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(article starts on next page)

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A pointwise norm on a non-reduced analytic space

Mats Andersson [1]

Department of Mathematics, Chalmers University of Technology and University of Gothenburg, S-412 96 Göteborg, Sweden

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Let X be a possibly non-reduced space of pure dimension. We introduce a pointwise Hermitian norm on smooth $(0, q)$-forms, in particular on holomorphic functions, on X. The norm is canonical, up to equivalence, where the underlying reduced space is a manifold. We prove that the space of holomorphic functions is complete with respect to the natural topology induced by this norm.

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1. Introduction

Starting with papers by Pardon and Stern, [29,30], in the early 90s, a lot of research on the $\bar{\partial}$-equation on a reduced singular space X has been conducted during the last decades, e.g., [14,20,28,31,26,22,23,8,12] and many others. In most of them estimates for solutions are discussed. There are also works, e.g., [1], on estimates of holomorphic extensions from a singular subvariety. Given a local embedding of a reduced X into a smooth manifold $\mathcal{U}$, a pointwise norm of functions and forms on X is inherited from a Hermitian norm on $\mathcal{U}$. Any two such local norms are equivalent, and thus one gets a global pointwise norm that is unique, up equivalence, on any compact subset of X.

E-mail address: matsa@chalmers.se.

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Only quite recently there has been some work about analysis on non-reduced spaces. The celebrated Ohsawa-Takegoshi theorem, [27], has been generalized to encompass extensions of holomorphic functions defined on non-reduced subvarieties X defined by certain multiplier ideal sheaves of a manifold Y, see, e.g., [17,19]. In this case the L^2-norm of a function (or form) ϕ on the subvariety is defined as a limit of L^2-norms of an arbitrary extension of ϕ over small neighborhoods of X in Y. A pointwise, but not canonical, norm of holomorphic functions on a non-reduced X is used by Sznajdman in [33], where he proved an analytically formulated Briançon-Skoda-Huneke type theorem on a non-reduced X of pure dimension.

In this paper we introduce, given a non-reduced space X of pure dimension n, a pointwise Hermitian norm $|\cdot|_X$ on $\mathcal{O}_X$ such that $|\phi|_X^2$ is a smooth function on the underlying reduced space Z for any holomorphic ϕ. The norm is canonical (up to local equivalence) on the regular part of Z, whereas the extension across Z_{sing} possibly depends on some choices. The norm extends to smooth $(0,q)$-forms on X.

Given any point $x \in X$ there is a local embedding $i\colon X \to \mathcal{U} \subset \mathbb{C}^N$, where $\mathcal{U} \subset \mathbb{C}^N$ is an open subset and $x \in X \cap \mathcal{U}$. This means that we have an ordinary local embedding $\iota\colon Z \to \mathcal{U}$ and a coherent ideal sheaf $\mathcal{J}$ in $\mathcal{U}$ with zero set $Z \cap \mathcal{U}$ such that the structure sheaf $\mathcal{O}_X$, the sheaf of holomorphic functions on X, is isomorphic to $\mathcal{O}_{\mathcal{U}}/\mathcal{J}$. Thus we have a natural surjective mapping $i^*\colon \mathcal{O}_{\mathcal{U}} \to \mathcal{O}_X$ with kernel $\mathcal{J}$.

Recall that a holomorphic differential operator L in $\mathcal{U}$ is Noetherian with respect to $\mathcal{J}$ if $L\Phi = 0$ on Z for all Φ in $\mathcal{J}$. It is well-known that locally one can find a finite set $L_1, \dots, L_m$ of Noetherian operators such that $L_j\Phi = 0$ on Z if and only if Φ is in $\mathcal{J}$. The analogous statement for a polynomial ideal is a keystone in the celebrated Fundamental principle due to Ehrenpreis and Palamodov, see, e.g., [15,24]. Each Noetherian operator with respect to $\mathcal{J}$ defines an intrinsic mapping $\mathcal{L}\colon \mathcal{O}_X \to \mathcal{O}_Z$ by

$$\mathcal{L}(i^*\Phi) = \iota^* L\Phi. \tag{1.1}$$

We say that $\mathcal{L}$ is a *Noetherian operator on* X. It follows that locally there are Noetherian operators $\mathcal{L}_0, \dots, \mathcal{L}_m$ on X such that

$$\mathcal{L}_j\phi = 0 \text{ in } \mathcal{O}_Z,\ j = 1, \dots, m, \quad \text{if and only if} \quad \phi = 0 \text{ in } \mathcal{O}_X. \tag{1.2}$$

Given $\mathcal{L}_j$ as in (1.2), following [33] let us consider

$$|\phi(z)|^2 = \sum_0^m |\mathcal{L}_j\phi(z)|^2. \tag{1.3}$$

Clearly $|\phi| = 0$ in an open set if and only if $\phi = 0$ there so (1.3) is a Hermitian norm. However, it depends on the choice of $\mathcal{L}_j$. For instance, (1.2) still holds if $\mathcal{L}_j$ are multiplied by any h in $\mathcal{O}_Z$ that is generically non-vanishing on Z. The set of all Noetherian operators on X is a (left) $\mathcal{O}_Z$-module,[2] but it is not locally finitely generated since any derivation along Z is Noetherian. We will define our norm from a suitable subsheaf. The construction relies on the close connection between Noetherian operators and so-called Coleff-Herrera currents established by J-E Björk, [16].

Assume for the moment that Z is smooth and that we have a local embedding $i\colon X \to \mathcal{U}$. Let $\mathcal{H}om_{\mathcal{O}_\mathcal{U}}(\mathcal{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}_X^Z)$ be the $\mathcal{O}_\mathcal{U}$-module of Coleff-Herrera currents in $\mathcal{U}$ that are annihilated by $\mathcal{J}$. This sheaf, introduced by J-E Björk, [16], consists of all $\bar\partial$-closed $(N, N-n)$-currents in $\mathcal{U}$, with support on $Z \cap \mathcal{U}$, such that $\bar h\mu = 0$ for all holomorphic h that vanish on Z, and[3] $\Phi\mu = 0$ for Φ in $\mathcal{J}$.

Proposition 1.1. *Let $\pi\colon \mathcal{U} \to Z \cap \mathcal{U}$ be a submersion and let ω_z be a non-vanishing holomorphic n-form on $Z \cap \mathcal{U}$. Each μ in $\mathcal{H}om_{\mathcal{O}_\mathcal{U}}(\mathcal{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}_X^Z)$ induces a Noetherian operator $\mathcal{L}\colon \mathcal{O}_X \to \mathcal{O}_Z$ by*

$$\mathcal{L}\phi\,\omega_z = \pi_*(\phi\mu). \tag{1.4}$$

The set of $\mathcal{L}$ so obtained is a coherent $\mathcal{O}_Z$-module $\mathcal{N}_{X,\pi}$ on $Z \cap \mathcal{U}$, and any set of local generators satisfies (1.2).

Clearly $\mathcal{N}_{X,\pi}$ is independent of the choice of ω_z. After shrinking $\mathcal{U}$, if needed, we can assume that $\mathcal{N}_{X,\pi}$ is finitely generated in $\mathcal{U}$. A finite set of generators, cf. (1.3), therefore gives a pointwise norm $|\cdot|_{X,\pi}$ in $\mathcal{U}$. If $|\cdot|'_{X,\pi}$ is obtained in this way from another finite set of generators, then $|\cdot|'_{X,\pi}$ is equivalent to $|\cdot|_{X,\pi}$ on (compact subsets of) $\mathcal{U}$, which we write as $|\cdot|'_{X,\pi} \sim |\cdot|_{X,\pi}$.

Definition 1.2. Let $\mathcal{N}_X$ be the $\mathcal{O}_Z$-module generated by all local Noetherian operators $\mathcal{L}$ on X obtained from local embeddings and submersions as in (1.4).

Theorem 1.3. *Let X be a reduced space of pure dimension such that its underlying reduced space Z is smooth. Then $\mathcal{N}_X$ is a coherent $\mathcal{O}_Z$-module on Z, and any set of local generators of $\mathcal{N}_X$ satisfies (1.2).*

Any finite set of local generators, cf. (1.3), gives rise to a local pointwise Hermitian norm. Moreover, any two norms obtained in this way are locally equivalent. It turns out, see Proposition 4.3, that if $\mathcal{U}$ is small enough, then $\mathcal{N}_X$ is generated in $Z \cap \mathcal{U}$ by the sheaves $\mathcal{N}_{X,\pi^\ell}$ for a suitable finite set of submersions $\pi^\ell\colon \mathcal{U} \to Z \cap \mathcal{U}$. Thus $|\cdot|_X$ is equivalent in $X \cap \mathcal{U}$ to the finite sum of the norms $|\phi|_{X,\pi^\ell}$. Patching together we get a global pointwise Hermitian norm $|\cdot|_X$ on X.

[2] In this paper '$\mathcal{O}_Z$-module' means 'sheaf of $\mathcal{O}_Z$-modules'.
[3] If Z has singular points an additional regularity assumption is required, see Section 2.1 below.

To describe the norm $|\cdot|_X$ more concretely, assume that we have a local embedding $i: \mathcal{U} \to \Omega \subset \mathbb{C}^N$ and coordinates $(z, w) = (z_1, \ldots, z_n, w_1, \ldots, w_\kappa)$ in $\mathcal{U}$, where $\kappa = N - n$, such that $Z = \{w = 0\}$. By the Nullstellensatz,

$$\mathcal{I} := \langle w_1^{M_1+1}, \ldots, w_\kappa^{M_\kappa+1} \rangle \subset \mathcal{J} \tag{1.5}$$

if M_j are large enough natural numbers. For multiindices $m = (m_1, \ldots, m_\kappa) \in \mathbb{N}^\kappa$, let $|m| = m_1 + \cdots + m_\kappa$. If $M = (M_1, \ldots, M_\kappa)$, then $m \le M$ means that $m_j \le M_j$ for $j = 1, \ldots, \kappa$. We will use the short-hand notation $\partial^{|m|}/\partial w^m = (\partial^{m_1}/\partial w_\kappa^m) \cdots (\partial^{m_\kappa}/\partial w_\kappa^m)$, and define $\partial^{|\beta|}/\partial z^\beta$ similarly for $\beta = (\beta_1, \ldots, \beta_n)$.

Theorem 1.4. *With the notation above, if $\mathcal{U}$ is small enough and (1.5) holds, then there is a finite set of holomorphic functions $a_1, \ldots, a_\nu$ in $\mathcal{U}$ such that the operators*

$$\phi \mapsto \mathcal{L}_{m,\beta,j}\phi := \frac{\partial^{|m|+|\beta|}(\phi a_j)}{\partial z^\beta \partial w^m}(\cdot, 0), \quad m \le M, \ |\beta| \le |M| - |m|, \ j = 1, \ldots, \nu, \tag{1.6}$$

are Noetherian on $X \cap \mathcal{U}$ and generate the $\mathcal{O}_Z$-module $\mathcal{N}_X$ on $Z \cap \mathcal{U}$.

The precise requirement of the functions a_j is that they generate the coherent $\mathcal{O}_\mathcal{U}$-module $(\mathcal{I} : \mathcal{J})/\mathcal{I}$, see Remark 4.1. An immediate consequence of the theorem is that

$$|\phi(z)|_X^2 \sim \sum_{j=1}^{\nu} \sum_{m \le M} \sum_{|\beta| \le |M|-|m|} \Big| \frac{\partial^{|m|+|\beta|}(\phi a_j)}{\partial z^\beta \partial w^m}(z, 0) \Big|^2 \tag{1.7}$$

in $\mathcal{U}$. It follows from (1.7) that

$$|\xi\phi|_X \le C|\phi|_X,$$

locally in $\mathcal{U}$ where C only depends on $\xi \in \mathcal{O}_X$. Notice that if in addition ξ is invertible in $\mathcal{O}_X$, then $|\phi|_X \sim |\xi\phi|_X$ since $|\phi|_X = |\xi^{-1}\xi\phi|_X \le C|\xi\phi|_X$.

We say that a point $x \in X$ is *regular* if Z is smooth at x and in addition $\mathcal{O}_X$ is Cohen-Macaulay. The set of regular points is a Zariski-open dense subset of Z. In a neighborhood of a regular point we can represent $\mathcal{O}_X$ as a free $\mathcal{O}_Z$-module (in a non-canonical way): Let $i\colon X \to \mathcal{U}$ be a local embedding at x and assume that we have local coordinates (z, w) in $\mathcal{U}$ such that $Z = \{w = 0\}$. To each multiindex $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{i\kappa}) \in \mathbb{N}^\kappa$ we associate the monomial $w^{\alpha_i} := w_1^{\alpha_{i1}} \cdots w_\kappa^{\alpha_{i\kappa}}$. After possibly shrinking $\mathcal{U}$ there is a (not unique) set of monomials $1, w^{\alpha_1}, \ldots, w^{\alpha_{\tau-1}}$ such that each ϕ in $\mathcal{O}_X$ in $\mathcal{U}$ has a unique representative

$$\hat{\phi}(z, w) = \hat{\phi}_0(z) \otimes 1 + \hat{\phi}_1(z) \otimes w^{\alpha_1} + \cdots + \hat{\phi}_{\nu-1}(z) \otimes w^{\alpha_{\tau-1}}, \tag{1.8}$$

in $\mathcal{O}_\mathcal{U}$, where $\hat{\phi}_i$ are in $\mathcal{O}_Z$. Let $\pi\colon \mathcal{U} \to Z \cap \mathcal{U}$ be the submersion $(z, w) \mapsto z$.

Theorem 1.5. *Assume that $x \in X$ is a regular point, and let $i\colon X \to \mathcal{U}$ be a local embedding at x as above. Then*

$$\left(|\hat{\phi}_0(z)|^2 + \cdots + |\hat{\phi}_{\tau-1}(z)|^2\right)^{1/2} \tag{1.9}$$

is a pointwise norm in $X \cap \mathcal{U}$ that is equivalent to $|\phi(z)|_{X,\pi}$ in $Z \cap \mathcal{U}$.

It follows that (1.9) only depends, up to equivalence, on the submersion π. Moreover, the sum of the norms (1.9) obtained from a suitable finite set of submersions $\mathcal{U} \to Z \cap \mathcal{U}$ is equivalent to $|\phi|_X$.

In Section 7 we consider an arbitrary pure-dimensional singular space X and prove that at each point x on the singular locus Z_{sing} there is a local embedding $i\colon X \to \mathcal{U}$ such that $\mathcal{N}_X$, a priori defined on $(Z \setminus Z_{sing}) \cap \mathcal{U}$, admits a coherent extension to $Z \cap \mathcal{U}$. Patching together we get a global pointwise Hermitian norm on X.

Definitions of the sheaf of smooth $(0,q)$-forms $\mathcal{E}_X^{0,q}$ on a non-reduced X and of an associated $\bar\partial$-operator were recently given in [7]. In Section 8 we point out that the Noetherian operators in $\mathcal{N}_X$ extend to mappings $\mathcal{E}_X^{0,q} \to \mathcal{E}_Z^{0,q}$. In this way we get an extension of the norm $|\cdot|_X$ to smooth $(0,q)$-forms. Thus one, e.g., can discuss norm estimates for possible solutions to the $\bar\partial$-equation on X, but this question is not pursued in this paper. In the recent paper [6] we find L^p-estimates of extensions of holomorphic functions ϕ defined on a non-reduced subvariety X of a strictly pseudoconvex domain D, given that certain L^p-norms of $|\phi|_X$ over $Z \cap D$ are finite. This generalizes results in [2,18], see also [1], in the case when X is reduced. In this paper we prove the following.

Theorem 1.6. *Assume that ϕ_j is a sequence of holomorphic functions on X that is a Cauchy sequence on each compact subset with respect to the uniform norm induced by $|\cdot|_X$. Then there is a holomorphic function ϕ on X such that $\phi_j \to \phi$ uniformly on compact subsets of X.*

This statement is well-known but non-trivial in the reduced case, see, e.g., [24, Theorem 7.4.9].

The plan of the paper is as follows. In Section 2 we recall the definition of Coleff-Herrera currents as well as some basic facts. Proposition 1.1 and Theorems 1.3 and 1.4 are proved in Sections 3 and 4. Theorem 1.5 is proved in Section 5. Section 6 is devoted to a non-trivial example where the $\mathcal{N}_X$ and the norm $|\cdot|_X$ are computed explicitly. The content of Sections 7 and 8 is already mentioned.

The proof of Theorem 1.6 relies on some further residue theory that we recall in Sections 9 and 10. In the latter one we also provide a proof of Theorem 1.6 in case Z is smooth. For the general case we need a kind of resolution of X that is described in Section 11, and in Section 12 the proof of Theorem 1.6 is concluded.

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2. Some preliminaries

In this section we have collected a few definitions and results that will be used.

2.1. Coleff-Herrera currents

Assume that $j\colon Z \to \mathcal{U} \subset \mathbb{C}^N$ is an embedding of a reduced variety Z of pure dimension n. A germ of a current μ in $\mathcal{U}$ of bidegree $(N, N-n)$ is a Coleff-Herrera current with support on Z, $\mu \in \mathcal{CH}_{\mathcal{U}}^Z$, if it is $\bar\partial$-closed, is annihilated by $\bar{\mathcal{J}}_Z$ (i.e., $\bar h\mu = 0$ for h in $\mathcal{J}_Z$) and in addition has the *standard extension property* (SEP). The latter condition can be expressed in the following way: Let χ be any smooth function on the real axis that is 0 close to the origin and 1 in a neighborhood of ∞. Then μ has the SEP if for any holomorphic function h (or tuple h of holomorphic functions) whose zero set $Z(h)$ has positive codimension on Z, $\chi(|h|/\epsilon)\mu \to \mu$ when $\epsilon \to 0$. The intuitive meaning is that μ does not carry any mass on the set $Z \cap Z(h)$. See, e.g., [3, Section 5] for a discussion.

Example 2.1 *(Coleff-Herrera product).* If $f_1, \dots, f_{N-n}$ are holomorphic functions in $\mathcal{U}$ with common zero set Z, then the Coleff-Herrera product

$$\bar\partial\frac{1}{f} := \bar\partial\frac{1}{f_{N-n}}\wedge\cdots\wedge\bar\partial\frac{1}{f_1} \tag{2.1}$$

can be defined in various ways by suitable limit processes. Its annihilator is precisely the ideal (sheaf) $\mathcal{J}(f) = \langle f_1, \dots, f_{N-n}\rangle$. If A is a holomorphic N-form, then $A\wedge\bar\partial(1/f)$ is a Coleff-Herrera current. $\square$

Proposition 2.2. *If f_j are as in Example 2.1, μ is in $\mathcal{CH}_{\mathcal{U}}^Z$ and $\mathcal{J}(f)\mu = 0$, then there is (locally) a holomorphic N-form A such that*

$$\mu = A\wedge\bar\partial\frac{1}{f}. \tag{2.2}$$

The statements in Example 2.1 are due to Coleff and Herrera, Dickenstein and Sessa, and Passare in the '80s, whereas Proposition 2.2 is due to Björk, [16]. Proofs and further discussions and references can be found in [16] and [3, Sections 3 and 4].

2.2. Embeddings of a non-reduced space

Let $i\colon X \to \mathcal{U} \subset \mathbb{C}^N$ be a local embedding of a non-reduced space of pure dimension n and consider the sheaf $\mathcal{H}om_{\mathcal{O}_\mathcal{U}}(\mathcal{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}^Z_\mathcal{U})$, i.e., the sheaf of currents μ in $\mathcal{CH}^Z_\mathcal{U}$ such that $\mathcal{J}\mu = 0$. It is indeed a sheaf over $\mathcal{O}_X = \mathcal{O}_\mathcal{U}/\mathcal{J}$; for the rest of this paper we will omit the lower index and write just $\mathcal{H}om\,(\mathcal{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}^Z_\mathcal{U})$. The duality principle,

$$\Phi \in \mathcal{J} \text{ if and only if } \Phi\mu = 0 \text{ for all } \mu \in \mathcal{H}om\,(\mathcal{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}^Z_\mathcal{U}), \tag{2.3}$$

is known since long ago, see, e.g., [5, (1.6)].

Given a point x on X there is a minimal number $\hat{N}$ such that there is a local embedding $i'\colon X \to \mathcal{U}' \subset \mathbb{C}^{\hat{N}}_\zeta$ at x. Such a minimal embedding is unique up to biholomorphisms. Moreover, any embedding $i\colon X \to \mathcal{U} \subset \mathbb{C}^N$ factorizes so that, in a neighborhood of x,

$$X \xrightarrow{i'} \mathcal{U}' \xrightarrow{j} \mathcal{U} := \mathcal{U}' \times \mathcal{U}'' \subset \mathbb{C}^{\hat{N}}_\zeta \times \mathbb{C}^{N-\hat{N}}_{w''} = \mathbb{C}^N, \quad i = j \circ i', \tag{2.4}$$

where i' is minimal, $\mathcal{U}''$ is an open subset of $\mathbb{C}^{N-\hat{N}}_{w''}$, $j(\zeta) = (\zeta, 0)$, and the ideal in $\mathcal{U}$ is $\mathcal{J} = \mathcal{J}' \otimes 1 + (w''_1, \dots, w''_m)$, where $\mathcal{O}_X \simeq \mathcal{O}_{\mathcal{U}'}/\mathcal{J}'$. It follows from [7, Lemma 4] that the mapping

$$j_*\colon \mathcal{H}om\,(\mathcal{O}_{\mathcal{U}'}/\mathcal{J}', \mathcal{CH}^Z_{\mathcal{U}'}) \to \mathcal{H}om\,(\mathcal{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}^Z_\mathcal{U}) \tag{2.5}$$

is an $\mathcal{O}_X$-linear isomorphism. It is naturally expressed as $\mu' \mapsto \mu = \mu' \otimes [w'' = 0]$, where $[w'' = 0]$ denotes the current of integration over $\{w'' = 0\}$.

Remark 2.3. The equivalence classes in (2.5) can be considered as elements of an intrinsic $\mathcal{O}_X$-module ω^n_X of $\bar\partial$-closed $(n, 0)$-form on X, introduced in [7], so that $i_*\colon \omega^n_X \to \mathcal{H}om\,(\mathcal{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}^Z_\mathcal{U})$ is an isomorphism. In case X is reduced, ω^n_X is the classical Barlet sheaf, [13], consisting of $\bar\partial$-closed meromorphic n-forms. □

If Z is smooth, $\pi\colon \mathcal{U} \to Z$ is a (holomorphic) submersion, and μ is in $\mathcal{H}om\,(\mathcal{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}^Z_\mathcal{U})$, then $\pi_*\mu$ is a holomorphic n-form on Z.

Lemma 2.4. *With the notation above, there is a submersion $\pi'\colon \mathcal{U}' \to Z$ such that $\pi'_*\mu' = \pi_*\mu$.*

Proof. Let (z, w') be coordinates in $\mathcal{U}'$ such that $Z = \{w' = 0\}$. Then the fiber of π over $z \in Z$ must be of the form $(w', w'') \mapsto (z + b'w' + b''w'', w', w'')$, where

$$b'w' = b'(z, w', w'')w' = \sum_{i=1}^{\hat{N}-n} b'_i(z, w', w'')w'_i,$$

$$b''w'' = b''(z, w', w'')w'' = \sum_{j=1}^{m} b''_j(z, w', w'')w''_j,$$

and b'_i and b''_j are holomorphic. Now, since $\mu = \mu' \otimes [w'' = 0]$,

$$\pi_*\mu(z) = \int\limits_{w',w''} \mu(z + b'w' + b''w'', w', w'') = \int\limits_{w'} \mu'(z + b'|_{w''=0}w', w').$$

This is precisely $\pi'_*\mu'(z)$, where π' is the submersion with fiber $w' \mapsto (z+b'(z, w', 0)w', w')$ over z. $\square$

2.3. Local representation of certain currents

Consider an open set $\mathcal{U} \subset \mathbb{C}^n_z \times \mathbb{C}^\kappa_w$, let $Z = \mathbb{C}^n_z \times \{0\}$, and let $\pi\colon \mathcal{U} \to Z \cap \mathcal{U}$ be the submersion $(z, w) \mapsto z$. We use the short-hand notation

$$dz = dz_1\wedge \ldots \wedge dz_n, \quad dw = dw_1\wedge \ldots \wedge dw_\kappa, \tag{2.6}$$

and

$$\bar\partial \frac{dw}{w^{m+\mathbf{1}}} = \bar\partial \frac{dw_1}{w_1^{m_1+1}}\wedge\bar\partial \frac{dw_2}{w_2^{m_2+1}}\wedge \ldots \wedge\bar\partial \frac{dw_\kappa}{w_\kappa^{m_\kappa+1}}, \tag{2.7}$$

if $m = (m_1, \ldots, m_\kappa) \in \mathbb{N}^\kappa$ is a multiindex. It is well-known, and follows immediately from the one-variable case, that if $\xi_J(z, w)d\bar z_J$ is a smooth $(0, k)$-form in $\mathcal{U}$, then

$$\pi_*\Big(\xi_J(z, w)d\bar z_J \frac{1}{(2\pi i)^\kappa}\bar\partial \frac{dw}{w^{m+\mathbf{1}}}\wedge dz\Big) = \frac{1}{m!}\frac{\partial^{|m|}}{\partial w^m}\xi_J(z, 0)\, d\bar z_J\wedge dz. \tag{2.8}$$

If τ is any $(N, N - n + k)$-current in $\mathcal{U}$ with support on Z that is annihilated by all $\bar w_j$ and $d\bar w_\ell$, then it has the unique representation (as a locally finite sum)

$$\tau = \sum_\gamma \tau_\gamma(z)\frac{1}{(2\pi i)^\kappa}\bar\partial \frac{dw}{w^{\gamma+\mathbf{1}}}\wedge dz, \tag{2.9}$$

where τ_γ are $(0, k)$-currents on $Z \cap \mathcal{U}$ and

$$\tau_\gamma\wedge dz = \pi_*(w^\gamma\tau) \tag{2.10}$$

on $Z \cap \mathcal{U}$, cf. [7, (2.11)]. Clearly $\bar\partial\tau = 0$ if and only if $\bar\partial\tau_\alpha = 0$ for all α. In particular, τ is a Coleff-Herrera current if and only if all τ_α are holomorphic functions.

3. The sheaf $\mathcal{N}_X$ in a special case

Let $\iota\colon Z \to \mathcal{U} \subset \mathbb{C}^{n+\kappa}$ be a smooth submanifold of dimension n, let $w_1, \dots, w_\kappa$ be functions in $\mathcal{U}$ that generate $\mathcal{J}_Z$, and let $M \in \mathbb{N}^\kappa$ be a multiindex. In this section we prove Proposition 1.1 and Theorems 1.3 and 1.4 for the space $i\colon X' \to \mathcal{U} \subset \mathbb{C}^{n+\kappa}$ with structure sheaf $\mathcal{O}_{X'} = \mathcal{O}_{\mathcal{U}}/\mathcal{I}$, where

$$\mathcal{I} = \langle w_1^{M_1+1}, \dots, w_\kappa^{M_\kappa+1} \rangle.$$

Proof of Proposition 1.1 for X'. Assume that $\pi\colon \mathcal{U} \to Z$ is a submersion. In a neighborhood $\mathcal{V}$ of a given point $x \in Z$, there are coordinates (z, w) such that π is $(z, w) \mapsto z$ there. Since the proposition is local it is enough to prove it in $\mathcal{V}$. Given these coordinates, each function ϕ in $\mathcal{O}_{X'}$ has a unique representation

$$\phi = \sum_{m \le M} \phi_m(z) w^m, \tag{3.1}$$

where ϕ_m are in $\mathcal{O}_Z$. Using the notation (2.6) and (2.7), let

$$\hat{\mu} = \frac{1}{(2\pi i)^\kappa} \bar{\partial} \frac{dw}{w^{M+\mathbf{1}}} \wedge dz. \tag{3.2}$$

It follows from Proposition 2.2 that each μ in $\mathcal{H}om\,(\mathcal{O}_{\mathcal{U}}/\mathcal{I}, \mathcal{CH}^Z_{X'})$ is $a\hat{\mu}$ for some holomorphic a, i.e., the $\mathcal{O}_{\mathcal{U}}$-module $\mathcal{H}om\,(\mathcal{O}_{\mathcal{U}}/\mathcal{I}, \mathcal{CH}^Z_{X'})$ is generated by $\hat{\mu}$. If $\mu = a\hat{\mu}$, then $\pi_*(\psi\mu) = \pi_*(\psi a \hat{\mu})$ so in view of (2.8) we have

$$\mathcal{L}\psi\, dz = \pi_*(\psi a \hat{\mu}) = \frac{1}{M!} \frac{\partial^{|M|}}{\partial w^M} (\psi a)(z, 0)\, dz = \sum_{m \le M} c_m(z) \frac{\partial^{|m|}}{\partial w^m} \psi(z, 0)\, dz, \tag{3.3}$$

where c_m are functions in $\mathcal{O}_Z$. More precisely,

$$c_m(z) = \frac{1}{M!} \binom{M}{m} \frac{\partial^{|M-m|}}{\partial w^{M-m}} a(z, 0)$$

with suitable multiindex notation. It follows that $\mathcal{N}_{X',\pi}$ is the $\mathcal{O}_Z$-module in $Z \cap \mathcal{V}$ generated by the Noetherian operators $(\partial^{|m|}/\partial w^m)|_{w=0}$ for $m \le M$. By the uniqueness of the representations (3.1) these generators are independent, so $\mathcal{N}_{X',\pi}$ is a free $\mathcal{O}_Z$-module in $\mathcal{V}$ and hence coherent, and clearly (1.2) holds. $\square$

The next result is a main technical point in this paper.

Proposition 3.1. *Assume that (z, w) are coordinates in $\mathcal{U}$. The $\mathcal{O}_Z$-module $\mathcal{N}_{X'}$ is generated by the Noetherian operators*

$$\mathcal{L}_{m,\beta} := \frac{\partial^{|m|+|\beta|}}{\partial w^m \partial z^\beta}\Big|_{w=0}, \quad m \le M,\ |\beta| \le |M - m|. \tag{3.4}$$

Noting that $|M-m|=|M|-|m|$, Proposition 3.1 is precisely Theorem 1.4 for X'. In view of the unique representations (3.1) one sees that $\mathcal{N}_{X'}$ is a free $\mathcal{O}_Z$-module and therefore coherent. Furthermore, (1.2) holds since it does already for $\mathcal{N}_{X',\pi}$. Thus also Theorem 1.3 follows for X'.

Remark 3.2. If $M_j=1$ for some j then (3.4) means that there are no derivatives with respect to w_j. Then the embedding $i\colon X'\to\mathcal{U}$ is not minimal so one can delete this variable, cf. Section 2.2. In view of Lemma 2.4 this does not affect the definition of $\mathcal{N}_{X'}$. With no loss of generality one can therefore assume that $M_j>1$ for all j. $\square$

Proof of Proposition 3.1. Let us temporarily denote the $\mathcal{O}_Z$-module generated by the operators $\mathcal{L}_{m,\beta}$ in (3.4) by $\mathcal{M}$. Fix a point $x\in Z$. Any local submersion $\pi\colon\mathcal{U}\to Z$ at x (thus possibly just defined in a neighborhood $\mathcal{V}$ of x) is a trivial projection $(\zeta,\eta)\mapsto\zeta$ via the local change of coordinates

$$w_k=\eta_k,\ k=1,\dots,\kappa,\quad z_j=\zeta_j+\sum_{i=1}^{\kappa}b_{ji}\eta_i,\ j=1,\dots,n, \tag{3.5}$$

where b_{jk} are holomorphic functions. In fact, if $\pi(z,w)=(\pi_1(z,w),\dots,\pi_n(z,w))$ in the coordinates (z,w), then $\pi_j(z,0)=z_j$. Thus $\pi_j(z,w)=z_j+\mathcal{O}(w)$, where each $\mathcal{O}(w)$ denotes a function that vanishes on Z, i.e., contains some factor w_i, and so we get (3.5) with $\eta_k=w_k$ and $\zeta_j=\pi_j(z,w)$. We have

$$\frac{\partial}{\partial\eta_k}=\frac{\partial}{\partial w_k}+\sum_{j=1}^{n}(b_{jk}+\mathcal{O}(w))\frac{\partial}{\partial z_j},\quad k=1,\dots,\kappa. \tag{3.6}$$

In these new coordinates $\mathcal{I}=\langle\eta^{M+1}\rangle$. It thus follows from the argument above that this submersion π gives rise to the Noetherian operators

$$\Big(\frac{\partial}{\partial\eta}\Big)^{\gamma}:=\Big(\frac{\partial}{\partial\eta_\kappa}\Big)^{\gamma_\kappa}\cdots\Big(\frac{\partial}{\partial\eta_1}\Big)^{\gamma_1} \tag{3.7}$$

for $\gamma\le M$, generating $\mathcal{N}_{X',\pi}$. The notation in (3.7) will be used for the rest of this section. We will also suppress the distinction between a Noetherian operator L in $\mathcal{U}$ with respect to $\mathcal{I}$ and its induced operator $\mathcal{L}$ on X'.

Lemma 3.3. *Each operator $(\partial/\partial\eta)^\gamma$, $\gamma\le M$, belongs to (i.e., induces an element in) $\mathcal{M}$.*

Proof. We will proceed by induction over the number of factors $k\le\kappa$ involved in (3.7). Therefore, assume that $\gamma=(\gamma_1,\dots,\gamma_k)\le(M_1,\dots,M_k)$,

$$\gamma'=(\gamma_1,\dots,\gamma_{k-1})\le M':=(M_1,\dots,M_{k-1})$$

and let

$$\left(\frac{\partial}{\partial\eta}\right)^{\gamma'} := \left(\frac{\partial}{\partial\eta_{k-1}}\right)^{\gamma_{k-1}} \cdots \left(\frac{\partial}{\partial\eta_1}\right)^{\gamma_1}.$$

Assume also that we have proved that there are holomorphic functions $c_{m,\alpha}$, depending on both z and w, such that

$$\left(\frac{\partial}{\partial\eta}\right)^{\gamma'} = \sum_{m'\le M'} \sum_{|\alpha|\le|M'-m'|} c_{m',\alpha}\left(\frac{\partial}{\partial z}\right)^{\alpha}\left(\frac{\partial}{\partial w}\right)^{m'}. \tag{3.8}$$

If we apply $(\partial/\partial\eta_k)^{\gamma_k}$ to (3.8) a simple computation gives us (3.8) for k instead of $k-1$. By induction therefore (3.8) holds for $k=\kappa$ and so the lemma follows. $\square$

Proposition 3.4. *One can choose a finite number of submersions π^ℓ at x such that the corresponding operators $(\partial/\partial\eta^\ell)^\gamma$ for $\gamma\le M$ together generate $\mathcal{M}$ at x.*

Taking this proposition for granted, we can conclude the proof of Proposition 3.1. In fact, Lemma 3.3 means that $\mathcal{N}_{X',\pi}\subset\mathcal{M}$ for an arbitrary submersion π at x. By definition thus $\mathcal{N}_{X'}\subset\mathcal{M}$. On the other hand, Proposition 3.4 implies, cf. Remark 3.2, that $\mathcal{M}\subset\mathcal{N}_{X'}$. Thus $\mathcal{N}_{X'}=\mathcal{M}$ and so Proposition 3.1 is proved. $\square$

The rest of this section is devoted to the proof of Proposition 3.4. It will be apparent that one can choose the π^ℓ as arbitrarily small perturbations of any fixed submersion at x. First notice that if we choose a submersion so that b_{jk} are constant in the associated change of variables in (3.5), then

$$\frac{\partial}{\partial\eta_k} = \frac{\partial}{\partial w_k} + \sum_{j=1}^{n} b_{jk}\frac{\partial}{\partial z_j}, \quad k=1,\dots,\kappa. \tag{3.9}$$

We will choose our π^ℓ in this way. Then $\partial/\partial\eta_k$ is independent of $w_{k'}$ for $k'\neq k$ which makes it possible to proceed by induction over the codimension κ.

Let us first assume that $\kappa=1$, i.e., that we have just one variable w. Each point $a^\ell=(a_1^\ell,\dots,a_n^\ell)\in\mathbb{C}^n$ gives rise to a change of coordinates, with $b_{j1}=a_j^\ell$, and thus a submersion π^ℓ. The associated non-tangential derivative is, cf. (3.9),

$$\frac{\partial}{\partial\eta^\ell} = \frac{\partial}{\partial w} + \sum_{j=1}^{n} a_j^\ell\frac{\partial}{\partial z_j}. \tag{3.10}$$

Recall that

$$C_m := \binom{n+m}{m} \tag{3.11}$$

is the number of multiindices $\alpha=(\alpha_1,\dots,\alpha_n)$ such that $|\alpha|\le m$.

Lemma 3.5. *If we choose C_m generic points $a^\ell \in \mathbb{C}^n$, then for each α with $|\alpha| \le m$ there are unique $d_{\ell,\alpha}$ such that*

$$\Big(\frac{\partial}{\partial z}\Big)^{\alpha}\Big(\frac{\partial}{\partial w}\Big)^{m-|\alpha|}\psi = \sum_{\ell} d_{\ell,\alpha}\Big(\frac{\partial}{\partial \eta^\ell}\Big)^{m}\psi.$$

Proof. In view of (3.10) we have

$$\Big(\frac{\partial}{\partial \eta^\ell}\Big)^{m}\psi = \Big(\frac{\partial}{\partial w} + \sum_j a_j^\ell \frac{\partial}{\partial z_j}\Big)^{m}\psi = \sum_{|\alpha|\le m} (a^\ell)^{\alpha}\binom{m}{\alpha}\Big(\frac{\partial}{\partial z}\Big)^{\alpha}\Big(\frac{\partial}{\partial w}\Big)^{m-|\alpha|}\psi,$$

where

$$(a^\ell)^\alpha = (a_1^\ell)^{\alpha_1}\cdots(a_n^\ell)^{\alpha_n}.$$

We claim that the $C_m \times C_m$-matrix $A = (a^\ell)^\alpha$ is invertible if the a^ℓ are generic. If $n = 1$ then A is a Vandermonde matrix, and it is well-known that it is invertible if the $C_m = m+1$ points a^ℓ in $\mathbb{C}$ are distinct, so the claim follows. For the general case one can argue as follows: Given $x_\alpha \in \mathbb{C}^{C_m}$, consider the polynomial

$$p(t) = \sum_{|\alpha|\le m} x_\alpha t^\alpha$$

in $\mathbb{C}^n_t$. We get the action of the matrix A on x_α by evaluating $p(t)$ at the various points a^ℓ. Now $A(x_\alpha) = 0$ means that $p(t)$ vanishes at these C_m generic points, and hence $p(t)$ must vanish identically. This means that $(x_\alpha) = 0$ and since (x_α) is arbitrary, A is invertible. Now the lemma follows by taking

$$x_\alpha = \binom{m}{\alpha}\Big(\frac{\partial}{\partial \eta^\ell}\Big)^{m}\psi. \quad \square$$

Proof of Proposition 3.4. For $k = 1, \dots, \kappa$, let L_k be a set of C_{M_k} generic points in $\mathbb{C}^n$. For each $\ell = (\ell_1, \cdots, \ell_\kappa) \in \mathbb{L} := \oplus_{k=1}^{\kappa} L_k$ we get a change of coordinates, and an associated submersion π^ℓ, determined by $b^\ell_{jk} = b^{\ell_k}_j$. The associated differential operators $\partial/\partial\eta_k^\ell$, $k = 1, \dots, \kappa$, only depend, cf. (3.9), on the components $\ell_k \in L_\kappa$, respectively, so we can denote them by $\partial/\partial\eta_k^{\ell_k}$.

We claim that if $m \le M$ and $|\beta| \le |M - m|$, then there are complex numbers $c_{\alpha,m,\ell,\gamma}$ for $\gamma \le M$ such that, for any ψ in $\mathcal{O}_\mathcal{U}/\mathcal{I}$,

$$\Big(\frac{\partial}{\partial z}\Big)^{\alpha}\Big(\frac{\partial}{\partial w}\Big)^{m}\psi = \sum_{\ell\in\mathbb{L}}\sum_{\gamma\le M} c_{\alpha,m,\ell,\gamma}\Big(\frac{\partial}{\partial \eta^\ell}\Big)^{\gamma}\psi. \tag{3.12}$$

Clearly the claim implies the proposition. If $\kappa = 0$ the claim is trivially true. Assume now that the claim is proved for $\kappa - 1$. We can write, in a non-unique way,

$$\Big(\frac{\partial}{\partial z}\Big)^{\beta}\Big(\frac{\partial}{\partial w}\Big)^{m} = \Big(\frac{\partial}{\partial z}\Big)^{\beta_\kappa}\Big(\frac{\partial}{\partial w_\kappa}\Big)^{m_\kappa}\Big(\frac{\partial}{\partial z}\Big)^{\beta'}\Big(\frac{\partial}{\partial w'}\Big)^{m'}$$

where $w' = (w_1, \ldots, w_{\kappa-1})$, $m' = (m'_1, \ldots, m_{\kappa-1}) \le M' = (M_1, \ldots, M_{\kappa-1})$, $|\alpha'| \le |M' - m'|$, $m_\kappa \le M_\kappa$ and $|\beta_\kappa| \le M_\kappa - m_k$. By the induction hypothesis

$$\omega := \Big(\frac{\partial}{\partial z}\Big)^{\beta'}\Big(\frac{\partial}{\partial w'}\Big)^{m'}\psi$$

is a linear combination of

$$\Big(\frac{\partial}{\partial \eta^{\ell}}\Big)^{\gamma'}\psi = \Big(\frac{\partial}{\partial \eta_{\kappa-1}^{\ell_{\kappa-1}}}\Big)^{\gamma_{\kappa-1}}\cdots\Big(\frac{\partial}{\partial \eta_1^{\ell_1}}\Big)^{\gamma_1}\psi$$

for $\gamma' = (\gamma_1, \ldots, \gamma_{\kappa-1}) \le M'$ and $\ell' = (\ell_1, \ldots, \ell_{\kappa-1}) \in \oplus_{j=1}^{\kappa-1} L_k$. Lemma 3.5 implies that

$$\Big(\frac{\partial}{\partial z}\Big)^{\beta_\kappa}\Big(\frac{\partial}{\partial w_\kappa}\Big)^{m_\kappa}\omega$$

is a linear combination of

$$\Big(\frac{\partial}{\partial \eta_\kappa^{\ell_\kappa}}\Big)^{\gamma_\kappa}\omega$$

for $\gamma_\kappa \le M_\kappa$ and $\ell_\kappa \in L_\kappa$. Now the claim follows. $\square$

The proof requires $C_{M_1}\cdots C_{M_\kappa}$ different projections π^ℓ to generate the entire $\mathcal{O}_Z$-module $\mathcal{N}_{X'}$, and we think that this is the optimal number.

4. The sheaf $\mathcal{N}_X$ when Z is smooth

We shall now prove Proposition 1.1 and Theorems 1.3 and 1.4 in the general case. They are local, so let us assume that we at a given point $x \in X$ have an embedding

$$i\colon X \to \mathcal{U} \subset \mathbb{C}^N \tag{4.1}$$

and that the underlying reduced space Z is smooth.

Proof of Proposition 1.1. We can assume that we have coordinates (z, w) in $\mathcal{U}$ so that $\pi(z, w) = z$. We first claim that $\mathcal{N}_{X,\pi}$ is an $\mathcal{O}_Z$-module at x. In fact, if $\mathcal{L}$ is defined by (1.2) for some μ in $\mathcal{H}om\,(\mathcal{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}_\mathcal{U}^Z)$ and ξ is in $\mathcal{O}_Z$, then

$$\xi\mathcal{L}\phi\,\omega_z = \xi\pi_*(\phi\mu) = \pi_*(\phi\pi^*\xi\,\mu). \tag{4.2}$$

Since $\pi^*\xi\,\mu$ is in $\mathcal{H}om\,(\mathcal{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}_\mathcal{U}^Z)$ as well, $\xi\mathcal{L}$ is in $\mathcal{N}_{X,\pi}$, and so the claim follows.

Let us now prove that $\mathcal{N}_{X,\pi}$ is finitely generated at x. By the Nullstellensatz there is a multi-index $M = (M_1, \dots, M_\kappa) \in \mathbb{N}^\kappa$ such that (1.5) holds. If μ is in $\mathcal{H}om\,(\mathcal{O}_X, \mathcal{CH}^Z_{\mathcal{U}})$, therefore $w_j^{M_j+1}\mu = 0$ for each j. Shrinking $\mathcal{U}$ if necessary we can find $\mu_1, \dots, \mu_\nu$ that generate the $\mathcal{O}_{\mathcal{U}}$-module $\mathcal{H}om\,(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}^Z_{\mathcal{U}})$. If μ is any current in this sheaf thus

$$\mu = \sum_1^\nu c_j(z,w)\mu_j$$

for some holomorphic c_j. Since

$$c_j(z,w) = \sum_{m \le M} c_{jk}(z)w^m, \quad j = 1, \dots, \nu, \tag{4.3}$$

in $\mathcal{O}_{X'} = \mathcal{O}_{\mathcal{U}}/\mathcal{I}$, the equalities (4.3) hold in $\mathcal{O}_X = \mathcal{O}_{\mathcal{U}}/\mathcal{J}$ as well. Thus

$$\mu = \sum_{j=1}^\nu \sum_{m \le M} c_{jk} w^m \mu_j.$$

Notice that each $w^m\mu_j$ is in $\mathcal{H}om\,(\mathcal{O}_X, \mathcal{CH}^Z_{\mathcal{U}})$. If ϕ is in $\mathcal{O}_X$,

$$\pi_*(\phi\mu) = \sum_{j=1}^\nu \sum_{m \le M} c_{jk}\pi_*(\phi w^m \mu_j). \tag{4.4}$$

Thus the $\mathcal{O}_Z$-module $\mathcal{N}_{X,\pi}$ is generated in $\mathcal{U}$ by $\mathcal{L}_{j,m}$, where

$$\mathcal{L}_{j,m}\phi\,\omega_z = \pi_*(\phi w^m \mu_j), \quad m \le M,\ j = 1, \dots, \nu. \tag{4.5}$$

With no loss of generality we can assume that $\omega_z = dz$. Let $\hat\mu$ be as in (3.2). After possibly shrinking $\mathcal{U}$ further, there are holomorphic a_j in $\mathcal{U}$ such that $\mu_j = a_j\hat\mu$. Thus

$$\mathcal{L}_{j,m}\phi\,\omega_z = \pi_*(\phi w^m \mu_j) = \pi_*(\phi a_j w^m \hat\mu).$$

It follows from (3.3) that $\mathcal{L}_{j,m}$ are induced by differential operators $L_{j,m}\Phi$ in $\mathcal{U}$ which are Noetherian with respect to $\mathcal{J}$ since Φa_j is in $\mathcal{I}$ if Φ is in $\mathcal{J}$.

We now claim that (1.2) holds for the set of generators $\mathcal{L}_{j,m}$ above, cf. (4.5), i.e., that $\mathcal{L}_{j,m}\phi = 0$ for $m \le M$ and $j = 1, \dots, \nu$, if and only if $\phi = 0$ in $\mathcal{O}_X$. By possibly shrinking $\mathcal{U}$ further, there are holomorphic a_j in $\mathcal{U}$ such that $\mu_j = a_j\hat\mu$. For each fixed j, $\mathcal{L}_{j,m}\phi = 0$ for all $m \le M$ if and only if $0 = \pi_*(\phi w^m \mu_j) = \pi_*(\phi a_j \hat\mu)$ for all $m \le M$, and this holds if and only if $\phi a_j = 0$ in $\mathcal{O}_{X'}$, which in turn holds if and only if $\phi a_j \hat\mu = 0$, i.e., $\phi\mu_j = 0$. Now the claim follows from the duality principle (2.3).

Finally we notice that $\mathcal{N}_{X,\pi}$ is a finitely generated submodule at x of the free $\mathcal{O}_Z$-module $\mathcal{N}_{X',\pi}$, generated by $(\partial^{|m[}/\partial w^m)|_{w=0}$, $m \le M$, and therefore $\mathcal{N}_{X,\pi}$ is coherent. $\square$

Remark 4.1. Using the setup and notation in the preceding proof we have

$$\mathcal{H}om\,(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}^Z_{\mathcal{U}}) = \{a\hat{\mu};\ a \in (\mathcal{I} : \mathcal{J})/\mathcal{I}\}; \tag{4.6}$$

the colon ideal (sheaf) $(\mathcal{I} : \mathcal{J})$ by definition consists of all a in $\mathcal{O}_{\mathcal{U}}$ such that $a\mathcal{J} \subset \mathcal{I}$. In fact, we already know that each μ on the left hand side has the form $a\hat{\mu}$ for some a. Recall that $a\hat{\mu} = 0$ if and only if $a \in \mathcal{I}$, cf. Example 2.1. Thus $\mathcal{J}a\hat{\mu} = 0$ if and only if $\mathcal{J}a \subset \mathcal{I}$. Now (4.6) follows. □

Notice that the generators for the $\mathcal{O}_Z$-module $\mathcal{N}_{X,\pi}$, cf. (4.5) are, if $\omega_z = dz$, precisely $\mathcal{L}_{j,m}\phi = (\partial^{|m|}(a_j\phi)/\partial w^m)|_{m=0}$, $m \le M$, $j = 1, \dots, \nu$, where $a_1, \dots, a_\nu$ is a generating set for the coherent $\mathcal{O}_{\mathcal{U}}$-module $(\mathcal{I} : \mathcal{J})/\mathcal{I}$.

Proof of Theorems 1.3 and 1.4. Recall that the $\mathcal{O}_Z$-module $\mathcal{N}_X$ at a point $x \in X$ is by definition generated by $\mathcal{N}_{X,\pi}$ obtained from all submersions in any local embeddings at x. In view of Lemma 2.4 it is however enough to take all $\mathcal{N}_{X,\pi}$ obtained from one single embedding, so let us fix (4.1). We will use the notation from the proof of Proposition 1.1. Assume that

$$\mathcal{L}\phi\, dz = \pi_*(\phi\mu),$$

where π is a local submersion and μ is in $\mathcal{H}om\,(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}^Z_{\mathcal{U}})$. In view of (4.4) and (4.5),

$$\mu = \sum_{j=1}^{\nu} \sum_{\gamma \le M} c_{j\gamma}(z) w^\gamma \mu_j = \sum_{j=1}^{\nu} \sum_{\gamma \le M} c_{j\gamma}(z) w^\gamma a_j \hat{\mu}. \tag{4.7}$$

By Theorem 1.4 for X', i.e., so that $\mathcal{O}_{X'} = \mathcal{O}_{\mathcal{U}}/\mathcal{I}$, we have

$$\pi_*(\psi w^\gamma \mu) = \sum_{m \le M} \sum_{|\beta| \le |M-m|} d_{m,\gamma,\beta}(z) \frac{\partial^{|\beta|+|m|}}{\partial z^\beta \partial w^m} \psi(z, 0)\, dz. \tag{4.8}$$

Combining (4.7) and (4.8) we get

$$\mathcal{L}\phi\, dz = \pi_*(\phi\mu) = \sum_{j=1}^{\nu} \sum_{m \le M} \sum_{|\beta| \le |M-m|} c'_{j,m,\beta}(z) \frac{\partial^{|\beta|+|m|}}{\partial z^\beta \partial w^m} (a_j\phi)(z, 0).$$

Thus the $\mathcal{O}_Z$-module $\mathcal{N}_X$ is generated by the finite set (1.6) of Noetherian operators on X and so Theorem 1.4 follows. Moreover, each of these differential operators belongs to the free $\mathcal{O}_Z$-module $\mathcal{N}_{X'}$ and hence $\mathcal{N}_X$ is coherent. Since (1.2) holds already for $\mathcal{N}_{X,\pi}$, by Proposition 1.1, now also Theorem 1.3 is proved. □

Remark 4.2. To compute the norm $|\cdot|_X$ locally at x by means of Theorem 1.5 one has to choose suitable coordinates, the ideal $\mathcal{I} \subset \mathcal{J}$, and find $a_1, \dots, a_\nu$ that generate $(\mathcal{I} : \mathcal{J})/\mathcal{I}$, i.e., so that $a_1\hat{\mu}, \dots, a_\nu\hat{\mu}$ generate $\mathcal{H}om\,(\mathcal{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}^Z_\mathcal{U})$, in $\mathcal{U}$. Then the norm is given by (1.7). $\square$

Proposition 4.3. *Let (4.1) be a local embedding at $x \in X$ and let $\pi^\ell \colon \mathcal{U} \to Z\cap\mathcal{U}$ be a finite number of independent local submersions as in Proposition 3.4. Then the submodules $\mathcal{N}_{X,\pi^\ell}$ generate $\mathcal{N}_X$ in a neighborhood of x.*

Proof. Assume that $\mathcal{L}\phi\,\omega_z = \pi_*(\phi\mu)$ for a local submersion π and μ in $\mathcal{H}om\,(\mathcal{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}^Z_\mathcal{U})$. Let (z, w) be coordinates in $\mathcal{U}$ such that π is $(z, w) \mapsto z$, and choose $\mathcal{I} \subset \mathcal{J}$ and the associated $\hat{\mu}$ as before. Moreover, cf. Proposition 2.2, let a be a holomorphic function in $\mathcal{U}$ such that $\mu = a\hat{\mu}$. In view of (3.3) and Proposition 3.4 we have

$$\mathcal{L}\phi\,dz = \frac{1}{M!}\frac{\partial^{|M|}}{\partial w^M}\Big|_{w=0}(\phi a)\,dz = \sum_\ell\sum_{\gamma\le M} c_{\ell,\gamma}(z)\Big(\frac{\partial}{\partial\eta^\ell}\Big)^\gamma\Big|_{w=0}(\phi a)\,dz. \tag{4.9}$$

If $d\zeta^\ell$ is the non-vanishing holomorphic n-form associated with coordinates defining π^ℓ, then $dz = c_\ell(z)d\zeta^\ell$, so

$$\frac{1}{\gamma!}\Big(\frac{\partial}{\partial\eta^\ell}\Big)^\gamma\Big|_{w=0}(\phi a)\,dz = c_\ell\frac{1}{\gamma!}\Big(\frac{\partial}{\partial\eta^\ell}\Big)^\gamma\Big|_{w=0}(\phi a)\,d\zeta^\ell = c_\ell\pi^\ell_*(\phi a w^{M-\gamma}\hat{\mu}). \tag{4.10}$$

Thus each $\mathcal{L}_{\ell,\gamma}\phi = (\partial/\partial\eta^\ell)^\gamma)|_{w=0}(\phi a)$ is in $\mathcal{N}_{X,\pi^\ell}$, so the proposition follows from (4.9). $\square$

5. The sheaf $\mathcal{N}_X$ at regular points

Let $i \colon X \to \mathcal{U}$ be a local embedding at $x \in X$ with coordinates (z, w) in $\mathcal{U}$ so that $Z = \{w = 0\}$. If (1.5) holds and Φ is holomorphic in $\mathcal{U}$, then

$$\Phi(z) = \sum_{m\le M} c_m(z) w^m$$

in $\mathcal{O}_X = \mathcal{O}_\mathcal{U}/\mathcal{J}$, where c_m are in $\mathcal{O}_Z$, cf. (4.3). Thus the right hand side is a representative of $\phi = i^*\Phi$ in $\mathcal{O}_X$. Therefore the set of monomials $\{w^m;\ m \le M\}$ generates $\mathcal{O}_X$ as an $\mathcal{O}_Z$-module. Let us extract a minimal generating set $1, w^{\alpha_1}, \dots, w^{\alpha_{\tau-1}}$ at x (clearly 1 must be one of the generators). Then each element ϕ in $\mathcal{O}_{X,x}$ has a representative $\hat{\phi}$ of the form (1.8), where $\hat{\phi}_j$ are in $\mathcal{O}_{Z,x}$.

Proposition 5.1. *Given such a minimal generating set at x, the representation (1.8) of ϕ is unique for all ϕ in $\mathcal{O}_{X,x}$ if and only if $\mathcal{O}_{X,x}$ is Cohen-Macaulay.*

For a proof of Proposition 5.1, see, e.g., [7, Proposition 3.1].

Assume now that x is a regular point, i.e., Z is smooth at x and $\mathcal{O}_{X,x}$ is Cohen-Macaulay. Given a minimal generating set w^{α_j}, thus $\mathcal{O}_{X,x}$ is a free $\mathcal{O}_{Z,x}$-module, i.e.,

$$\mathcal{O}_{Z,x'}^{\tau} \to \mathcal{O}_{X,x'}, \quad (\hat{\phi}_j) \mapsto \hat{\phi} := \hat{\phi}_0 + \hat{\phi}_1 w^{\alpha_1} + \cdots \hat{\phi}_{\tau-1} w^{\alpha_{\tau-1}} \tag{5.1}$$

is an isomorphism for $x' = x$. By coherence, it is an isomorphism for all $x' \in Z$ in a neighborhood of x, say, in $Z \cap \mathcal{U}$, after shrinking $\mathcal{U}$. Thus $\phi = 0$ in $\mathcal{O}_{X,x'}$ if and only if $\hat{\phi}_j = 0$ in $\mathcal{O}_{Z.x'}$ for $j = 0, \ldots, \tau - 1$, so the expression (1.9) is a pointwise norm of ϕ in $X \cap \mathcal{U}$.

Proof of Theorem 1.5. Let us choose $i\colon X \to \mathcal{U}$ at x so that (5.1) is an isomorphism for $x \in X \cap \mathcal{U}$. We have to relate (1.9) to our norm $|\cdot|_X$, and we proceed as follows: Assume that $\mu_1, \ldots, \mu_\nu$ is a generating set for the $\mathcal{O}_\mathcal{U}$-module $\mathcal{H}om\,(\mathcal{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}_\mathcal{U}^Z)$. If ϕ is in $\mathcal{O}_X$, then $\phi\mu_j$ are well-defined elements in $\mathcal{H}om\,(\mathcal{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}_\mathcal{U}^Z)$. With the notation in the proof of Proposition 1.1, cf. (2.9) and (2.10), we have the unique representations

$$\phi\mu_j = \sum_{m \le M} b_{j,m}(z) \frac{1}{(2\pi i)^\kappa} \bar{\partial} \frac{dw}{w^{m+\mathbf{1}}} \wedge dz, \quad j = 1, \ldots, \nu, \tag{5.2}$$

where

$$b_{j,m} \wedge dz = \pi_*(\phi w^m \mu_j). \tag{5.3}$$

If we represent ϕ by $\hat{\phi}$ in (5.1), then

$$b_{j,m} = \hat{\phi}_0 \pi_*(w^m \mu_j) + \hat{\phi}_1 \pi_*(w^{m+\alpha_1} \mu_j) + \cdots + \hat{\phi}_{\nu-1} \pi_*(w^{m+\alpha_{\nu-1}} \mu_j)$$

and thus $b_{j,m}$ are $\mathcal{O}_Z$-linear combinations of the $\hat{\phi}_j$. Hence the mapping

$$\phi \mapsto \phi \wedge \mu_j, \quad j = 1, \ldots, \nu,$$

via the isomorphism (5.1), induces an $\mathcal{O}_Z$-linear holomorphic morphism

$$T\colon \mathcal{O}_Z^\tau \to \mathcal{O}_Z^{\nu C_M},$$

where $C_M = (M + \mathbf{1})!$ is the number of $m \in \mathbb{N}^\kappa$ such that $m \le M$.

In view of the duality principle (2.3), T is injective. In fact, the image of T being zero, means that $\phi\mu = 0$, i.e., $\phi\mu_j = 0$ for $j = 1, \ldots, \nu$, and so $\phi = 0$ in $\mathcal{O}_X$ which in turn, cf. (5.1), means that $\hat{\phi}_j = 0$ for $j = 0, \ldots, \tau - 1$. By [7, Lemma 4.11], the matrix T is pointwise injective. If $(b_{j,m})$ is in the image of T therefore

$$\sum_{j=0} |\hat{\phi}_j|^2 \sim \sum_{j=1}^{\nu} \sum_{m \le M} |b_{j,m}|^2. \tag{5.4}$$

From (4.5) and (5.3) we see that

$$\sum_{j=1}^{\nu}\sum_{m\le M}|b_{j,m}|^2 \sim \sum_{j=1}^{\nu}\sum_{m\le M}|\mathcal{L}_{j,m}\phi|^2 = |\phi|^2_{X,\pi}. \tag{5.5}$$

Now Theorem 1.5 follows from (5.4) and (5.5). $\square$

6. An example

Consider the 2-plane $Z = \{w_1 = w_2 = 0\}$ in $\mathcal{U} \subset \mathbb{C}^4_{z_1,z_2,w_1,w_2}$, where $\mathcal{U}$ is the product of balls $\{|z| < 1,\ |w| < 1\}$ in $\mathbb{C}^4$, and let

$$\mathcal{J} = \langle w_1^2, w_2^2, w_1w_2, w_1z_2 - w_2z_1\rangle.$$

Then $\mathcal{O}_{\mathcal{U}}/\mathcal{J}$ has pure dimension 2 and is Cohen-Macaulay except at the point $0 \in \mathcal{U}$, see, [7, Example 6.9]. It is also shown there, notice that $\mathcal{I} = \langle w_1^2, w_2^2\rangle \subset \mathcal{J}$, that $\mathcal{H}om\,(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}^Z_{\mathcal{U}})$ is generated by

$$\mu_1 = w_1w_2\hat{\mu}, \quad \mu_2 = (z_1w_2 + z_2w_1)\hat{\mu},$$

where

$$\hat{\mu} = \frac{1}{(2\pi i)^2}\bar{\partial}\frac{dw_1}{w_1^2}\wedge\bar{\partial}\frac{dw_2}{w_2^2}\wedge dz_1\wedge dz_2.$$

Following the recipe in Theorem 1.4 and Remark 4.2 we get a generating set for $\mathcal{N}_X$ by applying each of the differential operators

$$1, \frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}, \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial^2}{\partial z_1\partial w_1}, \frac{\partial^2}{\partial z_1\partial w_2}, \frac{\partial^2}{\partial z_2\partial w_1}, \frac{\partial^2}{\partial z_2\partial w_2}, \frac{\partial^2}{\partial w_1\partial w_2}$$

to $a_1\phi = w_1w_2\phi$ and $a_2\phi = (z_1w_2 + z_2w_1)\phi$, respectively, and evaluate at $w = 0$. Then a_1 only contributes with the Noetherian operator 1, whereas a_2 gives rise to

$$z_1, z_2, 0, 0, z_2\frac{\partial}{\partial z_1}, (1 + z_1\frac{\partial}{\partial z_1}), (1 + z_2\frac{\partial}{\partial z_2}), z_1\frac{\partial}{\partial z_2}, (z_1\frac{\partial}{\partial w_1} + z_2\frac{\partial}{\partial w_2}). \tag{6.1}$$

Because of the operator 1 from the a_1, we can forget about z_j and replace $1 + z_j\frac{\partial}{\partial z_j}$ by $z_j\frac{\partial}{\partial z_j}$. Thus we get

$$|\phi|^2_X \sim |\phi|^2 + |z|^2\Big|\frac{\partial\phi}{\partial z_1}\Big|^2 + |z|^2\Big|\frac{\partial\phi}{\partial z_2}\Big|^2 + \Big|z_1\frac{\partial\phi}{\partial w_1} + z_2\frac{\partial\phi}{\partial w_2}\Big|^2.$$

6.1. Functions in $X \setminus \{0\}$

Let $\mathcal{L}_0 = 1$ and let $\mathcal{L}$ denote the right-most operator in (6.1). If ϕ is an $\mathcal{O}_X$-function defined in $Z \setminus \{0\}$, then both $\mathcal{L}_0\phi$ and $\mathcal{L}\phi$ are holomorphic functions in $Z \setminus \{0\}$. Since

$\{0\}$ has codimension 2 in Z, they both have holomorphic extensions across 0 that we denote by $\phi_0(z)$ and $h(z)$, respectively.

Notice that $1, w_1$ is a basis for $\mathcal{O}_X$ over $\mathcal{O}_Z$ where $z_1 \neq 0$, and similarly, $1, w_2$ is a basis for $\mathcal{O}_X$ over $\mathcal{O}_Z$ where $z_2 \neq 0$. Given any ϕ_0 and h in $\mathcal{U}$ we get a $\mathcal{O}_X$-function ϕ in $\mathcal{U} \setminus \{0\}$, defined as

$$\phi = \phi_0 + (h/z_1)w_1, \ z_1 \neq 0; \quad \phi = \phi_0 + (h/z_2)w_2, \ z_2 \neq 0. \tag{6.2}$$

It is readily checked that $\mathcal{L}_0\phi = \phi_0$ and $\mathcal{L}\phi = h$. In other words, there is a $1-1$ correspondence between $\mathcal{O}_X$-functions ϕ in $Z \setminus \{0\}$ and $\mathcal{O}_Z^2$.

Lemma 6.1. *The $\mathcal{O}_X$-function ϕ has an extension across 0 if and only if $h(0) = 0$.*

Proof. If ϕ is defined in $\mathcal{U}$ then $h = \mathcal{L}\phi$ in $\mathcal{U}$ and then clearly $h(0) = 0$. Conversely, if $h(0) = 0$, then $h(z) = c_1(z)z_1 + c_2(z)z_2$ for some functions c_1, c_2 in $\mathcal{U}$. It is readily checked that indeed ϕ, defined by (6.2), coincides with

$$\phi_0(z) + c_1(z)w_1 + c_2(z)w_2$$

in $\mathcal{U} \setminus \{0\}$. Thus ϕ extends across 0. $\square$

In view of this lemma, if we take, e.g., $h = 1$ in (6.2), we get an $\mathcal{O}_X$-function ϕ in $\mathcal{U} \setminus \{0\}$ that does not extend across 0.

7. Extension of $\mathcal{N}_X$ across Z_{sing}

We now drop the assumption that the underlying space Z is smooth.

Lemma 7.1. *Let x be a fixed point on the singular locus Z_{sing} of Z and let $X \to \mathcal{U} \subset \mathbb{C}^N$ be a local embedding at x. If $\mathcal{U}$ is small enough there are holomorphic functions $f_1, \ldots, f_\kappa$ so that $Z(f) = \{f_1 = \cdots = f_\kappa = 0\}$ has codimension κ, and contains $Z \cap \mathcal{U}$ and such that $df := df_1 \wedge \ldots \wedge df_\kappa$ is non-vanishing on $Z_{reg} \setminus Z(f)_{sing}$. If $x' \in \mathcal{U} \setminus Z$ is given we can choose f_j so that $x' \notin Z(f)$.*

That is, $Z(f)$ is a complete intersection that may have "unnecessary" irreducible components, but $df \neq 0$ at each point on Z_{reg} that is not hit by any of these components. This is of course a well-known result and follows, e.g., from the more precise statement in the lemma on page 72 in [21]. However, we provide a simple argument here for the reader's convenience.

Proof. If $\mathcal{U}$ is small enough we can find a finite number of functions $g_1, \ldots, g_m$ that generate $\mathcal{J}_Z$. For each irreducible component Z^ℓ of Z we choose a point $x^\ell \in Z^\ell_{reg} \cap \mathcal{U}$. Notice that dg_j span the annihilator of the tangent bundle at x^ℓ for each ℓ. If $f_1, \cdots, f_\kappa$

are generic linear combinations of the g_j, then df_j span these spaces as well for each ℓ, and f_j define a complete intersection $Z(f)$ that avoids x'. Clearly $Z \subset Z(f)$ and $df \neq 0$ at x^ℓ for each ℓ. It is not hard to see (cf. [25, Theorem 4.3.6]) that df is non-vanishing on the regular part of the irreducible component of $Z(f)$ that contains x^ℓ; i.e., on $Z^\ell_{reg} \setminus Z(f)_{sing}$, for each ℓ. $\square$

Let x, f and $\mathcal{U}$ be as in Lemma 7.1 and let us write Z rather than $Z \cap \mathcal{U}$. Since df is generically non-vanishing on Z we can choose coordinates $(\zeta, \eta) = (\zeta_1, \cdots, \zeta_n; \eta_1 \cdots, \eta_\kappa)$ in $\mathcal{U}$ such that, with suitable matrix notation, $H = \partial f/\partial\eta$ is generically invertible on Z. Let $h = \det H$. If

$$w = f(\zeta, \eta), \;\; z = \zeta, \tag{7.1}$$

then, cf. (2.6), $dw \wedge dz = h d\eta \wedge d\zeta$ and hence (z, w) are local coordinates at each point on $Z \setminus \{h = 0\}$. Notice that

$$\frac{\partial}{\partial w} = H^{-1}\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial \zeta} - G\frac{\partial}{\partial w}, \tag{7.2}$$

where $G = \partial f/\partial\zeta$ is holomorphic. Since $H^{-1} = \Theta/h$, where Θ is holomorphic, therefore

$$h\frac{\partial}{\partial w} = \Theta\frac{\partial}{\partial \eta}, \qquad h\frac{\partial}{\partial z} = h\frac{\partial}{\partial \zeta} - G\Theta\frac{\partial}{\partial \eta}. \tag{7.3}$$

For a sufficiently large multiindex $M = (M_1, \ldots, M_\kappa)$ the complete intersection ideal $\langle f_1^{M_1+1}, \ldots, f_\kappa^{M_\kappa+1}\rangle$ is contained in $\mathcal{J}$. Possibly after shrinking the neighborhood $\mathcal{U}$ of x there are generators $\mu_1, \ldots, \mu_\nu$ for $\mathcal{H}om\,(\mathcal{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}^Z_\mathcal{U})$ and holomorphic functions $a_1, \ldots, a_\kappa$ in $\mathcal{U}$, cf. Proposition 2.2, such that

$$\mu_j = a_j \frac{1}{(2\pi i)^\kappa}\bar\partial\frac{1}{f^{M+1}}\wedge d\eta\wedge d\zeta. \tag{7.4}$$

Notice that a_j must vanish on the “unnecessary” irreducible components of $Z(f)$. For the rest of this section we will use the notation (3.7).

Proposition 7.2. *With the notation above, the differential operators*

$$\Phi \mapsto L_{m,\beta,j}\Phi := \Big(h\frac{\partial}{\partial w}\Big)^m \Big(h\frac{\partial}{\partial z}\Big)^\beta (a_j\Phi), \quad m \le M, \; |\beta| \le |M - m|, \; j = 1, \ldots, \nu, \tag{7.5}$$

a priori defined on $\mathcal{U} \cap \{h \neq 0\}$, have holomorphic extensions to $\mathcal{U}$. They are Noetherian with respect to $\mathcal{J}$ and the induced operators $\mathcal{L}_{m,\beta,j}$, cf. (1.1), belong to $\mathcal{N}_X$ on Z_{reg} and generate the $\mathcal{O}_Z$-module $\mathcal{N}_X$ where $h \neq 0$.

Proof. From (7.3) it is clear that $L_{m,\beta,j}$ have holomorphic extensions to $\mathcal{U}$. Since (z,w) are local coordinates at a point on Z_{reg} where $h \neq 0$ it follows from (7.2) and Theorem 1.4, cf. Remark 4.1, that the induced operators $\mathcal{L}_{m,\beta,j}$ are in $\mathcal{N}_X$ there. By a simple induction argument it follows from the same theorem that they actually generate $\mathcal{N}_X$ there. It also follows that $L_{m,\beta,j}$ are Noetherian there with respect to $\mathcal{J}$ in $\mathcal{U} \cap \{h \neq 0\}$ and by continuity their extensions are Noetherian as well. Thus $\mathcal{L}_{m,\beta,j}$ are Noetherian on $X \cap \mathcal{U}$.

We have to prove that $\mathcal{L}_{m,\beta,j}$ are in $\mathcal{N}_X$ on Z_{reg} where $h = 0$. Let $x' \in X_{reg}$ be such a point and assume that $df \neq 0$. For a generic choice of constant matrices b, c we have that $df \wedge d(c\zeta + b\eta) \neq 0$. Thus we can choose new coordinates

$$w' = f(\zeta, \eta), \quad z' = c\zeta + b\eta$$

in a neighborhood $\mathcal{V} \subset \mathcal{U}$ of x'. It follows that

$$\phi \mapsto \Big(\frac{\partial}{\partial w'}\Big)^m \Big(\frac{\partial}{\partial z'}\Big)^\alpha \Big|_{w'=0} (\phi a_j), \quad m \leq M, \ |\alpha| \leq |M - m|, \tag{7.6}$$

are in $\mathcal{N}_X$ in $Z \cap \mathcal{V}$. Since $z' = bz + cw$, $w' = w$, in $\mathcal{V} \setminus \{h = 0\}$, we have

$$\frac{\partial}{\partial w} = \frac{\partial}{\partial w'} + \frac{\partial z'}{\partial w}\frac{\partial}{\partial z'}, \quad \frac{\partial}{\partial z} = \frac{\partial z'}{\partial z}\frac{\partial}{\partial z'},$$

so by (7.3), applied to z', w' instead of ζ, η,

$$h\frac{\partial}{\partial w_k} = h\frac{\partial}{\partial w'_k} + \sum_i d_{ki}\frac{\partial}{\partial z'_i}, \quad h\frac{\partial}{\partial z_k} = \sum_i d'_{ki}\frac{\partial}{\partial z'_i}, \tag{7.7}$$

where d_{ki}, d'_{ki} have holomorphic extensions to $\mathcal{V}$. Thus $\mathcal{L}_{m,\beta,j}$ are $\mathcal{O}_Z$-linear combinations of (7.6), and hence belong to $\mathcal{N}_X$.

Let us now consider a point $x' \in Z_{reg}$ where $df = 0$, i.e., some "unnecessary" component of $Z(f)$ passes through x'. Then certainly $h(x') = 0$. Let π be the projection $(z,w) \mapsto z$. By (7.1), (7.4), and (3.3),

$$\frac{1}{\gamma!}\Big(\frac{\partial}{\partial w}\Big)^\gamma (a_j\phi)\, dz = \pi_*\Big(\phi a_j df \wedge f^{M-\gamma}\frac{1}{(2\pi i)^\kappa}\bar\partial\frac{1}{f^{M+\mathbf{1}}}\wedge dz\Big), \quad \gamma \leq M, \tag{7.8}$$

in $\mathcal{U} \setminus \{h = 0\}$. Let $v = (v_1, \ldots, v_\kappa)$ generate $\mathcal{J}_Z$ at x' and assume first that $dv \wedge dz \neq 0$ so that (z, v) are local coordinates in a neighborhood $\mathcal{V} \subset \mathcal{U}$ of x'. Since $\mathcal{J}(f) \subset \mathcal{J}_Z$, $f = Av$ for a holomorphic matrix A in $\mathcal{V}$. At points $z \in Z \cap \mathcal{V} \setminus \{h = 0\}$ both f_j and v_j are minimal sets of generators for $\mathcal{J}_Z$ so A is invertible there. Therefore also (z, v) define the submersion π in $\mathcal{V} \setminus \{h = 0\}$. Since $f^{M-\gamma} df \wedge dz = \alpha d\eta \wedge d\zeta$, where α is holomorphic, the right hand side of (7.8) is $\pi_*(\phi\alpha\mu_k)$ which is $d\zeta$ times an element in $\mathcal{N}_X$ in $\mathcal{V}$. It follows that the left hand side of (7.8) is $d\zeta$ times $\mathcal{L}\phi$, where $\mathcal{L}$ extends to an element in $\mathcal{N}_X$ in $\mathcal{V}$.

Notice that if b^ℓ is a small constant $n \times \kappa$-matrix, then $\eta' = \eta$, $\zeta' = \zeta + b^\ell f$ is a change of variables in $\mathcal{V}$, possibly after shrinking our neighborhood $\mathcal{V}$ of x'. In fact, $d\eta' \wedge d\zeta' = d\eta \wedge (d\zeta + b^\ell df)$ is non-vanishing if b^ℓ is small enough. Taking $w^\ell = f$, $z^\ell = \zeta'$, we get that $w^\ell = w$, $z^\ell = z + b^\ell w$, and hence

$$dw^\ell \wedge dz^\ell = df \wedge d(\zeta + b^\ell df) = df \wedge d\zeta = h d\eta \wedge d\zeta,$$

where h is the same function as in (7.7). As in the preceding step of the proof we conclude that

$$\phi \mapsto \Big(\frac{\partial}{\partial w^\ell}\Big)^\gamma (a_j \phi), \quad \gamma \le M,$$

a priori defined in $\mathcal{V} \setminus \{h = 0\}$, have extensions to elements in $\mathcal{N}_X$ in $Z \cap \mathcal{V}$. By Proposition 3.4 there is a finite set of such b^ℓ and holomorphic $d_{m,\beta,\ell,j}$ in a possibly even smaller neighborhood $\mathcal{V}$ of x' such that

$$\Big(\frac{\partial}{\partial w}\Big)^m \Big(\frac{\partial}{\partial z}\Big)^\beta (a_j \phi) = \sum_\ell \sum_{\gamma \le M} d_{m,\beta,\ell,\gamma} \Big(\frac{\partial}{\partial w^\ell}\Big)^\gamma (a_j \phi), \quad m \le M, \ |\beta| \le |M - m|. \tag{7.9}$$

It follows that all the operators on the left hand side of (7.9) are in $\mathcal{N}_X$ in $\mathcal{V}$.

Finally, if $dv \wedge d\zeta = 0$ at x' we introduce new coordinates $z' = c\zeta + b\eta, w' = w$ as before so that $dv \wedge dz' \neq 0$. From what we have just proved, then all

$$\Big(\frac{\partial}{\partial w'}\Big)^m \Big(\frac{\partial}{\partial z'}\Big)^\beta (a_j \phi)$$

are holomorphic at x'. It now follows from (7.7) that $\mathcal{L}_{m,\beta,j}$ are in $\mathcal{N}_X$ at x'. Thus Proposition 7.2 is proved. □

We can now formulate our main result of this section.

Theorem 7.3. *Given a point $x \in Z_{sing}$ there is a local embedding at x $i\colon X \to \mathcal{U} \subset \mathbb{C}^N$ and a finite number of Noetherian operators $\mathcal{L}_1, \dots, \mathcal{L}_r$ on $X \cap \mathcal{U}$ that generate $\mathcal{N}_X$ on $\mathcal{U} \cap Z_{reg}$.*

Clearly, such a set $\mathcal{L}_1, \dots, \mathcal{L}_r$ defines a coherent extension of $\mathcal{N}_X$ to $\mathcal{U} \cap Z$.

Proof. Choose the embedding $i\colon X \to \mathcal{U} \subset \mathbb{C}^N$ at x small enough so that we have a complete intersection $f = (f_1, \dots, f_\kappa)$ as above, global coordinates (ζ, η), and so that Proposition 7.2 applies, with $h = \det(\partial f/\partial \eta)$. We then get $\mathcal{L}_j$ in $\mathcal{N}_X$ on $X \cap Z_{reg}$ that have holomorphic extensions across Z_{sing} and that generate $\mathcal{N}_X$ in $Z_{reg} \cap (\mathcal{U} \setminus \{h = 0\})$. Choosing η' as other sets of κ coordinates we get another set of such $\mathcal{L}_j$ that generate $\mathcal{N}_X$ on $Z_{reg} \cap \mathcal{U}$ except where $h' = \det(\partial f/\partial \eta')$ vanishes. Repeating a finite number of times we get a finite set of $\mathcal{L}_j$ that generate $\mathcal{N}_X$ on $Z_{reg} \cap (\mathcal{U} \setminus \{df = 0\})$.

Possibly after shrinking $\mathcal{U}$, we can make the same construction for a finite number $f^1, \dots, f^\rho$ of complete intersections such that $Z \cap \mathcal{U} = Z(f^1) \cap \cdots \cap Z(f^\rho) \cap \mathcal{U}$, see Lemma 7.1, and we thus get a finite set $\mathcal{L}_j$ as desired. $\square$

Example 7.4. Let $Z = \{f = 0\}$ be a reduced subvariety of $\mathcal{U} \subset \mathbb{C}^2$ and assume that $df \neq 0$ on Z_{reg}. If X is defined by $\mathcal{J} = \langle f^2 \rangle$, then

$$\mu = \bar{\partial}\frac{1}{f^2}\wedge d\eta\wedge d\zeta$$

is a generator for $\mathcal{H}om\,(\mathcal{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}^Z_\mathcal{U})$. Let us choose coordinates (ζ, η) on $\mathbb{C}^2$ so that neither $h := \partial f/\partial\eta$ nor $\partial f/\partial\zeta$ vanish identically on Z. If we let $w = f(\zeta, \eta)$ and $z = \zeta$, then

$$h\frac{\partial}{\partial w} = \frac{\partial}{\partial \eta}, \quad h\frac{\partial}{\partial z} = h\frac{\partial}{\partial \zeta} - \frac{\partial f}{\partial \zeta}\frac{\partial}{\partial \eta}.$$

Thus Proposition 7.2 gives us the Noetherian operators $1,\ \partial/\partial\eta,\ h(\partial/\partial\zeta)$. If we add the operators obtained by interchanging the roles of η and ζ we find that the extension of $\mathcal{N}_X$ across Z_{sing} generated by $1, \partial/\partial\eta, \partial/\partial\zeta$. Clearly, this extension is independent of the choice of coordinates in $\mathcal{U}$. $\square$

7.1. Global pointwise norm on X

In Example 7.4 the extension of $\mathcal{N}_X$ across Z_{sing} is invariant. We do not know whether this is true in general. In any case we can define a global pointwise norm in the following way: Each point $x \in Z_{sing}$ has a neighborhood $\mathcal{U}_x$ where we have a coherent extension by Theorem 7.3 and in $\mathcal{U}_x$ we thus have a pointwise norm $|\cdot|_{X,x}$. We can choose a locally finite open covering $\{\mathcal{U}_{x_j}\}$ of X, and a partition of unity χ_j subordinate to this covering and define the global norm

$$|\cdot|_X^2 = \sum_j \chi_j |\cdot|^2_{X,x_j}. \tag{7.10}$$

8. Pointwise norm of smooth $(0, q)$-forms

In [7] was introduced a notion of smooth $(0, q)$-form on a non-reduced space X. We will recall this definition and show that our pointwise norm $|\cdot|_X$ extends to a pointwise norm on such forms.

Consider a local embedding $i\colon X \to \mathcal{U} \subset \mathbb{C}^N$ as before. If Φ is a smooth $(0, q)$-form in $\mathcal{U}$, $\Phi \in \mathcal{E}^{0,q}_\mathcal{U}$, following [7, Section 4] we say that $i^*\Phi = 0$, or equivalently $\Phi \in \mathcal{K}er\, i^*$, if

$$\Phi\wedge\mu = 0, \quad \mu \in \mathcal{H}om\,(\mathcal{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}^Z_\mathcal{U}).$$

In case Φ is holomorphic, this is equivalent to that $\Phi \in \mathcal{J}$ in view of the duality principle (2.3). We let

$$\mathcal{E}_X^{0,q} := \mathcal{E}_{\mathcal{U}}^{0,q}/\mathcal{K}er\, i^*$$

be the sheaf of smooth $(0,k)$-forms on X. Thus we have a well-defined surjective mapping $i^*\colon \mathcal{E}_{\mathcal{U}}^{0,q} \to \mathcal{E}_X^{0,q}$. By a standard argument, cf. Section 2.2, one checks that this definition is independent of the local embedding. For ϕ in $\mathcal{E}_X^{0,q}$ and μ in $\mathcal{H}om\,(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}_{\mathcal{U}}^Z)$ thus $\phi\wedge\mu$ is well-defined, and it vanishes for all such μ if and only if $\phi = 0$.

To extend our norm to forms in $\mathcal{E}_X^{0,q}$ let us first assume that the underlying reduced space Z is smooth. Assume that we have a local embedding $i\colon X \to \mathcal{U}$, and a submersion $\pi\colon \mathcal{U} \to Z \cap \mathcal{U}$. If μ is in $\mathcal{H}om\,(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}_{\mathcal{U}}^Z)$ and ϕ is in $\mathcal{E}_X^{0,q}$, then $\pi_*(\phi\wedge\mu)$ is a well-defined $(0,q)$-current on $Z \cap \mathcal{U}$ so we have

$$\mathcal{L}\phi\, \omega_z = \pi_*(\phi\wedge\mu). \tag{8.1}$$

Lemma 8.1. *An operator $\mathcal{L}$ so defined maps $\mathcal{L}\colon \mathcal{E}_X^{0,q} \to \mathcal{E}_Z^{0,q}$ and it is determined by its action on $\mathcal{O}_X$. If $\mathcal{L}\phi = 0$ for all μ in $\mathcal{H}om\,(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}_{\mathcal{U}}^Z)$, then $\phi = 0$.*

Since the $\mathcal{O}_Z$-module $\mathcal{N}_X$ is generated by operators of the form (8.1), it follows that any $\mathcal{L}$ in $\mathcal{N}_X$ extends to an operator $\mathcal{L}\colon \mathcal{E}_X^{0,q} \to \mathcal{E}_Z^{0,q}$.

Proof. Choose local coordinates (z,w) in $\mathcal{U}$ such that π is $(z,w) \mapsto z$. Then, cf. (2.9), each μ in $\mathcal{H}om\,(\mathcal{O}_{\mathcal{U}}/\mathcal{J}, \mathcal{CH}_{\mathcal{U}}^Z)$ has the form

$$\mu = \sum_m c_m(z) \frac{1}{(2\pi i)^\kappa} \bar\partial \frac{dw}{w^{m+\mathbf{1}}} \wedge dz. \tag{8.2}$$

Let us first assume that ϕ is a function in $\mathcal{E}_X^{0,0}$. Choose a smooth function Φ in $\mathcal{E}_U^{0,0}$ such that $\phi = i^*\Phi$. Then

$$\Phi\mu = \sum_m c_m(z)\Phi(z,w) \frac{1}{(2\pi i)^\kappa} \bar\partial \frac{dw}{w^{m+\mathbf{1}}} \wedge dz$$

and by (2.8) thus

$$\pi_*(\phi\mu) = \pi_*(\Phi\mu) = \sum_m c_m(z) \frac{1}{m!} \frac{\partial^{|m|}}{\partial z^m} \Phi(z,0)\, dz. \tag{8.3}$$

This differential operator is determined by its action on holomorphic functions and so the first statement of the lemma is proved for $q = 0$. If ϕ is in $\mathcal{E}_X^{0,q}$, $q \geq 1$, then $\phi = i^*\Phi$ for some form

$$\sum_{|J|=q}' \Phi_J(z,w)\, d\bar{z}_J$$

in $\mathcal{U}$, since any term with a factor $d\bar{w}_j$ belongs to $\mathcal{K}er\, i^*$. If $\phi_J = i^*\Phi_J$, we see that

$$\mathcal{L}\phi = \sum_{|J|=q}' \mathcal{L}\phi_J\, d\bar{z}_J.$$

Thus the first part of the lemma is proved. The second statement follows since $\pi_*(\phi\wedge w^m\mu) = 0$ for all m and μ implies that $\phi\wedge\mu = 0$ for all μ so by definition $\phi = 0$. $\square$

Remark 8.2. Notice that if L is the differential operator on the right hand side of (8.3), and $\phi = i^*\Phi$, then, observing that $L\Phi$ is well-defined for $(0,q)$-forms in $\mathcal{U}$,

$$\mathcal{L}\phi = \iota^* L\Phi, \tag{8.4}$$

where $\iota\colon Z \to \mathcal{U}$ is the underlying embedding. $\square$

Proposition 8.3. *Let x be a fixed point $x \in Z_{sing}$ and let $i\colon X \to \mathcal{U} \subset \mathbb{C}^N$ be a local embedding at x as in Theorem 7.3. All the operators $\mathcal{L}_1, \dots, \mathcal{L}_\rho$ extend to operators $\mathcal{E}_X^{0,q} \to \mathcal{E}_Z^{0,q}$. Moreover, $\phi = 0$ if (and only if) $\mathcal{L}_j\phi = 0$ for $j = 1, \dots, \rho$.*

Proof. We first prove the extension for the operators $\mathcal{L}_{m,\beta,j}$ in Proposition 7.2. By definition a smooth $(0,q)$-form ϕ on $X \cap \mathcal{U}$ is represented by a smooth $(0,q)$-form Φ in $\mathcal{U}$ and thus each $L_{m,\beta,j}\Phi$ is a smooth $(0,q)$-form in $\mathcal{U}$. Moreover, since it is Noetherian with respect to $\mathcal{J}$ in $\mathcal{U} \setminus Z_{sing}$, i.e., $L_{m,\beta,j}\Phi = 0$ if Φ is in $\mathcal{J}$ it follows by continuity that this holds also across Z_{sing}. By Remark 8.2, $\mathcal{L}_{m,\beta,j}\phi := \iota^* L_{m,\beta,j}\Phi$ in $Z_{reg} \cap \mathcal{U}$, and the same formula defines a smooth extension across $Z_{sing} \cap \mathcal{U}$. By continuity this extension is unique. All the operators $\mathcal{L}_1, \dots, \mathcal{L}_\rho$ are obtained in this way so the first statement in Proposition 8.3 is proved. Since $\mathcal{L}_j$ generate $\mathcal{N}_X$ at each point outside Z_{sing} it follows that $\phi = 0$ there if $\mathcal{L}_j\phi = 0$ for $j = 1, \dots, \rho$. By continuity then $\phi = 0$ in $X \cap \mathcal{U}$. $\square$

Assuming that we have chosen a Hermitian norm on Z, cf. the beginning of the introduction, we now get a pointwise norm

$$|\phi|^2_{\mathcal{U}} = \sum_{j=1}^{\rho} |\mathcal{L}_j\phi|^2_Z$$

on $\mathcal{U}$ of ϕ in $\mathcal{E}_X^{0,q}$. Patching together as in Section 7.1 we get a global norm on X.

Remark 8.4. Proposition 1.1 as well as Theorem 1.5 have analogues for smooth $(0,q)$-forms, and they are proved in basically the same way. We omit the details. $\square$

9. Pseudomeromorphic currents

Let Y be a reduced analytic space. The $\mathcal{O}_Y$-module $\mathcal{P}M_Y$ of pseudomeromorphic currents on Y was introduced in [10,8]. Roughly speaking, it consists of currents that locally are finite sums of direct images (under possibly nonproper mappings) of products of simple principal value currents and $\bar\partial$ of such currents. See [11] for a precise definition and for the properties stated in this section. The sheaf $\mathcal{P}M_Y$ is closed under $\bar\partial$ and under multiplication by smooth forms. If τ is pseudomeromorphic in an open subset $\mathcal{U} \subset Y$ and $W \subset \mathcal{U}$ is a subvariety then there is a well-defined pseudomeromorphic current $\mathbf{1}_{\mathcal{U}\setminus W}\tau$ in $\mathcal{U}$ obtained by extending the natural restriction of τ to $\mathcal{U} \setminus W$ in the trivial way. With the notation in Section 2.1, $\mathbf{1}_{\mathcal{U}\setminus W}\tau = \lim_\epsilon \chi(|h|/\epsilon)\tau$ if h is a tuple of holomorphic functions with common zero set W. Thus $\mathbf{1}_W\tau := \tau - \mathbf{1}_{\mathcal{U}\setminus W}\tau$ is a pseudomeromorphic current with support on W. If $W' \subset \mathcal{U}$ is another subvariety, then

$$\mathbf{1}_{W'}\mathbf{1}_W\tau = \mathbf{1}_{W'\cap W}\tau. \tag{9.1}$$

We can rephrase the standard extension property, cf. Section 2.1: If τ has support on a subvariety Z of pure dimension, then τ has the SEP with respect to Z if for each open subset $\mathcal{U} \subset Y$ and subvariety $W \subset \mathcal{U} \cap Z$ with positive codimension in Z, $\mathbf{1}_W\tau = 0$.

An important property is the dimension principle: *If τ in $\mathcal{P}M_Y$ has bidegree $(*, q)$ and support on a variety of codimension larger than q, then τ must vanish.*

Recall that a current τ on a manifold is semi-meromorphic if there are a smooth form ω with values in a line bundle L, and a non-trivial holomorphic section f of L, such that $\tau = \omega/f$, considered as a principal value current. We say that a current α on Y is *almost semi-meromorphic* if there is a smooth modification $\pi\colon \tilde Y \to Y$ and a semi-meromorphic current $\tilde\alpha$ in $\tilde Y$ such that $\alpha = \pi_*\tilde\alpha$. Notice that an almost semi-meromorphic α is smooth outside an analytic set W of positive codimension in Y.

Example 9.1. Coleff-Herrera currents in $\mathcal{U} \subset \mathbb{C}^N$ are pseudomeromorphic. Almost semi-meromorphic currents are pseudomeromorphic and have the SEP on $\mathcal{U}$. □

In general one cannot multiply pseudomeromorphic currents. However, assume that τ is pseudomeromorphic and α is almost semi-meromorphic in $\mathcal{U}$ and let W be the analytic set where α is not smooth. There is a unique pseudomeromorphic current T in $\mathcal{U}$ that coincides with the natural product $\alpha\wedge\mu$ in $\mathcal{U}\setminus W$ and such that $\mathbf{1}_W T = 0$. For simplicity we denote this current by $\alpha\wedge\mu$. If α' is another almost semi-meromorphic current in $\mathcal{U}$, then the expression $\alpha'\wedge\alpha\wedge\tau$ means $\alpha'\wedge(\alpha\wedge\tau)$. The equality

$$\alpha'\wedge\alpha\wedge\tau = \alpha\wedge\alpha'\wedge\tau \tag{9.2}$$

always holds. However, in general it is *not* true that $\alpha'\wedge\alpha\wedge\tau = (\alpha'\wedge\alpha)\wedge\tau$.

Example 9.2. Let f be a holomorphic function with non-empty zero set, let $\alpha = 1/f$, $\alpha' = f$, and $\tau = \bar\partial(1/f)$. Then $\alpha'\alpha\tau = 0$, but $\alpha'\alpha = 1$ and so $(\alpha'\alpha)\tau = \tau$. $\square$

Assume that τ is pseudomeromorphic, α is almost semi-meromorphic, ξ is smooth, and V is any subvariety. Then we have

$$\mathbf{1}_V \alpha\wedge\tau = \alpha\wedge\mathbf{1}_V\tau. \tag{9.3}$$

In particular: *If τ has support on and the SEP with respect to Z, then also $\alpha\wedge\tau$ has (support on and) the SEP with respect to Z.*

10. Uniform limits of holomorphic functions

Let X be a possibly non-reduced space of pure dimension n and let $i\colon X \to \mathcal{U} \subset \mathbb{C}^N$, so that $\mathcal{O}_X = \mathcal{O}_{\mathcal{U}}/\mathcal{J}$ as before. If $\mathcal{U}$ is small enough, then there are trivial Hermitian vector bundles E_k in $\mathcal{U}$, $E_0 = \mathbb{C}$ a line bundle, with morphisms $f_k\colon E_k \to E_{k-1}$, so that

$$0 \to \mathcal{O}(E_N) \stackrel{f_N}{\to} \cdots \stackrel{f_2}{\to} \mathcal{O}(E_1) \stackrel{f_1}{\to} \mathcal{O}(E_0) \to \mathcal{O}(E_0)/\mathcal{J} \to 0 \tag{10.1}$$

is a free resolution of $\mathcal{O}_{\mathcal{U}}/\mathcal{J}$. In [9] was introduced a residue current $R = R_\kappa + \cdots + R_N$ with support on Z, where R_k have bidegree $(0,k)$ and take values in $\mathrm{Hom}\,(E_0, E_k) \simeq E_k$, such that $f_{k+1}R_{k+1} - \bar\partial R_k = 0$ for each k, which can be written more compactly as

$$(f - \bar\partial)R = 0,$$

where $f := f_1 + \cdots + f_N$. The current R has the additional property that a holomorphic function Φ in $\mathcal{U}$ belongs to $\mathcal{J}$ if and only if the current $\Phi R = 0$. In particular, ϕR is a well-defined current for ϕ in $\mathcal{O}_X$. The assumption that X has pure dimension implies that R has the SEP with respect to $Z \cap \mathcal{U}$, see [8, Section 3] or [7, Section 6] for a proof.

Recall that ϕ is a meromorphic function on X if $\phi = g/h$, where h is not nilpotent, i.e., a representative of h does not vanish identically on Z, and $g/h = g'/h'$ if $gh' - g'h = 0$ in $\mathcal{O}_X$. Because of the SEP the product ϕR is a well-defined pseudomeromorphic current in $\mathcal{U}$ if ϕ is meromorphic on $X \cap \mathcal{U}$. The following criterion for holomorphicity was proved in [4].

Theorem 10.1. *Assume that $i\colon X \to \mathcal{U}$ has pure dimension and R is an associated current as above. If ϕ is meromorphic on X, then it is holomorphic if and only if*

$$(f - \bar\partial)(\phi R) = 0. \tag{10.2}$$

To give the idea for the general case let us first sketch a proof of Theorem 1.6, relying on Theorem 10.1, in case X is reduced.

Proof of Theorem 1.6 when X is reduced. The statement is elementary on X_{reg}; moreover it is clear that $\phi_j \to \phi$ where ϕ is bounded (weakly holomorphic) and thus meromorphic on X.

There is a (unique) almost semi-meromorphic current ω on X of bidegree $(n, *)$ such that $i_*\omega = R \wedge dz$, where $(z_1, \dots, z_N)$ are coordinates in $\mathcal{U}$, see [8, Proposition 3.3]. In particular, ω has the SEP on X. Let $\pi\colon X' \to X$ be a smooth modification so that $\omega = \pi_*\omega'$, where ω' is semi-meromorphic. Since $\pi^*\phi_j \to \pi^*\phi$ in $\mathcal{O}_{X'}$ and X' is smooth, indeed $\pi^*\phi_j \to \pi^*\phi$ in $\mathcal{E}_{X'}$. Therefore $\pi^*\phi_j\, \omega' \to \pi^*\phi\, \omega'$. Since ϕ_j are smooth, $\pi_*(\pi^*\phi_j\, \omega') = \phi_j\omega$. Combining we find that

$$\phi_j\omega \to \pi_*(\pi^*\phi\, \omega'). \tag{10.3}$$

Since ω' has the SEP, so have $\pi^*\phi\, \omega'$ and $\pi_*(\pi^*\phi\, \omega')$. Moreover,

$$\pi_*(\pi^*\phi\, \omega') = \phi\omega \tag{10.4}$$

on the open subset of X where ϕ is holomorphic, thus on X_{reg}. Since both sides of (10.4) have the SEP and coincide outside a set of positive codimension, they indeed coincide on X. By (10.3) thus $\phi_j\omega \to \phi\omega$. Applying i_* we get $\phi_j R \to \phi R$ and hence $(f - \bar\partial)(\phi_j R) \to (f - \bar\partial)(\phi R)$. It now follows from Theorem 10.1 that ϕ is indeed holomorphic. $\square$

For the rest of this section we will discuss the proof Theorem 1.6 when X is non-reduced but Z is smooth. We begin with

Lemma 10.2. *Theorem 1.6 is true when Z is smooth and $\mathcal{O}_X$ is Cohen-Macaulay.*

Proof. Given a point $x \in Z$, let $i\colon W \to \mathcal{U}$ be an embedding at x as in Section 5, so that we have unique representatives

$$\hat\phi_j(z, w) = \sum_{\ell=0}^{\tau-1} \hat\phi_{j\ell}(z) w^{\alpha_\ell}$$

in $\mathcal{U}$ of ϕ_j in Theorem 1.6. By the hypothesis and Theorem 1.5 it follows that $\hat\phi_{j\ell}$ is a Cauchy sequence in $Z \cap \mathcal{U}$ for each fixed ℓ, and hence we have holomorphic limits $\hat\phi_\ell = \lim_j \hat\phi_{j\ell}$ for each ℓ. Let us define the function

$$\hat\phi(z, w) := \sum_{\ell=0}^{\tau-1} \hat\phi_\ell(z) w^{\alpha_\ell}$$

in $\mathcal{U}$ and let ϕ be its pullback to $\mathcal{O}_X$. Since the convergence holds for all derivatives of $\hat\phi_{j\ell}$ as well, it follows from (1.7) that $|\phi_j - \phi|_X \to 0$. $\square$

The non-Cohen-Macaulay case is trickier. Let us first look at an example.

Example 10.3. Consider the space X in Section 6. If ϕ_j is a sequence as in Theorem 1.6, it follows from Lemma 10.2 that ϕ_j has a holomorphic limit ϕ in $X \setminus \{0\}$. Let $\mathcal{L}$ be the Noetherian operator in Section 6.1 and recall that $\mathcal{L}\phi$ is a well-defined function on Z. By the hypothesis in Theorem 1.6, $\mathcal{L}\phi_j$ is a Cauchy sequence on Z and since $\mathcal{L}\phi_j \to \mathcal{L}\phi$ in $Z \setminus \{0\}$ we conclude that $\mathcal{L}\phi_j \to \mathcal{L}\phi$ uniformly in Z. Since $\mathcal{L}\phi_j(0) = 0$ therefore $\mathcal{L}\phi(0) = 0$, and thus ϕ is $\mathcal{O}_X$-holomorphic in X, cf. Lemma 6.1. It follows that $|\phi_j - \phi|_X \to 0$ on X. $\square$

We cannot see how the argument in Example 10.3 can be extended directly, so we have to go back to the relation between our $\mathcal{L}_j$ and Coleff-Herrera currents.

Proof of Theorem 1.6 when Z is smooth. Given any point $x \in X$ let us choose a local embedding $i\colon X \to \mathcal{U}$ at x such that there is a Hermitian free resolution (10.1) and the associated residue current R in $\mathcal{U}$. Since Theorem 1.6 is local it is enough to prove it in $X \cap \mathcal{U}$. We will use, [7, Lemma 6.2]:

Proposition 10.4. *There is a trivial vector bundle $F \to \mathcal{U}$ and an F-valued Coleff-Herrera current μ such that its entries generate $\mathcal{H}om\,(\mathcal{O}_\mathcal{U}/\mathcal{J}, \mathcal{CH}^Z_\mathcal{U})$, and an almost semi-meromorphic current $\alpha = \alpha_0 + \cdots + \alpha_n$, where α_k have bidegree $(0,k)$ and take values in* $\mathrm{Hom}\,(F, E_{\kappa+k})$, *such that*

$$R\wedge dz = \alpha\mu, \quad R_{\kappa+k}\wedge dz = \alpha_k\mu, \;\; k = 0, 1, \ldots, n.$$

Moreover, α is smooth where $\mathcal{O}_X$ is Cohen-Macaulay.

Let W be the subset of $Z \cap \mathcal{U}$ where $\mathcal{O}_X$ is not Cohen-Macaulay. Since $\mathcal{O}_X$ has pure dimension W has codimension at least 2 in $Z \cap \mathcal{U}$, see, e.g., [7].

Lemma 10.5. *If ϕ is holomorphic in $(X \cap \mathcal{U}) \setminus W$, then ϕ has a meromorphic extension to $X \cap \mathcal{U}$.*

This result should be well-known but we provide a proof since we could not find any reference.

Proof. Since Z is smooth we can assume that we have coordinates (z, w) in $\mathcal{U}$ so that $Z \cap \mathcal{U} = \{w = 0\}$. Let $\mu = (\mu_1, \ldots, \mu_\nu)$ be the tuple in Proposition 10.4 and consider the representations (5.2). Here M must be chosen so that (1.5) holds. Fix $x' \in Z \cap \mathcal{U}$ where $\mathcal{O}_X$ is Cohen-Macaulay and a monomial basis $1, \ldots, w^{\alpha_{\tau-1}}$ for $\mathcal{O}_X$ over $\mathcal{O}_Z$ in a neighborhood $\mathcal{U}'$ of x', cf. Section 5. We then have (letting $R = \nu C_M$) the $R \times \nu$-matrix T in $\mathcal{U}'$ that for each holomorphic ϕ in $\mathcal{O}(X \cap \mathcal{U}')$ maps the coefficients $(\hat{\phi}_\ell)$ of its representative $\hat{\phi}$ given by (5.1) in this monomial basis onto the coefficients of the expansions (5.2) of $\phi\mu_j$, cf. Section 5.

Notice that the entries in T are $\mathbb{C}$-linear combinations of the coefficients of the representations (5.2) for $\phi = 1$ in $\mathcal{U}$. Thus T has a holomorphic extension to $Z \cap \mathcal{U}$ (we may assume that $Z \cap \mathcal{U}$ is connected). As pointed out in Section 5, T is pointwise injective in $Z \cap \mathcal{U}'$ and hence, after reordering the rows, $T = (T' \;\; T'')^t$ where T' is a $\nu \times \nu$-matrix that is invertible in $\mathcal{U}'$. Thus T' has a meromorphic inverse S' in $Z \cap \mathcal{U}$ and if $S = (S' \;\; 0)$, then $ST = I$ in $Z \cap \mathcal{U}$.

Since ϕ is holomorphic outside W, it defines a tuple $(b_{j,m})$ in $\mathcal{O}_Z^R$ in $(Z \cap \mathcal{U}) \setminus W$ via the representation (5.2) of $\phi\mu$. Since W has at least codimension 2 in $Z \cap \mathcal{U}$, the tuple $(b_{j,m})$ extends to $Z \cap \mathcal{U}$. Now

$$\tilde{\Phi} := \sum_{\ell=0}^{\tau-1} (Sb)_\ell(z) w^{\alpha_\ell}$$

is a meromorphic function in $\mathcal{U}$ that defines a meromorphic function $\tilde{\phi}$ on $X \cap \mathcal{U}$, since $(Sb)_\ell(z)$ are meromorphic on $Z \cap \mathcal{U}$. Moreover, $\tilde{\Phi} = \hat{\phi}$ in $\mathcal{U}'$ and so $\tilde{\phi}$ coincides with ϕ in $X \cap \mathcal{U}'$. By uniqueness $\tilde{\phi} = \phi$ in $\mathcal{U} \setminus W$ and thus $\tilde{\phi}$ is the desired meromorphic extension. $\square$

If ϕ_j is a Cauchy sequence in $|\cdot|_X$-norm and $Z \cap \mathcal{U}$ is smooth, then $\phi_j \to \phi$ uniformly on compact subsets of $\mathcal{U} \setminus W$ by Lemma 10.2 and ϕ has a meromorphic extension to $X \cap \mathcal{U}$ by Lemma 10.5.

Lemma 10.6. *With this notation $\phi_j R \to \phi R$ in $\mathcal{U}$.*

Proof. Let $\mathcal{I}$ and μ be as in Proposition 10.4 and the proof of Lemma 10.2. Assume that a is an F-valued holomorphic function in $\mathcal{U}$ such that $\mu = a\hat{\mu}$, cf. (3.2). Recall that

$$|\phi|_X = |\phi a|_{X'}, \tag{10.5}$$

where $\mathcal{O}_{X'} = \mathcal{O}/\mathcal{I}$. Define the F-valued $\mathcal{O}_{X'}$-functions $\psi_j = a\phi_j$. It follows from the hypothesis and (10.5) that ψ_j is a Cauchy sequence with respect to $|\cdot|_{X'}$. Since $\mathcal{O}_{X'}$ is Cohen-Macaulay it follows from the proof of Lemma 10.2 that there is ψ in $\mathcal{O}_{X'}$ and representatives $\hat{\psi}_j$ and $\hat{\psi}$ in $\mathcal{U}$ such that $\hat{\psi}_j \to \hat{\psi}$ in $\mathcal{E}(\mathcal{U})$. Let Φ_j be representatives of ϕ_j in $\mathcal{U}$. By Proposition 10.4 and (9.2) we have

$$\phi_j R \wedge dz = \Phi_j R \wedge dz = \Phi_j \alpha\mu = \Phi_j \alpha a \hat{\mu} = \alpha \Phi_j a \hat{\mu} = \alpha(\Phi_j a)\hat{\mu} = \alpha \hat{\psi}_j \hat{\mu}, \tag{10.6}$$

where the fifth equality holds since both Φ_j and a are holomorphic, and the last equality holds since both $\Phi_j a$ and $\hat{\psi}_j$ are representatives in $\mathcal{U}$ of the class ψ_j in $\mathcal{O}_{X'}$. Since $\hat{\psi}_j \to \hat{\psi}$ in $\mathcal{E}(\mathcal{U})$, $\alpha\hat{\psi}_j\hat{\mu} = \hat{\psi}_j \alpha\hat{\mu} \to \hat{\psi}\alpha\hat{\mu} = \alpha\hat{\psi}\hat{\mu} = \alpha\psi\hat{\mu}$. By (10.6) thus

$$\phi_j R \wedge dz \to \alpha\psi\hat{\mu}. \tag{10.7}$$

Let Φ be a representative in $\mathcal{U}$ of ϕ. Since Φ, α and a are almost semi-meromorphic in $\mathcal{U}$, by (9.2),

$$\phi R \wedge dz = \Phi R \wedge dz = \Phi \alpha \mu = \alpha \Phi \mu = \alpha \Phi a \hat{\mu}. \tag{10.8}$$

We claim that

$$\alpha \Phi a \hat{\mu} = \alpha(\Phi a)\hat{\mu}. \tag{10.9}$$

In fact, both Φ and α are almost semi-meromorphic in $\mathcal{U}$ and smooth in a neighborhood of each point on $Z \cap \mathcal{U}$ where $\mathcal{O}_X$ is Cohen-Macaulay, cf. Lemma 10.2 and Proposition 10.4. Therefore (10.9) holds in $\mathcal{U} \setminus W$, and $W \subset Z \cap \mathcal{U}$ has positive codimension in Z. Both sides of (10.9) have the SEP with respect to Z, see Section 9, so (10.9) holds everywhere. The right hand side of (10.9) is equal to $\alpha\psi\hat{\mu}$, and so Lemma 10.6 follows from (10.7), (10.8), and (10.9). $\square$

Since ϕ_j are holomorphic, we have by Theorem 10.1 and Lemma 10.6 that $0 = \nabla_f(\phi_j R) \to \nabla_f(\phi R)$, and hence ϕ is holomorphic in view of Theorem 10.1. Now take $\mathcal{L}$ in $\mathcal{N}_X$. By the hypothesis and definition of $|\cdot|_X$, $\mathcal{L}\phi_j$ is a holomorphic Cauchy sequence in $\mathcal{U}$ so it converges to a holomorphic limit H. On the other hand we know that $\mathcal{L}\phi_j \to \mathcal{L}\phi$ where $\mathcal{O}_X$ is Cohen-Macaulay. Thus $\mathcal{L}\phi_j \to \mathcal{L}\phi$ uniformly. We conclude that $|\phi_j - \phi|_X \to 0$ uniformly in $\mathcal{U}$. Thus Theorem 1.6 is proved in $X \cap \mathcal{U}$ and hence in general if Z is smooth. $\square$

11. Resolution of X

Assume that our X of pure dimension n is embedded in the smooth manifold Y of dimension N as before, and let Z denote the underlying reduced space. There exists a modification $\pi\colon Y' \to Y$ that is a biholomorphism $Y' \setminus \pi^{-1}Z_{sing} \simeq Y \setminus Z_{sing}$ and such that the strict transform Z' of Z is smooth and the restriction of π to Z' is a modification of Z. Such a π is called a strong resolution. Let $\widetilde{\mathcal{J}}$ be the ideal sheaf on Y' generated by pullbacks of generators of $\mathcal{J}$ and consider the relative gap sheaf $\mathcal{J}' = \widetilde{\mathcal{J}}[\pi^{-1}Z_{sing}]$, which is coherent, cf. [32, Theorem 2]. In fact, one obtains $\mathcal{J}'$ by extending $\widetilde{\mathcal{J}}$ so that one gets rid of all primary components corresponding to the exceptional divisor, and also possible embedded primary ideals in $Z' \cap \pi^{-1}Z_{sing}$. Thus $\mathcal{J}'$ is the smallest coherent sheaf of pure dimension n that contains $\widetilde{\mathcal{J}}$ and such that $\mathcal{O}_{Y'}/\mathcal{J}'$ has support on Z'. We let X' denote the analytic space with structure sheaf $\mathcal{O}_{X'} = \mathcal{O}_{Y'}/\mathcal{J}'$. Notice that we have the induced mapping

$$p^*\colon \mathcal{O}_X \to \mathcal{O}_{X'}. \tag{11.1}$$

In fact, if $\Phi \in \mathcal{J}$, then $\pi^*\Phi \in \tilde{\mathcal{J}} \subset \mathcal{J}'$ and thus p^* in (11.1) is well-defined. We say that $p\colon X' \to X$ is a resolution of X. Notice that p^* extends to map meromorphic functions on X to meromorphic functions on X'. Let $p_0 = \pi|_{Z'}$ and let

$$V := p_0^{-1} Z_{sing} = \pi^{-1} Z_{sing} \cap Z'. \tag{11.2}$$

Lemma 11.1. *Assume that ϕ' is meromorphic on X' and holomorphic on $X' \setminus V$. Then there is a unique meromorphic ϕ on X, holomorphic in $X \setminus Z_{sing}$, such that $\phi' = p^*\phi$.*

Proof. Since π is proper it follows from Grauert's theorem that the direct image $\mathcal{F} = \pi_*(\mathcal{O}_{Y'}/\mathcal{J}')$ is coherent, and clearly it coincides with $\mathcal{O}_Y/\mathcal{J}$ outside $Z_{sing} \subset Y$. Moreover, it contains $\mathcal{O}_X = \mathcal{O}_Y/\mathcal{J}$ since $\pi_*\pi^{-1}\phi = \phi$ for ϕ in $\mathcal{O}_X$. Thus $\mathcal{F}/\mathcal{O}_X$ has support on Z_{sing}. Let h be a function that vanishes on Z_{sing} but not identically in Z. Then $h^\nu \mathcal{F}/\mathcal{O}_X = 0$ if ν is large enough. If ϕ' is a section of $\mathcal{O}_{X'}$, therefore $g := h^\nu \pi_* \phi'$ is holomorphic. Thus $\phi := g/h^\nu$ is meromorphic and $\phi' = p^*\phi$. $\square$

Lemma 11.2. *Let μ be a tuple of currents that generate the $\mathcal{O}_X$-module $\mathcal{H}om\,(\mathcal{O}_Y/\mathcal{J}, \mathcal{CH}_Y^Z)$.*

(i) There is a unique tuple μ' of pseudomeromorphic (N, κ)-currents in Y' with support on Z' such that $\pi_*\mu' = \mu$.

(ii) A holomorphic function Φ' defined in a neighborhood in Y' of a point on Z' is in $\mathcal{J}'$ if and only if $\Phi'\mu' = 0$.

In view of (ii) thus $\phi'\mu'$ is well-defined for ϕ' in $\mathcal{O}_{X'}$. It is not necessarily true that μ' is $\bar\partial$-closed. Since π is a biholomorphism outside $\pi^{-1}Z_{sing}$ it follows however that $\bar\partial\mu' = 0$ there. Moreover, since μ' is pseudomeromorphic it has the SEP, by virtue of the dimension principle. In the literature such a μ' is often said to be a Coleff-Herrera current with poles at $V \subset Z'$. If h' is holomorphic and vanishes to enough order on V then $0 = h'\bar\partial\mu' = \bar\partial(h'\mu')$, and hence $h'\mu'$ is in $\mathcal{H}om\,(\mathcal{O}_{Y'}/\mathcal{J}', \mathcal{CH}_{Y'}^{Z'})$.

Proof. Recall that μ is pseudomeromorphic, cf. Section 9. By [11, Theorem 2.15] there is a pseudomeromorphic current T in Y' such that $\pi_*T = \mu$. Since π is a biholomorphism outside $\pi^{-1}Z_{sing}$ the current T must be unique there, in particular it must have support on $\pi^{-1}Z$. Thus $T = \mathbf{1}_{Z'}T + \mathbf{1}_{\pi^{-1}Z\setminus Z'}T$. Since $\pi_*(\mathbf{1}_{\pi^{-1}Z\setminus Z'})$ has support on Z_{sing} that has codimension at least 1 in Z, it vanishes by the dimension principle. If $\mu' := \mathbf{1}_{Z'}T$, therefore $\pi_*\mu' = \pi_*T = \mu$. Moreover, since μ' is unique outside $Z' \cap \pi^{-1}Z_{sing} = V$ it is unique, again by the dimension principle, since V has codimension at least 1 in Z'. Thus (i) is proved.

Since $\mathcal{J}'$ has no embedded components, Φ' is in $\mathcal{J}'$ if and only if Φ' is in $\mathcal{J}'$ on $Z' \setminus V$. This in turn holds if and only if $\Phi = \pi_*\Phi'$ belongs to $\mathcal{J}$ on Z_{reg} which holds if and only if $\Phi\mu = 0$ on Z_{reg}. However this holds if and only if $\Phi'\mu' = 0$ on $Z' \setminus V$ which by the SEP of μ' holds if and only if $\Phi'\mu' = 0$ on Z'. Thus (ii) holds. $\square$

Let R be a current in Y with support on Z and the SEP as in Section 10. Recall, Proposition 10.4, that there is an almost semi-meromorphic current α in Y such that $R = \alpha\mu$, where μ is a tuple of Coleff-Herrera currents that generate $\mathcal{H}om\,(\mathcal{O}_Y/\mathcal{J}, \mathcal{CH}_Y^Z)$.

Lemma 11.3. *There is an almost semi-meromorphic current α' in Y' such that $R' = \alpha'\mu'$ has the SEP and $\pi_* R' = R$.*

Proof. By definition there is a modification $\tau: W \to Y$ such that $\alpha = \tau_*\gamma$, where γ is semi-meromorphic. There is a modification $W' \to Y$ that factors over both V and Y'. If we pull back γ to W', then its direct image α' in Y' is almost semi-meromorphic and $\pi_*\alpha' = \alpha$. It follows from (9.3) that $R' := \alpha'\mu'$ has the SEP. Moreover, $\pi_* R' = R$ where π is a biholomorphism, i.e., outside Z_{sing}. Since both currents have the SEP, the equality holds in Y. $\square$

12. Proof of Theorem 1.6

Lemma 12.1. *Assume that Z is smooth and that $\mathcal{L}$ is a holomorphic differential operator on X that belongs to $\mathcal{N}_X$ in $Z \setminus W$, where W has positive codimension. If $Z(h) \supset W$, then $h^r\mathcal{L}$ is in $\mathcal{N}_X$ for large enough r.*

Proof. Recall that the sheaf $\mathcal{N}_X$ locally can be considered as a coherent submodule of $\mathcal{O}_Z^\nu$ for some large ν. Then also $\mathcal{L}$ can be considered as an element in $\mathcal{O}_Z^\nu$. If $\mathcal{L}$ is not in $\mathcal{N}_X$, then $\mathcal{M}' = \langle \mathcal{N}_X, \mathcal{L}\rangle/\mathcal{N}_X$ is a coherent sheaf with support on W. By the Nullstellensatz $h^r\mathcal{M}' = 0$ for large enough r. Thus $h^r\mathcal{L} \in \mathcal{N}_X$ for such r. $\square$

It remains to prove Theorem 1.6 in a neighborhood of a point $x \in Z_{sing}$, cf. (7.10). Let $i: X \to \mathcal{U} \subset \mathbb{C}^N$ be a local embedding at x. We can assume that $\mathcal{N}_X$ admits a coherent extension to $X \cap \mathcal{U}$, cf. Theorem 7.3, that we denote by $\mathcal{N}_X$ as well. Recall that the $\mathcal{O}_Z$ module $\mathcal{N}_X$ is generated in $\mathcal{U}$ by a finite number of operators $\mathcal{L}_1, \ldots, \mathcal{L}_r$ that are induced by Noetherian operators $L_1, \ldots, L_r$ with respect to $\mathcal{J}$ in $\mathcal{U}$.

Let $\pi: \mathcal{U}' \to \mathcal{U}$ be a modification as in Section 11, with $\mathcal{U}'$ and $\mathcal{U}$ instead of Y' and Y, respectively. Thus we have the space $i': X' \to \mathcal{U}'$, $p^*: \mathcal{O}_X \to \mathcal{O}_{X'}$ and the induced mapping $p_0: Z' \to Z \cap \mathcal{U}$. Since Z' is smooth we have the well-defined $\mathcal{O}_{X'}$-module $\mathcal{N}_{X'}$ of Noetherian operators on X'.

We say that $\mathcal{L}'$ is a meromorphic Noetherian operator on X' with poles on $V := p_0^{-1}Z_{sing} \subset Z'$ if $\xi^\rho L'$ is a Noetherian operator on X' as soon as ξ in $\mathcal{O}_{Z'}$ vanishes on V and ρ is large enough.

Lemma 12.2. *There are meromorphic operators $\mathcal{L}_1', \ldots, \mathcal{L}_r$ on X' such that*

$$\mathcal{L}_j'(p^*\phi) = p_0^*(\mathcal{L}_j\phi) \tag{12.1}$$

on $Z' \setminus V$. Moreover, there is a holomorphic (nontrivial) function h on Z such that $h'\mathcal{L}_j'$ are in $\mathcal{N}_{X'}$ if $h' = p_0 h$.

Proof. Given a holomorphic differential operator T on $\mathcal{U}$ there is a holomorphic differential operator $\widetilde{T}$ in $\mathcal{U}'$ with values in a power $N^\nu_{\mathcal{U}'/\mathcal{U}}$ of the relative canonical bundle, and a holomorphic section s of $N_{\mathcal{U}'/\mathcal{U}}$, vanishing on $\pi^{-1}Z_{sing}$, such that $\pi^*(T\Phi) = s^{-\nu}\widetilde{T}(\pi^*\Phi)$. See, e.g., the discussion preceding [11, Corollary 4.26]. Thus $T' = s^{-\nu}\widetilde{T}$ is a meromorphic differential operator, with poles at $\pi^{-1}Z_{sing}$, such that $\pi^*(T\Phi) = T'(\pi^*\Phi)$.

Let $L'_1, \dots, L'_r$ be meromorphic differential operators on $\mathcal{U}'$ such that $\pi^*(L_j\Psi) = L'_j(\pi^*\Psi)$, $j = 1, \dots, r$. If Φ' is in $\mathcal{J}'$, then $\Phi' = \pi^*\Phi$ for some Φ in $\mathcal{J}$ outside Z_{sing} and hence $L'_j\Phi' = 0$ outside V. By continuity $L'_j\Phi' = 0$. Thus we have induced meromorphic operators $\mathcal{L}'_1, \dots, \mathcal{L}'_r$ on X' with poles at V and (12.1) holds.

Since π is a biholomorphism outside $\pi^{-1}Z_{sing}$ it follows that $\mathcal{L}'_j$ belong to $\mathcal{N}_{X'}$ there. By Lemma 12.1, $h'\mathcal{L}_j$ are in $\mathcal{N}_{X'}$ if $h' = p_0^*h$, where h is a holomorphic function in Z that vanishes to high enough order on Z_{sing}. $\square$

Lemma 12.3. *After possibly shrinking $\mathcal{U}$ there is a holomorphic function H in $\mathcal{U}$, not vanishing identically on Z, such that*

$$|p^*(H\phi)(z')|_{X'} \le C|\phi(p_0(z'))|_X, \quad z' \in Z'. \tag{12.2}$$

Proof. Let $\widehat{\mathcal{N}}_{X'}$ be the $\mathcal{O}_{Z'}$-module generated by the $h'\mathcal{L}'_j$ from Lemma 12.2. Then $\widehat{\mathcal{N}}_{X'} \subset \mathcal{N}_{X'}$ with equality outside $Z(h')$. Therefore $\mathcal{N}_{X'}/\widehat{\mathcal{N}}_{X'}$ is annihilated by $H' = \pi^*H$ if H is a high power of h. That is, if $\mathcal{T}$ is in $\mathcal{N}_{X'}$, then $H'\mathcal{T}$ is in $\widehat{\mathcal{N}}_{X'}$ and thus $H'\mathcal{T}$ is an $\mathcal{O}_{Z'}$-linear combination of the $h'\mathcal{L}'_j$.

Fix a point $x' \in \pi^{-1}(x) \cap Z'$. Let $\mathcal{T}_\ell$ be a set of generators for $\mathcal{N}_{X'}$ in a neighborhood $\mathcal{V}$ of x'. For any ϕ we have, with $\phi' = p^*\phi$, and $z' \in \mathcal{V}$,

$$|H'(z')||\phi'(z')|_{X'} \sim \sum_\ell \big|(H'\mathcal{T}_\ell\phi')(z')\big| \lesssim \sum_j \big|(h'\mathcal{L}'_j p^*\phi')(z')\big| \le \sum_j \big|p_0^*(\mathcal{L}_j\phi)(z')\big| \sim |\phi(\pi(z'))|_X.$$

On the other hand, if ν is large enough, $|\mathcal{T}_\ell((H')^\nu\phi')| \lesssim |H'\mathcal{T}_\ell\phi'|$ for each ℓ and hence $|(H')^\nu\phi'|_{X'} \lesssim |H'||\phi'|_{X'}$. Denoting H^ν by H thus (12.2) holds for $z' \in \mathcal{V}$. Since $\pi^{-1}(x)$ is compact, (12.2) holds for all z' in an open neighborhood of $\pi^{-1}(x)$. Hence the lemma follows. $\square$

Assume that ϕ_j is a sequence as in Theorem 1.6 and let $\phi'_j = p^*\phi_j$. It follows from Lemma 12.3, and Theorem 1.6 in case that Z is smooth, see Section 10, that there is a holomorphic function ξ' on $X' \cap \mathcal{U}'$ such that $H'\phi'_j \to \xi'$ uniformly in the $|\cdot|_{X'}$-norm. Notice that ξ'/H' is meromorphic on $X' \cap \mathcal{U}'$.

Lemma 12.4. *With the notation above, $\phi'_j R' \to (\xi'/H')R'$ in $\mathcal{U}'$.*

Proof. Let μ' be as in Lemma 11.3. Since $\bar\partial\mu'$ has support on V, $\bar\partial(g'\mu') = 0$ for a suitable $g' = \pi^*g$ not vanishing identically on Z'. From Lemma 11.2 we conclude that

$g'\mu'$ is a tuple in $\mathcal{H}om\,(\mathcal{O}_{\mathcal{U}'}/\mathcal{J}', \mathcal{CH}^{Z'}_{\mathcal{U}'})$. Since Z' is smooth, if $\mathcal{V} \subset \mathcal{U}'$ is a small enough open neighborhood of any given point in $\mathcal{U}'$, then we have coordinates (z, w) such that $Z' \cap \mathcal{V} = \{w = 0\}$. Then $g'\mu' = adz \wedge \hat{\mu}$, cf. (3.2), for a suitable holomorphic tuple a in $\mathcal{V}$. Using (9.2) and Lemma 11.3 we can now prove Lemma 12.4 in $\mathcal{V}$ in the same way as Lemma 10.6. Now Lemma 12.4 follows in $\mathcal{U}'$ since the statement is local. $\square$

By Lemma 11.1 there is a meromorphic ξ on $X \cap \mathcal{U}$ such that $\xi' = p^*\xi$. Define the meromorphic function $\phi = \xi/H$ on $X \cap \mathcal{U}$. Clearly $p^*\phi = \xi'/H'$ so that

$$\pi_*((\xi'/H')R') = \phi R \tag{12.3}$$

in $\mathcal{U} \setminus (Z(H) \cap Z)$. However, both sides of (12.3) have the SEP with respect to $Z \cap \mathcal{U}$ so the equality holds in $\mathcal{U}$. Since $\pi_*(\phi'_j R') = \pi_*(p^*\phi_j R') = \phi_j R$ we conclude from Lemma 12.4 that $\phi_j R \to \phi R$. In view of Theorem 10.1 now Theorem 1.6 follows as in the smooth case in Section 10.

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