $\pi^*\xi$ is a sum of terms with a factor $\mathcal{O}(|s_1|^{t_1(\mu+r-1)}\dots|s_k|^{t_k(\mu+r-1)})$, and each such term annihilates $\bar\partial(1/s_1^{t_1(\mu+r-1)}\dots s_k^{t_k(\mu+r-1)})$. □

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