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Completely Compact Herz–Schur Multipliers of Dynamical Systems

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Abstract
We prove that if G is a discrete group and (A, G, α) is a C*-dynamical system such that the reduced crossed product $A \rtimes_{r,\alpha} G$ possesses property (SOAP) then every completely compact Herz–Schur (A, G, α)-multiplier can be approximated in the completely bounded norm by Herz–Schur (A, G, α)-multipliers of finite rank. As a consequence, if G has the approximation property (AP) then the completely compact Herz–Schur multipliers of $A(G)$ coincide with the closure of $A(G)$ in the completely bounded multiplier norm. We study the class of invariant completely compact Herz–Schur multipliers of $A \rtimes_{r,\alpha} G$ and provide a description of this class in the case of the irrational rotation algebra.

Keywords Crossed product · Multiplier · Completely compact map

Mathematics Subject Classification Primary 37A55 · Secondary 46L07 · 43A55

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1 Introduction

Fourier multipliers are transformations on function spaces associated with abelian topological groups that act term-wise on the Fourier series of their elements. In the case of the group $\mathbb{Z}$, such mappings have a natural significance in classical Analysis, where the approximation of a given function by trigonometric polynomials has been of paramount importance. When such truncations are realised as Fourier multipliers, these mappings have finite rank and are hence compact. In subsequent developments, compactness of Fourier multipliers has been studied from other perspectives as well, for example, in relation with the compactness of pseudo-differential operators (see e.g. [8]).

For non-commutative locally compact groups G, where Fourier transform is not readily available, the natural setting for the study of Fourier multipliers is provided by the Fourier algebra $A(G)$ of the group G—a commutative Banach algebra consisting of all coefficients of the left regular representation of G whose Gelfand spectrum can be canonically identified with G. This study was initiated in [6], with illustrious subsequent history and some far-reaching applications to approximation techniques in operator algebra theory, where finite rank multipliers have played a cornerstone role (see [5]). In [22], Lau showed that the Fourier algebra $A(G)$ has a non-zero weakly compact left multiplier if and only if G is discrete and that, for discrete amenable groups, $A(G)$ coincides with the algebra of its weakly compact multipliers. We refer the reader to [7, 11, 15] for further related results.

In the case of non-abelian groups, the new property of complete boundedness—brought about by non-commutativity—becomes a natural requirement, and the associated multipliers of $A(G)$ are known as Herz–Schur multipliers [6]. Herz–Schur multipliers are related to Schur multipliers—transformations on the algebra of all bounded operators on $L^2(G)$ that extend point-wise multiplication of integral kernels by a given fixed function—via operator transference, pioneered in the area by Bożejko and Fendler [3]. We refer the reader to [34, 35] for a survey of these results and ideas.

A natural quantised version of the notion of compactness in the non-commutative setting—complete compactness—was first introduced by Saar in his Diplomarbeit [32] under G. Wittstock's supervision and further developed in unpublished paper [37] by Webster. More recently it was also studied in the context of operator multipliers in [16] and completely almost periodic functionals on completely contractive Banach algebras in [31]. The notion has been important for the study of various operator space analogues of the Grothendieck approximation property and, in particular, for questions revolving around finite rank approximation. The main aim of this paper is to initiate the study of complete compactness for Herz–Schur multipliers. We work in the higher generality of dynamical systems. Herz–Schur multipliers of crossed products were introduced, and their relation with the surrounding class of operator-valued Schur multipliers investigated, in [25] (see also [1] for discrete dynamical systems). The completely compact operator-valued Schur multipliers were characterised in [18]. Here we provide a characterisation of completely compact Herz–Schur multipliers of the reduced crossed product of (unital) C*-algebras A by the action of a discrete group G, under a mild approximation property.

The paper is organised as follows. In Sect. 2, we recall some results about operator-valued Schur multipliers, Herz–Schur multipliers of C*-dynamical systems (A, G, α) and their interrelation [26] that will be needed in the sequel. In Sect. 3, we associate with every completely bounded map Φ with domain $A \rtimes_{\alpha,r} G$ a Herz–Schur multiplier F_Φ, and identify some properties of the map $\Phi \to F_\Phi$. This is the main technical tool, used in several subsequent results. We focus on actions of discrete groups, since the dynamical systems of non-discrete groups have no non-trivial compact Herz–Schur multipliers (see Proposition 3.2). Theorem 3.7 provides a characterisation of completely compact Herz–Schur (A, G, α)-multipliers in the case where $A \rtimes_{\alpha,r} G$ possesses the strong operator approximation property (SOAP) [9]. As a consequence, one can choose the maps, approximating the identity, to be in this case Herz–Schur multipliers of finite rank. As an immediate corollary, we obtain that for groups with the approximation property (AP) [17], a Herz–Schur multiplier $u : G \to \mathbb{C}$ defines a completely compact map $S_u : C_r^*(G) \to C_r^*(G)$ if and only if u belongs to the closure $A_{\mathrm{cb}}(G)$ of $A(G)$ in the space $M^{\mathrm{cb}}A(G)$ of Herz-Schur multipliers. Using a result of Bożejko [2], we prove that the classes of compact and completely compact Herz–Schur multipliers $u : G \to \mathbb{C}$ are in general different.

The rest of the paper is devoted to special classes of completely compact (A, G, α)-multipliers. Namely, in Sect. 4 we examine the subclass of invariant multipliers of a C*-dynamical system (A, G, α), obtained by lifting completely bounded maps on the C*-algebra A that satisfy a natural covariance property, and exhibit a canonical way of constructing completely compact multipliers. We provide a description of the latter class in the case of the irrational rotation algebra. Finally, in Sect. 5, we consider completely compact multipliers of the dynamical system $(c_0(G), G, \alpha)$, where α is induced by left translations. Such multipliers induce, via the Stone-von Neumann Theorem, mappings on the space $\mathcal{K}$ of all compact operators on $\ell^2(G)$ that are characterised in terms of the Haagerup tensor product $\mathcal{K} \otimes \mathcal{K}$. In the opposite direction, we show that any compact Schur multiplier on $\mathcal{K}$ gives rise to a natural completely compact $(c_0(G), G, \alpha)$-multiplier.

We finish this introduction with a general comment about a separability assumption. Many of our results rely on the development of the theory of Herz–Schur multipliers of C*-dynamical systems (A, G, α), undertaken in [25]. In [25], the C*-algebra A of the dynamical system is assumed to be separable. However, if the group G of the dynamical system is discrete, as pointed out before [26,Theorem 2.1], an inspection of the proofs from [25] reveals that the separability assumption on A can be lifted. In the present paper, all C*-dynamical systems (A, G, α) will be assumed to be over discrete groups and arbitrary C*-algebras A.

2 Schur and Herz–Schur Multipliers

We denote by $\mathcal{B}(H)$ the algebra of all bounded linear operators acing on a Hilbert space H, and by I_H (or I when H is clear from the context) the identity operator on H. For an operator space $\mathcal{X} \subseteq \mathcal{B}(H)$, we let $M_n(\mathcal{X})$ be the space of all n by n matrices with entries in $\mathcal{X}$, and identify it with a subspace of $\mathcal{B}(H^n)$ (where H^n is the direct sum of n copies of H). If $\mathcal{X}$ and $\mathcal{Y}$ are operator spaces, acting on Hilbert spaces H and K,

respectively, and $\varphi : \mathcal{X} \to \mathcal{Y}$ is a linear map, we let as usual $\varphi^{(n)} : M_n(\mathcal{X}) \to M_n(\mathcal{Y})$ be the map given by $\varphi^{(n)}\left((x_{i,j})_{i,j}\right) = \left(\varphi(x_{i,j})\right)_{i,j}$. The map φ is called *completely bounded* if $\|\varphi\|_{\mathrm{cb}} := \sup_{n\in\mathbb{N}} \left\|\varphi^{(n)}\right\| < \infty$. We write $\mathrm{CB}(\mathcal{X}, \mathcal{Y})$ for the space of all completely bounded maps from $\mathcal{X}$ into $\mathcal{Y}$. We let $\mathcal{X} \otimes_{\min} \mathcal{Y}$ be the *minimal tensor product* of $\mathcal{X}$ and $\mathcal{Y}$, that is, the closure of the algebraic tensor product $\mathcal{X} \otimes \mathcal{Y}$ when considered as a subspace of $\mathcal{B}(H \otimes K)$. We will use throughout the paper basic results from operator space theory, and we refer the reader to the monographs [10, 29] for the necessary background.

For a locally compact group G, we let λ^0 be its left regular representation on $L^2(G)$; thus,

$$\lambda^0_t g(s) = g(t^{-1}s), \quad g \in L^2(G), s, t \in G.$$

We use the same symbol, λ^0, to denote the left regular representation of $L^1(G)$ on $L^2(G)$. Let

$$C^*_r(G) = \overline{\{\lambda^0(f) : f \in L^1(G)\}} \subseteq \mathcal{B}(L^2(G))$$

be the *reduced C*-algebra* of G, $\mathrm{VN}(G) := \overline{C^*_r(G)}^{w^*}$ the von Neumann algebra of G (here w^* denotes the weak* topology of $\mathcal{B}(L^2(G))$), and $A(G)$ be the *Fourier algebra* of G, that is, the collection of the functions on G of the form $s \to (\lambda^0_s \xi, \eta)$, where $\xi, \eta \in L^2(G)$. The algebra $A(G)$ will be equipped with the operator space structure arising from its identification with the predual of $\mathrm{VN}(G)$; its norm will be denoted by $\|\cdot\|_A$, and by $\|\cdot\|_{A(G)}$ in cases where we need to emphasise the group. We also write $B(G)$ (resp. $B_r(G)$) for the Fourier-Stieltjes (resp. the reduced Fourier-Stieltjes) algebra of G. The space $B(G)$ (resp. $B_r(G)$) is generated by continuous positive-definite functions on G (resp. by positive-definite functions weakly associated to the left regular representation λ^0 of G), and one has the inclusions $A(G) \subseteq B_r(G) \subseteq B(G)$. By [12,Proposition 2.1], $B(G)$ and $B_r(G)$ can be identified with the dual space of the full C*-algebra $C^*(G)$ and reduced C*-algebra $C^*_r(G)$ of G, respectively. Moreover, when $B(G)$ is equipped with the norm arising from the identification $B(G) = C^*(G)^*$, it becomes a Banach algebra with respect to the pointwise multiplication, and $A(G)$ and $B_r(G)$ are closed ideals of $B(G)$. The norms on $A(G)$ and $B_r(G)$ inherited from $B(G)$ coincide with the norms arising from the identifications $A(G)^* = \mathrm{VN}(G)$ and $B_r(G) = C^*_r(G)^*$. We refer the reader to the monograph [21] for necessary further background from Abstract Harmonic Analysis.

A function $u : G \to \mathbb{C}$ is called a *multiplier* of $A(G)$ if $uv \in A(G)$ for every $v \in A(G)$. We denote by $MA(G)$ the algebra of all multipliers of $A(G)$. An element $u \in MA(G)$ is called a *Herz–Schur multiplier* of $A(G)$ [6] if the map $v \to uv$ on $A(G)$ is completely bounded (here, and in the sequel, we equip $A(G)$ and $B(G)$ with the operator space structures, arising from the identifications $A(G)^* = \mathrm{VN}(G)$ and $B(G) = C^*(G)^*$). We let $M^{\mathrm{cb}}A(G)$ be the algebra of all Herz–Schur multipliers of $A(G)$. We note that $u \in M^{\mathrm{cb}}A(G)$ if and only if the map $S_u : C^*_r(G) \to C^*_r(G)$, $\lambda^0(f) \mapsto \lambda^0(uf)$, $f \in L^1(G)$, is completely bounded, which is proved using similar arguments to the ones in [6,Proposition 1.2].

We henceforth fix a Hilbert space H and a non-degenerate C*-algebra $A \subseteq \mathcal{B}(H)$. Let G be a discrete group and $\alpha : G \to \mathrm{Aut}(A)$ be a point-norm continuous homomorphism; thus, (A, G, α) is a C*-dynamical system. We write $\{\delta_s : s \in G\}$ for the canonical orthonormal basis of $\ell^2(G)$. We let $\ell^1(G, A)$ be the convolution *-algebra of all summable functions $f : G \to A$, set $\mathcal{H} := \ell^2(G) \otimes H$ and identify it with the Hilbert space $\ell^2(G, H)$ of all square summable H-valued functions on G. Let $\lambda : G \to \mathcal{B}(\mathcal{H}), t \to \lambda_t$, be the unitary representation of G given by

$$\lambda_t \xi(s) = \xi(t^{-1}s), \quad s, t \in G, \xi \in \mathcal{H};$$

note that $\lambda_t = \lambda_t^0 \otimes I$. Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be the *-representation given by

$$\pi(a)\xi(s) = \alpha_{s^{-1}}(a)(\xi(s)), \quad a \in A, \xi \in \mathcal{H}, s \in G.$$

We note the covariance relation

$$\pi(\alpha_t(a)) = \lambda_t \pi(a) \lambda_t^*, \quad a \in A, t \in G. \tag{1}$$

The pair (π, λ) gives rise to a *-representation $\tilde{\pi} : \ell^1(G, A) \to \mathcal{B}(\mathcal{H})$, given by

$$\tilde{\pi}(f) = \sum_{s \in G} \pi(f(s))\lambda_s, \quad f \in \ell^1(G, A). \tag{2}$$

(Note that the series on the right hand side of (2) converges in norm for every $f \in \ell^1(G, A)$). The reduced crossed product $A \rtimes_{\alpha,r} G$ is defined by letting

$$A \rtimes_{\alpha,r} G = \overline{\tilde{\pi}(\ell^1(G, A))},$$

where the closure is taken in the operator norm of $\mathcal{B}(\mathcal{H})$. Note that, after identifying A with $\pi(A)$, we may consider A as a C*-subalgebra of $A \rtimes_{\alpha,r} G$. It is well-known that if $\rho : A \to \mathcal{B}(K)$ is a faithful non-degenerate *-representation and α' is the canonical action of G on $\rho(A)$, arising from α, then $A \rtimes_{\alpha,r} G \cong \rho(A) \rtimes_{\alpha',r} G$ canonically (see e.g. [30,Theorem 7.7.5]).

Identifying $\mathcal{H}$ with $\oplus_{s \in G} H$, we associate to every operator $x \in \mathcal{B}(\mathcal{H})$ a matrix $(x_{p,q})_{p,q}$, where $x_{p,q} \in \mathcal{B}(H)$, $p, q \in G$; thus,

$$\langle x_{p,q}\xi, \eta \rangle = \langle x(\delta_q \otimes \xi), \delta_p \otimes \eta \rangle, \quad \xi, \eta \in H, p, q \in G.$$

In particular, if $a \in A$ and $t \in G$ then

$$(\pi(a)\lambda_t)_{p,q} = \begin{cases} \alpha_{p^{-1}}(a) & \text{if } pq^{-1} = t \\ 0 & \text{if } pq^{-1} \neq t. \end{cases} \tag{3}$$

Equation (3) implies that

$$(\pi(a)\lambda_t)_{e,qp^{-1}} = \delta_{t,pq^{-1}} a$$

(here $\delta_{s,r}$ is the Kronecker symbol). By linearity and continuity,

$$x_{p,q} = \alpha_{p^{-1}}(x_{e,qp^{-1}}), \quad x \in A \rtimes_{\alpha,r} G. \tag{4}$$

Let $\mathcal{E} : \mathcal{B}(\ell^2(G)) \otimes_{\min} A \to A$ be the (unital completely positive) map given by $\mathcal{E}(x) = x_{e,e}$; note that $\mathcal{E}(\pi(a)) = a$, $a \in A$. Equation (3) implies that $x_{e,r} = \mathcal{E}(x\lambda_r)$, $x \in A \rtimes_{\alpha,r} G$, $r \in G$. Now (4) can be rewritten as

$$x_{p,q} = \alpha_{p^{-1}}(\mathcal{E}(x\lambda_{qp^{-1}})), \quad x \in A \rtimes_{\alpha,r} G, \ \ p, q \in G. \tag{5}$$

To every operator $x \in A \rtimes_{\alpha,r} G$, one can associate the family $(a_s)_{s\in G} \subseteq A$, where $a_t = \mathcal{E}(x\lambda_{t^{-1}})$; we call $\sum_{t\in G} \pi(a_t)\lambda_t$ the *Fourier series* of x and write $x \sim \sum_{t\in G} \pi(a_t)\lambda_t$ (no convergence is assumed). Equation (5) shows that

$$x \sim \sum_{t\in G} \pi(a_t)\lambda_t \implies x_{p,q} = \alpha_{p^{-1}}(a_{pq^{-1}}). \tag{6}$$

Thus, if $x \in A \rtimes_{\alpha,r} G$ then the diagonal of its matrix coincides with the family $(\alpha_{r^{-1}}(\mathcal{E}(x)))_{r\in G}$.

We note that

$$\alpha_t(\mathcal{E}(x)) = \mathcal{E}(\lambda_t x \lambda_t^*), \quad t \in G, x \in A \rtimes_{\alpha,r} G. \tag{7}$$

The latter equality follows from (1) in the case where $x = \pi(a)\lambda_s$ and follows by linearity and continuity for a general $x \in A \rtimes_{\alpha,r} G$.

If $F : G \to \mathrm{CB}(A)$ is a bounded map and $f \in \ell^1(G, A)$, let $F \cdot f \in \ell^1(G, A)$ be the function given by

$$(F \cdot f)(t) = F(t)(f(t)), \quad t \in G.$$

Recall [25,Definition 3.1] that F is called a *Herz–Schur* (A, G, α)*-multiplier* if the map S_F, given by

$$S_F(\tilde{\pi}(f)) = \tilde{\pi}(F \cdot f), \quad f \in \ell^1(G, A), \tag{8}$$

is completely bounded. If F is a Herz–Schur (A, G, α)-multiplier, then S_F has a (unique) extension to a completely bounded map on $A \rtimes_{\alpha,r} G$, which will be denoted in the same way. We write $\mathfrak{S}(A, G, \alpha)$ for the space of all Herz–Schur (A, G, α)-multipliers, and set $\|F\|_{\mathrm{m}} = \|S_F\|_{\mathrm{cb}}$, $F \in \mathfrak{S}(A, G, \alpha)$. Note that, if $u \in M^{\mathrm{cb}}A(G)$ and $\tilde{u} : G \to \mathrm{CB}(A)$ is the function given by $\tilde{u}(t) = u(t)\mathrm{id}_A$, then $\tilde{u} \in \mathfrak{S}(A, G, \alpha)$ and its norm in $\mathfrak{S}(A, G, \alpha)$ coincides with its norm in $M^{\mathrm{cb}}A(G)$ [25,Remark 3.2 (ii)]. Without risk of confusion, for brevity we will denote by S_u the map $S_{\tilde{u}}$.

Let $\varphi : G \times G \to \mathrm{CB}(A, \mathcal{B}(H))$ be a bounded function and S_φ be the map from the space of all $G \times G$ A-valued matrices into the space of all $G \times G$ $\mathcal{B}(H)$-valued matrices, given by

$$S_\varphi\left((x_{p,q})_{p,q}\right) = \left(\varphi(q,p)(x_{p,q})\right)_{p,q}.$$

The map φ is called a *Schur A-multiplier* if S_φ is a completely bounded map from $\mathcal{B}(\ell^2(G)) \otimes_{\min} A$ into $\mathcal{B}(\mathcal{H})$. By the paragraph after [26,Theorem 2.1], this definition is equivalent, for a discrete group G, to the definition of Schur A-multipliers in [25,Definition 2.1]. For a Schur A-multiplier φ, we write $\|\varphi\|_{\mathrm{m}} = \|S_\varphi\|_{\mathrm{cb}}$. Schur $\mathbb{C}$-multipliers are referred to as *Schur multipliers*. For a bounded function $F : G \to \mathrm{CB}(A)$, let $\mathcal{N}(F) : G \times G \to \mathrm{CB}(A)$ be the function given by

$$\mathcal{N}(F)(s,t)(a) = \alpha_{t^{-1}}(F(ts^{-1})(\alpha_t(a))), \quad s,t \in G,\ a \in A.$$

In the case where $A = \mathbb{C}$, we write $N(u)$ in the place of $\mathcal{N}(u)$. The following statement [25,Theorem 3.18] is a crossed product version of a classical result of Bożejko and Fendler [3] in the case $A = \mathbb{C}$.

Theorem 2.1 *The map $\mathcal{N}$ is an isometric injection from $\mathfrak{S}(A,\alpha,G)$ into the algebra of Schur A-multipliers.*

Note that the image $\mathcal{N}(\mathfrak{S}(A,\alpha,G))$ coincides with the so-called invariant Schur-A-multipliers, denoted by $\mathfrak{S}_{\mathrm{inv}}(G,G,A)$ in [25,p. 413].

We remark that in [25] Herz–Schur (A,G,α)-multipliers and Schur A-multipliers were defined for general locally compact group G and separable C*-algebra A. But in the case of discrete G the separability condition on A can be removed without changing the statements from [25] (see the comment before [26,Theorem 2.1]).

The following observation will be frequently used: If G is discrete then $S_{\mathcal{N}(F)}|_{A\rtimes_{\alpha,r}G} = S_F$ for any $F \in \mathfrak{S}(A,\alpha,G)$. To see this, it suffices to show that

$$S_{\mathcal{N}(F)}(\pi(a)\lambda_s) = \pi(F(s)(a))\lambda_s, \quad a \in A, s \in G.$$

Write $\pi(a)\lambda_s = (x_{p,q})_{p,q}$ for $x_{p,q} = \delta_{s,pq^{-1}}\alpha_{p^{-1}}(a)$ (see (3)) and

$$\begin{aligned}(S_{\mathcal{N}(F)}(\pi(a)\lambda_s)_{p,q} &= \delta_{s,pq^{-1}}\mathcal{N}(F)(q,p)(\alpha_{p^{-1}}(a))\\ &= \delta_{s,pq^{-1}}\alpha_{p^{-1}}(F(pq^{-1})(a))\\ &= \delta_{s,pq^{-1}}\alpha_{p^{-1}}(F(s)(a)) = (\pi(F(s)(a))\lambda_s)_{p,q}.\end{aligned}$$

3 Characterisation of Complete Compactness

Let $\mathcal{X}$ and $\mathcal{Z}$ be operator spaces. A completely bounded map $\Psi : \mathcal{X} \to \mathcal{Z}$ is called *completely compact* if for very $\epsilon > 0$ there exists a finite dimensional subspace $\mathcal{Y} \subseteq \mathcal{Z}$ such that

$$\mathrm{dist}(\Psi^{(m)}(x), M_m(\mathcal{Y})) < \epsilon, \text{ for all } x \in M_m(\mathcal{X}), \|x\| \le 1, \text{ and all } m \in \mathbb{N}. \quad (9)$$

We denote by $\mathrm{CC}(\mathcal{X},\mathcal{Z})$ the space of all completely compact linear maps from $\mathcal{X}$ to $\mathcal{Z}$, and write $\mathrm{CC}(\mathcal{X})$ for $\mathrm{CC}(\mathcal{X},\mathcal{X})$. By $\mathrm{F}(\mathcal{X})$ we denote the subspace of all (linear)

maps of finite rank on $\mathcal{X}$. Clearly, $\mathrm{F}(\mathcal{X}) \subseteq \mathrm{CC}(\mathcal{X})$ and any completely compact map on $\mathcal{X}$ is compact.

A net $(\Phi_i)_{i\in\mathbb{I}} \subseteq \mathrm{CB}(\mathcal{X})$ is said to converge to $\Phi \in \mathrm{CB}(\mathcal{X})$ in the *strongly stable point norm topology* [9,p. 197] if

$$\|(\mathrm{id} \otimes \Phi_i)(x) - (\mathrm{id} \otimes \Phi)(x)\| \to_{i\in\mathbb{I}} 0, \quad x \in \mathcal{B}(K) \otimes_{\min} \mathcal{X},$$

for any Hilbert space K. An operator space $\mathcal{X}$ is said to possess the *strong operator approximation property (SOAP)* if the identity map on $\mathcal{X}$ can be approximated by finite rank maps in the strongly stable point norm topology. The notion was introduced by Effros and Ruan in [9] as an operator space analogue of the approximation property for Banach spaces. It admits a characterisation using complete compactness. Namely, $\mathcal{X}$ has the SOAP if for any operator space $\mathcal{Z}$ and any completely compact map $\Psi \in \mathrm{CC}(\mathcal{Z}, \mathcal{X})$, Ψ can be approximated by finite rank maps completely uniformly, i.e. in the $\|\cdot\|_{\mathrm{cb}}$-norm, see [37,Theorem 5.9] or [36,Proposition 4.4.6]. In particular, if $\mathcal{X}$ has the SOAP then $\overline{\mathrm{F}(\mathcal{X})}^{\mathrm{cb}} = \mathrm{CC}(\mathcal{X})$.

Lemma 3.1 *Let $\mathcal{X}$ be an operator space, $\Psi \in \mathrm{CC}(\mathcal{X})$, $\Phi \in \mathrm{CB}(\mathcal{X})$ and $(\Phi_i)_{i\in\mathbb{I}} \subseteq \mathrm{CB}(\mathcal{X})$ be a net such that $\Phi_i \to_{i\in\mathbb{I}} \Phi$ in the strongly stable point-norm topology. Then $\|\Phi_i \circ \Psi - \Phi \circ \Psi\|_{\mathrm{cb}} \to_{i\in\mathbb{I}} 0$.*

Proof By [37,Proposition 5.6] $\Phi_i \to_{i\in\mathbb{I}} \Phi$ completely uniformly on completely compact sets of $\mathcal{X}$ (see [37,Definitions 5.1 and 5.2] for the terminology). If Ψ is completely compact then $(\{\Psi^{(n)}(x) : x \in M_n(\mathcal{X}), \|x\| \le 1\})_{n\in\mathbb{N}}$ is a subset of a completely compact set and hence $\|\Phi_i \circ \Psi - \Phi \circ \Psi\|_{\mathrm{cb}} \to_{i\in\mathbb{I}} 0$. □

The following observation shows that, when studying compact or completely compact multipliers, the interest lies only in discrete groups.

Proposition 3.2 *Let G be a non-discrete locally compact group and $u : G \to \mathbb{C}$ be a Herz–Schur multiplier. If the map $S_u : C_r^*(G) \to C_r^*(G)$ is compact then $u = 0$.*

Proof Suppose that S_u is compact. Let $B_r(G)$ be the reduced Fourier-Stieltjes algebra of G. Identifying the dual $C_r^*(G)^*$ of $C_r^*(G)$ with $B_r(G)$, we have that the adjoint map $S_u^* : B_r(G) \to B_r(G)$ is compact (see e.g. [28,Theorem 1.4.4]). Note that $S_u^*(v) = uv$, $v \in B_r(G)$. Since $A(G) \subseteq B_r(G)$, $A(G)$ is the closed hull of its compactly supported elements and $A(G) \cap C_c(G) = B_r(G) \cap C_c(G)$ (see [21,Proposition 2.3.3]), the map S_u^* leaves $A(G)$ invariant. The restriction of S_u^* to $A(G)$ is thus compact, and by [22,Proposition 6.9], $u = 0$. □

We henceforth assume that the group G is discrete. Recall that $A \subseteq \mathcal{B}(H)$ is a fixed non-degenerate C*-algebra, equipped with an action $\alpha : G \to \mathrm{Aut}(A)$. It will be convenient to set

$$\mathcal{V}_{A,G,\alpha} = \mathrm{span}\,\{\pi(a)\lambda_s : a \in A, s \in G\};$$

thus, $\mathcal{V}_{A,G,\alpha}$ is a (dense) subalgebra of $A \rtimes_{\alpha,r} G$. We will write

$$\mathfrak{S}_{\mathrm{cc}}(A, G, \alpha) = \{F \in \mathfrak{S}(A, G, \alpha) : S_F \text{ is completely compact}\}.$$

If $\Phi : A \rtimes_{\alpha,r} G \to \mathcal{B}(\ell^2(G)) \otimes_{\min} A$ is a bounded linear map, we let $F_\Phi : G \to \mathcal{B}(A)$ be the function given by

$$F_\Phi(s)(a) = \mathcal{E}(\Phi(\pi(a)\lambda_s)\lambda_s^*). \tag{10}$$

Proposition 3.3 *Let $\Phi : A \rtimes_{\alpha,r} G \to \mathcal{B}(\ell^2(G)) \otimes_{\min} A$ be a completely bounded map.*

(i) *The map F_Φ is a Herz–Schur (A, G, α)-multiplier and $\|F_\Phi\|_{\mathrm{m}} \le \|\Phi\|_{\mathrm{cb}}$;*
(ii) *If $F \in \mathfrak{S}(A, G, \alpha)$ then $F_{S_F} = F$;*
(iii) *If Φ completely compact then $F_\Phi(s)$ is completely compact for every $s \in G$;*
(iv) *If Φ has finite rank and $\mathrm{ran}\Phi \subseteq \mathcal{V}_{A,G,\alpha}$ then $F_\Phi(s)$ has finite rank for every $s \in G$ and $S_{F_\Phi} \in \mathrm{F}(A \rtimes_{\alpha,r} G)$;*
(v) *If φ is a Schur A-multiplier and Ψ is the restriction of S_φ to $A \rtimes_{\alpha,r} G$ then $F_\Psi(s) = \varphi(s^{-1}, e)$, $s \in G$.*

Proof (i) By the Haagerup-Paulsen-Wittstock Theorem (see e.g. [29,Theorem 8.4]), there exist a Hilbert space K, operators $V, W : H \to K$ and a *-representation $\rho : A \rtimes_{\alpha,r} G \to \mathcal{B}(K)$ such that

$$\Phi(x) = W^*\rho(x)V, \quad x \in A \rtimes_{\alpha,r} G. \tag{11}$$

Using (7), we have

$$\begin{aligned}
\mathcal{N}(F_\Phi)(s,t)(a) &= \alpha_{t^{-1}}\left(F_\Phi(ts^{-1})(\alpha_t(a))\right) \\
&= \alpha_{t^{-1}}\left(\mathcal{E}(\Phi(\pi(\alpha_t(a))\lambda_{ts^{-1}})\lambda_{ts^{-1}}^*)\right) \\
&= \alpha_{t^{-1}}\left(\mathcal{E}(\Phi(\lambda_t\pi(a)\lambda_{t^{-1}}\lambda_{ts^{-1}})\lambda_{ts^{-1}}^*)\right) \\
&= \mathcal{E}\left(\lambda_{t^{-1}}\Phi((\lambda_t\pi(a)\lambda_{s^{-1}})\lambda_s\lambda_{t^{-1}})\lambda_t\right) \\
&= \mathcal{E}\left(\lambda_{t^{-1}}\Phi(\lambda_t\pi(a)\lambda_{s^{-1}})\lambda_s\right) \\
&= \mathcal{E}\left(\lambda_t^*W^*\rho(\lambda_t)\rho(\pi(a))\rho(\lambda_s)^*V\lambda_s\right).
\end{aligned} \tag{12}$$

Note that, if $E_{p,q}$ denotes the matrix unit in $\mathcal{B}(\ell^2(G, H))$ with I at the (p, q)-entry, $p, q \in G$, and zero elsewhere, then $\mathcal{E}(x) = E_{e,e}xE_{e,e}$, $x \in A \rtimes_{r,\alpha} G$. Let $\tilde{V}(s) = \rho(\lambda_s)^*V\lambda_sE_{e,e}$ and $\tilde{W}(t) = \rho(\lambda_t)^*W\lambda_tE_{e,e}$, $s, t \in G$; thus, $\tilde{V}, \tilde{W} : G \to \mathcal{B}(H, K)$, and

$$\sup_{s\in G}\|\tilde{V}(s)\| \sup_{t\in G}\|\tilde{W}(t)\| \le \|V\|\|W\|. \tag{13}$$

By (12),

$$\mathcal{N}(F_\Phi)(s,t)(a) = \tilde{W}(t)^*(\rho\circ\pi)(a)\tilde{V}(s), \quad a \in A, s, t \in G. \tag{14}$$

By [25,Theorems 2.6 and 3.8], F_Φ is a Herz–Schur (A, G, α)-multiplier. Equations (13), (14) and Theorem 2.1 show that

$$\|F_\Phi\|_{\mathrm{m}} = \|S_{\mathcal{N}(F_\Phi)}\|_{\mathrm{cb}} \leq \|V\|\|W\|.$$

Taking the infimum over all possible choices of V and W in the representation (11) of Φ, we obtain that $\|F_\Phi\|_{\mathrm{m}} \leq \|\Phi\|_{\mathrm{cb}}$.

(ii) Let $F \in \mathfrak{S}(A, G, \alpha)$. The fact that $F_{S_F} = F$ follows from the definition (8) of S_F and the definition (10) of F_Φ.

(iii) follows from the definition (10) of F_Φ and the fact that $\mathcal{E}$ is completely bounded.

(iv) Since Φ has finite rank, there exists a finite subset $E \subseteq G$ and a finite dimensional subspace $\mathcal{U} \subseteq A$, $s \in E$, such that

$$\mathrm{ran}\,\Phi \subseteq \mathrm{span}\{\pi(a)\lambda_s : s \in E, a \in \mathcal{U}\}.$$

A direct verification shows that

$$\mathrm{ran}\,F_\Phi(s) \subseteq \begin{cases} \mathcal{U} & \text{if } s \in E, \\ \{0\} & \text{if } s \notin E. \end{cases}$$

Clearly, $\mathrm{ran}S_{F_\Phi} \subseteq \mathrm{span}\{\pi(a)\lambda_s : s \in E, a \in \mathcal{U}\}$.

(v) According to (3),

$$\left(\Psi(\pi(a)\lambda_s)\lambda_{s^{-1}}\right)_{p,q} = (S_\varphi(\pi(a)\lambda_s)_{p,s^{-1}q} = \begin{cases} \varphi(s^{-1}q, p)(\alpha_{p^{-1}}(a)) & \text{if } p = q \\ 0 & \text{if } p \neq q. \end{cases}$$

Thus,

$$F_\Psi(s)(a) = \left(\Psi(\pi(a)\lambda_s)\lambda_{s^{-1}}\right)_{e,e} = \varphi(s^{-1}, e)(a), \quad a \in A. \qquad \square$$

Proposition 3.3 (ii) and (iii) give the following immediate corollary.

Corollary 3.4 *If $F \in \mathfrak{S}_{\mathrm{cc}}(A, G, \alpha)$ then $F(s)$ is completely compact for every $s \in G$;*

We will call a (possibly vector-valued) function φ defined on $G \times G$ *band finite* if there exists a finite set $E \subseteq G$ such that $\varphi(s, t) = 0$ if $ts^{-1} \notin E$. Let $V : \ell^2(G)\otimes H \to \ell^2(G) \otimes \ell^2(G) \otimes H$ be the isometry given by

$$V(\delta_s \otimes \xi) = \delta_s \otimes \delta_s \otimes \xi, \quad s \in G, \xi \in H,$$

and $\tau : A \rtimes_{r,\alpha} G \to C_r^*(G) \otimes_{\min} (A \rtimes_{r,\alpha} G)$ be the dual co-action to α, that is, the *-homomorphism, given by

$$\tau\,(\pi(a)\lambda_t) = \lambda_t^0 \otimes \pi(a)\lambda_t, \quad a \in A, t \in G,$$

(see e.g. [20]).

Lemma 3.5 *The following hold:*

(i) $V^*\tau(x)V = x$, $x \in A \rtimes_{r,\alpha} G$;
(ii) *If* $\Phi \in \mathrm{CB}(A \rtimes_{r,\alpha} G)$ *then*

$$S_{F_\Phi}(x) = V^*(\mathrm{id} \otimes \Phi)(\tau(x))V, \quad x \in A \rtimes_{r,\alpha} G. \tag{15}$$

Proof (i) If $a \in A$, $s, r, t \in G$ and $\xi, \eta \in H$ then

$$\begin{aligned}
&\left\langle V^*\tau(\pi(a)\lambda_t)V(\delta_s \otimes \xi), \delta_r \otimes \eta \right\rangle \\
&\quad = \langle (\lambda_t \otimes \pi(a)\lambda_t)(\delta_s \otimes \delta_s \otimes \xi), \delta_r \otimes \delta_r \otimes \eta \rangle \\
&\quad = \langle \delta_{ts} \otimes \pi(a)\lambda_t(\delta_s \otimes \xi), \delta_r \otimes \delta_r \otimes \eta \rangle = \langle \pi(a)\lambda_t(\delta_s \otimes \xi), \delta_r \otimes \eta \rangle ,
\end{aligned}$$

by (3).

(ii) Let $a \in A$, $s, r, t \in G$ and $\xi, \eta \in H$. Using (3) and (5), we have

$$\begin{aligned}
&\langle V^*(\mathrm{id} \otimes \Phi)(\tau(\pi(a)\lambda_t))V(\delta_s \otimes \xi), \delta_r \otimes \eta \rangle \\
&\qquad = \langle (\mathrm{id} \otimes \Phi)(\lambda_t^0 \otimes \pi(a)\lambda_t)(\delta_s \otimes \delta_s \otimes \xi), \delta_r \otimes \delta_r \otimes \eta \rangle \\
&\qquad = \langle \left(\lambda_t^0 \otimes \Phi(\pi(a)\lambda_t)\right)(\delta_s \otimes \delta_s \otimes \xi), \delta_r \otimes \delta_r \otimes \eta \rangle \\
&\qquad = \langle \delta_{ts} \otimes \Phi(\pi(a)\lambda_t)(\delta_s \otimes \xi), \delta_r \otimes \delta_r \otimes \eta \rangle \\
&\qquad = \begin{cases} 0 & \text{if } ts \neq r, \\ \langle \Phi(\pi(a)\lambda_t)(\delta_s \otimes \xi), \delta_r \otimes \eta \rangle & \text{if } ts = r. \end{cases} \\
&\qquad = \begin{cases} 0 & \text{if } ts \neq r, \\ \left\langle \alpha_{(ts)^{-1}}\left(\mathcal{E}(\Phi(\pi(a)\lambda_t)\lambda_t^*)\right)\xi, \eta \right\rangle & \text{if } ts = r. \end{cases} \\
&\qquad = \left\langle \pi\left(\mathcal{E}(\Phi(\pi(a)\lambda_t)\lambda_t^*)\right)(\delta_{ts} \otimes \xi), \delta_r \otimes \eta \right\rangle \\
&\qquad = \left\langle S_{F_\Phi}(\pi(a)\lambda_t)(\delta_s \otimes \xi), \delta_r \otimes \eta \right\rangle .
\end{aligned}$$

□

Lemma 3.6 *Let G be a discrete group and (A, G, α) be a C*-dynamical system such that $A \rtimes_{r,\alpha} G$ has the SOAP. Then there exists a net $(F_i)_{i\in\mathbb{I}}$ of finitely supported Herz–Schur (A, G, α)-multipliers with $F_i(s) \in \mathrm{F}(A)$, $i \in \mathbb{I}$, $s \in G$, such that $(S_{F_i})_{i\in\mathbb{I}}$ converges to the identity map on $A \rtimes_{r,\alpha} G$ in the strongly stable point norm topology.*

Proof Write $\mathfrak{A} = A \rtimes_{r,\alpha} G$ and let $(\Phi_i)_{i\in\mathbb{I}} \subseteq \mathrm{F}(\mathfrak{A})$ be a net such that

$$\|(\mathrm{id} \otimes \Phi_i)(x) - x\| \to_{i\in\mathbb{I}} 0, \quad x \in \mathcal{B}(\ell^2) \otimes_{\min} \mathfrak{A}.$$

Following the proof of [24,Theorem 4.3], given $\varepsilon > 0$, for each $i \in \mathbb{I}$ there exists a finite rank map $\Phi_{i,\varepsilon} \in \mathrm{CB}(\mathfrak{A})$ whose range lies in $\mathcal{V}_{A,G,\alpha}$, such that

$$\|\Phi_i - \Phi_{i,\varepsilon}\|_{\mathrm{cb}} < \varepsilon, \quad i \in \mathbb{I}.$$

Hence

$$\|(\mathrm{id}\otimes\Phi_{i,\varepsilon})(x)-x\|\to_{(i,\epsilon)}0,\quad x\in\mathcal{B}(H)\otimes_{\min}\mathfrak{A},$$

along the product directed set.

The function $F_{i,\varepsilon}:=F_{\Phi_{i,\varepsilon}}$ is finitely supported. By Proposition 3.3 (i) and (iv), $F_{i,\varepsilon}$ is a Herz–Schur multiplier and $F_{i,\varepsilon}(s)\in\mathrm{F}(A)$, $s\in G$, $i\in\mathbb{I}$. If $y\in\mathcal{B}(H)\otimes_{\min}\mathfrak{A}$, using Lemma 3.5 we have

$$\begin{aligned}\|\mathrm{id}\otimes S_{F_{i,\varepsilon}}(y)-y\| &= \|(\mathrm{id}\otimes V^*)(\mathrm{id}\otimes\mathrm{id}\otimes\Phi_{i,\varepsilon}(\mathrm{id}\otimes\tau(y))(\mathrm{id}\otimes V)\\ &\quad-(\mathrm{id}\otimes V^*)(\mathrm{id}\otimes\tau(y)(\mathrm{id}\otimes V)\|\\ &\le \|(\mathrm{id}\otimes\mathrm{id}\otimes\Phi_{i,\varepsilon})((\mathrm{id}\otimes\tau)(y))-(\mathrm{id}\otimes\tau)(y)\|\to_{(i,\epsilon)}0,\end{aligned}$$

that is, $S_{F_{i,\varepsilon}}$ converges to the identity map in the strongly stable point norm topology. □

Theorem 3.7 *Let F be a Herz–Schur multiplier of (A,G,α). Assume that $A\rtimes_{r,\alpha}G$ possesses the SOAP. The following are equivalent:*

(i) *the map S_F is completely compact;*
(ii) *there exist a net $(F_i)_{i\in\mathbb{I}}\subseteq\mathfrak{S}(A,G,\alpha)$ and a finite dimensional subspace $\mathcal{U}_i\subseteq A$, such that F_i is finitely supported, $\mathrm{ran}F_i(s)\subseteq\mathcal{U}_i$ for each $i\in\mathbb{I}$ and each $s\in G$, and $\|F_i-F\|_{\mathrm{m}}\to_{i\in\mathbb{I}}0$;*
(iii) *there exists a net $(\varphi_i)_{i\in\mathbb{I}}$ of band finite Schur A-multipliers and a finite dimensional subspace $\mathcal{U}_i\subseteq A$, $i\in\mathbb{I}$, such that $\mathrm{ran}\alpha_s\circ\varphi_i(s,e)\subseteq\mathcal{U}_i$ for all $i\in\mathbb{I}$ and all $s\in G$, and $\|S_{\mathcal{N}(F)}-S_{\varphi_i}\|_{\mathrm{cb}}\to_{i\in\mathbb{I}}0$.*

Proof (i)⇒(ii) Let $(\tilde{F}_i)_{i\in\mathbb{I}}$ be a net as in Lemma 3.6 and $F_i=\tilde{F}_i\circ F$, $i\in\mathbb{I}$. By Lemma 3.1, $(F_i)_{i\in\mathbb{I}}$ satisfies the conditions of (ii).

(ii)⇒(iii) Set $\varphi_i=\mathcal{N}(F_i)$; by Theorem 2.1, φ_i is a Schur A-multiplier, $i\in\mathbb{I}$. Since F_i is finitely supported, φ_i is band finite. Moreover,

$$\varphi_i(s,t)(a)=\alpha_{t^{-1}}(F_i(ts^{-1})(\alpha_t(a))),\quad a\in A,$$

and hence

$$\mathrm{ran}\alpha_s\circ\varphi_i(s,e)\subseteq\mathrm{ran}F_i(s^{-1})\subseteq\mathcal{U}_i,\quad s\in G.$$

(iii)⇒(i) Let Φ_i be the restriction of the map S_{φ_i} to $A\rtimes_{r,\alpha}G$, $i\in\mathbb{I}$. By Proposition 3.3 (i) and (ii), $\|F-F_{\Phi_i}\|_{\mathrm{m}}\to_{i\in\mathbb{I}}0$. Since φ_i is band finite, $F_i:=F_{\Phi_i}$ is supported on a finite set, say $E_i\subseteq G$. By Proposition 3.3 (v),

$$\mathrm{ran}F_i(s)\subseteq\mathrm{ran}\varphi_i(s^{-1},e)\subseteq\alpha_s(\mathcal{U}_i),\quad s\in G.$$

Let $\mathcal{V}_i=\mathrm{span}\left(\cup_{s\in E_i}\alpha_s(\mathcal{U}_i)\right)$; then $\mathcal{V}_i$ is finite dimensional and

$$\mathrm{ran}S_{F_i}\subseteq\mathrm{span}\{\pi(a)\lambda_s:a\in\mathcal{V}_i,s\in E_i\}.$$

Thus, S_{F_i} has finite rank, $i \in \mathbb{I}$. Since the set $\mathrm{CC}(A \rtimes_{r,\alpha} G)$ is closed in the completely bounded norm, $S_F \in \mathfrak{S}_{\mathrm{cc}}(A \rtimes_{r,\alpha} G)$. □

Corollary 3.8 *If $A \rtimes_{r,\alpha} G$ possesses the SOAP then every completely compact Herz–Schur (A, G, α)-multiplier is a limit of (A, G, α)-multipliers of finite rank.*

Remark By [37,Theorem 5.9], any completely compact map on a C^*-algebra with SOAP is a limit of finite rank maps. Corollary 3.8 is a refinement of this statement.

Let $A_{\mathrm{cb}}(G)$ be the closure of $A(G)$ within $M^{\mathrm{cb}}A(G)$ with respect to $\|\cdot\|_{\mathrm{m}}$. The algebra was first introduced and studied in [13]. It is a regular commutative Tauberian Banach algebra whose Gelfand space can be canonically identified with G (see [14,Proposition 2.2]).

It is known that $M^{\mathrm{cb}}A(G)$ is a dual Banach space [6,Proposition 1.10]. Recall that G has the *approximation property (AP)* [17] if there exists a net $(u_i)_{i\in\mathbb{I}}$ of finitely supported Herz–Schur multipliers such that $u_i \to_{i\in\mathbb{I}} 1$ in the weak* topology of $M^{\mathrm{cb}}A(G)$. By [17,Theorem 1.9] (see also [5,Theorem 12.4.9], G has (AP) if and only if $C_r^*(G)$ has SOAP. We note that, by [33,Theorem 3.6], $A \rtimes_{r,\alpha} G$ has the SOAP if A has the SOAP and G has the AP. The SOAP for $A \rtimes_{r,\alpha} G$ was also recently studied in [27].

Theorem 3.7 has the following immediate consequence.

Corollary 3.9 *Let G be a discrete group possessing property (AP) and $u : G \to \mathbb{C}$ be a Herz–Schur multiplier. The following are equivalent:*

(i) *the map $S_u : C_r^*(G) \to C_r^*(G)$ is completely compact;*
(ii) *there exists a net $(u_i)_{i\in\mathbb{I}}$ of finitely supported elements of $A(G)$ such that $\|S_u - S_{u_i}\|_{\mathrm{cb}} \to_{i\in\mathbb{I}} 0$;*
(iii) *there exists a net $(\varphi_i)_{i\in\mathbb{N}}$ of band finite Schur multipliers such that $\|S_{N(u)} - S_{\varphi_i}\| \to_{i\in\mathbb{I}} 0$;*
(iv) $u \in A_{\mathrm{cb}}(G)$.

Proof The equivalences (i)⇔(ii)⇔(iii) follow from Theorem 3.7. The equivalence (ii)⇔(iv) follows from the facts that the finitely supported functions in $A(G)$ form a dense set in $A(G)$ and $\|S_u\|_{\mathrm{cb}} = \|u\|_{\mathrm{m}}$. □

Remark 3.10 Let G be a discrete group possessing (AP) and $u \in MA(G)$. The map S_u is compact if and only if $u \in A_M(G) := \overline{A(G)}^{\|\cdot\|_{MA(G)}}$.

Proof By [17,Theorem 1.9], there exists a net $(u_i)_{i\in\mathbb{I}}$ consisting of finitely supported functions such that $S_{u_i}(x) \to_{i\in\mathbb{I}} x$, $x \in C_r^*(G)$. If S_u is compact then $S_{u_i}S_u \to_{i\in\mathbb{I}} S_u$ in norm; thus, $\|u - uu_i\|_{MA(G)} \to_{i\in\mathbb{I}} 0$.

Conversely, suppose that $u \in A_M(G)$. Since $\|\cdot\|_{MA(G)} \le \|\cdot\|_{A(G)}$, there exists a net $(u_i)_{i\in\mathbb{I}}$ of finitely supported functions such that $\|u_i - u\|_{MA(G)} \to_{i\in\mathbb{I}} 0$. Thus, $\|S_{u_i} - S_u\| \to_{i\in\mathbb{I}} 0$; since S_{u_i} has finite rank, $i \in \mathbb{I}$, we have that S_u is compact. □

If the group G is amenable then $B(G) = M^{\mathrm{cb}}A(G) = MA(G)$ and the norms on these three spaces coincide, ([6,Corollary 1.8 (ii)]). It was shown in [22,Proposition 6.10] that, in this case, the map S_u is compact precisely when $u \in A(G)$. By Corollary 3.9, automatic complete compactness holds:

Corollary 3.11 *Let $u : G \to \mathbb{C}$ be a Herz–Schur multiplier. If G is a discrete amenable group then the following are equivalent:*

(i) *the map S_u is completely compact;*
(ii) $u \in A(G)$.

Proposition 3.12 *Let G be a discrete group containing the free group F_∞ on infinitely many generators. If G has the (AP), then there exists a multiplier $u \in MA(G)$, for which the map S_u is compact but not completely compact.*

Proof Write $H = F_\infty$, considering it as a subgroup of G. We note first that if $\varphi \in A(H)$ and $\tilde{\varphi}$ is the extension by zero of φ to G, then $\|\varphi\|_{A(H)} = \|\tilde{\varphi}\|_{A(G)}$. In fact, we have $\varphi(s) = (\lambda_H(s)\xi, \eta)$ for some $\xi, \eta \in \ell^2(H)$ such that $\|\varphi\|_{A(H)} = \|\xi\|_2\|\eta\|_2$. Considering $\ell^2(H)$ as a subspace of $\ell^2(G)$ and letting $\tilde{\xi}$ and $\tilde{\eta}$ be the extensions by zero to $\ell^2(G)$ of ξ and η, respectively, we have that $\tilde{\varphi}(s) = (\lambda_G(s)\tilde{\xi}, \tilde{\eta})$ and hence $\|\varphi\|_{A(G)} \leq \|\tilde{\xi}\|_2\|\tilde{\eta}\|_2 = \|\varphi\|_{A(H)}$. As the restriction map $r : A(G) \to A(H)$, $u \mapsto u|_H$, is contractive [12,Proposition 3.21], we have also $\|\varphi\|_{A(H)} \leq \|\tilde{\varphi}\|_{A(G)}$, giving $\|\varphi\|_{A(H)} = \|\tilde{\varphi}\|_{A(G)}$.

In the proof of [2,Theorem 2], Bożejko constructed functions $\varphi_n \in A(H)$ with finite supports $E_n \subseteq H \subseteq G$ such that $\|\varphi_n\|_{MA(H)} = 1$ but $\|\varphi_n\|_{M^{\mathrm{cb}}A(H)} \geq C\sqrt{n}$, for some constant $C > 0$. Given $u \in A(G)$, we now have

$$\|\varphi_n u\|_{A(G)} = \|\varphi_n u|_H\|_{A(H)} \leq \|u|_H\|_{A(H)} \leq \|u\|_{A(G)};$$

for the last inequality we use the contractability of r. Thus, $\|\varphi_n\|_{MA(G)} \leq 1$. Let $\tilde{\varphi}_n$ be the extension by zero of φ_n to a function on G, $n \in \mathbb{N}$, and $\psi_n = N(\tilde{\varphi}_n)$. By [3], ψ_n is a Schur multiplier, $n \in \mathbb{N}$, and

$$\|\varphi_n\|_{M^{\mathrm{cb}}A(G)} = \|S_{\psi_n}\|_{\mathrm{cb}} \geq \|S_{\psi_n \chi_{H\otimes H}}\|_{\mathrm{cb}} = \|\varphi_n\|_{M^{\mathrm{cb}}A(H)} \geq C\sqrt{n}.$$

This shows that the norms $\|\cdot\|_{MA(G)}$ and $\|\cdot\|_{M^{\mathrm{cb}}A(G)}$ are not equivalent on $A(G)$. As the completely bounded norm dominates the multiplier norm, by applying the open mapping theorem one obtains that $A_{\mathrm{cb}}(G) \neq A_M(G)$. By Corollary 3.9 and Remark 3.10, there exists a compact multiplier which is not completely compact. □

We remark that, for the free group $\mathbb{F}_2$ on two generators, the inequality $A_{\mathrm{cb}}(\mathbb{F}_2) \neq A_M(\mathbb{F}_2)$ was proved in [4,Proposition 4.2].

4 Subclasses of Completely Compact Herz–Schur Multipliers

In this section, we exhibit some canonical ways to construct completely compact Herz–Schur multipliers of dynamical systems, and describe them explicitly in an important special case. We assume throughout the section that (A, G, α) is a C*-dynamical system, where G is a discrete group. A linear map $T : A \to A$ will be called α-*invariant* if

$$\alpha_t \circ T = T \circ \alpha_t, \quad t \in G. \tag{16}$$

Note that, if $T \in \mathrm{CB}(A)$ is α-invariant then

$$(T \otimes \mathrm{id}_{\mathcal{B}(\ell^2(G))})(\pi(a)) = \pi(T(a)), \quad a \in A.$$

It follows that the map $T \otimes \mathrm{id}_{\mathcal{B}(\ell^2(G))} \in \mathrm{CB}(\mathcal{B}(\mathcal{H}))$ leaves $A \rtimes_{\alpha,r} G$ invariant, and hence its restriction to $A \rtimes_{\alpha,r} G$, which will be denoted by $\tilde{T}$, is well-defined. We have that

$$\tilde{T}(\pi(a)\lambda_t) = \pi(T(a))\lambda_t, \quad a \in A, t \in G. \tag{17}$$

Thus, letting $F_T : G \to \mathrm{CB}(A)$ be the function given by $F_T(s) = T$, $s \in G$, we have that $\tilde{T} = S_{F_T}$, and that $F_T(s)$ is α-invariant for every $s \in G$. This leads us to introducing the following classes of Herz–Schur multipliers:

$$\mathfrak{S}^{\mathrm{inv}}(A, G, \alpha) = \{F \in \mathfrak{S}(A, G, \alpha) : F(s) \text{ is } \alpha-invariant for every s \in G\},$$

and

$$\mathfrak{S}^{\mathrm{inv}}_{\mathrm{cc}}(A, G, \alpha) = \mathfrak{S}^{\mathrm{inv}}(A, G, \alpha) \cap \mathfrak{S}_{\mathrm{cc}}(A, G, \alpha).$$

We remark that $\mathfrak{S}^{\mathrm{inv}}(A, G, \alpha)$ should not be confused with $\mathfrak{S}_{\mathrm{inv}}(G, G, A)$ which appears in [25] and denotes the so-called invariant Schur-A-multipliers (see the remark after Theorem 2.1); while invariance is a property of the map $\varphi : G \times G \to \mathrm{CB}(A)$, α-invariance is a property of the maps $F(s)$, $s \in G$.

We will denote by $Z(A)$ the centre of A, and by $Z(A)^+$ the cone of its positive elements. Recall that the action of a discrete group G on A is *amenable* [5,Definition 4.3.1] if there exists a net $(\xi_i)_{i\in\mathbb{I}}$ of finitely supported functions $\xi_i : G \to Z(A)^+$, such that $\sum_{i\in\mathbb{I}} \xi_i(t)^2 = 1_A$ and

$$\left\| \sum_{s\in G} \left(\xi_i(s) - \alpha_t(\xi_i(t^{-1}s)) \right)^2 \right\| \to_{i\in\mathbb{I}} 0$$

for every $t \in G$.

Theorem 4.1 *Let α be an amenable action of a discrete group G on a unital C*-algebra A. Let $u \in M^{\mathrm{cb}}A(G)$, $T \in \mathrm{CB}(A)$ be α-invariant and $F_{T,u} : G \to \mathrm{CB}(A)$ be given by $F_{T,u}(t)(a) = u(t)T(a)$, $a \in A$, $t \in G$. The following hold:*

(i) $F_{T,u} \in \mathfrak{S}^{\mathrm{inv}}(A, G, \alpha)$;
(ii) $F_{T,u} \in \mathfrak{S}^{\mathrm{inv}}_{\mathrm{cc}}(A, G, \alpha)$ *if* $u \in A_{\mathrm{cb}}(G)$ *and* $T \in \mathrm{CC}(A)$;
(iii) $F_{T,u} \in \overline{\mathrm{F}(A \rtimes_{\alpha,r} G) \cap \mathfrak{S}(A, G, \alpha)}^{\mathrm{cb}}$ *if* $u \in A_{\mathrm{cb}}(G)$ *and* $T \in \overline{\mathrm{F}(A)}^{\mathrm{cb}}$.

Proof (i) Let $\tilde{u} : G \to \mathrm{CB}(A)$ be given by $\tilde{u}(t) = u(t)\mathrm{id}_A$. The map $S_{\tilde{u}}$ is the restriction to $A \rtimes_{\alpha,r} G$ of the map $\mathrm{id}_{\mathcal{B}(H)} \otimes S_u \in \mathrm{CB}(\mathcal{B}(H) \otimes_{\min} C^*_r(G))$ and hence $\tilde{u} \in \mathfrak{S}(A, G, \alpha)$. Since $\tilde{T} = S_{F_T}$ and $F_T \in \mathfrak{S}^{\mathrm{inv}}(A, G, \alpha)$, we conclude that

$$F_{T,u} = \tilde{T} \circ S_{\tilde{u}} \in \mathfrak{S}^{\mathrm{inv}}(A, G, \alpha).$$

(ii) Since the space of finitely supported functions in $A(G)$ is dense in $A(G)$ [12] and $\|\cdot\|_{\mathrm{m}} \leq \|\cdot\|_A$, given $\varepsilon > 0$, there exists $v \in A(G)$ supported in a finite set $E \subseteq G$ such that $\|u - v\|_{\mathrm{m}} < \varepsilon/\|T\|_{\mathrm{cb}}$. Let $\mathcal{F} \subseteq A$ be a finite dimensional space such that $\mathrm{dist}(T^{(n)}(a), M_n(\mathcal{F})) < \varepsilon/|E|$ for every $a \in M_n(A)$, $\|a\| \leq 1$, and every $n \in \mathbb{N}$.

Set

$$\mathcal{Y} = \left\{ \sum_{s\in E} \pi(a_s)\lambda_s : a_s \in \mathcal{F} \right\} \text{ and } \mathcal{Z} = \left\{ \sum_{s\in E} \pi(a_s)\lambda_s : a_s \in A \right\};$$

we have that $\mathcal{Y}$ is a finite-dimensional subspace of $\mathcal{Z}$. Let $x = [x_{k,l}] \in M_n(\mathcal{Z})$, $\|x\| \leq 1$, and write $L_s = \mathrm{diag}(\lambda_s, \ldots, \lambda_s) \in M_n(\mathcal{Z})$. Then

$$[x_{k,l}] = \sum_{s\in E} (\pi \circ \mathcal{E})^{(n)} \left([x_{k,l}] L_{s^{-1}} \right) L_s;$$

as $\mathcal{E}$ is completely contractive, $\left\| \mathcal{E}^{(n)} \left([x_{k,l}] L_{s^{-1}} \right) \right\| \leq 1$. For each $s \in E$, choose $y_s = [y^s_{k,l}] \in M_n(\mathcal{F})$ such that

$$\left\| T^{(n)} \left(\mathcal{E}^{(n)} \left([x_{k,l}] L_{s^{-1}} \right) \right) - y_s \right\| < \varepsilon/|E| \tag{18}$$

and set $y = \sum_{s\in E} \pi^{(n)}(y_s) L_s \in M_n(\mathcal{Y})$. By (18),

$$\left\| \tilde{T}^{(n)} \left([x_{k,l}] \right) - y \right\| = \left\| \sum_{s\in E} \left[T \left(\mathcal{E} \left(x_{k,l} \lambda_{s^{-1}} \right) \right) \lambda_s - y^s_{k,l} \lambda_s \right] \right\| \leq \varepsilon.$$

Thus, $\tilde{T}|_{\mathcal{Z}}$ is completely compact. Since the image of S_v is in $\mathcal{Z}$ we obtain that $\tilde{T} \circ S_v$ is completely compact. The statement now follows from the inequalities

$$\|\tilde{T} \circ S_v - \tilde{T} \circ S_u\|_{\mathrm{cb}} \leq \|\tilde{T}\|_{\mathrm{cb}} \|S_v - S_u\|_{\mathrm{cb}} = \|\tilde{T}\|_{\mathrm{cb}} \|u - v\|_{\mathrm{m}} \leq \varepsilon$$

and the fact that the space of completely compact maps is closed with respect to the completely bounded norm.

(iii) Let $(T_k)_{k\in\mathbb{N}} \subseteq \mathrm{F}(A)$ be a sequence such that $\|T_k - T\|_{\mathrm{cb}} \to_{k\to\infty} 0$. We follow the idea in the proof of [26,Corollary 4.6]. Let $(\xi_i)_{i\in I}$ be a net as in definition of amenable action and the maps $F_{i,k}(s) : A \to A$, $s \in G$, $i \in \mathbb{I}$, $k \in \mathbb{N}$, be given by

$$F_{i,k}(s)(a) = \sum_{q\in G} \xi_i(q) \alpha_q(T_k(\alpha_q^{-1}(a)) \alpha_s(\xi_i(s^{-1}q)), \quad a \in A. \tag{19}$$

Note that if $E_i \subseteq G$ is a finite set with $\mathrm{supp}\xi_i \subseteq E_i$ then the sum on the right hand side of (19) is over the (finite) set $E_i \cap sE_i$, for every $k \in \mathbb{N}$. We have

$$\begin{aligned}\mathcal{N}(F_{i,k})(s,t)(a) &= \alpha_{t^{-1}}\left(\sum_{q\in G}\xi_i(q)\alpha_q(T_k(\alpha_q^{-1}(\alpha_t(a)))\alpha_{ts^{-1}}(\xi_i(st^{-1}q))\right)\\ &= \sum_{q\in G}\alpha_{t^{-1}}(\xi_i(q))\alpha_{t^{-1}q}(T_k(\alpha_{q^{-1}t}(a)))\alpha_{s^{-1}}(\xi_i(st^{-1}q))\\ &= \sum_{p\in G}\alpha_{t^{-1}}(\xi_i(tp))\alpha_p(T_k(\alpha_{p^{-1}}(a)))\alpha_{s^{-1}}(\xi_i(sp)).\end{aligned}$$

Since $T_k : A \to A$ is a completely bounded map, by Haagerup-Paulsen-Wittstock Theorem, there exist a Hilbert space $H_{p,k}$, a *-representation $\pi_{p,k} : A \to \mathcal{B}(H_{p,k})$ and bounded operators $V_{p,k}, W_{p,k} : H \to H_{p,k}$, such that

$$\alpha_p(T_k(\alpha_p^{-1}(a))) = W_{p,k}^*\pi_{p,k}(a)V_{p,k}, \quad a \in A,$$

and $\|T_k\|_{\mathrm{cb}} = \|\alpha_p \circ T_k \circ \alpha_p^{-1}\|_{\mathrm{cb}} = \|V_{p,k}\|\|W_{p,k}\|$. Renormalising we may assume that $\|V_{p,k}\| = \|W_{p,k}\| = \|T_k\|_{\mathrm{cb}}^{1/2}$ for all p.

Set $\rho_k := \oplus_{p\in G}\pi_{p,k}$ and let $\mathbf{V}_{i,k}(s) : H \to \oplus_{p\in G}H_{p,k}$ and $\mathbf{W}_{i,k}(t) : H \to \oplus_{p\in G}H_{p,k}$ be the column operators given by

$$\mathbf{V}_{i,k}(s) = (V_{p,k}\alpha_s^{-1}(\xi_i(sp)))_{p\in G} \text{ and } \mathbf{W}_{i,k}(t) = (W_{p,k}\alpha_t^{-1}(\xi_i(tp)))_{p\in G}.$$

Then

$$\mathcal{N}(F_{i,k})(s,t)(a) = \mathbf{W}_{i,k}^*(t)\rho_k(a)\mathbf{V}_{i,k}(s)$$

and

$$\begin{aligned}\|\mathbf{W}_{i,k}(t)\|\|\mathbf{V}_{i,k}(s)\| &= \left\|\sum_{p\in G}\alpha_{t^{-1}}(\xi_i(tp))W_{p,k}^*W_{p,k}\alpha_{t^{-1}}(\xi_i(tp))\right\|^{1/2}\\ &\quad\times\left\|\sum_{p\in G}\alpha_{s^{-1}}(\xi_i(sp))V_{p,k}^*V_{p,k}\alpha_{s^{-1}}(\xi_i(sp))\right\|^{1/2}\\ &\le \|T_k\|_{\mathrm{cb}}\left\|\sum_{p\in G}\alpha_{s^{-1}}(\xi_i(sp))^2\right\|\\ &= \|T_k\|_{\mathrm{cb}}\left\|\sum_{p\in G}\xi_i(p)^2\right\| = \|T_k\|_{\mathrm{cb}}.\end{aligned}$$

Hence, by [25,Theorem 2.6], $\mathcal{N}(F_{i,k})$ is a Schur A-multiplier, for all $i \in \mathbb{I}$ and all $k \in \mathbb{N}$. By Theorem 2.1, $F_{i,k}$ is a Herz–Schur (A, G, α)-multiplier; moreover, as the sequence $(\|T_k\|_{\mathrm{cb}})_{k\in\mathbb{N}}$ is bounded, there exists $C > 0$ such that $\|S_{F_{i,k}}\|_{\mathrm{cb}} \leq C$, $i \in \mathbb{I}$, $k \in \mathbb{N}$.

Next we prove that $\|F_{i,k}(s) - T\|_{\mathrm{cb}} \to 0$ for each $s \in G$. As T is α-invariant, if $a = [a_{r,l}] \in M_n(A)$ then

$$\begin{aligned}
&\|F_{i,k}(s)^{(n)}(a) - T^{(n)}(a)\| \\
&= \left\| \sum_{p\in G} [\xi_i(p)\alpha_s(\xi_i(s^{-1}p))\alpha_p(T_k(\alpha_p^{-1}(a_{r,l}))) - \alpha_p(T(\alpha_p^{-1}(a_{r,l})))] \right\| \\
&\leq \sum_{p\in G} \left\| \xi_i(p)\alpha_s(\xi_i(s^{-1}p)) \right\| \left\| [\alpha_p(T_k(\alpha_p^{-1}(a_{r,l})) - \alpha_p(T(\alpha_p^{-1}(a_{r,l})))] \right\| \\
&+ \left\| \sum_{p\in G} \xi_i(p)\alpha_s(\xi_i(s^{-1}p)) - 1 \right\| \left\| T^{(n)}(a) \right\| \\
&\leq \left\| \sum_{p\in G} \xi_i(p)^2 \right\| \|T_k - T\|_{\mathrm{cb}} \|a\| + \left\| \sum_{p\in G} \xi_i(p)\alpha_s(\xi_i(s^{-1}p)) - 1 \right\| \|T\|_{\mathrm{cb}} \|a\|,
\end{aligned}$$

where in the last line we used the Cauchy-Schwarz inequality and the fact $\xi_i(p) \in Z(A)^+$, $p \in G$. By [5,Lemma 4.3.2], $\| \sum_p \xi_i(p)\alpha_s(\xi_i(s^{-1}p)) - 1\| \to_{i\in\mathbb{I}} 0$; the facts that $\|T_k - T\|_{\mathrm{cb}} \to_{k\to\infty} 0$ and $\sum_{p\in G} \xi_i(p)^2 = 1$ now imply that

$$\|F_{i,k}(s) - T\|_{\mathrm{cb}} \to_{(i,k)\in\mathbb{I}\times\mathbb{N}} 0, \quad s \in G. \tag{20}$$

Since $T_k \in \mathrm{F}(A)$ and the maps $F_{i,k}$ are finitely supported, the map $S_{F_{i,k}}$ on $A \rtimes_{\alpha,G} G$ has finite rank. In order to prove the statement, it hence suffices to show that

$$\|S_{F_{i,k}} \circ S_u - S_{F_{T,u}}\|_{\mathrm{cb}} \to_{(i,k)\in\mathbb{I}\times\mathbb{N}} 0. \tag{21}$$

Let $\varepsilon > 0$. Since $u \in A_{\mathrm{cb}}(G)$, there exists $v \in A(G)$ with finite support E such that

$$\|S_u - S_v\|_{\mathrm{cb}} \leq \|u - v\|_A \leq \varepsilon.$$

Set $\mathcal{Z} = \left\{ \sum_{s\in E} \pi(a_s)\lambda_s : a_s \in A \right\}$ and let $x = [x_{r,l}] \in M_n(\mathcal{Z})$. As in the proof of (ii), write L_s for $\mathrm{diag}(\lambda_s, \ldots, \lambda_s) \in M_n(\mathcal{Z})$. Then $x = \sum_{s\in E} (\pi \circ \mathcal{E})^{(n)}(xL_{s^{-1}})L_s$ and

$$\|S^{(n)}_{F_{i,k}}(x) - S^{(n)}_T(x)\| = \|\sum_{s\in E}[\pi((F_{i,k}(s) - T)(\mathcal{E}(x_{r,l}\lambda_{s^{-1}})))\lambda_s]\|$$
$$\leq \sum_{s\in E}\|[(F_{i,k}(s) - T)(\mathcal{E}(x_{r,l}\lambda_{s^{-1}})]\| \leq \sum_{s\in E}\|(F_{i,k}(s) - T)^{(n)}\|\|[\mathcal{E}(x_{r,l}\lambda_{s^{-1}})]\|$$
$$\leq \sum_{s\in E}\|(F_{i,k}(s) - T)^{(n)}\|\|x\| \leq |E|\|x\|(\max_{s\in E}\|F_{i,k}(s)^{(n)} - T^{(n)}\|).$$

By (20),

$$\|(S_{F_{i,k}} - S_T)|_{\mathcal{Z}}\|_{\mathrm{cb}} \to_{(i,k)\in\mathbb{I}\times\mathbb{N}} 0.$$

We now have

$$\|S_{F_{i,k}} \circ S_u - S_{F_{T,u}}\|_{\mathrm{cb}}$$
$$\leq \|S_{F_{i,k}} \circ (S_u - S_v)\|_{\mathrm{cb}} + \|S_{F_{i,k}} \circ S_v - S_T \circ S_v\|_{\mathrm{cb}} + \|S_T \circ (S_v - S_u)\|_{\mathrm{cb}}$$
$$\leq \|(S_{F_{i,k}} - S_T)|_{\mathcal{Z}}\|_{\mathrm{cb}}\|v\| + \varepsilon(\|S_{F_{i,k}}\|_{\mathrm{cb}} + \|S_T\|_{\mathrm{cb}}),$$

which implies (21). □

Theorem 4.1 exhibits a large class of elements of $\mathfrak{S}^{\mathrm{inv}}_{\mathrm{cc}}(A, G, \alpha)$. In the next theorem, we provide a precise description of the latter class of Herz–Schur multipliers in the case of the irrational rotation algebra. Let $\theta \in \mathbb{R}$ be irratrional and $\alpha : \mathbb{Z} \to \mathrm{Aut}(C(\mathbb{T}))$ be given by

$$\alpha_n(f)(z) = f(e^{2\pi in\theta}z), \quad f \in C(\mathbb{T}), z \in \mathbb{T}.$$

Let $M(\mathbb{T})$ denote the Banach algebra of all complex Borel measures on the unit circle $\mathbb{T}$, and note that $L^1(\mathbb{T})$ is a (closed) ideal of $M(\mathbb{T})$. For $\mu \in M(\mathbb{T})$, let $T_\mu : C(\mathbb{T}) \to C(\mathbb{T})$ be the completely bounded map given by $T_\mu(f) = \mu * f$. Note that T_μ is α-invariant; indeed,

$$\alpha_n(\mu * f)(z) = (\mu * f)(e^{2\pi in\theta}z) = \int f(e^{2\pi in\theta}w^{-1}z)d\mu(w)$$
$$= \int \alpha_n(f)(w^{-1}z)d\mu(w) = \mu * \alpha_n(f)(z).$$

Theorem 4.2 *The functions $F_{T_f,u}$, where $f \in L^1(\mathbb{T})$ and $u \in A(\mathbb{Z})$, have a dense linear span in $\mathfrak{S}^{\mathrm{inv}}_{\mathrm{cc}}(A, G, \alpha)$.*

Proof Let F be a completely compact Herz–Schur $(C(\mathbb{T}), \mathbb{Z}, \alpha)$-multiplier such that $F(n) \circ \alpha_m = \alpha_m \circ F(n)$ for all $m, n \in \mathbb{Z}$. We show that $F(n) = T_{f_n}$ for some $f_n \in L^1(\mathbb{T})$. In fact, let $\Phi_n : M(\mathbb{T}) \to M(\mathbb{T})$ be the dual map of $F(n)$. As $\alpha_n(f)(z) =$

$f(e^{2\pi in\theta}z) = (\delta_{e^{-2\pi in\theta}} * f)(z)$, where δ_s is the point mass measure at $s \in \mathbb{T}$, we obtain

$$\begin{aligned}\left\langle \Phi_n(\delta_{e^{2\pi im\theta}} * \mu),\, f\right\rangle &= \left\langle \delta_{e^{2\pi im\theta}} * \mu,\, F(n)(f)\right\rangle = \langle \mu, \alpha_{-m}(F(n)(f))\rangle \\ &= \langle \mu, F(n)(\alpha_{-m}(f))\rangle = \left\langle \delta_{e^{2\pi im\theta}} * \Phi_n(\mu),\, f\right\rangle,\end{aligned}$$

giving

$$\Phi_n(\delta_{e^{2\pi im\theta}} * \mu) = \delta_{e^{2\pi im\theta}} * \Phi_n(\mu), \quad m, n \in \mathbb{Z},\, \mu \in M(\mathbb{T}).$$

In particular, $\Phi_n(\delta_{e^{2\pi im\theta}}) = \delta_{e^{2\pi im\theta}} * \Phi_n(\delta_1)$. Using the weak* continuity of Φ_n, the density of $\{e^{2\pi im\theta} : m \in \mathbb{Z}\}$ in $\mathbb{T}$ and the fact that every measure μ is a weak* limit of linear combinations of point mass measures, we obtain

$$\Phi_n(\mu) = \mu * \Phi_n(\delta_1), \quad \mu \in M(\mathbb{T}).$$

It now follows that, if $\mu_n = \Phi_n(\delta_1)$ then $F(n)(f) = \mu_n * f$, $f \in C(\mathbb{T}) \simeq C_r^*(\mathbb{Z})$. By Corollary 3.4, $F(n)$ is completely compact for every $n \in \mathbb{Z}$; by Corollary 3.11, $\mu_n \in L^1(\mathbb{T})$.

Fix $F \in \mathfrak{S}_{\mathrm{cc}}^{\mathrm{inv}}(A, G, \alpha)$; then $F(n)$ is α-invariant for every $n \in \mathbb{Z}$. We show that $S_F \in \overline{[S_u \circ S_{F(n)} : u \in A(\mathbb{Z}), n \in \mathbb{Z}]}^{\|\cdot\|_{\mathrm{cb}}}$. By the amenability of $\mathbb{Z}$, there exists a bounded sequence $(u_i)_{i\in\mathbb{N}} \subseteq A(\mathbb{Z})$ of finitely supported functions such that $S_{u_i} \to \mathrm{id}$ in the strong point norm topology (see [17,Theorem 1.12, Theorem 1.9]). Since u_i is finitely supported,

$$S_{u_i} \circ S_F \in \mathrm{span}\{S_u \circ S_{F(n)} : u \in A(\mathbb{Z}), n \in \mathbb{Z}\}.$$

Since S_F is completely compact, Lemma 3.1 implies $\|S_{u_i} \circ S_F - S_F\|_{\mathrm{cb}} \to_{i\to\infty} 0$. □

5 Herz–Schur Multipliers of $\mathcal{K}$

Let G be a discrete group and $\alpha : G \to \mathrm{Aut}(c_0(G))$ be the homomorphism given by $\alpha_t(f)(s) = f(t^{-1}s)$, $s \in G$, $f \in c_0(G)$. For $a \in c_0(G)$, we write M_a for the operator on $\ell^2(G)$ given by $(M_a\xi)(s) = a(s)\xi(s)$, $\xi \in \ell^2(G)$, $s \in G$, and let

$$\mathcal{C} = \{M_a : a \in c_0(G)\};$$

the map $\iota : c_0(G) \to \mathcal{C}$ given by $\iota(a) = M_a$ is a *-isomorphism. By abuse of notation, we write α_t for the corresponding automorphism of $\mathcal{C}$. Recall that, for $t \in G$, we denote by λ_t^0 the left regular unitary acting on $\ell^2(G)$. The pair (ι, λ^0) of representations is covariant, and hence gives rise to a (faithful) representation $\iota \rtimes \lambda^0 : \mathcal{C} \rtimes_{\alpha,r} G \to \mathcal{B}(\ell^2(G))$. According to the Stone-von Neumann Theorem [38,Theorem 4.24], the image of $\iota \rtimes \lambda^0$ coincides with the C*-algebra $\mathcal{K}$ of all compact operators on $\ell^2(G)$. Thus, the Herz–Schur $(\mathcal{C}, G, \alpha)$-multipliers give rise, in a canonical way, to a certain class of completely bounded maps on $\mathcal{K}$. The aim of this section is to formalise this correspondence and examine the complete compactness of the resulting maps on $\mathcal{K}$.

Set $\mathcal{B} = (\pi \rtimes \lambda)(\mathcal{C} \rtimes_{\alpha,r} G)$; thus $\mathcal{B}$ is a C*-subalgebra of $\mathcal{B}(\ell^2(G \times G))$. By [38,Theorem 4.24], the map

$$\theta : \pi(a)\lambda_t \longrightarrow M_a\lambda_t^0, \quad a \in \mathcal{C}, t \in G,$$

is a *-isomorphism from $\mathcal{B}$ onto $\mathcal{K}$. For an element $\Phi \in \mathrm{CB}(\mathcal{K})$, let $\widetilde{\Phi} = \theta^{-1} \circ \Phi \circ \theta$; thus, $\widetilde{\Phi} \in \mathrm{CB}(\mathcal{B})$, and the correspondence $\Phi \longrightarrow \widetilde{\Phi}$ between $\mathrm{CB}(\mathcal{K})$ and $\mathrm{CB}(\mathcal{B})$ is bijective. For a Herz–Schur $(\mathcal{C}, G, \alpha)$-multiplier F, let $\Phi_F = \theta \circ S_F \circ \theta^{-1}$ note that $\Phi_F \in \mathrm{CB}(\mathcal{K})$ and

$$\Phi_F(M_a\lambda_t^0) = M_{F(t)(a)}\lambda_t^0, \quad a \in \mathcal{C}, t \in G.$$

We call the maps of the form Φ_F the *Herz–Schur $\mathcal{K}$-multipliers*. We let

$$\mathfrak{S}(\mathcal{K}) = \{\Phi_F : F \text{ is a Herz–Schur } (\mathcal{C}, G, \alpha)\text{-multiplier}\},$$

and $\mathfrak{S}_{\mathrm{cc}}(\mathcal{K}) = \mathfrak{S}(\mathcal{K}) \cap \mathrm{CC}(\mathcal{K})$. We note that the elements of $\mathfrak{S}(\mathcal{K})$ are precisely those completely bounded maps on $\mathcal{K}$ which leave its diagonals globally invariant, when elements of $\mathcal{K}$ are viewed as $G \times G$-matrices.

We recall that, if K is a Hilbert space and $\mathcal{A} \subseteq \mathcal{B}(K)$ is a C*-algebra, the Haagerup tensor product $\mathcal{A} \otimes_{\mathrm{h}} \mathcal{A}$ consists of (convergent) series $u = \sum_{i=1}^{\infty} a_i \otimes b_i$, where $(a_i)_{i\in\mathbb{N}} \subseteq \mathcal{A}$ and $(b_i)_{i\in\mathbb{N}} \subseteq \mathcal{A}$ are sequences for which the series $\sum_{i=1}^{\infty} a_i a_i^*$ and $\sum_{i=1}^{\infty} b_i^* b_i$ converge in norm. Each such u gives rise to a completely bounded map $\Gamma_u : \mathcal{B}(K) \to \mathcal{B}(K)$ given by

$$\Gamma_u(x) = \sum_{i=1}^{\infty} a_i x b_i, \quad x \in \mathcal{B}(K).$$

The following fact was noted in [16,Corollary 3.6].

Theorem 5.1 *Any completely compact map on $\mathcal{K}$ has the form Γ_u for some element $u \in \mathcal{K} \otimes_{\mathrm{h}} \mathcal{K}$.*

Theorem 5.2 *The mapping*

$$\Phi \longrightarrow \Phi_{F_{\widetilde{\Phi}}} \tag{22}$$

is a linear contractive surjection

(i) *from* $\mathrm{CB}(\mathcal{K})$ *onto* $\mathfrak{S}(\mathcal{K})$*;*
(ii) *from* $\mathrm{CC}(\mathcal{K})$ *onto* $\mathfrak{S}_{\mathrm{cc}}(\mathcal{K})$.

Proof (i) By Proposition 3.3 (i), if $\Phi \in \mathrm{CB}(\mathcal{K})$ then $F_{\widetilde{\Phi}}$ is a Herz–Schur $(\mathcal{C}, G, \alpha)$-multiplier and $\|F_{\widetilde{\Phi}}\|_{\mathrm{m}} \leq \|\Phi\|_{\mathrm{cb}}$. Thus, $\Phi_{F_{\widetilde{\Phi}}} \in \mathfrak{S}(\mathcal{K})$ and $\|\Phi_{F_{\widetilde{\Phi}}}\|_{\mathrm{cb}} \leq \|\Phi\|_{\mathrm{cb}}$. On the other hand, if F is a Herz–Schur $(\mathcal{C}, G, \alpha)$-multiplier then, by Proposition 3.3 (ii), $F_{S_F} = F$ and hence Φ_F is the image of itself under the map (22). The linearity of the map (22) is straighforward.

(ii) Let $\Phi \in \mathrm{CC}(\mathcal{K})$. Since $\mathcal{K}$ has SOAP, Lemma 3.1 shows that Φ can be approximated in the completely bounded norm by finite rank maps $\Phi_n : \mathcal{K} \to \mathcal{K}$, $n \in \mathbb{N}$. Clearly, $\tilde{\Phi}_n$ has finite rank, $n \in \mathbb{N}$. As in the proof of Lemma 3.6, we can assume that the maps $\tilde{\Phi}_n$ have range in $\mathrm{span}\{\pi(a)\lambda_s : s \in G, a \in \mathcal{C}\}$. By Proposition 3.3 (iv), $S_{F_{\tilde{\Phi}_n}}$ has finite rank and hence so has $\Phi_{F_{\tilde{\Phi}_n}}$. Moreover, by Proposition 3.3 (i), $\|F_{\tilde{\Phi}} - F_{\tilde{\Phi}_n}\|_{\mathrm{m}} \to 0$, giving $\|\Phi_{F_{\tilde{\Phi}}} - \Phi_{F_{\tilde{\Phi}_n}}\|_{\mathrm{cb}} \to_{n\to\infty} 0$. Hence $\Phi_{F_{\tilde{\Phi}}}$ is completely compact. The rest is similar to (i). □

For $b \in \mathcal{K}$ and $s \in G$, let $b_s = \lambda_s^0 b \lambda_{s^{-1}}^0$. For $u = \sum_{i=1}^{\infty} a_i \otimes b_i \in \mathcal{K} \otimes_{\mathrm{h}} \mathcal{K}$, write $u_s = \sum_{i=1}^{\infty} a_i \otimes (b_i)_s$; it is clear that u_s is a well-defined element of $\mathcal{K} \otimes_{\mathrm{h}} \mathcal{K}$.

Theorem 5.3 *Let $F : G \to \mathrm{CB}(\mathcal{C})$. The following are equivalent:*

(i) *F is a completely compact Herz–Schur $(\mathcal{C}, G, \alpha)$-multiplier;*
(ii) *there exists $u \in \mathcal{K} \otimes_{\mathrm{h}} \mathcal{K}$ such that $\Gamma_{u_s}(a) = F(s)(a)$, for all $a \in \mathcal{C}$ and $s \in G$.*

Proof (i)⇒(ii) The map $M_a\lambda_s^0 \mapsto M_{F(s)(a))}\lambda_s^0$ extends to a completely compact map on $\mathcal{K}$. By Theorem 5.1, there exists $u = \sum_{i=1}^{\infty} a_i \otimes b_i \in \mathcal{K} \otimes_{\mathrm{h}} \mathcal{K}$ such that

$$M_{F(s)(a)}\lambda_s^0 = \sum_{i=1}^{\infty} a_i M_a \lambda_s^0 b_i = \sum_{i=1}^{\infty} a_i M_a \lambda_s^0 b_i (\lambda_s^0)^* \lambda_s^0, \quad a \in \mathcal{C}.$$

Thus, $\mathcal{C}$ is an invariant subspace for Γ_{u_s} and $\Gamma_{u_s}(a) = F(s)(a)$, $a \in \mathcal{C}$.

(ii)⇒(i) By the previous paragraph, $\Gamma_u(M_a\lambda_s^0) = \Gamma_{u_s}(M_a)\lambda_s^0$. Thus, $\theta^{-1} \circ \Gamma_u \circ \theta$ coincides with S_F and so F is a completely compact Herz–Schur multiplier. □

Suppose that F is a Herz Schur $(\mathcal{C}, G, \alpha)$-multiplier of the form $F(s)(M_a) = M_{h_s a}$, where $h_s : G \to \mathbb{C}$ is a function, $s \in G$. We can associate with it the function $\psi : G \times G \to \mathbb{C}$, given by $\psi(s,t) = h_s(t)$. This is the identification made the next statement.

Corollary 5.4 *Let $\varphi : G \times G \to \mathbb{C}$ be a compact Schur multiplier. Then the function $\psi : G \times G \to \mathbb{C}$, given by $\psi(s,t) = \varphi(t, s^{-1}t)$, is a completely compact Herz Schur $(\mathcal{C}, G, \alpha)$-multiplier.*

Proof By [19], there exists an element $u = \sum_{i=1}^{\infty} a_i \otimes b_i \in c_0(G) \otimes_{\mathrm{h}} c_0(G)$ such that $\varphi(p,q) = \sum_{i=1}^{\infty} a_i(p) b_i(q)$, $p, q \in G$. By the injectivity of the Haagerup tensor product [10,Proposition 9.2.5], $u \in \mathcal{K} \otimes_{\mathrm{h}} \mathcal{K}$. Note that

$$\Gamma_{u_s}(M_a)(t) = \sum_{i=1}^{\infty} a_i(t) b_i(s^{-1}t) a(t), \quad a \in c_0(G), t \in G;$$

in other words, $\Gamma_{u_s}(M_a) = M_{h_s a}$, where $h_s(t) = \varphi(t, s^{-1}t)$, $t \in G$. The conclusion follows from Theorem 5.3. □

6 Some Remarks and Open Questions

In Corollary 3.11, we showed that the amenability of a discrete group G is a sufficient condition for automatic complete compactness: every compact multiplier is in this case completely compact. For such automatic complete compactness it suffices, instead of amenability, to assume that the completely bounded multiplier norm is equivalent to the multiplier norm. By a result of Losert [23], there exist non-amenable groups such that $M^{\mathrm{cb}}A(G) = MA(G)$, for instance $SL(2, \mathbb{R})$. However we do not know whether there exists a *discrete* group with this property.

In Proposition 3.12, we exhibited an example of a multiplier $u \in MA(G)$, for which the map S_u is compact but not completely compact. This multiplier however may not be completely bounded, as we only guarantee the *boundedness* of S_u if $u \in MA(G)$. We do not know if there exists a completely bounded compact multiplier which is not completely compact.

Finally, in Corollary 3.9, we showed that if G is a discrete group possessing property (AP) then every completely compact multiplier on G is the limit, in the completely bounded norm, of *finitely supported* multipliers. It would be interesting to know if (AP) is in fact equivalent to the latter approximation property; we do not know if this holds true.

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