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# Analytic Bertini theorem

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## Abstract

We prove an analytic Bertini theorem, generalizing a previous result of Fujino and Matsumura.

**Keywords** Bertini theorem · Multiplier ideal sheaf · Pluri-subharmonic function · Hodge metric

**Mathematics Subject Classification** 32U15 · 32Q15

## 1 Introduction

Let  $X$  be a connected complex projective manifold of dimension  $n \geq 1$ . Given any base-point free linear system  $\Lambda$  on  $X$ , it follows from the classical Bertini theorem [9] that a general hyperplane  $H$  of  $\Lambda$  is smooth. Let  $\varphi$  be a quasi-plurisubharmonic (quasi-psh) function on  $X$ . For a general member  $H \in \Lambda$ , the multiplier ideal sheaf  $\mathcal{I}(\varphi|_H)$  makes sense. It is natural to wonder if

$$\mathcal{I}(\varphi|_H) = \mathcal{I}(\varphi)|_H \quad (1.1)$$

holds for general  $H$ . It is well-known that the  $\mathcal{I}(\varphi|_H) \subseteq \mathcal{I}(\varphi)|_H$  direction always holds for a general  $H$ , as a consequence of the Ohsawa–Takegoshi  $L^2$ -extension theorem. Conversely, it is easy to construct examples such that the set  $\mathcal{B}$  of  $H \in \Lambda$  where the equality fails is not contained in any proper Zariski closed subset of  $\Lambda$ . A natural question arises: is the set  $\mathcal{B}$  small in a suitable sense? This kind of problem was first studied by Fujino and Matsumura, see [4, 5]. They proved that the complement of  $\mathcal{B}$  is dense with respect to the complex topology of  $\Lambda$  (regarded as a projective space). More recently, Meng and Zhou [11] proved that the complement of  $\mathcal{B}$  has zero Lebesgue measure. In this paper, we prove the following refinement:

**Theorem 1.1** *There is a pluripolar set  $\Sigma \subseteq \Lambda$  such that for all  $H \in \Lambda \setminus \Sigma$ ,  $H$  is smooth and (1.1) holds.*

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This result affirmatively answers a problem of Boucksom, see [5, Question 1.2]. From the point of view of pluripotential theory, this theorem is quite natural: a small set in pluripotential theory just means a pluripolar set. As shown in [4, Example 3.12], the exceptional set is not contained in a countable union of proper Zariski closed subsets in general, so Theorem 1.1 seems to be the optimal result. We also prove a more general analytic Bertini type result for fibrations Corollary 2.9.

Let us mention a key advantage of Theorem 1.1: our theorem can be applied to a countable family of quasi-psh functions at the same time, see Corollary 2.10. This corollary makes it possible to perform induction on the dimension when studying psh singularities.

## 2 Analytic Bertini theorem

In this section, varieties or algebraic varieties mean reduced separated schemes of finite type over  $\mathbb{C}$ .

**Definition 2.1** Let  $Y$  be a complex projective manifold. A subset  $A \subseteq Y$  is

- (1) *co-pluripolar* if  $Y \setminus A$  is pluripolar. When  $\dim Y = 1$ , we also say  $A \subseteq Y$  is co-polar.
- (2) *co-meager* if  $Y \setminus A$  is contained in a countable union of proper Zariski closed sets.

We say a condition in  $y \in Y$  is satisfied *quasi-everywhere* if there is a co-pluripolar subset  $Y_0 \subseteq Y$  such that the condition is satisfied for  $y \in Y_0$ .

Clearly, a co-meager set is co-pluripolar. Both classes are preserved by countable intersections.

**Lemma 2.2** Let  $\pi: Y \rightarrow X$  be a smooth morphism of smooth algebraic varieties. Let  $\varphi$  be a quasi-plurisubharmonic function on  $X$ , then

$$\pi^* \mathcal{I}(\varphi) = \mathcal{I}(\pi^* \varphi). \quad (2.1)$$

Here  $\mathcal{I}(\varphi)$  denotes the multiplier ideal sheaf of  $\varphi$  in the sense of Nadel. Observe that as  $\pi$  is flat,  $\pi^* \mathcal{I}(\varphi)$  is a subsheaf of  $\mathcal{O}_Y$ , so in (2.1) equality makes sense, the two sheaves are actually equal, not just isomorphic.

**Proof** As pointed out by the referee,  $\pi^* \mathcal{I}(\varphi) \supseteq \mathcal{I}(\pi^* \varphi)$  is proved in [1, Proposition 14.3]. So it suffices to prove the reverse inclusion.

By decomposing  $\pi$  into the composition of an étale morphism and a projection locally, it suffices to deal with the two cases separately. Fix a local section  $f$  of  $\mathcal{I}(\varphi)$ .

Assume that  $\pi: X \times \mathbb{C}^n \rightarrow X$  is the natural projection. Fix a volume form  $dV$  on  $X$ . Take the product volume form  $dV \otimes d\lambda$  on  $X \times \mathbb{C}^n$ , where  $d\lambda$  denotes the Lebesgue measure. It follows from Fubini theorem that  $|\pi^* f|^2 e^{-\pi^* \varphi}$  is locally integrable with respect to  $dV \otimes d\lambda$ .

Now assume that  $\pi$  is étale. The change of variable formula shows that  $|\pi^* f|^2 e^{-\pi^* \varphi}$  is locally integrable.  $\square$

In Lemma 2.2, we do not really need the algebraic structures on  $X$  and  $Y$ . For general complex manifolds, it suffices to apply the co-area formula.

We recall the notion of positive metrics on a torsion-free coherent sheaf.

**Definition 2.3** Let  $X$  be a smooth complex algebraic variety. Let  $\mathcal{F}$  be a torsion-free (algebraic) coherent sheaf on  $X$ . Let  $Z \subseteq X$  be the smallest Zariski closed set such that  $\mathcal{F}|_{X \setminus Z} = \mathcal{O}_{X \setminus Z}(F)$  for some vector bundle  $F$  on  $X \setminus Z$ . A *singular Hermitian metric* (resp. *positive singular Hermitian metric*) on  $\mathcal{F}$  is a singular Hermitian metric (resp. Griffiths positively curved singular Hermitian metric) on  $F$  in the sense of [13].

**Theorem 2.4** *Let  $X$  be a connected projective manifold of dimension  $n \geq 1$ . Let  $\varphi$  be a quasi-plurisubharmonic function on  $X$ . Let  $p: X \rightarrow \mathbb{P}^N$  be a morphism ( $N \geq 1$ ). Define*

$$\mathcal{G} := \{ H \in |\mathcal{O}_{\mathbb{P}^N}(1)| : H' := H \cap X \text{ is smooth and } \mathcal{I}(\varphi|_{H'}) = \mathcal{I}(\varphi)|_{H'} \}.$$

*Then  $\mathcal{G} \subseteq |\mathcal{O}_{\mathbb{P}^N}(1)|$  is co-pluripolar.*

**Remark 2.5** Here and in the sequel, we slightly abuse the notation by writing  $H \cap X$  for  $p^{-1}H$ , the scheme-theoretic inverse image of  $H$ . In other words,  $H \cap X := H \times_{\mathbb{P}^N} X$ .

By definition, any  $H \in |\mathcal{O}_{\mathbb{P}^N}(1)|$  such that  $p^{-1}H = \emptyset$  lies in  $\mathcal{G}$ .

We briefly sketch the argument. We need to show that for quasi-every  $H \in |\mathcal{O}_{\mathbb{P}^N}(1)|$ , the restriction formula (1.1) holds. A standard argument in algebraic geometry allows us to reduce the proof of (1.1) to proving the corresponding equality on global sections, after tensoring with a sufficiently ample line bundle  $L$ . In this case, we group the pairs consisting of  $H \in \Lambda$  and points on  $H \cap X$  as a single fibration  $\pi_1: U \rightarrow \Lambda$ . The theory of positivity of direct images allows us to construct a coherent sheaf  $\mathcal{F}$  on  $\Lambda$  endowed with a positive metric  $h_{\mathcal{F}}$  out of  $\pi_1$  and  $\varphi$ . By the construction of  $h_{\mathcal{F}}$ , the locus where the restriction formula (1.1) fails is contained in the singular locus of  $h_{\mathcal{F}}$ , which finishes the proof.

**Proof** Take an ample line bundle  $L$  with a smooth Hermitian metric  $h$  such that  $c_1(L, h) + \text{dd}^c \varphi \geq 0$ , where  $c_1(L, h)$  is the first Chern form of  $(L, h)$ , namely the curvature form of  $h$ . Let  $\mathcal{L}$  be the invertible sheaf corresponding to  $L$ . We introduce  $\Lambda := |\mathcal{O}_{\mathbb{P}^N}(1)|$  to simplify our notations.

**Step 2.6** We prove that the following set is co-pluripolar:

$$\begin{aligned} \mathcal{G}_{\mathcal{L}} &:= \{ H \in \Lambda : H \cap X \text{ is smooth and } H^0(H \cap X, \omega_{H \cap X} \otimes \mathcal{L}|_{H \cap X} \otimes \mathcal{I}(\varphi|_{H \cap X})) \\ &= H^0(H \cap X, \omega_{H \cap X} \otimes \mathcal{L}|_{H \cap X} \otimes \mathcal{I}(\varphi)|_{H \cap X}) \}. \end{aligned}$$

Here  $\omega_{H \cap X}$  denotes the dualizing sheaf of  $H \cap X$ .

Let  $U \subseteq \Lambda \times X$  be the closed subvariety whose  $\mathbb{C}$ -points correspond to pairs  $(H, x) \in \Lambda \times X$  with  $p(x) \in H$ . Let  $\pi_1: U \rightarrow \Lambda$  be the natural projection. We may assume that  $\pi_1$  is surjective, as otherwise there is nothing to prove.

Observe that  $U$  is a local complete intersection scheme by Krulls *Hauptidealsatz* and a *fortiori* a Cohen–Macaulay scheme. It follows from miracle flatness [10, Theorem 23.1] that the natural projection  $\pi_2: U \rightarrow X$  is flat. As the fibers of  $\pi_2$  over closed points of  $X$  are isomorphic to  $\mathbb{P}^{N-1}$ , it follows that  $\pi_2$  is smooth. Thus,  $U$  is smooth as well.

In the following, we will construct pluripolar sets  $\Sigma_1 \subseteq \Sigma_2 \subseteq \Sigma_3 \subseteq \Sigma_4 \subseteq \Lambda$  such that the behaviour of  $\pi_1$  is improved successively on the complement of  $\Sigma_i$ .

**Step 2.6.1** The usual Bertini theorem shows that there is a proper Zariski closed set  $\Sigma_1 \subseteq \Lambda$  such that  $\pi_1$  has smooth fibres outside  $\Sigma_1$ . This is slightly more general than the version that one finds in [7], see [9, Théorème 6.3] for a proof.

**Step 2.6.2** By Kollár’s torsion-free theorem [4, Theorem C],

$$\mathcal{F}^i := R^i \pi_{1*} (\omega_{U/\Lambda} \otimes \pi_2^* \mathcal{L} \otimes \mathcal{I}(\pi_2^* \varphi))$$

is torsion-free for all  $i$ . Here  $\omega_{U/\Lambda}$  denotes the relative dualizing sheaf of the morphism  $U \rightarrow \Lambda$ . Thus, there is a proper Zariski closed set  $\Sigma_2 \subseteq \Lambda$  such that

- (1)  $\Sigma_2 \supseteq \Sigma_1$ .
- (2) The  $\mathcal{F}^i$ ’s are locally free outside  $\Sigma_2$ .
- (3)  $\omega_{U/\Lambda} \otimes \pi_2^* \mathcal{L} \otimes \mathcal{I}(\pi_2^* \varphi)$  is  $\pi_1$ -flat on  $\pi_1^{-1}(\Lambda \setminus \Sigma_2)$  [3, Théorème 6.9.1].

We write  $\mathcal{F} = \mathcal{F}^0$ . By cohomology and base change [7, Theorem III.12.11], for any  $H \in \Lambda \setminus \Sigma_2$ , the fibre  $\mathcal{F}|_H$  of  $\mathcal{F}$  is given by

$$\mathcal{F}|_H = H^0(\pi_{1,H}, \omega_{U/\Lambda}|_{\pi_{1,H}} \otimes \pi_2^* \mathcal{L}|_{\pi_{1,H}} \otimes \mathcal{I}(\pi_2^* \varphi)|_{\pi_{1,H}}).$$

Here  $\pi_{1,H}$  denotes the fibre of  $\pi_1$  at  $H$ .

**Step 2.6.3** In order to proceed, we need to make use of the Hodge metric  $h_{\mathcal{H}}$  on  $\mathcal{F}$  defined in [8]. We briefly recall its definition in our setting. By [8, Section 22], we can find a proper Zariski closed set  $\Sigma_3 \subseteq \Lambda$  such that

- (1)  $\Sigma_3 \supseteq \Sigma_2$ .
- (2)  $\pi_1$  is submersive outside  $\Sigma_3$ .
- (3) Both  $\mathcal{F}$  and  $\pi_{1*}(\omega_{U/\Lambda} \otimes \pi_2^* \mathcal{L})/\mathcal{F}$  are locally free outside  $\Sigma_3$ .
- (4) For each  $i$ ,

$$R^i \pi_{1*}(\omega_{U/\Lambda} \otimes \pi_2^* \mathcal{L})$$

is locally free outside  $\Sigma_3$ .

Then for any  $H \in \Lambda \setminus \Sigma_3$ ,

$$H^0(H \cap X, \omega_{H \cap X} \otimes \mathcal{L}|_{H \cap X} \otimes \mathcal{I}(\varphi|_{H \cap X})) \subseteq \mathcal{F}|_H \subseteq H^0(H \cap X, \omega_{H \cap X} \otimes \mathcal{L}|_{H \cap X}).$$

See [8, Lemma 22.1].

Now we can give the definition of the Hodge metric on  $\Lambda \setminus \Sigma_3$ . Given any  $H \in \Lambda \setminus \Sigma_3$ , any  $\alpha \in \mathcal{F}|_H$ , the Hodge metric is defined as

$$h_{\mathcal{H}}(\alpha, \alpha) := \int_{X \cap H} |\alpha|_{h_{\varphi}|_{X \cap H}}^2 \in [0, \infty].$$

Observe that  $h_{\mathcal{H}}(\alpha, \alpha) < \infty$  if and only if  $\alpha \in H^0(H \cap X, \omega_{H \cap X} \otimes \mathcal{L}|_{H \cap X} \otimes \mathcal{I}(\varphi|_{H \cap X}))$ . Moreover,  $h_{\mathcal{H}}(\alpha, \alpha) > 0$  if  $\alpha \neq 0$ . It is shown in [8] (c.f. [12, Theorem 3.3.5]) that  $h_{\mathcal{H}}$  is indeed a singular Hermitian metric and it extends to a positive metric on  $\mathcal{F}$ .

**Step 2.6.4.** The determinant  $\det h_{\mathcal{H}}$  is singular at all  $H \in \Lambda \setminus \Sigma_3$  such that

$$H^0(H \cap X, \omega_{H \cap X} \otimes \mathcal{L}|_{H \cap X} \otimes \mathcal{I}(\varphi|_{H \cap X})) \neq \mathcal{F}|_H.$$

As the map  $\pi_2$  is smooth, we have  $\pi_2^* \mathcal{I}(\varphi) = \mathcal{I}(\pi_2^* \varphi)$  by Lemma 2.2. Under the identification  $\pi_{1,H} \cong H \cap X$ , we have

$$\pi_2^* \mathcal{I}(\varphi)|_{\pi_{1,H}} \cong \mathcal{I}(\varphi)|_{H \cap X}.$$

Thus we have the following inclusions:

$$H^0(H \cap X, \omega_{H \cap X} \otimes \mathcal{L}|_{H \cap X} \otimes \mathcal{I}(\varphi|_{H \cap X})) \subseteq H^0(H \cap X, \omega_{H \cap X} \otimes \mathcal{L}|_{H \cap X} \otimes \mathcal{I}(\varphi)|_{H \cap X}) = \mathcal{F}|_H.$$

Recall that the first inclusion follows from the Ohsawa–Takegoshi  $L^2$ -extension theorem. Hence  $\det h_{\mathcal{H}}$  is singular at all  $H \in |\mathcal{O}_{\mathbb{P}^N}(1)| \setminus \Sigma_3$  such that

$$H^0(H \cap X, \omega_{H \cap X} \otimes \mathcal{L}|_{H \cap X} \otimes \mathcal{I}(\varphi|_{H \cap X})) \neq H^0(H \cap X, \omega_{H \cap X} \otimes \mathcal{L}|_{H \cap X} \otimes \mathcal{I}(\varphi)|_{H \cap X}).$$

Let  $\Sigma_4$  be the union of  $\Sigma_3$  and the set of all such  $H$ . Since the Hodge metric  $h_{\mathcal{H}}$  is positive ([12, Theorem 3.3.5] and [8, Theorem 21.1]), its determinant  $\det h_{\mathcal{H}}$  is also positive ([13, Proposition 1.3] and [8, Proposition 25.1]), it follows that  $\Sigma_4$  is pluripolar. As a consequence,  $\mathcal{G}_{\mathcal{L}}$  is co-pluripolar.

**Step 2.7** Fix an ample invertible sheaf  $\mathcal{S}$  on  $X$ . The same result holds with  $\mathcal{L} \otimes \mathcal{S}^{\otimes a}$  in place of  $\mathcal{L}$ . Thus the set

$$A := \bigcap_{a=0}^{\infty} \mathcal{G}_{\mathcal{L} \otimes \mathcal{S}^{\otimes a}}$$

is co-pluripolar. For each  $H \in W$  such that  $X \cap H$  is smooth and  $\mathcal{I}(\varphi|_{X \cap H}) \neq \mathcal{I}(\varphi)|_{X \cap H}$ , let  $\mathcal{K}$  be the following cokernel:

$$0 \rightarrow \mathcal{I}(\varphi|_{X \cap H}) \rightarrow \mathcal{I}(\varphi)|_{X \cap H} \rightarrow \mathcal{K} \rightarrow 0.$$

By Serre vanishing theorem, taking  $a$  large enough, we may guarantee that

$$H^1(X \cap H, \omega_{X \cap H} \otimes (\mathcal{L} \otimes \mathcal{S}^{\otimes a})|_{X \cap H} \otimes \mathcal{I}(\varphi|_{X \cap H})) = 0$$

and

$$H^0(X \cap H, \omega_{X \cap H} \otimes (\mathcal{L} \otimes \mathcal{S}^{\otimes a})|_{X \cap H} \otimes \mathcal{K}) \neq 0.$$

Then

$$H^0(X \cap H, \omega_{X \cap H} \otimes (\mathcal{L} \otimes \mathcal{S}^{\otimes a})|_{X \cap H} \otimes \mathcal{I}(\varphi|_{X \cap H})) \neq H^0(X \cap H, \omega_{X \cap H} \otimes (\mathcal{L} \otimes \mathcal{S}^{\otimes a})|_{X \cap H} \otimes \mathcal{I}(\varphi)|_{X \cap H}).$$

Thus  $H \notin A$ . We conclude that  $\mathcal{G}$  is co-pluripolar.  $\square$

**Remark 2.8** As pointed out by the referee, in [4], Fujino and Matsumura also treated the case when  $X$  is not projective. It is of interest to understand if Theorem 2.4 can be extended to non-projective complex manifolds as well.

Note that the argument for  $\pi: U \rightarrow \Lambda$  in the proof of Theorem 2.4 works for more general fibrations. With essentially the same proof, we can similarly prove an analytic Bertini type theorem for fibrations.

**Corollary 2.9** *Let  $\pi: U \rightarrow W$  be a surjective morphism of projective varieties. Let  $(L, \phi)$  be a Hermitian pseudo-effective line bundle on  $U$ , namely  $L$  is a holomorphic line bundle on  $U$  and  $\phi$  is a plurisubharmonic metric on  $L$ . Then there is a pluripolar subset  $\Sigma \subseteq W$  such that for all  $w \in W \setminus \Sigma$ ,  $U_w := \pi^{-1}(w)$  is smooth and we have  $\mathcal{I}(\phi|_{U_w}) = \mathcal{I}(\phi)|_{U_w}$ .*

**Corollary 2.10** *Let  $X$  be a projective manifold of pure dimension  $n \geq 1$ . Let  $\Lambda$  be a base-point free linear system. Let  $\varphi$  be a quasi-psh function on  $X$ . Then there is a pluripolar subset  $\Sigma \subseteq \Lambda$  such that for any  $H \in \Lambda \setminus \Sigma$  and any real number  $k > 0$ ,*

$$\mathcal{I}(k\varphi|_H) = \mathcal{I}(k\varphi)|_H \quad (2.2)$$

and we have a short exact sequence for all  $k > 0$ ,

$$0 \rightarrow \mathcal{I}(k\varphi) \otimes \mathcal{O}_X(-H) \rightarrow \mathcal{I}(k\varphi) \rightarrow \mathcal{I}(k\varphi|_H) \rightarrow 0. \quad (2.3)$$

**Proof** First observe that by the strong openness theorem [6] in order to verify (2.2) for all real  $k > 0$ , it suffices to verify it for  $k$  lying in a countable subset  $K \subseteq \mathbb{R}_{>0}$ .

Applying Theorem 2.4 to each  $k\varphi$  with  $k \in K$  and each connected component of  $X$ , we find that there is a pluripolar set  $\Sigma_1 \subseteq \Lambda$  such that for any  $H \in \Lambda \setminus \Sigma_1$  and any  $k \in K$ , (2.2) holds. On the other hand, the union of the sets of associated primes of  $\mathcal{I}(k\varphi)$  for  $k > 0$  is a countable set, hence the set  $A$  of  $H \in \Lambda$  that avoids them is co-meager. It suffices to take  $\Sigma = \Sigma_1 \cup (\Lambda \setminus A)$ .  $\square$

Following the terminology of [2], given quasi-psh functions  $\varphi$  and  $\psi$  on  $X$ , we say  $\varphi \sim_{\mathcal{I}} \psi$  if for all real  $k > 0$ ,  $\mathcal{I}(k\varphi) = \mathcal{I}(k\psi)$ .

**Corollary 2.11** *Let  $X$  be a projective manifold of pure dimension  $n \geq 1$ . Let  $\varphi, \psi$  be quasi-psh functions on  $X$  such that  $\varphi \sim_{\mathcal{I}} \psi$ . Let  $\Lambda$  be a base-point free linear system. Then there is a pluripolar subset  $\Sigma \subseteq \Lambda$  such that for any  $H \in \Lambda \setminus \Sigma$ ,  $\varphi|_H$  and  $\psi|_H$  are both quasi-psh functions on  $H$  and we have  $\varphi|_H \sim_{\mathcal{I}} \psi|_H$ .*

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## Declaration

**Competing interests** I declare that I have no competing interests related to this paper.

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