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Analytic Bertini theorem

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Abstract
We prove an analytic Bertini theorem, generalizing a previous result of Fujino and Matsumura.

Keywords Bertini theorem · Multiplier ideal sheaf · Pluri-subharmonic function · Hodge metric

Mathematics Subject Classification 32U15 · 32Q15

1 Introduction

Let X be a connected complex projective manifold of dimension $n \geq 1$. Given any base-point free linear system Λ on X, it follows from the classical Bertini theorem [9] that a general hyperplane H of Λ is smooth. Let φ be a quasi-plurisubharmonic (quasi-psh) function on X. For a general member $H \in \Lambda$, the multiplier ideal sheaf $\mathcal{I}(\varphi|_H)$ makes sense. It is natural to wonder if

$$\mathcal{I}(\varphi|_H) = \mathcal{I}(\varphi)|_H \tag{1.1}$$

holds for general H. It is well-known that the $\mathcal{I}(\varphi|_H) \subseteq \mathcal{I}(\varphi)|_H$ direction always holds for a general H, as a consequence of the Ohsawa–Takegoshi L^2-extension theorem. Conversely, it is easy to construct examples such that the set $\mathcal{B}$ of $H \in \Lambda$ where the equality fails is not contained in any proper Zariski closed subset of Λ. A natural question arises: is the set $\mathcal{B}$ small in a suitable sense? This kind of problem was first studied by Fujino and Matsumura, see [4, 5]. They proved that the complement of $\mathcal{B}$ is dense with respect to the complex topology of Λ (regarded as a projective space). More recently, Meng and Zhou [11] proved that the complement of $\mathcal{B}$ has zero Lebesgue measure. In this paper, we prove the following refinement:

Theorem 1.1 *There is a pluripolar set $\Sigma \subseteq \Lambda$ such that for all $H \in \Lambda \setminus \Sigma$, H is smooth and* (1.1) *holds.*

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This result affirmatively answers a problem of Boucksom, see [5, Question 1.2]. From the point of view of pluripotential theory, this theorem is quite natural: a small set in pluripotential theory just means a pluripolar set. As shown in [4, Example 3.12], the exceptional set is not contained in a countable union of proper Zariski closed subsets in general, so Theorem 1.1 seems to be the optimal result. We also prove a more general analytic Bertini type result for fibrations Corollary 2.9.

Let us mention a key advantage of Theorem 1.1: our theorem can be applied to a countable family of quasi-psh functions at the same time, see Corollary 2.10. This corollary makes it possible to perform induction on the dimension when studying psh singularities.

2 Analytic Bertini theorem

In this section, varieties or algebraic varieties mean reduced separated schemes of finite type over $\mathbb{C}$.

Definition 2.1 Let Y be a complex projective manifold. A subset $A \subseteq Y$ is

(1) *co-pluripolar* if $Y \setminus A$ is pluripolar. When $\dim Y = 1$, we also say $A \subseteq Y$ is co-polar.
(2) *co-meager* if $Y \setminus A$ is contained in a countable union of proper Zariski closed sets.

We say a condition in $y \in Y$ is satisfied *quasi-everywhere* if there is a co-pluripolar subset $Y_0 \subseteq Y$ such that the condition is satisfied for $y \in Y_0$.

Clearly, a co-meager set is co-pluripolar. Both classes are preserved by countable intersections.

Lemma 2.2 *Let $\pi : Y \to X$ be a smooth morphism of smooth algebraic varieties. Let φ be a quasi-plurisubharmonic function on X, then*

$$\pi^* \mathcal{I}(\varphi) = \mathcal{I}(\pi^* \varphi). \tag{2.1}$$

Here $\mathcal{I}(\varphi)$ denotes the multiplier ideal sheaf of φ in the sense of Nadel. Observe that as π is flat, $\pi^* \mathcal{I}(\varphi)$ is a subsheaf of $\mathcal{O}_Y$, so in (2.1) equality makes sense, the two sheaves are actually equal, not just isomorphic.

Proof As pointed out by the referee, $\pi^* \mathcal{I}(\varphi) \supseteq \mathcal{I}(\pi^* \varphi)$ is proved in [1, Proposition 14.3]. So it suffices to prove the reverse inclusion.

By decomposing π into the composition of an étale morphism and a projection locally, it suffices to deal with the two cases separately. Fix a local section f of $\mathcal{I}(\varphi)$.

Assume that $\pi : X \times \mathbb{C}^n \to X$ is the natural projection. Fix a volume form $\mathrm{d}V$ on X. Take the product volume form $\mathrm{d}V \otimes \mathrm{d}\lambda$ on $X \times \mathbb{C}^n$, where $\mathrm{d}\lambda$ denotes the Lebesgue measure. It follows from Fubini theorem that $|\pi^* f|^2 e^{-\pi^* \varphi}$ is locally integrable with respect to $\mathrm{d}V \otimes \mathrm{d}\lambda$.

Now assume that π is étale. The change of variable formula shows that $|\pi^* f|^2 e^{-\pi^* \varphi}$ is locally integrable. □

In Lemma 2.2, we do not really need the algebraic structures on X and Y. For general complex manifolds, it suffices to apply the co-area formula.

We recall the notion of positive metrics on a torsion-free coherent sheaf.

Definition 2.3 Let X be a smooth complex algebraic variety. Let $\mathcal{F}$ be a torsion-free (algebraic) coherent sheaf on X. Let $Z \subseteq X$ be the smallest Zariski closed set such that $\mathcal{F}|_{X \setminus Z} = \mathcal{O}_{X \setminus Z}(F)$ for some vector bundle F on $X \setminus Z$. A *singular Hermitian metric* (resp. *positive singular Hermitian metric*) on $\mathcal{F}$ is a singular Hermitian metric (resp. Griffiths positively curved singular Hermitian metric) on F in the sense of [13].

Theorem 2.4 *Let X be a connected projective manifold of dimension $n \geq 1$. Let φ be a quasi-plurisubharmonic function on X. Let $p\colon X \to \mathbb{P}^N$ be a morphism ($N \geq 1$). Define*

$$\mathcal{G} := \left\{ H \in |\mathcal{O}_{\mathbb{P}^N}(1)| \colon H' := H \cap X \text{ is smooth and } \mathcal{I}(\varphi|_{H'}) = \mathcal{I}(\varphi)|_{H'} \right\}.$$

Then $\mathcal{G} \subseteq |\mathcal{O}_{\mathbb{P}^N}(1)|$ is co-pluripolar.

Remark 2.5 Here and in the sequel, we slightly abuse the notation by writing $H \cap X$ for $p^{-1}H$, the scheme-theoretic inverse image of H. In other words, $H \cap X := H \times_{\mathbb{P}^N} X$.

By definition, any $H \in |\mathcal{O}_{\mathbb{P}^N}(1)|$ such that $p^{-1}H = \emptyset$ lies in $\mathcal{G}$.

We briefly sketch the argument. We need to show that for quasi-every $H \in |\mathcal{O}_{\mathbb{P}^N}(1)|$, the restriction formula (1.1) holds. A standard argument in algebraic geometry allows us to reduce the proof of (1.1) to proving the corresponding equality on global sections, after tensoring with a sufficiently ample line bundle L. In this case, we group the pairs consisting of $H \in \Lambda$ and points on $H \cap X$ as a single fibration $\pi_1\colon U \to \Lambda$. The theory of positivity of direct images allows us to construct a coherent sheaf $\mathcal{F}$ on Λ endowed with a positive metric $h_{\mathcal{H}}$ out of π_1 and φ. By the construction of $h_{\mathcal{H}}$, the locus where the restriction formula (1.1) fails is contained in the singular locus of $h_{\mathcal{H}}$, which finishes the proof.

Proof Take an ample line bundle L with a smooth Hermitian metric h such that $c_1(L, h) + \mathrm{dd^c}\varphi \geq 0$, where $c_1(L, h)$ is the first Chern form of (L, h), namely the curvature form of h. Let $\mathcal{L}$ be the invertible sheaf corresponding to L. We introduce $\Lambda := |\mathcal{O}_{\mathbb{P}^N}(1)|$ to simplify our notations.

Step 2.6 We prove that the following set is co-pluripolar:

$$\begin{aligned}\mathcal{G}_{\mathcal{L}} := \{H \in \Lambda : H \cap X \text{ is smooth and } & H^0(H \cap X, \omega_{H\cap X} \otimes \mathcal{L}|_{H\cap X} \otimes \mathcal{I}(\varphi|_{H\cap X})) \\ & = H^0(H \cap X, \omega_{H\cap X} \otimes \mathcal{L}|_{H\cap X} \otimes \mathcal{I}(\varphi)|_{H\cap X})\}.\end{aligned}$$

Here $\omega_{H\cap X}$ denotes the dualizing sheaf of $H \cap X$.

Let $U \subseteq \Lambda \times X$ be the closed subvariety whose $\mathbb{C}$-points correspond to pairs $(H, x) \in \Lambda \times X$ with $p(x) \in H$. Let $\pi_1\colon U \to \Lambda$ be the natural projection. We may assume that π_1 is surjective, as otherwise there is nothing to prove.

Observe that U is a local complete intersection scheme by *Krulls Hauptidealsatz* and *a fortiori* a Cohen–Macaulay scheme. It follows from miracle flatness [10, Theorem 23.1] that the natural projection $\pi_2\colon U \to X$ is flat. As the fibers of π_2 over closed points of X are isomorphic to $\mathbb{P}^{N-1}$, it follows that π_2 is smooth. Thus, U is smooth as well.

In the following, we will construct pluripolar sets $\Sigma_1 \subseteq \Sigma_2 \subseteq \Sigma_3 \subseteq \Sigma_4 \subseteq \Lambda$ such that the behaviour of π_1 is improved successively on the complement of Σ_i.

Step 2.6.1 The usual Bertini theorem shows that there is a proper Zariski closed set $\Sigma_1 \subseteq \Lambda$ such that π_1 has smooth fibres outside Σ_1. This is slightly more general than the version that one finds in [7], see [9, Thèorème 6.3] for a proof.

Step 2.6.2 By Kollár's torsion-free theorem [4, Theorem C],

$$\mathcal{F}^i := R^i\pi_{1*}\left(\omega_{U/\Lambda} \otimes \pi_2^*\mathcal{L} \otimes \mathcal{I}(\pi_2^*\varphi)\right)$$

is torsion-free for all i. Here $\omega_{U/\Lambda}$ denotes the relative dualizing sheaf of the morphism $U \to \Lambda$. Thus, there is a proper Zariski closed set $\Sigma_2 \subseteq \Lambda$ such that

(1) $\Sigma_2 \supseteq \Sigma_1$.
(2) The $\mathcal{F}^i$'s are locally free outside Σ_2.
(3) $\omega_{U/\Lambda} \otimes \pi_2^*\mathcal{L} \otimes \mathcal{I}(\pi_2^*\varphi)$ is π_1-flat on $\pi_1^{-1}(\Lambda \setminus \Sigma_2)$ [3, Thèoréme 6.9.1].

We write $\mathcal{F} = \mathcal{F}^0$. By cohomology and base change [7, Theorem III.12.11], for any $H \in \Lambda\backslash\Sigma_2$, the fibre $\mathcal{F}|_H$ of $\mathcal{F}$ is given by

$$\mathcal{F}|_H = H^0\left(\pi_{1,H}, \omega_{U/\Lambda}|_{\pi_{1,H}} \otimes \pi_2^*\mathcal{L}|_{\pi_{1,H}} \otimes \mathcal{I}(\pi_2^*\varphi)|_{\pi_{1,H}}\right).$$

Here $\pi_{1,H}$ denotes the fibre of π_1 at H.

Step 2.6.3 In order to proceed, we need to make use of the Hodge metric $h_{\mathcal{H}}$ on $\mathcal{F}$ defined in [8]. We briefly recall its definition in our setting. By [8, Section 22], we can find a proper Zariski closed set $\Sigma_3 \subseteq \Lambda$ such that

(1) $\Sigma_3 \supseteq \Sigma_2$.
(2) π_1 is submersive outside Σ_3.
(3) Both $\mathcal{F}$ and $\pi_{1*}\left(\omega_{U/\Lambda} \otimes \pi_2^*\mathcal{L}\right)/\mathcal{F}$ are locally free outside Σ_3.
(4) For each i,

$$R^i\pi_{1*}\left(\omega_{U/\Lambda} \otimes \pi_2^*\mathcal{L}\right)$$

is locally free outside Σ_3.

Then for any $H \in \Lambda\backslash\Sigma_3$,

$$H^0(H\cap X, \omega_{H\cap X} \otimes \mathcal{L}|_{H\cap X} \otimes \mathcal{I}(\varphi|_{H\cap X})) \subseteq \mathcal{F}|_H \subseteq H^0(H\cap X, \omega_{H\cap X} \otimes \mathcal{L}|_{H\cap X}).$$

See [8, Lemma 22.1].

Now we can give the definition of the Hodge metric on $\Lambda\backslash\Sigma_3$. Given any $H \in \Lambda\backslash\Sigma_3$, any $\alpha \in \mathcal{F}|_H$, the Hodge metric is defined as

$$h_{\mathcal{H}}(\alpha,\alpha) := \int_{X\cap H} |\alpha|^2_{he^{-\varphi}|_{X\cap H}} \in [0,\infty].$$

Observe that $h_{\mathcal{H}}(\alpha,\alpha) < \infty$ if and only if $\alpha \in H^0(H\cap X, \omega_{H\cap X} \otimes \mathcal{L}|_{H\cap X} \otimes \mathcal{I}(\varphi|_{H\cap X}))$. Moreover, $h_{\mathcal{H}}(\alpha,\alpha) > 0$ if $\alpha \neq 0$. It is shown in [8] (c.f. [12, Theorem 3.3.5]) that $h_{\mathcal{H}}$ is indeed a singular Hermitian metric and it extends to a positive metric on $\mathcal{F}$.

Step 2.6.4. The determinant $\det h_{\mathcal{H}}$ is singular at all $H \in \Lambda\backslash\Sigma_3$ such that

$$H^0(H\cap X, \omega_{H\cap X} \otimes \mathcal{L}|_{H\cap X} \otimes \mathcal{I}(\varphi|_{H\cap X})) \neq \mathcal{F}|_H.$$

As the map π_2 is smooth, we have $\pi_2^*\mathcal{I}(\varphi) = \mathcal{I}(\pi_2^*\varphi)$ by Lemma 2.2. Under the identification $\pi_{1,H} \cong H\cap X$, we have

$$\pi_2^*\mathcal{I}(\varphi)|_{\pi_{1,H}} \cong \mathcal{I}(\varphi)|_{H\cap X}.$$

Thus we have the following inclusions:

$$H^0(H\cap X, \omega_{H\cap X} \otimes \mathcal{L}|_{H\cap X} \otimes \mathcal{I}(\varphi|_{H\cap X})) \subseteq H^0(H\cap X, \omega_{H\cap X} \otimes \mathcal{L}|_{H\cap X} \otimes \mathcal{I}(\varphi)|_{H\cap X}) = \mathcal{F}|_H.$$

Recall that the first inclusion follows from the Ohsawa–Takegoshi L^2-extension theorem. Hence $\det h_{\mathcal{H}}$ is singular at all $H \in |\mathcal{O}_{\mathbb{P}^N}(1)|\backslash\Sigma_3$ such that

$$H^0(H\cap X, \omega_{H\cap X} \otimes \mathcal{L}|_{H\cap X} \otimes \mathcal{I}(\varphi|_{H\cap X})) \neq H^0(H\cap X, \omega_{H\cap X} \otimes \mathcal{L}|_{H\cap X} \otimes \mathcal{I}(\varphi)|_{X\cap H}).$$

Let Σ_4 be the union of Σ_3 and the set of all such H. Since the Hodge metric $h_{\mathcal{H}}$ is positive ([12, Theorem 3.3.5] and [8, Theorem 21.1]), its determinant $\det h_{\mathcal{H}}$ is also positive ([13, Proposition 1.3] and [8, Proposition 25.1]), it follows that Σ_4 is pluripolar. As a consequence, $\mathcal{G}_{\mathcal{L}}$ is co-pluripolar.

Step 2.7 Fix an ample invertible sheaf $\mathcal{S}$ on X. The same result holds with $\mathcal{L}\otimes\mathcal{S}^{\otimes a}$ in place of $\mathcal{L}$. Thus the set

$$A := \bigcap_{a=0}^{\infty} \mathcal{G}_{\mathcal{L}\otimes\mathcal{S}^{\otimes a}}$$

is co-pluripolar. For each $H \in W$ such that $X\cap H$ is smooth and $\mathcal{I}(\varphi|_{X\cap H}) \neq \mathcal{I}(\varphi)|_{X\cap H}$, let $\mathcal{K}$ be the following cokernel:

$$0 \to \mathcal{I}(\varphi|_{X\cap H}) \to \mathcal{I}(\varphi)|_{X\cap H} \to \mathcal{K} \to 0.$$

By Serre vanishing theorem, taking a large enough, we may guarantee that

$$H^1(X\cap H, \omega_{X\cap H}\otimes(\mathcal{L}\otimes\mathcal{S}^{\otimes a})|_{X\cap H}\otimes\mathcal{I}(\varphi|_{X\cap H})) = 0$$

and

$$H^0(X\cap H, \omega_{X\cap H}\otimes(\mathcal{L}\otimes\mathcal{S}^{\otimes a})|_{X\cap H}\otimes\mathcal{K}) \neq 0.$$

Then

$$H^0(X\cap H, \omega_{X\cap H}\otimes(\mathcal{L}\otimes\mathcal{S}^{\otimes a})|_{X\cap H}\otimes\mathcal{I}(\varphi|_{X\cap H})) \neq H^0(X\cap H, \omega_{X\cap H}\otimes(\mathcal{L}\otimes\mathcal{S}^{\otimes a})|_{X\cap H} \otimes\mathcal{I}(\varphi)|_{X\cap H}).$$

Thus $H \notin A$. We conclude that $\mathcal{G}$ is co-pluripolar. □

Remark 2.8 As pointed out by the referee, in [4], Fujino and Matsumura also treated the case when X is not projective. It is of interest to understand if Theorem 2.4 can be extended to non-projective complex manifolds as well.

Note that the argument for $\pi: U \to \Lambda$ in the proof of Theorem 2.4 works for more general fibrations. With essentially the same proof, we can similarly prove an analytic Bertini type theorem for fibrations.

Corollary 2.9 *Let $\pi: U \to W$ be a surjective morphism of projective varieties. Let (L,ϕ) be a Hermitian pseudo-effective line bundle on U, namely L is a holomorphic line bundle on U and ϕ is a plurisubharmonic metric on L. Then there is a pluripolar subset $\Sigma \subseteq W$ such that for all $w \in W\setminus\Sigma$, $U_w := \pi^{-1}(w)$ is smooth and we have $\mathcal{I}(\phi|_{U_w}) = \mathcal{I}(\phi)|_{U_w}$.*

Corollary 2.10 *Let X be a projective manifold of pure dimension $n \geq 1$. Let Λ be a base-point free linear system. Let φ be a quasi-psh function on X. Then there is a pluripolar subset $\Sigma \subseteq \Lambda$ such that for any $H \in \Lambda\setminus\Sigma$ and any real number $k > 0$,*

$$\mathcal{I}(k\varphi|_H) = \mathcal{I}(k\varphi)|_H \tag{2.2}$$

and we have a short exact sequence for all $k > 0$,

$$0 \to \mathcal{I}(k\varphi)\otimes\mathcal{O}_X(-H) \to \mathcal{I}(k\varphi) \to \mathcal{I}(k\varphi|_H) \to 0. \tag{2.3}$$

Proof First observe that by the strong openness theorem [6] in order to verify (2.2) for all real $k > 0$, it suffices to verify it for k lying in a countable subset $K \subseteq \mathbb{R}_{>0}$.

Applying Theorem 2.4 to each $k\varphi$ with $k \in K$ and each connected component of X, we find that there is a pluripolar set $\Sigma_1 \subseteq \Lambda$ such that for any $H \in \Lambda\setminus\Sigma_1$ and any $k \in K$, (2.2) holds. On the other hand, the union of the sets of associated primes of $\mathcal{I}(k\varphi)$ for $k > 0$ is a countable set, hence the set A of $H \in \Lambda$ that avoids them is co-meager. It suffices to take $\Sigma = \Sigma_1 \cup (\Lambda\setminus A)$. □

Following the terminology of [2], given quasi-psh functions φ and ψ on X, we say $\varphi \sim_{\mathcal{I}} \psi$ if for all real $k > 0$, $\mathcal{I}(k\varphi) = \mathcal{I}(k\psi)$.

Corollary 2.11 *Let X be a projective manifold of pure dimension $n \geq 1$. Let φ, ψ be quasi-psh functions on X such that $\varphi \sim_{\mathcal{I}} \psi$. Let Λ be a base-point free linear system. Then there is a pluripolar subset $\Sigma \subseteq \Lambda$ such that for any $H \in \Lambda \backslash \Sigma$, $\varphi|_H$ and $\psi|_H$ are both quasi-psh functions on H and we have $\varphi|_H \sim_{\mathcal{I}} \psi|_H$.*

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Declaration

Competing interests I declare that I have no competing interests related to this paper.

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