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LOW-LYING ZEROS IN FAMILIES OF HOLOMORPHIC CUSP FORMS: THE WEIGHT ASPECT

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Abstract

We study low-lying zeros of L-functions attached to holomorphic cusp forms of level 1 and large even weight. In this family, the Katz–Sarnak heuristic with orthogonal symmetry type was established in the work of Iwaniec, Luo and Sarnak for test functions ϕ satisfying the condition $\mathrm{supp}(\widehat{\phi}) \subset (-2, 2)$. We refine their density result by uncovering lower-order terms that exhibit a sharp transition when the support of $\widehat{\phi}$ reaches the point 1. In particular, the first of these terms involves the quantity $\widehat{\phi}(1)$ which appeared in the previous work of Fouvry–Iwaniec and Rudnick in symplectic families. Our approach involves a careful analysis of the Petersson formula and circumvents the assumption of the Generalized Riemann Hypothesis (GRH) for higher-degree automorphic L-functions. Finally, when $\mathrm{supp}(\widehat{\phi}) \subset (-1, 1)$ we obtain an unconditional estimate which is significantly more precise than the prediction of the L-functions ratios conjecture.

1. Introduction

Katz and Sarnak [13] conjectured that the distribution of low-lying zeros in a family $\mathcal{F}$ of L-functions is governed by a certain random matrix model $G(\mathcal{F})$ called the symmetry type of $\mathcal{F}$. This symmetry type has been determined in many families; see for example [3, 8, 12, 16, 21, 29], as well as the references in [25]. Sarnak et al. [25] recently refined the Katz–Sarnak heuristics and introduced invariants which allow for a conjectural determination of the symmetry type.

In the current paper we focus on the family of classical holomorphic cusp forms of level 1 and large even weight k. As in [12, Chapter 10], this will ease the exposition and allow for a more transparent analysis. For this family, the predictions of Katz and Sarnak were confirmed in the influential work

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of Iwaniec et al. [12] for a certain class of test functions, under the assumption of the Riemann Hypothesis for Dirichlet L-functions and higher-degree automorphic L-functions. Our main goal is to refine the Iwaniec–Luo–Sarnak density result by determining lower-order terms up to an arbitrary negative power of $\log k$.

More precisely, we fix a basis B_k of Hecke eigenforms in the space H_k of holomorphic cusp forms of level 1 and even weight k. We normalize so that for every

$$f(z) = \sum_{n=1}^{\infty} a_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z} \in B_k,$$

the first coefficient satisfies $a_f(1) = 1$. Hence, the Hecke eigenvalues of f are given by $\lambda_f(n) = a_f(n)$ and for $\Re(s) > 1$ the L-function of f takes the form

$$L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}.$$

This classically extends to an entire function and satisfies a functional equation relating the value at s to that at $1-s$. In the sums over $f \in B_k$ to be considered in this paper, we will scale each term with the harmonic weight

$$\omega_f := \frac{\Gamma(k-1)}{(4\pi)^{k-1}(f,f)},$$

where

$$(f,f) := \int_{SL_2(\mathbb{Z})\backslash\mathbb{H}} y^{k-2} |f(z)|^2 dx dy.$$

Note that $k^{-1-\varepsilon} \ll_\varepsilon \omega_f \ll_\varepsilon k^{-1+\varepsilon}$ (see [12, Lemma 2.5], [7, p. 164] and [9, Theorem 2]), and moreover

$$\Omega_k := \sum_{f \in B_k} \omega_f = 1 + O(2^{-k}) \tag{1.1}$$

(take $m = n = 1$ in Proposition 3.3). The use of these essentially constant weights is standard (see for instance [12, Chapter 10], [17]). Note that the situation can be drastically different with arithmetic weights as in [14].

For an even Schwartz test function ϕ, we define the 1-level density

$$\mathcal{D}_k(\phi; X) := \frac{1}{\Omega_k} \sum_{f \in B_k} \omega_f \sum_{\gamma_f} \phi\Big(\gamma_f \frac{\log X}{2\pi}\Big),$$

where $\gamma_f = -i(\rho_f - \frac{1}{2})$, with ρ_f running through the non-trivial zeros of $L(s,f)$ (the Riemann Hypothesis for $L(s,f)$ states that $\gamma_f \in \mathbb{R}$). Here, X is a parameter which will later be chosen to be approximately equal to the average conductor of the L-functions $L(s,f)$ in the relevant family of cusp forms f (note that the conductor of $L(s,f)$ for $f\in B_k$ is equal to k^2). This standard choice ensures that the normalized zeros $\gamma_f\frac{\log X}{2\pi}$ have mean spacing asymptotically equal to 1. Moreover, for h a non-negative and not identically zero smooth weight function with compact support in $\mathbb{R}_{>0}$, we define the following averages of the 1-level densities over families of constant sign of the functional equation: (Note that for any $f\in H_k$, the sign of the functional equation of $L(s,f)$ is given by $(-1)^{\frac{k}{2}}$.)

$$\mathscr{D}^{\pm}_{K,h}(\phi):=\frac{1}{H^{\pm}(K)}\sum_{k\equiv 3\pm 1 \bmod 4} h\Big(\frac{k-1}{K}\Big)\mathcal{D}_k(\phi;K^2),$$

where $H^{\pm}(K):=\sum_{k\equiv 3\pm 1 \bmod 4} h\big(\frac{k-1}{K}\big)$. The Katz–Sarnak prediction for this family (see [**12**], [**25**, Conjecture 2 and Section 2.7]) states that

$$\lim_{K\to\infty}\mathscr{D}^{\pm}_{K,h}(\phi)=\int_{\mathbb{R}}\widehat{\phi}\cdot\widehat{W}^{\pm}, \tag{1.2}$$

with

$$\widehat{W}^{+}(t)=\widehat{W}(SO(\text{even}))(t)=\delta_0(t)+\frac{\eta(t)}{2};\qquad \widehat{W}^{-}(t)=\widehat{W}(SO(\text{odd}))(t)=\delta_0(t)-\frac{\eta(t)}{2}+1,$$

where δ_0 is the Dirac distribution, $\eta(t)=1$ for $|t|<1$, $\eta(\pm1)=\frac12$ and $\eta(t)=0$ for $|t|>1$, and $\widehat{\phi}(\xi):=\int_{\mathbb{R}}\phi(x)e^{-2\pi i\xi x}dx$. Under the Riemann Hypothesis for Dirichlet and symmetric square L-functions, the estimate (1.2) was confirmed in [**12**, Theorem 1.3] under the condition $\operatorname{supp}(\widehat{\phi})\subset(-2,2)$. Note also that this work has been extended to families of more general automorphic L-functions in [**26**].

We now state our main theorem which, in the case when the level $N=1$, refines the estimate in [**12**, Theorem 1.3] by weakening its assumptions and obtaining lower-order terms which contain a phase transition as the support of $\widehat{\phi}$ reaches 1.

THEOREM 1.1 *Let ϕ be an even Schwartz test function for which supp$(\widehat{\phi})\subset(-2,2)$. Assuming the Riemann Hypothesis for Dirichlet L-functions, we have the estimate*

$$\mathscr{D}^{\pm}_{K,h}(\phi)=\int_{\mathbb{R}}\widehat{\phi}\cdot\widehat{W}^{\pm}+\sum_{1\le j\le J}\frac{R_{j,h}\widehat{\phi}^{(j-1)}(0)\pm S_{j,h}\widehat{\phi}^{(j-1)}(1)}{(\log K)^j}+O_{\phi,h,J}\Big(\frac{1}{(\log K)^{J+1}}\Big), \tag{1.3}$$

where the constants $R_{j,h}$ and $S_{j,h}$ appearing in the lower-order terms only depend on the weight function h (see 6.9, 6.10 and 6.11).

We deduce Theorem 1.1 from a power-saving formula for the 1-level density (see 6.1), which we combine with an asymptotic evaluation of the resulting terms (see Theorem 6.6). In [**12**], Iwaniec, Luo and Sarnak obtain the main term in Theorem 1.1 assuming the Riemann Hypothesis both for Dirichlet L-functions and symmetric square L-functions. The first is to evaluate a term appearing when splitting signs, and the second allows them to bound the contribution from the terms involving the coefficients $\lambda_f(p^2)$. However, in [**12**] they claim that the symmetric square Riemann Hypothesis can be removed using the Petersson formula. Applying [12, Corollary 2.2] shows that this is possible whenever the support of $\widehat{\phi}$ is contained in $(-\frac{5}{3},\frac{5}{3})$. In this paper, we refine [12, Corollary 2.2] (see Proposition 3.3) in order to achieve the same result with the extended support interval $(-2,2)$. Our improvement ultimately boils down to a more precise decomposition of the involved ranges and a careful application of bounds on Bessel functions. The specific estimate that we obtain is the following:

$$\sum_{f\in B_k}\omega_f\lambda_f(m)\lambda_f(n)=\delta(m,n)+O_\varepsilon\left(\frac{(m,n)^{\frac{1}{2}}(mn)^{\frac{1}{4}+\varepsilon}}{k}+\frac{k^{\frac{1}{6}}(m,n)^{\frac{1}{2}}}{(mn)^{\frac{1}{4}-\varepsilon}}\right) \tag{1.4}$$

(note that in the range $mn\le k^2/(4\pi e)^2$ the above error term can be replaced by a term with exponential decay in k). In particular, when $(m,n)=1$ this estimate is non-trivial in the range $k^{\frac{2}{3}+\varepsilon}\ll mn\ll k^{4-\varepsilon}$, whereas [12, Corollary 2.2] is nontrivial up to $mn\ll k^{\frac{10}{3}-\varepsilon}$.

The terms involving $\widehat{\phi}^{(j)}(1)$ in (1.3) are responsible for a sharp transition at 1 in these orthogonal families and are analogous to those obtained in symplectic families in [3, 5, 6, 24, 28]. Indeed, in the family of real Dirichlet characters considered in [5], after applying the explicit formula and treating the resulting sums over primes by repeatedly using the Poisson summation formula, one obtains lower-order terms involving $\widehat{\phi}^{(j)}(1)$. This work was inspired by the function field case considered in [24], in which, using Poisson summation, the 1-level density is turned into an average of the trace of the Frobenius class in the hyperelliptic ensemble, from which a transition term is isolated using the explicit formula. Transition terms also surface in predictions coming from the L-function ratios conjecture [6, 18, 28]; in this case one needs to compute averages of ratios of local factors at infinity. In the current situation, these terms come from a significantly different source, namely from a careful analysis of averages of Bessel functions and Kloosterman sums coming from the Petersson trace formula. In the related situation of families of holomorphic cusp forms in the level aspect, the first transition term was previously isolated using an integral identity for the Bessel function [18]. Independent of the use of different methods, this seems to indicate that a transition in lower-order terms should exist whenever the symmetry type of a family is even or odd orthogonal or symplectic.

Interestingly, averaging over all even values of the weight k, we find that

$$\frac{1}{H(K)}\sum_{k\equiv 0 \bmod 2}h\Big(\frac{k-1}{K}\Big)\mathcal{D}_k(\phi;K^2)=\int_{\mathbb{R}}\widehat{\phi}\cdot\widehat{W}+\sum_{1\le j\le J}\frac{R_{j,h}\widehat{\phi}^{(j-1)}(0)}{(\log K)^j}+O_{\phi,h,J}\Big(\frac{1}{(\log K)^{J+1}}\Big),$$

where $\widehat{W}=\widehat{W}(O):=\frac{1}{2}+\delta_0$ and $H(K):=H^+(K)+H^-(K)$. Hence, as expected we see that there is no transition at 1 in this mixed signs family (see also [17, Theorem 1.6]). We should point out that for similar reasons, there is no transition in mixed sign families of holomorphic cusp forms of fixed weight and of large level [17, 23].

Remark 1.2 One can explicitly compute the constants $R_{j,h}$ and $S_{j,h}$ in Theorem 1.1. In particular, the first of these are given by

$$R_{1,h}=S_{1,h}=-\gamma+\frac{\int_0^\infty h\cdot\log}{\int_0^\infty h}-\log(4\pi)-\sum_p\frac{\log p}{p(p-1)}.$$

Constants similar to $R_{1,h}$ have appeared previously in the literature on low-lying zeros; cf., for example, [23] and the references therein.

We now state our results for test functions whose Fourier transform is supported in the interval $(-1,1)$. Under this restriction our estimates are substantially more precise. Indeed, we do not need the GRH assumption, the error term is exponentially small in the weight k, and we do not need the average over k (we set $X=k^2$ in $\mathcal{D}_k(\phi;X)$).

THEOREM 1.3 *Let ϕ be an even Schwartz test function for which supp$(\widehat{\phi})\subset(-1,1)$. Then the (unaveraged) 1-level density satisfies the estimate*

$$\mathcal{D}_k(\phi;k^2)=\frac{1}{\log(k^2)}\int_{\mathbb{R}}\left(\frac{\Gamma'}{\Gamma}\Big(\frac14+\frac{k+1}{4}+\frac{\pi it}{\log(k^2)}\Big)+\frac{\Gamma'}{\Gamma}\Big(\frac14+\frac{k-1}{4}+\frac{\pi it}{\log(k^2)}\Big)\right)\phi(t)dt \quad (1.5)$$
$$+2\sum_p\frac1p\widehat{\phi}\Big(\frac{2\log p}{\log(k^2)}\Big)\frac{\log p}{\log(k^2)}-\widehat{\phi}(0)\frac{\log\pi}{\log k}+O\big(k^{\frac32}2^{-k}\big).$$

REMARK 1.4

(1) We emphasize that Theorem 1.3 is unconditional. Moreover, the error term in (1.5) is exponentially small, in particular this is significantly more precise than predictions from the ratios conjecture [1, 2, 17]. This comes from exponential bounds on the Bessel functions occurring in the Petersson trace formula (see 3.2).
(2) The Katz–Sarnak main term in this case is given by $\widehat{\phi}(0)+\frac{\phi(0)}{2}$. One can extract this term from (1.5) by applying Lemmas 2.2 and 4.1.
(3) The first estimate for $\mathcal{D}_k(\phi;k^2)$ was obtained by Iwaniec, Luo and Sarnak [12, Theorem 1.2], who showed that the Katz–Sarnak prediction holds in this family under the condition supp$(\widehat{\phi})\subset(-1,1)$. Their estimate was refined by Miller [17, Lemmas 4.2 and 4.4], who obtained a formula with the error term $O_\varepsilon(k^{\frac{\sigma}{2}-\frac56+\varepsilon})$, under the same condition.

The paper is divided as follows. In Sections 2 and 3 we discuss prerequisites, establish (1.4) and discard higher prime powers in the explicit formula. Section 4 is dedicated to the proof of Theorem 1.3. Finally, in Section 5 we apply estimates on averages of Bessel functions to isolate a transition term, which we carefully evaluate in Section 6.

2. Explicit formula

We begin by recalling the explicit formula for holomorphic cusp form L-functions in the case where the level equals 1.

LEMMA 2.1 *Let ϕ be an even Schwartz test function whose Fourier transform has compact support. We have the formula*

$$\begin{aligned}\mathcal{D}_k(\phi;X) = -2\widehat{\phi}(0)\frac{\log\pi}{\log X} + \frac{1}{\log X}\int_{\mathbb{R}}\Big(\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{k+1}{4}+\frac{\pi it}{\log X}\Big)\\ + \frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{k-1}{4}+\frac{\pi it}{\log X}\Big)\Big)\phi(t)dt \\ - \frac{2}{\Omega_k}\sum_{f\in B_k}\omega_f\sum_{p,\nu}\frac{\alpha_f^\nu(p)+\beta_f^\nu(p)}{p^{\frac{\nu}{2}}}\widehat{\phi}\Big(\frac{\nu\log p}{\log X}\Big)\frac{\log p}{\log X}.\end{aligned} \tag{2.1}$$

Here, $\alpha_f(p)$ and $\beta_f(p)$ are the local coefficients of the L-function

$$L(s,f) = \prod_p\Big(1-\frac{\alpha_f(p)}{p^s}\Big)^{-1}\Big(1-\frac{\beta_f(p)}{p^s}\Big)^{-1} \qquad (\Re(s)>1);$$

in particular we have that $|\alpha_f(p)| = |\beta_f(p)| = 1$.

Proof. For $f\in B_k$, the formula [**12**, (4.11) with a typo corrected] reads

$$\begin{aligned}\sum_{\gamma_f}\phi\Big(\gamma_f\frac{\log X}{2\pi}\Big) = \frac{1}{\log X}\int_{\mathbb{R}}\Big(\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{k+1}{4}+\frac{\pi it}{\log X}\Big)+\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{k-1}{4}+\frac{\pi it}{\log X}\Big)\Big)\phi(t)dt\\ -2\widehat{\phi}(0)\frac{\log\pi}{\log X} - 2\sum_{p,\nu}\frac{\alpha_f^\nu(p)+\beta_f^\nu(p)}{p^{\frac{\nu}{2}}}\widehat{\phi}\Big(\frac{\nu\log p}{\log X}\Big)\frac{\log p}{\log X}.\end{aligned}$$

Summing over $f\in B_k$ against the weight ω_f we obtain the desired formula. □

We now estimate the integral involving the logarithmic derivative of the gamma function in (2.1).

LEMMA 2.2 *Let $\varepsilon>0$ and let ϕ be an even Schwartz test function. Then we have the estimate*

$$\begin{aligned}\frac{1}{\log X}\int_{\mathbb{R}}\Big(\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{k+1}{4}+\frac{\pi it}{\log X}\Big)+\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{k-1}{4}+\frac{\pi it}{\log X}\Big)\Big)\phi(t)dt = \widehat{\phi}(0)\Big(\frac{\log(k^2)-\log 16}{\log X}\Big)\\ + O_\varepsilon(k^{-1+\varepsilon}).\end{aligned}$$

Proof. Using more terms in the Stirling approximation, the estimate in Lemma 2.2 can be refined to an asymptotic series in descending powers of k. Applying Stirling's formula

$$\frac{\Gamma'}{\Gamma}(z) = \log z + O(|z|^{-1})$$

in the region $\Re(z) > 0$, we see that

$$\begin{aligned}
&\frac{1}{\log X}\int_{\mathbb{R}}\left(\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{k+1}{4}+\frac{\pi it}{\log X}\Big)+\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{k-1}{4}+\frac{\pi it}{\log X}\Big)\right)\phi(t)dt\\
&=\frac{1}{\log X}\int_{\mathbb{R}}\left(\log\Big(\frac{1}{4}+\frac{k+1}{4}+\frac{\pi it}{\log X}\Big)+\log\Big(\frac{1}{4}+\frac{k-1}{4}+\frac{\pi it}{\log X}\Big)\right)\phi(t)dt+O(k^{-1})\\
&=\frac{1}{\log X}\int_{-k^{\varepsilon}}^{k^{\varepsilon}}\left(\log\Big(\frac{1}{4}+\frac{k+1}{4}+\frac{\pi it}{\log X}\Big)+\log\Big(\frac{1}{4}+\frac{k-1}{4}+\frac{\pi it}{\log X}\Big)\right)\phi(t)dt+O_{\varepsilon}(k^{-1})\\
&=\frac{1}{\log X}\int_{-k^{\varepsilon}}^{k^{\varepsilon}}\log\Big(\frac{k^2}{16}\Big)\phi(t)dt+O_{\varepsilon}(k^{-1+\varepsilon}).
\end{aligned}$$

The result follows from extending the integral to $\mathbb{R}$. □

3. The Petersson trace formula and related estimates

In order to handle the term involving sums over prime powers in (2.1), we will apply the Petersson trace formula. For $m,n \in \mathbb{Z}$ and $c \in \mathbb{N}$, we define the Kloosterman sum

$$S(m,n;c) := \sum_{\substack{x \bmod c\\ (x,c)=1}} e\Big(\frac{mx+n\bar{x}}{c}\Big),$$

where $\bar{x}$ denotes the multiplicative inverse of x modulo c. We will repeatedly use the classical Weil bound (see for instance [11, Corollary 11.12])

$$|S(m,n;c)| \le \tau(c)(m,n,c)^{\frac{1}{2}}c^{\frac{1}{2}}.$$

LEMMA 3.1 *Let $m,n,k \in \mathbb{N}$, with $2 \mid k$. We have the exact formula*

$$\sum_{f\in B_k}\omega_f\lambda_f(m)\lambda_f(n) = \delta(m,n) + 2\pi i^k\sum_{c\ge 1}c^{-1}S(m,n,c)J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big), \tag{3.1}$$

where J_{k-1} is the Bessel function of order $k-1$.

Proof. See [22] or [11, Proposition 14.5]. □

We recall the following bound on the Bessel function.

LEMMA 3.2 *Let $k \in \mathbb{N}$. We have the bound*

$$J_{k-1}(x) \ll \min\Big(\frac{1}{(k-1)!}\Big(\frac{x}{2}\Big)^{k-1}, x^{-\frac{1}{4}}(|x-k+1|+k^{\frac{1}{3}})^{-\frac{1}{4}}\Big).$$

Proof. See [12, (2.11') and (2.11")], which for the range $x \geq k^2$ follows from [27], specifically equations (1) p.49, (2) p.77, (6) p.78, (1) and (3) p.199, (1) p.202, (4) p.250, (5) p.252, and for the remaining range follows from [15, Theorem 2]. □

In [12, Corollary 2.2], this bound is shown to imply the estimate

$$\sum_{f \in B_k} \omega_f \lambda_f(m)\lambda_f(n) = \delta(m,n) + O\big(k^{-\frac{5}{6}}(mn)^{\frac{1}{4}}\tau_3((m,n))\log(2mn)\big),$$

which is non-trivial in the range $mn \ll k^{\frac{10}{3}-\varepsilon}$. By a more careful decomposition of the sum over c in (3.1), we establish a more precise estimate which for coprime m and n is non-trivial in the wider range $mn \ll k^{4-\varepsilon}$.

PROPOSITION 3.3 *Let $\varepsilon > 0$, and let $m,n,k \in \mathbb{N}$, with $2 \mid k$. We have the estimate*

$$\sum_{f \in B_k} \omega_f \lambda_f(m)\lambda_f(n) = \delta(m,n) + O_\varepsilon\Big(\frac{(m,n)^{\frac{1}{2}}(mn)^{\frac{1}{4}+\varepsilon}}{k} + \frac{k^{\frac{1}{6}}(m,n)^{\frac{1}{2}}}{(mn)^{\frac{1}{4}-\varepsilon}}\Big).$$

Moreover, in the range $mn \leq k^2/(4\pi e)^2$, we have the exponentially precise estimate

$$\sum_{f \in B_k} \omega_f \lambda_f(m)\lambda_f(n) = \delta(m,n) + O\Big(2^{-k}(mn)^{\frac{1}{4}}\log(2mn)\prod_{p\mid(m,n)}\Big(1+\frac{3}{\sqrt{p}}\Big)\Big). \tag{3.2}$$

Proof. We bound the rightmost term in the statement of Lemma 3.1, by combining the Weil bound with Lemma 3.2, as follows:

$$\begin{aligned}
\sum_{c\geq 1} c^{-1}S(m,n,c)J_{k-1}\Big(\frac{4\pi\sqrt{mn}}{c}\Big) \ll & \sum_{c \leq \frac{4\pi\sqrt{mn}}{k-1}-\frac{4\pi\sqrt{mn}}{k^{5/3}}} c^{-\frac{1}{4}}\tau(c)\frac{(m,n,c)^{\frac{1}{2}}}{(mn)^{\frac{1}{8}}}\Big|\frac{4\pi\sqrt{mn}}{c}-k+1\Big|^{-\frac{1}{4}} \\
&+ \sum_{\frac{4\pi\sqrt{mn}}{k-1}-\frac{4\pi\sqrt{mn}}{k^{5/3}} < c < \frac{4\pi\sqrt{mn}}{k-1}+\frac{4\pi\sqrt{mn}}{k^{5/3}}} c^{-\frac{1}{4}}\tau(c)\frac{(m,n,c)^{\frac{1}{2}}}{(mn)^{\frac{1}{8}}}k^{-\frac{1}{12}} \\
&+ \sum_{\frac{4\pi\sqrt{mn}}{k-1}+\frac{4\pi\sqrt{mn}}{k^{5/3}} \leq c < \frac{4e\pi\sqrt{mn}}{k}} c^{-\frac{1}{4}}\tau(c)\frac{(m,n,c)^{\frac{1}{2}}}{(mn)^{\frac{1}{8}}}\Big|\frac{4\pi\sqrt{mn}}{c}-k+1\Big|^{-\frac{1}{4}} \\
&+ \sum_{c \geq \frac{4e\pi\sqrt{mn}}{k}} (m,n,c)^{\frac{1}{2}}\tau(c)c^{-\frac{1}{2}}\frac{1}{(k-1)!}\Big(\frac{2\pi\sqrt{mn}}{c}\Big)^{k-1} = S_1+S_2+S_3+S_4.
\end{aligned}$$

We first bound S_4. To do so, note that

$$\begin{aligned} S_4 &= \frac{1}{(k-1)!}(2\pi\sqrt{mn})^{k-1}\sum_{d|(m,n)} d^{\frac12} \sum_{\substack{c\geq \frac{4e\pi\sqrt{mn}}{k}\\ (c,(m,n))=d}} \tau(c)c^{-k+\frac12} \\ &\leq \frac{1}{(k-1)!}(2\pi\sqrt{mn})^{k-1}\sum_{d|(m,n)} \tau(d)d^{-k+1} \sum_{\substack{f\geq \frac{4e\pi\sqrt{mn}}{dk}\\ (f,(m,n)/d)=1}} \tau(f)f^{-k+\frac12} \\ &\ll \frac{1}{(k-1)!}(2\pi\sqrt{mn})^{k-1}\Big(\frac{k}{4e\pi\sqrt{mn}}\Big)^{k-\frac32}\log(2mn)\sum_{d|(m,n)}\frac{\tau(d)}{d^{\frac12}} \\ &\ll 2^{-k}(mn)^{\frac14}\log(2mn)\prod_{p|(m,n)}\Big(1+\frac{3}{\sqrt{p}}\Big), \end{aligned}$$

by the bound $\sum_{\ell\geq 1}\tau(p^\ell)p^{-\frac{\ell}{2}}\leq 3/\sqrt{p}$ which holds for large enough p as well as Stirling's approximation in the form $(k-1)!\sim\sqrt{2\pi(k-1)}(\frac{k-1}{e})^{k-1}$. Note also that S_1, S_2 and S_3 are all empty whenever $mn\leq k^2/(4\pi e)^2$ and hence (3.2) follows.

We now assume that $mn > k^2/(4\pi e)^2$. A straightforward computation shows that

$$\begin{aligned} S_2 &\ll_\varepsilon k^{-\frac{1}{12}}(m,n)^{\frac12}(mn)^{-\frac18+\varepsilon}\sum_{\frac{4\pi\sqrt{mn}}{k-1}-\frac{4\pi\sqrt{mn}}{k^{5/3}}<c<\frac{4\pi\sqrt{mn}}{k-1}+\frac{4\pi\sqrt{mn}}{k^{5/3}}} c^{-\frac14} \\ &\ll k^{-\frac32}(m,n)^{\frac12}(mn)^{\frac14+\varepsilon}+k^{\frac16}(m,n)^{\frac12}(mn)^{-\frac14+\varepsilon}, \end{aligned}$$

where the second term accounts for the possibility that the sum contains only one term. As for S_1, we compute that

$$\begin{aligned} S_1 &\ll \sum_{c\leq\frac{2\pi\sqrt{mn}}{k-1}}\tau(c)\frac{(m,n,c)^{\frac12}}{(mn)^{\frac14}} + \sum_{\frac{2\pi\sqrt{mn}}{k-1}<c\leq\frac{4\pi\sqrt{mn}}{k-1}-\frac{4\pi\sqrt{mn}}{k^{5/3}}}\tau(c)\frac{(m,n,c)^{\frac12}}{(mn)^{\frac18}}\Big|4\pi\sqrt{mn}-c(k-1)\Big|^{-\frac14} \\ &\ll_\varepsilon \frac{(mn)^{\frac14}\log(2mn)}{k}\prod_{p|(m,n)}\Big(1+\frac{3}{\sqrt{p}}\Big)+k^{\frac16}(m,n)^{\frac12}(mn)^{-\frac14+\varepsilon} \\ &\quad + \sum_{\frac{2\pi\sqrt{mn}}{k-1}<c\leq\lfloor\frac{4\pi\sqrt{mn}}{k-1}\rfloor-\frac{4\pi\sqrt{mn}}{k^{5/3}}}\tau(c)\frac{(m,n,c)^{\frac12}}{(mn)^{\frac18}}\Big|4\pi\sqrt{mn}-c(k-1)\Big|^{-\frac14}. \end{aligned}$$

Making the change of variables $b=\lfloor\frac{4\pi\sqrt{mn}}{k-1}\rfloor-c$, we see that the sum over c is

$$\ll_\varepsilon (m,n)^{\frac12}(mn)^{-\frac18+\varepsilon}\sum_{\frac{4\pi\sqrt{mn}}{k^{5/3}}\leq b<\frac{2\pi\sqrt{mn}}{k-1}}|b(k-1)|^{-\frac14}\ll k^{-1}(m,n)^{\frac12}(mn)^{\frac14+\varepsilon}.$$

In a similar way we see that $S_3 \ll_\varepsilon k^{-1}(m,n)^{\frac12}(mn)^{\frac14+\varepsilon} + k^{\frac16}(m,n)^{\frac12}(mn)^{-\frac14+\varepsilon}$, and the proof is finished. □

In the next lemma, we apply Proposition 3.3 in order to discard higher prime powers in the explicit formula (2.1).

LEMMA 3.4 *Assume that $k \in 2\mathbb{N}$, $X \in \mathbb{R}_{\geq 2}$ and the even Schwartz test function ϕ are such that $X^\sigma < k^4$, where $\sigma :=$sup(supp($\widehat{\phi}$)). Then we have the following estimate on the 1-level density:*

$$\mathcal{D}_k(\phi;X) = \widehat{\phi}(0)\Big(\frac{\log(k^2)-\log(16\pi^2)}{\log X}\Big) + 2\sum_p \frac1p \widehat{\phi}\Big(\frac{2\log p}{\log X}\Big)\frac{\log p}{\log X}$$
$$-2\sum_{f\in B_k}\omega_f \sum_p \frac{\lambda_f(p)}{p^{\frac12}}\widehat{\phi}\Big(\frac{\log p}{\log X}\Big)\frac{\log p}{\log X} + O_\varepsilon\Big(\frac{X^{\frac{\sigma}{4}+\varepsilon}}{k} + \frac{1}{k^{\frac13-\varepsilon}}\Big). \tag{3.3}$$

Assuming the stronger condition $X^\sigma < (k/4\pi e)^2$, we have the more precise estimate

$$\mathcal{D}_k(\phi;X) = \frac{1}{\log X}\int_{\mathbb{R}}\Big(\frac{\Gamma'}{\Gamma}\Big(\frac14+\frac{k+1}{4}+\frac{\pi i t}{\log X}\Big)+\frac{\Gamma'}{\Gamma}\Big(\frac14+\frac{k-1}{4}+\frac{\pi i t}{\log X}\Big)\Big)\phi(t)dt$$

$$-2\widehat{\phi}(0)\frac{\log \pi}{\log X} + 2\sum_p \frac1p\widehat{\phi}\Big(\frac{2\log p}{\log X}\Big)\frac{\log p}{\log X} - 2\sum_{f\in B_k}\omega_f\sum_p\frac{\lambda_f(p)}{p^{\frac12}}\widehat{\phi}\Big(\frac{\log p}{\log X}\Big)\frac{\log p}{\log X} + O_\varepsilon\Big(\frac{k^{\frac12+\varepsilon}}{2^k}\Big). \tag{3.4}$$

Proof. The goal of this proof is to estimate the terms with $p, \nu \geq 2$ in (2.1). By the Hecke relations, the sum of those terms is equal to

$$-\frac{2}{\Omega_k}\sum_{f\in B_k}\omega_f\sum_{p,\nu\geq 2}\frac{\lambda_f(p^\nu)-\lambda_f(p^{\nu-2})}{p^{\frac{\nu}{2}}}\widehat{\phi}\Big(\frac{\nu\log p}{\log X}\Big)\frac{\log p}{\log X}.$$

From Proposition 3.3 and (1.1), we see that

$$\begin{aligned}
&-\frac{2}{\Omega_k}\sum_{f\in B_k}\omega_f\sum_{p,\nu\geq 2}\frac{\lambda_f(p^\nu)}{p^{\frac{\nu}{2}}}\widehat{\phi}\Big(\frac{\nu\log p}{\log X}\Big)\frac{\log p}{\log X}\\
&\ll_\varepsilon 2^{-k}\sum_{\substack{p,\nu\geq 2\\ p\leq \min(X^{\sigma/\nu},(k/4\pi e)^{2/\nu})}} p^{-\frac{\nu}{4}+\varepsilon} + \sum_{\substack{p,\nu\geq 2\\ \min(X^{\sigma/\nu},(k/4\pi e)^{2/\nu})<p\leq X^{\sigma/\nu}}} p^{-\frac{\nu}{2}+\varepsilon}\big(k^{-1}p^{\frac{\nu}{4}}+k^{\frac16}p^{-\frac{\nu}{4}}\big)\\
&\ll_\varepsilon k^{\frac12+\varepsilon}2^{-k} + I_{[X^\sigma>(k/4\pi e)^2]}\cdot\big(k^{-1}X^{\frac{\sigma}{4}+\varepsilon}+k^{-\frac13+\varepsilon}\big),
\end{aligned}$$

where I_P is 1 if P is true and 0 otherwise. Similarly,

$$\frac{2}{\Omega_k}\sum_{f\in B_k}\omega_f\sum_{\substack{p\\ \nu\geq 3}}\frac{\lambda_f(p^{\nu-2})}{p^{\frac{\nu}{2}}}\widehat{\phi}\Big(\frac{\nu\log p}{\log X}\Big)\frac{\log p}{\log X}\ll_\varepsilon 2^{-k}+I_{[X^\sigma>(k/4\pi e)^2]}\cdot k^{-\frac{1}{3}+\varepsilon}.$$

The only terms left are

$$\frac{2}{\Omega_k}\sum_{f\in B_k}\omega_f\sum_{p}\frac{\lambda_f(1)}{p}\widehat{\phi}\Big(\frac{2\log p}{\log X}\Big)\frac{\log p}{\log X}=2\sum_{p}\frac{1}{p}\widehat{\phi}\Big(\frac{2\log p}{\log X}\Big)\frac{\log p}{\log X}.$$

We conclude the proof by applying Lemmas 2.1 and 2.2, and (1.1). □

4. 1-Level density: unconditional results

In this section we evaluate the 1-level density $\mathcal{D}_k(\phi;X)$, for test functions satisfying $\sup(\mathrm{supp}(\phi)) < 1$, unconditionally. We begin by asymptotically evaluating the second term on the right-hand side of (3.3).

LEMMA 4.1 *Let ϕ be an even Schwartz test function. For any fixed $J\geq 1$, we have the estimate*

$$2\sum_{p}\frac{1}{p}\widehat{\phi}\Big(2\frac{\log p}{\log X}\Big)\frac{\log p}{\log X}=\frac{\phi(0)}{2}+\sum_{1\leq j\leq J}\frac{c_j\widehat{\phi}^{(j-1)}(0)}{(\log X)^j}+O_J\Big(\frac{1}{(\log X)^{J+1}}\Big),$$

where

$$c_1:=2+2\int_1^\infty\frac{\theta(t)-t}{t^2}dt$$

and

$$c_j:=\frac{2^j}{(j-2)!}\int_1^\infty(\log t)^{j-2}\Big(\frac{\log t}{j-1}-1\Big)\frac{\theta(t)-t}{t^2}dt$$

for $j\geq 2$, with $\theta(t):=\sum_{p\leq t}\log p$.

Proof. Performing summation by parts, we reach the following identity:

$$\begin{aligned}\frac{2}{\log X}\sum_{p}\frac{\log p}{p}\widehat{\phi}\Big(2\frac{\log p}{\log X}\Big)&=\frac{2}{\log X}\int_1^\infty\frac{1}{t}\widehat{\phi}\Big(2\frac{\log t}{\log X}\Big)d\theta(t)\\&=\frac{\phi(0)}{2}+\frac{2\widehat{\phi}(0)}{\log X}-\frac{2}{\log X}\int_1^\infty\Bigg(2\frac{\widehat{\phi}'\big(2\frac{\log t}{\log X}\big)}{\log X}-\widehat{\phi}\Big(2\frac{\log t}{\log X}\Big)\Bigg)\frac{\theta(t)-t}{t^2}dt.\end{aligned}$$

By the prime number theorem in the form $\theta(t)-t \ll t\exp(-2c\sqrt{\log t})$, we see that for any $0<\xi<1$,

$$\int_{X^{\xi/2}}^{\infty}\left(2\frac{\widehat{\phi}'\left(2\frac{\log t}{\log X}\right)}{\log X}-\widehat{\phi}\left(2\frac{\log t}{\log X}\right)\right)\frac{\theta(t)-t}{t^2}dt \ll \exp\left(-c\sqrt{\xi\log X}\right).$$

Moreover, expanding into Taylor series and applying the prime number theorem, we see that

$$\begin{aligned}
&\frac{2\widehat{\phi}(0)}{\log X}-\frac{2}{\log X}\int_1^{X^{\frac{\xi}{2}}}\left(2\frac{\widehat{\phi}'\left(2\frac{\log t}{\log X}\right)}{\log X}-\widehat{\phi}\left(2\frac{\log t}{\log X}\right)\right)\frac{\theta(t)-t}{t^2}dt\\
&=\frac{2\widehat{\phi}(0)}{\log X}+2\sum_{0\le j\le J}\frac{1}{j!}\left(\widehat{\phi}^{(j)}(0)-2\frac{\widehat{\phi}^{(j+1)}(0)}{\log X}\right)\int_1^{X^{\frac{\xi}{2}}}\frac{(2\log t)^j}{(\log X)^{j+1}}\frac{\theta(t)-t}{t^2}dt\\
&\qquad+O_J\left(\int_1^{X^{\frac{\xi}{2}}}\frac{(2\log t)^{J+1}}{(\log X)^{J+2}}\frac{\theta(t)-t}{t^2}dt\right)\\
&=\sum_{1\le j\le J+1}\frac{c_j\widehat{\phi}^{(j-1)}(0)}{(\log X)^j}+O_J\left(\frac{1}{(\log X)^{J+2}}+\exp(-c\sqrt{\xi\log X})\right).
\end{aligned}$$

The result follows from selecting $\xi=(\log X)^{-1+\delta}$ for some $\delta>0$. □

We now set $X=k^2$ and prove Theorem 1.3.

Proof of Theorem 1.3. We apply Proposition 3.3 and obtain that the second prime sum in (3.4) satisfies the bound

$$2\sum_{f\in B_k}\omega_f\sum_p\frac{\lambda_f(p)}{p^{\frac12}}\widehat{\phi}\left(\frac{\log p}{\log(k^2)}\right)\frac{\log p}{\log(k^2)}\ll k^{\frac{3\sigma}{2}}2^{-k}.$$

Now, the desired result follows immediately from Lemma 3.4. □

5. 1-Level density averaged over the weight: extended support

In this section we study the quantities $\mathscr{D}^{+}_{K,h}(\phi)$ and $\mathscr{D}^{-}_{K,h}(\phi)$, that is we average the 1-level density $\mathcal{D}_k(\phi;K^2)$ over $k\asymp K$ against the weight $h(\frac{k-1}{K})$.

Lemma 5.1 ([10, Lemma 5.8], [12, Corollary 8.2]) *For h a non-negative smooth function with compact support in $\mathbb{R}_{>0}$ and for any $K\ge 2$, we have the estimates*

$$2\sum_{k\equiv 0 \bmod 2}h\left(\frac{k-1}{K}\right)J_{k-1}(x)=h\left(\frac{x}{K}\right)+O\left(\frac{x}{K^3}\right);$$

$$2\sum_{k\equiv 0 \bmod 2}i^kh\left(\frac{k-1}{K}\right)J_{k-1}(x)=-\frac{K}{\sqrt{x}}\Im\left(\overline{\zeta_8}e^{ix}\hbar\left(\frac{K^2}{2x}\right)\right)+O\left(\frac{x}{K^4}\right),$$

where $\zeta_8 = e^{2\pi i/8}$ *and* $\hbar(x) = \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}} e^{ixu} du$ *(as noted in* [12, p. 102], $\hbar(x)$ *is a Schwartz function).*

In the next lemma we estimate the total weight $H^{\pm}(K) = \sum_{k\equiv 3\pm 1 \bmod 4} h\left(\frac{k-1}{K}\right)$ and a related sum.

LEMMA 5.2 *For h a non-negative smooth function with compact support in $\mathbb{R}_{>0}$ and for any $K, N \geq 2$, we have the estimates*

$$H^{\pm}(K) = \frac{K\int_{\mathbb{R}^+} h}{4} + O_N(K^{-N})$$

and

$$4\sum_{k\equiv 3\pm 1 \bmod 4} h\Big(\frac{k-1}{K}\Big)\log k = K\log K\int_{\mathbb{R}^+} h + K\int_{\mathbb{R}^+} h\cdot\log + \sum_{\ell=1}^{N}\frac{(-1)^{\ell+1}}{\ell K^{\ell-1}}\int_{\mathbb{R}^+} t^{-\ell}h(t)dt + O_N(K^{-N}).$$

Proof. More generally, we will show that for any a mod 4,

$$\sum_{k\equiv a \bmod 4} h\Big(\frac{k}{K}\Big) = \frac{K\int_{\mathbb{R}^+} h}{4} + O_N(K^{-N}); \tag{5.1}$$

$$4\sum_{k\equiv a \bmod 4} h\Big(\frac{k}{K}\Big)\log(k+1) = K\log K\int_{\mathbb{R}^+} h + K\int_{\mathbb{R}^+} h\cdot\log + \sum_{\ell=1}^{N}\frac{(-1)^{\ell+1}}{\ell K^{\ell-1}}\int_{\mathbb{R}^+} t^{-\ell}h(t)dt + O_N(K^{-N}). \tag{5.2}$$

Now, for any b mod 4, Poisson summation gives

$$\sum_{k\in\mathbb{Z}} e\Big(\frac{bk}{4}\Big)h\Big(\frac{k}{K}\Big) = K\sum_{k\in\mathbb{Z}}\widehat{h}\Big(\Big(k-\frac{b}{4}\Big)K\Big) = K\widehat{h}(0)\delta_{b=0} + O_N(K^{-N}).$$

The estimate (5.1) follows by orthogonality of additive characters. Similarly, we see that

$$\sum_{k\equiv a \bmod 4} h\Big(\frac{k}{K}\Big)\log(k+1) = \frac{\int_{\mathbb{R}^+} h\left(\frac{t}{K}\right)\log(t+1)dt}{4} + O_N(K^{-N}). \tag{5.3}$$

Indeed, integration by parts shows that

$$\int_{\mathbb{R}^+} h\Big(\frac{t}{K}\Big)\log(t+1)e(-\xi t)dt \ll_M \frac{K\log K}{(|\xi|K)^M}.$$

Finally, the integral on the right-hand side of (5.3) equals

$$K\int_{\mathbb{R}^+} h(t)\log(Kt+1)dt = K\log K\int_{\mathbb{R}^+} h + K\int_{\mathbb{R}^+} h\cdot\log + \sum_{\ell=1}^{N}\frac{(-1)^{\ell+1}}{\ell K^{\ell-1}}\int_{\mathbb{R}^+} t^{-\ell}h(t)dt + O_N(K^{-N}),$$

and (5.2) follows. □

In the next lemma we estimate the average of (3.3) over k. In order to do so, we will apply Lemmas 5.1 and 5.2.

LEMMA 5.3 *Let ϕ be an even Schwartz test function and let h be a non-negative and not identically zero smooth function with compact support in $\mathbb{R}_{>0}$. Under the condition $\sigma = \sup(\mathrm{supp}(\widehat{\phi})) < 2$ and for $K \geq 2$, we have the estimate*

$$\mathscr{D}^{\pm}_{K,h}(\phi) = \widehat{\phi}(0)\left(1 + \frac{\frac{\int_{\mathbb{R}^+} h\cdot\log}{\int_{\mathbb{R}^+} h} - \log(4\pi)}{\log K}\right) + 2\sum_p \frac{1}{p}\widehat{\phi}\Big(\frac{2\log p}{\log(K^2)}\Big)\frac{\log p}{\log(K^2)}$$

$$\mp \frac{\pi}{\log(K^2)H^{\pm}(K)}\sum_p \frac{\log p}{p^{\frac{1}{2}}}\widehat{\phi}\Big(\frac{\log p}{\log(K^2)}\Big)\sum_{c=1}^{\infty}\frac{S(p,1;c)}{c}h\Big(\frac{4\pi\sqrt{p}}{cK}\Big) + O_{\varepsilon}\big(K^{\frac{\sigma}{2}-1+\varepsilon} + K^{-\frac{1}{3}+\varepsilon}\big).$$

Proof. From combining Lemma 3.4 with $X = K^2$ and Lemma 5.2, we have that

$$\mathscr{D}^{\pm}_{K,h}(\phi) = \widehat{\phi}(0)\left(1 + \frac{\frac{\int_{\mathbb{R}^+} h\cdot\log}{\int_{\mathbb{R}^+} h} - \log(4\pi)}{\log K}\right) + 2\sum_p \frac{1}{p}\widehat{\phi}\Big(\frac{2\log p}{\log(K^2)}\Big)\frac{\log p}{\log(K^2)}$$

$$-\frac{2}{H^{\pm}(K)}\sum_{k\equiv 3\pm 1 \bmod 4} h\Big(\frac{k-1}{K}\Big)\sum_{f\in B_k}\omega_f\sum_p \frac{\lambda_f(p)}{p^{\frac{1}{2}}}\widehat{\phi}\Big(\frac{\log p}{\log(K^2)}\Big)\frac{\log p}{\log(K^2)} + O_{\varepsilon}\big(K^{\frac{\sigma}{2}-1+\varepsilon} + K^{-\frac{1}{3}+\varepsilon}\big).$$

By the Petersson trace formula (Lemma 3.1), the third term is equal to

$$-\frac{2\pi}{H^{\pm}(K)}\sum_{k\equiv 0 \bmod 2}(i^k \pm 1)h\Big(\frac{k-1}{K}\Big)\sum_p \frac{1}{p^{\frac{1}{2}}}\widehat{\phi}\Big(\frac{\log p}{\log(K^2)}\Big)\frac{\log p}{\log(K^2)}\sum_{c\geq 1}c^{-1}S(p,1,c)J_{k-1}\Big(\frac{4\pi\sqrt{p}}{c}\Big). \tag{5.4}$$

Applying Lemma 5.1, we see that

$$-\frac{2}{H^{\pm}(K)}\sum_{k\equiv 0 \bmod 2} i^k h\Big(\frac{k-1}{K}\Big)J_{k-1}\Big(\frac{4\pi\sqrt{p}}{c}\Big) = \frac{Kc^{\frac{1}{2}}}{H^{\pm}(K)2\pi^{\frac{1}{2}}p^{\frac{1}{4}}}\Im\Big(\overline{\zeta_8}e^{\frac{i4\pi\sqrt{p}}{c}}\check{h}\Big(\frac{K^2c}{8\pi\sqrt{p}}\Big)\Big) + O\Big(\frac{\sqrt{p}}{cK^5}\Big). \tag{5.5}$$

Since $p \leq K^{4-\varepsilon}$, we see by the rapid decay of $\check{h}$ that for any $A > 1$, the first term in this expression is

$$\ll_A \frac{c^{\frac{1}{2}}}{p^{\frac{1}{4}}}\Big(\frac{K^2c}{\sqrt{p}}\Big)^{-A},$$

and hence by the Weil bound the contribution of this term to (5.4) is

$$\ll_A K^{A(\sigma-2)+\frac{\sigma}{2}}.$$

As for the sum of the error terms in (5.5), the contribution is $\ll K^{2\sigma-5}$ (by the Weil bound), which is an admissible error term. Moreover, applying Lemma 5.1 once more,

$$-\frac{2}{H^{\pm}(K)}\sum_{k\equiv 0 \bmod 2} h\Big(\frac{k-1}{K}\Big)J_{k-1}\Big(\frac{4\pi\sqrt{p}}{c}\Big) = -\frac{1}{H^{\pm}(K)}h\Big(\frac{4\pi\sqrt{p}}{cK}\Big)+O\Big(\frac{\sqrt{p}}{cK^4}\Big),$$

resulting in a main term as well as the admissible error term $O(K^{2\sigma-4})$ (once more by the Weil bound). □

We now end this section by evaluating the second sum over primes in Lemma 5.3, under GRH for Dirichlet L-functions. This term will be responsible for the phase transition at 1 and will be investigated more closely in Section 6.

LEMMA 5.4 *Let ϕ be an even Schwartz test function and suppose that $\sigma =$sup(supp($\widehat{\phi}$)) < 2. Let h be a non-negative smooth function with compact support in $\mathbb{R}_{>0}$ and assume the Riemann Hypothesis for Dirichlet L-functions. Then for any $K\geq 2$, we have the estimate*

$$\sum_{c\geq 1}\frac{1}{c}\sum_{p}\frac{\log p}{p^{1/2}}\widehat{\phi}\Big(\frac{\log p}{\log(K^2)}\Big)S(p,1;c)h\Big(\frac{4\pi\sqrt{p}}{cK}\Big)=\log(K^2)\int_0^{\sigma}K^u\widehat{\phi}(u)\sum_{c\geq 1}\frac{\mu^2(c)}{c\varphi(c)}h\Big(\frac{4\pi K^{u-1}}{c}\Big)du$$
$$+O\big(K^{\sigma-1}(\log K)^3\big), \tag{5.6}$$

where φ is Euler's totient function.

Proof. If $\sigma<1$, then for large enough K the left-hand side of (5.6) is identically zero. We may thus assume that $\sigma\geq 1$. The sum over p equals

$$\int_0^{\infty}\frac{1}{t^{\frac12}}\widehat{\phi}\Big(\frac{\log t}{\log(K^2)}\Big)h\Big(\frac{4\pi\sqrt{t}}{cK}\Big)dT(t)=-\int_0^{\infty}\bigg(\frac{1}{t^{\frac12}}\widehat{\phi}\Big(\frac{\log t}{\log(K^2)}\Big)h\Big(\frac{4\pi\sqrt{t}}{cK}\Big)\bigg)'T(t)dt, \tag{5.7}$$

where, by [12, Lemma 6.1],

$$T(t):=\sum_{p\leq t}S(p,1;c)\log p = t\frac{\mu^2(c)}{\varphi(c)}+O\big(\varphi(c)t^{\frac12}(\log(ct))^2\big).$$

Note that our restriction on the support of h implies that $c\asymp\sqrt{t}/K$, and hence the restriction on the support of $\widehat{\phi}$ implies that for square-free values of c and for $t\leq K^{4-\epsilon}$, the main term in this estimate is always larger than the error term. The total contribution of the main term in this estimate is given by

$$-\sum_{c\geq 1}\frac{\mu^2(c)}{c\varphi(c)}\int_0^\infty \left(\frac{1}{t^{\frac12}}\widehat{\phi}\Big(\frac{\log t}{\log(K^2)}\Big)h\Big(\frac{4\pi\sqrt{t}}{cK}\Big)\right)' tdt=\sum_{c\geq 1}\frac{\mu^2(c)}{c\varphi(c)}\int_0^\infty \frac{1}{t^{\frac12}}\widehat{\phi}\Big(\frac{\log t}{\log(K^2)}\Big)h\Big(\frac{4\pi\sqrt{t}}{cK}\Big)dt,$$

which is equal to the claimed main term by a change of variables. As for the error term, we compute the derivative in (5.7) and find that the contribution of this term is $\ll K^{\sigma-1}(\log K)^3$, finishing the proof. □

6. Evaluation of the transition term

The goal of this section is to evaluate the integral in Lemma 5.4. This will be done using different techniques depending on the range of the variable u. To this end, for $a,b\in\mathbb{R}_{\geq 0}\cup\{\infty\}$, we define

$$I_{a,b}:=\frac{\pi}{H^{\pm}(K)}\int_a^b K^u\widehat{\phi}(u)\sum_{c\geq 1}\frac{\mu^2(c)}{c\varphi(c)}h\Big(\frac{4\pi K^{u-1}}{c}\Big)du.$$

Notice that the inner sum is long only when u is larger and far away from 1. By Lemmas 5.3 and 5.4, we see that when $\sigma=\sup(\mathrm{supp}(\widehat{\phi}))<2$ and under the assumption of GRH for Dirichlet L-functions,

$$\mathscr{D}^{\pm}_{K,h}(\phi)=\widehat{\phi}(0)\left(1+\frac{\frac{\int_{\mathbb{R}^+}h\cdot\log}{\int_{\mathbb{R}^+}h}-\log(4\pi)}{\log K}\right)+2\sum_p\frac{1}{p}\widehat{\phi}\Big(\frac{2\log p}{\log(K^2)}\Big)\frac{\log p}{\log(K^2)}$$
$$\mp I_{0,\infty}+O_\epsilon\big(K^{\frac{\sigma}{2}-1+\varepsilon}+K^{-\frac13+\varepsilon}\big). \quad (6.1)$$

We now move on to evaluating the integral $I_{0,\infty}$. We let δ_K be a positive parameter which satisfies $\delta_K\gg_h 1/\log K$. Recall that h is supported in $\mathbb{R}_{>0}$, and hence for K large enough the integrand in $I_{0,\infty}$ is zero in the interval $[0,1-\delta_K)$. Hence,

$$I_{0,\infty}=I_{1-\delta_K,\sigma},$$

where, as before, $\sigma=\sup(\mathrm{supp}(\widehat{\phi}))$.

Lemma 6.1 *We have the unconditional estimate*

$$\sum_{c\leq x}\frac{c\mu^2(c)}{\varphi(c)}=x+O(x^{\frac12}).$$

Proof. The error term in Lemma 6.1 can be improved by replacing (6.4) with the stronger estimate obtained from using [19, Exercise 19, §6.2.1]. We first establish the following estimate for square-free values of d:

$$S_d(x) := \sum_{\substack{m\leq \frac{x}{d}\\(m,d)=1}} \frac{\mu^2(m)}{m} = C_1(d)\log x + C_2(d) + O\Big(x^{-\frac12} d^{\frac12} \prod_{p\mid d}\big(1-p^{-\frac12}\big)^{-1}\Big), \tag{6.2}$$

where

$$C_1(d) := \frac{1}{\zeta(2)}\prod_{p\mid d}\Big(1-\frac{1}{p+1}\Big); \qquad C_2(d) := C_1(d)\Big(\gamma - 2\frac{\zeta'}{\zeta}(2) - \sum_{p\mid d}\frac{p\log p}{p+1}\Big).$$

To do so, note that

$$S_d(x) = \sum_{\ell_1\mid d} \frac{\lambda(\ell_1)}{\ell_1} S_{\ell_1}\Big(\frac{x}{d}\Big),$$

where $\lambda(n)$ denotes the Liouville function. Applying this equality iteratively, we reach the identity

$$S_d(x) = \sum_{\substack{\ell_1\mid d\\ \ell_2\mid \ell_1\\ \vdots\\ \ell_k\mid \ell_{k-1}}} \frac{\lambda(\ell_1)\cdots\lambda(\ell_k)}{\ell_1\cdots\ell_k} S_{\ell_k}\Big(\frac{x}{d\ell_1\cdots\ell_{k-1}}\Big) = \sum_{\ell\mid d^\infty}\frac{\lambda(\ell)}{\ell} S_1\Big(\frac{x}{d\ell}\Big). \tag{6.3}$$

A summation by parts combined with [20, Theorem 8.25] yields that

$$S_1(x) = \frac{1}{\zeta(2)}\Big(\log x + \gamma - 2\frac{\zeta'}{\zeta}(2)\Big) + O(x^{-\frac12}). \tag{6.4}$$

Indeed, the precise value of the constant is deduced from writing $S_1(x) = \frac{1}{2\pi i}\int_{(1)} \frac{\zeta(s+1)}{\zeta(2s+2)}\frac{x^s}{s}ds$ and shifting the contour of integration to the left. Inserting (6.4) into (6.3), we are left with an error term which is

$$\ll d^{\frac12}x^{-\frac12}\sum_{\ell\mid d^\infty}\frac{1}{\ell^{\frac12}} = d^{\frac12}x^{-\frac12}\prod_{p\mid d}\sum_{\alpha\geq 0}\frac{1}{p^{\frac{\alpha}{2}}} = d^{\frac12}x^{-\frac12}\prod_{p\mid d}\big(1-p^{-\frac12}\big)^{-1},$$

and (6.2) follows. The claimed estimate then follows from the convolution identity

$$\sum_{c\leq x}\frac{\mu^2(c)}{\varphi(c)} = \sum_{c\leq x}\frac{\mu^2(c)}{c}\sum_{d\mid c}\frac{\mu^2(d)}{\varphi(d)} = \sum_{d\leq x}\frac{\mu^2(d)}{d\varphi(d)}\sum_{\substack{m\leq \frac{x}{d}\\(m,d)=1}}\frac{\mu^2(m)}{m}$$

and a straightforward summation by parts. □

We now evaluate the part of the integral $I_{0,\infty}$ for which u is slightly larger than 1. In this range, the sum over c is fairly long and we can effectively apply Lemma 6.1.

LEMMA 6.2 *(The range $u > 1+\delta_K$) Let ϕ be an even Schwartz test function and let $K \geq 2$. Then we have*

$$I_{1+\delta_K,\infty} = \int_{1+\delta_K}^{\infty} \widehat{\phi} + O\big(K^{-\frac{\delta_K}{2}}\big).$$

Proof. By Lemma 6.1, we have that

$$S(y) := \sum_{c\leq y} \frac{c\mu^2(c)}{\varphi(c)} = y + O(y^{\frac{1}{2}}).$$

Hence, for $u > 1+\delta_K$,

$$\begin{aligned}\sum_{c\geq 1} \frac{\mu^2(c)}{c\varphi(c)} h\Big(\frac{4\pi K^{u-1}}{c}\Big) &= -\int_0^{\infty} S(y)\Big(\frac{1}{y^2} h\Big(\frac{4\pi K^{u-1}}{y}\Big)\Big)' dy \\ &= \int_0^{\infty} \frac{1}{y^2} h\Big(\frac{4\pi K^{u-1}}{y}\Big) dy \\ &\quad + O\bigg(\int_0^{\infty} \Big(\frac{1}{y^3} h\Big(\frac{4\pi K^{u-1}}{y}\Big) + \frac{K^{u-1}}{y^4} h'\Big(\frac{4\pi K^{u-1}}{y}\Big)\Big) y^{\frac{1}{2}} dy\bigg) \\ &= \frac{1}{4\pi K^{u-1}} \int_0^{\infty} h + O\big(K^{-\frac{3}{2}(u-1)}\big).\end{aligned}$$

The desired estimate follows by integrating over u against $K^u \widehat{\phi}(u)$ and applying Lemma 5.2. □

We now evaluate the part of the integral $I_{0,\infty}$ in which u is close to 1. In this range we can expand $\widehat{\phi}(u)$ into Taylor series around $u=1$ and recover the transition terms $\widehat{\phi}^{(j)}(1)$ (see Lemma 6.5). The resulting integrals are evaluated in Lemma 6.4 by applying the inverse Mellin transform, shifting the contours of integration and estimating Mellin transforms on the appropriate contours. We recall that the Mellin transform of a function $f : \mathbb{R}_{>0} \to \mathbb{C}$ is defined by

$$\mathcal{M}f(s) := \int_0^{\infty} x^{s-1} f(x) dx,$$

whenever this integral converges.

LEMMA 6.3 *Whenever $x \in \mathbb{R}_{\geq 0}, j \in \mathbb{N}, \Re(s) > 0$ and $x|s| \geq 2$, we have*

$$\int_x^{\infty} u^j e^{-us} du \ll \frac{j! e^{-\Re(s)x} x^j}{|s|},$$

where the implied constant is absolute.

Proof. Applying integration by parts, we reach the exact formula

$$\int_x^\infty u^j e^{-us} du = \frac{e^{-xs}x^j}{s} \sum_{0\le \ell \le j} \frac{j!}{(j-\ell)!(xs)^\ell},$$

from which the lemma immediately follows. □

LEMMA 6.4 *For $j \geq 0$, $K \geq 2$ and $20(\log K)^{-1} \leq \delta_K \leq \frac{1}{2}$, we have the estimate*

$$\sum_{c\geq 1} \frac{\mu^2(c)}{\varphi(c)} \int_{K^{-\delta_K}}^{K^{\delta_K}} \frac{(\log v)^j}{c} h\Big(\frac{4\pi v}{c}\Big) dv = \mathcal{M}h(1)\frac{(\delta_K \log K)^{j+1}}{4\pi(j+1)} + C_{j,h} + O_\varepsilon\Big(j!(\delta_K\log K)^j K^{\delta_K(-\frac{1}{2}+\varepsilon)}\Big),$$

where

$$C_{j,h} := \frac{(-1)^j}{j+1}\frac{d^{j+1}}{(ds)^{j+1}}\Big(sZ(s)(4\pi)^{s-1}\mathcal{M}h(1-s)\Big)\Big|_{s=0} \tag{6.5}$$

with

$$Z(s) = \zeta(s+1)\prod_p \Big(1+\frac{1}{p-1}\Big(\frac{1}{p^{s+1}}-\frac{1}{p^{2s+1}}\Big)\Big).$$

Proof. Define

$$f_{K,j}(c) := \int_{K^{-\delta_K}}^{K^{\delta_K}} \frac{(\log v)^j}{c} h\Big(\frac{4\pi v}{c}\Big) dv.$$

By the restriction on the support of h, the function $f_{K,j}$ also has compact support in $\mathbb{R}_{>0}$. We conclude that its Mellin transform $\varphi_{K,j}(s)$ is entire. Moreover,

$$\begin{aligned}\varphi_{K,j}(s) := \int_0^\infty x^{s-1} f_{K,j}(x)dx &= (4\pi)^{s-1}\mathcal{M}h(1-s)\int_{K^{-\delta_K}}^{K^{\delta_K}} (\log v)^j v^{s-1} dv \\ &= \varphi^+_{K,j}(s) + \varphi^-_{K,j}(s),\end{aligned}$$

where

$$\varphi^\pm_{K,j}(s) = \pm(4\pi)^{s-1}\mathcal{M}h(1-s)\int_1^{K^{\pm\delta_K}} (\log v)^j v^{s-1} dv$$

are also entire. For any $N \geq 1$, applying [4, Lemma 2.1] yields the crude bound

$$\varphi^\pm_{K,j}(s) \ll_{N,j} K|s|^{-N} \qquad (|\Im(s)| \geq 1, |\Re(s)| \leq 1). \tag{6.6}$$

Now, Mellin inversion gives the formula

$$f_{K,j}(c) = \frac{1}{2\pi i}\int_{(\frac12)} c^{-s}\big(\varphi^+_{K,j}(s) + \varphi^-_{K,j}(s)\big)ds.$$

Hence, by absolute convergence,

$$\sum_{c\geq 1}\frac{\mu^2(c)}{\varphi(c)} f_{K,j}(c) = \frac{1}{2\pi i}\int_{(\frac12)} Z(s)\big(\varphi^+_{K,j}(s) + \varphi^-_{K,j}(s)\big)ds, \tag{6.7}$$

where

$$Z(s) := \sum_{c=1}^{\infty}\frac{\mu^2(c)}{c^s\varphi(c)} = \zeta(s+1)\prod_p\Big(1+\frac{1}{p-1}\Big(\frac{1}{p^{s+1}}-\frac{1}{p^{2s+1}}\Big)\Big).$$

Next, by applying Lemma 6.3, we obtain the estimate

$$\varphi^-_{K,j}(s) = (4\pi)^{s-1}\mathcal{M}h(1-s)\bigg(\int_0^1(\log v)^j v^{s-1}dv + O\Big(j!(\delta_K\log K)^j K^{-\delta_K\Re(s)}\Big)\bigg) \quad (\Re(s)>0, |s|\geq \tfrac{1}{10}).$$

Moreover,

$$\int_0^1 (\log v)^j v^{s-1}dv = \frac{(-1)^j j!}{s^{j+1}} \quad (\Re(s)>0).$$

Hence, by the rapid decay of $\mathcal{M}h(1-s)$ on vertical lines (see [4, Lemma 2.1]), we obtain

$$\frac{1}{2\pi i}\int_{(\frac12)} Z(s)\varphi^-_{K,j}(s)ds = \frac{(-1)^j j!}{2\pi i}\int_{(\frac12)} Z(s)(4\pi)^{s-1}\mathcal{M}h(1-s)\frac{ds}{s^{j+1}} + O\Big(j!(\delta_K\log K)^j K^{-\frac{\delta_K}{2}}\Big).$$

As for the first part of the integral in (6.7), by applying (6.6) we can shift the countour to the left until the line $\Re(s) = -\frac12+\varepsilon$ and reach the identity

$$\frac{1}{2\pi i}\int_{(\frac12)} Z(s)\varphi^+_{K,j}(s)ds = (4\pi)^{-1}\mathcal{M}h(1)\int_1^{K^{\delta_K}}(\log v)^j v^{-1}dv + \frac{1}{2\pi i}\int_{(-\frac12+\varepsilon)} Z(s)\varphi^+_{K,j}(s)ds.$$

In a similar fashion as before, we see that

$$\varphi^+_{K,j}(s) = (4\pi)^{s-1}\mathcal{M}h(1-s)\bigg(\frac{(-1)^{j+1}j!}{s^{j+1}} + O\Big(j!(\delta_K\log K)^j K^{\delta_K\Re(s)}\Big)\bigg) \quad (\Re(s)<0, |s|\geq \tfrac{1}{10})$$

and deduce that

$$\frac{1}{2\pi i}\int_{(\frac12)} Z(s)\varphi^+_{K,j}(s)ds = \mathcal{M}h(1)\frac{(\delta_K\log K)^{j+1}}{4\pi(j+1)} + \frac{(-1)^{j+1}j!}{2\pi i}\int_{(-\frac12+\varepsilon)} Z(s)(4\pi)^{s-1}\mathcal{M}h(1-s)\frac{ds}{s^{j+1}}$$
$$+ O_\varepsilon\Big(j!(\delta_K\log K)^j K^{\delta_K(-\frac12+\varepsilon)}\Big).$$

Putting these estimates together, we conclude that

$$\sum_{c\geq 1}\frac{\mu^2(c)}{\varphi(c)}f_{K,j}(c)=\mathcal{M}h(1)\frac{(\delta_K\log K)^{j+1}}{4\pi(j+1)}+\frac{(-1)^j j!}{2\pi i}\Big(\int_{(\frac{1}{2})}-\int_{(-\frac{1}{2}+\varepsilon)}\Big)Z(s)(4\pi)^{s-1}\mathcal{M}h(1-s)\frac{ds}{s^{j+1}}$$
$$+O_\varepsilon\Big(j!(\delta_K\log K)^j K^{\delta_K(-\frac{1}{2}+\varepsilon)}\Big).$$

The result follows. □

LEMMA 6.5 (*The range* $1-\delta_K<u<1+\delta_K$) *Let* ϕ *be an even Schwartz test function. We have, for* $K\geq 2$ *and odd* $J\geq 1$, *that*

$$I_{1-\delta_K,1+\delta_K}=\int_1^{1+\delta_K}\widehat{\phi}+\frac{\pi K}{H^{\pm}(K)}\sum_{0\leq j\leq J}\frac{\widehat{\phi}^{(j)}(1)C_{j,h}}{j!(\log K)^{j+1}}+O_{\varepsilon,J}\Big(\delta_K^{J+2}+\frac{1}{(\log K)^{J+2}}+\frac{K^{\delta_K(-\frac{1}{2}+\varepsilon)}}{\log K}\Big),$$

where the constants $C_{j,h}$ *are defined in* (6.5).

Proof. By the definition of $I_{1-\delta_K,1+\delta_K}$, we need to evaluate the sum

$$\sum_{c\geq 1}\frac{\mu^2(c)}{\varphi(c)}\int_{1-\delta_K}^{1+\delta_K}\frac{K^u}{c}\widehat{\phi}(u)h\Big(\frac{4\pi K^{u-1}}{c}\Big)du=\frac{K}{\log K}\sum_{c\geq 1}\frac{\mu^2(c)}{\varphi(c)}\int_{K^{-\delta_K}}^{K^{\delta_K}}\frac{1}{c}\widehat{\phi}\Big(\frac{\log v}{\log K}+1\Big)h\Big(\frac{4\pi v}{c}\Big)dv.$$

Expanding into Taylor series and applying Lemma 6.4, this is

$$=K\sum_{0\leq j\leq J}\frac{\widehat{\phi}^{(j)}(1)}{j!(\log K)^{j+1}}\sum_{c\geq 1}\frac{\mu^2(c)}{\varphi(c)}\int_{K^{-\delta_K}}^{K^{\delta_K}}\frac{(\log v)^j}{c}h\Big(\frac{4\pi v}{c}\Big)dv$$
$$+O_{\varepsilon,J}\Big(K\delta_K^{J+2}+\frac{K}{(\log K)^{J+2}}+\frac{K^{1+\delta_K(-\frac{1}{2}+\varepsilon)}}{\log K}\Big).$$

Note that the error term obtained after the Taylor series expansion contains the expression

$$\sum_{c\geq 1}\frac{\mu^2(c)}{\varphi(c)}\int_{K^{-\delta_K}}^{K^{\delta_K}}c^{-1}|\log v|^{J+1}h\Big(\frac{4\pi v}{c}\Big)dv,$$

which can be evaluated using Lemma 6.4 whenever $J+1$ is even. Applying Lemma 6.4 once more, we reach the expression

$$\frac{K\mathcal{M}h(1)}{4\pi}\sum_{0\leq j\leq J}\frac{\widehat{\phi}^{(j)}(1)\delta_K^{j+1}}{(j+1)!}+K\sum_{0\leq j\leq J}\frac{\widehat{\phi}^{(j)}(1)C_{j,h}}{j!(\log K)^{j+1}}+O_{\varepsilon,J}\Big(K\delta_K^{J+2}+\frac{K}{(\log K)^{J+2}}+\frac{K^{1+\delta_K(-\frac{1}{2}+\varepsilon)}}{\log K}\Big).$$

Finally, the desired result follows from an application of Lemma 5.2. □

Collecting the estimates in this section, we reach the following theorem.

THEOREM 6.6 *Let ϕ be an even Schwartz test function for which supp$(\widehat{\phi}) \subset (-2,2)$. Assuming the Riemann Hypothesis for Dirichlet L-functions, for $K \geq 2$ we have the estimate*

$$\mathscr{D}^{\pm}_{K,h}(\phi) = \widehat{\phi}(0)\left(1 + \frac{\frac{\int_0^\infty h\cdot\log}{\int_0^\infty h} - \log(4\pi)}{\log K}\right) + \frac{\phi(0)}{2} + \sum_{1\leq j\leq J} \frac{c_j \widehat{\phi}^{(j-1)}(0)}{2^j(\log K)^j}$$
$$\mp \int_1^\infty \widehat{\phi} \pm \sum_{1\leq j\leq J} \frac{D_{j,h}\widehat{\phi}^{(j-1)}(1)}{(\log K)^j} + O_{\varepsilon,J}\Big(\frac{1}{(\log K)^{J+1}}\Big),$$

where the c_j are defined in Lemma 4.1, and

$$D_{j,h} = -\frac{4\pi}{\int_{\mathbb{R}} h\cdot(j-1)!}C_{j-1,h} = \frac{4\pi(-1)^j}{j!\int_{\mathbb{R}+} h}\frac{d^j}{(ds)^j}\Big(sZ(s)(4\pi)^{s-1}\mathcal{M}h(1-s)\Big)\Big|_{s=0}$$

with

$$Z(s) = \zeta(s+1)\prod_p\Big(1 + \frac{1}{p-1}\Big(\frac{1}{p^{s+1}} - \frac{1}{p^{2s+1}}\Big)\Big). \tag{6.8}$$

Proof. Recall (6.1), which is valid for σ =supp(sup$(\widehat{\phi})) < 2$:

$$\mathscr{D}^{\pm}_{K,h}(\phi) = \widehat{\phi}(0)\left(1 + \frac{\frac{\int_{\mathbb{R}+} h\cdot\log}{\int_{\mathbb{R}+} h} - \log(4\pi)}{\log K}\right) + 2\sum_p \frac{1}{p}\widehat{\phi}\Big(\frac{2\log p}{\log(K^2)}\Big)\frac{\log p}{\log(K^2)}$$
$$\mp I_{0,\infty} + O_\varepsilon\big(K^{\frac{\sigma}{2}-1+\varepsilon} + K^{-\frac{1}{3}+\varepsilon}\big).$$

We can clearly assume, without loss of generality, that J is odd. The sum over primes is estimated in Lemma 4.1. Moreover, we recall that for K large enough $I_{0,1-\delta_K} = 0$ and therefore we have that

$$I_{0,\infty} = I_{1-\delta_K,1+\delta_K} + I_{1+\delta_K,\infty},$$

which together with Lemmas 6.2 and 6.5 and the choice $\delta_K = 3(J+3)\log\log(K+3)/\log K$ implies the desired result. □

Proof of Theorem 1.1. The result follows immediately from Theorem 6.6 with

$$S_{j,h} = D_{j,h} = \frac{4\pi(-1)^j}{j!\int_{\mathbb{R}+} h}\frac{d^j}{(ds)^j}\Big(sZ(s)(4\pi)^{s-1}\mathcal{M}h(1-s)\Big)\Big|_{s=0} \tag{6.9}$$

(see 6.8 for the definition of $Z(s)$);

$$R_{1,h} = \frac{\int_{\mathbb{R}+} h\cdot\log}{\int_{\mathbb{R}+} h} - \log(4\pi) + 1 + \int_1^\infty \frac{\theta(t)-t}{t^2}dt; \tag{6.10}$$

and

$$R_{j,h} = \frac{1}{(j-2)!} \int_1^\infty (\log t)^{j-2} \Big(\frac{\log t}{j-1} - 1 \Big) \frac{\theta(t) - t}{t^2} dt \tag{6.11}$$

for $j \geq 2$ (note that these constants do not depend on h). □

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