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Robust optimization of a bi-objective tactical resource allocation problem with uncertain qualification costs

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Abstract

In the presence of uncertainties in the parameters of a mathematical model, optimal solutions using nominal or expected parameter values can be misleading. In practice, robust solutions to an optimization problem are desired. Although robustness is a key research topic within single-objective optimization, little attention is received within multi-objective optimization, i.e. robust multi-objective optimization.This work builds on recent work within robust multi-objective optimization and presents a new robust efficiency concept for bi-objective optimization problems with one uncertain objective. Our proposed concept and algorithmic contribution are tested on a real-world *multi-item capacitated resource planning* problem, appearing at a large aerospace company manufacturing high precision engine parts. Our algorithm finds all the robust efficient solutions required by the decision-makers in significantly less time than the approach of Kuhn et al. (Eur J Oper Res 252(2):418–431, 2016) on 28 of the 30 industrial instances.

Keywords Robust optimization · Bi-objective mixed integer programming · Robust efficient (RE)solutions · Capacity planning · Decision support system

Mathematics Subject Classification 90-C29 · 90-B50 · 90-C15

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1 Introduction

Two common challenges while dealing with complex real-world decision-making problems are (i) uncertain input parameters and (ii) multiple conflicting goals. The former may be a consequence of measurement errors, imprecise forecasts, unwanted disturbances, and a lack of historical data. The latter can be due to the involvement of multiple stakeholders/agents with competing goals. This work deals with bi-objective discrete optimization problems where one of the objective functions contains uncertain parameters. Few applications of uncertain bi-objective optimization are considered in the literature. Kuhn et al. [21, Sect. 8.1], consider a flight route planning problem (first presented in [20]), the two objectives being route efficiency—in terms of minimizing travel time—and risk—in terms of minimizing exposure to weather conditions; the latter (weather condition) is considered uncertain. A hazardous materials transportation problem is studied by Kuhn et al. [21, Sect. 8.2], with the objectives to minimize travel times and the number of people exposed in case of an accident, the travel times being considered uncertain, especially in urban areas.

One such application is presented in this work: it is focused on the planning of machining capacity for a large aerospace tier-1 [5, p. 64] engine system manufacturer with uncertain *qualification costs*, i.e. non-recurring fixed costs for verifying or assisting a machine in performing a specific task. The problem is the so-called tactical resource allocation problem (TRAP), a multi-item capacitated resource planning problem with a medium–to–long term planning horizon. A corresponding model is discussed in detail in [14, Chap. 2] and [13]. A solution to the TRAP is the allocation of tasks to machines and a schedule for the qualifications to be performed. The two objectives considered are minimizing the maximum excess resource loading above a given threshold and minimizing the total qualification costs incurred due to man-hours needed to verify machines for tasks or buying new fixtures/tools to adjust the machine for a given task. The uncertainty of the coefficients of latter objective function is not considered in [13]. In this work, we assess the impact of uncertainty in the qualification costs on the efficient frontier for 30 industrial instances.

Various methods have been proposed for solving uncertain multi-objective optimization problems (MOOPs) using stochastic programming and robust optimization. Stochastic programming for MOOP (see [16]) is used when enough data is available. A drawback of the stochastic approach is that for some problems so-called *long-run optimality* is not relevant, as the *repeatability element* of the decisions is missing; the decision-maker has to live with the consequences of the decisions made once. Since in the TRAP, qualification costs are incurred only once, it is evident that combining robust optimization and multi-objective optimization has certain benefits over other approaches. Moreover, the absence of historical data or expert opinions as bases for assumptions about the underlying probability distributions does not encourage us to use stochastic programming. Although a lot of work has been done in the field of single-objective robust optimization, its generalization to MOOP is relatively new. [1, 3] present some basic concepts for single-objective robust optimization which have been used for MOOPs (e.g. [15, 29, 30]). Each of these relies, however, on some form of a priori scalarization and does not assess the impact of uncertainty on the efficient frontier in different scenarios. Consequently, so-called *robust efficiency* concepts for generalizing *efficiency* in MOOPs to uncertain MOOPs have been developed (see the discussion in Sect. 2). Most of the main concepts are described in the survey [18]. Our focus in this work is on bi-objective optimization problems with one uncertain objective function and a deterministic feasible set. Specifically, we build on the contribution of

Table 1 Payoff matrix of the commuting MONFG with uncertain travel times

$\{(\text{reward}(X)), (\text{reward}(Y))\}$	Scenario	agent Y: taxi	agent Y: train
agent X: taxi	nom	$\{(-10,-10),(-10,-10)\}$	$\{(-20,-10),(-5,-30)\}$
	wc	$\{(-10,-15),(-10,-15)\}$	$\{(-20,-15),(-5,-32)\}$
agent X: train	nom	$\{(-5,-30),(-20,-10)\}$	$\{(-5,-30),(-5,-30)\}$
	wc	$\{(-5,-32),(-20,-15)\}$	$\{(-5,-32),(-5,-32)\}$

[21] by suggesting a new robust efficiency concept and an algorithm that shows improved results on the industrial instances considered.

1.1 Relevance to multi-objective multi-agent decision problems

A multi-agent system (MAS) contains multiple agents deployed in a shared environment. Even though several multi-agent systems have multiple objectives, they are commonly (often erroneously) modeled as a single-objective decision problem. There has been some interest in using multi-objective approaches for MAS. [26] present several multi-objective approaches used for decision problems in MAS. To establish a relation between our contribution—on a bi-objective optimization problem with an uncertain objective function—to multi-objective MAS, we present an example of the Multi-Objective Normal Form Game (MONFG).[1] Suppose two agents wish to commute from an origin to a final destination. There are two modes of transportation: taxi and train. When both agents take a taxi, they split the cost. However, if both choose to take a train, they pay for their individual tickets. The exact costs of the train tickets and the taxi fare are available, but the travel time is uncertain for both modes of transportation. We assume two scenarios: expected/nominal (nom) and a worst-case (wc). The costs and travel times are given by $\begin{pmatrix} & \text{cost} & \text{time(nom)} & \text{time(wc)} \\ \text{taxi} & 20 & 10 & 15 \\ \text{train} & 5 & 30 & 32 \end{pmatrix}$ and a corresponding individual payoff matrix for the commuting MONFG is given in Table 1 (the negative values representing costs).[2]

The actions of the two agents result in a *reward vector*, representing *cost* and *travel time*. For instance, if agent X takes a taxi and agent Y takes a train, the reward received by the two agents in the nominal scenario is $(-20, -10)$ and $(-5, -30)$, respectively. In the worst-case scenario, the two agents receive the reward $(-20, -15)$ and $(-5, -32)$, respectively. Assuming that the two agents are cooperating (considering a team reward) and that there is no predefined utility function, it would make sense to identify actions that are equally good. However, if the two agents choose different modes of transportation the individual reward vectors of both agents have lower values than if both agents take a taxi, in the nominal as well as the worst-case scenarios. Hence, the actions of both agents choosing the same mode of transportation are of interest to the system planner. This type of problem can be modeled as bi-objective optimization problems with one uncertain objective (here, the team reward for travel time is uncertain), which is the topic of our work. Since we assume that the two agents are cooperating, the rewards for the two agents should be combined, e.g. summing costs and travel times, respectively. For instance, when both agents take a taxi, the total reward is $(-20, -20)$ in the

[1] A modification of the example presented in [26, Sect. 1.1]

[2] Table 1 differs from [26, Tab. 3] as we consider two travel time scenarios

nominal scenario and $(-20, -30)$ in the worst-case scenario. In real-world problems there are often several possible actions (or feasible solutions); hence, an explicit payoff matrix cannot be established, which calls for a combination of multi-objective and robust optimization.

1.2 Contributions and outline

The contribution of this work is in the field of bi-objective optimization with one uncertain objective function with well-defined nominal (most likely) and worst-case scenarios. Our first contribution is the new robust efficient (RE) concept of *positive robustness* ε*-representative lightly RE* solutions and a measure to evaluate the net gain by the inclusion of a worst-case scenario to the optimization problem. Our second contribution is the suggestion of a new approach to identify relevant RE solutions to the so-called *tactical resource allocation problem* (TRAP), leading to computational gains over existing approaches such as the algorithm presented in [21].

In Sect. 1.3, we present a mathematical model for the TRAP. In Sect. 2 we present some of the existing robust efficiency concepts from the literature. In Sect. 3, we present a new robust efficiency concept for bi-objective robust optimization problems with one uncertain objective function. Further, we present a measure for assessing the net gain by the inclusion of a worst-case scenario. In Sect. 4, we present a so-called 3-*stage method* and perform numerical tests on 30 industrial instances of the TRAP.

1.3 Problem description

Formally, the TRAP is defined as follows (see Table 2 for notations).

Definition 1 (TRAP) Given a set $\mathcal{J}$ of job types (tasks) and a set $\mathcal{K}$ of machines, let p_{jk} be the average processing time (including set-up time) of job type $j \in \mathcal{J}$ when performed in a compatible machine $k \in \mathcal{K}_j \subseteq \mathcal{K}$. Each machine $k \in \mathcal{K}$ has the capacity C_{kt} (time units) in time period $t \in \mathcal{T}$ and a relative loading threshold $\zeta_k \in [0, 1]$. The demand a_{jt} of each job type $j \in \mathcal{J}$ in time period $t \in \mathcal{T}$ must be met. The number of machines allocated to the same job type in each time period must not exceed the value of the parameter $\tau \in \mathbb{Z}_+$. For assignments (j, k), such that $k \in \mathcal{N}_j \subseteq \mathcal{K}_j$ and $j \in \mathcal{J}$, so-called *qualifications* are required which generate additional one-time costs (β^q_{jk}, where $q \in \mathcal{Q}$ is the index of a scenario). For a job type $j \in \mathcal{J}$, the machines in the set $\mathcal{K}_j \setminus \mathcal{N}_j$ do not require any qualifications. The total number of qualifications performed per time period t must not exceed the value of the parameter $\gamma \in \mathbb{Z}_+$.

1.3.1 Model description

The two objectives considered and the constraints defining the feasible set for the TRAP is expressed as (the index $q \in \mathcal{Q}$ denotes a scenario)

$$\underset{\mathbf{x},\mathbf{s},\mathbf{n},\mathbf{z}}{\text{minimize}} \quad g_1(\mathbf{x}, s, n, z) := \sum_{t \in \mathcal{T}} n_t, \tag{1a}$$

Table 2 Notations for the tactical resource allocation model

Sets	Description
$\mathcal{J} = \{1, \dots, J\}$	Set of job types to be performed on the products/parts
$\mathcal{K} = \{1, \dots, K\}$	Set of machines
$\mathcal{K}_j \subseteq \mathcal{K}$	Set of machines feasible for job type $j \in \mathcal{J}$
$\mathcal{N}_j \subseteq \mathcal{K}_j$	Set of machines feasible, but not qualified for job type $j \in \mathcal{J}$
$\mathcal{T} = \{1, \dots, T\}$	Set of time periods
$\mathcal{Q} = \{\hat{q}, \tilde{q}\}$	Set of indices of scenarios (nominal and worst-case) for the qualification costs
Variables	Description
$x_{jkt} \in \mathbb{Z}_+$	Number of orders of job type $j \in \mathcal{J}$ performed in machine $k \in \mathcal{K}_j$ in time period $t \in \mathcal{T}$
$s_{jkt} \in \{0, 1\}$	= 1 if an order of job type $j \in \mathcal{J}$ is allocated to machine $k \in \mathcal{K}_j$ in time period $t \in \mathcal{T}$
$z_{jkt} \in \{0, 1\}$	= 1 if machine $k \in \mathcal{N}_j$ is qualified for job type $j \in \mathcal{J}$ in time period $t \in \mathcal{T}$
$n_t \in \mathbb{R}_+$	Maximum resource loading above thresholds ζ_k, $k \in \mathcal{K}$, in time period $t \in \mathcal{T}$
$\mathbf{y} = (\mathbf{x}, \mathbf{s}, \mathbf{n}, \mathbf{z})$	Bold notations representing vectors of the corresponding indexed variables
Parameters	Description
$a_{jt} \in \mathbb{Z}_+$	Demand of orders of job type $j \in \mathcal{J}$ in time period $t \in \mathcal{T}$
$p_{jk} \in \mathbb{Q}_+$	Average machining time in machine $k \in \mathcal{K}_j$ for job type $j \in \mathcal{J}$
$C_{kt} \in \mathbb{Z}_+$	Capacity (hours) available in machine $k \in \mathcal{K}$ in time period $t \in \mathcal{T}$
$\beta^q_{jk} \in \mathbb{Z}_+$	Qualification cost in scenario $q \in \mathcal{Q}$, for machine $k \in \mathcal{N}_j$ for job type $j \in \mathcal{J}$
$\gamma \in \mathbb{Z}_+$	Upper limit on the number of qualifications in a single time period
$\tau \in \mathbb{Z}_+$	Upper limit on number of alternative machines for each job type in a single time period
$\zeta_k \in [0, 1]$	Loading threshold for machine $k \in \mathcal{K}$

$$\underset{\mathbf{x},\mathbf{s},\mathbf{n},\mathbf{z}}{\text{minimize}} \quad g_2(\mathbf{x}, s, n, z, q) := \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{N}_j} \beta^q_{jk} z_{jkt}, \quad q \in \mathcal{Q}, \tag{1b}$$

subject to

$$\sum_{k \in \mathcal{K}_j} x_{jkt} = a_{jt}, \quad j \in \mathcal{J}, t \in \mathcal{T}, \tag{2a}$$

$$x_{jkt} \leq \min \left\{ a_{jt}, \left\lfloor \frac{C_{kt}}{p_{jk}} \right\rfloor \right\} s_{jkt}, \quad k \in \mathcal{K}_j, j \in \mathcal{J}, t \in \mathcal{T}, \tag{2b}$$

$$\sum_{k \in \mathcal{K}_j} s_{jkt} \leq \tau, \quad j \in \mathcal{J}, t \in \mathcal{T}, \tag{2c}$$

$$\frac{1}{C_{kt}} \sum_{j \in \mathcal{J}} p_{jk} x_{jkt} \leq n_t + \zeta_k \leq 1, \quad k \in \mathcal{K}, t \in \mathcal{T}, \tag{2d}$$

$$\sum_{t \in \mathcal{T}: t \leq l} z_{jkt} \geq s_{jkl}, \quad k \in \mathcal{N}_j, j \in \mathcal{J}, l \in \mathcal{T}, \tag{2e}$$

$$\sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{N}_j} z_{jkt} \leq \gamma, \quad t \in \mathcal{T}, \tag{2f}$$

$$x_{jkt} \in \mathbb{Z}_+, \quad k \in \mathcal{K}_j, j \in \mathcal{J}, t \in \mathcal{T}, \tag{2g}$$

$$s_{jkt} \in \{0, 1\}, \quad k \in \mathcal{K}_j, j \in \mathcal{J}, t \in \mathcal{T}, \tag{2h}$$

$$z_{jkt} \in \{0, 1\}, \quad k \in \mathcal{N}_j, j \in \mathcal{J}, t \in \mathcal{T}, \tag{2i}$$

$$n_t \geq 0, \quad t \in \mathcal{T}. \tag{2j}$$

We denote the set of feasible solutions as

$$Y := \left\{ \mathbf{y} = (\mathbf{x}, \mathbf{s}, \mathbf{n}, \mathbf{z}) : \text{the constraints (2a)–(2j) hold} \right\}. \tag{3}$$

The set of *job types* ($\mathcal{J}$) contains unique tasks (operations such as milling, turning, and grinding) to be performed on parts/products. Each job type $j \in \mathcal{J}$ has the demand a_{jt} in time period $t \in \mathcal{T}$, and the production should equal this demand, as expressed in (2a). The constraints (2b) ensure that no job is performed in any machine and time period to which it is not allocated. The constraints (2c) limit, for each job type and time period, the number of alternative machines to τ, the value of which is given as an input by the user; a too small value of τ may result in an empty set of feasible solutions and a too high value may be inconvenient for the production planners due to a resulting increased complexity of the product route. The constraints (2d) and (2j) make sure that the allocated machining hours do not exceed the capacity of the machines at any time period. The constraints (2d) are also used to define the objective function (1a), which is to minimize the sum over the time periods $t \in \mathcal{T}$ of the *excess resource loading of the machines* (i.e. $n_t \geq 0$), which is defined as the maximum (over the machines) ratio between the allocated machining hours and the available hours (i.e. $\frac{1}{C_{kt}} \sum_{j \in \mathcal{J}} p_{jk} x_{jkt}$) minus the loading threshold $\zeta_k \in [0, 1]$ for the machine. The constraints (2e) implies that if a job type j is scheduled in machine k during time period t, then qualification of the machine for this job type must be done once during the time periods $\{1, \ldots, t\}$. The constraints (2f) limit the number of qualifications allowed to be scheduled in each time period to γ; this is due to a limited number of skilled professionals for completing new qualifications. The constraints (2g)–(2j) define the allowed values of the variables.

1.3.2 Qualification cost

The qualification cost is the one-time cost incurred by the company to qualify a machine for a job type. Once qualified, the new allocation can be used in all the subsequent time periods. For each allocation (j, k), $j \in \mathcal{J}$, $k \in \mathcal{N}_j$, in scenario q, the qualification cost parameter β^q_{jk} is represented by a natural number; hence, the uncertainty set is finite. It is common in the robust optimization literature to assume a nominal or most-likely scenario (see [3]). We denote the uncertainty set by $\mathcal{Q} := \{\hat{q}, \tilde{q}\}$, where the indices $\hat{q}$ and $\tilde{q}$ represent the nominal and worst-case scenario, respectively; for each $j \in \mathcal{J}$ and $k \in \mathcal{N}_j$ it holds that $\beta^{\hat{q}}_{jk} \leq \beta^{\tilde{q}}_{jk}$. Since the qualification costs of any two allocations (j', k') and (j, k) are independent, the nominal and worst-case values, $\boldsymbol{\beta}^{\hat{q}}$ and $\boldsymbol{\beta}^{\tilde{q}}$, respectively, are *well-defined*. For uncertainty sets with more than two scenarios, most of the robust efficiency concepts to be defined will be retained, unless otherwise mentioned.

2 Preliminaries and an example to highlight robust efficiency concepts

Both robust and multi-objective optimization have, respectively, rich sets of literature. We introduce most of the key concepts in robust multi-objective optimization, aided by an instance of the TRAP, starting with some necessary notations ([10]). For any two vectors $\mathbf{z}, \mathbf{w} \in \mathbb{R}^b$ it holds that

$$\mathbf{z} \leqq \mathbf{w} \iff w_i \in [z_i, \infty) \quad \forall i \in \{1, \ldots, b\}; \tag{4a}$$

$$\mathbf{z} \leq \mathbf{w} \iff w_i \in [z_i, \infty) \quad \forall i \in \{1, \ldots, b\} \quad \text{and} \quad \mathbf{z} \neq \mathbf{w}; \tag{4b}$$

$$\mathbf{z} < \mathbf{w} \iff w_i \in (z_i, \infty) \quad \forall i \in \{1, \ldots, b\}. \tag{4c}$$

In a robust counterpart to a deterministic version of the TRAP, the objective function is defined as $\mathbf{g} : Y \times \mathcal{Q} \mapsto \mathbb{R}^2_+$, i.e. the scenarios corresponding to the indices in $\mathcal{Q}$ affect the objective values. An uncertain bi-objective TRAP is defined as a set of parametrized problems, as

$$\mathcal{P}(\mathcal{Q}) := \{\mathcal{P}(q) : q \in \mathcal{Q}\}, \tag{5a}$$

where $\mathcal{P}(q)$ denotes the instance (corresponding to scenario q)

$$\min_{\mathbf{y} \in Y} \mathbf{g}(\mathbf{y}, q) := \min_{\mathbf{y} \in Y} \big(g_1(\mathbf{y}), g_2(\mathbf{y}, q)\big) \tag{5b}$$

of a bi-objective optimization problem, and the functions $g_1 : Y \mapsto \mathbb{R}_+$ and $g_2 : Y \times \mathcal{Q} \mapsto \mathbb{Z}_+$ are defined by

$$g_1(\mathbf{y}) := \sum_{t \in \mathcal{T}} n_t; \tag{5c}$$

$$g_2(\mathbf{y}, q) := \sum_{t \in \mathcal{T}} \sum_{j \in \mathcal{J}} \sum_{k \in \mathcal{N}_j} \beta^q_{jk} z_{jkt}, \qquad q \in \mathcal{Q}. \tag{5d}$$

Note that the feasible set Y is independent of the scenario q. As compared to the cases of single-objective robust optimization[3] (SO-RO) and single-scenario (hence, deterministic) multi-objective optimization[4] (SS-MOOP), it is not evident what properties characterize *good solutions*. Consequently, we evaluate some of the key concepts mentioned in [18, 21], and [11], and relate these to our model. However, most of the concepts apply to general cases as well i.e. for robust multi-objective optimization problems (robust MOOPs) with more than two objective functions. All set notations defined are listed in Table 4.

[3] A field of optimization theory that requires a certain measure of solution robustness against uncertainty in parameters with a single objective function

[4] A multi-objective optimization problem with no uncertain parameters

Table 3 The objective vectors $\mathbf{g}(\mathbf{y}^i, q)$ for $i \in \{1, \ldots, 8\}$ and $q \in \{\hat{q}, \tilde{q}\}$

q	$\mathbf{y}^1$	$\mathbf{y}^2$	$\mathbf{y}^3$	$\mathbf{y}^4$	$\mathbf{y}^5$	$\mathbf{y}^6$	$\mathbf{y}^7$	$\mathbf{y}^8$
$\hat{q}$	$\begin{pmatrix}0\\30\end{pmatrix}$	$\begin{pmatrix}0.05\\32\end{pmatrix}$	$\begin{pmatrix}0.21\\28\end{pmatrix}$	$\begin{pmatrix}0.3\\25\end{pmatrix}$	$\begin{pmatrix}0.33\\27\end{pmatrix}$	$\begin{pmatrix}0.35\\26\end{pmatrix}$	$\begin{pmatrix}0.4\\25\end{pmatrix}$	$\begin{pmatrix}0.4\\27\end{pmatrix}$
$\tilde{q}$	$\begin{pmatrix}0\\58\end{pmatrix}$	$\begin{pmatrix}0.05\\55\end{pmatrix}$	$\begin{pmatrix}0.21\\53\end{pmatrix}$	$\begin{pmatrix}0.3\\54\end{pmatrix}$	$\begin{pmatrix}0.33\\56\end{pmatrix}$	$\begin{pmatrix}0.35\\53\end{pmatrix}$	$\begin{pmatrix}0.4\\48\end{pmatrix}$	$\begin{pmatrix}0.4\\48\end{pmatrix}$

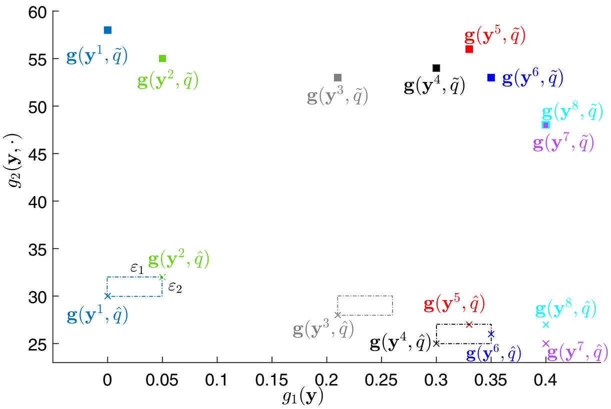


Fig. 1 Objective vectors $\mathbf{g}(\mathbf{y}^i, q)$, $i \in \{1, \ldots, 8\}$, $q \in \{\hat{q}, \tilde{q}\}$, in the criterion space. Each solution represented by a specific color; cross- and square-markers denote the nominal ($\hat{q}$) and worst-case ($\tilde{q}$) scenario, respectively. The sets of efficient solutions are $Y^{\mathbf{g}}_{\text{eff}}(\hat{q}) = \{\mathbf{y}^1, \mathbf{y}^3, \mathbf{y}^4\}$ and $Y^{\mathbf{g}}_{\text{eff}}(\tilde{q}) = \{\mathbf{y}^1, \mathbf{y}^2, \mathbf{y}^3, \mathbf{y}^7, \mathbf{y}^8\}$

Definition 2 (Efficient solutions for $\mathcal{P}(q)$) A point $\bar{\mathbf{y}} \in Y$ is an *efficient solution* to the TRAP with scenario q, i.e. $\mathcal{P}(q)$ as defined in (5b), if $\nexists\, \mathbf{y} \in Y$ such that $\mathbf{g}(\mathbf{y}, q) \preceq \mathbf{g}(\bar{\mathbf{y}}, q)$ holds. If $\bar{\mathbf{y}}$ is efficient in the solution space Y w.r.t. $\mathcal{P}(q)$, then $\mathbf{g}(\bar{\mathbf{y}}, q)$ is a *non-dominated point* in the criterion space corresponding to the objective functions $g_1(\cdot)$ and $g_2(\cdot, q)$. The set of efficient solutions to the bi-objective optimization problem $\mathcal{P}(q)$ is denoted as $Y^{\mathbf{g}}_{\text{eff}}(q)$, for the objective functions $\mathbf{g}(\cdot, q) = (g_1(\cdot), g_2(\cdot, q))$.

As an illustrative example, we assume eight feasible solutions to a given instance of the TRAP denoted as $\mathbf{y}^i = (\mathbf{x}^i, \mathbf{s}^i, \mathbf{n}^i, \mathbf{z}^i) \in Y$, $i \in \{1, \ldots, 8\}$, and two scenarios $\mathcal{Q} := \{\hat{q}, \tilde{q}\}$. The corresponding objective vectors are listed in Table 3 and visualized—in the criterion space—in Fig. 1.

Since in a bi-objective problem with one uncertain objective function with two scenarios each solution maps to two points in the criterion space, the concept of efficiency is not well-defined (see Fig. 1). We next present the main concepts of *robust efficiency*, the counterparts of *efficiency* for SS-MOOP.

Definition 3 (Flimsily Robust Efficient (FRE); [4]) A point $\mathbf{y} \in \mathbf{Y}$ is called *flimsily RE* (FRE) if it is an efficient solution to the deterministic MOOP $\mathcal{P}(q)$, for at least one scenario $q \in \mathcal{Q}$. The set of FRE solutions is defined as

$$Y_{\text{FRE}} := \bigcup_{q \in \mathcal{Q}} Y^{\mathbf{g}}_{\text{eff}}(q), \tag{6}$$

where $Y^{\mathbf{g}}_{\text{eff}}(q)$ is the set of efficient solutions for the SS-MOOP $\mathcal{P}(q)$ (see (5b)).

In our recurring example, $Y_{\mathrm{FRE}} = \{\mathbf{y}^1, \mathbf{y}^2, \mathbf{y}^3, \mathbf{y}^4, \mathbf{y}^7, \mathbf{y}^8\}$ (cf. Fig. 1).

Definition 4 (Highly Robust Efficient (HRE); [18]) A point $\mathbf{y} \in \mathbf{Y}$ is called *highly RE* (HRE) if it is an efficient solution to each of the deterministic MOOPs $\mathcal{P}(q)$, $q \in \mathcal{Q}$. The set of HRE solutions is defined as

$$Y_{\mathrm{HRE}} := \bigcap_{q \in \mathcal{Q}} Y^{\mathbf{g}}_{\mathrm{eff}}(q) \tag{7}$$

Unless some necessary conditions are satisfied [18, Le. 9], it is not guaranteed that every instance of a robust MOOP possesses HRE solutions.

In our recurring example, $Y_{\mathrm{HRE}} = \{\mathbf{y}^1, \mathbf{y}^3\}$ (cf. Fig. 1). We next present a concept that mitigates the conservativeness of choosing the best solutions for the worst-case scenario. It is based on the *light robustness* concept for SO-RO problems ([12]). It characterizes solutions that, w.r.t. an ε-neighbourhood (in the criterion space) of an efficient solution in the nominal scenario, are efficient in the worst-case scenario.

Definition 5 (ε-lightly RE; [27]) For a parameterized uncertain MOOP $\mathcal{P}(\mathcal{Q})$ with pre-defined nominal scenario $(\hat{q})$, and $\boldsymbol{\varepsilon} \in \mathbb{R}^2_+$, the set $Y_{\mathrm{light}}(\hat{\mathbf{y}}, \boldsymbol{\varepsilon})$ of *ε-lightly robust efficient* solutions w.r.t. an efficient solution $\hat{\mathbf{y}} \in Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q})$ to the deterministic MOOP $\mathcal{P}(\hat{q})$ is defined as the set of efficient solutions to

$$\min_{\mathbf{y} \in Y_{\mathrm{nb}}(\hat{\mathbf{y}}, \boldsymbol{\varepsilon})} \left(g_1(\mathbf{y}),\ \max_{q \in \mathcal{Q}} g_2(\mathbf{y}, q) \right) \equiv \min_{\mathbf{y} \in Y_{\mathrm{nb}}(\hat{\mathbf{y}}, \boldsymbol{\varepsilon})} \mathbf{g}(\mathbf{y}, \tilde{q}), \tag{8}$$

where the equivalence holds when the worst-case scenario $\tilde{q}$ is well-defined, and

$$Y_{\mathrm{nb}}(\hat{\mathbf{y}}, \boldsymbol{\varepsilon}) := \{\, \mathbf{y} \in Y : \mathbf{g}(\hat{\mathbf{y}}, \hat{q}) \leqq \mathbf{g}(\mathbf{y}, \hat{q}) \leqq \mathbf{g}(\hat{\mathbf{y}}, \hat{q}) + \boldsymbol{\varepsilon} \,\}, \quad \hat{\mathbf{y}} \in Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}), \quad \boldsymbol{\varepsilon} \in \mathbb{R}^2_+, \tag{9}$$

defines a neighborhood of $\hat{\mathbf{y}}$.

The set of all ε-lightly RE solutions is defined as

$$Y_{\mathrm{light}}(Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}), \boldsymbol{\varepsilon}) := \bigcup_{\hat{\mathbf{y}} \in Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q})} Y_{\mathrm{light}}(\hat{\mathbf{y}}, \boldsymbol{\varepsilon}). \tag{10}$$

In our recurring example, $Y_{\mathrm{light}}(Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}), \boldsymbol{\varepsilon}) = \{\mathbf{y}^1, \mathbf{y}^2, \mathbf{y}^3, \mathbf{y}^4, \mathbf{y}^6\}$, where $\boldsymbol{\varepsilon} = (.05, 2)^\top$ (dashed rectangles); see Fig. 1. By Def. 5, it holds that $\hat{\mathbf{y}} \in Y_{\mathrm{light}}(\hat{\mathbf{y}}, \boldsymbol{\varepsilon})$ for all $\hat{\mathbf{y}} \in Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q})$ and $\boldsymbol{\varepsilon} \in \mathbb{R}^2_+$. Note that even though $\mathbf{y}^5 \in Y_{\mathrm{nb}}(\mathbf{y}^4, \boldsymbol{\varepsilon})$ holds, $\mathbf{y}^5 \notin Y_{\mathrm{light}}(\mathbf{y}^4, \boldsymbol{\varepsilon})$ because $\mathbf{g}(\mathbf{y}^4, \tilde{q}) \preceq \mathbf{g}(\mathbf{y}^5, \tilde{q})$.

For each efficient solution in the nominal scenario, there may exist multiple ε-lightly RE solutions. Hence, the number of solutions to be evaluated by the DM may be quite large if the entire set $Y_{\mathrm{light}}(Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}), \boldsymbol{\varepsilon})$ is considered. To reduce this number, the following definition identifies a representative set of ε-lightly robust efficient solutions.

Definition 6 (ε-representative lightly RE; [21]) For a parameterized uncertain MOOP $\mathcal{P}(\mathcal{Q})$ with pre-defined nominal $(\hat{q})$ and worst-case $(\tilde{q})$ scenarios, and $\boldsymbol{\varepsilon} \in \mathbb{R}^2_+$, the set of *$\varepsilon$-representative lightly robust efficient* solutions w.r.t. an efficient solution $\hat{\mathbf{y}} \in Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q})$ to the deterministic MOOP $\mathcal{P}(\hat{q})$ is defined as

$$Y_{\mathrm{r-light}}(\hat{\mathbf{y}}, \boldsymbol{\varepsilon}) := \operatorname*{arg\,min}_{\mathbf{y} \in \mathbf{Y}_{\mathrm{nb}}(\hat{\mathbf{y}}, \boldsymbol{\varepsilon})} g_2(\mathbf{y}, \tilde{q}). \tag{11}$$

The set of all the ε-representative lightly RE solutions is defined as

$$Y_{\mathrm{r-light}}(Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}), \boldsymbol{\varepsilon}) := \bigcup_{\hat{\mathbf{y}} \in Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q})} Y_{\mathrm{r-light}}(\hat{\mathbf{y}}, \boldsymbol{\varepsilon}).$$

In our recurring example (Fig. 1), $Y_{\mathrm{r-light}}(\mathbf{y}^1,\boldsymbol{\varepsilon})=\mathbf{y}^2$, $Y_{\mathrm{r-light}}(\mathbf{y}^3,\boldsymbol{\varepsilon})=\mathbf{y}^3$, $Y_{\mathrm{r-light}}(\mathbf{y}^4,\boldsymbol{\varepsilon})=\mathbf{y}^6$; $Y_{\mathrm{r-light}}(Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}),\boldsymbol{\varepsilon})=\{\mathbf{y}^2,\mathbf{y}^3,\mathbf{y}^6\}$. Note that the inequality $|Y_{\mathrm{r-light}}(Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}), \boldsymbol{\varepsilon})| \le |Y_{\mathrm{light}}(Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}), \boldsymbol{\varepsilon})|$ holds.

[17] prove that the solutions obtained using only the traditional concept of strict robustness for SO-RO,[5] which focuses solely on the worst-case scenario can be dominated; instead, they introduce the concept of *Pareto Robust Optimal* (PRO) solutions for SO-RO problems. [21] suggest a generalization of PRO solutions from SO-RO to robust MOOPs, called *PRO Robust Efficient* (PRO-RE) solutions.

Defining a vector valued function $\boldsymbol{\psi} : Y \mapsto \mathbb{R}_+^{|Q|+1}$ for TRAP as $\boldsymbol{\psi}(\mathbf{y}) := (g_1(\mathbf{y}), g_2(\mathbf{y}, q)_{q\in Q})$ [21, p. 423], PRO-RE is defined next.

Definition 7 (Pareto robust optimal robust efficient (PRO-RE)) The set of PRO-RE solutions to the TRAP (5) with a vector valued function $\boldsymbol{\psi}$ is defined as

$$Y^{\boldsymbol{\psi}}_{\mathrm{eff}} := \{\bar{\mathbf{y}} \in Y : \nexists \mathbf{y} \in Y \text{ s.t. } \boldsymbol{\psi}(\mathbf{y}) \preceq \boldsymbol{\psi}(\bar{\mathbf{y}})\}. \tag{12}$$

By Def. 7, $Y^{\boldsymbol{\psi}}_{\mathrm{eff}}$ contains solutions that are non-dominated in the criterion space corresponding to $\boldsymbol{\psi}$ and the solution space Y; it can thus be used to filter out dominated solutions. For instance, in our recurring example, $\mathbf{y}^7$ and $\mathbf{y}^8$ are both FRE solutions as they belong to the set $Y^{\mathbf{g}}_{\mathrm{eff}}(\tilde{q})$ (see Def. 2). Since $\psi(\mathbf{y}^7) = (0.4, 25, 48)^\top$ and $\psi(\mathbf{y}^8) = (0.4, 27, 48)^\top$ (see Table 3), it holds that $\psi(\mathbf{y}^7) \preceq \psi(\mathbf{y}^8)$. Hence, $\mathbf{y}^8$ is *not* a PRO-RE solution. HRE solutions are, however, PRO-RE, since they are efficient in all scenarios.

The RE concepts presented in [17, 21, 1, 11, 27, 7], and [18] provide a sufficient basis for our contributions. Furthermore, as per [21, Def. 9]—for bi-objective—and [7]—for general multi-objective— robust optimization problems, flimsily, highly, ε-lightly, and ε -representative lightly RE solutions must be PRO-RE, (cf. Def. 7). The sets corresponding to each type of RE while satisfying the constraints (12) are defined as

$$Y^{PRO}_{\mathrm{FRE}} := Y_{\mathrm{FRE}} \cap Y^{\boldsymbol{\psi}}_{\mathrm{eff}}, \tag{13a}$$

$$Y^{PRO}_{\mathrm{HRE}} := Y_{\mathrm{HRE}} \cap Y^{\boldsymbol{\psi}}_{\mathrm{eff}}, \tag{13b}$$

$$Y^{PRO}_{\mathrm{light}}(Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}), \boldsymbol{\varepsilon}) := Y_{\mathrm{light}}(Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}), \boldsymbol{\varepsilon}) \cap Y^{\boldsymbol{\psi}}_{\mathrm{eff}}, \tag{13c}$$

$$Y^{PRO}_{\mathrm{r-light}}(Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}), \boldsymbol{\varepsilon}) := Y_{\mathrm{r-light}}(Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}), \boldsymbol{\varepsilon}) \cap Y^{\boldsymbol{\psi}}_{\mathrm{eff}}. \tag{13d}$$

Similarly, the corresponding set of PRO solutions that are efficient in scenario q is defined as

[5] as presented in [18, Def. 11]

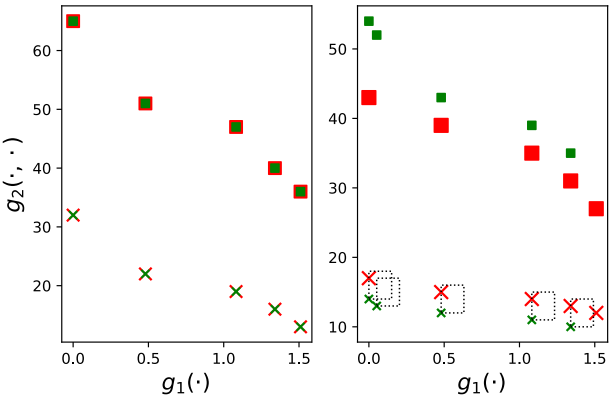


Fig. 2 Nominal and worst-case PRO-RE solutions for instance #19 (left) and #18 (right). Green-cross: $\mathbf{g}(\mathbf{y},\hat{q}) : \mathbf{y} \in Y_{\mathrm{eff}}^{\mathrm{PRO}}(\hat{q})$; green-square: $\mathbf{g}(\mathbf{y},\tilde{q}) : \mathbf{y} \in Y_{\mathrm{eff}}^{PRO}(\hat{q})$; red-cross: $\mathbf{g}(\mathbf{y},\hat{q}) : \mathbf{y} \in Y_{\mathrm{eff}}^{PRO}(\tilde{q})$; red-square: $\mathbf{g}(\mathbf{y},\tilde{q}) : \mathbf{y} \in Y_{\mathrm{eff}}^{PRO}(\tilde{q})$

$$Y_{\mathrm{eff}}^{PRO}(q) := Y_{\mathrm{eff}}^{\mathbf{g}}(q) \cap Y_{\mathrm{eff}}^{\boldsymbol{\psi}}, q \in \mathcal{Q}. \tag{14}$$

3 Significance of uncertainty

As the numbers of uncertain objective functions or scenarios increase, the number of objective functions in the corresponding deterministic MOOP increases [17], hence increasing the computational effort needed to find all PRO-RE solutions. E.g., the bi-objective TRAP with one uncertain objective and two scenarios results in solving a tri-objective deterministic MOOP to find all PRO-RE solutions (see Def. 7). [8] established theoretically that the number of non-dominated points grows exponentially with the number of objective functions. Hence, restricting the number of uncertain objective functions or scenarios make the problem more tractable. However, ignoring uncertainties may also result in sub-optimal or too sensitive solutions. Consequently, it is important to understand whether the inclusion of scenarios reveals any deficiency caused by only using the nominal scenario for instances of a given problem class.

3.1 Motivation

To assess the gain from the inclusion of a worst-case scenario one should perform numerical tests (over a fairly large number of numerical instances representing various possible realizations of the TRAP). We consider two instances of the TRAP, illustrated[6] in Fig. 2.

In Fig. 2 (left), all the efficient solutions $\mathbf{y} \in Y_{\mathrm{eff}}^{PRO}(\hat{q})$ are highly PRO-RE (i.e. $Y_{\mathrm{eff}}^{PRO}(\hat{q}) \subseteq Y_{\mathrm{HRE}}^{\mathrm{PRO}}$). Thus, nothing is gained by including the worst-case scenario for this particular instance. However, for the instance #18 (see Sect. 5.1 and 7.2 for details on instances) in Fig.2 (right), none of the efficient solutions (i.e. $Y_{\mathrm{eff}}^{PRO}(\hat{q})$) are in $Y_{\mathrm{HRE}}^{\mathrm{PRO}}$. Figure 2 considers $\hat{\mathbf{y}} \in Y_{\mathrm{eff}}^{\mathrm{PRO}}(\hat{q})$ and $\tilde{\mathbf{y}} \in Y_{\mathrm{eff}}^{\mathrm{PRO}}(\tilde{q})$, corresponding to the vectors $\boldsymbol{\psi}(\hat{\mathbf{y}}) = (0, 14, 54)^{\top}$ and $\boldsymbol{\psi}(\tilde{\mathbf{y}}) = (0, 17, 43)^{\top}$, respectively (see Def. 7). Hence, the solution $\hat{\mathbf{y}}$ maps to the

[6] In all forthcoming bi-objective illustrations, solutions from the sets $Y_{\mathrm{eff}}^{PRO}(\hat{q})$ (green) and $Y_{\mathrm{eff}}^{PRO}(\tilde{q})$ (red) are plotted. Each solution $\mathbf{y} \in Y_{\mathrm{eff}}^{PRO}(\hat{q}) \cup Y_{\mathrm{eff}}^{\mathrm{PRO}}(\tilde{q})$ yields the two points $\mathbf{g}(\mathbf{y},\hat{q})$ (cross-marker) and $\mathbf{g}(\mathbf{y},\tilde{q})$ (square-marker).

two points $\mathbf{g}(\hat{\mathbf{y}}, \hat{q}) = (0, 14)^\top$ (left-most green-cross) and $\mathbf{g}(\hat{\mathbf{y}}, \tilde{q}) = (0, 54)^\top$ (left-most green-square), while the solution $\tilde{\mathbf{y}}$ yields $\mathbf{g}(\tilde{\mathbf{y}}, \hat{q}) = (0, 17)^\top$ (left-most red cross) and $\mathbf{g}(\tilde{\mathbf{y}}, \tilde{q}) = (0, 43)^\top$ (left-most red square). It follows that three units of qualification cost are lost in the nominal scenario, while in the worst-case, the qualification cost is reduced by eleven units. Hence, a total net reduction of eight units of qualification cost is obtained by this swap. A DM relying on its tolerance for loss of nominal quality may find this swap useful. We formalize this idea in the next section.

3.2 Positive robustness

We next present a new RE concept for the TRAP to identify solutions having a positive effect on mitigating risks due to the worst-case scenario.

Definition 8 (positive robustness ε-representative lightly RE solution) For a parameterized uncertain bi-objective MOOP $\mathcal{P}(\mathcal{Q})$ with one uncertain objective function g_2, predefined nominal ($\hat{q}$) and worst-case ($\tilde{q}$) scenarios, $\boldsymbol{\varepsilon} \in \mathbb{R}^2_+$, and $0 < \kappa \ll 1$, the set of *positive robustness ε-representative lightly robust efficient* solutions for the solution $\hat{\mathbf{y}} \in Y^{\mathbf{g}}_{\text{eff}}(\hat{q})$ is defined as

$$Y_{\text{r-light}}(\hat{\mathbf{y}}, \boldsymbol{\varepsilon}, \kappa) := \operatorname{argmin}_{\mathbf{y} \in \mathbf{Y}_{\text{nb}}(\hat{\mathbf{y}}, \boldsymbol{\varepsilon})} \{ g_2(\mathbf{y}, \tilde{q}) \,:\, \nu(\mathbf{y}, \hat{\mathbf{y}}) \geq \kappa \}, \tag{15}$$

where $\nu(\mathbf{y}, \hat{\mathbf{y}}) = [g_2(\hat{\mathbf{y}}, \tilde{q}) - g_2(\mathbf{y}, \tilde{q})] - [g_2(\mathbf{y}, \hat{q}) - g_2(\hat{\mathbf{y}}, \hat{q})]$. The set of all positive robustness ε-representative lightly RE solutions is defined as

$$Y_{\text{r-light}}(Y^{\mathbf{g}}_{\text{eff}}(\hat{q}), \boldsymbol{\varepsilon}, \kappa) := \bigcup_{\hat{\mathbf{y}} \in Y^{\mathbf{g}}_{\text{eff}}(\hat{q})} Y_{\text{r-light}}(\hat{\mathbf{y}}, \boldsymbol{\varepsilon}, \kappa).$$

Positive robustness refers to the expression $\nu(\mathbf{y}, \hat{\mathbf{y}}) > 0$, measuring the net reduction of qualification costs from swapping $\hat{\mathbf{y}} \in Y^{\mathbf{g}}_{\text{eff}}(\hat{q})$ and $\mathbf{y} \in Y_{\text{nb}}(\hat{\mathbf{y}}, \boldsymbol{\varepsilon})$. The constraint $\mathbf{y} \in Y_{\text{nb}}(\hat{\mathbf{y}}, \boldsymbol{\varepsilon})$ limits the loss of nominal value while also yielding a representative set for each efficient solution in the nominal scenario, which holds also for ε-representative lightly RE solutions (cf. Def. 6).

A corresponding PRO-RE set is denoted as

$$Y^{PRO}_{\text{r-light}}(Y^{\mathbf{g}}_{\text{eff}}(\hat{q}), \boldsymbol{\varepsilon}, \kappa) := Y_{\text{r-light}}(Y^{\mathbf{g}}_{\text{eff}}(\hat{q}), \boldsymbol{\varepsilon}, \kappa) \cap Y^{\boldsymbol{\Psi}}_{\text{eff}}. \tag{16}$$

Note that for $\kappa = 0$, it holds that $Y^{PRO}_{\text{r-light}}(Y^{\mathbf{g}}_{\text{eff}}(\hat{q}, 0), \boldsymbol{\varepsilon}, 0) = Y^{PRO}_{\text{r-light}}(Y^{\mathbf{g}}_{\text{eff}}(\hat{q}), \boldsymbol{\varepsilon})$.

Remark 1 The set $Y^{\text{PRO}}_{\text{r-light}}(Y^{\mathbf{g}}_{\text{eff}}(\hat{q}), \boldsymbol{\varepsilon}, \kappa)$ can be considered by the DM instead of the ε-representative lightly PRO-RE set $Y^{PRO}_{\text{r-light}}(Y^{\mathbf{g}}_{\text{eff}}(\hat{q}), \boldsymbol{\varepsilon})$, as solutions with *positive robustness* are not guaranteed to exist in the latter. In Fig. 3, three solutions are in the set $Y^{\text{PRO}}_{\text{eff}}(\tilde{q}) \setminus Y^{\text{PRO}}_{\text{eff}}(\hat{q})$ (corresponding points in the criterion space are distinct red-cross and -square marks) two of which belong to $Y_{\text{r-light}}(Y^{\text{PRO}}_{\text{eff}}(\hat{q}), \boldsymbol{\varepsilon})$; $\boldsymbol{\varepsilon}$ marked by black-dashed rectangles. None of these two solutions, however, possess the so-called *positive robustness* as compared with the respective closest nominal PRO-RE solutions (marked by a green cross at the lower-left vertex of the respective rectangle). Hence, solutions without positive robustness have a limited utility for the DM.

3.3 Inclusion of a worst-case scenario

The topic of assessing the importance of including a worst-case scenario for an uncertain MOOP, over a fairly large number of numerical instances has, to the best of our knowledge, not been discussed in the existing literature. While the positive robustness ε-representative lightly RE solutions rely on the values of $\boldsymbol{\varepsilon}$, we next present a measure that does not require specifying $\boldsymbol{\varepsilon}$.

A worst-case scenario and the corresponding efficient solutions in $Y^{PRO}_{\text{eff}}(\tilde{q})$ add a significant value to a robust bi-objective optimization problem if the set $\cup_{\mathbf{y}\in\mathbf{Y}^{\mathbf{PRO}}_{\text{eff}}(\tilde{\mathbf{q}})}\{\mathbf{g}(\mathbf{y},\hat{q})\}$ is a good *approximation*[7] of the efficient frontier $\cup_{\mathbf{y}\in\mathbf{Y}^{\mathbf{PRO}}_{\text{eff}}(\hat{\mathbf{q}})}\{\mathbf{g}(\mathbf{y},\hat{q})\}$. Further, $\cup_{\mathbf{y}\in Y^{PRO}_{\text{eff}}(\hat{q})}\{\mathbf{g}(\mathbf{y},\tilde{q})\}$ is *not* a good approximation of the efficient frontier $\cup_{\mathbf{y}\in Y^{PRO}_{\text{eff}}(\tilde{q})}\{\mathbf{g}(\mathbf{y},\tilde{q})\}$ for the worst-case scenario.

Definition 9 (Scenario approximation) For each $r, q \in \{\hat{q},\tilde{q}\}$, the region in the criterion space of $(g_1(\cdot), g_2(\cdot, r))$, defined by the set

$$A(r,q) := \left\{ \bigcup_{\mathbf{y}\in Y^{PRO}_{\text{eff}}(q)} \{\mathbf{g}(\mathbf{y},r)\} + \mathbb{R}^2_+ \right\} \bigcap \left\{ \begin{pmatrix} g_1^{\text{BOT}}(r) \\ g_2^{\text{TOP}}(r) \end{pmatrix} - \mathbb{R}^2_+ \right\}, \tag{17}$$

where

$$g_1^{\text{BOT}}(r) = \max\left\{ g_1(\mathbf{y},r) \,:\, \mathbf{y} \in \cup_{u\in\{\hat{q},\tilde{q}\}} Y^{\text{PRO}}_{\text{eff}}(u) \right\}; \tag{18a}$$

$$g_2^{\text{TOP}}(r) = \max\left\{ g_2(\mathbf{y},r) \,:\, \mathbf{y} \in \cup_{u\in\{\hat{q},\tilde{q}\}} Y^{PRO}_{\text{eff}}(u) \right\}, \tag{18b}$$

is dominated by the worst-case ($r = \tilde{q}$) or nominal ($r = \hat{q}$) objective vectors corresponding to PRO-RE solutions that are efficient in the worst-case ($q = \tilde{q}$) and nominal ($q = \hat{q}$) scenario, respectively.

The *nominal* ($\hat{q}$) *and worst-case* ($\tilde{q}$) *scenario approximation area is defined as* $\texttt{area}(A(\hat{q},\hat{q}) \setminus A(\hat{q},\tilde{q}))$ *and* $\texttt{area}(A(\tilde{q},\tilde{q}) \setminus A(\tilde{q},\hat{q}))$, *respectively*, *where* $\texttt{area}(B)$ *measures the area covered by a set* $B \subset \mathbb{R}^2$.

In Fig. 4 the grey shaded area illustrates the set $A(\hat{q},\hat{q}) \setminus A(\hat{q},\tilde{q})$, i.e. the region in the criterion space that is not dominated by any non-dominated point in the worst-case scenario (the set $\cup_{\mathbf{y}\in\mathbf{Y}^{\mathbf{PRO}}_{\text{eff}}(\tilde{\mathbf{q}})}\{\mathbf{g}(\mathbf{y},\hat{q})\}$), but is dominated by non-dominated points in the nominal scenario (the set $\cup_{\mathbf{y}\in\mathbf{Y}^{\mathbf{PRO}}_{\text{eff}}(\hat{\mathbf{q}})}\{\mathbf{g}(\mathbf{y},\hat{q})\}$). Analogously, the red shaded area illustrates the set $A(\tilde{q},\tilde{q}) \setminus A(\tilde{q},\hat{q})$, i.e. the region that is not dominated by any non-dominated point in the nominal scenario (the set $\cup_{\mathbf{y}\in\mathbf{Y}^{\mathbf{PRO}}_{\text{eff}}(\hat{\mathbf{q}})}\{\mathbf{g}(\mathbf{y},\tilde{q})\}$), but is dominated by non-dominated points in the worst-case scenario (the set $\cup_{\mathbf{y}\in\mathbf{Y}^{\mathbf{PRO}}_{\text{eff}}(\tilde{\mathbf{q}})}\{\mathbf{g}(\mathbf{y},\tilde{q})\}$).

Definition 10 The difference between the worst-case and nominal scenario approximations indicates the utility of including the worst-case scenario. We suggest a measure

[7] The concept of approximations is also used to benchmark performance of in-exact methods for approximating the efficient frontier in MOOPs; see [32] for details

defined as the ratio of the difference in the areas of the worst-case and nominal scenario approximations and the maximum of these two values, as[8]

$$\Delta_{\text{diff}} := \frac{\texttt{area}(A(\tilde{q},\tilde{q}) \setminus A(\tilde{q},\hat{q})) - \texttt{area}(A(\hat{q},\hat{q}) \setminus A(\hat{q},\tilde{q}))}{\max\{\ \texttt{area}(A(\tilde{q},\tilde{q}) \setminus A(\tilde{q},\hat{q}))\ ;\texttt{area}(A(\hat{q},\hat{q}) \setminus A(\hat{q},\tilde{q}))\ \}}.$$

If $\Delta_{\text{diff}} \leq 0$, the worst-case scenario is not expected to add significant gains, whereas if $\Delta_{\text{diff}} > 0$, it is expected to result in solutions that provide significant gain. We validate this claim empirically in Sect. 5.3.

4 Solution approaches for bi-objective robust optimization with one uncertain objective function

From the definitions in (13), RE solutions are required to be PRO-RE. To the best of our knowledge, the only fully autonomous algorithm suggested for robust bi-objective optimization problems with one uncertain objective function is the one presented in [21, Sect. 6]. Some interactive methods—relying on users providing their preferences—are, however, suggested in [23, 28], and [24, pp. 27–57].

The algorithm by [21] first finds a *minimal set*[9] of efficient solutions in $Y^{\boldsymbol{\psi}}_{\text{eff}}$ to a deterministic tri-objective (for $|Q| = 2$) optimization problem (cf. Def. 7). Then, it filters the resulting minimal subset of $Y^{\boldsymbol{\psi}}_{\text{eff}}$ for flimsily/highly/ε-representative lightly PRO-RE solutions.

Note that it requires computationally expensive tri-objective optimization to identify all the PRO-RE solutions. Further, the set $Y^{\boldsymbol{\psi}}_{\text{eff}}$ may contain more solutions than required by the DMs (typically the ones in (13)).

4.1 A new approach to find required PRO-RE solutions

We propose a 3-stage method (Alg. 1): (i) compute $Y^{\text{PRO}}_{\text{eff}}(\tilde{q})$ and $Y^{\text{PRO}}_{\text{eff}}(\hat{q})$; (ii) identify $Y^{\text{PRO}}_{\text{HRE}}$ and $Y^{\text{PRO}}_{\text{FRE}}$ from solutions identified in (i); (iii) for a given $\boldsymbol{\varepsilon}$, compute $Y_{\text{r-light}}(Y^{\text{PRO}}_{\text{eff}}(\hat{q}), \boldsymbol{\varepsilon}, \kappa)$. The set $Y_{\text{r-light}}(Y^{\text{PRO}}_{\text{eff}}(\hat{q}), \boldsymbol{\varepsilon}, \kappa)$ obtained is not guaranteed to be PRO-RE.[10] For instance, in Fig. 5 the second worst-case efficient solution from the left (marked red-cross and red-square) dominates the solution corresponding to the second positive robustness ε-representative lightly RE solution from the left (marked blue-cross and blue-square).[11] In Sect. 4.3 we suggest a procedure to prevent this issue.

[8] For the case when both areas are 0, $\Delta_{\text{diff}} := 0$

[9] A set $\{\mathbf{y}^i\}_{i\in\mathcal{I}}$ is minimal w.r.t. a vector valued function $\boldsymbol{\psi}$ if for any two indices $i,j \in \mathcal{I}$ such that $i \neq j$, $\boldsymbol{\psi}(\mathbf{y}^i) \neq \boldsymbol{\psi}(\mathbf{y}^j)$ holds

[10] Note that $Y^{\text{PRO}}_{\text{eff}}(\hat{q}) \cup Y^{\text{PRO}}_{\text{eff}}(\tilde{q}) \subseteq Y^{\boldsymbol{\psi}}_{\text{eff}}$

[11] i.e. $(0.48, 18, 40) \preceq (0.81, 18, 40)$

Algorithm 1 3-stage method

Output: $Y^{\text{PRO}}_{\text{FRE}}, Y^{\text{PRO}}_{\text{HRE}}, Y^{\text{PRO}}_{\text{r-light}}(Y^{\text{PRO}}_{\text{eff}}(\hat{q}), \varepsilon, \kappa)$

$Y^{\text{PRO}}_{\text{eff}}(\hat{q}), Y^{\text{PRO}}_{\text{eff}}(\tilde{q}) \leftarrow$ `Lex-BiObjective` ▷ Stage 1: Section 4.2
`Filter` $Y^{\text{PRO}}_{\text{HRE}}$ and $Y^{\text{PRO}}_{\text{FRE}}$ ▷ Stage 2
for $\hat{\mathbf{y}} \in Y^{\text{PRO}}_{\text{eff}}(\hat{q})$ **do** ▷ Stage 3 starts
  minValue $= \infty$, minSol $= \emptyset$
  for $\tilde{\mathbf{y}} \in Y^{\text{PRO}}_{\text{eff}}(\tilde{q})$ **do**
    if $\big((g(\hat{\mathbf{y}}, \hat{q}) \leq g(\tilde{\mathbf{y}}, \hat{q}) \leq g(\hat{\mathbf{y}}, \hat{q}) + \varepsilon) \wedge (v(\tilde{\mathbf{y}}, \hat{\mathbf{y}}) \geq \kappa)\big)$ **then**
      if minValue $> g_2(\tilde{\mathbf{y}}, \tilde{q})$ **then**
        minValue $= g_2(\tilde{\mathbf{y}}, \tilde{q})$
        minSol $= \{\tilde{\mathbf{y}}\}$
      end if
    end if
  end for
  if minValue $\neq \infty$ **then**
    $Y^{\text{PRO}}_{\text{r-light}}(Y^{\text{PRO}}_{\text{eff}}(\hat{q}), \varepsilon, \kappa) \cup$ minSol
  else
    $Y^{\text{PRO}}_{\text{r-light}}(Y^{\text{PRO}}_{\text{eff}}(\hat{q}), \varepsilon, \kappa) \cup \{\varepsilon$-`Search`$(\hat{\mathbf{y}})\}$ ▷ Alg. 2 and Sec. 4.3
  end if
end for

4.2 Finding nominal and worst-case PRO-RE solutions: `Lex-BiObjective`

In the first stage of Alg. 1 two bi-objective optimization problems are solved, to find $Y^{PRO}_{\text{eff}}(\hat{q})$ and $Y^{PRO}_{\text{eff}}(\tilde{q})$, respectively. The corresponding bi-objective optimization problem is denoted as $\mathcal{P}^{\text{PRO}}(q)$ and is defined as

$$\min_{\mathbf{y} \in \mathbf{Y}^{\psi}_{\text{eff}}} \big(g_1(\mathbf{y}), g_2(\mathbf{y}, q)\big), \quad q \in \mathcal{Q}. \tag{19}$$

The two minimization problems in (19) corresponding to $\hat{q}$ and $\tilde{q}$ are bi-objective mixed-integer programming problems;[12] any efficient solution to the two is PRO-RE, as per Def. 7. Thus, a non-dominated point must possess a minimal value w.r.t. the other objective (i.e. $g_2(., \tilde{q})$ for $\mathcal{P}^{\text{PRO}}(\hat{q})$ and $g_2(., \hat{q})$ for $\mathcal{P}^{\text{PRO}}(\tilde{q})$).

A scalarized problem within an ε-constraint method to solve an instance of $\mathcal{P}^{\text{PRO}}(r)$, for $r \in \mathcal{Q}$, is given by

$$\mathbf{y}(r) \in \operatorname*{lexmin}_{\mathbf{y} \in Y} \big\{g_1(\mathbf{y}) + w(r)\, g_2(\mathbf{y}, r), g_2(\mathbf{y}, u) \,:\, g_2(\mathbf{y}, r) \leq \lambda\big\}, \tag{20}$$

where $u \in \mathcal{Q} \setminus r$, the constant λ is an upper bound used to explore the criterion space during each iteration of the ε-constraint method (updated in each iteration), and $w(r) > 0$ is appropriately small to ensure at least a weakly efficient solution w.r.t. g_1 and $g_2(\mathbf{y}, r)$ (see [22]). Lexicographic minimization (lexmin) is performed to obtain a PRO-RE solution (for details on lexicographic minimization, see [10, Sect. 5.1]).

[12] Since $\mathbf{x}$, $\mathbf{s}$, and $\mathbf{z}$ are integral, in an efficient solution, the variables $\mathbf{n}$ will take a finite number of discrete values. Hence, the efficient frontier contains a finite number of non-dominated points and no continuous line segments

4.3 Stage 3: Search for positive robustness ε-representative lightly PRO-RE solutions

Stage 3 of Alg. 1 searches for solutions in the set $Y^{PRO}_{\mathrm{r-light}}(\hat{\mathbf{y}}, \boldsymbol{\varepsilon}, \kappa)$, where $\hat{\mathbf{y}} \in Y^{PRO}_{\mathrm{eff}}(\hat{q})$. The details of stage 3 are mentioned in Alg. 1. It starts by checking if any of the solutions in $Y^{PRO}_{\mathrm{eff}}(\tilde{q})$ (already computed in stage 1) have positive robustness; such solutions are stored. Figure 6 shows an instance for which some of the efficient solutions for the worst-case are in the set $Y^{PRO}_{\mathrm{r-light}}(\hat{\mathbf{y}}, \boldsymbol{\varepsilon}, \kappa)$. Further, two solutions in $Y^{PRO}_{\mathrm{eff}}(\tilde{q})$ are not in the set $Y^{PRO}_{\mathrm{r-light}}(Y^{PRO}_{\mathrm{eff}}(\hat{q}), \boldsymbol{\varepsilon}, \kappa)$, due to not having positive robustness.

For a given $\hat{\mathbf{y}} \in Y^{PRO}_{\mathrm{eff}}(\hat{q})$ (identified in stage 1 of Alg. 1) if none of the solutions in $Y^{PRO}_{\mathrm{eff}}(\tilde{q}) \cap Y_{\mathrm{nb}}(\hat{y}, \boldsymbol{\varepsilon})$ have positive robustness, the following optimization problem is solved:

$$\mathbf{y}^* \in \underset{\mathbf{y} \in \mathbf{Y}_{\mathrm{nb}}(\hat{\mathbf{y}}, \boldsymbol{\varepsilon})}{\arg\min} \{ g_2(\mathbf{y}, \tilde{q}) \mid v(\mathbf{y}, \hat{\mathbf{y}}) \geq \kappa \}, \tag{21}$$

where $0 < \kappa \ll 1$ and $v(\mathbf{y}, \hat{\mathbf{y}})$ corresponds to positive robustness (see Def. 8). If the model (21) is feasible, the optimal solution $\mathbf{y}^*$ can be used to identify an efficient solution (a result of [6, Prop. 5]) in the set $Y_{\mathrm{nb}}(\hat{\mathbf{y}}, \boldsymbol{\varepsilon})$ w.r.t. $\boldsymbol{\psi}(\cdot) = (g_1(\cdot), g_2(\cdot, \hat{q}), g_2(\cdot, \tilde{q}))$ by solving

$$\bar{\mathbf{y}} \in \underset{\mathbf{y} \in \mathbf{Y}_{\mathrm{nb}}(\hat{\mathbf{y}}, \boldsymbol{\varepsilon})}{\arg\min} \left\{ \sum_{i=1}^{3} \psi_i(\mathbf{y}) \,\middle|\, \psi_3(\mathbf{y}) \leq \psi_3(\mathbf{y}^*), v(\mathbf{y}, \hat{\mathbf{y}}) \geq \kappa \right\}. \tag{22}$$

Constraint generation. To check if the optimal solution to (22) is indeed PRO-RE, i.e. non-dominated w.r.t. $\boldsymbol{\psi}$ but in the entire feasible set Y, the following optimization model is solved (a result of [2]) for the first iteration (of the while loop in Alg. 2; i.e. $f = 1$).

$$(\boldsymbol{\ell}^1, \mathbf{y}^1) \in \mathrm{argmax}_{\boldsymbol{\ell} \in \mathbb{R}^3_+,\, \mathbf{y} \in Y} \{ \mathbf{1}^\top \boldsymbol{\ell} \mid \psi_i(\bar{\mathbf{y}}) - \ell_i - \psi_i(\mathbf{y}) = 0,\ i \in \{1, 2, 3\} \}. \tag{23}$$

If the solution to (23) from the first iteration of constraint generation is such that $\boldsymbol{\ell}^1 = \mathbf{0}^3$, then the solution $\bar{\mathbf{y}}$ is PRO-RE; otherwise, the solution $\mathbf{y}^1$ (for $f = 1$) fulfills $\boldsymbol{\psi}(\mathbf{y}^1) \preceq \boldsymbol{\psi}(\bar{\mathbf{y}})$. In the latter case, the variables $m_{if} \in \{0, 1\}$ and the constraints (24) are added to the models (21) and (22), for the k^{th} iteration of the constraint generation:

$$\psi_i(\mathbf{y}) + \delta_i \leq m_{if} \psi_i(\mathbf{y}^f), \quad i \in \{1, 2, 3\}, f \in \{1, \dots, k\}, \tag{24a}$$

$$\sum_{i=1}^{3} m_{if} \geq 1, \quad f \in \{1, \dots, k\}, \tag{24b}$$

$$m_{if} \in \{0, 1\}, \quad i \in \{1, 2, 3\}, f \in \{1, \dots, k\}, \tag{24c}$$

where $\delta_1 = 0.005$, $\delta_2 = \delta_3 = 1$, and $\mathbf{y}^f$ is the optimal solution to (23) in the f^{th} iteration.[13] This ensures that a feasible/optimal solution of (21) and (22) is neither weakly nor strictly dominated by any point corresponding to $\{\mathbf{y}^f\}_{f=1,\dots,k}$. The constraint generation continues until the model (21) becomes infeasible or (23) yields a solution with $\boldsymbol{\ell}^{k+1} = \mathbf{0}^3$.

[13] (a) The function $g_1(\cdot)$ (or $\psi_1(\cdot)$) is real valued; smallest decrease can be safely approximated by $\delta_1 = .005$, since other two functions are integer valued, and $\delta_2 = \delta_3 = 1$; (b) at iteration $k + 1$, it implies that $\boldsymbol{\ell}^f \neq \mathbf{0}^3$ holds for $f \in \{1, \dots, k\}$ in the constraint generation procedure.

Algorithm 2 ε-`Search` $(\hat{\mathbf{y}})$

Output: $Y^{\text{PRO}}_{\text{r-light}}(\hat{\mathbf{y}}, \varepsilon, \kappa)$

$k = 1$
while True **do**
  $\mathbf{y}^* \leftarrow$ `solve` model (21)
  if optimal **then**
    $\bar{\mathbf{y}} \leftarrow$ `solve` model (22)
  else
    `break` $Y^{\text{PRO}}_{\text{r-light}}(\hat{\mathbf{y}}, \varepsilon, \kappa) = \emptyset$
  end if
  $(\boldsymbol{\ell}^k, \mathbf{y}^k) \leftarrow$ `solve` model (23) and store $\mathbf{y}^k$
  if $\boldsymbol{\ell}^k = \mathbf{0}^3$ **then**
    $Y^{\text{PRO}}_{\text{r-light}}(\hat{\mathbf{y}}, \varepsilon, \kappa) = \bar{\mathbf{y}}$
    `break`
  else ▷ constraint generation
    add constraints and variables to (21) and (22), according to (24)
    $k = k + 1$
  end if
end while

5 Tests and results

To investigate the computational efficiency of our proposed approach, we generated 30 instances, which are expected to represent some of the possible realizations of actual data that the model might encounter. Hence, allowing for confident conclusions about the benefits of including worst-case scenarios and computational performance of our proposed approach.

All the computations are done in `Python 3.7` using `Gurobi 9` on a system with 1.70GHz processor, 16 GB RAM, and 4 cores. We set a time limit of 5000 seconds for each scalarized optimization problem, and a global time limit on the algorithm is set to 20000 seconds for the entire process. We also terminate if the MIP duality gap is less than 0.05%. Another termination criterion records and updates the best incumbent solution after every 200 node explorations; if there is no improvement in the optimality gap, the solver terminates the process. However, this termination criterion is only activated after 1500 seconds. For a fair comparison these termination criteria are implemented for all the algorithms used to benchmark against our proposed approach. Moreover, all computed sets reported are *minimal sets*.

5.1 Industrial instances

We have used real industrial data for most of the parameters and sets indicated in Table 2. However, processing times p_{jk} of job type $j \in \mathcal{J}$ in machines $k \in \mathcal{N}_j$ (qualification required), and the qualification cost parameters β^q_{jk}, $j \in \mathcal{J}$, $k \in \mathcal{N}_j$, $q \in \mathcal{Q}$, were not available. In order to generate instances which represent possible realizations of the actual data we introduce the following distributions, which are based on knowledge of the managers.

Skewness of processing times. The *skew normal distribution* is a generalized normal distribution allowing for non-zero skewness; [31]. We generate processing times p_{jk}, $k \in \mathcal{N}_j$, $j \in \mathcal{J}$, for newly qualified machines from three differently skewed normal distributions with mean μ and skewness/shape parameter α: positive skew ($\alpha > 0$), negative skew ($\alpha < 0$), and zero skew ($\alpha = 0$). A location parameter/mean μ is based on the expected processing time of a given job type j on an already qualified machine $k \in \mathcal{K}_j \setminus \mathcal{N}_j$ and which is similar to that of the machine being qualified. For all these distributions, we set $\sigma := 0.1\mu$; according to the internal statistical process control data (and managerial experience) processing times of newly qualified allocations have a standard deviation of 10% of the expected value.

Range of values for the qualification cost: The exact cost for qualifying a machine for a job type is not known a priori, and for prediction the engineering team has to spend time on simulations. Hence, the input received is the so-called *cost levels*, assigned to each qualification. For testing our model and proposed modifications, we use two sets of cost levels: $\mathcal{H} = \{1, \dots, \beta_{\max}\}$, with $\beta_{\max} \in \{20, 50\}$. The qualification costs are selected from different discrete distributions over the discrete domain $\mathcal{H}$.

Nominal qualification cost. Letting π_h be the frequency of cost level $h \in \mathcal{H}$, its relative frequency is $\hat{\pi}_h := (\sum_{i \in \mathcal{H}} \pi_i)^{-1} \pi_h$; we also define $\hat{\pi}_0 = 0$. To determine a cost $\beta^{\hat{q}}_{jk}$, a sample α is drawn from the interval [0, 1]. Then,

$$\beta^{\hat{q}}_{jk} := \begin{cases} h \in \mathcal{H} : \sum_{i=0}^{h-1} \hat{\pi}_i \leq \alpha < \sum_{i=0}^{h} \hat{\pi}_i, & \alpha \in [0, 1), \\ |\mathcal{H}|, & \alpha = 1. \end{cases}$$

The frequency distributions are defined as follows. For each $h \in \mathcal{H}$, $\pi_h = 1$ (`Uniform`), $\pi_h = h$ (`Right`), $\pi_h = |\mathcal{H}| - (h-1)$ (`Left`), $\pi_h = \min\{h; |\mathcal{H}| - (h-1)\}$ (`Symmetric`), and $\pi_h = \min\{h; |\mathcal{H}| - (h-1); \max\{h - \lceil \frac{|\mathcal{H}|-1}{2} \rceil; \lfloor \frac{|\mathcal{H}|+1}{2} \rfloor - (h-1)\}\}$ (`Bimodal`). For the worst-case values $\beta^{\tilde{q}}_{jk}$ a random integer value is drawn from a uniform discrete distribution over the domain [0,10], and added to the nominal qualification cost $\beta^{\hat{q}}_{jk}$.

5.2 Constant data for the 30 instances

The demand is from quarterly forecasts made at GKN Aerospace in January 2015 (J_{15}) for the period 2016–2017. The minimum, maximum, and median values of the demand are 1, 172, and 11, respectively. For processing times $p_{jk}, j \in \mathcal{J}, k \in \mathcal{K}_j$, the minimum, maximum, and median are 0.1, 89.7, and 5.63 hours, respectively. Each machine has a yearly capacity of 5000 hours which can be equally divided among four quarters in a year. The planning period of two years with quarterly time buckets yields $T = 8$ time periods. There are $K = 125$ machines, and about $J = 510$ unique job types, each having integral demand per time period. The number of possible assignments of job types to machines during the entire planning period thus amounts to $\sim 10^5$. The parameter values $\tau = 3$, $\gamma = 4$, and $\zeta_k = 0.7$, $k \in \mathcal{K}$, are used for all 30 instances. The instances are denoted as $(\bar{\beta}, \bar{p}, \beta_{\max})$, where $\bar{\beta} \in \{\text{Left}, \text{Right}, \text{Symmetric}, \text{Uniform}, \text{Bimodal}\}$ (representing distributions for the qualification costs), $\bar{p} \in \{\text{skew+}, \text{skew−}, \text{skew0}\}$ (representing distributions for processing times p_{jk}, $k \in \mathcal{N}_j$), and lastly, $\beta_{\max} \in \{20, 50\}$. The details of each instance are indicated in the captions of the subfigures of Fig. 11. All 30 public instances are available at https://www.shorturl.at/uGVX3.

5.3 Results

We present two main results, the first regarding the difference of the areas of the worst-case and nominal scenario approximations (Def. 9). The second result concerns the benefits of using the 3-stage method as opposed to the approach of [21] for identifying relevant PRO-RE solutions.

5.3.1 Difference between worst-case and nominal scenario approximations

Figure 7 shows the values of the differences (Δ_{diff}, cf. Def. 10) on the left axis, while the right axis (orange) indicates the number of positive robustness ε-representative lightly PRO-RE solutions. There are two main conclusions to be drawn: (a) among the 18 instances possessing a positive $\Delta_{\text{diff}} > 0$ (instances 21, 17, 6, 2, 14, 18, 30, 29, 4, 10, 23, 20, 26, 5, 14, 15, 27, and 22), 17 instances have $|Y^{\text{PRO}}_{\text{r-light}}(Y^{\text{PRO}}_{\text{eff}}(\hat{q}), \boldsymbol{\varepsilon}, \kappa)| > 0$, and only instance 14 has $|Y^{\text{PRO}}_{\text{r-light}}(Y^{\text{PRO}}_{\text{eff}}(\hat{q}), \boldsymbol{\varepsilon}, \kappa)| = 0$. Hence, $\Delta_{\text{diff}} > 0$ indicates that there may exist solutions that are good replacements for some of the efficient solutions in the nominal scenario (i.e. $|Y^{\text{PRO}}_{\text{r-light}}(Y^{\text{PRO}}_{\text{eff}}(\hat{q}), \boldsymbol{\varepsilon}, \kappa)| > 0$); (b) when $\Delta_{\text{diff}} \leq 0$, which is the case for remaining twelve instances (the twelve left-most in Fig. 7), only four instances (16, 11, 1, and 7) have solutions with positive robustness values ($|Y^{\text{PRO}}_{\text{r-light}}(Y^{\text{PRO}}_{\text{eff}}(\hat{q}), \boldsymbol{\varepsilon}, \kappa)| > 0$). A possible explanation is that Δ_{diff} is an aggregate measure, hence these instances have positive robustness values even though $\Delta_{\text{diff}} \leq 0$ (see Fig. 11). Hence, Fig. 7 indicates that the worst-case scenario does have an effect for a significant proportion of the instances of the TRAP. Note that the values of $\boldsymbol{\varepsilon}$ influence the value of $|Y^{\text{PRO}}_{\text{r-light}}(Y^{\mathbf{g}}_{\text{eff}}(\hat{q}), \boldsymbol{\varepsilon}, \kappa)|$. For our problem it is based on the range of values for the two objective functions ($\varepsilon_1 = 0.12 \cdot (g_1^{\text{BOT}}(\hat{q}) - g_1^{\text{TOP}}(\hat{q}))$ and $\varepsilon_2 = 0.12 \cdot (g_2^{\text{TOP}}(\hat{q}) - g_2^{\text{BOT}}(\hat{q}))$).

5.3.2 Solution time comparisons

We compare the solution times obtained from using 3-stage method (Alg. 1) with the results obtained if algorithm from [21] is used for the 30 instances. As mentioned in Sect. 4 §2 the first step in [21] requires a tri-objective optimization. For this purpose a tri-objective criterion space search method called Quadrant Shrinking Method (QSM) [6] is used. We implemented the QSM [6, Alg. 1, p. 877] in `Python 3.7` using the data structures suggested in [6].[14] The QSM generates all the PRO-RE solutions, whereas our 3-stage method identifies only the PRO-RE solutions required by the DMs (i.e. $Y^{\text{PRO}}_{\text{FRE}}$, $Y^{\text{PRO}}_{\text{r-light}}(Y^{\text{PRO}}_{\text{eff}}(\hat{q}), \boldsymbol{\varepsilon}, \kappa)$, and $Y^{\text{PRO}}_{\text{HRE}}$). Both algorithms identify the same number of solutions in the sets $Y^{\text{PRO}}_{\text{FRE}}$, $Y^{\text{PRO}}_{\text{HRE}}$, and—most importantly—$Y^{\text{PRO}}_{\text{r-light}}(Y^{\text{PRO}}_{\text{eff}}(\hat{q}), \boldsymbol{\varepsilon}, \kappa)$.

Figure 8 highlights two conclusions. (a) For 17 out of the 30 instances the number $|Y^{\Psi}_{\text{eff}}|$ of PRO-RE solutions identified by the two methods are equal. For 15 out of these 17 instances our 3-stage method is computationally superior. This validates (empirically) that solving two bi-objective optimization problems is faster than solving one tri-objective optimization problem for the given instances of the TRAP. (b) For the remaining 13 instances the two methods found different numbers of PRO-RE solutions; only for instance 11 the

[14] Numerous modern implementations of tri-objective criterion space search methods are proposed in the literature, but we confine ourselves with just one of the good ones, others to be found in, e.g. [25] and [19]

Table 4 Summary of set notations

Set	Description	Ref.
Y	Feasible set corresponding to the TRAP	(3)
$Y^{\mathbf{g}}_{\mathrm{eff}}(q)$	Set of efficient solutions in scenario q for objective functions $\mathbf{g}(\cdot,q) := (g_1, g_2(\cdot,q))$,	Def. 2
Y_{FRE}	$= \bigcup_{q\in\mathcal{Q}} Y^{\mathbf{g}}_{\mathrm{eff}}(q)$; set of flimsily robust efficient solutions	Def. 3
Y_{HRE}	$= \bigcap_{q\in\mathcal{Q}} Y^{\mathbf{g}}_{\mathrm{eff}}(q)$; set of highly robust efficient solutions	Def. 4
$Y_{\mathrm{nb}}(\hat{\mathbf{y}},\boldsymbol{\varepsilon})$	$= \{\, \mathbf{y}\in Y \,:\, \mathbf{g}(\hat{\mathbf{y}},\hat{q}) \leqq \mathbf{g}(\mathbf{y},\hat{q}) \leqq \mathbf{g}(\hat{\mathbf{y}},\hat{q}) + \boldsymbol{\varepsilon} \,\}$; ε-neighborhood of $\hat{\mathbf{y}} \in Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q})$; $\boldsymbol{\varepsilon}\in\mathbb{R}^2_+$	(9)
$Y_{\mathrm{light}}(\hat{\mathbf{y}},\boldsymbol{\varepsilon})$	Set of efficient solutions to $\min_{\mathbf{y}\in Y_{\mathrm{nb}}(\hat{\mathbf{y}},\boldsymbol{\varepsilon})} \mathbf{g}(\mathbf{y},\tilde{q})$	Def. 5
$Y_{\mathrm{light}}(Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}),\boldsymbol{\varepsilon})$	$= \bigcup_{\hat{\mathbf{y}}\in Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q})} Y_{\mathrm{light}}(\hat{\mathbf{y}},\boldsymbol{\varepsilon})$; set of all ε-lightly RE solutions	Def. 5
$Y_{\mathrm{r-light}}(\hat{\mathbf{y}},\boldsymbol{\varepsilon})$	$= \mathrm{argmin}_{\mathbf{y}\in \mathbf{Y}_{\mathrm{nb}}(\hat{\mathbf{y}},\boldsymbol{\varepsilon})} g_2(\mathbf{y},\tilde{q})$	Def. 6
$Y_{\mathrm{r-light}}(Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}),\boldsymbol{\varepsilon})$	$= \bigcup_{\hat{\mathbf{y}}\in Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q})} Y_{\mathrm{r-light}}(\hat{\mathbf{y}},\boldsymbol{\varepsilon})$; set of ε-representative lightly RE solutions	Def. 6
$Y^{\boldsymbol{\psi}}_{\mathrm{eff}}$	set of efficient solutions with objective functions $\boldsymbol{\psi} := (g_1(\cdot), g_2(\cdot,\hat{q}), g_2(\cdot,\tilde{q}))$; also called PRO-RE solutions	Def. 7
Y^{PRO}_i	$= Y_i \cap Y^{\boldsymbol{\psi}}_{\mathrm{eff}}$; $i\in\{\mathrm{FRE},\mathrm{HRE}\}$	(13)
$Y^{PRO}_j(Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}),\boldsymbol{\varepsilon})$	$= Y_j(Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}),\boldsymbol{\varepsilon}) \cap Y^{\boldsymbol{\psi}}_{\mathrm{eff}}$; $j\in\{\mathrm{light},\mathrm{r\text{-}light}\}$	(13)
$Y^{PRO}_{\mathrm{eff}}(q)$	$= Y^{\mathbf{g}}_{\mathrm{eff}}(q) \cap Y^{\boldsymbol{\psi}}_{\mathrm{eff}}$	(14)
$Y_{\mathrm{r-light}}(\hat{\mathbf{y}},\boldsymbol{\varepsilon},\kappa)$	$= \mathrm{argmin}_{\mathbf{y}\in \mathbf{Y}_{\mathrm{nb}}(\hat{\mathbf{y}},\boldsymbol{\varepsilon})} \{ g_2(\mathbf{y},\tilde{q}) \,:\, v(\mathbf{y},\hat{\mathbf{y}}) \ge \kappa \}$; $\hat{\mathbf{y}}\in Y^{\mathbf{g}}_{\mathrm{eff}}(q)$; $v(\mathbf{y},\hat{\mathbf{y}}) = [g_2(\hat{\mathbf{y}},\tilde{q}) - g_2(\mathbf{y},\tilde{q})] - [g_2(\mathbf{y},\hat{q}) - g_2(\hat{\mathbf{y}},\hat{q})]$	(15)
$Y_{\mathrm{r-light}}(Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}),\boldsymbol{\varepsilon},\kappa)$	$= \bigcup_{\hat{\mathbf{y}}\in Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q})} Y_{\mathrm{r-light}}(\hat{\mathbf{y}},\boldsymbol{\varepsilon},\kappa)$; set of positive robustness ε-representative lightly RE solutions	Def. 8
$Y^{PRO}_{\mathrm{r-light}}(Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}),\boldsymbol{\varepsilon},\kappa)$	$= Y_{\mathrm{r-light}}(Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}),\boldsymbol{\varepsilon},\kappa) \cap Y^{\boldsymbol{\psi}}_{\mathrm{eff}}$	(16)

QSM (tri-objective) method found fewer PRO-RE solutions than the 3-stage method.[15] Our approach does not find all the PRO-RE solutions, but manages to find all those that are required by the DMs in significantly less time than the solution approach mentioned in [21] (using the QSM in [6] for tri-objective optimization).

5.4 Computational complexity

We compare the computational complexity of the algorithm in [21, Sect. 6] with that of our 3-stage method Alg. 1;[16] see Table 5 for a summary.

The method of [21] (see Sect. 4 §2) involves two main steps: (i) compute $Y^{\boldsymbol{\psi}}_{\mathrm{eff}}$; (ii) filter solutions into the sets of flimsily, highly, and ε-representative lightly PRO-RE solutions. Computing $Y^{\boldsymbol{\psi}}_{\mathrm{eff}}$ requires the solution of $O(3|Y^{\boldsymbol{\psi}}_{\mathrm{eff}}| + 1)$ single-objective scalarized MILPs[17] [6, Thm. 24]; the filtering requires $O(|Y^{\boldsymbol{\psi}}_{\mathrm{eff}}| \cdot |\mathcal{Q}| \cdot (\log|Y^{\boldsymbol{\psi}}_{\mathrm{eff}}| + 1) + |Y^{\boldsymbol{\psi}}_{\mathrm{eff}}|^2)$ elementary operations [21, Sect. 6].

15 The solution time limit of 20000 seconds was reached for instance 11

16 For the set notations, see Table 4

17 Using the QSM method

Alg. 1 involves three stages: In stage 1 the sets $Y^{PRO}_{\text{eff}}(q)$, $q \in \mathcal{Q}$, are computed, i.e. $|\mathcal{Q}|$ bi-objective optimization problems are solved; the use of an ϵ-constraint method then leads to solving $O(2\sum_{q\in\mathcal{Q}} |Y^{PRO}_{\text{eff}}(q)| + |\mathcal{Q}|)$ single-objective MILPs.[18] (see [9]). Stage 2 filters the sets Y^{PRO}_{FRE} and Y^{PRO}_{HRE}, by identifying the intersection and union, respectively, of the sets $Y^{PRO}_{\text{eff}}(q)$, $q \in \mathcal{Q}$; since both procedures utilize a sorting step, the total number of elementary operations is $O(\sum_{q\in\mathcal{Q}}(|Y^{PRO}_{\text{eff}}(q)| \cdot \log |Y^{PRO}_{\text{eff}}(q)|))$. Stage 3 makes, say b, calls to Alg. 2, each of which comprising an iterative algorithm that requires solving $\phi \in [1, |Y_{\text{nb}}(\hat{\mathbf{y}}, \boldsymbol{\epsilon})|]$ MILPs (if $\boldsymbol{\epsilon} \neq \mathbf{0}$ else $\phi = 0$), where $\hat{\mathbf{y}} \in Y^{\mathbf{g}}_{\text{eff}}(\hat{q})$; for fairly small values of $\boldsymbol{\epsilon}$, the upper bound on ϕ is not expected to be large (see (9)).

Obtaining estimates of b and ϕ (number of calls to Alg. 2 and iterations, respectively) is quite tricky. These measures are illustrated for our instances in Fig. 9, with the values of b on the left axis, and the share of the total solution time of Alg. 1 spent on stages 1 and 2 on the right axis. For 15 out of the 30 instances, there are no calls to Alg. 2 (hence, the share equals 1). For 25 instances the share is greater than 0.9. However, for a few instances (6, 11, 18, 23, 28, and 30) the calls to Alg. 2 (i.e. stage 3) constitute a significant portion of the solution time. E.g. for instance #11, stages 1 and 2 use only 56.55% of the total time, the remaining 43.45% being spent on calls to Alg. 2 (stage 3).

6 Conclusion

We propose an approach for solving robust bi-objective optimization problems, with one uncertain objective function including two well-defined scenarios; a nominal (most likely) and a worst-case scenario. We present a new measure to assess the net gain from the inclusion of the worst-case scenario.

For understanding the general applicability, we discuss the two ideas separately. We first consider the positive robustness ϵ-representative lightly robust efficient (RE) concept, which is similar to that of ϵ-representative lightly RE, except that the former satisfies an additional constraint (cf. (15)). As ϵ-representative lightly RE solutions [21], our proposed RE solutions apply to MOOPs with one uncertain objective function and any number of scenarios as long as nominal and worst-case scenarios are well-defined. Second, the difference between worst-case and nominal scenario approximations can be used to filter out instances that are expected not to gain significantly from the addition of a worst-case scenario, and thus avoiding the need to find solutions in the set $Y^{PRO}_{\text{r-light}}(Y^{PRO}_{\text{eff}}(\hat{q}), \boldsymbol{\epsilon}, \kappa)$ (Def. 8). Our proposed measure Δ_{diff} (Def. 10) is applicable to bi-objective optimization problems, with one uncertain objective function having any number of scenarios, provided that the worst-case and nominal scenarios are well-defined.

Extensions and future research. For most of our 30 instances, the computation time is too long (see details at https://www.shorturl.at/nrIJQ) and may discourage decision-makers from performing multiple what-if/sensitivity analyses. Hence, in-exact computing approaches could be investigated, especially if more scenarios are included. There is also a potential for generalizing our concepts to general robust multi-objective problems with several uncertain objectives.

[18] Note that the inequality $\sum_{q\in\mathcal{Q}} |Y^{PRO}_{\text{eff}}(q)| \leq |Y^{\Psi}_{\text{eff}}|$ follows from (14)

Appendix

Practical significance

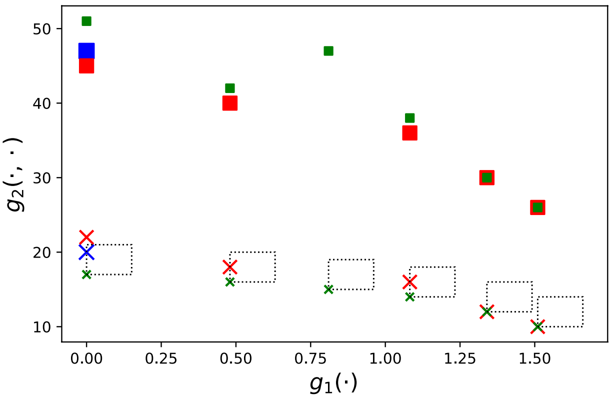
We here investigate the effect of using our proposed positive robustness ε-representative lightly PRO-RE solutions, instead of nominal PRO-RE or worst-case PRO-RE solutions. We assume two scenarios of interest, i.e. $\mathcal{Q} = \{\hat{q}, \tilde{q}\}$, and consider three different policies. The nominal (nom) and worst-case (wc) policies use solutions in the sets $Y^{PRO}_{\text{eff}}(\hat{q})$ and $Y^{PRO}_{\text{eff}}(\tilde{q})$, respectively. The P-$\varepsilon$ policy uses the set $Y^{\text{PRO}}_{\text{r-light}}(Y^{PRO}_{\text{eff}}(\hat{q}), \boldsymbol{\varepsilon}, \kappa)$, $\kappa > 0$ and $\boldsymbol{\varepsilon} > \mathbf{0}$ being user-defined.

A summary of the effects of applying the three policies to instance 18 is provided in Table 6 (see Fig. 11xvi). First, we assess the effectiveness of wc and P-ε policies when the realized scenario is $\hat{q}$ (i.e. nom), and analogously, the nom and P-ε policies when the realized scenario is $\tilde{q}$ (i.e. wc). The *closest point* to $\hat{\mathbf{y}} \in Y^{\text{PRO}}_{\text{eff}}(\hat{q})$ in the set $Y^{\text{PRO}}_{\text{eff}}(\tilde{q})$ is defined as $\mathbf{y}^*(\hat{\mathbf{y}}) := \arg\min_{\mathbf{y} \in \mathbf{Y}^{\text{PRO}}_{\text{eff}}(\tilde{\mathbf{q}})} ||\bar{\boldsymbol{\psi}}(\mathbf{y}) - \bar{\boldsymbol{\psi}}(\hat{\mathbf{y}})||_\infty$, where $\bar{\boldsymbol{\psi}}(\mathbf{y}) := (\bar{g}_1(\mathbf{y}), \bar{g}_2(\mathbf{y}, \hat{q}), \bar{g}_2(\mathbf{y}, \tilde{q}))$, $\bar{g}_1(\mathbf{y}) := g_1(\mathbf{y}) / \max\left\{g_1^{\text{BOT}}(\hat{q}), g_1^{\text{BOT}}(\tilde{q})\right\}$, and $\bar{g}_2(\mathbf{y}, q) := g_2(\mathbf{y}, q) / g_2^{\text{TOP}}(q)$, $q \in \{\hat{q}, \tilde{q}\}$ (the objectives of (18) being normalized). The closest point to $\tilde{\mathbf{y}} \in Y^{\text{PRO}}_{\text{eff}}(\tilde{q})$ in the set $Y^{\text{PRO}}_{\text{eff}}(\hat{q})$ is analogously defined as $\mathbf{y}^*(\tilde{\mathbf{y}}) := \arg\min_{\mathbf{y} \in \mathbf{Y}^{\text{PRO}}_{\text{eff}}(\hat{\mathbf{q}})} ||\bar{\boldsymbol{\psi}}(\mathbf{y}) - \bar{\boldsymbol{\psi}}(\tilde{\mathbf{y}})||_\infty$. The idea of closest point is based on the concept *price of robustness* [28, Def. 14].

All the non-dominated points in the nominal scenario are given in the first column of Table 6. The wc policy implies a swap of (0, 14, 54) (the first green-cross and square from the left) with (0, 17, 43) (the first red-cross and square from the left is the closest point in $Y^{\text{PRO}}_{\text{eff}}(\tilde{q})$). Hence, the net increase from this swap is 0 for g_1 and 3 for g_2 (as it is known that this is a nominal scenario the qualification cost in the nominal scenario is increased). The net increase in g_1 ($g_2(\cdot, \hat{q})$) is presented in column two (three). The policy P- ε yields a swap if there exists a solution in $Y^{PRO}_{\text{r-light}}(\mathbf{y}, \boldsymbol{\varepsilon}, \kappa)$, where $\boldsymbol{\psi}(\mathbf{y}) = (0, 14, 54)$ and $\boldsymbol{\varepsilon} = (0.15, 4)$. The point (0, 17, 43) satisfies (15) for $\psi(\mathbf{y}) = (0, 14, 54)$. The points corresponding to $Y^{\text{PRO}}_{\text{eff}}(\tilde{q})$ are presented in the sixth column, for the case when the worst-case scenario is realized. Hence, the nom and P-ε policies are compared. For the nom policy we conclude that (0, 14, 54) is the closest point to (0, 17, 43) in $Y^{PRO}_{\text{eff}}(\hat{q})$. The increase in qualification cost equals $54 - 43 = 11$ (the realized scenario being $\tilde{q}$). When the worst-case scenario is realized, the P- ε policy checks whether the closest point to (0, 17, 43) in $Y^{\text{PRO}}_{\text{eff}}(\hat{q})$, i.e. (0, 14, 54) can be swapped with a solution from $Y^{PRO}_{\text{r-light}}(\mathbf{y}, \boldsymbol{\varepsilon}, \kappa)$, where $\boldsymbol{\psi}(\mathbf{y}) = (0, 14, 54)$. The point (0, 17, 43) (first (from left) blue-cross and-square in Fig. 11xvi) is such a point (efficient also in the worst-case scenario). Hence, there is no increase in g_1 and g_2 (see the two last columns of the first row). The resulting swaps are provided in Table 7. The column sums (cf. Table 6) are plotted in Fig. 10, for 21 of our 30 instances such that[19] $Y^{\text{PRO}}_{\text{r-light}}(Y^{PRO}_{\text{eff}}(\hat{q}), \kappa) \neq \emptyset$. For 13 (21) of these 21 instances the P-ε policy is superior to the wc policy in the nominal scenario for g_1 ($g_2(\cdot, \hat{q})$) (Fig. 10i–10ii). For 16 (21) of the 21 instances the P-ε policy is superior to the nom policy in the worst-case scenario for g_1 ($g_2(\cdot, \tilde{q})$) (Fig. 10iii–iv).

[19] When $Y^{PRO}_{\text{r-light}}(Y^{PRO}_{\text{eff}}(\hat{q}), \kappa) = \emptyset$ the P-ε policy is equivalent to using $Y^{\text{PRO}}_{\text{eff}}(\hat{q})$.

Fig. 3 Objective vectors for instance #6. Green/red-cross/square: see Fig. 2; blue-cross: $\mathbf{g}(\mathbf{y}, \hat{q}) : \mathbf{y} \in Y^{PRO}_{\mathrm{r-light}}(Y^{\mathbf{g}}_{\mathrm{eff}}(\hat{q}), \boldsymbol{\varepsilon}, \kappa)$; blue-square: $\mathbf{g}(\mathbf{y}, \tilde{q}) : \mathbf{y} \in Y^{PRO}_{\mathrm{r-light}}(Y^{PRO}_{\mathrm{eff}}(\hat{q}), \boldsymbol{\varepsilon}, \kappa)$



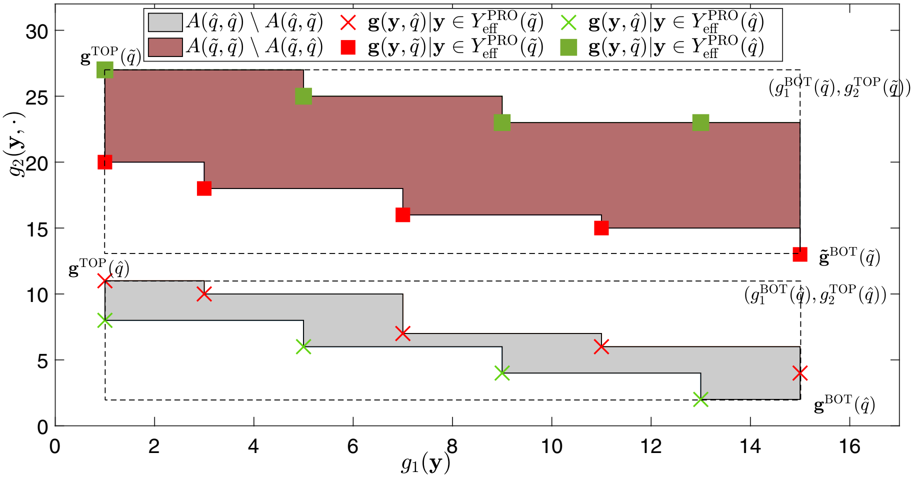
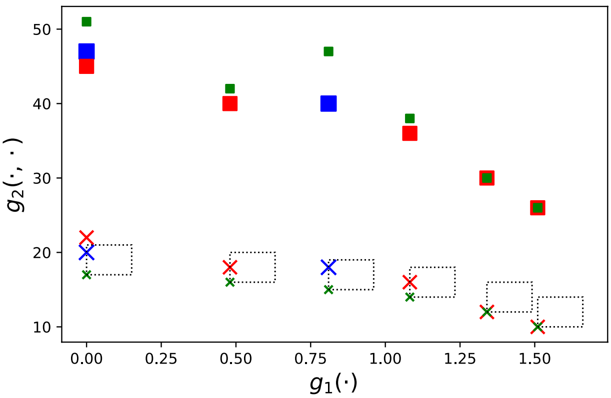


Fig. 4 Approximation graph. The grey- and red-shaded region indicates the nominal and worst-case scenario approximation, respectively

Fig. 5 Objective values for instance #6; cf. Fig. 3. Green/red-cross/square: see Fig. 2; blue-cross: $\mathbf{g}(\mathbf{y}, \hat{q}) : Y_{\mathrm{r-light}}(Y^{PRO}_{\mathrm{eff}}(\hat{q}), \boldsymbol{\varepsilon}, \kappa)$; blue-square: $\mathbf{g}(\mathbf{y}, \tilde{q}) : Y_{\mathrm{r-light}}(Y^{PRO}_{\mathrm{eff}}(\hat{q}), \boldsymbol{\varepsilon}, \kappa)$



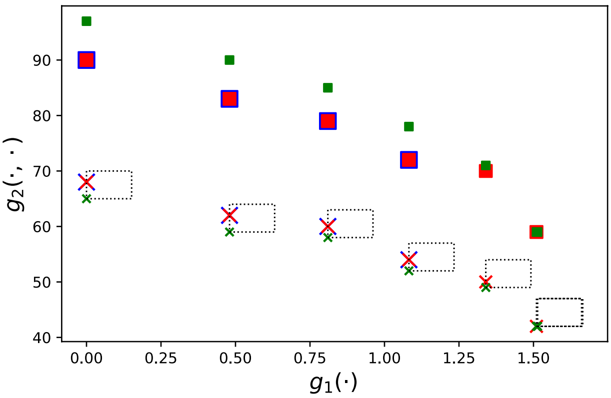


Fig. 6 Objective values for instance #10. For definitions of the markers see the caption of Fig. 3

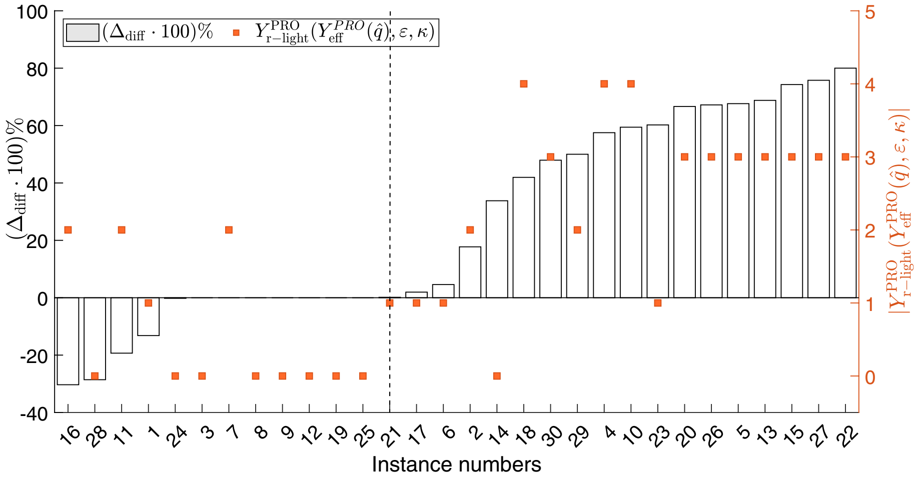


Fig. 7 Left axis: Difference between worst-case and nominal scenario approximations (Δ_{diff}). Right axis: $|Y^{PRO}_{\text{r-light}}(Y^{PRO}_{\text{eff}}(\hat{q}), \varepsilon, \kappa)|$. For all instances to the left of the black dashed line $\Delta_{\text{diff}} \leq 0$. All computed sets are minimal sets

Instances and summary of all set notations

Results for all 30 instances (#1–30) illustrated in Fig. 11, except #6 (L, skew0, 50), #10 (R, skew−, 50), and #19 (S, skew+, 20), which are shown in Fig. 3, 6, and 2 (left), respectively. Notations: L: Left, R: Right, S: Symmetric, U: Uniform, M: Bimodal.

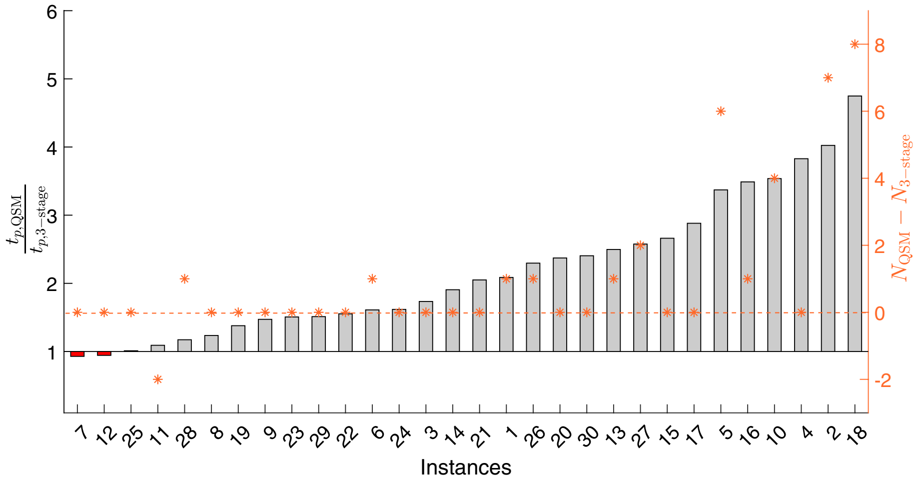


Fig. 8 Left axis: Ratio of solutions times (grey bars; red bars for negative values). Right axis: Difference between the number of PRO-RE solutions ($|Y^{\Psi}_{\mathrm{eff}}|$) identified by the QSM (N_{QSM}) and the 3-stage method ($N_{3\text{-stage}}$) (orange rhombuses). All computed sets are minimal sets

Table 5 Computational complexity. b: number of calls to Alg. 2; ϕ: number of iterations per call

Algorithm	# of Scalarized MILPs	Filtering/additional operationss
[21]	$O(3\|Y^{\Psi}_{\mathrm{eff}}\| + 1)$	$O(\|Y^{\Psi}_{\mathrm{eff}}\| \cdot \|\mathcal{Q}\| \cdot (\log(\|Y^{\Psi}_{\mathrm{eff}}\|) + 1) + \|Y^{\Psi}_{\mathrm{eff}}\|^2)$
Alg. 1	$O(2\|Y^{\Psi}_{\mathrm{eff}}\| + \|\mathcal{Q}\|) + 3b\phi$	$O(\sum_{q\in\mathcal{Q}}(\|Y^{PRO}_{\mathrm{eff}}(q)\| \cdot \log\|Y^{\mathrm{PRO}}_{\mathrm{eff}}(q)\|))$

Fig. 9 For the 30 instances: the number, b, of calls to Alg. 2 (blue); the ratio of solution times of the first two stages and all three stages of Alg. 1 (orange). Note that the calls to Alg. 2 are in stage 3

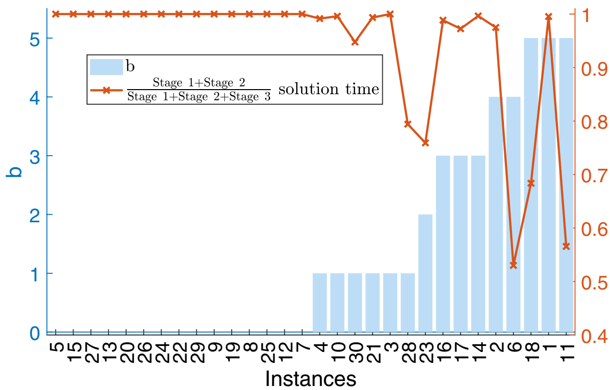
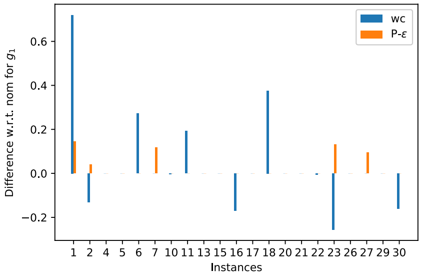


Table 6 Effect of the policies on instance 18

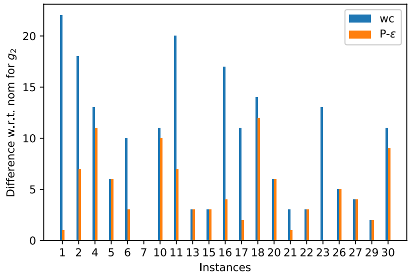
Nominal scenario ($\hat{q}$)					Worst-case scenario ($\tilde{q}$)				
$\boldsymbol{\psi}(Y_{\mathrm{eff}}^{PRO}(\hat{q}))$	wc policy		P-ε policy		$\boldsymbol{\psi}(Y_{\mathrm{eff}}^{PRO}(\tilde{q}))$	nom policy		P-ε policy	
	Δg_1	Δg_2	Δg_1	Δg_2		Δg_1	Δg_2	Δg_1	Δg_2
(0, 14, 54)	0	3	0	3	(0, 17, 43)	0	11	0	0
(0.052, 13, 52)	−.052	4	0	0	(0.48, 15, 39)	0	4	0	0
(0.48, 12, 43)	0	3	0	3	(1.082, 14, 35)	0	4	0	0
(1.082, 11, 39)	.258	2	0	3	(1.34, 13, 31)	−.258	8	−.258	4
(1.34, 10, 35)	.17	2	0	3	(1.51, 12, 27)	−.169	8	−.169	4
sums	0.376	14	0	12	sums	−.427	35	−.427	8

Table 7 Swaps in different policies for instance 18

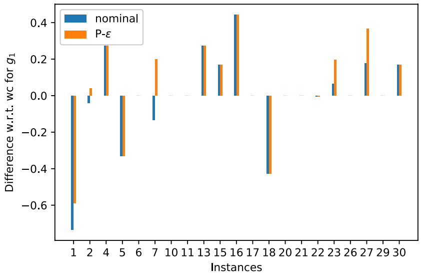
Nominal scenario ($\hat{q}$)			Worst-case scenario ($\tilde{q}$)		
$\boldsymbol{\psi}(Y_{\mathrm{eff}}^{PRO}(\hat{q}))$	wc policy	P-ϵ policy	$\boldsymbol{\psi}(Y_{\mathrm{eff}}^{PRO}(\tilde{q}))$	nom policy	P-ϵ policy
(0, 14, 54)	(0, 17, 43)	(0, 17, 43)	(0, 17, 43)	(0, 14, 54)	(0, 17, 43)
(0.052, 13, 52)	(0, 17, 43)	(0.052, 13, 52)	(0.48, 15, 39)	(0.48, 12, 43)	(0.48, 15, 39)
(0.48, 12, 43)	(0.48, 15, 39)	(0.48, 15, 39)	(1.082, 14, 35)	(1.082, 11, 39)	(1.082, 14, 35)
(1.082, 11, 39)	(1.34, 13, 31)	(1.082, 14, 35)	(1.34, 13, 31)	(1.082, 11, 39)	(1.082, 14, 35)
(1.34, 10, 35)	(1.51, 12, 27)	(1.34, 13, 31)	(1.51, 12, 27)	(1.34, 10, 35)	(1.34, 13, 31)



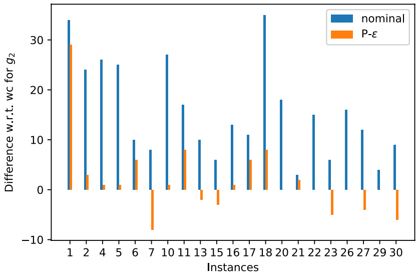
(i) Effect of policies in the nominal scenario for g_1 (Δg_1)



(ii) Effect of policies in the nominal scenario for g_2 (Δg_2)

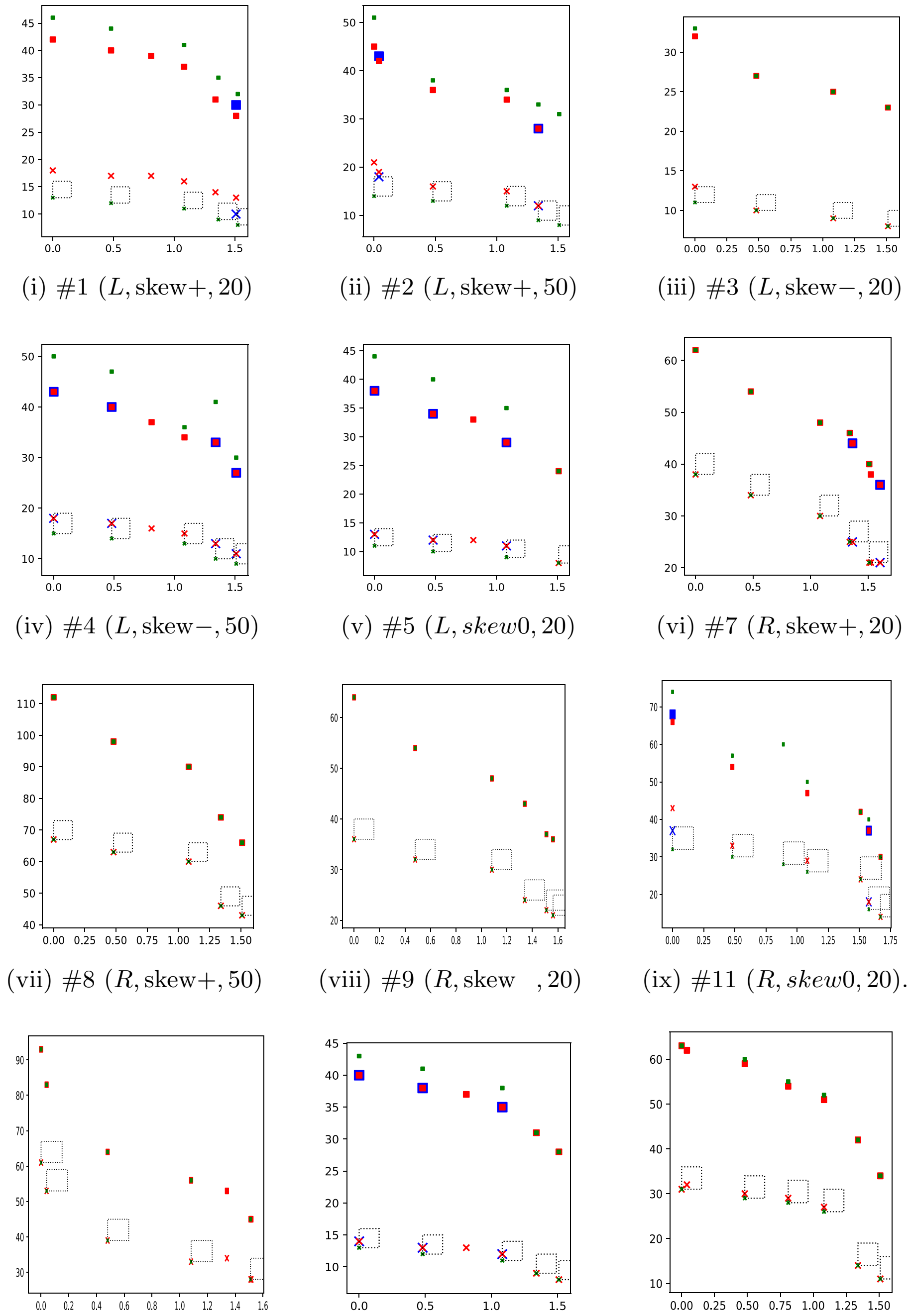


(iii) Effect of policies in the worst-case scenario for g_1 (Δg_1)



(iv) Effect of policies in the worst-case scenario for g_2 (Δg_2)

Fig. 10 Effect of policies on difference instances



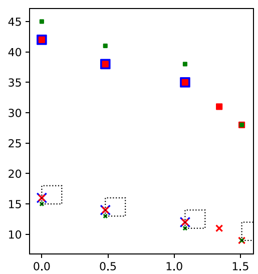
(i) #1 $(L, \text{skew}+, 20)$ (ii) #2 $(L, \text{skew}+, 50)$ (iii) #3 $(L, \text{skew}-, 20)$

(iv) #4 $(L, \text{skew}-, 50)$ (v) #5 $(L, skew0, 20)$ (vi) #7 $(R, \text{skew}+, 20)$

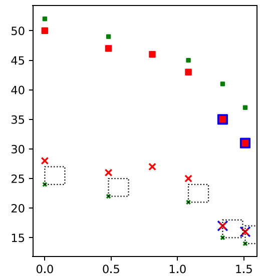
(vii) #8 $(R, \text{skew}+, 50)$ (viii) #9 $(R, \text{skew} \quad , 20)$ (ix) #11 $(R, skew0, 20)$.

(x) #12 $(R, skew0, 50)$ (xi) #13 $(U, \text{skew}+, 20)$ (xii) #14 $(U, \text{skew}+, 50)$

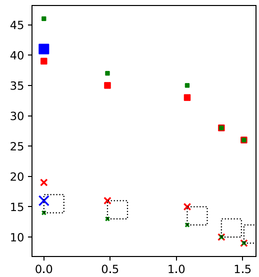
Fig. 11 Instances #1–5, #7–9, #11–18, #20–30 (axes and labels as in Fig. 3)



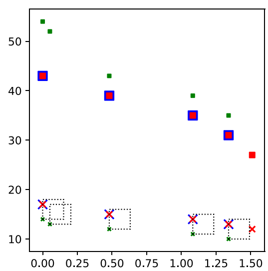
(xiii) #15 $(U, \text{skew}-, 20)$



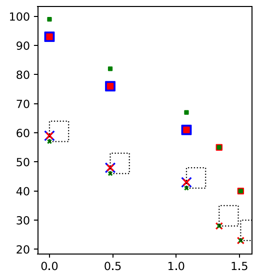
(xiv) #16 $(U, \text{skew} \quad, 50)$



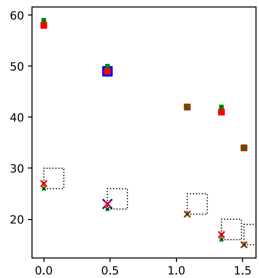
(xv) #17 $(U, \text{skew}0, 20)$



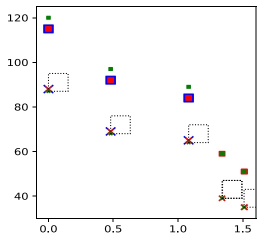
(xvi) #18 $(U, \text{skew}0, 50)$



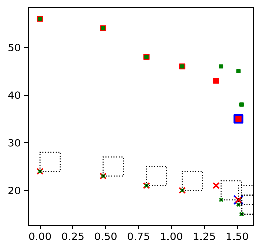
(xvii) #20 $(S, \text{skew}+, 50)$



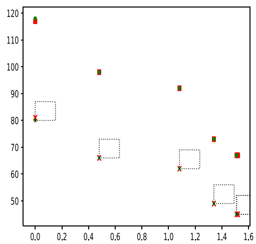
(xviii) #21 $(S, \text{skew}-, 20)$



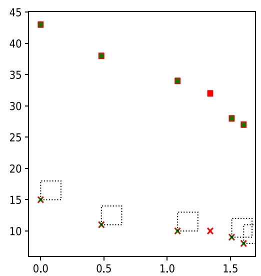
(xix) #22 $(S, \text{skew} \quad, 50)$



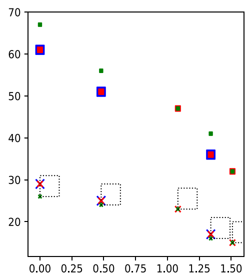
(xx) #23 $(S, \text{skew}0, 20)$



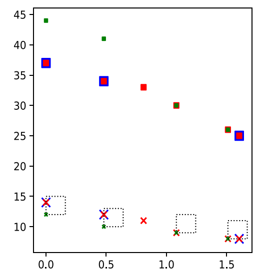
(xxi) #24 $(S, \text{skew}0, 50)$



(xxii) #25 $(M, \text{skew}+, 20)$

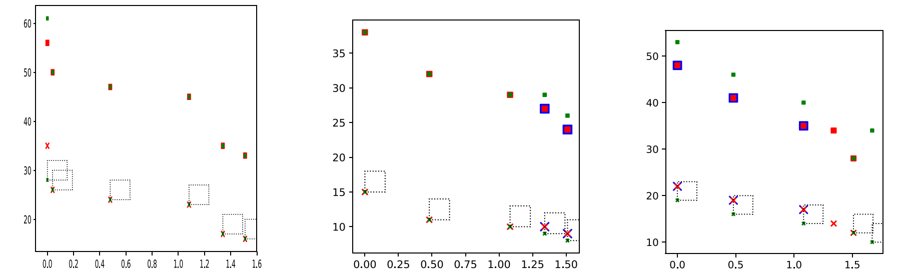


(xxiii) #26 $(M, \text{skew}+, 50)$



(xxiv) #27 $(M, \text{skew}-, 20)$

Fig. 11 (continued)



(xxv) #28 $(M, \text{skew}-, 50)$ (xxvi) #29 $(M, \text{skew}0, 20)$ (xxvii) #30 $(M, \text{skew}0, 50)$

Fig. 11 (continued)

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