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A pointwise norm on a non-reduced analytic space

Mats Andersson¹

Department of Mathematics, Chalmers University of Technology and University of Gothenburg, S-412 96 Göteborg, Sweden



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ABSTRACT

Let X be a possibly non-reduced space of pure dimension. We introduce a pointwise Hermitian norm on smooth $(0, q)$ -forms, in particular on holomorphic functions, on X . The norm is canonical, up to equivalence, where the underlying reduced space is a manifold. We prove that the space of holomorphic functions is complete with respect to the natural topology induced by this norm.

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1. Introduction

Starting with papers by Pardon and Stern, [29,30], in the early 90s, a lot of research on the $\bar{\partial}$ -equation on a reduced singular space X has been conducted during the last decades, e.g., [14,20,28,31,26,22,23,8,12] and many others. In most of them estimates for solutions are discussed. There are also works, e.g., [1], on estimates of holomorphic extensions from a singular subvariety. Given a local embedding of a reduced X into a smooth manifold \mathcal{U} , a pointwise norm of functions and forms on X is inherited from a Hermitian norm on \mathcal{U} . Any two such local norms are equivalent, and thus one gets a global pointwise norm that is unique, up to equivalence, on any compact subset of X .

E-mail address: matsa@chalmers.se.

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Only quite recently there has been some work about analysis on non-reduced spaces. The celebrated Ohsawa-Takegoshi theorem, [27], has been generalized to encompass extensions of holomorphic functions defined on non-reduced subvarieties X defined by certain multiplier ideal sheaves of a manifold Y , see, e.g., [17,19]. In this case the L^2 -norm of a function (or form) ϕ on the subvariety is defined as a limit of L^2 -norms of an arbitrary extension of ϕ over small neighborhoods of X in Y . A pointwise, but not canonical, norm of holomorphic functions on a non-reduced X is used by Sznajdman in [33], where he proved an analytically formulated Briançon-Skoda-Huneke type theorem on a non-reduced X of pure dimension.

In this paper we introduce, given a non-reduced space X of pure dimension n , a pointwise Hermitian norm $|\cdot|_X$ on \mathcal{O}_X such that $|\phi|_X^2$ is a smooth function on the underlying reduced space Z for any holomorphic ϕ . The norm is canonical (up to local equivalence) on the regular part of Z , whereas the extension across Z_{sing} possibly depends on some choices. The norm extends to smooth $(0, q)$ -forms on X .

Given any point $x \in X$ there is a local embedding $i: X \rightarrow \mathcal{U} \subset \mathbb{C}^N$, where $\mathcal{U} \subset \mathbb{C}^N$ is an open subset and $x \in X \cap \mathcal{U}$. This means that we have an ordinary local embedding $\iota: Z \rightarrow \mathcal{U}$ and a coherent ideal sheaf \mathcal{J} in \mathcal{U} with zero set $Z \cap \mathcal{U}$ such that the structure sheaf \mathcal{O}_X , the sheaf of holomorphic functions on X , is isomorphic to $\mathcal{O}_{\mathcal{U}}/\mathcal{J}$. Thus we have a natural surjective mapping $i^*: \mathcal{O}_{\mathcal{U}} \rightarrow \mathcal{O}_X$ with kernel \mathcal{J} .

Recall that a holomorphic differential operator L in \mathcal{U} is Noetherian with respect to \mathcal{J} if $L\Phi = 0$ on Z for all Φ in \mathcal{J} . It is well-known that locally one can find a finite set L_1, \dots, L_m of Noetherian operators such that $L_j\Phi = 0$ on Z if and only if Φ is in \mathcal{J} . The analogous statement for a polynomial ideal is a keystone in the celebrated Fundamental principle due to Ehrenpreis and Palamodov, see, e.g., [15,24]. Each Noetherian operator with respect to \mathcal{J} defines an intrinsic mapping $\mathcal{L}: \mathcal{O}_X \rightarrow \mathcal{O}_Z$ by

$$\mathcal{L}(i^*\Phi) = \iota^*L\Phi. \quad (1.1)$$

We say that \mathcal{L} is a *Noetherian operator* on X . It follows that locally there are Noetherian operators $\mathcal{L}_0, \dots, \mathcal{L}_m$ on X such that

$$\mathcal{L}_j\phi = 0 \text{ in } \mathcal{O}_Z, \quad j = 1, \dots, m, \quad \text{if and only if} \quad \phi = 0 \text{ in } \mathcal{O}_X. \quad (1.2)$$

Given \mathcal{L}_j as in (1.2), following [33] let us consider

$$|\phi(z)|^2 = \sum_0^m |\mathcal{L}_j\phi(z)|^2. \quad (1.3)$$

Clearly $|\phi| = 0$ in an open set if and only if $\phi = 0$ there so (1.3) is a Hermitian norm. However, it depends on the choice of \mathcal{L}_j . For instance, (1.2) still holds if \mathcal{L}_j are multiplied by any h in \mathcal{O}_Z that is generically non-vanishing on Z . The set of all Noetherian operators

on X is a (left) \mathcal{O}_Z -module,² but it is not locally finitely generated since any derivation along Z is Noetherian. We will define our norm from a suitable subsheaf. The construction relies on the close connection between Noetherian operators and so-called Coleff-Herrera currents established by J-E Björk, [16].

Assume for the moment that Z is smooth and that we have a local embedding $i: X \rightarrow \mathcal{U}$. Let $\text{Hom}_{\mathcal{O}_{\mathcal{U}}}(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}_X^{\mathbb{Z}})$ be the $\mathcal{O}_{\mathcal{U}}$ -module of Coleff-Herrera currents in \mathcal{U} that are annihilated by \mathcal{J} . This sheaf, introduced by J-E Björk, [16], consists of all $\bar{\partial}$ -closed $(N, N-n)$ -currents in \mathcal{U} , with support on $Z \cap \mathcal{U}$, such that $h\mu = 0$ for all holomorphic h that vanish on Z , and³ $\Phi\mu = 0$ for Φ in \mathcal{J} .

Proposition 1.1. *Let $\pi: \mathcal{U} \rightarrow Z \cap \mathcal{U}$ be a submersion and let ω_z be a non-vanishing holomorphic n -form on $Z \cap \mathcal{U}$. Each μ in $\text{Hom}_{\mathcal{O}_{\mathcal{U}}}(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}_X^{\mathbb{Z}})$ induces a Noetherian operator $\mathcal{L}: \mathcal{O}_X \rightarrow \mathcal{O}_Z$ by*

$$\mathcal{L}\phi\omega_z = \pi_*(\phi\mu). \quad (1.4)$$

The set of \mathcal{L} so obtained is a coherent \mathcal{O}_Z -module $\mathcal{N}_{X,\pi}$ on $Z \cap \mathcal{U}$, and any set of local generators satisfies (1.2).

Clearly $\mathcal{N}_{X,\pi}$ is independent of the choice of ω_z . After shrinking \mathcal{U} , if needed, we can assume that $\mathcal{N}_{X,\pi}$ is finitely generated in \mathcal{U} . A finite set of generators, cf. (1.3), therefore gives a pointwise norm $|\cdot|_{X,\pi}$ in \mathcal{U} . If $|\cdot|'_{X,\pi}$ is obtained in this way from another finite set of generators, then $|\cdot|'_{X,\pi}$ is equivalent to $|\cdot|_{X,\pi}$ on (compact subsets of) \mathcal{U} , which we write as $|\cdot|'_{X,\pi} \sim |\cdot|_{X,\pi}$.

Definition 1.2. Let \mathcal{N}_X be the \mathcal{O}_Z -module generated by all local Noetherian operators \mathcal{L} on X obtained from local embeddings and submersions as in (1.4).

Theorem 1.3. *Let X be a reduced space of pure dimension such that its underlying reduced space Z is smooth. Then \mathcal{N}_X is a coherent \mathcal{O}_Z -module on Z , and any set of local generators of \mathcal{N}_X satisfies (1.2).*

Any finite set of local generators, cf. (1.3), gives rise to a local pointwise Hermitian norm. Moreover, any two norms obtained in this way are locally equivalent. It turns out, see Proposition 4.3, that if \mathcal{U} is small enough, then \mathcal{N}_X is generated in $Z \cap \mathcal{U}$ by the sheaves \mathcal{N}_{X,π^ℓ} for a suitable finite set of submersions $\pi^\ell: \mathcal{U} \rightarrow Z \cap \mathcal{U}$. Thus $|\cdot|_X$ is equivalent in $X \cap \mathcal{U}$ to the finite sum of the norms $|\phi|_{X,\pi^\ell}$. Patching together we get a global pointwise Hermitian norm $|\cdot|_X$ on X .

² In this paper ‘ \mathcal{O}_Z -module’ means ‘sheaf of \mathcal{O}_Z -modules’.

³ If Z has singular points an additional regularity assumption is required, see Section 2.1 below.

To describe the norm $|\cdot|_X$ more concretely, assume that we have a local embedding $i: \mathcal{U} \rightarrow \Omega \subset \mathbb{C}^N$ and coordinates $(z, w) = (z_1, \dots, z_n, w_1, \dots, w_\kappa)$ in \mathcal{U} , where $\kappa = N - n$, such that $Z = \{w = 0\}$. By the Nullstellensatz,

$$\mathcal{I} := \langle w_1^{M_1+1}, \dots, w_\kappa^{M_\kappa+1} \rangle \subset \mathcal{J} \quad (1.5)$$

if M_j are large enough natural numbers. For multiindices $m = (m_1, \dots, m_\kappa) \in \mathbb{N}^\kappa$, let $|m| = m_1 + \dots + m_\kappa$. If $M = (M_1, \dots, M_\kappa)$, then $m \leq M$ means that $m_j \leq M_j$ for $j = 1, \dots, \kappa$. We will use the short-hand notation $\partial^{|m|}/\partial w^m = (\partial^{m_1}/\partial w_\kappa^{m_1}) \dots (\partial^{m_\kappa}/\partial w_\kappa^{m_\kappa})$, and define $\partial^{|\beta|}/\partial z^\beta$ similarly for $\beta = (\beta_1, \dots, \beta_n)$.

Theorem 1.4. *With the notation above, if \mathcal{U} is small enough and (1.5) holds, then there is a finite set of holomorphic functions a_1, \dots, a_ν in \mathcal{U} such that the operators*

$$\phi \mapsto \mathcal{L}_{m,\beta,j}\phi := \frac{\partial^{|m|+|\beta|}(\phi a_j)}{\partial z^\beta \partial w^m}(\cdot, 0), \quad m \leq M, \quad |\beta| \leq |M| - |m|, \quad j = 1, \dots, \nu, \quad (1.6)$$

are Noetherian on $X \cap \mathcal{U}$ and generate the \mathcal{O}_Z -module \mathcal{N}_X on $Z \cap \mathcal{U}$.

The precise requirement of the functions a_j is that they generate the coherent $\mathcal{O}_\mathcal{U}$ -module $(\mathcal{I} : \mathcal{J})/\mathcal{I}$, see Remark 4.1. An immediate consequence of the theorem is that

$$|\phi(z)|_X^2 \sim \sum_{j=1}^\nu \sum_{m \leq M} \sum_{|\beta| \leq |M| - |m|} \left| \frac{\partial^{|m|+|\beta|}(\phi a_j)}{\partial z^\beta \partial w^m}(z, 0) \right|^2 \quad (1.7)$$

in \mathcal{U} . It follows from (1.7) that

$$|\xi\phi|_X \leq C|\phi|_X,$$

locally in \mathcal{U} where C only depends on $\xi \in \mathcal{O}_X$. Notice that if in addition ξ is invertible in \mathcal{O}_X , then $|\phi|_X \sim |\xi\phi|_X$ since $|\phi|_X = |\xi^{-1}\xi\phi|_X \leq C|\xi\phi|_X$.

We say that a point $x \in X$ is *regular* if Z is smooth at x and in addition \mathcal{O}_X is Cohen-Macaulay. The set of regular points is a Zariski-open dense subset of Z . In a neighborhood of a regular point we can represent \mathcal{O}_X as a free \mathcal{O}_Z -module (in a non-canonical way): Let $i: X \rightarrow \mathcal{U}$ be a local embedding at x and assume that we have local coordinates (z, w) in \mathcal{U} such that $Z = \{w = 0\}$. To each multiindex $\alpha_i = (\alpha_{i1}, \dots, \alpha_{i\kappa}) \in \mathbb{N}^\kappa$ we associate the monomial $w^{\alpha_i} := w_1^{\alpha_{i1}} \dots w_\kappa^{\alpha_{i\kappa}}$. After possibly shrinking \mathcal{U} there is a (not unique) set of monomials $1, w^{\alpha_1}, \dots, w^{\alpha_{\tau-1}}$ such that each ϕ in \mathcal{O}_X in \mathcal{U} has a unique representative

$$\hat{\phi}(z, w) = \hat{\phi}_0(z) \otimes 1 + \hat{\phi}_1(z) \otimes w^{\alpha_1} + \dots + \hat{\phi}_{\nu-1}(z) \otimes w^{\alpha_{\tau-1}}, \quad (1.8)$$

in $\mathcal{O}_\mathcal{U}$, where $\hat{\phi}_i$ are in \mathcal{O}_Z . Let $\pi: \mathcal{U} \rightarrow Z \cap \mathcal{U}$ be the submersion $(z, w) \mapsto z$.

Theorem 1.5. *Assume that $x \in X$ is a regular point, and let $i: X \rightarrow \mathcal{U}$ be a local embedding at x as above. Then*

$$(|\hat{\phi}_0(z)|^2 + \cdots + |\hat{\phi}_{\tau-1}(z)|^2)^{1/2} \quad (1.9)$$

is a pointwise norm in $X \cap \mathcal{U}$ that is equivalent to $|\phi(z)|_{X,\pi}$ in $Z \cap \mathcal{U}$.

It follows that (1.9) only depends, up to equivalence, on the submersion π . Moreover, the sum of the norms (1.9) obtained from a suitable finite set of submersions $\mathcal{U} \rightarrow Z \cap \mathcal{U}$ is equivalent to $|\phi|_X$.

In Section 7 we consider an arbitrary pure-dimensional singular space X and prove that at each point x on the singular locus Z_{sing} there is a local embedding $i: X \rightarrow \mathcal{U}$ such that \mathcal{N}_X , a priori defined on $(Z \setminus Z_{\text{sing}}) \cap \mathcal{U}$, admits a coherent extension to $Z \cap \mathcal{U}$. Patching together we get a global pointwise Hermitian norm on X .

Definitions of the sheaf of smooth $(0, q)$ -forms $\mathcal{E}_X^{0,q}$ on a non-reduced X and of an associated $\bar{\partial}$ -operator were recently given in [7]. In Section 8 we point out that the Noetherian operators in \mathcal{N}_X extend to mappings $\mathcal{E}_X^{0,q} \rightarrow \mathcal{E}_Z^{0,q}$. In this way we get an extension of the norm $|\cdot|_X$ to smooth $(0, q)$ -forms. Thus one, e.g., can discuss norm estimates for possible solutions to the $\bar{\partial}$ -equation on X , but this question is not pursued in this paper. In the recent paper [6] we find L^p -estimates of extensions of holomorphic functions ϕ defined on a non-reduced subvariety X of a strictly pseudoconvex domain D , given that certain L^p -norms of $|\phi|_X$ over $Z \cap D$ are finite. This generalizes results in [2,18], see also [1], in the case when X is reduced. In this paper we prove the following.

Theorem 1.6. *Assume that ϕ_j is a sequence of holomorphic functions on X that is a Cauchy sequence on each compact subset with respect to the uniform norm induced by $|\cdot|_X$. Then there is a holomorphic function ϕ on X such that $\phi_j \rightarrow \phi$ uniformly on compact subsets of X .*

This statement is well-known but non-trivial in the reduced case, see, e.g., [24, Theorem 7.4.9].

The plan of the paper is as follows. In Section 2 we recall the definition of Coleff-Herrera currents as well as some basic facts. Proposition 1.1 and Theorems 1.3 and 1.4 are proved in Sections 3 and 4. Theorem 1.5 is proved in Section 5. Section 6 is devoted to a non-trivial example where the \mathcal{N}_X and the norm $|\cdot|_X$ are computed explicitly. The content of Sections 7 and 8 is already mentioned.

The proof of Theorem 1.6 relies on some further residue theory that we recall in Sections 9 and 10. In the latter one we also provide a proof of Theorem 1.6 in case Z is smooth. For the general case we need a kind of resolution of X that is described in Section 11, and in Section 12 the proof of Theorem 1.6 is concluded.

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2. Some preliminaries

In this section we have collected a few definitions and results that will be used.

2.1. Coleff-Herrera currents

Assume that $j: Z \rightarrow \mathcal{U} \subset \mathbb{C}^N$ is an embedding of a reduced variety Z of pure dimension n . A germ of a current μ in \mathcal{U} of bidegree $(N, N - n)$ is a Coleff-Herrera current with support on Z , $\mu \in \mathcal{CH}_{\mathcal{U}}^Z$, if it is $\bar{\partial}$ -closed, is annihilated by $\bar{\mathcal{J}}_Z$ (i.e., $\bar{h}\mu = 0$ for h in \mathcal{J}_Z) and in addition has the *standard extension property* (SEP). The latter condition can be expressed in the following way: Let χ be any smooth function on the real axis that is 0 close to the origin and 1 in a neighborhood of ∞ . Then μ has the SEP if for any holomorphic function h (or tuple h of holomorphic functions) whose zero set $Z(h)$ has positive codimension on Z , $\chi(|h|/\epsilon)\mu \rightarrow \mu$ when $\epsilon \rightarrow 0$. The intuitive meaning is that μ does not carry any mass on the set $Z \cap Z(h)$. See, e.g., [3, Section 5] for a discussion.

Example 2.1 (*Coleff-Herrera product*). If f_1, \dots, f_{N-n} are holomorphic functions in \mathcal{U} with common zero set Z , then the Coleff-Herrera product

$$\bar{\partial} \frac{1}{f} := \bar{\partial} \frac{1}{f_{N-n}} \wedge \dots \wedge \bar{\partial} \frac{1}{f_1} \quad (2.1)$$

can be defined in various ways by suitable limit processes. Its annihilator is precisely the ideal (sheaf) $\mathcal{J}(f) = \langle f_1, \dots, f_{N-n} \rangle$. If A is a holomorphic N -form, then $A \wedge \bar{\partial}(1/f)$ is a Coleff-Herrera current. \square

Proposition 2.2. *If f_j are as in Example 2.1, μ is in $\mathcal{CH}_{\mathcal{U}}^Z$ and $\mathcal{J}(f)\mu = 0$, then there is (locally) a holomorphic N -form A such that*

$$\mu = A \wedge \bar{\partial} \frac{1}{f}. \quad (2.2)$$

The statements in Example 2.1 are due to Coleff and Herrera, Dickenstein and Sessa, and Passare in the '80s, whereas Proposition 2.2 is due to Björk, [16]. Proofs and further discussions and references can be found in [16] and [3, Sections 3 and 4].

2.2. Embeddings of a non-reduced space

Let $i: X \rightarrow \mathcal{U} \subset \mathbb{C}^N$ be a local embedding of a non-reduced space of pure dimension n and consider the sheaf $\mathcal{H}om_{\mathcal{O}_{\mathcal{U}}}(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}_{\mathcal{U}}^Z)$, i.e., the sheaf of currents μ in $\mathcal{CH}_{\mathcal{U}}^Z$ such that $\mathcal{J}\mu = 0$. It is indeed a sheaf over $\mathcal{O}_X = \mathcal{O}_{\mathcal{U}}/\mathcal{J}$; for the rest of this paper we will omit the lower index and write just $\mathcal{H}om(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}_{\mathcal{U}}^Z)$. The duality principle,

$$\Phi \in \mathcal{J} \text{ if and only if } \Phi\mu = 0 \text{ for all } \mu \in \mathcal{H}om(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}_{\mathcal{U}}^Z), \quad (2.3)$$

is known since long ago, see, e.g., [5, (1.6)].

Given a point x on X there is a minimal number \hat{N} such that there is a local embedding $i': X \rightarrow \mathcal{U}' \subset \mathbb{C}_{\zeta}^{\hat{N}}$ at x . Such a minimal embedding is unique up to biholomorphisms. Moreover, any embedding $i: X \rightarrow \mathcal{U} \subset \mathbb{C}^N$ factorizes so that, in a neighborhood of x ,

$$X \xrightarrow{i'} \mathcal{U}' \xrightarrow{j} \mathcal{U} := \mathcal{U}' \times \mathcal{U}'' \subset \mathbb{C}_{\zeta}^{\hat{N}} \times \mathbb{C}_{w''}^{N-\hat{N}} = \mathbb{C}^N, \quad i = j \circ i', \quad (2.4)$$

where i' is minimal, \mathcal{U}'' is an open subset of $\mathbb{C}_{w''}^{N-\hat{N}}$, $j(\zeta) = (\zeta, 0)$, and the ideal in \mathcal{U} is $\mathcal{J} = \mathcal{J}' \otimes 1 + (w_1'', \dots, w_m'')$, where $\mathcal{O}_X \simeq \mathcal{O}_{\mathcal{U}'}/\mathcal{J}'$. It follows from [7, Lemma 4] that the mapping

$$j_*: \mathcal{H}om(\mathcal{O}_{\mathcal{U}'}/\mathcal{J}', \mathcal{CH}_{\mathcal{U}'}^Z) \rightarrow \mathcal{H}om(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}_{\mathcal{U}}^Z) \quad (2.5)$$

is an \mathcal{O}_X -linear isomorphism. It is naturally expressed as $\mu' \mapsto \mu = \mu' \otimes [w'' = 0]$, where $[w'' = 0]$ denotes the current of integration over $\{w'' = 0\}$.

Remark 2.3. The equivalence classes in (2.5) can be considered as elements of an intrinsic \mathcal{O}_X -module ω_X^n of $\bar{\partial}$ -closed $(n, 0)$ -form on X , introduced in [7], so that $i_*: \omega_X^n \rightarrow \mathcal{H}om(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}_{\mathcal{U}}^Z)$ is an isomorphism. In case X is reduced, ω_X^n is the classical Barlet sheaf, [13], consisting of $\bar{\partial}$ -closed meromorphic n -forms. \square

If Z is smooth, $\pi: \mathcal{U} \rightarrow Z$ is a (holomorphic) submersion, and μ is in $\mathcal{H}om(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}_{\mathcal{U}}^Z)$, then $\pi_*\mu$ is a holomorphic n -form on Z .

Lemma 2.4. *With the notation above, there is a submersion $\pi': \mathcal{U}' \rightarrow Z$ such that $\pi'_*\mu' = \pi_*\mu$.*

Proof. Let (z, w') be coordinates in \mathcal{U}' such that $Z = \{w' = 0\}$. Then the fiber of π over $z \in Z$ must be of the form $(w', w'') \mapsto (z + b'w' + b''w'', w', w'')$, where

$$b'w' = b'(z, w', w'')w' = \sum_{i=1}^{\hat{N}-n} b'_i(z, w', w'')w'_i,$$

$$b''w'' = b''(z, w', w'')w'' = \sum_{j=1}^m b_j''(z, w', w'')w_j'',$$

and b'_i and b''_j are holomorphic. Now, since $\mu = \mu' \otimes [w'' = 0]$,

$$\pi_*\mu(z) = \int_{w', w''} \mu(z + b'w' + b''w'', w', w'') = \int_{w'} \mu'(z + b'|_{w''=0}w', w').$$

This is precisely $\pi'_*\mu'(z)$, where π' is the submersion with fiber $w' \mapsto (z + b'(z, w', 0)w', w')$ over z . \square

2.3. Local representation of certain currents

Consider an open set $\mathcal{U} \subset \mathbb{C}_z^n \times \mathbb{C}_w^\kappa$, let $Z = \mathbb{C}_z^n \times \{0\}$, and let $\pi: \mathcal{U} \rightarrow Z \cap \mathcal{U}$ be the submersion $(z, w) \mapsto z$. We use the short-hand notation

$$dz = dz_1 \wedge \dots \wedge dz_n, \quad dw = dw_1 \wedge \dots \wedge dw_\kappa, \quad (2.6)$$

and

$$\bar{\partial} \frac{dw}{w^{m+1}} = \bar{\partial} \frac{dw_1}{w_1^{m_1+1}} \wedge \bar{\partial} \frac{dw_2}{w_2^{m_2+1}} \wedge \dots \wedge \bar{\partial} \frac{dw_\kappa}{w_\kappa^{m_\kappa+1}}, \quad (2.7)$$

if $m = (m_1, \dots, m_\kappa) \in \mathbb{N}^\kappa$ is a multiindex. It is well-known, and follows immediately from the one-variable case, that if $\xi_J(z, w)d\bar{z}_J$ is a smooth $(0, k)$ -form in \mathcal{U} , then

$$\pi_* \left(\xi_J(z, w) d\bar{z}_J \frac{1}{(2\pi i)^\kappa} \bar{\partial} \frac{dw}{w^{m+1}} \wedge dz \right) = \frac{1}{m!} \frac{\partial^{|m|}}{\partial w^m} \xi_J(z, 0) d\bar{z}_J \wedge dz. \quad (2.8)$$

If τ is any $(N, N - n + k)$ -current in \mathcal{U} with support on Z that is annihilated by all \bar{w}_j and $d\bar{w}_\ell$, then it has the unique representation (as a locally finite sum)

$$\tau = \sum_{\gamma} \tau_{\gamma}(z) \frac{1}{(2\pi i)^\kappa} \bar{\partial} \frac{dw}{w^{\gamma+1}} \wedge dz, \quad (2.9)$$

where τ_{γ} are $(0, k)$ -currents on $Z \cap \mathcal{U}$ and

$$\tau_{\gamma} \wedge dz = \pi_*(w^{\gamma} \tau) \quad (2.10)$$

on $Z \cap \mathcal{U}$, cf. [7, (2.11)]. Clearly $\bar{\partial}\tau = 0$ if and only if $\bar{\partial}\tau_{\alpha} = 0$ for all α . In particular, τ is a Coleff-Herrera current if and only if all τ_{α} are holomorphic functions.

3. The sheaf \mathcal{N}_X in a special case

Let $\iota: Z \rightarrow \mathcal{U} \subset \mathbb{C}^{n+\kappa}$ be a smooth submanifold of dimension n , let w_1, \dots, w_κ be functions in \mathcal{U} that generate \mathcal{J}_Z , and let $M \in \mathbb{N}^\kappa$ be a multiindex. In this section we prove Proposition 1.1 and Theorems 1.3 and 1.4 for the space $i: X' \rightarrow \mathcal{U} \subset \mathbb{C}^{n+\kappa}$ with structure sheaf $\mathcal{O}_{X'} = \mathcal{O}_{\mathcal{U}}/\mathcal{I}$, where

$$\mathcal{I} = \langle w_1^{M_1+1}, \dots, w_\kappa^{M_\kappa+1} \rangle.$$

Proof of Proposition 1.1 for X' . Assume that $\pi: \mathcal{U} \rightarrow Z$ is a submersion. In a neighborhood \mathcal{V} of a given point $x \in Z$, there are coordinates (z, w) such that π is $(z, w) \mapsto z$ there. Since the proposition is local it is enough to prove it in \mathcal{V} . Given these coordinates, each function ϕ in $\mathcal{O}_{X'}$ has a unique representation

$$\phi = \sum_{m \leq M} \phi_m(z) w^m, \quad (3.1)$$

where ϕ_m are in \mathcal{O}_Z . Using the notation (2.6) and (2.7), let

$$\hat{\mu} = \frac{1}{(2\pi i)^\kappa} \bar{\partial} \frac{dw}{w^{M+1}} \wedge dz. \quad (3.2)$$

It follows from Proposition 2.2 that each μ in $\mathcal{H}om(\mathcal{O}_{\mathcal{U}}/\mathcal{I}, \mathcal{CH}_{X'}^Z)$ is $a\hat{\mu}$ for some holomorphic a , i.e., the $\mathcal{O}_{\mathcal{U}}$ -module $\mathcal{H}om(\mathcal{O}_{\mathcal{U}}/\mathcal{I}, \mathcal{CH}_{X'}^Z)$ is generated by $\hat{\mu}$. If $\mu = a\hat{\mu}$, then $\pi_*(\psi\mu) = \pi_*(\psi a\hat{\mu})$ so in view of (2.8) we have

$$\mathcal{L}\psi dz = \pi_*(\psi a\hat{\mu}) = \frac{1}{M!} \frac{\partial^{|M|}}{\partial w^M} (\psi a)(z, 0) dz = \sum_{m \leq M} c_m(z) \frac{\partial^{|m|}}{\partial w^m} \psi(z, 0) dz, \quad (3.3)$$

where c_m are functions in \mathcal{O}_Z . More precisely,

$$c_m(z) = \frac{1}{M!} \binom{M}{m} \frac{\partial^{|M-m|}}{\partial w^{M-m}} a(z, 0)$$

with suitable multiindex notation. It follows that $\mathcal{N}_{X', \pi}$ is the \mathcal{O}_Z -module in $Z \cap \mathcal{V}$ generated by the Noetherian operators $(\partial^{|m|}/\partial w^m)|_{w=0}$ for $m \leq M$. By the uniqueness of the representations (3.1) these generators are independent, so $\mathcal{N}_{X', \pi}$ is a free \mathcal{O}_Z -module in \mathcal{V} and hence coherent, and clearly (1.2) holds. \square

The next result is a main technical point in this paper.

Proposition 3.1. Assume that (z, w) are coordinates in \mathcal{U} . The \mathcal{O}_Z -module $\mathcal{N}_{X'}$ is generated by the Noetherian operators

$$\mathcal{L}_{m, \beta} := \frac{\partial^{|m|+|\beta|}}{\partial w^m \partial z^\beta} \Big|_{w=0}, \quad m \leq M, \quad |\beta| \leq |M-m|. \quad (3.4)$$

Noting that $|M - m| = |M| - |m|$, Proposition 3.1 is precisely Theorem 1.4 for X' . In view of the unique representations (3.1) one sees that $\mathcal{N}_{X'}$ is a free \mathcal{O}_Z -module and therefore coherent. Furthermore, (1.2) holds since it does already for $\mathcal{N}_{X',\pi}$. Thus also Theorem 1.3 follows for X' .

Remark 3.2. If $M_j = 1$ for some j then (3.4) means that there are no derivatives with respect to w_j . Then the embedding $i: X' \rightarrow \mathcal{U}$ is not minimal so one can delete this variable, cf. Section 2.2. In view of Lemma 2.4 this does not affect the definition of $\mathcal{N}_{X'}$. With no loss of generality one can therefore assume that $M_j > 1$ for all j . \square

Proof of Proposition 3.1. Let us temporarily denote the \mathcal{O}_Z -module generated by the operators $\mathcal{L}_{m,\beta}$ in (3.4) by \mathcal{M} . Fix a point $x \in Z$. Any local submersion $\pi: \mathcal{U} \rightarrow Z$ at x (thus possibly just defined in a neighborhood \mathcal{V} of x) is a trivial projection $(\zeta, \eta) \mapsto \zeta$ via the local change of coordinates

$$w_k = \eta_k, \quad k = 1, \dots, \kappa, \quad z_j = \zeta_j + \sum_{i=1}^{\kappa} b_{ji} \eta_i, \quad j = 1, \dots, n, \quad (3.5)$$

where b_{jk} are holomorphic functions. In fact, if $\pi(z, w) = (\pi_1(z, w), \dots, \pi_n(z, w))$ in the coordinates (z, w) , then $\pi_j(z, 0) = z_j$. Thus $\pi_j(z, w) = z_j + \mathcal{O}(w)$, where each $\mathcal{O}(w)$ denotes a function that vanishes on Z , i.e., contains some factor w_i , and so we get (3.5) with $\eta_k = w_k$ and $\zeta_j = \pi_j(z, w)$. We have

$$\frac{\partial}{\partial \eta_k} = \frac{\partial}{\partial w_k} + \sum_{j=1}^n (b_{jk} + \mathcal{O}(w)) \frac{\partial}{\partial z_j}, \quad k = 1, \dots, \kappa. \quad (3.6)$$

In these new coordinates $\mathcal{I} = \langle \eta^{M+1} \rangle$. It thus follows from the argument above that this submersion π gives rise to the Noetherian operators

$$\left(\frac{\partial}{\partial \eta} \right)^\gamma := \left(\frac{\partial}{\partial \eta_\kappa} \right)^{\gamma_\kappa} \cdots \left(\frac{\partial}{\partial \eta_1} \right)^{\gamma_1} \quad (3.7)$$

for $\gamma \leq M$, generating $\mathcal{N}_{X',\pi}$. The notation in (3.7) will be used for the rest of this section. We will also suppress the distinction between a Noetherian operator L in \mathcal{U} with respect to \mathcal{I} and its induced operator \mathcal{L} on X' .

Lemma 3.3. *Each operator $(\partial/\partial \eta)^\gamma$, $\gamma \leq M$, belongs to (i.e., induces an element in) \mathcal{M} .*

Proof. We will proceed by induction over the number of factors $k \leq \kappa$ involved in (3.7). Therefore, assume that $\gamma = (\gamma_1, \dots, \gamma_k) \leq (M_1, \dots, M_k)$,

$$\gamma' = (\gamma_1, \dots, \gamma_{k-1}) \leq M' := (M_1, \dots, M_{k-1})$$

and let

$$\left(\frac{\partial}{\partial\eta}\right)^{\gamma'} := \left(\frac{\partial}{\partial\eta_{k-1}}\right)^{\gamma_{k-1}} \cdots \left(\frac{\partial}{\partial\eta_1}\right)^{\gamma_1}.$$

Assume also that we have proved that there are holomorphic functions $c_{m,\alpha}$, depending on both z and w , such that

$$\left(\frac{\partial}{\partial\eta}\right)^{\gamma'} = \sum_{m' \leq M'} \sum_{|\alpha| \leq |M' - m'|} c_{m',\alpha} \left(\frac{\partial}{\partial z}\right)^{\alpha} \left(\frac{\partial}{\partial w}\right)^{m'}. \quad (3.8)$$

If we apply $(\partial/\partial\eta_k)^{\gamma_k}$ to (3.8) a simple computation gives us (3.8) for k instead of $k-1$. By induction therefore (3.8) holds for $k = \kappa$ and so the lemma follows. \square

Proposition 3.4. *One can choose a finite number of submersions π^ℓ at x such that the corresponding operators $(\partial/\partial\eta^\ell)^\gamma$ for $\gamma \leq M$ together generate \mathcal{M} at x .*

Taking this proposition for granted, we can conclude the proof of Proposition 3.1. In fact, Lemma 3.3 means that $\mathcal{N}_{X',\pi} \subset \mathcal{M}$ for an arbitrary submersion π at x . By definition thus $\mathcal{N}_{X'} \subset \mathcal{M}$. On the other hand, Proposition 3.4 implies, cf. Remark 3.2, that $\mathcal{M} \subset \mathcal{N}_{X'}$. Thus $\mathcal{N}_{X'} = \mathcal{M}$ and so Proposition 3.1 is proved. \square

The rest of this section is devoted to the proof of Proposition 3.4. It will be apparent that one can choose the π^ℓ as arbitrarily small perturbations of any fixed submersion at x . First notice that if we choose a submersion so that b_{jk} are constant in the associated change of variables in (3.5), then

$$\frac{\partial}{\partial\eta_k} = \frac{\partial}{\partial w_k} + \sum_{j=1}^n b_{jk} \frac{\partial}{\partial z_j}, \quad k = 1, \dots, \kappa. \quad (3.9)$$

We will choose our π^ℓ in this way. Then $\partial/\partial\eta_k$ is independent of $w_{k'}$ for $k' \neq k$ which makes it possible to proceed by induction over the codimension κ .

Let us first assume that $\kappa = 1$, i.e., that we have just one variable w . Each point $a^\ell = (a_1^\ell, \dots, a_n^\ell) \in \mathbb{C}^n$ gives rise to a change of coordinates, with $b_{j1} = a_j^\ell$, and thus a submersion π^ℓ . The associated non-tangential derivative is, cf. (3.9),

$$\frac{\partial}{\partial\eta^\ell} = \frac{\partial}{\partial w} + \sum_{j=1}^n a_j^\ell \frac{\partial}{\partial z_j}. \quad (3.10)$$

Recall that

$$C_m := \binom{n+m}{m} \quad (3.11)$$

is the number of multiindices $\alpha = (\alpha_1, \dots, \alpha_n)$ such that $|\alpha| \leq m$.

Lemma 3.5. *If we choose C_m generic points $a^\ell \in \mathbb{C}^n$, then for each α with $|\alpha| \leq m$ there are unique $d_{\ell,\alpha}$ such that*

$$\left(\frac{\partial}{\partial z}\right)^\alpha \left(\frac{\partial}{\partial w}\right)^{m-|\alpha|} \psi = \sum_{\ell} d_{\ell,\alpha} \left(\frac{\partial}{\partial \eta^\ell}\right)^m \psi.$$

Proof. In view of (3.10) we have

$$\left(\frac{\partial}{\partial \eta^\ell}\right)^m \psi = \left(\frac{\partial}{\partial w} + \sum_j a_j^\ell \frac{\partial}{\partial z_j}\right)^m \psi = \sum_{|\alpha| \leq m} (a^\ell)^\alpha \binom{m}{\alpha} \left(\frac{\partial}{\partial z}\right)^\alpha \left(\frac{\partial}{\partial w}\right)^{m-|\alpha|} \psi,$$

where

$$(a^\ell)^\alpha = (a_1^\ell)^{\alpha_1} \cdots (a_n^\ell)^{\alpha_n}.$$

We claim that the $C_m \times C_m$ -matrix $A = (a^\ell)^\alpha$ is invertible if the a^ℓ are generic. If $n = 1$ then A is a Vandermonde matrix, and it is well-known that it is invertible if the $C_m = m + 1$ points a^ℓ in \mathbb{C} are distinct, so the claim follows. For the general case one can argue as follows: Given $x_\alpha \in \mathbb{C}^{C_m}$, consider the polynomial

$$p(t) = \sum_{|\alpha| \leq m} x_\alpha t^\alpha$$

in \mathbb{C}_t^n . We get the action of the matrix A on x_α by evaluating $p(t)$ at the various points a^ℓ . Now $A(x_\alpha) = 0$ means that $p(t)$ vanishes at these C_m generic points, and hence $p(t)$ must vanish identically. This means that $(x_\alpha) = 0$ and since (x_α) is arbitrary, A is invertible. Now the lemma follows by taking

$$x_\alpha = \binom{m}{\alpha} \left(\frac{\partial}{\partial \eta^\ell}\right)^m \psi. \quad \square$$

Proof of Proposition 3.4. For $k = 1, \dots, \kappa$, let L_k be a set of C_{M_k} generic points in \mathbb{C}^n . For each $\ell = (\ell_1, \dots, \ell_\kappa) \in \mathbb{L} := \bigoplus_{k=1}^\kappa L_k$ we get a change of coordinates, and an associated submersion π^ℓ , determined by $b_{jk}^\ell = b_j^{\ell_k}$. The associated differential operators $\partial/\partial \eta_k^\ell$, $k = 1, \dots, \kappa$, only depend, cf. (3.9), on the components $\ell_k \in L_\kappa$, respectively, so we can denote them by $\partial/\partial \eta_k^{\ell_k}$.

We claim that if $m \leq M$ and $|\beta| \leq |M - m|$, then there are complex numbers $c_{\alpha,m,\ell,\gamma}$ for $\gamma \leq M$ such that, for any ψ in $\mathcal{O}_U/\mathcal{I}$,

$$\left(\frac{\partial}{\partial z}\right)^\alpha \left(\frac{\partial}{\partial w}\right)^m \psi = \sum_{\ell \in \mathbb{L}} \sum_{\gamma \leq M} c_{\alpha,m,\ell,\gamma} \left(\frac{\partial}{\partial \eta^\ell}\right)^\gamma \psi. \quad (3.12)$$

Clearly the claim implies the proposition. If $\kappa = 0$ the claim is trivially true. Assume now that the claim is proved for $\kappa - 1$. We can write, in a non-unique way,

$$\left(\frac{\partial}{\partial z}\right)^\beta \left(\frac{\partial}{\partial w}\right)^m = \left(\frac{\partial}{\partial z}\right)^{\beta_\kappa} \left(\frac{\partial}{\partial w_\kappa}\right)^{m_\kappa} \left(\frac{\partial}{\partial z}\right)^{\beta'} \left(\frac{\partial}{\partial w'}\right)^{m'}$$

where $w' = (w_1, \dots, w_{\kappa-1})$, $m' = (m'_1, \dots, m'_{\kappa-1}) \leq M' = (M_1, \dots, M_{\kappa-1})$, $|\alpha'| \leq |M' - m'|$, $m_\kappa \leq M_\kappa$ and $|\beta_\kappa| \leq M_\kappa - m_\kappa$. By the induction hypothesis

$$\omega := \left(\frac{\partial}{\partial z}\right)^{\beta'} \left(\frac{\partial}{\partial w'}\right)^{m'} \psi$$

is a linear combination of

$$\left(\frac{\partial}{\partial \eta^\ell}\right)^{\gamma'} \psi = \left(\frac{\partial}{\partial \eta_{\kappa-1}^{\ell_{\kappa-1}}}\right)^{\gamma_{\kappa-1}} \cdots \left(\frac{\partial}{\partial \eta_1^{\ell_1}}\right)^{\gamma_1} \psi$$

for $\gamma' = (\gamma_1, \dots, \gamma_{\kappa-1}) \leq M'$ and $\ell' = (\ell_1, \dots, \ell_{\kappa-1}) \in \oplus_{j=1}^{\kappa-1} L_k$. Lemma 3.5 implies that

$$\left(\frac{\partial}{\partial z}\right)^{\beta_\kappa} \left(\frac{\partial}{\partial w_\kappa}\right)^{m_\kappa} \omega$$

is a linear combination of

$$\left(\frac{\partial}{\partial \eta_\kappa^{\ell_\kappa}}\right)^{\gamma_\kappa} \omega$$

for $\gamma_\kappa \leq M_\kappa$ and $\ell_\kappa \in L_\kappa$. Now the claim follows. \square

The proof requires $C_{M_1} \cdots C_{M_\kappa}$ different projections π^ℓ to generate the entire \mathcal{O}_Z -module $\mathcal{N}_{X'}$, and we think that this is the optimal number.

4. The sheaf \mathcal{N}_X when Z is smooth

We shall now prove Proposition 1.1 and Theorems 1.3 and 1.4 in the general case. They are local, so let us assume that we at a given point $x \in X$ have an embedding

$$i: X \rightarrow \mathcal{U} \subset \mathbb{C}^N \quad (4.1)$$

and that the underlying reduced space Z is smooth.

Proof of Proposition 1.1. We can assume that we have coordinates (z, w) in \mathcal{U} so that $\pi(z, w) = z$. We first claim that $\mathcal{N}_{X, \pi}$ is an \mathcal{O}_Z -module at x . In fact, if \mathcal{L} is defined by (1.2) for some μ in $\text{Hom}(\mathcal{O}_U/\mathcal{I}, \mathcal{CH}_U^Z)$ and ξ is in \mathcal{O}_Z , then

$$\xi \mathcal{L} \phi \omega_z = \xi \pi_*(\phi \mu) = \pi_*(\phi \pi^* \xi \mu). \quad (4.2)$$

Since $\pi^* \xi \mu$ is in $\text{Hom}(\mathcal{O}_U/\mathcal{I}, \mathcal{CH}_U^Z)$ as well, $\xi \mathcal{L}$ is in $\mathcal{N}_{X, \pi}$, and so the claim follows.

Let us now prove that $\mathcal{N}_{X,\pi}$ is finitely generated at x . By the Nullstellensatz there is a multi-index $M = (M_1, \dots, M_\kappa) \in \mathbb{N}^\kappa$ such that (1.5) holds. If μ is in $\text{Hom}(\mathcal{O}_X, \mathcal{CH}_\mathcal{U}^Z)$, therefore $w_j^{M_j+1}\mu = 0$ for each j . Shrinking \mathcal{U} if necessary we can find μ_1, \dots, μ_ν that generate the $\mathcal{O}_\mathcal{U}$ -module $\text{Hom}(\mathcal{O}_\mathcal{U}/\mathcal{I}, \mathcal{CH}_\mathcal{U}^Z)$. If μ is any current in this sheaf thus

$$\mu = \sum_1^\nu c_j(z, w)\mu_j$$

for some holomorphic c_j . Since

$$c_j(z, w) = \sum_{m \leq M} c_{jk}(z)w^m, \quad j = 1, \dots, \nu, \quad (4.3)$$

in $\mathcal{O}_{X'} = \mathcal{O}_\mathcal{U}/\mathcal{I}$, the equalities (4.3) hold in $\mathcal{O}_X = \mathcal{O}_\mathcal{U}/\mathcal{I}$ as well. Thus

$$\mu = \sum_{j=1}^\nu \sum_{m \leq M} c_{jk}w^m\mu_j.$$

Notice that each $w^m\mu_j$ is in $\text{Hom}(\mathcal{O}_X, \mathcal{CH}_\mathcal{U}^Z)$. If ϕ is in \mathcal{O}_X ,

$$\pi_*(\phi\mu) = \sum_{j=1}^\nu \sum_{m \leq M} c_{jk}\pi_*(\phi w^m\mu_j). \quad (4.4)$$

Thus the \mathcal{O}_Z -module $\mathcal{N}_{X,\pi}$ is generated in \mathcal{U} by $\mathcal{L}_{j,m}$, where

$$\mathcal{L}_{j,m}\phi\omega_z = \pi_*(\phi w^m\mu_j), \quad m \leq M, \quad j = 1, \dots, \nu. \quad (4.5)$$

With no loss of generality we can assume that $\omega_z = dz$. Let $\hat{\mu}$ be as in (3.2). After possibly shrinking \mathcal{U} further, there are holomorphic a_j in \mathcal{U} such that $\mu_j = a_j\hat{\mu}$. Thus

$$\mathcal{L}_{j,m}\phi\omega_z = \pi_*(\phi w^m\mu_j) = \pi_*(\phi a_j w^m\hat{\mu}).$$

It follows from (3.3) that $\mathcal{L}_{j,m}$ are induced by differential operators $L_{j,m}\Phi$ in \mathcal{U} which are Noetherian with respect to \mathcal{I} since Φa_j is in \mathcal{I} if Φ is in \mathcal{I} .

We now claim that (1.2) holds for the set of generators $\mathcal{L}_{j,m}$ above, cf. (4.5), i.e., that $\mathcal{L}_{j,m}\phi = 0$ for $m \leq M$ and $j = 1, \dots, \nu$, if and only if $\phi = 0$ in \mathcal{O}_X . By possibly shrinking \mathcal{U} further, there are holomorphic a_j in \mathcal{U} such that $\mu_j = a_j\hat{\mu}$. For each fixed j , $\mathcal{L}_{j,m}\phi = 0$ for all $m \leq M$ if and only if $0 = \pi_*(\phi w^m\mu_j) = \pi_*(\phi a_j\hat{\mu})$ for all $m \leq M$, and this holds if and only if $\phi a_j = 0$ in $\mathcal{O}_{X'}$, which in turn holds if and only if $\phi a_j\hat{\mu} = 0$, i.e., $\phi\mu_j = 0$. Now the claim follows from the duality principle (2.3).

Finally we notice that $\mathcal{N}_{X,\pi}$ is a finitely generated submodule at x of the free \mathcal{O}_Z -module $\mathcal{N}_{X',\pi}$, generated by $(\partial^{|m|}/\partial w^m)|_{w=0}$, $m \leq M$, and therefore $\mathcal{N}_{X,\pi}$ is coherent. \square

Remark 4.1. Using the setup and notation in the preceding proof we have

$$\mathrm{Hom}(\mathcal{O}_{\mathcal{U}}/\mathcal{I}, \mathcal{CH}_{\mathcal{U}}^Z) = \{a\hat{\mu}; a \in (\mathcal{I} : \mathcal{J})/\mathcal{I}\}; \quad (4.6)$$

the colon ideal (sheaf) $(\mathcal{I} : \mathcal{J})$ by definition consists of all a in $\mathcal{O}_{\mathcal{U}}$ such that $a\mathcal{J} \subset \mathcal{I}$. In fact, we already know that each μ on the left hand side has the form $a\hat{\mu}$ for some a . Recall that $a\hat{\mu} = 0$ if and only if $a \in \mathcal{I}$, cf. Example 2.1. Thus $\mathcal{J}a\hat{\mu} = 0$ if and only if $\mathcal{J}a \subset \mathcal{I}$. Now (4.6) follows. \square

Notice that the generators for the \mathcal{O}_Z -module $\mathcal{N}_{X,\pi}$, cf. (4.5) are, if $\omega_z = dz$, precisely $\mathcal{L}_{j,m}\phi = (\partial^{|m|}(a_j\phi)/\partial w^m)|_{m=0}$, $m \leq M$, $j = 1, \dots, \nu$, where a_1, \dots, a_ν is a generating set for the coherent $\mathcal{O}_{\mathcal{U}}$ -module $(\mathcal{I} : \mathcal{J})/\mathcal{I}$.

Proof of Theorems 1.3 and 1.4. Recall that the \mathcal{O}_Z -module \mathcal{N}_X at a point $x \in X$ is by definition generated by $\mathcal{N}_{X,\pi}$ obtained from all submersions in any local embeddings at x . In view of Lemma 2.4 it is however enough to take all $\mathcal{N}_{X,\pi}$ obtained from one single embedding, so let us fix (4.1). We will use the notation from the proof of Proposition 1.1. Assume that

$$\mathcal{L}\phi dz = \pi_*(\phi\mu),$$

where π is a local submersion and μ is in $\mathrm{Hom}(\mathcal{O}_{\mathcal{U}}/\mathcal{I}, \mathcal{CH}_{\mathcal{U}}^Z)$. In view of (4.4) and (4.5),

$$\mu = \sum_{j=1}^{\nu} \sum_{\gamma \leq M} c_{j\gamma}(z) w^{\gamma} \mu_j = \sum_{j=1}^{\nu} \sum_{\gamma \leq M} c_{j\gamma}(z) w^{\gamma} a_j \hat{\mu}. \quad (4.7)$$

By Theorem 1.4 for X' , i.e., so that $\mathcal{O}_{X'} = \mathcal{O}_{\mathcal{U}}/\mathcal{I}$, we have

$$\pi_*(\psi w^{\gamma} \mu) = \sum_{m \leq M} \sum_{|\beta| \leq |M-m|} d_{m,\gamma,\beta}(z) \frac{\partial^{|\beta|+|m|}}{\partial z^{\beta} \partial w^m} \psi(z, 0) dz. \quad (4.8)$$

Combining (4.7) and (4.8) we get

$$\mathcal{L}\phi dz = \pi_*(\phi\mu) = \sum_{j=1}^{\nu} \sum_{m \leq M} \sum_{|\beta| \leq |M-m|} c'_{j,m,\beta}(z) \frac{\partial^{|\beta|+|m|}}{\partial z^{\beta} \partial w^m} (a_j\phi)(z, 0).$$

Thus the \mathcal{O}_Z -module \mathcal{N}_X is generated by the finite set (1.6) of Noetherian operators on X and so Theorem 1.4 follows. Moreover, each of these differential operators belongs to the free \mathcal{O}_Z -module $\mathcal{N}_{X'}$ and hence \mathcal{N}_X is coherent. Since (1.2) holds already for $\mathcal{N}_{X,\pi}$, by Proposition 1.1, now also Theorem 1.3 is proved. \square

Remark 4.2. To compute the norm $|\cdot|_X$ locally at x by means of Theorem 1.5 one has to choose suitable coordinates, the ideal $\mathcal{I} \subset \mathcal{J}$, and find a_1, \dots, a_ν that generate $(\mathcal{I} : \mathcal{J})/\mathcal{I}$,

i.e., so that $a_1\hat{\mu}, \dots, a_\nu\hat{\mu}$ generate $\mathcal{H}om(\mathcal{O}_{\mathcal{U}}/\mathcal{I}, \mathcal{CH}_{\mathcal{U}}^Z)$, in \mathcal{U} . Then the norm is given by (1.7). \square

Proposition 4.3. *Let (4.1) be a local embedding at $x \in X$ and let $\pi^\ell: \mathcal{U} \rightarrow Z \cap \mathcal{U}$ be a finite number of independent local submersions as in Proposition 3.4. Then the submodules $\mathcal{N}_{X, \pi^\ell}$ generate \mathcal{N}_X in a neighborhood of x .*

Proof. Assume that $\mathcal{L}\phi\omega_z = \pi_*(\phi\mu)$ for a local submersion π and μ in $\mathcal{H}om(\mathcal{O}_{\mathcal{U}}/\mathcal{I}, \mathcal{CH}_{\mathcal{U}}^Z)$. Let (z, w) be coordinates in \mathcal{U} such that π is $(z, w) \mapsto z$, and choose $\mathcal{I} \subset \mathcal{I}$ and the associated $\hat{\mu}$ as before. Moreover, cf. Proposition 2.2, let a be a holomorphic function in \mathcal{U} such that $\mu = a\hat{\mu}$. In view of (3.3) and Proposition 3.4 we have

$$\mathcal{L}\phi dz = \frac{1}{M!} \frac{\partial^{|M|}}{\partial w^M} \Big|_{w=0} (\phi a) dz = \sum_{\ell} \sum_{\gamma \leq M} c_{\ell, \gamma}(z) \left(\frac{\partial}{\partial \eta^\ell} \right)^\gamma \Big|_{w=0} (\phi a) dz. \quad (4.9)$$

If $d\zeta^\ell$ is the non-vanishing holomorphic n -form associated with coordinates defining π^ℓ , then $dz = c_\ell(z)d\zeta^\ell$, so

$$\frac{1}{\gamma!} \left(\frac{\partial}{\partial \eta^\ell} \right)^\gamma \Big|_{w=0} (\phi a) dz = c_\ell \frac{1}{\gamma!} \left(\frac{\partial}{\partial \eta^\ell} \right)^\gamma \Big|_{w=0} (\phi a) d\zeta^\ell = c_\ell \pi_*^\ell (\phi a w^{M-\gamma} \hat{\mu}). \quad (4.10)$$

Thus each $\mathcal{L}_{\ell, \gamma}\phi = (\partial/\partial \eta^\ell)^\gamma|_{w=0}(\phi a)$ is in $\mathcal{N}_{X, \pi^\ell}$, so the proposition follows from (4.9). \square

5. The sheaf \mathcal{N}_X at regular points

Let $i: X \rightarrow \mathcal{U}$ be a local embedding at $x \in X$ with coordinates (z, w) in \mathcal{U} so that $Z = \{w = 0\}$. If (1.5) holds and Φ is holomorphic in \mathcal{U} , then

$$\Phi(z) = \sum_{m \leq M} c_m(z) w^m$$

in $\mathcal{O}_X = \mathcal{O}_{\mathcal{U}}/\mathcal{I}$, where c_m are in \mathcal{O}_Z , cf. (4.3). Thus the right hand side is a representative of $\phi = i^*\Phi$ in \mathcal{O}_X . Therefore the set of monomials $\{w^m; m \leq M\}$ generates \mathcal{O}_X as an \mathcal{O}_Z -module. Let us extract a minimal generating set $1, w^{\alpha_1}, \dots, w^{\alpha_{\tau-1}}$ at x (clearly 1 must be one of the generators). Then each element ϕ in $\mathcal{O}_{X, x}$ has a representative $\hat{\phi}$ of the form (1.8), where $\hat{\phi}_j$ are in $\mathcal{O}_{Z, x}$.

Proposition 5.1. *Given such a minimal generating set at x , the representation (1.8) of ϕ is unique for all ϕ in $\mathcal{O}_{X, x}$ if and only if $\mathcal{O}_{X, x}$ is Cohen-Macaulay.*

For a proof of Proposition 5.1, see, e.g., [7, Proposition 3.1].

Assume now that x is a regular point, i.e., Z is smooth at x and $\mathcal{O}_{X, x}$ is Cohen-Macaulay. Given a minimal generating set w^{α_j} , thus $\mathcal{O}_{X, x}$ is a free $\mathcal{O}_{Z, x}$ -module, i.e.,

$$\mathcal{O}_{Z,x'}^\tau \rightarrow \mathcal{O}_{X,x'}, \quad (\hat{\phi}_j) \mapsto \hat{\phi} := \hat{\phi}_0 + \hat{\phi}_1 w^{\alpha_1} + \cdots + \hat{\phi}_{\tau-1} w^{\alpha_{\tau-1}} \quad (5.1)$$

is an isomorphism for $x' = x$. By coherence, it is an isomorphism for all $x' \in Z$ in a neighborhood of x , say, in $Z \cap \mathcal{U}$, after shrinking \mathcal{U} . Thus $\phi = 0$ in $\mathcal{O}_{X,x'}$ if and only if $\hat{\phi}_j = 0$ in $\mathcal{O}_{Z,x'}$ for $j = 0, \dots, \tau - 1$, so the expression (1.9) is a pointwise norm of ϕ in $X \cap \mathcal{U}$.

Proof of Theorem 1.5. Let us choose $i: X \rightarrow \mathcal{U}$ at x so that (5.1) is an isomorphism for $x \in X \cap \mathcal{U}$. We have to relate (1.9) to our norm $|\cdot|_X$, and we proceed as follows: Assume that μ_1, \dots, μ_ν is a generating set for the $\mathcal{O}_{\mathcal{U}}$ -module $\mathcal{H}om(\mathcal{O}_{\mathcal{U}}/\mathcal{I}, \mathcal{CH}_{\mathcal{U}}^Z)$. If ϕ is in \mathcal{O}_X , then $\phi\mu_j$ are well-defined elements in $\mathcal{H}om(\mathcal{O}_{\mathcal{U}}/\mathcal{I}, \mathcal{CH}_{\mathcal{U}}^Z)$. With the notation in the proof of Proposition 1.1, cf. (2.9) and (2.10), we have the unique representations

$$\phi\mu_j = \sum_{m \leq M} b_{j,m}(z) \frac{1}{(2\pi i)^\kappa} \bar{\partial} \frac{dw}{w^{m+1}} \wedge dz, \quad j = 1, \dots, \nu, \quad (5.2)$$

where

$$b_{j,m} \wedge dz = \pi_*(\phi w^m \mu_j). \quad (5.3)$$

If we represent ϕ by $\hat{\phi}$ in (5.1), then

$$b_{j,m} = \hat{\phi}_0 \pi_*(w^m \mu_j) + \hat{\phi}_1 \pi_*(w^{m+\alpha_1} \mu_j) + \cdots + \hat{\phi}_{\nu-1} \pi_*(w^{m+\alpha_{\nu-1}} \mu_j)$$

and thus $b_{j,m}$ are \mathcal{O}_Z -linear combinations of the $\hat{\phi}_j$. Hence the mapping

$$\phi \mapsto \phi \wedge \mu_j, \quad j = 1, \dots, \nu,$$

via the isomorphism (5.1), induces an \mathcal{O}_Z -linear holomorphic morphism

$$T: \mathcal{O}_Z^\tau \rightarrow \mathcal{O}_Z^{\nu C_M},$$

where $C_M = (M+1)!$ is the number of $m \in \mathbb{N}^\kappa$ such that $m \leq M$.

In view of the duality principle (2.3), T is injective. In fact, the image of T being zero, means that $\phi\mu = 0$, i.e., $\phi\mu_j = 0$ for $j = 1, \dots, \nu$, and so $\phi = 0$ in \mathcal{O}_X which in turn, cf. (5.1), means that $\hat{\phi}_j = 0$ for $j = 0, \dots, \tau - 1$. By [7, Lemma 4.11], the matrix T is pointwise injective. If $(b_{j,m})$ is in the image of T therefore

$$\sum_{j=0} |\hat{\phi}_j|^2 \sim \sum_{j=1}^\nu \sum_{m \leq M} |b_{j,m}|^2. \quad (5.4)$$

From (4.5) and (5.3) we see that

$$\sum_{j=1}^{\nu} \sum_{m \leq M} |b_{j,m}|^2 \sim \sum_{j=1}^{\nu} \sum_{m \leq M} |\mathcal{L}_{j,m} \phi|^2 = |\phi|_{X,\pi}^2. \quad (5.5)$$

Now Theorem 1.5 follows from (5.4) and (5.5). \square

6. An example

Consider the 2-plane $Z = \{w_1 = w_2 = 0\}$ in $\mathcal{U} \subset \mathbb{C}_{z_1, z_2, w_1, w_2}^4$, where \mathcal{U} is the product of balls $\{|z| < 1, |w| < 1\}$ in \mathbb{C}^4 , and let

$$\mathcal{J} = \langle w_1^2, w_2^2, w_1 w_2, w_1 z_2 - w_2 z_1 \rangle.$$

Then $\mathcal{O}_{\mathcal{U}}/\mathcal{J}$ has pure dimension 2 and is Cohen-Macaulay except at the point $0 \in \mathcal{U}$, see, [7, Example 6.9]. It is also shown there, notice that $\mathcal{I} = \langle w_1^2, w_2^2 \rangle \subset \mathcal{J}$, that $\text{Hom}(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}_{\mathcal{U}}^Z)$ is generated by

$$\mu_1 = w_1 w_2 \hat{\mu}, \quad \mu_2 = (z_1 w_2 + z_2 w_1) \hat{\mu},$$

where

$$\hat{\mu} = \frac{1}{(2\pi i)^2} \bar{\partial} \frac{dw_1}{w_1^2} \wedge \bar{\partial} \frac{dw_2}{w_2^2} \wedge dz_1 \wedge dz_2.$$

Following the recipe in Theorem 1.4 and Remark 4.2 we get a generating set for \mathcal{N}_X by applying each of the differential operators

$$1, \frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}, \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial^2}{\partial z_1 \partial w_1}, \frac{\partial^2}{\partial z_1 \partial w_2}, \frac{\partial^2}{\partial z_2 \partial w_1}, \frac{\partial^2}{\partial z_2 \partial w_2}, \frac{\partial^2}{\partial w_1 \partial w_2}$$

to $a_1 \phi = w_1 w_2 \phi$ and $a_2 \phi = (z_1 w_2 + z_2 w_1) \phi$, respectively, and evaluate at $w = 0$. Then a_1 only contributes with the Noetherian operator 1, whereas a_2 gives rise to

$$z_1, z_2, 0, 0, z_2 \frac{\partial}{\partial z_1}, (1 + z_1 \frac{\partial}{\partial z_1}), (1 + z_2 \frac{\partial}{\partial z_2}), z_1 \frac{\partial}{\partial z_2}, (z_1 \frac{\partial}{\partial w_1} + z_2 \frac{\partial}{\partial w_2}). \quad (6.1)$$

Because of the operator 1 from the a_1 , we can forget about z_j and replace $1 + z_j \frac{\partial}{\partial z_j}$ by $z_j \frac{\partial}{\partial z_j}$. Thus we get

$$|\phi|_X^2 \sim |\phi|^2 + |z|^2 \left| \frac{\partial \phi}{\partial z_1} \right|^2 + |z|^2 \left| \frac{\partial \phi}{\partial z_2} \right|^2 + \left| z_1 \frac{\partial \phi}{\partial w_1} + z_2 \frac{\partial \phi}{\partial w_2} \right|^2.$$

6.1. Functions in $X \setminus \{0\}$

Let $\mathcal{L}_0 = 1$ and let \mathcal{L} denote the right-most operator in (6.1). If ϕ is an \mathcal{O}_X -function defined in $Z \setminus \{0\}$, then both $\mathcal{L}_0 \phi$ and $\mathcal{L} \phi$ are holomorphic functions in $Z \setminus \{0\}$. Since

$\{0\}$ has codimension 2 in Z , they both have holomorphic extensions across 0 that we denote by $\phi_0(z)$ and $h(z)$, respectively.

Notice that $1, w_1$ is a basis for \mathcal{O}_X over \mathcal{O}_Z where $z_1 \neq 0$, and similarly, $1, w_2$ is a basis for \mathcal{O}_X over \mathcal{O}_Z where $z_2 \neq 0$. Given any ϕ_0 and h in \mathcal{U} we get a \mathcal{O}_X -function ϕ in $\mathcal{U} \setminus \{0\}$, defined as

$$\phi = \phi_0 + (h/z_1)w_1, \quad z_1 \neq 0; \quad \phi = \phi_0 + (h/z_2)w_2, \quad z_2 \neq 0. \quad (6.2)$$

It is readily checked that $\mathcal{L}_0\phi = \phi_0$ and $\mathcal{L}\phi = h$. In other words, there is a 1 – 1 correspondence between \mathcal{O}_X -functions ϕ in $Z \setminus \{0\}$ and \mathcal{O}_Z^2 .

Lemma 6.1. *The \mathcal{O}_X -function ϕ has an extension across 0 if and only if $h(0) = 0$.*

Proof. If ϕ is defined in \mathcal{U} then $h = \mathcal{L}\phi$ in \mathcal{U} and then clearly $h(0) = 0$. Conversely, if $h(0) = 0$, then $h(z) = c_1(z)z_1 + c_2(z)z_2$ for some functions c_1, c_2 in \mathcal{U} . It is readily checked that indeed ϕ , defined by (6.2), coincides with

$$\phi_0(z) + c_1(z)w_1 + c_2(z)w_2$$

in $\mathcal{U} \setminus \{0\}$. Thus ϕ extends across 0. \square

In view of this lemma, if we take, e.g., $h = 1$ in (6.2), we get an \mathcal{O}_X -function ϕ in $\mathcal{U} \setminus \{0\}$ that does not extend across 0.

7. Extension of \mathcal{N}_X across Z_{sing}

We now drop the assumption that the underlying space Z is smooth.

Lemma 7.1. *Let x be a fixed point on the singular locus Z_{sing} of Z and let $X \rightarrow \mathcal{U} \subset \mathbb{C}^N$ be a local embedding at x . If \mathcal{U} is small enough there are holomorphic functions f_1, \dots, f_κ so that $Z(f) = \{f_1 = \dots = f_\kappa = 0\}$ has codimension κ , and contains $Z \cap \mathcal{U}$ and such that $df := df_1 \wedge \dots \wedge df_\kappa$ is non-vanishing on $Z_{\text{reg}} \setminus Z(f)_{\text{sing}}$. If $x' \in \mathcal{U} \setminus Z$ is given we can choose f_j so that $x' \notin Z(f)$.*

That is, $Z(f)$ is a complete intersection that may have “unnecessary” irreducible components, but $df \neq 0$ at each point on Z_{reg} that is not hit by any of these components. This is of course a well-known result and follows, e.g., from the more precise statement in the lemma on page 72 in [21]. However, we provide a simple argument here for the reader’s convenience.

Proof. If \mathcal{U} is small enough we can find a finite number of functions g_1, \dots, g_m that generate \mathcal{J}_Z . For each irreducible component Z^ℓ of Z we choose a point $x^\ell \in Z_{\text{reg}}^\ell \cap \mathcal{U}$. Notice that dg_j span the annihilator of the tangent bundle at x^ℓ for each ℓ . If f_1, \dots, f_κ

are generic linear combinations of the g_j , then df_j span these spaces as well for each ℓ , and f_j define a complete intersection $Z(f)$ that avoids x' . Clearly $Z \subset Z(f)$ and $df \neq 0$ at x^ℓ for each ℓ . It is not hard to see (cf. [25, Theorem 4.3.6]) that df is non-vanishing on the regular part of the irreducible component of $Z(f)$ that contains x^ℓ ; i.e., on $Z_{reg}^\ell \setminus Z(f)_{sing}$, for each ℓ . \square

Let x , f and \mathcal{U} be as in Lemma 7.1 and let us write Z rather than $Z \cap \mathcal{U}$. Since df is generically non-vanishing on Z we can choose coordinates $(\zeta, \eta) = (\zeta_1, \dots, \zeta_n; \eta_1, \dots, \eta_\kappa)$ in \mathcal{U} such that, with suitable matrix notation, $H = \partial f / \partial \eta$ is generically invertible on Z . Let $h = \det H$. If

$$w = f(\zeta, \eta), \quad z = \zeta, \quad (7.1)$$

then, cf. (2.6), $dw \wedge dz = h d\eta \wedge d\zeta$ and hence (z, w) are local coordinates at each point on $Z \setminus \{h = 0\}$. Notice that

$$\frac{\partial}{\partial w} = H^{-1} \frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial \zeta} - G \frac{\partial}{\partial w}, \quad (7.2)$$

where $G = \partial f / \partial \zeta$ is holomorphic. Since $H^{-1} = \Theta / h$, where Θ is holomorphic, therefore

$$h \frac{\partial}{\partial w} = \Theta \frac{\partial}{\partial \eta}, \quad h \frac{\partial}{\partial z} = h \frac{\partial}{\partial \zeta} - G \Theta \frac{\partial}{\partial \eta}. \quad (7.3)$$

For a sufficiently large multiindex $M = (M_1, \dots, M_\kappa)$ the complete intersection ideal $\langle f_1^{M_1+1}, \dots, f_\kappa^{M_\kappa+1} \rangle$ is contained in \mathcal{J} . Possibly after shrinking the neighborhood \mathcal{U} of x there are generators μ_1, \dots, μ_ν for $\text{Hom}(\mathcal{O}_{\mathcal{U}/\mathcal{J}}, \mathcal{CH}_{\mathcal{U}}^Z)$ and holomorphic functions a_1, \dots, a_κ in \mathcal{U} , cf. Proposition 2.2, such that

$$\mu_j = a_j \frac{1}{(2\pi i)^\kappa} \bar{\partial} \frac{1}{f^{M+1}} \wedge d\eta \wedge d\zeta. \quad (7.4)$$

Notice that a_j must vanish on the “unnecessary” irreducible components of $Z(f)$. For the rest of this section we will use the notation (3.7).

Proposition 7.2. *With the notation above, the differential operators*

$$\Phi \mapsto L_{m,\beta,j} \Phi := \left(h \frac{\partial}{\partial w} \right)^m \left(h \frac{\partial}{\partial z} \right)^\beta (a_j \Phi), \quad m \leq M, \quad |\beta| \leq |M - m|, \quad j = 1, \dots, \nu, \quad (7.5)$$

a priori defined on $\mathcal{U} \cap \{h \neq 0\}$, have holomorphic extensions to \mathcal{U} . They are Noetherian with respect to \mathcal{J} and the induced operators $\mathcal{L}_{m,\beta,j}$, cf. (1.1), belong to \mathcal{N}_X on Z_{reg} and generate the \mathcal{O}_Z -module \mathcal{N}_X where $h \neq 0$.

Proof. From (7.3) it is clear that $L_{m,\beta,j}$ have holomorphic extensions to \mathcal{U} . Since (z, w) are local coordinates at a point on Z_{reg} where $h \neq 0$ it follows from (7.2) and Theorem 1.4, cf. Remark 4.1, that the induced operators $\mathcal{L}_{m,\beta,j}$ are in \mathcal{N}_X there. By a simple induction argument it follows from the same theorem that they actually generate \mathcal{N}_X there. It also follows that $L_{m,\beta,j}$ are Noetherian there with respect to \mathcal{J} in $\mathcal{U} \cap \{h \neq 0\}$ and by continuity their extensions are Noetherian as well. Thus $\mathcal{L}_{m,\beta,j}$ are Noetherian on $X \cap \mathcal{U}$.

We have to prove that $\mathcal{L}_{m,\beta,j}$ are in \mathcal{N}_X on Z_{reg} where $h = 0$. Let $x' \in X_{reg}$ be such a point and assume that $df \neq 0$. For a generic choice of constant matrices b, c we have that $df \wedge d(c\zeta + b\eta) \neq 0$. Thus we can choose new coordinates

$$w' = f(\zeta, \eta), \quad z' = c\zeta + b\eta$$

in a neighborhood $\mathcal{V} \subset \mathcal{U}$ of x' . It follows that

$$\phi \mapsto \left(\frac{\partial}{\partial w'} \right)^m \left(\frac{\partial}{\partial z'} \right)^\alpha \Big|_{w'=0} (\phi a_j), \quad m \leq M, \quad |\alpha| \leq |M - m|, \quad (7.6)$$

are in \mathcal{N}_X in $Z \cap \mathcal{V}$. Since $z' = bz + cw$, $w' = w$, in $\mathcal{V} \setminus \{h = 0\}$, we have

$$\frac{\partial}{\partial w} = \frac{\partial}{\partial w'} + \frac{\partial z'}{\partial w} \frac{\partial}{\partial z'}, \quad \frac{\partial}{\partial z} = \frac{\partial z'}{\partial z} \frac{\partial}{\partial z'},$$

so by (7.3), applied to z', w' instead of ζ, η ,

$$h \frac{\partial}{\partial w_k} = h \frac{\partial}{\partial w'_k} + \sum_i d_{ki} \frac{\partial}{\partial z'_i}, \quad h \frac{\partial}{\partial z_k} = \sum_i d'_{ki} \frac{\partial}{\partial z'_i}, \quad (7.7)$$

where d_{ki}, d'_{ki} have holomorphic extensions to \mathcal{V} . Thus $\mathcal{L}_{m,\beta,j}$ are \mathcal{O}_Z -linear combinations of (7.6), and hence belong to \mathcal{N}_X .

Let us now consider a point $x' \in Z_{reg}$ where $df = 0$, i.e., some “unnecessary” component of $Z(f)$ passes through x' . Then certainly $h(x') = 0$. Let π be the projection $(z, w) \mapsto z$. By (7.1), (7.4), and (3.3),

$$\frac{1}{\gamma!} \left(\frac{\partial}{\partial w} \right)^\gamma (a_j \phi) dz = \pi_* \left(\phi a_j df \wedge f^{M-\gamma} \frac{1}{(2\pi i)^\kappa} \bar{\partial} \frac{1}{f^{M+1}} \wedge dz \right), \quad \gamma \leq M, \quad (7.8)$$

in $\mathcal{U} \setminus \{h = 0\}$. Let $v = (v_1, \dots, v_\kappa)$ generate \mathcal{J}_Z at x' and assume first that $dv \wedge dz \neq 0$ so that (z, v) are local coordinates in a neighborhood $\mathcal{V} \subset \mathcal{U}$ of x' . Since $\mathcal{J}(f) \subset \mathcal{J}_Z$, $f = Av$ for a holomorphic matrix A in \mathcal{V} . At points $z \in Z \cap \mathcal{V} \setminus \{h = 0\}$ both f_j and v_j are minimal sets of generators for \mathcal{J}_Z so A is invertible there. Therefore also (z, v) define the submersion π in $\mathcal{V} \setminus \{h = 0\}$. Since $f^{M-\gamma} df \wedge dz = \alpha d\eta \wedge d\zeta$, where α is holomorphic, the right hand side of (7.8) is $\pi_*(\phi \alpha \mu_k)$ which is $d\zeta$ times an element in \mathcal{N}_X in \mathcal{V} . It follows that the left hand side of (7.8) is $d\zeta$ times $\mathcal{L}\phi$, where \mathcal{L} extends to an element in \mathcal{N}_X in \mathcal{V} .

Notice that if b^ℓ is a small constant $n \times \kappa$ -matrix, then $\eta' = \eta$, $\zeta' = \zeta + b^\ell f$ is a change of variables in \mathcal{V} , possibly after shrinking our neighborhood \mathcal{V} of x' . In fact, $d\eta' \wedge d\zeta' = d\eta \wedge (d\zeta + b^\ell df)$ is non-vanishing if b^ℓ is small enough. Taking $w^\ell = f$, $z^\ell = \zeta'$, we get that $w^\ell = w$, $z^\ell = z + b^\ell w$, and hence

$$dw^\ell \wedge dz^\ell = df \wedge d(\zeta + b^\ell df) = df \wedge d\zeta = h d\eta \wedge d\zeta,$$

where h is the same function as in (7.7). As in the preceding step of the proof we conclude that

$$\phi \mapsto \left(\frac{\partial}{\partial w^\ell} \right)^\gamma (a_j \phi), \quad \gamma \leq M,$$

a priori defined in $\mathcal{V} \setminus \{h = 0\}$, have extensions to elements in \mathcal{N}_X in $Z \cap \mathcal{V}$. By Proposition 3.4 there is a finite set of such b^ℓ and holomorphic $d_{m,\beta,\ell,j}$ in a possibly even smaller neighborhood \mathcal{V} of x' such that

$$\left(\frac{\partial}{\partial w} \right)^m \left(\frac{\partial}{\partial z} \right)^\beta (a_j \phi) = \sum_{\ell} \sum_{\gamma \leq M} d_{m,\beta,\ell,\gamma} \left(\frac{\partial}{\partial w^\ell} \right)^\gamma (a_j \phi), \quad m \leq M, \quad |\beta| \leq |M - m|. \quad (7.9)$$

It follows that all the operators on the left hand side of (7.9) are in \mathcal{N}_X in \mathcal{V} .

Finally, if $dv \wedge d\zeta = 0$ at x' we introduce new coordinates $z' = c\zeta + b\eta$, $w' = w$ as before so that $dv \wedge dz' \neq 0$. From what we have just proved, then all

$$\left(\frac{\partial}{\partial w'} \right)^m \left(\frac{\partial}{\partial z'} \right)^\beta (a_j \phi)$$

are holomorphic at x' . It now follows from (7.7) that $\mathcal{L}_{m,\beta,j}$ are in \mathcal{N}_X at x' . Thus Proposition 7.2 is proved. \square

We can now formulate our main result of this section.

Theorem 7.3. *Given a point $x \in Z_{\text{sing}}$ there is a local embedding at x $i: X \rightarrow \mathcal{U} \subset \mathbb{C}^N$ and a finite number of Noetherian operators $\mathcal{L}_1, \dots, \mathcal{L}_r$ on $X \cap \mathcal{U}$ that generate \mathcal{N}_X on $\mathcal{U} \cap Z_{\text{reg}}$.*

Clearly, such a set $\mathcal{L}_1, \dots, \mathcal{L}_r$ defines a coherent extension of \mathcal{N}_X to $\mathcal{U} \cap Z$.

Proof. Choose the embedding $i: X \rightarrow \mathcal{U} \subset \mathbb{C}^N$ at x small enough so that we have a complete intersection $f = (f_1, \dots, f_\kappa)$ as above, global coordinates (ζ, η) , and so that Proposition 7.2 applies, with $h = \det(\partial f / \partial \eta)$. We then get \mathcal{L}_j in \mathcal{N}_X on $X \cap Z_{\text{reg}}$ that have holomorphic extensions across Z_{sing} and that generate \mathcal{N}_X in $Z_{\text{reg}} \cap (\mathcal{U} \setminus \{h = 0\})$. Choosing η' as other sets of κ coordinates we get another set of such \mathcal{L}_j that generate \mathcal{N}_X on $Z_{\text{reg}} \cap \mathcal{U}$ except where $h' = \det(\partial f / \partial \eta')$ vanishes. Repeating a finite number of times we get a finite set of \mathcal{L}_j that generate \mathcal{N}_X on $Z_{\text{reg}} \cap (\mathcal{U} \setminus \{df = 0\})$.

Possibly after shrinking \mathcal{U} , we can make the same construction for a finite number f^1, \dots, f^p of complete intersections such that $Z \cap \mathcal{U} = Z(f^1) \cap \dots \cap Z(f^p) \cap \mathcal{U}$, see Lemma 7.1, and we thus get a finite set \mathcal{L}_j as desired. \square

Example 7.4. Let $Z = \{f = 0\}$ be a reduced subvariety of $\mathcal{U} \subset \mathbb{C}^2$ and assume that $df \neq 0$ on Z_{reg} . If X is defined by $\mathcal{J} = \langle f^2 \rangle$, then

$$\mu = \bar{\partial} \frac{1}{f^2} \wedge d\eta \wedge d\zeta$$

is a generator for $\mathcal{H}om(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}_{\mathcal{U}}^Z)$. Let us choose coordinates (ζ, η) on \mathbb{C}^2 so that neither $h := \partial f / \partial \eta$ nor $\partial f / \partial \zeta$ vanish identically on Z . If we let $w = f(\zeta, \eta)$ and $z = \zeta$, then

$$h \frac{\partial}{\partial w} = \frac{\partial}{\partial \eta}, \quad h \frac{\partial}{\partial z} = h \frac{\partial}{\partial \zeta} - \frac{\partial f}{\partial \zeta} \frac{\partial}{\partial \eta}.$$

Thus Proposition 7.2 gives us the Noetherian operators $1, \partial/\partial \eta, h(\partial/\partial \zeta)$. If we add the operators obtained by interchanging the roles of η and ζ we find that the extension of \mathcal{N}_X across Z_{sing} generated by $1, \partial/\partial \eta, \partial/\partial \zeta$. Clearly, this extension is independent of the choice of coordinates in \mathcal{U} . \square

7.1. Global pointwise norm on X

In Example 7.4 the extension of \mathcal{N}_X across Z_{sing} is invariant. We do not know whether this is true in general. In any case we can define a global pointwise norm in the following way: Each point $x \in Z_{sing}$ has a neighborhood \mathcal{U}_x where we have a coherent extension by Theorem 7.3 and in \mathcal{U}_x we thus have a pointwise norm $|\cdot|_{X,x}$. We can choose a locally finite open covering $\{\mathcal{U}_{x_j}\}$ of X , and a partition of unity χ_j subordinate to this covering and define the global norm

$$|\cdot|_X^2 = \sum_j \chi_j |\cdot|_{X,x_j}^2. \quad (7.10)$$

8. Pointwise norm of smooth $(0, q)$ -forms

In [7] was introduced a notion of smooth $(0, q)$ -form on a non-reduced space X . We will recall this definition and show that our pointwise norm $|\cdot|_X$ extends to a pointwise norm on such forms.

Consider a local embedding $i: X \rightarrow \mathcal{U} \subset \mathbb{C}^N$ as before. If Φ is a smooth $(0, q)$ -form in \mathcal{U} , $\Phi \in \mathcal{E}_{\mathcal{U}}^{0,q}$, following [7, Section 4] we say that $i^*\Phi = 0$, or equivalently $\Phi \in \text{Ker } i^*$, if

$$\Phi \wedge \mu = 0, \quad \mu \in \mathcal{H}om(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}_{\mathcal{U}}^Z).$$

In case Φ is holomorphic, this is equivalent to that $\Phi \in \mathcal{J}$ in view of the duality principle (2.3). We let

$$\mathcal{E}_X^{0,q} := \mathcal{E}_U^{0,q} / \text{Ker } i^*$$

be the sheaf of smooth $(0, k)$ -forms on X . Thus we have a well-defined surjective mapping $i^*: \mathcal{E}_U^{0,q} \rightarrow \mathcal{E}_X^{0,q}$. By a standard argument, cf. Section 2.2, one checks that this definition is independent of the local embedding. For ϕ in $\mathcal{E}_X^{0,q}$ and μ in $\text{Hom}(\mathcal{O}_U/\mathcal{J}, \mathcal{CH}_U^Z)$ thus $\phi \wedge \mu$ is well-defined, and it vanishes for all such μ if and only if $\phi = 0$.

To extend our norm to forms in $\mathcal{E}_X^{0,q}$ let us first assume that the underlying reduced space Z is smooth. Assume that we have a local embedding $i: X \rightarrow \mathcal{U}$, and a submersion $\pi: \mathcal{U} \rightarrow Z \cap \mathcal{U}$. If μ is in $\text{Hom}(\mathcal{O}_U/\mathcal{J}, \mathcal{CH}_U^Z)$ and ϕ is in $\mathcal{E}_X^{0,q}$, then $\pi_*(\phi \wedge \mu)$ is a well-defined $(0, q)$ -current on $Z \cap \mathcal{U}$ so we have

$$\mathcal{L}\phi \omega_z = \pi_*(\phi \wedge \mu). \quad (8.1)$$

Lemma 8.1. *An operator \mathcal{L} so defined maps $\mathcal{L}: \mathcal{E}_X^{0,q} \rightarrow \mathcal{E}_Z^{0,q}$ and it is determined by its action on \mathcal{O}_X . If $\mathcal{L}\phi = 0$ for all μ in $\text{Hom}(\mathcal{O}_U/\mathcal{J}, \mathcal{CH}_U^Z)$, then $\phi = 0$.*

Since the \mathcal{O}_Z -module \mathcal{N}_X is generated by operators of the form (8.1), it follows that any \mathcal{L} in \mathcal{N}_X extends to an operator $\mathcal{L}: \mathcal{E}_X^{0,q} \rightarrow \mathcal{E}_Z^{0,q}$.

Proof. Choose local coordinates (z, w) in \mathcal{U} such that π is $(z, w) \mapsto z$. Then, cf. (2.9), each μ in $\text{Hom}(\mathcal{O}_U/\mathcal{J}, \mathcal{CH}_U^Z)$ has the form

$$\mu = \sum_m c_m(z) \frac{1}{(2\pi i)^\kappa} \bar{\partial} \frac{dw}{w^{m+1}} \wedge dz. \quad (8.2)$$

Let us first assume that ϕ is a function in $\mathcal{E}_X^{0,0}$. Choose a smooth function Φ in $\mathcal{E}_U^{0,0}$ such that $\phi = i^*\Phi$. Then

$$\Phi\mu = \sum_m c_m(z) \Phi(z, w) \frac{1}{(2\pi i)^\kappa} \bar{\partial} \frac{dw}{w^{m+1}} \wedge dz$$

and by (2.8) thus

$$\pi_*(\phi\mu) = \pi_*(\Phi\mu) = \sum_m c_m(z) \frac{1}{m!} \frac{\partial^{|m|}}{\partial z^m} \Phi(z, 0) dz. \quad (8.3)$$

This differential operator is determined by its action on holomorphic functions and so the first statement of the lemma is proved for $q = 0$. If ϕ is in $\mathcal{E}_X^{0,q}$, $q \geq 1$, then $\phi = i^*\Phi$ for some form

$$\sum_{|J|=q}^I \Phi_J(z, w) d\bar{z}_J$$

in \mathcal{U} , since any term with a factor $d\bar{w}_j$ belongs to $\text{Ker } i^*$. If $\phi_J = i^* \Phi_J$, we see that

$$\mathcal{L}\phi = \sum_{|J|=q}^I \mathcal{L}\phi_J d\bar{z}_J.$$

Thus the first part of the lemma is proved. The second statement follows since $\pi_*(\phi \wedge w^m \mu) = 0$ for all m and μ implies that $\phi \wedge \mu = 0$ for all μ so by definition $\phi = 0$. \square

Remark 8.2. Notice that if L is the differential operator on the right hand side of (8.3), and $\phi = i^* \Phi$, then, observing that $L\Phi$ is well-defined for $(0, q)$ -forms in \mathcal{U} ,

$$\mathcal{L}\phi = \iota^* L\Phi, \quad (8.4)$$

where $\iota: Z \rightarrow \mathcal{U}$ is the underlying embedding. \square

Proposition 8.3. *Let x be a fixed point $x \in Z_{\text{sing}}$ and let $i: X \rightarrow \mathcal{U} \subset \mathbb{C}^N$ be a local embedding at x as in Theorem 7.3. All the operators $\mathcal{L}_1, \dots, \mathcal{L}_\rho$ extend to operators $\mathcal{E}_X^{0,q} \rightarrow \mathcal{E}_Z^{0,q}$. Moreover, $\phi = 0$ if (and only if) $\mathcal{L}_j \phi = 0$ for $j = 1, \dots, \rho$.*

Proof. We first prove the extension for the operators $\mathcal{L}_{m,\beta,j}$ in Proposition 7.2. By definition a smooth $(0, q)$ -form ϕ on $X \cap \mathcal{U}$ is represented by a smooth $(0, q)$ -form Φ in \mathcal{U} and thus each $L_{m,\beta,j}\Phi$ is a smooth $(0, q)$ -form in \mathcal{U} . Moreover, since it is Noetherian with respect to \mathcal{J} in $\mathcal{U} \setminus Z_{\text{sing}}$, i.e., $L_{m,\beta,j}\Phi = 0$ if Φ is in \mathcal{J} it follows by continuity that this holds also across Z_{sing} . By Remark 8.2, $\mathcal{L}_{m,\beta,j}\phi := \iota^* L_{m,\beta,j}\Phi$ in $Z_{\text{reg}} \cap \mathcal{U}$, and the same formula defines a smooth extension across $Z_{\text{sing}} \cap \mathcal{U}$. By continuity this extension is unique. All the operators $\mathcal{L}_1, \dots, \mathcal{L}_\rho$ are obtained in this way so the first statement in Proposition 8.3 is proved. Since \mathcal{L}_j generate \mathcal{N}_X at each point outside Z_{sing} it follows that $\phi = 0$ there if $\mathcal{L}_j \phi = 0$ for $j = 1, \dots, \rho$. By continuity then $\phi = 0$ in $X \cap \mathcal{U}$. \square

Assuming that we have chosen a Hermitian norm on Z , cf. the beginning of the introduction, we now get a pointwise norm

$$|\phi|_{\mathcal{U}}^2 = \sum_{j=1}^{\rho} |\mathcal{L}_j \phi|_Z^2$$

on \mathcal{U} of ϕ in $\mathcal{E}_X^{0,q}$. Patching together as in Section 7.1 we get a global norm on X .

Remark 8.4. Proposition 1.1 as well as Theorem 1.5 have analogues for smooth $(0, q)$ -forms, and they are proved in basically the same way. We omit the details. \square

9. Pseudomeromorphic currents

Let Y be a reduced analytic space. The \mathcal{O}_Y -module $\mathcal{P}M_Y$ of pseudomeromorphic currents on Y was introduced in [10,8]. Roughly speaking, it consists of currents that locally are finite sums of direct images (under possibly nonproper mappings) of products of simple principal value currents and $\bar{\partial}$ of such currents. See [11] for a precise definition and for the properties stated in this section. The sheaf $\mathcal{P}M_Y$ is closed under $\bar{\partial}$ and under multiplication by smooth forms. If τ is pseudomeromorphic in an open subset $\mathcal{U} \subset Y$ and $W \subset \mathcal{U}$ is a subvariety then there is a well-defined pseudomeromorphic current $\mathbf{1}_{\mathcal{U} \setminus W} \tau$ in \mathcal{U} obtained by extending the natural restriction of τ to $\mathcal{U} \setminus W$ in the trivial way. With the notation in Section 2.1, $\mathbf{1}_{\mathcal{U} \setminus W} \tau = \lim_{\epsilon} \chi(|h|/\epsilon) \tau$ if h is a tuple of holomorphic functions with common zero set W . Thus $\mathbf{1}_W \tau := \tau - \mathbf{1}_{\mathcal{U} \setminus W} \tau$ is a pseudomeromorphic current with support on W . If $W' \subset \mathcal{U}$ is another subvariety, then

$$\mathbf{1}_{W'} \mathbf{1}_W \tau = \mathbf{1}_{W' \cap W} \tau. \quad (9.1)$$

We can rephrase the standard extension property, cf. Section 2.1: If τ has support on a subvariety Z of pure dimension, then τ has the SEP with respect to Z if for each open subset $\mathcal{U} \subset Y$ and subvariety $W \subset \mathcal{U} \cap Z$ with positive codimension in Z , $\mathbf{1}_W \tau = 0$.

An important property is the dimension principle: *If τ in $\mathcal{P}M_Y$ has bidegree $(*, q)$ and support on a variety of codimension larger than q , then τ must vanish.*

Recall that a current τ on a manifold is semi-meromorphic if there are a smooth form ω with values in a line bundle L , and a non-trivial holomorphic section f of L , such that $\tau = \omega/f$, considered as a principal value current. We say that a current α on Y is *almost semi-meromorphic* if there is a smooth modification $\pi: \tilde{Y} \rightarrow Y$ and a semi-meromorphic current $\tilde{\alpha}$ in \tilde{Y} such that $\alpha = \pi_* \tilde{\alpha}$. Notice that an almost semi-meromorphic α is smooth outside an analytic set W of positive codimension in Y .

Example 9.1. Coleff-Herrera currents in $\mathcal{U} \subset \mathbb{C}^N$ are pseudomeromorphic. Almost semi-meromorphic currents are pseudomeromorphic and have the SEP on \mathcal{U} . \square

In general one cannot multiply pseudomeromorphic currents. However, assume that τ is pseudomeromorphic and α is almost semi-meromorphic in \mathcal{U} and let W be the analytic set where α is not smooth. There is a unique pseudomeromorphic current T in \mathcal{U} that coincides with the natural product $\alpha \wedge \mu$ in $\mathcal{U} \setminus W$ and such that $\mathbf{1}_W T = 0$. For simplicity we denote this current by $\alpha \wedge \mu$. If α' is another almost semi-meromorphic current in \mathcal{U} , then the expression $\alpha' \wedge \alpha \wedge \tau$ means $\alpha' \wedge (\alpha \wedge \tau)$. The equality

$$\alpha' \wedge \alpha \wedge \tau = \alpha \wedge \alpha' \wedge \tau \quad (9.2)$$

always holds. However, in general it is *not* true that $\alpha' \wedge \alpha \wedge \tau = (\alpha' \wedge \alpha) \wedge \tau$.

Example 9.2. Let f be a holomorphic function with non-empty zero set, let $\alpha = 1/f$, $\alpha' = f$, and $\tau = \bar{\partial}(1/f)$. Then $\alpha'\alpha\tau = 0$, but $\alpha'\alpha = 1$ and so $(\alpha'\alpha)\tau = \tau$. \square

Assume that τ is pseudomeromorphic, α is almost semi-meromorphic, ξ is smooth, and V is any subvariety. Then we have

$$\mathbf{1}_V \alpha \wedge \tau = \alpha \wedge \mathbf{1}_V \tau. \quad (9.3)$$

In particular: If τ has support on and the SEP with respect to Z , then also $\alpha \wedge \tau$ has (support on and) the SEP with respect to Z .

10. Uniform limits of holomorphic functions

Let X be a possibly non-reduced space of pure dimension n and let $i: X \rightarrow \mathcal{U} \subset \mathbb{C}^N$, so that $\mathcal{O}_X = \mathcal{O}_{\mathcal{U}}/\mathcal{J}$ as before. If \mathcal{U} is small enough, then there are trivial Hermitian vector bundles E_k in \mathcal{U} , $E_0 = \mathbb{C}$ a line bundle, with morphisms $f_k: E_k \rightarrow E_{k-1}$, so that

$$0 \rightarrow \mathcal{O}(E_N) \xrightarrow{f_N} \dots \xrightarrow{f_3} \mathcal{O}(E_1) \xrightarrow{f_1} \mathcal{O}(E_0) \rightarrow \mathcal{O}(E_0)/\mathcal{J} \rightarrow 0 \quad (10.1)$$

is a free resolution of $\mathcal{O}_{\mathcal{U}}/\mathcal{J}$. In [9] was introduced a residue current $R = R_{\kappa} + \dots + R_N$ with support on Z , where R_k have bidegree $(0, k)$ and take values in $\text{Hom}(E_0, E_k) \simeq E_k$, such that $f_{k+1}R_{k+1} - \bar{\partial}R_k = 0$ for each k , which can be written more compactly as

$$(f - \bar{\partial})R = 0,$$

where $f := f_1 + \dots + f_N$. The current R has the additional property that a holomorphic function Φ in \mathcal{U} belongs to \mathcal{J} if and only if the current $\Phi R = 0$. In particular, ϕR is a well-defined current for ϕ in \mathcal{O}_X . The assumption that X has pure dimension implies that R has the SEP with respect to $Z \cap \mathcal{U}$, see [8, Section 3] or [7, Section 6] for a proof.

Recall that ϕ is a meromorphic function on X if $\phi = g/h$, where h is not nilpotent, i.e., a representative of h does not vanish identically on Z , and $g/h = g'/h'$ if $gh' - g'h = 0$ in \mathcal{O}_X . Because of the SEP the product ϕR is a well-defined pseudomeromorphic current in \mathcal{U} if ϕ is meromorphic on $X \cap \mathcal{U}$. The following criterion for holomorphicity was proved in [4].

Theorem 10.1. Assume that $i: X \rightarrow \mathcal{U}$ has pure dimension and R is an associated current as above. If ϕ is meromorphic on X , then it is holomorphic if and only if

$$(f - \bar{\partial})(\phi R) = 0. \quad (10.2)$$

To give the idea for the general case let us first sketch a proof of Theorem 1.6, relying on Theorem 10.1, in case X is reduced.

Proof of Theorem 1.6 when X is reduced. The statement is elementary on X_{reg} ; moreover it is clear that $\phi_j \rightarrow \phi$ where ϕ is bounded (weakly holomorphic) and thus meromorphic on X .

There is a (unique) almost semi-meromorphic current ω on X of bidegree $(n, *)$ such that $i_*\omega = R \wedge dz$, where (z_1, \dots, z_N) are coordinates in \mathcal{U} , see [8, Proposition 3.3]. In particular, ω has the SEP on X . Let $\pi: X' \rightarrow X$ be a smooth modification so that $\omega = \pi_*\omega'$, where ω' is semi-meromorphic. Since $\pi^*\phi_j \rightarrow \pi^*\phi$ in $\mathcal{O}_{X'}$ and X' is smooth, indeed $\pi^*\phi_j \rightarrow \pi^*\phi$ in $\mathcal{E}_{X'}$. Therefore $\pi^*\phi_j \omega' \rightarrow \pi^*\phi \omega'$. Since ϕ_j are smooth, $\pi_*(\pi^*\phi_j \omega') = \phi_j \omega$. Combining we find that

$$\phi_j \omega \rightarrow \pi_*(\pi^*\phi \omega'). \quad (10.3)$$

Since ω' has the SEP, so have $\pi^*\phi \omega'$ and $\pi_*(\pi^*\phi \omega')$. Moreover,

$$\pi_*(\pi^*\phi \omega') = \phi \omega \quad (10.4)$$

on the open subset of X where ϕ is holomorphic, thus on X_{reg} . Since both sides of (10.4) have the SEP and coincide outside a set of positive codimension, they indeed coincide on X . By (10.3) thus $\phi_j \omega \rightarrow \phi \omega$. Applying i_* we get $\phi_j R \rightarrow \phi R$ and hence $(f - \bar{\partial})(\phi_j R) \rightarrow (f - \bar{\partial})(\phi R)$. It now follows from Theorem 10.1 that ϕ is indeed holomorphic. \square

For the rest of this section we will discuss the proof Theorem 1.6 when X is non-reduced but Z is smooth. We begin with

Lemma 10.2. *Theorem 1.6 is true when Z is smooth and \mathcal{O}_X is Cohen-Macaulay.*

Proof. Given a point $x \in Z$, let $i: W \rightarrow \mathcal{U}$ be an embedding at x as in Section 5, so that we have unique representatives

$$\hat{\phi}_j(z, w) = \sum_{\ell=0}^{\tau-1} \hat{\phi}_{j\ell}(z) w^{\alpha_\ell}$$

in \mathcal{U} of ϕ_j in Theorem 1.6. By the hypothesis and Theorem 1.5 it follows that $\hat{\phi}_{j\ell}$ is a Cauchy sequence in $Z \cap \mathcal{U}$ for each fixed ℓ , and hence we have holomorphic limits $\hat{\phi}_\ell = \lim_j \hat{\phi}_{j\ell}$ for each ℓ . Let us define the function

$$\hat{\phi}(z, w) := \sum_{\ell=0}^{\tau-1} \hat{\phi}_\ell(z) w^{\alpha_\ell}$$

in \mathcal{U} and let ϕ be its pullback to \mathcal{O}_X . Since the convergence holds for all derivatives of $\hat{\phi}_{j\ell}$ as well, it follows from (1.7) that $|\phi_j - \phi|_X \rightarrow 0$. \square

The non-Cohen-Macaulay case is trickier. Let us first look at an example.

Example 10.3. Consider the space X in Section 6. If ϕ_j is a sequence as in Theorem 1.6, it follows from Lemma 10.2 that ϕ_j has a holomorphic limit ϕ in $X \setminus \{0\}$. Let \mathcal{L} be the Noetherian operator in Section 6.1 and recall that $\mathcal{L}\phi$ is a well-defined function on Z . By the hypothesis in Theorem 1.6, $\mathcal{L}\phi_j$ is a Cauchy sequence on Z and since $\mathcal{L}\phi_j \rightarrow \mathcal{L}\phi$ in $Z \setminus \{0\}$ we conclude that $\mathcal{L}\phi_j \rightarrow \mathcal{L}\phi$ uniformly in Z . Since $\mathcal{L}\phi_j(0) = 0$ therefore $\mathcal{L}\phi(0) = 0$, and thus ϕ is \mathcal{O}_X -holomorphic in X , cf. Lemma 6.1. It follows that $|\phi_j - \phi|_X \rightarrow 0$ on X . \square

We cannot see how the argument in Example 10.3 can be extended directly, so we have to go back to the relation between our \mathcal{L}_j and Coleff-Herrera currents.

Proof of Theorem 1.6 when Z is smooth. Given any point $x \in X$ let us choose a local embedding $i: X \rightarrow \mathcal{U}$ at x such that there is a Hermitian free resolution (10.1) and the associated residue current R in \mathcal{U} . Since Theorem 1.6 is local it is enough to prove it in $X \cap \mathcal{U}$. We will use, [7, Lemma 6.2]:

Proposition 10.4. *There is a trivial vector bundle $F \rightarrow \mathcal{U}$ and an F -valued Coleff-Herrera current μ such that its entries generate $\text{Hom}(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}_{\mathcal{U}}^Z)$, and an almost semi-meromorphic current $\alpha = \alpha_0 + \dots + \alpha_n$, where α_k have bidegree $(0, k)$ and take values in $\text{Hom}(F, E_{\kappa+k})$, such that*

$$R \wedge dz = \alpha \mu, \quad R_{\kappa+k} \wedge dz = \alpha_k \mu, \quad k = 0, 1, \dots, n.$$

Moreover, α is smooth where \mathcal{O}_X is Cohen-Macaulay.

Let W be the subset of $Z \cap \mathcal{U}$ where \mathcal{O}_X is not Cohen-Macaulay. Since \mathcal{O}_X has pure dimension W has codimension at least 2 in $Z \cap \mathcal{U}$, see, e.g., [7].

Lemma 10.5. *If ϕ is holomorphic in $(X \cap \mathcal{U}) \setminus W$, then ϕ has a meromorphic extension to $X \cap \mathcal{U}$.*

This result should be well-known but we provide a proof since we could not find any reference.

Proof. Since Z is smooth we can assume that we have coordinates (z, w) in \mathcal{U} so that $Z \cap \mathcal{U} = \{w = 0\}$. Let $\mu = (\mu_1, \dots, \mu_\nu)$ be the tuple in Proposition 10.4 and consider the representations (5.2). Here M must be chosen so that (1.5) holds. Fix $x' \in Z \cap \mathcal{U}$ where \mathcal{O}_X is Cohen-Macaulay and a monomial basis $1, \dots, w^{\alpha_\tau-1}$ for \mathcal{O}_X over \mathcal{O}_Z in a neighborhood \mathcal{U}' of x' , cf. Section 5. We then have (letting $R = \nu C_M$) the $R \times \nu$ -matrix T in \mathcal{U}' that for each holomorphic ϕ in $\mathcal{O}(X \cap \mathcal{U}')$ maps the coefficients $(\hat{\phi}_\ell)$ of its representative $\hat{\phi}$ given by (5.1) in this monomial basis onto the coefficients of the expansions (5.2) of $\phi \mu_j$, cf. Section 5.

Notice that the entries in T are \mathbb{C} -linear combinations of the coefficients of the representations (5.2) for $\phi = 1$ in \mathcal{U} . Thus T has a holomorphic extension to $Z \cap \mathcal{U}$ (we may assume that $Z \cap \mathcal{U}$ is connected). As pointed out in Section 5, T is pointwise injective in $Z \cap \mathcal{U}'$ and hence, after reordering the rows, $T = (T' \ T'')^t$ where T' is a $\nu \times \nu$ -matrix that is invertible in \mathcal{U}' . Thus T' has a meromorphic inverse S' in $Z \cap \mathcal{U}$ and if $S = (S' \ 0)$, then $ST = I$ in $Z \cap \mathcal{U}$.

Since ϕ is holomorphic outside W , it defines a tuple $(b_{j,m})$ in \mathcal{O}_Z^R in $(Z \cap \mathcal{U}) \setminus W$ via the representation (5.2) of $\phi\mu$. Since W has at least codimension 2 in $Z \cap \mathcal{U}$, the tuple $(b_{j,m})$ extends to $Z \cap \mathcal{U}$. Now

$$\tilde{\Phi} := \sum_{\ell=0}^{\tau-1} (Sb)_\ell(z) w^{\alpha_\ell}$$

is a meromorphic function in \mathcal{U} that defines a meromorphic function $\tilde{\phi}$ on $X \cap \mathcal{U}$, since $(Sb)_\ell(z)$ are meromorphic on $Z \cap \mathcal{U}$. Moreover, $\tilde{\Phi} = \hat{\phi}$ in \mathcal{U}' and so $\tilde{\phi}$ coincides with ϕ in $X \cap \mathcal{U}'$. By uniqueness $\tilde{\phi} = \phi$ in $\mathcal{U} \setminus W$ and thus $\tilde{\phi}$ is the desired meromorphic extension. \square

If ϕ_j is a Cauchy sequence in $|\cdot|_X$ -norm and $Z \cap \mathcal{U}$ is smooth, then $\phi_j \rightarrow \phi$ uniformly on compact subsets of $\mathcal{U} \setminus W$ by Lemma 10.2 and ϕ has a meromorphic extension to $X \cap \mathcal{U}$ by Lemma 10.5.

Lemma 10.6. *With this notation $\phi_j R \rightarrow \phi R$ in \mathcal{U} .*

Proof. Let \mathcal{I} and μ be as in Proposition 10.4 and the proof of Lemma 10.2. Assume that a is an F -valued holomorphic function in \mathcal{U} such that $\mu = a\hat{\mu}$, cf. (3.2). Recall that

$$|\phi|_X = |\phi a|_{X'}, \quad (10.5)$$

where $\mathcal{O}_{X'} = \mathcal{O}/\mathcal{I}$. Define the F -valued $\mathcal{O}_{X'}$ -functions $\psi_j = a\phi_j$. It follows from the hypothesis and (10.5) that ψ_j is a Cauchy sequence with respect to $|\cdot|_{X'}$. Since $\mathcal{O}_{X'}$ is Cohen-Macaulay it follows from the proof of Lemma 10.2 that there is ψ in $\mathcal{O}_{X'}$ and representatives $\hat{\psi}_j$ and $\hat{\psi}$ in \mathcal{U} such that $\hat{\psi}_j \rightarrow \hat{\psi}$ in $\mathcal{E}(\mathcal{U})$. Let Φ_j be representatives of ϕ_j in \mathcal{U} . By Proposition 10.4 and (9.2) we have

$$\phi_j R \wedge dz = \Phi_j R \wedge dz = \Phi_j \alpha \mu = \Phi_j \alpha a \hat{\mu} = \alpha \Phi_j a \hat{\mu} = \alpha (\Phi_j a) \hat{\mu} = \alpha \hat{\psi}_j \hat{\mu}, \quad (10.6)$$

where the fifth equality holds since both Φ_j and a are holomorphic, and the last equality holds since both $\Phi_j a$ and $\hat{\psi}_j$ are representatives in \mathcal{U} of the class ψ_j in $\mathcal{O}_{X'}$. Since $\hat{\psi}_j \rightarrow \hat{\psi}$ in $\mathcal{E}(\mathcal{U})$, $\alpha \hat{\psi}_j \hat{\mu} = \hat{\psi}_j \alpha \hat{\mu} \rightarrow \hat{\psi} \alpha \hat{\mu} = \alpha \hat{\psi} \hat{\mu} = \alpha \psi \hat{\mu}$. By (10.6) thus

$$\phi_j R \wedge dz \rightarrow \alpha \psi \hat{\mu}. \quad (10.7)$$

Let Φ be a representative in \mathcal{U} of ϕ . Since Φ , α and a are almost semi-meromorphic in \mathcal{U} , by (9.2),

$$\phi R \wedge dz = \Phi R \wedge dz = \Phi \alpha \mu = \alpha \Phi \mu = \alpha \Phi a \hat{\mu}. \quad (10.8)$$

We claim that

$$\alpha \Phi a \hat{\mu} = \alpha (\Phi a) \hat{\mu}. \quad (10.9)$$

In fact, both Φ and α are almost semi-meromorphic in \mathcal{U} and smooth in a neighborhood of each point on $Z \cap \mathcal{U}$ where \mathcal{O}_X is Cohen-Macaulay, cf. Lemma 10.2 and Proposition 10.4. Therefore (10.9) holds in $\mathcal{U} \setminus W$, and $W \subset Z \cap \mathcal{U}$ has positive codimension in Z . Both sides of (10.9) have the SEP with respect to Z , see Section 9, so (10.9) holds everywhere. The right hand side of (10.9) is equal to $\alpha \psi \hat{\mu}$, and so Lemma 10.6 follows from (10.7), (10.8), and (10.9). \square

Since ϕ_j are holomorphic, we have by Theorem 10.1 and Lemma 10.6 that $0 = \nabla_f(\phi_j R) \rightarrow \nabla_f(\phi R)$, and hence ϕ is holomorphic in view of Theorem 10.1. Now take \mathcal{L} in \mathcal{N}_X . By the hypothesis and definition of $|\cdot|_X$, $\mathcal{L}\phi_j$ is a holomorphic Cauchy sequence in \mathcal{U} so it converges to a holomorphic limit H . On the other hand we know that $\mathcal{L}\phi_j \rightarrow \mathcal{L}\phi$ where \mathcal{O}_X is Cohen-Macaulay. Thus $\mathcal{L}\phi_j \rightarrow \mathcal{L}\phi$ uniformly. We conclude that $|\phi_j - \phi|_X \rightarrow 0$ uniformly in \mathcal{U} . Thus Theorem 1.6 is proved in $X \cap \mathcal{U}$ and hence in general if Z is smooth. \square

11. Resolution of X

Assume that our X of pure dimension n is embedded in the smooth manifold Y of dimension N as before, and let Z denote the underlying reduced space. There exists a modification $\pi: Y' \rightarrow Y$ that is a biholomorphism $Y' \setminus \pi^{-1}Z_{\text{sing}} \simeq Y \setminus Z_{\text{sing}}$ and such that the strict transform Z' of Z is smooth and the restriction of π to Z' is a modification of Z . Such a π is called a strong resolution. Let $\tilde{\mathcal{J}}$ be the ideal sheaf on Y' generated by pullbacks of generators of \mathcal{J} and consider the relative gap sheaf $\mathcal{J}' = \tilde{\mathcal{J}}[\pi^{-1}Z_{\text{sing}}]$, which is coherent, cf. [32, Theorem 2]. In fact, one obtains \mathcal{J}' by extending $\tilde{\mathcal{J}}$ so that one gets rid of all primary components corresponding to the exceptional divisor, and also possible embedded primary ideals in $Z' \cap \pi^{-1}Z_{\text{sing}}$. Thus \mathcal{J}' is the smallest coherent sheaf of pure dimension n that contains $\tilde{\mathcal{J}}$ and such that $\mathcal{O}_{Y'}/\mathcal{J}'$ has support on Z' . We let X' denote the analytic space with structure sheaf $\mathcal{O}_{X'} = \mathcal{O}_{Y'}/\mathcal{J}'$. Notice that we have the induced mapping

$$p^*: \mathcal{O}_X \rightarrow \mathcal{O}_{X'}. \quad (11.1)$$

In fact, if $\Phi \in \mathcal{J}$, then $\pi^*\Phi \in \tilde{\mathcal{J}} \subset \mathcal{J}'$ and thus p^* in (11.1) is well-defined. We say that $p: X' \rightarrow X$ is a resolution of X . Notice that p^* extends to map meromorphic functions on X to meromorphic functions on X' . Let $p_0 = \pi|_{Z'}$ and let

$$V := p_0^{-1}Z_{\text{sing}} = \pi^{-1}Z_{\text{sing}} \cap Z'. \quad (11.2)$$

Lemma 11.1. *Assume that ϕ' is meromorphic on X' and holomorphic on $X' \setminus V$. Then there is a unique meromorphic ϕ on X , holomorphic in $X \setminus Z_{\text{sing}}$, such that $\phi' = p^*\phi$.*

Proof. Since π is proper it follows from Grauert's theorem that the direct image $\mathcal{F} = \pi_*(\mathcal{O}_{Y'}/\mathcal{J}')$ is coherent, and clearly it coincides with $\mathcal{O}_Y/\mathcal{J}$ outside $Z_{\text{sing}} \subset Y$. Moreover, it contains $\mathcal{O}_X = \mathcal{O}_Y/\mathcal{J}$ since $\pi_*\pi^{-1}\phi = \phi$ for ϕ in \mathcal{O}_X . Thus $\mathcal{F}/\mathcal{O}_X$ has support on Z_{sing} . Let h be a function that vanishes on Z_{sing} but not identically in Z . Then $h^\nu\mathcal{F}/\mathcal{O}_X = 0$ if ν is large enough. If ϕ' is a section of $\mathcal{O}_{X'}$, therefore $g := h^\nu\pi_*\phi'$ is holomorphic. Thus $\phi := g/h^\nu$ is meromorphic and $\phi' = p^*\phi$. \square

Lemma 11.2. *Let μ be a tuple of currents that generate the \mathcal{O}_X -module $\text{Hom}(\mathcal{O}_Y/\mathcal{J}, \mathcal{CH}_Y^Z)$.*

- (i) *There is a unique tuple μ' of pseudomeromorphic (N, κ) -currents in Y' with support on Z' such that $\pi_*\mu' = \mu$.*
- (ii) *A holomorphic function Φ' defined in a neighborhood in Y' of a point on Z' is in \mathcal{J}' if and only if $\Phi'\mu' = 0$.*

In view of (ii) thus $\phi'\mu'$ is well-defined for ϕ' in $\mathcal{O}_{X'}$. It is not necessarily true that μ' is $\bar{\partial}$ -closed. Since π is a biholomorphism outside $\pi^{-1}Z_{\text{sing}}$ it follows however that $\bar{\partial}\mu' = 0$ there. Moreover, since μ' is pseudomeromorphic it has the SEP, by virtue of the dimension principle. In the literature such a μ' is often said to be a Coleff-Herrera current with poles at $V \subset Z'$. If h' is holomorphic and vanishes to enough order on V then $0 = h'\bar{\partial}\mu' = \bar{\partial}(h'\mu')$, and hence $h'\mu'$ is in $\text{Hom}(\mathcal{O}_{Y'}/\mathcal{J}', \mathcal{CH}_{Y'}^Z)$.

Proof. Recall that μ is pseudomeromorphic, cf. Section 9. By [11, Theorem 2.15] there is a pseudomeromorphic current T in Y' such that $\pi_*T = \mu$. Since π is a biholomorphism outside $\pi^{-1}Z_{\text{sing}}$ the current T must be unique there, in particular it must have support on $\pi^{-1}Z$. Thus $T = \mathbf{1}_{Z'}T + \mathbf{1}_{\pi^{-1}Z \setminus Z'}T$. Since $\pi_*(\mathbf{1}_{\pi^{-1}Z \setminus Z'})$ has support on Z_{sing} that has codimension at least 1 in Z , it vanishes by the dimension principle. If $\mu' := \mathbf{1}_{Z'}T$, therefore $\pi_*\mu' = \pi_*T = \mu$. Moreover, since μ' is unique outside $Z' \cap \pi^{-1}Z_{\text{sing}} = V$ it is unique, again by the dimension principle, since V has codimension at least 1 in Z' . Thus (i) is proved.

Since \mathcal{J}' has no embedded components, Φ' is in \mathcal{J}' if and only if Φ' is in \mathcal{J}' on $Z' \setminus V$. This in turn holds if and only if $\Phi = \pi_*\Phi'$ belongs to \mathcal{J} on Z_{reg} which holds if and only if $\Phi\mu = 0$ on Z_{reg} . However this holds if and only if $\Phi'\mu' = 0$ on $Z' \setminus V$ which by the SEP of μ' holds if and only if $\Phi'\mu' = 0$ on Z' . Thus (ii) holds. \square

Let R be a current in Y with support on Z and the SEP as in Section 10. Recall, Proposition 10.4, that there is an almost semi-meromorphic current α in Y such that $R = \alpha\mu$, where μ is a tuple of Coleff-Herrera currents that generate $\text{Hom}(\mathcal{O}_Y/\mathcal{J}, \mathcal{CH}_Y^Z)$.

Lemma 11.3. *There is an almost semi-meromorphic current α' in Y' such that $R' = \alpha'\mu'$ has the SEP and $\pi_*R' = R$.*

Proof. By definition there is a modification $\tau: W \rightarrow Y$ such that $\alpha = \tau_*\gamma$, where γ is semi-meromorphic. There is a modification $W' \rightarrow Y$ that factors over both V and Y' . If we pull back γ to W' , then its direct image α' in Y' is almost semi-meromorphic and $\pi_*\alpha' = \alpha$. It follows from (9.3) that $R' := \alpha'\mu'$ has the SEP. Moreover, $\pi_*R' = R$ where π is a biholomorphism, i.e., outside Z_{sing} . Since both currents have the SEP, the equality holds in Y . \square

12. Proof of Theorem 1.6

Lemma 12.1. *Assume that Z is smooth and that \mathcal{L} is a holomorphic differential operator on X that belongs to \mathcal{N}_X in $Z \setminus W$, where W has positive codimension. If $Z(h) \supset W$, then $h^r\mathcal{L}$ is in \mathcal{N}_X for large enough r .*

Proof. Recall that the sheaf \mathcal{N}_X locally can be considered as a coherent submodule of \mathcal{O}_Z^ν for some large ν . Then also \mathcal{L} can be considered as an element in \mathcal{O}_Z^ν . If \mathcal{L} is not in \mathcal{N}_X , then $\mathcal{M}' = \langle \mathcal{N}_X, \mathcal{L} \rangle / \mathcal{N}_X$ is a coherent sheaf with support on W . By the Nullstellensatz $h^r\mathcal{M}' = 0$ for large enough r . Thus $h^r\mathcal{L} \in \mathcal{N}_X$ for such r . \square

It remains to prove Theorem 1.6 in a neighborhood of a point $x \in Z_{\text{sing}}$, cf. (7.10). Let $i: X \rightarrow \mathcal{U} \subset \mathbb{C}^N$ be a local embedding at x . We can assume that \mathcal{N}_X admits a coherent extension to $X \cap \mathcal{U}$, cf. Theorem 7.3, that we denote by \mathcal{N}_X as well. Recall that the \mathcal{O}_Z module \mathcal{N}_X is generated in \mathcal{U} by a finite number of operators $\mathcal{L}_1, \dots, \mathcal{L}_r$ that are induced by Noetherian operators L_1, \dots, L_r with respect to \mathcal{J} in \mathcal{U} .

Let $\pi: \mathcal{U}' \rightarrow \mathcal{U}$ be a modification as in Section 11, with \mathcal{U}' and \mathcal{U} instead of Y' and Y , respectively. Thus we have the space $i': X' \rightarrow \mathcal{U}'$, $p^*: \mathcal{O}_X \rightarrow \mathcal{O}_{X'}$ and the induced mapping $p_0: Z' \rightarrow Z \cap \mathcal{U}$. Since Z' is smooth we have the well-defined $\mathcal{O}_{X'}$ -module $\mathcal{N}_{X'}$ of Noetherian operators on X' .

We say that \mathcal{L}' is a meromorphic Noetherian operator on X' with poles on $V := p_0^{-1}Z_{\text{sing}} \subset Z'$ if $\xi^\rho L'$ is a Noetherian operator on X' as soon as ξ in $\mathcal{O}_{Z'}$ vanishes on V and ρ is large enough.

Lemma 12.2. *There are meromorphic operators $\mathcal{L}'_1, \dots, \mathcal{L}'_r$ on X' such that*

$$\mathcal{L}'_j(p^*\phi) = p_0^*(\mathcal{L}_j\phi) \quad (12.1)$$

on $Z' \setminus V$. Moreover, there is a holomorphic (nontrivial) function h on Z such that $h'\mathcal{L}'_j$ are in $\mathcal{N}_{X'}$ if $h' = p_0h$.

Proof. Given a holomorphic differential operator T on \mathcal{U} there is a holomorphic differential operator \tilde{T} in \mathcal{U}' with values in a power $N_{\mathcal{U}'/\mathcal{U}}^\nu$ of the relative canonical bundle, and a

holomorphic section s of $N_{\mathcal{U}'/\mathcal{U}}$, vanishing on $\pi^{-1}Z_{\text{sing}}$, such that $\pi^*(T\Phi) = s^{-\nu}\tilde{T}(\pi^*\Phi)$. See, e.g., the discussion preceding [11, Corollary 4.26]. Thus $T' = s^{-\nu}\tilde{T}$ is a meromorphic differential operator, with poles at $\pi^{-1}Z_{\text{sing}}$, such that $\pi^*(T\Phi) = T'(\pi^*\Phi)$.

Let L'_1, \dots, L'_r be meromorphic differential operators on \mathcal{U}' such that $\pi^*(L_j\Psi) = L'_j(\pi^*\Psi)$, $j = 1, \dots, r$. If Φ' is in \mathcal{J}' , then $\Phi' = \pi^*\Phi$ for some Φ in \mathcal{J} outside Z_{sing} and hence $L'_j\Phi' = 0$ outside V . By continuity $L'_j\Phi' = 0$. Thus we have induced meromorphic operators $\mathcal{L}'_1, \dots, \mathcal{L}'_r$ on X' with poles at V and (12.1) holds.

Since π is a biholomorphism outside $\pi^{-1}Z_{\text{sing}}$ it follows that \mathcal{L}'_j belong to $\mathcal{N}_{X'}$ there. By Lemma 12.1, $h'\mathcal{L}'_j$ are in $\mathcal{N}_{X'}$ if $h' = p_0^*h$, where h is a holomorphic function in Z that vanishes to high enough order on Z_{sing} . \square

Lemma 12.3. *After possibly shrinking \mathcal{U} there is a holomorphic function H in \mathcal{U} , not vanishing identically on Z , such that*

$$|p^*(H\phi)(z')|_{X'} \leq C|\phi(p_0(z'))|_X, \quad z' \in Z'. \quad (12.2)$$

Proof. Let $\hat{\mathcal{N}}_{X'}$ be the $\mathcal{O}_{Z'}$ -module generated by the $h'\mathcal{L}'_j$ from Lemma 12.2. Then $\hat{\mathcal{N}}_{X'} \subset \mathcal{N}_{X'}$ with equality outside $Z(h')$. Therefore $\mathcal{N}_{X'}/\hat{\mathcal{N}}_{X'}$ is annihilated by $H' = \pi^*H$ if H is a high power of h . That is, if \mathcal{T} is in $\mathcal{N}_{X'}$, then $H'\mathcal{T}$ is in $\hat{\mathcal{N}}_{X'}$ and thus $H'\mathcal{T}$ is an $\mathcal{O}_{Z'}$ -linear combination of the $h'\mathcal{L}'_j$.

Fix a point $x' \in \pi^{-1}(x) \cap Z'$. Let \mathcal{T}_ℓ be a set of generators for $\mathcal{N}_{X'}$ in a neighborhood \mathcal{V} of x' . For any ϕ we have, with $\phi' = p^*\phi$, and $z' \in \mathcal{V}$,

$$|H'(z')||\phi'(z')|_{X'} \sim \sum_{\ell} |(H'\mathcal{T}_\ell\phi')(z')| \lesssim \sum_j |(h'\mathcal{L}'_jp^*\phi')(z')| \leq \sum_j |p_0^*(\mathcal{L}_j\phi)(z')| \sim |\phi(\pi(z'))|_X.$$

On the other hand, if ν is large enough, $|\mathcal{T}_\ell((H')^\nu\phi')| \lesssim |H'\mathcal{T}_\ell\phi'|$ for each ℓ and hence $|((H')^\nu\phi')|_{X'} \lesssim |H'||\phi'|_{X'}$. Denoting H^ν by H thus (12.2) holds for $z' \in \mathcal{V}$. Since $\pi^{-1}(x)$ is compact, (12.2) holds for all z' in an open neighborhood of $\pi^{-1}(x)$. Hence the lemma follows. \square

Assume that ϕ_j is a sequence as in Theorem 1.6 and let $\phi'_j = p^*\phi_j$. It follows from Lemma 12.3, and Theorem 1.6 in case that Z is smooth, see Section 10, that there is a holomorphic function ξ' on $X' \cap \mathcal{U}'$ such that $H'\phi'_j \rightarrow \xi'$ uniformly in the $|\cdot|_{X'}$ -norm. Notice that ξ'/H' is meromorphic on $X' \cap \mathcal{U}'$.

Lemma 12.4. *With the notation above, $\phi'_j R' \rightarrow (\xi'/H')R'$ in \mathcal{U}' .*

Proof. Let μ' be as in Lemma 11.3. Since $\bar{\partial}\mu'$ has support on V , $\bar{\partial}(g'\mu') = 0$ for a suitable $g' = \pi^*g$ not vanishing identically on Z' . From Lemma 11.2 we conclude that

$g'\mu'$ is a tuple in $\mathcal{H}om(\mathcal{O}_{\mathcal{U}'}/\mathcal{I}', \mathcal{CH}_{\mathcal{U}'}^{Z'})$. Since Z' is smooth, if $\mathcal{V} \subset \mathcal{U}'$ is a small enough open neighborhood of any given point in \mathcal{U}' , then we have coordinates (z, w) such that $Z' \cap \mathcal{V} = \{w = 0\}$. Then $g'\mu' = adz \wedge \hat{\mu}$, cf. (3.2), for a suitable holomorphic tuple a in \mathcal{V} . Using (9.2) and Lemma 11.3 we can now prove Lemma 12.4 in \mathcal{V} in the same way as Lemma 10.6. Now Lemma 12.4 follows in \mathcal{U}' since the statement is local. \square

By Lemma 11.1 there is a meromorphic ξ on $X \cap \mathcal{U}$ such that $\xi' = p^*\xi$. Define the meromorphic function $\phi = \xi/H$ on $X \cap \mathcal{U}$. Clearly $p^*\phi = \xi'/H'$ so that

$$\pi_*((\xi'/H')R') = \phi R \quad (12.3)$$

in $\mathcal{U} \setminus (Z(H) \cap Z)$. However, both sides of (12.3) have the SEP with respect to $Z \cap \mathcal{U}$ so the equality holds in \mathcal{U} . Since $\pi_*(\phi'_j R') = \pi_*(p^*\phi_j R') = \phi_j R$ we conclude from Lemma 12.4 that $\phi_j R \rightarrow \phi R$. In view of Theorem 10.1 now Theorem 1.6 follows as in the smooth case in Section 10.

References

- [1] W. Alexandre, E. Mazzilli, Extension of holomorphic functions defined on singular complex hyper-surfaces with growth estimates, *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* (5) 14 (2015) 293–330.
- [2] E. Amar, Extension de fonctions holomorphes et courants, *Bull. Sci. Math.* 107 (1982) 25–48.
- [3] M. Andersson, Uniqueness and factorization of Coleff-Herrera currents, *Ann. Fac. Sci. Toulouse Math.* 18 (4) (2009) 651–661.
- [4] M. Andersson, A residue criterion for strong holomorphicity, *Ark. Mat.* 48 (1) (2010) 1–15.
- [5] M. Andersson, Coleff-Herrera currents, duality, and Noetherian operators, *Bull. Soc. Math. Fr.* 139 (2011) 535–554.
- [6] M. Andersson, L^p -estimates of extensions of holomorphic functions defined on a non-reduced subvariety, *Ann. Inst. Fourier*, in press, available at arXiv:2004.07152 [math.CV].
- [7] M. Andersson, R. Lärkäng, The $\bar{\partial}$ -equation on a non-reduced analytic space, *Math. Ann.* 374 (2019) 553–599.
- [8] M. Andersson, H. Samuelsson, A Dolbeault-Grothendieck lemma on complex spaces via Koppelman formulas, *Invent. Math.* 190 (2012) 261–297.
- [9] M. Andersson, E. Wulcan, Residue currents with prescribed annihilator ideals, *Ann. Sci. Éc. Norm. Supér.* 40 (2007) 985–1007.
- [10] M. Andersson, E. Wulcan, Decomposition of residue currents, *J. Reine Angew. Math.* 638 (2010) 103–118.
- [11] M. Andersson, E. Wulcan, Direct images of semi-meromorphic currents, *Ann. Inst. Fourier* 68 (2018) 875–900.
- [12] M. Andersson, R. Lärkäng, J. Ruppenthal, H. Samuelsson Kalm, E. Wulcan, Estimates for the $\bar{\partial}$ -equation on canonical surfaces, *J. Geom. Anal.* 30 (2020) 2974–3001.
- [13] D. Barlet, Le faisceau ω_X sur un espace analytique X de dimension pure, in: *Fonctions de plusieurs variables complexes, III (Sém. François Norguet, 1975–1977)*, in: *Lecture Notes in Math.*, vol. 670, Springer, Berlin, 1978, pp. 187–204.
- [14] B. Berndtsson, N. Sibony, The $\bar{\partial}$ -equation on a positive current, *Invent. Math.* 147 (2) (2002) 371–428.
- [15] J.-E. Björk, *Rings of Differential Operators*, North-Holland Mathematical Library, vol. 21, North-Holland Publishing Co., Amsterdam-New York, ISBN 0-444-85292-1, 1979, xvii+374 pp.
- [16] J.-E. Björk, Residues and \mathcal{D} -modules, in: *The Legacy of Niels Henrik Abel*, Springer, Berlin, 2004, pp. 605–651.
- [17] J. Cao, J.-P. Demailly, S. Matsumura, A general extension theorem for cohomology classes on non reduced analytic subspaces, *Sci. China Math.* 60 (2017) 949–962.
- [18] A. Cumenge, Extension dans des classes de Hardy de fonctions holomorphes et estimations de type “mesures de Carleson” pour l’équation $\bar{\partial}$, *Ann. Inst. Fourier (Grenoble)* 33 (1983) 59–97.

- [19] J-P. Demailly, Extension of holomorphic functions defined on non reduced analytic subvarieties, in: The Legacy of Bernhard Riemann After One Hundred and Fifty Years, Vol. I, in: Adv. Lect. Math. (ALM), vol. 35.1, Int. Press, Somerville, MA, 2016, pp. 191–222.
- [20] J.E. Fornæss, N. Ovrelid, S. Vassiliadou, Semiglobal results for $\bar{\partial}$ on a complex space with arbitrary singularities, Proc. Am. Math. Soc. 133 (8) (2005) 2377–2386.
- [21] H. Grauert, R. Remmert, Coherent Analytic Sheaves, Grundlehren der mathematischen Wissenschaften, vol. 265, Springer Verlag, 1984.
- [22] G. Henkin, P. Polyakov, Residual $\bar{\partial}$ -cohomology and the complex Radon transform on subvarieties of $\mathbb{C}P^n$, Math. Ann. 354 (2012) 497–527.
- [23] G. Henkin, P. Polyakov, Explicit Hodge-type decomposition on projective complete intersections, J. Geom. Anal. 26 (1) (2016) 672–713.
- [24] L. Hörmander, An Introduction to Complex Analysis in Several Variables, third edition, North Holland, 1990.
- [25] T. de Jong, G. Pfister, Local Analytic Geometry, Basic Theory and Applications, Advanced Lectures in Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 2000.
- [26] R. Lärkäng, J. Ruppenthal, Koppelman formulas on affine cones over smooth projective complete intersections, Indiana Univ. Math. J. 67 (2018) 753–780.
- [27] T. Ohsawa, H. Takegoshi, On the extension of L^2 holomorphic functions, Math. Z. 195 (1987) 197–204.
- [28] N. Ovrelid, S. Vassiliadou, L^2 - $\bar{\partial}$ -cohomology groups of some singular complex spaces, Invent. Math. 192 (2) (2013) 413–458.
- [29] W. Pardon, M. Stern, L^2 - $\bar{\partial}$ -cohomology of complex projective varieties, J. Am. Math. Soc. 4 (3) (1991) 603–621.
- [30] W. Pardon, M. Stern, Pure Hodge structure on the L^2 -cohomology of varieties with isolated singularities, J. Reine Angew. Math. 533 (2001) 55–80.
- [31] J. Ruppenthal, L^2 -theory for the $\bar{\partial}$ -operator on compact complex spaces, Duke Math. J. 163 (15) (2014) 2887–2934.
- [32] Y-T. Siu, G. Trautmann, Gap Sheaves and Extension of Coherent Analytic Subsheaves, Lecture Notes in Mathematics, vol. 172, Springer-Verlag, 1971.
- [33] J. Sznajdman, A Briançon-Skoda type result for a non-reduced analytic space, J. Reine Angew. Math. 742 (2018) 1–16.