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Homogenization of a Locally Periodic Oscillating Boundary

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Abstract
This paper deals with the homogenization of a mixed boundary value problem for the Laplace operator in a domain with locally periodic oscillating boundary. The Neumann condition is prescribed on the oscillating part of the boundary, and the Dirichlet condition on a separate part. It is shown that the homogenization result holds in the sense of weak L^2 convergence of the solutions and their flows, under natural hypothesis on the regularity of the domain. The strong L^2 convergence of average preserving extensions of the solutions and their flows is also considered.

Keywords Homogenization · Asymptotic analysis · Periodic unfolding · Locally periodic boundary · Oscillating boundary

Mathematics Subject Classification 80M35 · 80M40 · 35B27

1 Introduction

This paper is concerned with the homogenization of a boundary value problem for the Laplace operator on a domain in $\mathbb{R}^2$ with locally periodic oscillating boundary. Specifically, the domain is given by

$$\Omega^\varepsilon = \left\{x \in \mathbb{R}^2 : 0 < x_1 < 1,\ 0 < x_2 < \eta\big(x_1, \frac{x_1}{\varepsilon}\big)\right\},$$

where η is a positive Lipschitz continuous function which is periodic in the second variable, and ε is a small positive parameter (see Figure 1(a)). Certain further requirements are imposed on η to ensure a particularly simple structure of the homogenized

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problem. On the oscillating part of the boundary the Neumann condition is prescribed, and on a separate part the Dirichlet condition, and the data are assumed to be L^2. We are interested in the asymptotic behavior of the solutions u^ε to the boundary value problem, and their flows, as ε tends to zero.

Domains with oscillating boundaries have attracted particular interest in the case where the domain is thin, that is to say when homogenization and dimension reduction may take place. This is natural in the mathematical physics program of the derivation of lower-dimensional theories from three-dimensional (see e.g. [16]). There is a rich literature on thin heterogeneous domains (see [8, 50, 52] and the references therein). Asymptotic analysis in thin domains with locally periodic oscillating boundary was conducted in for example [3, 6–9, 20, 23, 24, 48, 54].

There are many works on homogenization in periodically oscillating domains with pillar type oscillations of fixed amplitude where the cross-section of each pillar is constant in the vertical direction. For the literature on pillar type oscillations, we refer to [31, 34, 35, 49] and the references therein. Oscillating boundary domains with non-uniform cylindrical pillars, that is when the cross-sections of the pillars are varying in the vertical direction, have been considered in [1, 32, 42, 45]. In the mentioned works, the top boundary of the pillars have been assumed to be flat, that is the measure of the cross-section of each pillar at the maximum height is assumed to be positive, and also the base of each pillar assumed to be flat. There are few works on non-flat top boundaries, and [2, 22] stand out. In [22], the authors restrict the boundary graph functions to be smooth, periodic, and to have a unique maximum in each period. In [33], the authors consider an oscillating domain without explicit periodicity assumption and the base of each uniform pillar is allowed to be non-flat. Locally periodic flat pillar type domains were considered in [28], with respect to width and height. The works of Mel'nyk and his collaborators (c.f. [29, 30, 36, 44, 46, 47]) appear to have had a strong influence on later developments, after the initial works of Brizzi and Chalot [21, 22].

In this paper, the oscillating domain Ω^ε is a bounded region partially bounded by the graph of a locally periodic Lipschitz function $\eta(x_1, x_1/\varepsilon)$, where $\eta : [0, 1] \times \mathbb{T} \to \mathbb{R}$, $\mathbb{T}$ is the 1-torus, and ε is a small parameter. These assumptions on η ensure that the domain is connected and Lipschitz. In particular, the domain is not thin in the direction normal to oscillation, and not of pillar-type, and no assumptions are made on its flatness. Under these assumptions, the domain naturally becomes asymptotically disconnected (in the x_1 direction) between two curves, one that appears as a part of the limiting boundary and one that appears as an interior interface, as ε tends to zero. The assumptions on η guarantee that these curves are graphs of Lipschitz functions.

The analysis simplifies considerably with the Brizzi-Chalot condition of a single bump in each period. It appears to be worthwhile to remark that it is not generally true that as soon as there is more than one bump in each period, even if one restricts to the smallest possible periodicity cell, the asymptotic limit will not be decoupled (fast and slow variables). An example that shows that the connectedness of the sections is not necessary is given.

The expected influence of the domain oscillations on the asymptotic behavior of the solutions is that since the Laplace operator is local and the periodicity of the domain makes a region asymptotically disconnected in the x_1 direction, $\partial/\partial x_1$ cannot be

present in the homogenized equation in that region. This will show in the homogenized boundary value problem.

The result of this paper is the homogenization of the domain for the particular boundary value problem in which the heterogeneity is only in the domain. Three properties of homogenization are shown. Namely, (i) the weak L^2 convergence of the zero-extended solutions and their flows (Theorem 6.1), (ii) that the error of the zeroth approximation of the solutions and their flows converge strongly to zero in weighted L^2 restricted to the oscillating domain (Theorem 7.1), and (iii) the strong L^2 convergence of average preserving extensions of the solutions and their flows (Theorem 8.1). In regard to (iii), in [22] reflection extensions were constructed and used, while the extension we use is the one used in [28, 41].

The analysis methods we use are standard techniques of asymptotic analysis and homogenization in particular. The method of homogenization is outlined in [38]. The method of periodic unfolding is described in [25–27], which is closely related to the notion of two-scale convergence [4, 51, 58], a generalization of weak convergence. Some works in which the unfolding method was used extensively in problems with oscillating boundary are [1, 18, 19, 28]. The homogenized problem is of degenerate elliptic type, as will be described below (c.f. [22]). The classical theory of Sobolev spaces for such is outlined in [39, 40]. For the method of asymptotic expansions we refer to [10–15, 17, 43, 53, 55, 56].

The rest of this paper is organized as follows. The problem statement and the results are presented in Sect. 2. The homogenized problem is derived using formal asymptotic expansions of the solutions in Sect. 3. In Sect. 4, an example is provided that shows that the first term in the asymptotic expansion to u^ε may be globally regular even in a degenerating case. In Sect. 5, the mean value property for the periodic unfolding is presented, adjusted to the present problem. The homogenization result of weak convergence of the solutions and their flows is established in Sect. 6. In Sect. 7, we prove that the energies in the problem converge. In Sect. 8, the strong convergence of the extended solutions and their flows using average preserving extensions is shown. A numerical example, illustrating the rate of convergence is presented in Sect. 9. In Sect. 10, some information about a case of non-connected sections is provided.

2 Problem Statement and Results

In this section we state the boundary value problem and present the results.

With strictly positive Lipschitz $\eta : [0, 1] \times \mathbb{T} \to \mathbb{R}$, the sequence of Lipschitz domains with periodically oscillating boundaries is for each $\varepsilon = 1/k$, $k = 1, 2, \ldots$, defined by

$$\Omega^\varepsilon = \left\{x \in \mathbb{R}^2 : 0 < x_1 < 1,\ 0 < x_2 < \eta\big(x_1, \frac{x_1}{\varepsilon}\big)\right\}.$$

Here, $\mathbb{T}$ denotes the one-dimensional torus realized as $(0, 1)$. See Figure 1(a) for an illustration of Ω^ε. Additional restrictions on η will be added below in order to ensure homogenization.

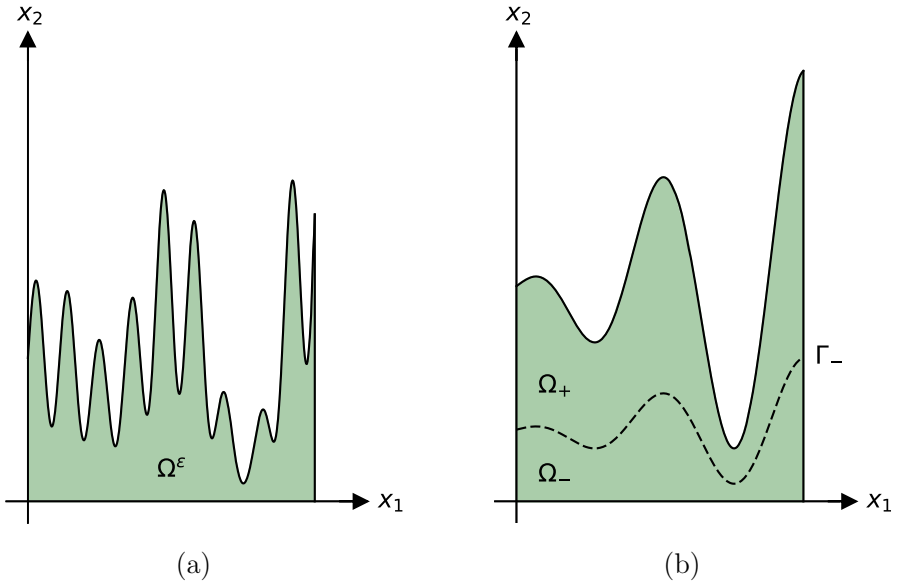

Fig. 1 A locally periodic domain Ω^ε (a), $\varepsilon = 1/8$, and the corresponding homogeneous domain Ω (b), with Γ_- marked with a dashed line separating the regions Ω_+ and Ω_-. The particular function η is $\eta(x_1, y) = (1 + x_1 \cos(4\pi x_1))(1 + \sin(2\pi(x_1 + y))/2)$

Let $u^\varepsilon \in H^1(\Omega^\varepsilon, \Gamma)$ be the sequence of solutions to the following mixed boundary value problem:

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega^\varepsilon, \\ u &= 0 \quad \text{on } \Gamma, \\ \nabla u \cdot \nu &= 0 \quad \text{on } \partial\Omega^\varepsilon \setminus \Gamma, \end{aligned} \tag{1}$$

with f in $L^2(\Omega)$, where $\Omega^\varepsilon \subset \Omega$. Here ν denotes the outward unit normal to the domain, and $H^1(\Omega^\varepsilon, \Gamma)$ the functions in $H^1(\Omega^\varepsilon)$ with zero trace on $\Gamma = (0, 1) \times \{0\}$. Our goal is to describe the asymptotic behavior of the solutions u^ε to (1) as ε tends to zero.

The solutions u^ε will be approximated in terms of the solution u^0 to the homogenized problem:

$$\begin{aligned} -\mathrm{div}(hA^0\nabla u) &= hf \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma, \\ hA^0\nabla u \cdot \nu &= 0 \quad \text{on } \partial\Omega \setminus \Gamma, \end{aligned} \tag{2}$$

where the coefficients A^0 and the domain Ω are defined as follows. In terms of the Lipschitz functions

$$\eta_-(x_1) = \min_y \eta(x_1, y), \qquad \eta_+(x_1) = \max_y \eta(x_1, y),$$

the domain

$$\Omega = \{x \in \mathbb{R}^2 : 0 < x_1 < 1,\ 0 < x_2 < \eta_+(x_1)\}$$

is separated into the regions

$$\Omega_- = \{x \in \mathbb{R}^2 : 0 < x_1 < 1,\ 0 < x_2 < \eta_-(x_1)\},$$
$$\Omega_+ = \{x \in \mathbb{R}^2 : 0 < x_1 < 1,\ \eta_-(x_1) < x_2 < \eta_+(x_1)\},$$

with interior interface $\Gamma_- = \partial\Omega_- \cap \partial\Omega_+$. The effective matrix is

$$A^0 = \begin{pmatrix} \chi_{\Omega_-} & 0 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

An illustration of Ω, with the regions Ω_+ and Ω_-, and the interface Γ_-, indicated is shown in Fig. 1b, corresponding to the domain Ω^ε in Figure 1(a).

Let h denote what we call the density of Ω^ε in Ω:

$$Y(x) = \{y : x_2 < \eta(x_1, y)\}, \tag{4}$$
$$h(x) = |Y(x)|. \tag{5}$$

In Ω_- the density is $h = 1$. A part of the upper boundary, denoted by Γ_a is given by

$$\Gamma_a = \{(x_1, x_2) \mid x_1 \in (0, 1),\ x_2 = \eta_+(x_1),\ h(x_1, x_2) = 0\}.$$

In order to ensure that homogenization takes place, the following hypotheses will be used:

(H1) $Y(x)$ is connected, $x \in \Omega$.
(H2) $Y(x) = k/N + Y(x)$, $x \in \Omega$, $k \in \mathbb{Z}$, for some natural number $N \geq 1$, and $Y(x) = y_0 - Y(x)$ for some $y_0 \in [0, 1/N)$, and $Y_0(x) = \{y \in Y(x) : y_0 \leq y \leq y_0 + 1/(2N)\}$ is connected, $x \in \Omega$.

The hypothesis (H1) means that there is only one so-called pillar or bump in each period, and (H2) means restricting to a fundamental symmetry cell with respect to some translations and mirror symmetry. The hypothesis (H1) is stronger than (H2), and it gives a sufficient condition for homogenization. The hypothesis (H2) is included here to illustrate that it is not necessary there is only one 'bump' in each period even if one uses the smallest periodicity cell. The graph of a function η that satisfies (H2) but not (H1) is illustrated in Figure 3.

The assumption that η is strictly positive ensures that the segment $\Gamma = (0, 1) \times \{0\}$ is separated from the graph of $\eta(x_1, x_1/\varepsilon)$, so Ω_- is a nonempty connected Lipschitz domain. The subdomains Ω_+ and Ω_- have been chosen such that Ω_+ covers the periodic region of Ω^ε, and Ω_+ is of positive measure if $\eta(x, y)$ is non-constant in y for at least one x.

Denote by $L^2(\Omega, h)$ the Lebesgue space $\{v : \int_\Omega v^2 h\, dx < \infty\}$, and $W(\Omega, \Gamma)$ the Sobolev space

$$W(\Omega, \Gamma) = \left\{v \in L^2(\Omega, h) : A^0 \nabla v \in L^2(\Omega, h),\ v = 0 \text{ on } \Gamma\right\}, \tag{6}$$

where A^0 is defined in (3) and h is defined in (4)-(5). The homogenized problem (2) has a unique solution $u^0 \in W(\Omega, \Gamma)$.

We will first establish the homogenization of (1), that is the convergence of the solutions u^ε and their flows ∇u^ε to the solution u^0 to the homogenized problem (2) and its flow $hA^0\nabla u^0$, under hypothesis that (H1) holds.

Let $u^\varepsilon \in H^1(\Omega^\varepsilon, \Gamma)$ be the solutions to (1), and let $u^0 \in W(\Omega, \Gamma)$ be the solution to (2). In Theorem 6.1, it is shown under hypothesis (H1) that

$$\begin{aligned} &\widetilde{u^\varepsilon} \rightharpoonup hu^0 \quad \text{weakly in } L^2(\Omega), \\ &\widetilde{\nabla u^\varepsilon} \rightharpoonup hA^0\nabla u^0 \quad \text{weakly in } L^2(\Omega), \end{aligned}$$

as ε tends to zero. Here tilde denotes extension by zero.

At this point we know that there will be oscillations in $\widetilde{u^\varepsilon}$ in the upper part Ω_+ due to the periodicity of Ω_ε, while no oscillations in the lower part Ω_- due to the compact embedding of $H^1(\Omega_-)$ into $L^2(\Omega_-)$. Moreover, we cannot at this point exclude the possibility of oscillations in the solutions $\widetilde{u^\varepsilon}$ due to something else, as the above weak convergences may be expressed as weak unfolding or two-scale convergence. A next step is to check whether the strong unfolding convergence holds for the solutions u^ε and their flows.

The weak convergence of the zero extensions in the upper part Ω_+ cannot be strong, unless the limit is zero. To this end we first describe the error measured in the oscillating domain. Not only are the unfoldings of the solutions u^ε and their flows strongly converging, there are no oscillations.

For u^ε the solutions to (1) and u^0 the solution to (2), in Theorem 7.1 it is shown under hypothesis (H1) that

$$\begin{aligned} &\|u^\varepsilon - u^0\|_{L^2(\Omega^\varepsilon, h)} \to 0, \\ &\|\nabla u^\varepsilon - A^0\nabla u^0\|_{L^2(\Omega^\varepsilon, h)} \to 0, \end{aligned}$$

as ε tends to zero.

Turning back to the question of homogenization, there is an extension of the solutions u^ε and their flows for which strong convergence holds in $L^2(\Omega)$.

For u^ε the solutions to (1) and u^0 the solution to (2), in Theorem 8.1 it is shown that when the functions are extended in a way preserving their average, under hypothesis (H1) (c.f. [28, 41]),

$$\begin{aligned} &\widetilde{u^\varepsilon}^m \to u^0 \quad \text{strongly in } L^2(\Omega, h), \\ &\widetilde{\nabla u^\varepsilon}^m \to A^0\nabla u^0 \quad \text{strongly in } L^2(\Omega, h), \end{aligned}$$

as ε tends to zero, when $\sim m$ denotes the particular extension ((26), (27) in Sect. 8). It is remarked that the above mentioned convergences also hold under hypothesis (H2).

Without the hypotheses (H1), (H2), under a slightly milder restriction on the domain, in Theorem 10.1 it is shown that the solutions u^ε to (1) converge to the solution u^0 to the limit problem (31), with A^0 given by (30), in the sense that

$$\widetilde{u^\varepsilon} \rightharpoonup \int_{Y(x)} u^0\,dy \quad \text{weakly in } L^2(\Omega),$$
$$\widetilde{\nabla u^\varepsilon} \rightharpoonup \int_{Y(x)} A^0\nabla_x u^0\,dy \quad \text{weakly in } L^2(\Omega),$$

as ε tends to zero. One also has strong unfolding convergence. We do not analyze further the convergence in the case of non-connected sections.

In the case of the homogeneous Dirichlet condition on the oscillating part of the boundary, the limit is trivial in the oscillating part. This case was studied in [5].

Remark 2.1 (The effect of anisotropy and oscillating coefficients) To illustrate the effect of anisotropy and oscillating coefficients on the asymptotic behavior of the solutions u^ε to (1), one can consider the following elliptic model problem:

$$\begin{aligned} -\mathrm{div}\big(A\big(x, \frac{x_1}{\varepsilon}\big)\nabla u\big) &= f \quad \text{in } \Omega^\varepsilon,\\ u &= 0 \quad \text{on } \Gamma,\\ A\big(x, \frac{x_1}{\varepsilon}\big)\nabla u\cdot\nu &= 0 \quad \text{on } \partial\Omega^\varepsilon\setminus\Gamma, \end{aligned}$$

under the assumption that the not necessarily symmetric matrix $A \in C^1(\overline{\Omega}, L^\infty(\mathbb{T}))$ satisfies the ellipticity condition

$$A(x, y)\xi\cdot\xi \geq C|\xi|^2, \quad x\in\Omega,\ y\in\mathbb{T},\ \xi\in\mathbb{R}^2.$$

Suppose further that hypothesis (H2) is satisfied, and that the coefficient matrix satisfies the corresponding symmetry conditions: $A(x, y) = A(x, k/N + y)$, $k \in \mathbb{Z}$, and $A(x, y) = A(x, y_0 - y)$, for $x \in \Omega_+$, where N and y_0 are the same as for $Y(x)$ in (H2). Then the weak limits of $\widetilde{u^\varepsilon}$ and $\widetilde{\nabla u^\varepsilon}$ are hu^0 and A^0u^0, respectively, in $L^2(\Omega)$, as ε tends to zero, where u^0 solves the limit problem

$$\begin{aligned} -\mathrm{div}(A^0\nabla u) &= hf \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \Gamma,\\ A^0\nabla u\cdot\nu &= 0 \quad \text{on } \partial\Omega\setminus\Gamma, \end{aligned}$$

where the entries A^0_{ij} of the effective matrix A^0 are given by, here including h,

$$A^0_{11} = \chi_{\Omega_-}\Big(\int_{Y(x)} \frac{1}{A_{11}}\,dy\Big)^{-1},$$

$$A^0_{12} = A^0_{11} \int_{Y(x)} \frac{A_{12}}{A_{11}}\, dy,$$
$$A^0_{21} = A^0_{11} \int_{Y(x)} \frac{A_{21}}{A_{11}}\, dy,$$
$$A^0_{22} = A^0_{11} \int_{Y(x)} \frac{A_{12}}{A_{11}}\, dy \int_{Y(x)} \frac{A_{21}}{A_{11}}\, dy + \int_{Y(x)} \frac{\det A}{A_{11}}\, dy,$$

where A_{ij} denote the entries of the local matrix A. In the fixed region Ω_-, A^0 is the classical effective matrix for layered materials because there $Y(x) = \mathbb{T}$, while in the oscillating region Ω_+ all but the last term in A^0_{22} vanish due to the insulation in the x_1 direction. In particular, for $-\Delta$ the effective matrix reduces to the matrix given in (3). A similar statement can be made for the case of non-connected sections analogous to Theorem 10.1.

3 Asymptotic Expansions for the Solutions

To derive the homogenized equation (2) for the solutions u^ε to the equation

$$-\Delta u^\varepsilon = f \quad \text{in } \Omega^\varepsilon, \tag{7}$$

asymptotic expansions may be used, in the form of a formal power series in the small parameter ε. In the problem (7), in the periodic region Ω^ε_+ of the domain Ω, ε represents both the periodicity in the x_1 direction, as well as the order of magnitude of the widths of the pillars with homogeneous Neumann condition on their sides. In the fixed region Ω_- of the domain, ε represents the periodicity on the interface Γ_-.

The above reasoning leads us to consider the following Bakhvalov ansatz for inner expansion in the periodic region Ω^ε_+:

$$u^\varepsilon(x) \sim \left(u^0_+(x_2, y) + \varepsilon u^1_+(x_2, y) + \varepsilon^2 u^2_+(x_2, y)\right)\Big|_{y = \frac{x_1}{\varepsilon}}, \tag{8}$$

where u^i_+ are assumed to be periodic in y.

Let $\chi(x, y)$ denote the characteristic function of the set

$$\{(x, y) : 0 < x_1 < 1,\ 0 < x_2 < \eta(x_1, y)\}.$$

Then the characteristic function of Ω^ε is

$$\chi_{\Omega^\varepsilon}(x) = \chi\left(x, \frac{x_1}{\varepsilon}\right).$$

We write (7) in the homogeneous domain Ω as follows:

$$-\sum_i \frac{\partial}{\partial x_i}\left(\chi\left(x, \frac{x_1}{\varepsilon}\right)\frac{\partial}{\partial x_i} u^\varepsilon\right) = \chi\left(x, \tfrac{x_1}{\varepsilon}\right) f \quad \text{in } \Omega. \tag{9}$$

With the ansatz (8),

$$\Delta \sim \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x_2^2}.$$

A substitution of (8) into (9) and collecting similar powers of ε result in the following equations for the initial powers of ε:

$$\begin{aligned}
\varepsilon^{-2}: &\quad -\frac{\partial}{\partial y}\big(\chi\frac{\partial u_+^0}{\partial y}\big) = 0,\\
\varepsilon^{-1}: &\quad -\frac{\partial}{\partial y}\big(\chi\frac{\partial u_+^1}{\partial y}\big) = 0,\\
\varepsilon^{0}: &\quad -\frac{\partial}{\partial y}\big(\chi\frac{\partial u_+^2}{\partial y}\big) = \frac{\partial}{\partial x_2}\big(\chi\frac{\partial u_+^0}{\partial x_2}\big) + \chi f.
\end{aligned}$$

The equation for power ε^{-2} suggests that u_+^0 is independent of y, under the assumption that $Y(x)$ is connected, or some discrete symmetry in the problem such as (H2). Viewing the above equations as definitions of u_+^i in $\mathbb{T}$ with x as a parameter, the compatibility condition for u_+^2 may be read off from the ε^0 equation:

$$\int_{\mathbb{T}} \big(\frac{\partial}{\partial x_2}\big(\chi\frac{\partial u_+^0}{\partial x_2}\big) + \chi f\big)\, dy = 0.$$

Because

$$\int_{\mathbb{T}} \chi(x, y)\, dy = \int_{Y(x)} dy = |Y(x)| = h(x),$$

where

$$\begin{aligned}
Y(x) &= \{y : x_2 < \eta(x_1, y)\},\\
h(x) &= |Y(x)|,
\end{aligned}$$

the compatibility condition is

$$-\frac{\partial}{\partial x_2}\big(h\frac{\partial u_+^0}{\partial x_2}\big) = hf \quad \text{in } \Omega_+. \tag{10}$$

The equation (10) is the homogenized equation for u^ε in the periodic part Ω_+ of Ω.

In the fixed region Ω_- of the domain, the equation (7) reads

$$-\Delta u^\varepsilon = f \quad \text{in } \Omega_-. \tag{11}$$

Consider the following ansatz for inner expansion in the fixed region Ω_-:

$$u^\varepsilon(x) \sim \left(u_-^0(x,y) + \varepsilon u_-^1(x,y) + \varepsilon^2 u_-^2(x,y)\right)\Big|_{y=\frac{x_1}{\varepsilon}}, \tag{12}$$

where u_-^i are assumed to be periodic in y. With the ansatz (12),

$$\Delta \sim \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial y^2} + \frac{2}{\varepsilon}\frac{\partial^2}{\partial y \partial x_1} + \Delta_x.$$

A substitution of (12) into (11) and collecting similar powers of ε result in the following equations for the initial powers of ε:

$$\begin{aligned}
\varepsilon^{-2}: &\quad -\frac{\partial^2 u_-^0}{\partial y^2} = 0,\\
\varepsilon^{-1}: &\quad -\frac{\partial^2 u_-^1}{\partial y^2} = 2\frac{\partial^2 u_-^0}{\partial y \partial x_1},\\
\varepsilon^{0}: &\quad -\frac{\partial^2 u_-^2}{\partial y^2} = 2\frac{\partial^2 u_-^1}{\partial y \partial x_1} + \Delta_x u_-^0 + f.
\end{aligned}$$

The equation for power ε^{-2} suggests that u_-^0 is independent of y. By the periodicity of u_-^1, the compatibility condition for u_-^2 reads

$$-\Delta u_-^0 = f \quad \text{in } \Omega_-.$$

In order $u^0 = \chi_{\Omega_-} u_-^0 + \chi_{\Omega_+} u_+^0$ to be globally defined, an interface condition is needed on $\Gamma_- = \partial\Omega_- \cap \partial\Omega_+$. If one requires continuity of u^0 and its flow on Γ_-, one must have $u_+^0 = u_-^0$ on Γ_-, as well as

$$\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix} \nabla u_+^0 \cdot \nu - \nabla u_-^0 \cdot \nu = 0,$$

where ν is one of the unit normals on Γ_-, and the upper left entry of the matrix in the first term is arbitrary. This condition becomes explicitly,

$$h\frac{\partial u_+^0}{\partial x_2}\nu_2 - \nabla u_-^0 \cdot \nu = 0,$$

which may be expressed as

$$[hA^0 \nabla u^0 \cdot \nu] = 0 \quad \text{on } \Gamma_-,$$

where $[\cdot]$ denotes the jump on Γ_-, and A^0 is given by (3).

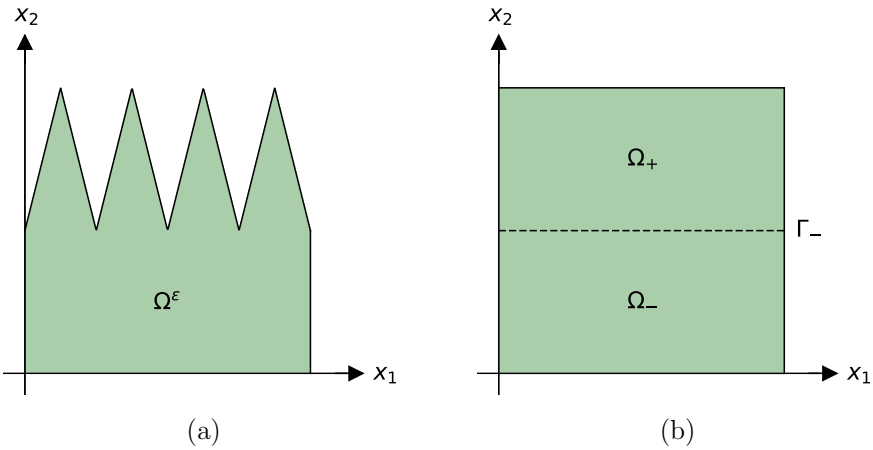

Fig. 2 A locally periodic domain Ω^ε (a), $\varepsilon = 1/4$, and the corresponding homogeneous domain Ω (b), with interface Γ_- marked as a dashed line separating the regions Ω_+ and Ω_-

4 Example of Behavior of the Solutions in a Degenerating Case

The solution u^0 to the homogenized problem (2) may belong to $H^1(\Omega)$ even if $h(x)$ tends to zero as x approaches $M = \{x : h(x) = 0\} \subset \partial\Omega$, as the following example illustrates. Consider the case $f = 1$ with $\eta(x, y) = \hat{\eta}(y)$ the piecewise linear function given by $\hat{\eta}(0) = 1$ and $\hat{\eta}(1/2) = 2$, that is $\hat{\eta}(y) = 2 - 2|y - 1/2|$. The homogeneous domains are $\Omega = (0, 1) \times (0, 2)$, $\Omega_+ = (0, 1) \times (1, 2)$, $\Omega_- = (0, 1) \times (0, 1)$, and the interface is $\Gamma_- = (0, 1) \times \{1\}$. In Figure 2, illustrations of the domains Ω^ε and Ω are shown. In this case,

$$h(x) = \begin{cases} 2 - x_2 & \text{in } \Omega_+, \\ 1 & \text{in } \overline{\Omega_-}, \end{cases}$$

and

$$\begin{aligned} h &= 0 \quad \text{on } M = \{x : h(x) = 0\} = (0, 1) \times \{2\}, \\ h &= 1 \quad \text{on } \Gamma_-. \end{aligned}$$

The solution to the homogenized problem (2) is

$$u^0(x) = \begin{cases} 1/4 + x_2 - x_2^2/4 & \text{in } \Omega_+, \\ 3x_2/2 - x_2^2/2 & \text{in } \overline{\Omega_-}. \end{cases}$$

In particular, $u^0 \in H^1(\Omega)$ and it has continuous gradient over the interface Γ_-.

5 Periodic Unfolding

The only apparent possible cause of oscillations in the solutions to (1) and their flows is the periodicity in the domain, in the x_1 direction. For the study of these oscillations we will use periodic unfolding.

The periodic unfolding at rate ε of a function $v : \mathbb{R}^2 \to \mathbb{R}$ along x_1 is

$$(T^\varepsilon v)(x, y) = v\big(\varepsilon\big[\frac{x_1}{\varepsilon}\big] + \varepsilon y, x_2\big),$$

where $[\cdot]$ denotes the integer part, and v is extended by zero when necessary. Using this change of variables for the periodic coefficient $\chi_{\Omega^\varepsilon_+}$ in (1), the characteristic function of the region of Ω^ε where coefficients are periodic,

$$\Omega^\varepsilon_+ = \big\{x \in \mathbb{R}^2 : 0 < x_1 < 1,\ \eta_-(x_1) < x_2 < \eta\big(x_1, \frac{x_1}{\varepsilon}\big)\big\},$$

gives $\chi_{\Omega^\varepsilon_u}$, the characteristic function of the domain

$$\Omega^\varepsilon_u = \big\{(x, y) \in \mathbb{R}^2 \times \mathbb{T} : 0 < x_1 < 1,\ \eta_-\big(\varepsilon\big[\frac{x_1}{\varepsilon}\big] + \varepsilon y\big) < x_2 < \eta\big(\varepsilon\big[\frac{x_1}{\varepsilon}\big] + \varepsilon y, y\big)\big\}.$$

There holds

$$T^\varepsilon \chi_{\Omega^\varepsilon_+} = \chi_{\Omega^\varepsilon_u} \to \chi_{\Omega_u} \quad \text{strongly in } L^p(\mathbb{R}^2 \times \mathbb{T}), \quad 1 \le p < \infty, \tag{13}$$

where

$$\begin{aligned}\Omega_u &= \{(x, y) \in \mathbb{R}^2 \times \mathbb{T} : 0 < x_1 < 1,\ \eta_-(x_1) < x_2 < \eta(x_1, y)\}\\ &= \{(x, y) \in \mathbb{R}^2 \times \mathbb{T} : x \in \Omega_+,\ y \in Y(x)\}.\end{aligned}$$

The property (13) is the strong unfolding convergence of the sequence and it will be used in the passage from periodic domain to a fixed domain in integrals. It expresses that $\chi_{\Omega^\varepsilon_+}$ converges weakly in $L^p(\mathbb{R}^2)$ while not strongly, and that the oscillation spectrum of the sequence belongs to the integers if not empty. To obtain (13) one uses the almost everywhere pointwise convergence of $\chi_{\Omega^\varepsilon_u}$ to χ_{Ω_u} and the Lebesgue dominated convergence theorem, or views it as a consequence of Lemma 5.1 below.

The cost of replacing in integrals the ε depending unfolded domain Ω^ε_u with the fixed domain Ω_u is described by the following lemma.

Lemma 5.1 *Let Ω contain Ω^ε_+. Suppose that $\|v^\varepsilon\|_{L^p(\Omega)} \le C$ and $p > 1$. Then*

$$\int_{\Omega^\varepsilon_+} v^\varepsilon\, dx = \int_{\Omega_u} T^\varepsilon v^\varepsilon\, dxdy + O(\varepsilon^{1-1/p}),$$

as ε tends to zero.

Proof Because $T^\varepsilon \chi_{\Omega^\varepsilon_+} = \chi_{\Omega^\varepsilon_u}$, the discrepancy can be computed as follows:

$$\int_{\Omega^\varepsilon_+} v^\varepsilon \, dx - \int_{\Omega_u} T^\varepsilon v^\varepsilon \, dxdy = \int_{\Omega\times\mathbb{T}} T^\varepsilon \chi_{\Omega^\varepsilon_+} T^\varepsilon v^\varepsilon \, dxdy - \int_{\Omega_u} T^\varepsilon v^\varepsilon \, dxdy$$
$$= \int_{\Omega\times\mathbb{T}} (\chi_{\Omega^\varepsilon_u} - \chi_{\Omega_u}) T^\varepsilon v^\varepsilon \, dxdy.$$

The differences $\Omega^\varepsilon_u \setminus \Omega_u$ and $\Omega_u \setminus \Omega^\varepsilon_u$ are contained in some strip of measure $O(\varepsilon)$:

$$\left\{(x, y) \in \mathbb{R}^2 \times \mathbb{T} : \operatorname{dist}((x, y), \partial\Omega_u) < C\varepsilon\right\},$$

where C may be chosen independent of ε by the Lipschitz continuity of η and η_-,

By the Hölder inequality,

$$\left| \int_{\Omega^\varepsilon_+} v^\varepsilon \, dx - \int_{\Omega_u} T^\varepsilon v^\varepsilon \, dxdy \right|$$
$$\leq \|\chi_{\Omega^\varepsilon_u} - \chi_{\Omega_u}\|_{L^{1/(1-1/p)}(\Omega\times\mathbb{T})} \|v^\varepsilon\|_{L^p(\Omega)} = O(\varepsilon^{(p-1)/p}),$$

which gives the desired estimate.

6 Homogenization

In this section we establish the homogenization of problem (1) to (2) in the sense of weak convergence of the solutions and their flows. The method we use is the unfolding Lemma 5.1 to pass to the fixed domain Ω_u, and the weak compactness in $L^2(\Omega_u)$ to characterize the asymptotic behavior of u^ε.

Throughout this section (H1) is assumed to hold.

Theorem 6.1 *Suppose that (H1) holds. Let $u^\varepsilon \in H^1(\Omega^\varepsilon, \Gamma)$ be the solutions to (1), and let $u^0 \in W(\Omega, \Gamma)$ be the solution to (2). Then*

$$\text{(i)} \quad \widetilde{u^\varepsilon} \rightharpoonup h u^0 \quad \textit{weakly in } L^2(\Omega),$$
$$\text{(ii)} \quad \widetilde{\nabla u^\varepsilon} \rightharpoonup h A^0 \nabla u^0 \quad \textit{weakly in } L^2(\Omega),$$

as ε tends to zero, where $\sim$ denotes extension by zero.

The convergence of the flows in Theorem 6.1(ii) means that u^ε converges weakly to u^0 in $H^1(\Omega_-, \Gamma)$, and strongly in $L^2(\Omega_-)$ by the Relich theorem.

Lemma 6.1 *For any $\varepsilon = 1/k$, $k = 1, 2, \ldots$, there exists a unique solution $u^\varepsilon \in H^1(\Omega^\varepsilon, \Gamma)$ to (1). For the solutions u^ε, the following a priori estimate holds:*

$$\|u^\varepsilon\|_{H^1(\Omega^\varepsilon, \Gamma)} \leq C,$$

where C is independent of ε.

Proof The variational form of (1) is: Find $u^\varepsilon \in H^1(\Omega^\varepsilon, \Gamma)$ such that

$$\int_{\Omega^\varepsilon} \nabla u \cdot \nabla \psi \, dx = \int_{\Omega^\varepsilon} f \psi \, dx, \tag{14}$$

for all $\psi \in H^1(\Omega^\varepsilon, \Gamma)$. Using the Poincaré inequality

$$\int_{\Omega^\varepsilon} v^2 \, dx \leq (\max \eta_+)^2 \int_{\Omega^\varepsilon} \left(\frac{\partial v}{\partial x_2}\right)^2 dx, \quad v \in H^1(\Omega^\varepsilon, \Gamma),$$

one verifies that left hand side in (14) is an inner product on $H^1(\Omega^\varepsilon, \Gamma)$. The right hand side in (14) is a bounded linear functional on $H^1(\Omega^\varepsilon, \Gamma)$. The Riesz theorem guarantees the existence of a unique solution $u^\varepsilon \in H^1(\Omega^\varepsilon, \Gamma)$. Using u^ε as a test function in (14) gives

$$\|u^\varepsilon\|_{H^1(\Omega^\varepsilon, \Gamma)} \leq C \|f\|_{L^2(\Omega)},$$

uniformly in ε by the uniform Poincaré constant, from which the desired a priori estimate is obtained.

Lemma 6.2 *For the sequence of solutions u^ε to (1), the following a priori estimates hold for the unfolded sequences:*

$$\|T^\varepsilon u^\varepsilon\|_{L^2(\Omega_u)} \leq C,$$
$$\|T^\varepsilon \nabla u^\varepsilon\|_{L^2(\Omega_u)} \leq C,$$

where C is independent of ε.

Proof By the definition of unfolding,

$$\int_{\Omega^\varepsilon_+} (u^\varepsilon)^2 \, dx = \int_{\Omega^\varepsilon_u} (T^\varepsilon u^\varepsilon)^2 \, dxdy = \int_{\Omega_u} (T^\varepsilon u^\varepsilon)^2 \, dxdy + \int_{\Omega^\varepsilon_u \setminus \Omega_u} (T^\varepsilon u^\varepsilon)^2 \, dxdy,$$

because $T^\varepsilon u^\varepsilon$ is zero outside Ω^ε_u. By using the estimate for the solutions u^ε in Lemma 6.1, the estimate $\|T^\varepsilon u^\varepsilon\|_{L^2(\Omega_u)} \leq C$ is obtained. The same computation with $|\nabla u^\varepsilon|$ in place of u^ε gives the estimate $\|T^\varepsilon \nabla u^\varepsilon\|_{L^2(\Omega_u)} \leq C$.

Regarding the function space, $W(\Omega, \Gamma)$ is associated to the homogenized problem (2). In the cover Ω_+ of the periodic region of Ω^ε, the $H^1(\Omega)$ ellipticity of the operator to the homogenized problem (2) may be violated. For

$$hA^0\xi \cdot \xi = \begin{cases} h\xi_2^2 & \text{in } \overline{\Omega_+}, \\ |\xi|^2 & \text{in } \Omega_-, \end{cases}$$

and $h(x)$ tends to zero as x approaches $M = \{x : h(x) = 0\}$, which might be nonempty. The properties

$$h(x) > 0 \quad \text{in } \Omega,$$
$$h, h^{-1} \in L^1_{\text{loc}}(\Omega),$$

ensure that $W(\Omega, \Gamma)$ is a Hilbert space when equipped with the inner product $(u, v) = \int_\Omega hA^0 \nabla u \cdot \nabla v \, dx$, and that $C_0^\infty(\Omega)$ is embedded into $W(\Omega, \Gamma)$. We denote by $C^\infty(\Omega, \Gamma)$, the $C^\infty(\Omega)$ functions vanishing in a neighborhood of Γ.

The existence of a solution $u^0 \in W(\Omega, \Gamma)$ to (2) is obtained by weak compactness in the proof of Theorem 6.1 below, with uniqueness by linearity. One might also obtain it in the direct way as follows.

Lemma 6.3 *There exists a unique solution $u^0 \in W(\Omega, \Gamma)$ to (2).*

Proof The variational form of (2) is: Find $u^0 \in W(\Omega, \Gamma)$ such that

$$\int_\Omega hA^0 \nabla u^0 \cdot \nabla \psi \, dx = \int_\Omega f \psi h \, dx, \tag{15}$$

for all $\psi \in W(\Omega, \Gamma)$.

The Hilbert space structure on $W(\Omega, \Gamma)$ has been chosen such that

$$\int_\Omega hA^0 \nabla v \cdot \nabla v \, dx = \|v\|^2_{W(\Omega, \Gamma)}.$$

By the Hölder and Poincaré inequalities, the left hand side of (15) defines an inner product on $W(\Omega, \Gamma)$. The right hand side of (15) is a bounded linear functional on $W(\Omega, \Gamma)$. By the Riesz theorem, there exists a unique $u^0 \in W(\Omega, \Gamma)$ satisfying (15).

Proof of Theorem 6.1 From Lemmas 6.1 and 6.2 we have the following a priori estimates for the solutions u^ε to problem (1) and their unfoldings:

$$\|u^\varepsilon\|_{H^1(\Omega^\varepsilon, \Gamma)} \le C,$$
$$\|T^\varepsilon u^\varepsilon\|_{L^2(\Omega_u)} \le C,$$
$$\|T^\varepsilon \nabla u^\varepsilon\|_{L^2(\Omega_u)} \le C.$$

By weak compactness, there exist $u^0_- \in H^1(\Omega_-, \Gamma)$, $u^0_+ \in L^2(\Omega_u)$, $p \in L^2(\Omega_u)$, and a subsequence of ε which we still denoted by ε, such that

$$u^\varepsilon \rightharpoonup u^0_- \qquad \text{weakly in } H^1(\Omega_-, \Gamma), \tag{16}$$
$$T^\varepsilon u^\varepsilon \rightharpoonup u^0_+ \qquad \text{weakly in } L^2(\Omega_u), \tag{17}$$
$$T^\varepsilon \nabla u^\varepsilon \rightharpoonup \Big(p, \frac{\partial u^0_+}{\partial x_2}\Big) \qquad \text{weakly in } L^2(\Omega_u), \tag{18}$$

where the equality of second component of the weak limit of $T^\varepsilon \nabla u^\varepsilon$ and $\frac{\partial u_+^0}{\partial x_2}$, and that u_+^0 does not depend on y, follow from the boundedness of the sequences and

$$\frac{\partial}{\partial x_2} T^\varepsilon u^\varepsilon = T^\varepsilon \frac{\partial u^\varepsilon}{\partial x_2}, \qquad \frac{\partial}{\partial y} T^\varepsilon u^\varepsilon = \varepsilon T^\varepsilon \frac{\partial u^\varepsilon}{\partial x_1}. \tag{19}$$

By (H1) and (19), u_+^0 does not depend on y.

Claim 1: The average of p in y is zero: $\int_{Y(x)} p\, dy = 0$, a.e. $x \in \Omega_+$.

Information about p may be obtained by using oscillating test functions in the equation for u^ε (c.f. [43, 57]). The prototype $\varphi^\varepsilon = \varepsilon\phi(x)\{x_1/\varepsilon\}$, with $\phi \in C_0^\infty(\Omega_+)$ and $\{\cdot\}$ denoting fractional part, would serve the purpose in view of (22) because $T^\varepsilon \varphi^\varepsilon \to 0$ and $T^\varepsilon \nabla \varphi^\varepsilon \to (\phi, 0)$ strongly in $L^2(\Omega_u)$. As φ^ε are not necessarily continuous on Ω_+^ε, appropriate shifts are introduced as follows. Note that

$$\varepsilon\Big\{\frac{x_1}{\varepsilon}\Big\} = x_1 - x_k^\varepsilon, \quad \text{if } x_1 \in [x_k^\varepsilon, x_{k+1}^\varepsilon) \text{ and } x_k^\varepsilon = k\varepsilon.$$

Replace $x_k^\varepsilon = k\varepsilon$ with a grid where the graphs of $\eta(x_1, x_1/\varepsilon)$ and $\eta_-(x_1)$ are close:

$$x_k^\varepsilon \in \arg\min_{x \in \varepsilon[k,k+1]} \eta\Big(x, \frac{x}{\varepsilon}\Big), \quad k = 0, \ldots, \frac{1}{\varepsilon} - 1. \tag{20}$$

Let $\phi \in C_0^\infty(\Omega_+)$. Then with x_k^ε as in (20),

$$\varphi^\varepsilon(x) = (x_1 - x_k^\varepsilon)\phi(x), \quad \text{if } x_1 \in [x_k^\varepsilon, x_{k+1}^\varepsilon), \tag{21}$$

belongs to $C^\infty(\Omega_+^\varepsilon)$ for all small enough ε. Indeed,

$$\begin{aligned}
&\eta\Big(x_k^\varepsilon, \frac{x_k^\varepsilon}{\varepsilon}\Big) - \eta_-(x_k^\varepsilon) \\
&\quad = \min_{x \in \varepsilon[k,k+1]} \eta\Big(x, \frac{x}{\varepsilon}\Big) - \min_y \eta(x_k^\varepsilon, y) \\
&\quad \le \max_{\xi \in \varepsilon[k,k+1]} \Big(\min_y \eta(\xi, y) - \min_y \eta(x_k^\varepsilon, y)\Big) \\
&\quad \le C\varepsilon,
\end{aligned}$$

by the Lipschitz continuity of η_-, and ϕ is compactly supported.

With φ^ε given by (20), (21) as test functions in the equation (22) for u^ε, and using that

$$\begin{aligned}
T^\varepsilon \varphi^\varepsilon &\to 0, \\
T^\varepsilon \nabla \varphi^\varepsilon &\to (\phi, 0),
\end{aligned}$$

strongly in $L^2(\Omega_u)$ as ε tends to zero because $|x_1 - x_k^\varepsilon| \le \varepsilon$, one obtains in the limit

$$\int_{\Omega_u} p\phi\, dxdy = \int_{\Omega_+} \int_{Y(x)} p\, dy\, \phi\, dx = 0, \quad \phi \in C_0^\infty(\Omega_+).$$

which gives the first claim.

Claim 2: $\chi_{\Omega_-} u_-^0 + \chi_{\Omega_+} u_+^0 \in W(\Omega, \Gamma)$ **and is the solution to the homogenized problem** (2).

To verify that $\chi_{\Omega_-} u_-^0 + \chi_{\Omega_+} u_+^0 \in W(\Omega, \Gamma)$ it suffices to check that $u_-^0 = u_+^0$ in $L^2(\Gamma_-)$. Note that $u_+^0 \in L^2(\Omega_u)$ is continuously traced into $L^2(\Gamma_-)$ because $h > 0$ in a neighborhood of Γ_- in Ω (see Remark 6.1).

Because $u^\varepsilon \rightharpoonup u_-^0$ weakly in $H^1(\Omega_-)$, the weak continuity of the trace gives $u^\varepsilon \rightharpoonup u_-^0$ weakly in $L^2(\Gamma_-)$, and so $T^\varepsilon u^\varepsilon(x_1, \eta_-(x_1)) \rightharpoonup u_-^0(x_1, \eta_-(x_1))$ weakly in $L^2((0,1) \times \mathbb{T})$, as ε tends to zero. Denote $\Gamma_\varepsilon = \Omega_\varepsilon \cap \Gamma_-$. Let $\phi \in C_0^\infty(\Omega)$. On the one hand,

$$\begin{aligned}
\int_{\Gamma_\varepsilon} u^\varepsilon \phi \nu_2\, d\sigma &= \int_0^1 (\chi_{\Gamma_\varepsilon} u^\varepsilon \phi \nu_2)(x_1, \eta_-(x_1))\, dx_1 \\
&= \int_0^1 \int_{\mathbb{T}} T^\varepsilon(\chi_{\Gamma_\varepsilon} u^\varepsilon \phi \nu_2)(x_1, \eta_-(x_1))\, dy\, dx_1 \\
&\to \int_0^1 \int_{\mathbb{T}} (\chi_{Y(x)} u_-^0 \phi \nu_2)(x_1, \eta_-(x_1))\, dy\, dx_1 \\
&= \int_{\Gamma_-} h u_-^0 \phi \nu_2\, d\sigma,
\end{aligned}$$

as ε tends to zero, because $T^\varepsilon \chi_{\Gamma_\varepsilon}(x_1, \eta_-(x_1))$ converges to $\chi_{Y(x_1, \eta_-(x_1))}(y)$ strongly in $L^2((0,1) \times \mathbb{T})$. On the other hand,

$$\begin{aligned}
\int_{\Gamma_\varepsilon} u^\varepsilon \phi \nu_2\, d\sigma &= \int_{\Omega_u} T^\varepsilon \frac{\partial u^\varepsilon}{\partial x_2} T^\varepsilon \phi\, dxdy + \int_{\Omega_u} T^\varepsilon u^\varepsilon T^\varepsilon \frac{\partial \phi}{\partial x_2}\, dxdy + o(1) \\
&\to \int_{\Omega_u} \frac{\partial u_+^0}{\partial x_2} \phi\, dxdy + \int_{\Omega_u} u_+^0 \frac{\partial \phi}{\partial x_2}\, dxdy \\
&= \int_{\Gamma_-} h u_+^0 \phi \nu_2\, d\sigma,
\end{aligned}$$

as ε tends to zero. Thus

$$\int_{\Gamma_-} (u_+^0 - u_-^0) \phi h \nu_2\, d\sigma = 0, \quad \phi \in C_0^\infty(\Omega).$$

It follows that $u_-^0 = u_+^0$ in $L^2(\Gamma_-)$, for by the Lipschitz continuity of η_-,

$$h\nu_2 \ge \frac{\min\{h(x) : x \in \Gamma_-\}}{\sqrt{1 + \max(\eta_-')^2}} > 0.$$

A split of the variational form (14) of problem (1) into the periodic Ω_+^ε and the fixed Ω_- reads

$$\int_{\Omega_-} \nabla u^\varepsilon \cdot \nabla \psi \, dx + \int_{\Omega_+^\varepsilon} \nabla u^\varepsilon \cdot \nabla \psi \, dx = \int_{\Omega^\varepsilon} f \psi \, dx.$$

After unfolding Ω_+^ε and using Lemma 5.1 one arrives with $\psi \in C^\infty(\overline{\Omega}, \Gamma)$ at

$$\int_{\Omega_-} \nabla u^\varepsilon \cdot \nabla \psi \, dx + \int_{\Omega_u} T^\varepsilon \nabla u^\varepsilon \cdot T^\varepsilon \nabla \psi \, dxdy = \int_{\Omega^\varepsilon} f \psi \, dx + o(1). \tag{22}$$

By passing to the limit in (22) with $\psi \in C^\infty(\overline{\Omega}, \Gamma)$, using the weak convergence of u^ε, $T^\varepsilon u^\varepsilon$, $T^\varepsilon \nabla u^\varepsilon$ (16)–(18), and that p is of average zero in y, one obtains that $\chi_{\Omega_-} u_-^0 + \chi_{\Omega_+} u_+^0$ satisfies

$$\int_{\Omega_-} \nabla u_-^0 \cdot \nabla \psi \, dx + \int_{\Omega_u} \frac{\partial u_+^0}{\partial x_2} \frac{\partial \psi}{\partial x_2} \, dy \, dx = \int_\Omega \int_{Y(x)} f \psi \, dy \, dx, \tag{23}$$

as ε tends to zero.

Let

$$\Omega_U = \{(x, y) : x \in \Omega, \ y \in Y(x)\},$$

and

$$W(\Omega_U, \Gamma \times \mathbb{T}) = \{v : v \in L^2(\Omega_U), \ \frac{\partial v}{\partial x_1} \in L^2(\Omega_- \times \mathbb{T}), \ \frac{\partial v}{\partial x_2} \in L^2(\Omega_U), \\ \nabla_y v = 0 \text{ in } \Omega_U, \ v = 0 \text{ on } \Gamma \times \mathbb{T}\}.$$

By the density of $C^\infty(\overline{\Omega}, \Gamma)$ in $W(\Omega_U, \Gamma \times \mathbb{T})$ under hypothesis (H1), (23) holds for any test function in $W(\Omega_U, \Gamma \times \mathbb{T})$:

$$\int_{\Omega_U} \left(\chi_{\Omega_- \times \mathbb{T}} \frac{\partial u_-^0}{\partial x_1} \frac{\partial \psi}{\partial x_1} + \frac{\partial u_+^0}{\partial x_2} \frac{\partial \psi}{\partial x_2} \right) dxdy = \int_{\Omega_U} f \psi \, dxdy, \tag{24}$$

for any $\psi \in W(\Omega_U, \Gamma \times \mathbb{T})$. The equation (24) is well-posed in the Hilbert space $W(\Omega_U, \Gamma \times \mathbb{T})$. Under (H1), the problem (24) is equivalent to (2), and v belongs to the weighted space $W(\Omega, \Gamma)$ if and only if it belongs to $W(\Omega_U, \Gamma \times \mathbb{T})$. One concludes that $u^0 = \chi_{\Omega_-} u_-^0 + \chi_{\Omega_+} u_+^0$.

Claim 3: $\widetilde{u^\varepsilon} \rightharpoonup h u^0$ weakly in $L^2(\Omega)$.

The uniqueness of the solution u^0 to the homogenized problem (2) ensures that the full ε sequences (16)–(18) converge.

The weak limit of a sequence is obtained from the weak unfolding limit (weak two-scale limit) by taking the average over the cell of periodicity. Because $T^\varepsilon u^\varepsilon$ converges weakly to u^0, and u^0 does not depend on y,

$$\widetilde{u^\varepsilon} \rightharpoonup \int_{Y(x)} u^0\, dy = hu^0,$$

weakly in $L^2(\Omega_+)$, as ε tends to zero. It follows that $\widetilde{u^\varepsilon} \rightharpoonup hu^0$ weakly in $L^2(\Omega)$, as ε tends to zero.

Claim 4: $\widetilde{\nabla u^\varepsilon} \rightharpoonup hA^0\nabla u^0$ **weakly in** $L^2(\Omega)$.

By the same property of weakly converging unfolding, as was used in the previous paragraph, and that p is of average zero in y,

$$\widetilde{\nabla u^\varepsilon} \rightharpoonup \int_{Y(x)} \big(p, \frac{\partial u^0}{\partial x_2}\big)\, dy = \big(0, h\frac{\partial u^0}{\partial x_2}\big),$$

weakly in $L^2(\Omega_+)$, as ε tends to zero. It follows that $\widetilde{\nabla u^\varepsilon} \rightharpoonup hA^0\nabla u^0$ weakly in $L^2(\Omega)$, as ε tends to zero.

Remark 6.1 (About the L^2 trace) Let $\gamma(v)(x) = v(x)$ for $x \in \Gamma_-$, $v \in C^\infty(\overline{\Omega_+})$. Then $\gamma : \{v \in L^2(\Omega_+) : \frac{\partial v}{\partial x_2} \in L^2(\Omega_+)\} \to L^2(\Gamma_-)$ is linear and bounded, where the natural Hilbert space structures are employed. Indeed, for $v \in C^\infty(\overline{\Omega_+})$,

$$v(x_1, \eta_-(x_1)) = -\int_{\eta_-(x_1)}^{x_2} \frac{\partial v}{\partial x_2}(x_1, x_2)\, dx_2 + v(x_1, x_2).$$

By the triangle inequality for the integral, the Bunyakovsky-Cauchy-Schwarz inequality, and the inequality of arithmetic and geometric means,

$$v(x_1, \eta_-(x_1))^2 \le C\Big(\int_{\eta_-(x_1)}^{\eta_+(x_1)} \big(\frac{\partial v}{\partial x_2}\big)^2(x_1, x_2)\, dx_2 + v^2(x_1, x_2)\Big).$$

An integration in x_2 over the interval $(\eta_-(x_1), \eta_+(x_1))$ gives

$$v(x_1, \eta_-(x_1))^2 \le C\Big(\int_{\eta_-(x_1)}^{\eta_+(x_1)} \big(\frac{\partial v}{\partial x_2}\big)^2(x_1, x_2)\, dx_2 + \int_{\eta_-(x_1)}^{\eta_+(x_1)} v^2(x_1, x_2)\, dx_2\Big).$$

An integration in x_1 over the interval $(0, 1)$ gives

$$\int_{\Gamma_-} v^2\, d\sigma \le C\Big(\int_{\Omega_+} \big(\frac{\partial v}{\partial x_2}\big)^2\, dx + \int_{\Omega_+} v^2\, dx\Big).$$

The set $C^\infty(\overline{\Omega_+})$ is dense in the domain of γ.

7 Justification

In this section it is shown that the error in approximating the solutions u^ε to (1) and their flows with the solution u^0 to the homogenized problem (2) and its flow tends to zero, measured in $L^2(\Omega^\varepsilon, h)$. The method we use is the convergence of energy.

Theorem 7.1 *Suppose that (H1) holds. Let $u^\varepsilon \in H^1(\Omega^\varepsilon, \Gamma)$ be the solutions to (1), and let $u^0 \in W(\Omega, \Gamma)$ be the solution to (2). Then*

$$\text{(i)} \quad \|u^\varepsilon - u^0\|_{L^2(\Omega^\varepsilon, h)} \to 0,$$
$$\text{(ii)} \quad \|\nabla u^\varepsilon - A^0 \nabla u^0\|_{L^2(\Omega^\varepsilon, h)} \to 0,$$

as ε tends to zero.

Proof By the weak convergence of u^ε, $T^\varepsilon u^\varepsilon$, $T^\varepsilon \nabla u^\varepsilon$ (16)–(18), the property that sum of lim inf is less than or equal to lim inf of sum, using that u^ε, u^0 solve (22), (15),

$$\begin{aligned}
&\int_{\Omega_u} p^2\, dxdy + \int_\Omega hA^0 \nabla u^0 \cdot \nabla u^0\, dx \\
&\quad \le \liminf_{\varepsilon \to 0} \int_{\Omega_u} |T^\varepsilon \nabla u^\varepsilon|^2\, dxdy + \liminf_{\varepsilon \to 0} \int_{\Omega_u^\varepsilon \setminus \Omega_u} |T^\varepsilon \nabla u^\varepsilon|^2\, dxdy \\
&\qquad + \liminf_{\varepsilon \to 0} \int_{\Omega_-} |\nabla u^\varepsilon|^2\, dx \\
&\quad \le \liminf_{\varepsilon \to 0} \Big(\int_{\Omega_u^\varepsilon} |T^\varepsilon \nabla u^\varepsilon|^2\, dxdy + \int_{\Omega_-} |\nabla u^\varepsilon|^2\, dx \Big) \\
&\quad = \liminf_{\varepsilon \to 0} \Big(\int_{\Omega_u^\varepsilon} T^\varepsilon f\, T^\varepsilon u^\varepsilon\, dxdy + \int_{\Omega_-} f u^\varepsilon\, dx \Big) \\
&\quad = \liminf_{\varepsilon \to 0} \Big(\int_{\Omega_u} T^\varepsilon f\, T^\varepsilon u^\varepsilon\, dxdy + \int_{\Omega_-} f u^\varepsilon\, dx \Big) \\
&\quad = \int_\Omega f u^0 h\, dx \\
&\quad = \int_\Omega hA^0 \nabla u^0 \cdot \nabla u^0\, dx.
\end{aligned}$$

This shows that the energies converge:

$$\begin{aligned}
&\int_{\Omega_u^\varepsilon} (T^\varepsilon u^\varepsilon)^2\, dxdy + \int_{\Omega_u^\varepsilon} |T^\varepsilon \nabla u^\varepsilon|^2\, dxdy + \int_{\Omega_-} |\nabla u^\varepsilon|^2\, dx \\
&\quad \to \int_{\Omega_u} (u^0)^2\, dxdy + \int_{\Omega_u} A^0 \nabla u^0 \cdot \nabla u^0\, dxdy + \int_{\Omega_-} |\nabla u^0|^2\, dx, \qquad (25)
\end{aligned}$$

as ε tends to zero, as well as

$$\lim_{\varepsilon \to 0} \int_{\Omega_u^\varepsilon \setminus \Omega_u} |T^\varepsilon \nabla u^\varepsilon|^2\, dxdy = 0.$$

It follows that each weak convergence in (16)–(18) is strong, and $p = 0$. For (i),

$$\int_{\Omega^\varepsilon} (u^\varepsilon - u^0)^2 h\, dx = \textstyle\int_{\Omega_u^\varepsilon} T^\varepsilon (u^\varepsilon - u^0)^2 T^\varepsilon h\, dxdy + \int_{\Omega_-} (u^\varepsilon - u^0)^2\, dx \to 0,$$

as ε tends to zero. The convergence in (ii) then follows by the same argument.

As $T^\varepsilon \frac{\partial u^\varepsilon}{\partial x_1}$ converges to zero strongly in $L^2(\Omega_u)$, the proof of Theorem 7.1 shows that

$$\widetilde{\frac{\partial u^\varepsilon}{\partial x_1}} \to 0 \quad \text{strongly in } L^2(\Omega_+),$$

as ε tends to zero, which justifies the first term in the asymptotic expansion under hypothesis (H1).

8 Average Preserving Extension

In this section we prove the homogenization Theorem 6.1 for an average preserving extension (see [28, 41]). The method we use is the strong unfolding convergence obtained in the proof of Theorem 7.1, which is stated explicitly in the form of Lemma 8.1 below.

Throughout this section (H1) is assumed to hold. By the end of the section, a remark about (H2) is included.

Let the local average of a function v in the x_1 direction over Ω_+^ε be denoted by

$$m_\varepsilon(v)(x) = \frac{1}{h(x)} \int_{Y(x)} (T^\varepsilon v)(x, y)\, dy, \quad x \in \Omega_+, \tag{26}$$

where $m_\varepsilon(v)(x)$ is set to zero at points where $h(x) = |Y(x)| = 0$. The average preserving extension is defined by

$$\widetilde{u^\varepsilon}^m(x) = \begin{cases} u^\varepsilon(x) & \text{if } x \in \Omega^\varepsilon, \\ m_\varepsilon(u^\varepsilon)(x) & \text{if } x \in \Omega \setminus \Omega^\varepsilon. \end{cases} \tag{27}$$

For $v \in H^1(\Omega^\varepsilon)$, $m_\varepsilon(v)$ and $m_\varepsilon(\nabla v)$ are well-defined, using the Sobolev space property that v has a representative that is absolutely continuous on almost all line segments parallel to the coordinate axes and with square integrable partial derivatives. In particular, v belongs to $H^1(\{(t, x_2) : t \in \mathbb{R}\} \cap \Omega_+^\varepsilon)$ for almost all x_2 of relevance.

Theorem 8.1 *Suppose that (H1) holds. Let $u^\varepsilon \in H^1(\Omega^\varepsilon, \Gamma)$ be the solutions to (1), and let $u^0 \in W(\Omega, \Gamma)$ be the solution to (2). Then*

$$\text{(i)} \quad \widetilde{u^\varepsilon}^m \to u^0 \quad \textit{strongly in } L^2(\Omega, h),$$
$$\text{(ii)} \quad \widetilde{\nabla u^\varepsilon}^m \to A^0 \nabla u^0 \quad \textit{strongly in } L^2(\Omega, h),$$

as ε tends to zero, where $\sim m$ denotes the extension (26), (27).

In the proof of Theorem 7.1, the following strong convergence of the unfolded sequences were obtained.

Lemma 8.1 *Suppose that (H1) holds. Let $u^\varepsilon \in H^1(\Omega^\varepsilon, \Gamma)$ be the solutions to (1), and let $u^0 \in W(\Omega, \Gamma)$ be the solution to (2). Then*

$$\text{(i)} \quad T^\varepsilon u^\varepsilon \to u^0 \quad \textit{strongly in } L^2(\Omega_u),$$

$$\text{(ii)} \quad T^\varepsilon \nabla u^\varepsilon \to \Big(0, \frac{\partial u^0}{\partial x_2}\Big) \quad \textit{strongly in } L^2(\Omega_u),$$

as ε tends to zero.

Proof of Theorem 8.1 Because the extension does not alter the functions u^ε in Ω_-, the Ω_- parts of (i) and (ii) are included in Theorem 7.1. In Ω_+, by definition,

$$\|\widetilde{u^\varepsilon}^m - u^0\|^2_{L^2(\Omega_+, h)} = \|u^\varepsilon - u^0\|^2_{L^2(\Omega^\varepsilon_+, h)} + \|m_\varepsilon(u^\varepsilon) - u^0\|^2_{L^2(\Omega_+ \setminus \Omega^\varepsilon_+, h)}.$$

The first term on the right hand side tends to zero as ε tends to zero according to Theorem 7.1. The second term may be estimated as follows. Because u^0 is independent of y, and $T^\varepsilon u^\varepsilon$ converges to u^0 strongly in $L^2(\Omega_u)$, by Theorem 6.1 and Lemma 8.1(i), the Hölder inequality gives

$$\begin{aligned}
&\|m_\varepsilon(u^\varepsilon) - u^0\|^2_{L^2(\Omega_+, h)} \\
&= \int_{\Omega_+} \Big(\frac{1}{h(x)} \int_{Y(x)} T^\varepsilon u^\varepsilon \, dy - u^0\Big)^2 h \, dx \\
&= \int_{\Omega_+} \Big(\frac{1}{h(x)} \int_{Y(x)} T^\varepsilon u^\varepsilon \, dy - \frac{1}{h(x)} \int_{Y(x)} u^0 \, dy\Big)^2 h \, dx \\
&\le \int_{\Omega_u} (T^\varepsilon u^\varepsilon - u^0)^2 \, dx dy \to 0,
\end{aligned}$$

as ε tends to zero, which gives part (i).

Part (ii) is obtained by repeating the above two steps for the components of ∇u^ε in place of u^ε, using the second parts of Theorem 7.1 and Lemma 8.1.

Remark 8.1 (Hypothesis (H2)) The Theorems 6.1, 7.1, 8.1, and Lemma 8.1, hold under the slightly weaker hypothesis (H2). The weak limit u^0_+ of $T^\varepsilon u^\varepsilon$ is the first component of the unique solution (u^0_+, u^0_-) in the Sobolev space

$$\begin{aligned}
W = \Big\{(v, w) : v \in L^2(\Omega_u), \ \frac{\partial v}{\partial x_2} \in L^2(\Omega_u), \ \frac{\partial v}{\partial y} = 0, \\
w \in H^1(\Omega_-, \Gamma), \ v = w \text{ on } \Gamma_- \times \mathbb{T}\Big\},
\end{aligned}$$

to the following problem: Find $(u_+, u_-) \in W$ such that

$$\int_{\Omega_+} \int_{Y(x)} \frac{\partial u_+}{\partial x_2} \frac{\partial \varphi}{\partial x_2} \, dy \, dx + \int_{\Omega_-} \nabla u_- \cdot \nabla \phi \, dx = \int_{\Omega_+} f \int_{Y(x)} \varphi \, dy \, dx + \int_{\Omega_-} f \phi \, dx, \tag{28}$$

for any $(\varphi, \phi) \in W$. This statement relies on the density of $C^\infty(\overline{\Omega}, \Gamma)$ in W (c.f. Sect. 10 below).

By (H2), $(u_+^0(x, k/N + y), u_-^0)$, $k \in \mathbb{Z}$, and $(u_+^0(x, y_0 - y), u_-^0)$ solve the problem (28). By uniqueness of solution, $u_+^0(x, y) = u_+^0(x, k/N + y)$, $k \in \mathbb{Z}$, and $u_+^0(x, y) = u_+^0(x, y_0 - y)$. Because all $Y_0(x)$ are connected, $\frac{\partial u_+^0}{\partial y} = 0$ implies that u_+^0 is constant in y.

9 A Numerical Example

To illustrate the rate of convergence in Theorem 7.1 under hypothesis (H2), we consider the following example. Let u^ε be the solutions to (1) with $f = 1$ and

$$\eta(x, y) = \frac{1}{4}(1 + x\cos(4\pi x))(3 - \cos(2\pi y))b(y), \tag{29}$$

where a bump is introduced at y_0 by

$$b(y) = \begin{cases} 1 - 2(\delta - |y - y_0|) & \text{if } |y - y_0| \le \delta, \\ 1 & \text{otherwise,} \end{cases}$$

with $\delta = 1/10$ and $y_0 = 1/2$, $y \in [0, 1)$. Then η satisfies (H2) but not (H1). Moreover, $h = 1$ on the graph Γ_- of η_-, and $h = 0$ on the graph of η_+. The domains Ω^ε and Ω are illustrated in Figure 3.

The solutions u^ε, u^0 to the problems (1), (2) are approximated by means of the finite element method using piecewise linear Lagrange elements. The numerical approximations are denoted by u_s^ε, u_s^0. The numerically computed rates of convergence for the approximation in Theorem 6.1, are illustrated in Figure 5, obtained using the numerical tool FreeFEM [37]. The data points were obtained using the values of ε and the number of degrees of freedom given in Table 1.

One observes that the rate of convergence for u^ε appears to be close to $\varepsilon^{3/4}$, and for ∇u^ε close to $\varepsilon^{1/4}$, for the approximation measured in $L^2(\Omega^\varepsilon, h)$ for the selected values of ε.

10 A Case of Non-connected Sections

In this section we consider a case where the sections $Y(x)$ are allowed to be disconnected, that is the hypotheses (H1) and (H2) are mildly relaxed. First we describe the weak unfolding limit of the zero extended solutions to problem (1), and then we describe the weak limit in terms of local domain densities using the ideas of Mel'nyk, splitting the oscillating part of the domain into branches (c.f. [29, 30, 46]).

Here, the full unfolded limit domain will be used, denoted by

$$\Omega_U = \{(x, y) : x \in \Omega,\ y \in Y(x)\},$$

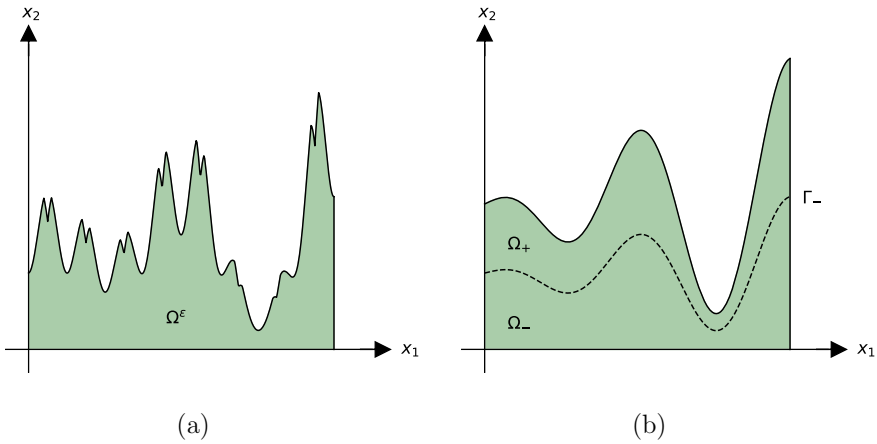

Fig. 3 A locally periodic domain Ω^ε (a) that satisfies (H2) but not (H1), $\varepsilon = 1/8$, and the corresponding homogeneous domain Ω (b), with Γ_- marked as a dashed line separating the regions Ω_+ and Ω_-. The particular function η is specified in (29)

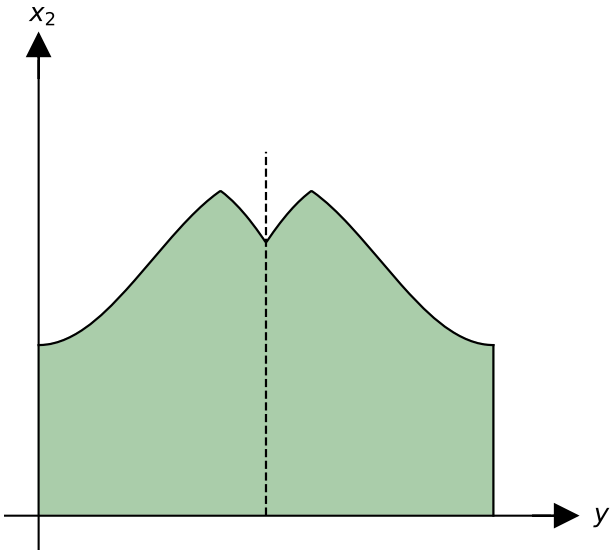

Fig. 4 The region below the graph $x_2 = \eta(1/2, y)$ for η given by (29) that satisfies (H2) but not (H1), with symmetry line $y = 1/2$ marked with a dashed line

and the effective matrix is given by

$$A^0 = \begin{pmatrix} \chi_{\Omega_- \times \mathbb{T}} & 0 \\ 0 & 1 \end{pmatrix}. \tag{30}$$

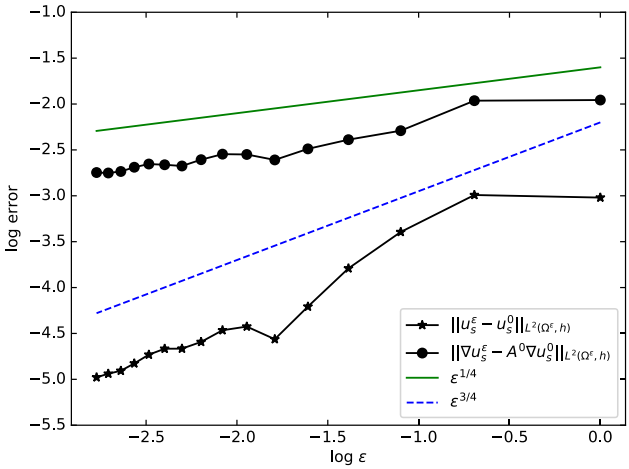

Fig. 5 The numerically computed rates of convergence for the approximations u^0 to u^ε, and $A^0\nabla u^0$ to ∇u^ε, in Theorem 7.1 for $f = 1$ and Ω^ε given by (29) and illustrated for $\varepsilon = 1/8$ in Figure 3

Table 1 The numerically computed errors for the approximation in Theorem 6.1, rounded to three significant digits

ε	$\|u_s^\varepsilon - u_s^0\|_{L^2(\Omega^\varepsilon,h)}$	$\|\nabla u_s^\varepsilon - A^0\nabla u_s^0\|_{L^2(\Omega^\varepsilon,h)}$	dof
0	–	–	$78 \cdot 10^2$
1/1	0.0488	0.141	$39 \cdot 10^3$
1/2	0.0502	0.140	$46 \cdot 10^3$
1/3	0.0336	0.101	$33 \cdot 10^3$
1/4	0.0226	0.0918	$27 \cdot 10^3$
1/5	0.0149	0.0830	$59 \cdot 10^3$
1/6	0.0104	0.0736	$69 \cdot 10^3$
1/7	0.0120	0.0781	$77 \cdot 10^3$
1/8	0.0115	0.0784	$11 \cdot 10^4$
1/9	0.0101	0.0738	$13 \cdot 10^4$
1/10	0.00941	0.0689	$15 \cdot 10^4$
1/11	0.00940	0.0698	$20 \cdot 10^4$
1/12	0.00882	0.0704	$23 \cdot 10^4$
1/13	0.00801	0.0680	$24 \cdot 10^4$
1/14	0.00740	0.0649	$35 \cdot 10^4$
1/15	0.00716	0.0638	$35 \cdot 10^4$
1/16	0.00688	0.0642	$37 \cdot 10^4$

The number of degrees of freedom, rounded to two significant digits, is abbreviated to dof. The row $\varepsilon = 0$ is for the numerical solution u_s^0 to the homogenized problem

The limit problem reads

$$\begin{aligned} -\mathrm{div}_x(A^0\nabla_x u) &= f \quad \text{in } \Omega_U, \\ \nabla_y u &= 0 \quad \text{in } \Omega_U, \\ A^0\nabla_x u\cdot\nu_x &= 0 \quad \text{on } \partial\Omega_U\setminus\Gamma\times\mathbb{T}, \\ u &= 0 \quad \text{on } \Gamma\times\mathbb{T}, \end{aligned} \tag{31}$$

where ν_x denotes the projection of the outward unit normal to Ω_U onto $\mathbb{R}^2$. The limit problem (31) is well-posed in the Sobolev space

$$\begin{aligned} &W(\Omega_U,\Gamma\times\mathbb{T}) \\ &\quad = \{v\in L^2(\Omega_U) : A^0\nabla_x v\in L^2(\Omega_U),\ \nabla_y v = 0 \text{ in } \Omega_U,\ v = 0 \text{ on } \Gamma\times\mathbb{T}\}, \end{aligned}$$

equipped with the natural Hilbert space structure.

Theorem 10.1 *Let $u^\varepsilon \in H^1(\Omega^\varepsilon,\Gamma)$ be the solutions to problem (1). Let $u^0 \in W(\Omega_U,\Gamma\times\mathbb{T})$ be the solution to (31). Then*

$$\begin{aligned} &\text{(i)}\quad \widetilde{u^\varepsilon} \rightharpoonup \int_{Y(x)} u^0\,dy \quad \textit{weakly in } L^2(\Omega), \\ &\text{(ii)}\quad \widetilde{\nabla u^\varepsilon} \rightharpoonup \int_{Y(x)} A^0\nabla_x u^0\,dy \quad \textit{weakly in } L^2(\Omega), \end{aligned}$$

as ε tends to zero.

Proof The following a priori estimate holds

$$\|u^\varepsilon\|_{H^1(\Omega_-)} + \|T^\varepsilon u^\varepsilon\|_{L^2(\Omega_u)} + \|T^\varepsilon\nabla u^\varepsilon\|_{L^2(\Omega_u)} \le C.$$

There exist

$$\begin{aligned} &u_-^0 \in H^1(\Omega_-,\Gamma), \\ &u_+^0 \in \Big\{v\in L^2(\Omega_u) : \frac{\partial v}{\partial x_2}\in L^2(\Omega_u),\ \nabla_y v = 0 \text{ in } \Omega_u\Big\}, \\ &p\in L^2(\Omega_u), \end{aligned}$$

such that, along a subsequence still denoted by ε,

$$\begin{aligned} u^\varepsilon &\rightharpoonup u_-^0 && \text{weakly in } H^1(\Omega_-,\ \Gamma), \\ T^\varepsilon u^\varepsilon &\rightharpoonup u_+^0 && \text{weakly in } L^2(\Omega_u), \\ T^\varepsilon\nabla u^\varepsilon &\rightharpoonup \Big(p,\ \frac{\partial u_+^0}{\partial x_2}\Big) && \text{weakly in } L^2(\Omega_u). \end{aligned}$$

At this point u_+^0 and p depend in general on the fast variable y.

The unfolded variational form of problem (7) is by Lemma 5.1,

$$\int_{\Omega_-} \nabla u^\varepsilon \cdot \nabla \psi \, dx + \int_{\Omega_u} T^\varepsilon \nabla u^\varepsilon \cdot T^\varepsilon \nabla \psi \, dxdy = \int_{\Omega_U} T^\varepsilon f \, T^\varepsilon \psi \, dxdy + o(1), \tag{32}$$

as ε tends to zero, for any sufficiently smooth $\psi \in H^1(\Omega^\varepsilon, \Gamma)$.

Step 1: $p = 0$ **a.e. in** Ω_u.

We will show that $p = 0$ almost everywhere in Ω_u. Let $\phi \in C_0^\infty(\Omega_u)$ and consider the sequence of test functions $\varphi^\varepsilon(x) = \varepsilon\phi\big(x, \frac{x_1}{\varepsilon}\big)$. Then

$$T^\varepsilon \varphi^\varepsilon \to 0,$$
$$T^\varepsilon \nabla \varphi^\varepsilon \to \big(\frac{\partial \phi}{\partial y}, 0\big),$$

strongly in $L^2(\Omega_u)$, as ε tends to zero. By passing to the limit in (32) with test functions φ^ε one obtains

$$\int_{\Omega_u} p \frac{\partial \phi}{\partial y} \, dxdy = 0, \quad \phi \in C_0^\infty(\Omega_u),$$

as ε tends to zero. That is, $\nabla_y p = 0$ in Ω_u.

Let $\phi \in C_0^\infty(\Omega_u)$ be such that $\nabla_y \phi = 0$. Then consider the sequence of test functions φ^ε defined by

$$\varphi^\varepsilon(x) = (x_1 - x_k^\varepsilon)\phi\big(x, \frac{x_1}{\varepsilon}\big), \quad \text{if } x_1 \in [x_k^\varepsilon, x_{k+1}^\varepsilon),$$

where x_k^ε is a tagging such as in (20). Then

$$T^\varepsilon \varphi^\varepsilon \to 0,$$
$$T^\varepsilon \nabla \varphi^\varepsilon \to (\phi, 0),$$

strongly in $L^2(\Omega_u)$, as ε tends to zero. By passing to the limit in (32) with test functions φ^ε one obtains

$$\int_{\Omega_u} p\phi \, dxdy = 0, \quad \phi \in C_0^\infty(\Omega_u, \nabla_y), \tag{33}$$

as ε tends to zero. By the density of $C_0^\infty(\Omega_u, \nabla_y)$ in the Sobolev space $\{v \in L^2(\Omega_u) : \nabla_y v = 0 \text{ in } \Omega_u\}$, one concludes that (33) holds for $\phi = p$, that is

$$\int_{\Omega_u} p^2 \, dxdy = 0.$$

Step 2: Transmission condition.

To determine the transmission condition for u_-^0 and u_+^0 on the internal interface

$$\Gamma_-^u = \{(x, y) : x \in \Gamma_-,\ y \in Y(x)\},$$

we will trace u^ε from either side.

Let $\phi \in C_0^\infty(\Omega_U)$, and set $\phi^\varepsilon(x, \frac{x_1}{\varepsilon})$.

On the one hand, from below,

$$\begin{aligned}\int_{\Gamma_\varepsilon} u^\varepsilon \phi^\varepsilon \nu_2 \, d\sigma &= \int_0^1 (\chi_{\Gamma_\varepsilon} u^\varepsilon \phi^\varepsilon \nu_2)(x_1, \eta_-(x_1)) \, dx_1 \\ &= \int_0^1 \int_{\mathbb{T}} T^\varepsilon (\chi_{\Gamma_\varepsilon} u^\varepsilon \phi \nu_2)(x_1, \eta_-(x_1)) \, dy \, dx_1 \\ &\to \int_0^1 \int_{\mathbb{T}} (\chi_{Y(x)} u_-^0 \phi \nu_2)(x_1, \eta_-(x_1), y) \, dy \, dx_1 \\ &= \int_{\Gamma_-} \int_{Y(x)} u_-^0 \phi \nu_2 \, dy \, d\sigma,\end{aligned}$$

as ε tends to zero, because $T^\varepsilon \chi_{\Gamma_\varepsilon}(x_1, \eta_-(x_1))$ converges to $\chi_{Y(x_1, \eta_-(x_1))}(y)$ strongly in $L^2((0, 1) \times \mathbb{T})$.

On the other hand, from above,

$$\begin{aligned}\int_{\Gamma_\varepsilon} u^\varepsilon \phi^\varepsilon \nu_2 \, d\sigma &= \int_{\Omega_u} T^\varepsilon \frac{\partial u^\varepsilon}{\partial x_2} T^\varepsilon \phi^\varepsilon \, dxdy + \int_{\Omega_u} T^\varepsilon u^\varepsilon T^\varepsilon \frac{\partial \phi^\varepsilon}{\partial x_2} \, dxdy + o(1) \\ &\to \int_{\Omega_u} \frac{\partial u_+^0}{\partial x_2} \phi \, dxdy + \int_{\Omega_u} u_+^0 \frac{\partial \phi}{\partial x_2} \, dxdy \\ &= \int_{\Gamma_-} \int_{Y(x)} u_+^0 \phi \nu_2 \, dy \, d\sigma,\end{aligned}$$

as ε tends to zero.

Thus

$$\int_{\Gamma_-} \int_{Y(x)} (u_+^0 - u_-^0) \phi \nu_2 \, dy \, d\sigma = 0, \quad \phi \in C_0^\infty(\Omega_U).$$

It follows that

$$u_-^0 = u_+^0 \quad \text{in } L^2(\Gamma_-^u), \tag{34}$$

for by the Lipschitz continuity of η_-,

$$\nu_2 \geq \frac{1}{\sqrt{1 + \max(\eta_-')^2}} > 0.$$

The transmission condition (34) means here that the trace of u_+^0 on each connected component of the sections $Y(x)$ for $x \in \Gamma_-$ is equal to the trace of u_-^0. That is, zero jump condition into each branch from below.

Step 3: Limit problem.

Let $\varphi \in C^\infty(\overline{\Omega_U})$ be such that $\nabla_y \varphi = 0$, and φ vanishes on $\Gamma \times \mathbb{T}$. Set $\psi^\varepsilon(x) = \varphi(x, \frac{x_1}{\varepsilon})$, which belongs to $H^1(\Omega^\varepsilon, \Gamma)$. Using ψ^ε as test functions in (32), one obtains

$$\int_{\Omega_U} A^0 \nabla_x u \cdot \nabla_x \varphi \, dxdy = \int_{\Omega_U} f \varphi \, dxdy, \tag{35}$$

in the limit as ε tends to zero, where $u = \chi_{\Omega_u} u_+^0 + \chi_{\Omega_- \times \mathbb{T}} u_-^0 \in W(\Omega_U, \Gamma \times \mathbb{T})$ by Step 1. In the upper part Ω_u one uses $p = 0$ by Step 2. By the density of the considered set of test functions in $W(\Omega_U, \Gamma \times \mathbb{T})$, as Ω_U is Lipschitz, one has that $u = u^0$ is the unique solution to problem (31). One concludes that the weak convergences (i) and (ii) have been established for the full sequences.

Remark that Lemma 8.1 holds without hypotheses (H1), (H2), by the same argument, and it takes the following form, which is the result that corresponds to Theorem 7.1 in this case.

Lemma 10.1 *Let $u^\varepsilon \in H^1(\Omega^\varepsilon, \Gamma)$ be the solutions to (1), and let $u^0 \in W(\Omega_U, \Gamma \times \mathbb{T})$ be the solution to (31). Then*

$$\text{(i)} \quad T^\varepsilon u^\varepsilon \to u^0 \quad \textit{strongly in } L^2(\Omega_U),$$

$$\text{(ii)} \quad T^\varepsilon \nabla u^\varepsilon \to \Big(\chi_{\Omega_- \times \mathbb{T}} \frac{\partial u^0}{\partial x_1}, \frac{\partial u^0}{\partial x_2}\Big) \quad \textit{strongly in } L^2(\Omega_U),$$

as ε tends to zero.

We will conclude this section by translating the limit problem (31) into a system in the original coordinates $x \in \Omega$, ($\overline{\Omega}$ is the Hausdorff limit of $\overline{\Omega^\varepsilon}$), in a particular case of a finite number of 'bumps' on the boundary.

Lemma 10.2 *Let ω be an open set in $\mathbb{R}^N$. Then there exists an open cover K' of ω such that for all $v \in L^2(\omega)$ with $\nabla_{x'} v = 0$ almost everywhere in ω, $(x, x') \in \omega$, one has that v is constant in x' almost everywhere in each $V \in K'$.*

Proof A subset V of ω is said to have the property (P) if the intersection of V and any k-plane parallel to the x'-coordinate k-planes is connected, $x' \in \mathbb{R}^k$. For $(x, x') \in \omega$, let $V(x, x')$ be a neighborhood of (x, x') in ω that is not contained in any distinct neighborhood of (x, x') in ω with the property (P). By the Zorn lemma, at least one such maximal element exists for the set inclusion partial order on the set of all neighborhoods of (x, x') with the property (P), because ω is open and if F is a totally ordered subset, $\cup_{V \in F} V$ is an upper bound for F. Then $K' = \{V(x, x') : (x, x') \in \omega\}$ is an open cover of ω.

Let $v \in L^2(\omega)$ be such that $\nabla_{x'} v = 0$ almost everywhere in ω. Then v has a representative $\tilde{v}$ that is absolutely continuous on almost all line segments parallel to the

x'-coordinate axes and whose classical partial derivatives parallel to the x'-coordinate axes belong to $L^2(\omega)$. It follows from Fubini's theorem that $\tilde{v}$ assumes constant value on each connected component of each k-plane parallel to the x'-coordinate k-planes when intersected with ω. Therefore v is constant in x' almost everywhere on each $V \in K'$.

Now we restrict to domains such that there is a finite open cover of Ω_U as in Lemma 10.2 with $\omega = \Omega_U$ and $x' = y$, effectively discarding domains with a countably infinite number of 'bumps' on the boundary. For instance, excluding $\eta(x, y) = x_1^2 \sin(\frac{y}{x_1})$, and $\eta(x, y) = \mathrm{dist}(y, K)$, where K is a Cantor set.

We say that K is a partition of an open set ω in $\mathbb{R}^N$ if K consists of disjoint nonempty open subsets of ω such that $\overline{\omega} = \cup_{V \in K} \overline{V}$. Under the assumption that there is a finite open cover K' of Ω_U as in Lemma 10.2, we can construct a finite partition K of Ω_U as follows. Given $K' = \{V_1, \ldots, V_k\}$, let $K = \{V_1, V_2 \setminus \overline{V_1}, \ldots, V_k \setminus \cup_{j=1}^{k-1} \overline{V_j}\}$.

Let K be a finite partition of Ω_U with the same property as the open cover in Lemma 10.2. Denote the measure of the connected components of $Y(x)$ by

$$h_V = |Y(x) \cap \{y : (x, y) \in V\}|,$$

for $V \in K$, that is the density of Ω^ε in Ω. Denote the common boundaries of the subdomains $U, V \in K$ of the partition by

$$\Gamma_{U,V} = \partial U \cap \partial V.$$

Let the effective matrix be given by

$$A^0 = \begin{pmatrix} \chi_{\Omega_-} & 0 \\ 0 & 1 \end{pmatrix}.$$

Let

$$\begin{aligned} W_K = \big\{ \{v_V\}_{V \in K} : \; & v_V \in L^2(\pi_{\mathbb{R}^2} V, h_V), \\ & A^0 \nabla v_V \in L^2(\pi_{\mathbb{R}^2} V, h_V), \\ & [v] = 0 \text{ on } \pi_{\mathbb{R}^2} \Gamma_{U,V}, \\ & v_V = 0 \text{ on } \Gamma \cap \pi_{\mathbb{R}^2} \overline{V}, \\ & U, V \in K \big\}, \end{aligned}$$

where $\pi_{\mathbb{R}^2}$ denotes the projection onto $\mathbb{R}^2$, equipped with the natural Hilbert space structure.

The limit problem

$$\begin{aligned} -\mathrm{div}(h_V A^0 \nabla u_V) &= h_V f \quad && \text{in } \pi_{\mathbb{R}^2} V, \\ u_V &= 0 && \text{on } \Gamma \cap \pi_{\mathbb{R}^2} \overline{V}, \\ h_V A^0 \nabla u_V \cdot \nu &= 0 && \text{on } (\partial\Omega \setminus \Gamma) \cap \pi_{\mathbb{R}^2} \overline{V}, \end{aligned} \tag{36}$$

$$[u] = 0 \qquad \text{on } \pi_{\mathbb{R}^2}\Gamma_{U,V},$$
$$[hA^0\nabla u \cdot \nu] = 0 \qquad \text{on } \pi_{\mathbb{R}^2}\Gamma_{U,V},$$

is then well-posed in W_K.

Corollary 10.1 *Let $u^\varepsilon \in H^1(\Omega^\varepsilon, \Gamma)$ be the solutions to (1), and let $u^0 \in W(\Omega_U, \Gamma \times \mathbb{T})$ be the solution to (31). Let K be a finite partition of Ω_U with the same property as the open cover in Lemma 10.2 with $\omega = \Omega_U$ and $x' = y$. Then*

$$\text{(i)} \quad \widetilde{u^\varepsilon} \rightharpoonup \sum_{V \in K} h_V \chi_{\pi_{\mathbb{R}^2} V} u^0\big|_V \quad \textit{weakly in } L^2(\Omega),$$
$$\text{(ii)} \quad \widetilde{\nabla u^\varepsilon} \rightharpoonup \sum_{V \in K} h_V \chi_{\pi_{\mathbb{R}^2} V} A^0 \nabla u^0\big|_V \quad \textit{weakly in } L^2(\Omega),$$

as ε tends to zero. Moreover, $u^0 = u^0_V$ in $V \in K$, where $\{u^0_V\}_{V \in K}$ in W_K is the solution to (36).

Proof It follows from the weak unfolding convergence $T^\varepsilon u^\varepsilon \rightharpoonup u^0$ in $L^2(\Omega_U)$, that

$$\widetilde{u^\varepsilon} \rightharpoonup \int_{Y(x)} u^0 \, dy = \int_{Y(x)} \sum_{V \in K} \chi_V u^0 \, dy = \sum_{V \in K} h_V \chi_{\pi_{\mathbb{R}^2} V} u^0\big|_V,$$

as ε tends to zero. The same computation gives the convergence of the flows. By writing $u^0 = \sum_{V \in K} \chi_V u^0$, one verifies that the limit problem (36) is equivalent to the limit problem (31) with $u^0_V = \chi_V u^0$ on V, for any suitable partition K.

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Declarations

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