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Non-pluripolar energy and the complex Monge–Ampère operator

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Abstract. Given a domain $\Omega \subset \mathbb{C}^n$ we introduce a class of plurisubharmonic (psh) functions $\mathcal{G}(\Omega)$ and Monge–Ampère operators $u \mapsto [dd^c u]^p$, $p \leq n$, on $\mathcal{G}(\Omega)$ that extend the Bedford–Taylor–Demailly Monge–Ampère operators. Here $[dd^c u]^p$ is a closed positive current of bidegree (p, p) that dominates the non-pluripolar Monge–Ampère current $\langle dd^c u\rangle^p$. We prove that $[dd^c u]^p$ is the limit of Monge–Ampère currents of certain natural regularizations of u.

On a compact Kähler manifold (X, ω) we introduce a notion of non-pluripolar energy and a corresponding finite energy class $\mathcal{G}(X, \omega) \subset \mathrm{PSH}(X, \omega)$ that is a global version of the class $\mathcal{G}(\Omega)$. From the local construction we get global Monge–Ampère currents $[dd^c\varphi + \omega]^p$ for $\varphi \in \mathcal{G}(X, \omega)$ that only depend on the current $dd^c\varphi + \omega$. The limits of Monge–Ampère currents of certain natural regularizations of φ can be expressed in terms of $[dd^c\varphi + \omega]^j$ for $j \leq p$. We get a mass formula involving the currents $[dd^c\varphi + \omega]^p$ that describes the loss of mass of the non-pluripolar Monge–Ampère measure $\langle dd^c\varphi + \omega\rangle^n$. The class $\mathcal{G}(X, \omega)$ includes ω-psh functions with analytic singularities and the class $\mathcal{E}(X, \omega)$ of ω-psh functions of finite energy and certain other convex energy classes, although it is not convex itself.

1. Introduction

Let Ω be a domain in $\mathbb{C}^n$, and let $u \in \mathrm{PSH}(\Omega)$, i.e. let u be a plurisubharmonic (psh) function on Ω. If u is C^2, then $dd^c u$ is a positive form, and the associated Monge–Ampère measure is defined as the top wedge power of this form with itself. This positive measure plays a fundamental role in pluripotential theory akin to the role played by the Laplacian in ordinary potential theory. If u is not C^2, then $dd^c u$ is no longer a form but a current. As is well known

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the wedge product of currents is typically not well-defined, which raises the question whether it is still possible to define a Monge–Ampère measure for more general psh functions.

Bedford and Taylor [6, 7] solved this problem when u is (locally) bounded. Their idea was to define the Monge–Ampère measure $(dd^c u)^n$ recursively. Assume that T is a closed positive current of bidegree (j, j). Then T has measure coefficients and since u is bounded, uT is a well-defined current and thus so is $dd^c(uT)$. Bedford and Taylor proved that this current is closed and positive. They could then recursively define closed positive currents

$$(dd^c u)^p := dd^c(u(dd^c u)^{p-1}).$$

The Monge–Ampère operators

$$u \mapsto (dd^c u)^p$$

have some essential continuity properties. Bedford and Taylor proved that if u_ℓ is any sequence of psh functions decreasing to u, then $(dd^c u_\ell)^p$ converges weakly to $(dd^c u)^p$.

We are interested in the situation when u is not locally bounded. Demailly [16, 17] showed that it is possible to extend the Bedford–Taylor Monge–Ampère operators to psh functions that are bounded outside "small" sets. Moreover, Błocki [10] and Cegrell [14] characterized the largest class $\mathcal{D}(\Omega)$ of psh functions on which there is a Monge–Ampère operator $u \mapsto (dd^c u)^n$ that is continuous under decreasing sequences. For instance, functions in $\mathrm{PSH}(\Omega)$ that are bounded outside a compact set in Ω are in $\mathcal{D}(\Omega)$, see, e.g., [10]. On the other hand psh functions with analytic singularities, i.e., locally of the form $u = c \log |f|^2 + b$, where $c > 0$, f is a tuple of holomorphic functions, and b is locally bounded, are not in $\mathcal{D}(\Omega)$ unless their unbounded locus is discrete, see [11] or Proposition 4.2.

To handle more singular psh functions Bedford and Taylor [8] introduced the notion of non-pluripolar Monge–Ampère currents. The idea is to capture the Monge–Ampère currents of the "bounded part" of $u \in \mathrm{PSH}(\Omega)$. Note that for any ℓ, $\max(u, -\ell)$ is psh and locally bounded, and thus $(dd^c \max(u, -\ell))^p$ is well-defined for any p. For each $p \leq n$,

$$\langle dd^c u\rangle^p := \lim_{\ell\to\infty} 1_{\{u>-\ell\}}\big(dd^c \max(u, -\ell)\big)^p \tag{1.1}$$

is a form with measure coefficients. The existence of the limit follows from the fact that the Monge–Ampère operators on bounded psh functions are local in the plurifine topology, i.e., if $u = v$ on a plurifine open set, then $(dd^c u)^p = (dd^c v)^p$ on that set. One serious issue is that the measure coefficients of $\langle dd^c u\rangle^p$ might be not locally finite, as an example due to Kiselman, [20], shows. If they are locally finite, however, by [12], $\langle dd^c u\rangle^p$ is a closed positive (p, p)-current. For instance, this is the case when u has analytic singularities. We refer to the currents $\langle dd^c u\rangle^p$ as *non-pluripolar Monge–Ampère currents*.

As the name suggests, the non-pluripolar Monge–Ampère currents do not charge pluripolar sets. Thus, since $\langle dd^c u\rangle^p$ cannot capture the behavior on the singular set of u, they do not coincide with Demailly's extensions of $(dd^c u)^p$ in general. In particular, it follows that the non-pluripolar Monge–Ampère operators $u \mapsto \langle dd^c u\rangle^p$ are far from being continuous under decreasing sequences in general. For instance, if $u = \log |f|^2$, where f is a holomorphic function, then $\langle dd^c u\rangle = 0$, whereas for any sequence u_ℓ decreasing to u, $dd^c u_\ell$ converges weakly to $dd^c u$, which by the Poincaré–Lelong formula is the current of integration $[f = 0]$ along the divisor of f.

The purpose of this paper is to introduce a new class of psh functions together with an extension of the Demailly–Bedford–Taylor Monge–Ampère operators that capture the singular behavior.

Definition 1.1. Let Ω be a domain in $\mathbb{C}^n$. We say $u \in \mathrm{PSH}(\Omega)$ has *locally finite non-pluripolar energy*, $u \in \mathcal{G}(\Omega)$ if, for each $j \le n-1$, $\langle dd^c u\rangle^j$ is locally finite and u is locally integrable with respect to $\langle dd^c u\rangle^j$.

If $u \in \mathcal{G}(\Omega)$ and $j \le n-1$, then $u\langle dd^c u\rangle^j$ is a well-defined current and thus by mimicking the original construction by Bedford and Taylor we can define generalized Monge–Ampère currents.

Definition 1.2. Given $u \in \mathcal{G}(\Omega)$, for $p = 1, \dots, n$ we define

$$[dd^c u]^p = dd^c(u\langle dd^c u\rangle^{p-1}) \tag{1.2}$$

and

$$S_p(u) = [dd^c u]^p - \langle dd^c u\rangle^p.$$

By using the locality of the non-pluripolar Monge–Ampère operators and the integrability, it follows that $[dd^c u]^p$ and $S_p(u)$ are closed positive currents. In particular, $[dd^c u]^p$ dominates $\langle dd^c u\rangle^p$.

Definitions 1.1 and 1.2 are inspired by the construction of Monge–Ampère currents in [1,5]. From [5, Proposition 4.1] it follows that psh functions with analytic singularities have locally finite non-pluripolar energy. Thus there are functions in $\mathcal{G}(\Omega)$ that are not in $\mathcal{D}(\Omega)$. If $u \in \mathrm{PSH}(\Omega)$ has analytic singularities, then the currents $[dd^c u]^p$ coincide with the Monge–Ampère currents $(dd^c u)^p$ introduced in [1,5]. In this case $S_p(u) = \mathbf{1}_Z[dd^c u]^p$, where Z is the unbounded locus of u. In [3,4] these Monge–Ampère currents are used to understand non-proper intersection theory in terms of currents. In particular, the Lelong numbers of the currents $[dd^c \log|f|^2]^p$ are certain local intersection numbers, so-called Segre numbers, associated with the ideal generated by f.

In Section 4 we provide other examples of functions in $\mathcal{G}(\Omega)$ and also psh functions that are not in $\mathcal{G}(\Omega)$. For instance, Example 4.3 shows that there are psh functions $u \le v$ such that $u \in \mathcal{G}(\Omega)$ but $v \notin \mathcal{G}(\Omega)$.

Given $u \in \mathcal{G}(\Omega)$, note that $\max(u, -\ell)$ is a natural sequence of locally bounded psh functions decreasing to u, cf. (1.1). Our first main theorem states that our new Monge–Ampère currents $[dd^c u]^p$ are the limits of the Monge–Ampère currents of this regularization.

Theorem 1.3. *Assume that $u \in \mathcal{G}(\Omega)$. Then*

$$\big(dd^c \max(u, -\ell)\big)^p \to [dd^c u]^p, \quad \ell \to \infty.$$

More generally, let $\chi_\ell : \mathbb{R} \to \mathbb{R}$ be a sequence of non-decreasing convex functions, bounded from below, that decreases to t as $\ell \to \infty$, and let $u_\ell = \chi_\ell \circ u$. Then

$$(dd^c u_\ell)^p \to [dd^c u]^p, \quad \ell \to \infty.$$

Note that $u_\ell = \max(u, -\ell)$ corresponds to $\chi_\ell(t) = \max(t, -\ell)$. Also note that Theorem 1.3 implies that $[dd^c u]^p$ coincides with the Bedford–Taylor–Demailly Monge–Ampère current when this is defined.

Example 1.4. Let $u = \log|f|^2$, where f is a tuple of holomorphic functions and let $\chi_\epsilon = \log(e^t + \epsilon)$. Then $\chi_\epsilon \circ u = \log(|f|^2 + \epsilon)$ and Theorem 1.3 asserts that

$$\lim_{\epsilon \to 0} \big(dd^c \log(|f|^2 + \epsilon)\big)^p = [dd^c u]^p.$$

This was proved in [1, Proposition 4.4].

The Monge–Ampère currents of the natural regularizations $\max(u, -\ell)$ do not always converge, see Example 5.9, and thus not all psh functions are in $\mathcal{G}(\Omega)$.

Since there are functions in $\mathcal{G}(\Omega)$ that are not in $\mathcal{D}(\Omega)$, we cannot expect continuity for all decreasing sequences. Our next result is a twisted version of Theorem 1.3 that illustrates that failure of continuity. Let v be a locally bounded psh function on Ω. Then $\max(u, v - \ell)$ is another natural sequence of locally bounded psh functions decreasing to u. Moreover, if χ_ℓ is as in Theorem 1.3, then also $\chi_\ell \circ (u - v) + v$ is a sequence of locally bounded psh functions decreasing to u.

Theorem 1.5. *Assume that $u \in \mathcal{G}(\Omega)$ and that v is a smooth psh function on Ω. Then*

$$\big(dd^c \max(u, v - \ell)\big)^p \to [dd^c u]^p + \sum_{j=1}^{p-1} S_j(u) \wedge (dd^c v)^{p-j}, \quad \ell \to \infty.$$

More generally, let $\chi_\ell : \mathbb{R} \to \mathbb{R}$ be a sequence of non-decreasing convex functions, bounded from below, that decreases to t as $\ell \to \infty$, and let $u_\ell = \chi_\ell \circ (u - v) + v$. Then

$$(dd^c u_\ell)^p \to [dd^c u]^p + \sum_{j=1}^{p-1} S_j(u) \wedge (dd^c v)^{p-j}, \quad \ell \to \infty.$$

Note that the lower degree Monge–Ampère currents $[dd^c u]^j$ come into play. Also note that Theorem 1.3 follows from Theorem 1.5 by setting $v = 0$.

When u has analytic singularities, Theorem 1.3 first appeared in [2, Theorem 1.1], and Theorem 1.5 appeared in [11, Theorem 1], although formulated slightly differently, cf. Remark 5.7 below. In those papers, using a Hironaka desingularization, the results are reduced to the case with divisorial singularities. Such a reduction is not available in the general case, and in this paper we instead rely on properties from [12] of the non-pluripolar Monge–Ampère operator. In particular, we get new proofs of the results in [2] and [11].

Let us now turn to the global setting. Assume that (X, ω) is a compact Kähler manifold of dimension n. Recall that a function φ is said to be ω-psh, $\varphi \in \mathrm{PSH}(X, \omega)$, if whenever h is a local potential for ω, i.e., $dd^c h = \omega$, $\varphi + h$ is psh. Then $dd^c\varphi + \omega$ is a closed positive current in $[\omega]$, and by the dd^c-lemma any closed positive current in $[\omega]$ can be written as $dd^c\varphi + \omega$ for some ω-psh φ, and this φ is unique up to adding of constants. Thus studying ω-psh functions is the same as studying closed positive currents in $[\omega]$.

If $\varphi \in \mathrm{PSH}(X, \omega)$ is bounded, then there are well-defined Monge–Ampère currents $(dd^c\varphi + \omega)^p$, locally defined as $(dd^c(\varphi + h))^p$, where h is a local potential for ω. It turns out, [12, Proposition 1.6], that for an unbounded φ the non-pluripolar Monge–Ampère currents $\langle dd^c\varphi + \omega\rangle^p$ are always well-defined. Moreover, [12, Proposition 1.20] showed that

$$\int_X \langle dd^c\varphi + \omega\rangle^p \wedge \omega^{n-p} \le \int_X \omega^n.$$

When φ is bounded we have equality

$$\int_X (dd^c\varphi + \omega)^p \wedge \omega^{n-p} = \int_X \omega^n \tag{1.3}$$

but in general the inequality can be strict.

Our definitions of $\mathcal{G}(X)$ and Monge–Ampère currents naturally lend themselves to the global setting.

Definition 1.6. Let (X, ω) be a compact Kähler manifold of dimension n. We say that $\varphi \in \mathrm{PSH}(X, \omega)$ has *finite non-pluripolar energy*, $\varphi \in \mathcal{G}(X, \omega)$, if, for each $j \le n-1$, φ is integrable with respect to $\langle dd^c\varphi + \omega\rangle^j$.

Definition 1.7. Given $\varphi \in \mathcal{G}(X, \omega)$, we define

$$[dd^c\varphi + \omega]^p = [dd^c(\varphi + h)]^p,$$

where h is a local potential for ω, and

$$S_p^\omega(\varphi) = [dd^c\varphi + \omega]^p - \langle dd^c\varphi + \omega\rangle^p.$$

Since two local potentials differ by a pluriharmonic function, it follows that $[dd^c\varphi + \omega]^p$ and $S_p^\omega(\varphi)$ are well-defined global positive closed currents on X. Note that whether an ω-psh function φ is in $\mathcal{G}(X, \omega)$ only depends on the current $dd^c\varphi + \omega$ and not on the choice of ω as a Kähler representative in the class $[\omega]$. Also the currents $[dd^c\varphi + \omega]^p$ and $S_p^\omega(\varphi)$ only depend on the current $dd^c\varphi + \omega$.

From Theorem 1.5 we get global regularization results. Given $\varphi \in \mathrm{PSH}(X, \omega)$, note that $\max(\varphi, -\ell)$ is a natural sequence of bounded ω-psh functions decreasing to φ.

Theorem 1.8. *Assume that $\varphi \in \mathcal{G}(X, \omega)$. Then*

$$\big(dd^c \max(\varphi, -\ell) + \omega\big)^p \to [dd^c\varphi + \omega]^p + \sum_{j=1}^{p-1} S_j^\omega(\varphi) \wedge \omega^{p-j}, \quad \ell \to \infty.$$

More generally, let $\chi_\ell : \mathbb{R} \to \mathbb{R}$ be a sequence of non-decreasing convex functions, bounded from below, that decreases to t as $\ell \to \infty$, and let $\varphi_\ell = \chi_\ell \circ \varphi$. Then

$$(dd^c\varphi_\ell + \omega)^p \to [dd^c\varphi + \omega]^p + \sum_{j=1}^{p-1} S_j^\omega(\varphi) \wedge \omega^{p-j}, \quad \ell \to \infty.$$

If η is another Kähler form in $[\omega]$, then $\eta = \omega + dd^c g$ for some smooth function g. There is an associated regularization of φ, namely $\varphi_\ell := \max(\varphi - g, -\ell) + g$, which corresponds to the max-regularization of the current $dd^c\varphi + \omega$ with respect to the alternative decomposition $dd^c(\varphi - g) + \eta$.

Theorem 1.9. *Assume that $\varphi \in \mathcal{G}(X, \omega)$, that η is a Kähler form in $[\omega]$, and that g and φ_ℓ are as above. Then*

$$(dd^c\varphi_\ell + \omega)^p \to [dd^c\varphi + \omega]^p + \sum_{j=1}^{p-1} S_j^\omega(\varphi) \wedge \eta^{p-j}, \quad \ell \to \infty. \tag{1.4}$$

More generally, let $\chi_\ell : \mathbb{R} \to \mathbb{R}$ *be a sequence of non-decreasing convex functions, bounded from below, that decreases to* t *as* $\ell \to \infty$*, and let* $\varphi_\ell = \chi_\ell \circ (\varphi - g) + g$*. Then* (1.4) *holds.*

Note that Theorem 1.8 follows immediately from Theorem 1.9 by setting $g = 0$. As in the local case, for φ with analytic singularities Theorems 1.8 and 1.9 follow from [11, Theorem 1], cf. Remark 6.9.

From (1.3) and Theorem 1.8 we get the following mass formula.

Theorem 1.10. *Assume that* $\varphi \in \mathcal{G}(X, \omega)$*. Then for each* $p \leq n$*,*

$$\int_X \langle dd^c\varphi + \omega\rangle^p \wedge \omega^{n-p} + \sum_{j=1}^{p} \int_X S_j^\omega(\varphi) \wedge \omega^{n-j} = \int_X \omega^n.$$

In fact, Theorem 1.10 is a cohomological consequence of the definition of $[dd^c\varphi + \omega]^p$; in Section 6.1 we provide a direct proof that does not rely on Theorem 1.8. For φ with analytic singularities this theorem appeared in [2, Theorem 1.2 and Proposition 5.2]. Note that, for $j \leq p$, the current $S_j^\omega(\varphi)$ captures the mass that “escapes” from $\langle dd^c\varphi + \omega\rangle^p$ at codimension j.

From the local case it follows that ω-psh functions with analytic singularities are in $\mathcal{G}(X, \omega)$, and in Section 11 we provide other examples. However, in the global setting we know more about the structure of the class $\mathcal{G}(X, \omega)$. In particular, it contains the Błocki–Cegrell class. Note that being in the Błocki–Cegrell class is a local statement, cf. Proposition 10.1 below. We say that $\varphi \in \mathrm{PSH}(X, \omega)$ is in $\mathcal{D}(X, \omega)$ if whenever g is a local dd^c-potential of ω in an open set $\mathcal{U} \subset X$, then $\varphi + g \in \mathcal{D}(\mathcal{U})$.

Theorem 1.11. *Let* (X, ω) *be a compact Kähler manifold of dimension* n*. Then*

$$\mathcal{D}(X, \omega) \subset \mathcal{G}(X, \omega).$$

Next, as the name suggests, the class $\mathcal{G}(X, \omega)$ of ω-psh functions with finite non-pluripolar energy can be understood as a finite energy class. Recall that the *Monge–Ampère energy* of $\varphi \in \mathrm{PSH}(X, \omega)$, introduced in [19], inspired by earlier work [13] in the local setting, is defined as

$$E(\varphi) = \frac{1}{n+1} \sum_{j=0}^{n} \int_X \varphi (dd^c\varphi + \omega)^j \wedge \omega^{n-j} \tag{1.5}$$

if φ is bounded and by

$$E(\varphi) = \inf\{E(\psi) : \psi \geq \varphi,\ \psi \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)\} \tag{1.6}$$

in general. The corresponding *finite energy class*

$$\mathcal{E}(X, \omega) := \{\varphi \in \mathrm{PSH}(X, \omega) : E(\varphi) > -\infty\} \tag{1.7}$$

is convex. Recall that, if $\varphi, \psi \in \mathrm{PSH}(X, \omega)$, then φ is said to be *less singular* than ψ, $\varphi \succeq \psi$, if $\varphi \geq \psi + O(1)$. If $\varphi \succeq \psi$ and $\psi \succeq \varphi$, we say that φ and ψ have the same *singularity type* and write $\varphi \sim \psi$. The class $\mathcal{E}(X,\omega)$ is closed under finite perturbations in the sense that if $\varphi \in \mathcal{E}(X,\omega)$ and $\psi \sim \varphi$, then $\psi \in \mathcal{E}(X,\omega)$. Moreover, $\mathcal{E}(X,\omega)$ is contained in the *full mass class*

$$\mathcal{F}(X,\omega) := \left\{\varphi \in \mathrm{PSH}(X,\omega) : \int_X \langle dd^c\varphi + \omega\rangle^n = \int_X \omega^n\right\}. \tag{1.8}$$

We introduce an alternative energy for $\varphi \in \mathrm{PSH}(X,\omega)$.

Definition 1.12. Let (X,ω) be a compact Kähler manifold of dimension n. For a function $\varphi \in \mathrm{PSH}(X,\omega)$ we define the *non-pluripolar energy*

$$E^{np}(\varphi) = \frac{1}{n}\sum_{j=0}^{n-1}\int_X \varphi\langle dd^c\varphi + \omega\rangle^j \wedge \omega^{n-j}.$$

Note that

$$\mathcal{G}(X,\omega) = \{\varphi \in \mathrm{PSH}(X,\omega) : E^{np}(\varphi) > -\infty\},$$

so that $\mathcal{G}(X,\omega)$ can be thought of as an finite energy class, cf. (1.7).

Theorem 1.13. *Let (X,ω) be a compact Kähler manifold. Then:*

(1) *if $\varphi \in \mathcal{G}(X,\omega)$ and $\psi \sim \varphi$, then $\psi \in \mathcal{G}(X,\omega)$,*

(2) $\mathcal{E}(X,\omega) \subset \mathcal{G}(X,\omega)$.

Although $\mathcal{G}(X,\omega)$ contains the convex subclass $\mathcal{E}(X,\omega)$ it is not convex itself. However, it contains certain other convex energy classes.

Definition 1.14. Let $\psi \in \mathcal{G}(X,\omega)$. For $\varphi \in \mathrm{PSH}(X,\omega)$ such that $\varphi \preceq \psi$, we define the *energy relative to ψ*

$$E^{\psi}(\varphi) = \inf\{E^{np}(\varphi') : \varphi' \geq \varphi,\ \varphi' \sim \psi\}.$$

We define the corresponding *finite relative energy classes*

$$\mathcal{E}^{\psi}(X,\omega) = \{\varphi \preceq \psi,\ E^{\psi}(\varphi) > -\infty\}$$

and the *relative full mass classes*

$$\mathcal{F}^{\psi}(X,\omega) = \left\{\varphi \in \mathrm{PSH}(X,\omega) : \varphi \preceq \psi,\ \sum_{j=0}^{n-1}\int_X \langle dd^c\varphi + \omega\rangle^j \wedge \omega^{n-j} = \sum_{j=0}^{n-1}\int_X \langle dd^c\psi + \omega\rangle^j \wedge \omega^{n-j}\right\}.$$

Note that if $\psi \in \mathrm{PSH}(X,\omega) \cap L^{\infty}(X)$, then $\mathcal{E}^{\psi}(X,\omega) \supset \mathcal{E}(X,\omega)$. The classes $\mathcal{E}^{\psi}(X,\omega)$ have the following properties, similar to $\mathcal{E}(X,\omega)$. Following [12], we say that $\varphi \in \mathrm{PSH}(X,\omega)$ has *small unbounded locus (sul)* if there exists a complete pluripolar closed subset $A \subset X$ such that φ is locally bounded outside A, cf. Section 2.2 below.

Theorem 1.15. *Let (X,ω) be a compact Kähler manifold. Then:*

(1) *if $\varphi \in \mathcal{E}^{\psi}(X,\omega)$ and $\varphi' \sim \varphi$, then $\varphi' \in \mathcal{E}^{\psi}(X,\omega)$,*

(2) *if ψ has sul, then $\mathcal{E}^{\psi}(X,\omega)$ is convex,*

(3) $\mathcal{E}^{\psi}(X,\omega) = \mathcal{G}(X,\omega) \cap \mathcal{F}^{\psi}(X,\omega)$.

The paper is organized as follows. In Section 2 we provide some background on the classical and the non-pluripolar Monge–Ampère operators in the local setting. In Section 3 we introduce the class $\mathcal{G}(\Omega)$, and more generally classes $\mathcal{G}_k(\Omega)$ of psh functions of *locally finite non-pluripolar energy of order* k, and our Monge–Ampère operators, and in Section 4 we provide various examples of functions in $\mathcal{G}(\Omega)$. In Section 5 we prove the regularity result Theorem 1.5. In fact, we prove a slightly more general version formulated in terms of $\mathcal{G}_k(\Omega)$.

In Section 6 we extend our definitions and regularity results to the global setting. In Section 7 we recall the classical Monge–Ampère energy and in Sections 8 and 9 we study the non-pluripolar energy and the relative energy, respectively; in particular, we prove Theorems 1.13 and 1.15. As in the local case we introduce more generally non-pluripolar and relative energies of order k and corresponding finite energy classes $\mathcal{G}_k(X,\omega)$ and $\mathcal{E}_k^{\psi}(X,\omega)$, and we prove versions of our results formulated in terms of these. In Section 10 we discuss the Błocki–Cegrell class and prove Theorem 1.11. Finally, in Section 11 we give various examples of functions with finite non-pluripolar energy.

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2. The complex Monge–Ampère product

Throughout this paper X is a domain in $\mathbb{C}^n$ or more generally a complex manifold of dimension n. All measures are assumed to be Borel measures. We let $d^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$, so that $dd^c \log|z_1|^2 = [z_1 = 0]$.

In this section we recall some basic facts about the (non-pluripolar) Monge–Ampère products. We refer to, e.g., [18, Chapter III] for the classical Bedford–Taylor–Demailly theory, see also [7, 8, 12].

First, the plurifine topology is the coarsest topology such that all psh functions on all open subsets of X are continuous. A basis for this topology is given by all sets of the form $V \cap \{u > 0\}$, where V is open and u is psh in V.

Let T be a closed positive current and let u be a locally bounded psh function on X. Then uT is a well-defined current and

$$dd^c u \wedge T := dd^c(uT)$$

is again a closed positive current. In particular, if $u_1, \dots, u_p$ are locally bounded psh functions, then the product $dd^c u_p \wedge \cdots \wedge dd^c u_1 \wedge T$ is defined inductively as

$$dd^c u_p \wedge \cdots \wedge dd^c u_1 \wedge T := dd^c(u_p dd^c u_{p-1} \wedge \cdots \wedge dd^c u_1 \wedge T). \tag{2.1}$$

It turns out that this product is commutative in the factors $dd^c u_j$ and multilinear in the factors $dd^c u_j$ and it does not charge pluripolar sets. Moreover, $dd^c u_p \wedge \cdots \wedge dd^c u_1$ is *local in the* *plurifine topology*, i.e., if O is a plurifine open set and $u_j = v_j$ pointwise on O, then

$$\mathbf{1}_O dd^c u_p \wedge \cdots \wedge dd^c u_1 = \mathbf{1}_O dd^c v_p \wedge \cdots \wedge dd^c v_1.$$

The products (2.1) satisfy the following continuity property.

Lemma 2.1. *Assume that $u_1, \dots, u_p$ are locally bounded psh functions and $u_1^\ell, \dots, u_p^\ell$ are decreasing sequences of psh functions converging to $u_1, \dots, u_p$, respectively, and that T is a closed positive current. Then*

$$dd^c u_p^\ell \wedge \cdots \wedge dd^c u_1^\ell \wedge T \to dd^c u_p \wedge \cdots \wedge dd^c u_1 \wedge T$$

weakly when $\ell \to \infty$.

We will use the following result.

Lemma 2.2. *Assume that for $j = 1, \dots, p$, u_s^j, $s \in \mathbb{R}$ (or some interval in $\mathbb{R}$), is a family of locally bounded psh functions on X such that $u_s^j \to u_t^j$ locally uniformly on X when $s \to t$. Then*

$$dd^c u_{s_p}^p \wedge \cdots \wedge dd^c u_{s_1}^1 \to dd^c u_{t_p}^p \wedge \cdots \wedge dd^c u_{t_1}^1$$

weakly when $(s_1, \dots, s_p) \to (t_1, \dots, t_p)$ in $\mathbb{R}^p$.

Proof. Note that

$$\begin{aligned} &dd^c u_{s_p}^p \wedge \cdots \wedge dd^c u_{s_1}^1 - dd^c u_{t_p}^p \wedge \cdots \wedge dd^c u_{t_1}^1 \\ &\quad = \sum_{j=1}^p \big(dd^c u_{s_p}^p \wedge \cdots \wedge dd^c u_{s_j}^j \wedge dd^c u_{t_{j-1}}^{j-1} \wedge \cdots \wedge dd^c u_{t_1}^1 \\ &\qquad\qquad - dd^c u_{s_p}^p \wedge \cdots \wedge dd^c u_{s_{j+1}}^{j+1} \wedge dd^c u_{t_j}^j \wedge \cdots \wedge dd^c u_{t_1}^1 \big). \end{aligned} \tag{2.2}$$

Let ξ be a test form. Since (2.1) is commutative in the factors $dd^c u_j$, we can write the action of the jth term on the right-hand side of (2.2) on ξ as

$$\int_X (u_{s_j}^j - u_{t_j}^j) dd^c u_{s_p}^p \wedge \cdots \wedge dd^c u_{s_{j+1}}^{j+1} \wedge dd^c u_{t_{j-1}}^{j-1} \wedge \cdots \wedge dd^c u_{t_1}^1 \wedge dd^c \xi. \tag{2.3}$$

Let $\mathcal{U} \subset X$ be a relatively compact neighborhood of the support of ξ. Then there is a neighborhood $\mathcal{V} \subset \mathbb{R}$ of t such that for $j = 1, \dots, p$, $s \in \mathcal{V}$, $|u_s^j - v_s^j| < \epsilon$ in $\overline{\mathcal{U}}$. Since u_s^j are locally bounded, there is an M such that $\|u_s^j\|_{L^\infty(\overline{\mathcal{U}})} \le M$. We may also assume that $\int_X |dd^c \xi| \le M$. Now by the Chern–Levine–Nirenberg inequalities there is constant C such that the absolute value of (2.3) is bounded by

$$\begin{aligned} &C \sup_{\overline{\mathcal{U}}} |u_{s_j}^j - u_{t_j}^j| \|u_{s_p}^p\|_{L^\infty(\overline{\mathcal{U}})} \cdots \|u_{s_{j+1}}^{j+1}\|_{L^\infty(\overline{\mathcal{U}})} \|u_{t_{j-1}}^{j-1}\|_{L^\infty(\overline{\mathcal{U}})} \cdots \|u_{t_1}^1\|_{L^\infty(\overline{\mathcal{U}})} \int_X |dd^c \xi| \\ &\quad \le C\epsilon (M + \epsilon)^{p-j} M^j \to 0. \end{aligned}$$

Since this holds for any ξ, (2.2) converges weakly to 0 when $(s_1, \dots, s_p) \to (t_1, \dots, t_p)$. □

2.1. The non-pluripolar Monge–Ampère product. Let $u_1, \dots, u_p$ be not necessarily locally bounded psh functions on X and let

$$O_\ell = \bigcap_{j=1}^{p} \{u_j > -\ell\}. \tag{2.4}$$

Then O_ℓ is a plurifine open set. Following [12, Definition 1.1], we say that the non-pluripolar Monge–Ampère product $\langle dd^c u_p \wedge \cdots \wedge dd^c u_1 \rangle$ is well-defined if for each compact subset $K \subset X$ we have

$$\sup_\ell \int_{K \cap O_\ell} \omega^{n-p} \wedge \bigwedge_{j=1}^{p} dd^c \max(u_j, -\ell) < \infty,$$

where ω is a smooth strictly positive $(1,1)$-form on X. This definition is clearly independent of ω.

Since the Monge–Ampère product for bounded functions is local in the plurifine topology, it follows that

$$\mathbf{1}_{O_\ell} \bigwedge_{j=1}^{p} dd^c \max(u_j, -\ell) = \mathbf{1}_{O_\ell} \bigwedge_{j=1}^{p} dd^c \max(u_j, -\ell'), \quad \ell' > \ell.$$

It follows that there is a well-defined positive (p, p)-current

$$\langle dd^c u_p \wedge \cdots \wedge dd^c u_1 \rangle := \lim_\ell \mathbf{1}_{O_\ell} \bigwedge_{j=1}^{p} dd^c \max(u_j, -\ell); \tag{2.5}$$

by [12, Theorem 1.8] it is closed.

Note that (2.5) is commutative in the factors $dd^c u_j$ since (2.1) is. By [12, Proposition 1.4] it is multilinear in the following sense: if v is another psh function, then

$$\left\langle dd^c(u_p + v) \wedge \bigwedge_{j=1}^{p-1} dd^c u_j \right\rangle = \left\langle dd^c u_p \wedge \bigwedge_{j=1}^{p-1} dd^c u_j \right\rangle + \left\langle dd^c v \wedge \bigwedge_{j=1}^{p-1} dd^c u_j \right\rangle \tag{2.6}$$

in the sense that the left-hand side is well-defined if and only if both terms on the right-hand side are, and equality holds in this case. Moreover, (2.5) only depends on the currents $dd^c u_j$, i.e., it is not affected by adding pluriharmonic functions to the u_j. Also, the operator

$$(u_1, \dots, u_p) \mapsto \langle dd^c u_p \wedge \cdots \wedge dd^c u_1 \rangle$$

is local in the plurifine topology whenever it is well-defined.

Lemma 2.3. *Assume that u is a psh function on X such that $\langle dd^c u \rangle^p$ is well-defined, and that u_λ is a sequence of psh functions on X decreasing to u, such that $\langle dd^c u_\lambda \rangle^p$ is well-defined for each λ. Moreover, assume that ω is a smooth positive $(1,1)$-form and that χ is a non-negative test function on X. Then*

$$\liminf_{\lambda \to \infty} \int_X \langle dd^c u_\lambda \rangle^p \wedge \chi \omega^{n-p} \geq \int_X \langle dd^c u \rangle^p \wedge \chi \omega^{n-p}. \tag{2.7}$$

Proof. Fix ℓ and let $O_\ell = \{u > -\ell\}$. Then

$$\max(u_\lambda, -\ell) \searrow \max(u, -\ell)$$

and by [8, Corollary 3.3],

$$\liminf_{\lambda\to\infty}\int_{O_\ell}\big(dd^c\max(u_\lambda,-\ell)\big)^p\wedge\chi\omega^{n-p}\geq\int_{O_\ell}\big(dd^c\max(u,-\ell)\big)^p\wedge\chi\omega^{n-p}. \tag{2.8}$$

Since $u_\lambda > u$, it follows that $u_\lambda = \max(u_\lambda, -\ell)$ and $u = \max(u, -\ell)$ in O_ℓ. Thus, since the non-pluripolar Monge–Ampère operator is local in the plurifine topology, (2.8) implies that

$$\liminf_{\lambda\to\infty}\int_{O_\ell}\langle dd^cu_\lambda\rangle^p\wedge\chi\omega^{n-p}\geq\int_{O_\ell}\langle dd^cu\rangle^p\wedge\chi\omega^{n-p}. \tag{2.9}$$

Now, (2.9) holds for all ℓ and since $\langle dd^cu\rangle^p$ does not charge pluripolar sets, in particular, not $V = \{u = -\infty\}$, we get (2.7). □

2.2. Psh functions with small unbounded locus. Following [12], we say that a psh function u on X has *small unbounded locus (sul)* if there exists a complete pluripolar closed subset $A \subset X$ such that u is locally bounded outside A.

Remark 2.4. Let $O = \bigcup_\ell O_\ell$, where O_ℓ is defined by (2.4). Note that if $u_1, \dots, u_p$ have sul and A is a closed complete pluripolar set such that $u_1, \dots, u_p$ are locally bounded outside A, then $X \setminus O \subset A$.

Remark 2.5. Assume that $u_1, \dots, u_p$ have sul, and that A is a closed complete pluripolar closed set such that each u_j is locally bounded outside A. Then $\langle dd^cu_p \wedge \cdots \wedge dd^cu_1\rangle$ is well-defined if and only if the Bedford–Taylor product $dd^cu_p \wedge \cdots \wedge dd^cu_1$, which is defined on $X \setminus A$, has locally finite mass near each point of A. Then $\langle dd^cu_p \wedge \cdots \wedge dd^cu_1\rangle$ is just the trivial extension of $dd^cu_p \wedge \cdots \wedge dd^cu_1$, cf. [12, p. 204].

3. Local Monge–Ampère currents – The classes $\mathcal{G}_k(X)$

We slightly extend the definition of $\mathcal{G}(X)$ from the introduction, cf. Definition 1.1.

Definition 3.1. Let X be a complex manifold of dimension n. For $1 \leq k \leq n-1$, we say that a psh function u on X has *locally finite non-pluripolar energy of order* k, $u \in \mathcal{G}_k(X)$, if, for each $j \leq k$, $\langle dd^cu\rangle^j$ is locally finite and u is locally integrable with respect to $\langle dd^cu\rangle^j$.

Note that if ω is a smooth strictly positive $(1,1)$-form, then $u \in \mathcal{G}_k(X)$ if and only if the measure

$$\sum_{j=0}^{k}\langle dd^cu\rangle^j\wedge\omega^{n-j} \tag{3.1}$$

is locally finite and u is locally integrable with respect to this measure. Clearly

$$\mathcal{G}_1(X) \supset \mathcal{G}_2(X) \supset \cdots \supset \mathcal{G}_{n-1}(X) = \mathcal{G}(X),$$

where $\mathcal{G}(X)$ is as in Definition 1.1.

If $u \in \mathcal{G}_k(X)$, then the Monge–Ampère currents

$$[dd^cu]^p = dd^c(u\langle dd^cu\rangle^{p-1}) \quad\text{and}\quad S_p(u) = [dd^cu]^p - \langle dd^cu\rangle^p$$

from Definition 1.2 are well-defined (p,p)-currents for $p = 1, \dots, k+1$.

Proposition 3.2. *The currents $[dd^c u]^p$ and $S_p(u)$ are closed and positive.*

Proof. Clearly $[dd^c u]^p$ is closed and, since $\langle dd^c u\rangle^p$ is closed, so is $S_p(u)$. Note that if u_ℓ is a sequence of smooth psh functions decreasing to u, then

$$[dd^c u]^p = \lim_{\ell\to\infty} dd^c(u_\ell \langle dd^c u\rangle^{p-1}).$$

Since $\langle dd^c u\rangle^{p-1}$ is closed and positive, the currents on the right-hand side are positive and thus so is the limit. To see that also $S_p(u)$ is positive, let $u_\ell = \max(u, -\ell)$ and $O_\ell = \{u > -\ell\}$. Since $u_\ell \to u$, we have

$$\begin{aligned}(3.2)\quad [dd^c u]^p &= \lim_{\ell\to\infty} dd^c(u_\ell\langle dd^c u\rangle^{p-1}) \\ &= \lim_{\ell\to\infty} \mathbf{1}_{X\setminus O_\ell} dd^c(u_\ell\langle dd^c u\rangle^{p-1}) + \lim_{\ell\to\infty} \mathbf{1}_{O_\ell} dd^c(u_\ell\langle dd^c u\rangle^{p-1}).\end{aligned}$$

Since the non-pluripolar Monge–Ampère operator is local in the plurifine topology, it follows that $\mathbf{1}_{O_\ell} dd^c(u_\ell\langle dd^c u\rangle^{p-1}) = \mathbf{1}_{O_\ell} dd^c u_\ell \wedge \langle dd^c u\rangle^{p-1} = \mathbf{1}_{O_\ell}\langle dd^c u_\ell\rangle^p$, and hence the last limit in (3.2) is equal to $\langle dd^c u\rangle^p$. Hence the first limit must exist as well and it is certainly positive. □

Proposition 3.3. *The currents $[dd^c u]^p$ and $S_p(u)$ only depend on $dd^c u$, i.e., they are not affected by adding a pluriharmonic function to u.*

Proof. Since $\langle dd^c u\rangle^p$ only depends on $dd^c u$, cf. Section 2.1, it is enough to prove the proposition for $[dd^c u]^p$. Assume that $u' = u + h$, where h is pluriharmonic. Then by (2.6),

$$\begin{aligned}[dd^c u']^p &= dd^c\big((u+h)\langle dd^c u\rangle^{p-1}\big) \\ &= dd^c(u\langle dd^c u\rangle^{p-1}) + dd^c(h\langle dd^c u\rangle^{p-1}) = [dd^c u]^p,\end{aligned}$$

where the last equality follows since $dd^c h = 0$ and $\langle dd^c u\rangle^{p-1}$ is closed. □

Remark 3.4. Assume that $u \in \mathcal{G}(X)$ has sul and is locally bounded outside the closed complete pluripolar set $A \subset X$. Note that $[dd^c u]^p$ coincides with the standard Monge–Ampère current $(dd^c u)^p$ outside A. It follows from Remark 2.5 that

$$\mathbf{1}_{X\setminus A}[dd^c u]^p = \langle dd^c u\rangle^p \quad \text{and} \quad S_p(u) = \mathbf{1}_A [dd^c u]^p.$$

4. Examples

Let us consider some examples of functions with locally finite non-pluripolar energy.

Example 4.1. Assume that $u \in \mathrm{PSH}(X)$ has analytic singularities, i.e., that u is locally of the form

$$u = c\log|f|^2 + b, \tag{4.1}$$

where $c > 0$, $f = (f_1, \dots, f_m)$ is a tuple of holomorphic functions,

$$|f|^2 = |f_1|^2 + \cdots + |f_m|^2,$$

and b is locally bounded. Then u is locally bounded outside the variety $Z \subset X$, locally defined as $\{f = 0\}$; in particular, u has sul.

As mentioned in the introduction, it follows from [5, Proposition 4.1] that $u \in \mathcal{G}(X)$. Moreover, the currents $[dd^c u]^p$ coincide with the Monge–Ampère currents $(dd^c u)^p$ defined inductively in [5] as $(dd^c u)^p = dd^c(u \mathbf{1}_{X \setminus Z}(dd^c u)^{p-1})$.

For the reader's convenience let us sketch an argument. Assume that (4.1) holds in the open set $\mathcal{U} \subset X$ and, for simplicity, that $c = 1$. By Hironaka's theorem there is a smooth modification $\pi : \mathcal{U}' \to \mathcal{U}$ such that $\pi^* f = f^0 f'$, where f^0 is a holomorphic section of a line bundle $L \to \mathcal{U}'$ and f' is a non-vanishing tuple of holomorphic sections of L^{-1}. Now

$$\pi^* u = \log |f^0|^2 + b',$$

where $b' = \log |f'|^2 + \pi^* b$ is locally bounded and psh in any local frame for L. It follows that, for any p, $(dd^c b')^p$ is a well-defined closed positive current on $\mathcal{U}'$ and one can check that

$$\pi_* (dd^c b')^p = \langle dd^c u \rangle^p.$$

Therefore, to see that $u \in \mathcal{G}(\mathcal{U})$ it is enough to verify that $\log |f^0|^2 (dd^c b')^p$ has locally bounded mass and this follows from a standard Chern–Levine–Nirenberg-type estimate, see, e.g., [18, Chapter III, Proposition 3.11]. By the Poincaré–Lelong formula,

$$S_p(u) = \pi_* \big(dd^c (\log |f^0|^2 (dd^c b')^p \big) = \pi_* \big([\operatorname{div} f^0] \wedge (dd^c b')^p \big),$$

where $[\operatorname{div} f^0]$ is the current of integration along the divisor of f^0.

Proposition 4.2. *Let Ω be a domain in $\mathbb{C}^n$. Assume that $u \in \mathrm{PSH}(\Omega)$ has analytic singularities and that the unbounded locus of u is not discrete. Then u is not in $\mathcal{D}(\Omega)$.*

This result was first noted in [11]. Here we provide a different argument.

Proof. Let $\kappa = \operatorname{codim} Z$. We claim that the current

$$S_\kappa(u) = \mathbf{1}_Z [dd^c u]^\kappa = \mathbf{1}_Z (dd^c u)^\kappa,$$

where $(dd^c u)^\kappa$ is the classical Bedford–Taylor–Demailly Monge–Ampère current, is non-zero. Taking this for granted, since $\kappa < n$ and all $S_j(u)$ are positive currents, it follows from Theorem 1.15 that we can find a decreasing sequence u_ℓ converging to u such that $(dd^c u_\ell)^n$ does not converge to $[dd^c u]^n$. We conclude that u is not in $\mathcal{D}(\mathcal{U})$, cf. the introduction.

To prove the claim, let us assume that (4.1) holds in the open set $\mathcal{U} \subset \Omega$. Now the Lelong numbers of $S_\kappa(u)$ and $\mathbf{1}_Z (dd^c \log |f|^2)^\kappa$ coincide at each point in $\mathcal{U}$, see, e.g., [5, (1.9)]. For dimension reasons, both currents must be Lelong currents, and thus

$$S_\kappa(u) = \mathbf{1}_Z (dd^c \log |f|^2)^\kappa.$$

By the classical King formula, see, e.g., [18, Chapter III, (8.18)], $\mathbf{1}_Z (dd^c \log |f|^2)^\kappa$ is the Lelong current of an effective cycle whose support is precisely the union of the irreducible components of Z of pure codimension κ. In particular, $S_\kappa(u)$ is non-zero. □

Next, let us consider some examples of functions in $\mathcal{G}(X)$ that do not have analytic singularities.

Example 4.3. Let f be a tuple of holomorphic functions in a domain $\Omega \subset \mathbb{C}^n$ such that $|f|^2 < 1$ and let

$$u = -(-\log|f|^2)^\epsilon$$

for some $\epsilon \in (0,1)$. Then u is psh in Ω and it is locally bounded outside $Z = \{f = 0\}$; in particular, u has sul. We claim that for each k, $u \in \mathcal{G}_k(\Omega)$ if and only if $\epsilon < \frac{1}{2}$. Moreover, although $v := \log|f|^2 \in \mathcal{G}(\Omega)$, cf. Example 4.1, $u + v$ is not in $\mathcal{G}_k(\Omega)$ for any ϵ.

To prove the claim, first note that

$$dd^c u = \frac{i}{2\pi}\epsilon(1-\epsilon)(-\log|f|^2)^{\epsilon-2}\frac{\partial|f|^2 \wedge \bar\partial|f|^2}{|f|^4} + \epsilon(-\log|f|^2)^{\epsilon-1} dd^c \log|f|^2.$$

Next, note that if $\pi : X \to \Omega$ is a smooth modification, then $u\langle dd^c u\rangle^j$ has locally finite mass if and only if $\pi^*(u\langle dd^c u\rangle^j) = \pi^* u\mathbf{1}_{X\setminus\pi^{-1}Z}(dd^c\pi^*u)^j$ has locally finite mass. By Hironaka's theorem there is such a modification so that $\pi^* f = f^0 f'$, where f^0 is a holomorphic section of a line bundle L and f' is a non-vanishing tuple of holomorphic sections of L^{-1}. Given a local frame, we may assume that f^0 and f' is a function and a tuple of functions, respectively. Let

$$\eta = \frac{\bar\partial|f'|^2}{|f'|^2}.$$

Then

$$\pi^*\left(\frac{\partial|f|^2 \wedge \bar\partial|f|^2}{|f|^4}\right) = \frac{df^0 \wedge d\overline{f^0}}{|f^0|^2} + \frac{df^0}{f^0} \wedge \eta + \overline{\frac{df^0}{f^0}} \wedge \eta + \bar\eta \wedge \eta$$

and, by the Poincaré–Lelong formula,

$$\pi^*(dd^c \log|f|^2) = [D] + \omega_f,$$

where D is the divisor defined by f^0 and $\omega_f := dd^c \log|f'|^2$ is smooth. It follows that

$$\pi^* u\mathbf{1}_{X\setminus\pi^{-1}Z} dd^c\pi^*u = C(-\log|f^0|^2 - \gamma)^{2\epsilon-2}\frac{df^0 \wedge d\overline{f^0}}{|f^0|^2} + \beta, \tag{4.2}$$

where C is a constant, $\gamma = |f'|^2$, and β has locally finite mass. Moreover, for each $j > 1$, $\pi^* u\mathbf{1}_{X\setminus\pi^{-1}Z}(dd^c\pi^*u)^j$ is a sum of terms that are integrable or of the form a smooth form times

$$(-\log|f^0|^2 - \gamma)^a \frac{df^0 \wedge d\overline{f^0}}{|f^0|^2}, \tag{4.3}$$

where $a \le 2\epsilon - 2$. By Hironaka's theorem we may assume that f^0 is a monomial, and then an elementary computation yields that (4.3) has locally finite mass if and only if $a < -1$. Hence (4.2), and thus $u\langle dd^c u\rangle$, have locally finite mass if and only if $\epsilon < \frac{1}{2}$. Moreover, if $\epsilon < \frac{1}{2}$, then $u\langle dd^c u\rangle^j$ has locally finite mass for $j > 1$. We conclude that $u \in \mathcal{G}(\Omega)$ if and only if $u \in \mathcal{G}_1(\Omega)$, which in turn holds if and only if $\epsilon < \frac{1}{2}$.

Finally, note that a necessary condition for $u + v$ to be in $\mathcal{G}_k(\Omega)$ for any k is that $v\langle dd^c u\rangle$ has locally finite mass. Now the pullback of $v\langle dd^c u\rangle$ contains a term of the form

$$C(-\log|f^0|^2 - \gamma)^{2\epsilon-2}\frac{df^0 \wedge d\overline{f^0}}{|f^0|^2},$$

where C is a constant. Since this does not have locally finite mass for any $\epsilon \in (0,1)$, it follows that $u + v \notin \mathcal{G}_k(\Omega)$ for any such ϵ.

Note in view of Example 4.3 that, in contrast to the case of $\mathcal{D}(\Omega)$, cf. [10, Theorem 1.2], it is not true that $v \in \mathcal{G}_k(\Omega)$ and $v \le u$ imply that $u \in \mathcal{G}_k(\Omega)$.

Remark 4.4. Let u and v be as in Example 4.3. It is not hard to check that $u + v$ has asymptotically analytic singularities in the sense of Rashkovskii, [22, Definition 3.4]. Indeed, this follows after noting that for each $\delta > 0$, there is a constant $C_\delta > 0$ such that $(1+\delta)v - C_\delta \le u + v \le v$. We saw above that $u + v \notin \mathcal{G}_k(\Omega)$ for any k. Hence we conclude that psh functions with asymptotically analytic singularities are not in $\mathcal{G}_k(\Omega)$ in general.

Note that u in Example 4.3 has sul; it is even locally bounded outside the analytic variety $\{f = 0\}$. Next, we will describe a way of constructing functions in $\mathcal{G}(X)$ that do not have sul. We will use the following lemma that follows as in Example 4.1, see also [21, Proposition 3.2].

Lemma 4.5. *Assume that u, v are psh functions with analytic singularities on X. Then, for any smooth positive $(1,1)$-form ω, test function $\chi \ge 0$, and $i \le j \le n-1$,*

$$\int_X u\langle dd^c u\rangle^i \wedge \langle dd^c v\rangle^{j-i} \wedge \chi\omega^{n-j} > -\infty.$$

Example 4.6. Let $\mathcal{U}$ be a neighborhood of the unit ball $\mathbf{B} \subset \mathbb{C}^n$ and let v_i, $i = 1, 2, \ldots,$ be negative psh functions with analytic singularities in $\mathcal{U}$. Let $u_\ell = \sum_{i=1}^{\ell} b_i v_i$, where $b_i > 0$, and let

$$u = \lim_{\ell\to\infty} u_\ell = \sum_{i=1}^{\infty} b_i v_i.$$

We claim that we can choose b_i so that the restriction of u to $\mathbf{B}$ is in $\mathcal{G}(\mathbf{B})$. Let χ be a smooth non-negative function with compact support in $\mathcal{U}$ such that $\chi \equiv 1$ in $\mathbf{B}$, and let ω be a smooth strictly positive $(1,1)$-form. It is enough to prove that, given $C > 0$, we can choose b_i so that

$$\int_{\mathcal{U}} u_\ell \langle dd^c u_\ell\rangle^j \wedge \chi\omega^{n-j} > -C, \quad \ell \ge 1,\ j \le n-1. \tag{4.4}$$

Since $v_i < 0$, we have $u_\ell \searrow u$ and it follows from Lemma 2.3 that

$$\int_{\mathcal{U}} u\langle dd^c u\rangle^j \wedge \chi\omega^{n-j} > -C, \quad j \le n-1,$$

and thus $u \in \mathcal{G}(\mathcal{U})$.

It remains to prove (4.4). Since (2.5) is multilinear, it follows that

$$\begin{aligned}\int_{\mathcal{U}} u_\ell\langle dd^c u_\ell\rangle^j \wedge \xi\omega^{n-j} &= \int_{\mathcal{U}} (u_{\ell-1} + b_\ell v_\ell)\big\langle dd^c(u_{\ell-1} + b_\ell v_\ell)\big\rangle^j \wedge \xi\omega^{n-j} \\ &= \int_{\mathcal{U}} u_{\ell-1}\langle dd^c u_{\ell-1}\rangle^j \wedge \xi\omega^{n-j} + \sum_{r=1}^{j+1} b_\ell^r T_r,\end{aligned}$$

where each T_r is a sum of terms of the form

$$\int_X \phi\langle dd^c u_{\ell-1}\rangle^\kappa \wedge \langle dd^c v_\ell\rangle^{j-\kappa} \wedge \xi\omega^{n-j},$$

where $\phi = u_{\ell-1}$ or $\phi = v_\ell$; in particular, they are independent of the choice of b_ℓ. By Lemma 4.5 each such integral is $> -\infty$. Thus by choosing b_ℓ small enough we can make the difference between $\int_X u_\ell \langle dd^c u_\ell \rangle^j \wedge \xi\omega^{n-j}$ and $\int_{\mathcal{U}} u_{\ell-1} \langle dd^c u_{\ell-1} \rangle^j \wedge \xi\omega^{n-j}$ arbitrarily small. In particular, for any $C > 0$ we can inductively choose b_i so that (4.4) holds.

Let us look at some explicit examples.

Example 4.7. Given $a = (a_1, \dots, a_k) \in (\mathbb{C}^n_x)^k$, let

$$|a \cdot x|^2 = |a_1 \cdot x|^2 + \cdots + |a_k \cdot x|^2,$$

and let $v_a = \log |a \cdot x|^2$. Then v_a is psh with analytic singularities and the unbounded locus of v_a equals $P_a := \bigcap_{i=1}^k \{a_i \cdot x = 0\}$. Note that for generic choices of a, P_a is a plane of codimension k. Next, choose $a^i \in (\mathbb{C}^n_x)^k$, $i = 1, 2, \dots$, so that $\bigcup_i P_{a^i}$ is dense in $\mathbb{C}^n$, let $v_i = v_{a^i}$, and let $u = \sum b_i v_i$ be constructed as in Example 4.6. Then the restriction of u to $\mathbf{B}$ is in $\mathcal{G}(\mathbf{B})$, but u is not locally bounded anywhere; in particular, u does not have sul.

Example 4.8. As in Example 4.6, let $\mathcal{U}$ be a neighborhood of the unit ball $\mathbf{B} \subset \mathbb{C}^n_x$. Given $a = (a_1, \dots, a_n) \in \mathcal{U}$, let

$$v_a = \log |x - a|^2 = \log(|x_1 - a_1|^2 + \cdots + |x_n - a_n|^2).$$

Then v_a is psh in $\mathcal{U}$ with analytic singularities and the unbounded locus of v_a equals a. Next, let a^i, $i = 1, 2, \dots$, be a dense subset of $\mathbf{B}$, let $v_i = v_{a^i}$, and let $u = \sum b_i v_i$ be constructed as in Example 4.6. Then the restriction of u to $\mathbf{B}$ is in $\mathcal{G}(\mathbf{B})$, but u is not locally bounded anywhere; in particular, u does not have sul. In fact, $u \in \mathcal{D}(\mathcal{U})$, see, e.g., [10, Theorem 2].

4.1. Direct products. On a direct product $X = X_1 \times X_2$ we can produce new examples of functions in $\mathcal{G}_k(X)$ by combining functions in $\mathcal{G}_k(X_1)$ and $\mathcal{G}_k(X_2)$.

Proposition 4.9. *Assume that for $i = 1, 2$, $u^i \in \mathcal{G}_k(X_i)$, where X_i is a complex manifold. Let $X = X_1 \times X_2$ and let $\pi_i : X \to X_i$, $i = 1, 2$, be the natural projections. Then*

$$u := \pi_1^* u^1 + \pi_2^* u^2 \in \mathcal{G}_k(X).$$

Proof. First note that $u \in \mathrm{PSH}(X)$. For $i = 1, 2$, let ω_i be a smooth strictly positive $(1, 1)$-form on X_i, so that

$$\omega := \pi_1^* \omega_1 + \pi_2^* \omega_2$$

is a smooth strictly positive $(1, 1)$-form on X. To prove that $u \in \mathcal{G}_k(X)$ it suffices to prove that, for any function χ of the form $\chi = \pi_1^* \chi_1 \cdot \pi_2^* \chi_2$, where χ_i is a non-negative test function in X_i, and for $0 \leq j \leq k$,

$$-\infty < \int_X u \langle dd^c u \rangle^j \wedge \chi\omega^{n-j} = \lim_{\ell \to \infty} \int_{O_\ell} u \langle dd^c u \rangle^j \wedge \chi\omega^{n-j},$$

where $O_\ell = \{u > -\ell\} \subset X$. Note that for λ large enough, $O_\ell \cap \operatorname{supp} \chi \subset O^1_\lambda \times O^2_\lambda$, where $O^i_\lambda = \{u_i > -\lambda\} \subset X_i$. In particular,

$$u = \pi_1^* u^1_\lambda + \pi_2^* u^2_\lambda$$

in $O_\ell \cap \operatorname{supp}\chi$, where $u^i_\lambda = \max(u^i, -\lambda)$. Thus, since the non-pluripolar Monge–Ampère operator is local in the plurifine topology,

$$\int_{O_\ell} u\langle dd^c u\rangle^j \wedge \chi\omega^{n-j} = \int_{O_\ell} (\pi_1^* u^1_\lambda + \pi_2^* u^2_\lambda)\big(dd^c(\pi_1^* u^1_\lambda + \pi_2^* u^2_\lambda)\big)^j \wedge \pi_1^*\chi_1 \pi_2^*\chi_2 (\pi_1^*\omega_1 + \pi_2^*\omega_2)^{n-j}. \tag{4.5}$$

Since (2.1) is multilinear, (4.5) is a finite sum of terms of the form

$$\begin{aligned}\int_{O_\ell} \pi_1^* u^1 (dd^c \pi_1^* u^1_\lambda)^{j_1} \wedge \pi_1^*\chi_1 (\pi_1^*\omega_1)^{n_1-j_1} \\ \wedge (dd^c \pi_2^* u^2_\lambda)^{j_2} \wedge \pi_2^*\chi_2 (\pi_2^*\omega_2)^{n_2-j_2} \\ \geq \int_{O^1_\lambda} u^1 (dd^c u^1_\lambda)^{j_1} \wedge \chi_1 \omega_1^{n_1-j_1} \int_{O^2_\lambda} (dd^c u^2_\lambda)^{j_2} \wedge \chi_2 \omega_2^{n_2-j_2}\end{aligned} \tag{4.6}$$

or of the form where the first factor $\pi_1^* u^1$ is replaced by $\pi_2^* u^2$. Since $u^i \in \mathcal{G}_k(X_i)$, each factor on the right-hand side of (4.6) is bounded uniformly in λ. □

Example 4.10. Let $\mathbf{B} \subset \mathbb{C}^n$ be the unit ball. Choose $u_1 \in \mathcal{D}(\mathbf{B}) \cap \mathcal{G}(\mathbf{B})$ that does not have analytic singularities, e.g., let u_1 be as in Example 4.8. Moreover, let u_2 be a psh function in $\mathbf{B}$ with analytic singularities that is not in $\mathcal{D}(\mathbf{B})$. Then by Proposition 4.1,

$$u = \pi_1^* u^1 + \pi_2^* u^2 \in \mathcal{G}(\mathbf{B} \times \mathbf{B}).$$

Now u neither has analytic singularities nor is in $\mathcal{D}(\mathbf{B} \times \mathbf{B})$.

5. Regularization

We will prove Theorem 1.5. In fact, we will prove the following slightly more general result.

Theorem 5.1. *Assume that $u \in \mathcal{G}_k(X)$ and that v is a smooth psh function on X. Then, for $p \leq k+1$,*

$$\big(dd^c \max(u, v-\ell)\big)^p \to [dd^c u]^p + \sum_{j=1}^{p-1} S_j(u) \wedge (dd^c v)^{p-j}, \quad \ell \to \infty. \tag{5.1}$$

More generally, let $\chi_\ell : \mathbb{R} \to \mathbb{R}$ be a sequence of non-decreasing convex functions, bounded from below, that decreases to t as $\ell \to \infty$, and let $u_\ell = \chi_\ell \circ (u - v) + v$. Then, for $p \leq k+1$,

$$(dd^c u_\ell)^p \to [dd^c u]^p + \sum_{j=1}^{p-1} S_j(u) \wedge (dd^c v)^{p-j}, \quad \ell \to \infty. \tag{5.2}$$

To illustrate the idea of the proof, let us start with a special case.

Proof of (5.1) *when $v = 0$.* Let $u_\ell = \max(u, -\ell)$. It is enough to prove that

$$(dd^c u_\ell)^p = dd^c u_\ell \wedge \langle dd^c u\rangle^{p-1} \tag{5.3}$$

since the right-hand side converges to $[dd^c u]^p$.

To prove (5.3), let ξ be a test form on X and consider

$$\int_X u_\ell\big((dd^c u_\ell)^{p-1} - \langle dd^c u\rangle^{p-1}\big) \wedge dd^c\xi. \tag{5.4}$$

Since $u_\ell = u$ in $O_\ell := \{u > -\ell\}$ and the non-pluripolar Monge–Ampère operator is local in the plurifine topology, we get that $u_\ell(dd^c u_\ell)^{p-1} = u_\ell\langle dd^c u\rangle^{p-1}$ in O_ℓ. Using that $u_\ell = -\ell$ in $X \setminus O_\ell$, we see that (5.4) equals

$$-\ell\int_X \big((dd^c u_\ell)^{p-1} - \langle dd^c u\rangle^{p-1}\big) \wedge dd^c\xi,$$

which vanishes by Stokes' theorem. Thus (5.3) follows. □

The proof of (5.1) for general v follows in the same way after replacing (5.3) by Lemma 5.4 below (with $\ell_1 = \cdots = \ell_p = \ell$). The general case follows by writing χ as a superposition of functions $\max(t, -s)$. For this we need some auxiliary results. Let us first consider an elementary lemma.

Lemma 5.2. *Assume that χ is non-decreasing, convex, and bounded on $(-\infty, 0]$ and that $\chi(0) = 0$ and $\chi'(0) = 1$. Let $g(s) = \chi''(-s)$. Then $g(s)\,ds$ is a probability measure on the interval $[0, \infty)$. Moreover,*

$$\chi(t) = \int_{s=0}^{\infty} \max(t, -s) g(s)\,ds.$$

Here χ' should be interpreted as the left derivative of χ, which is always well-defined since χ is convex.

Proof. Note that g is a (positive) measure since χ is convex.

First assume that χ is smooth. Notice that $\chi'(-s) \to 0$ when $s \to \infty$ since χ is bounded. Therefore

$$\int_0^\infty g(s)\,ds = \int_0^\infty \chi''(-s)\,ds = -\chi'(-s)\Big|_0^\infty = \chi'(0) = 1$$

and thus g is a probability measure. Moreover,

$$\begin{aligned}\int_0^\infty \max(t, -s)\chi''(-s)\,ds &= -\max(t, -s)\chi'(-s)\Big|_0^\infty + \int_0^\infty \frac{d}{ds}\max(t, -s)\cdot\chi'(-s)\,ds \\ &= -\int_0^{-t} \chi'(-s)\,ds = \chi(t).\end{aligned}$$

If χ is not smooth, the above arguments goes through verbatim if we understand χ' as the left derivative of χ, $g(s)\,ds$ as the corresponding Lebesque–Stieltjes measure, and the integrals as Lebesque–Stieltjes integrals. □

Assume that $T(s)$, $s \in \mathbb{R}^p_{\geq 0}$, is a continuous family of currents of order zero on X and $G(s)\,ds$ is a measure on $\mathbb{R}^p_{\geq 0}$. Then $\int_{\mathbb{R}^p_{\geq 0}} T(s)G(s)\,ds$ is a well-defined current of order zero on X, defined by

$$\int_X \int_{\mathbb{R}^p_{\geq 0}} T(s)G(s)\,ds \wedge \xi = \int_{\mathbb{R}^p_{\geq 0}} \int_X T(s) \wedge \xi G(s)\,ds,$$

if ξ is a (continuous) test form on X.

Lemma 5.3. *Assume that $T(t)$, $t \geq 0$, is a continuous family of positive currents that converges weakly to a current T_∞ on X when t tends to ∞. Moreover, assume that*

$$G_\ell(s) = G_\ell(s_1, \dots, s_p)$$

is a sequence of probability measures on $\mathbb{R}^p_{\geq 0}$ such that for all $R > 0$,

$$\lim_{\ell \to \infty} \int_{s_p > R} \cdots \int_{s_1 > R} G_\ell(s)\, ds = 1, \tag{5.5}$$

where $ds = ds_1 \cdots ds_p$. Finally, assume that $\rho : \mathbb{R}^p_{\geq 0} \to \mathbb{R}$ is a continuous function such that $\rho(s) \geq \min_j s_j$. Then

$$\int_{\mathbb{R}^p_{\geq 0}} T\big(\rho(s)\big) G_\ell(s)\, ds \to T_\infty$$

weakly when $\ell \to \infty$.

Proof. Let ξ be a test form on X. We need to prove that

$$\int_X \int_{\mathbb{R}^p_{\geq 0}} T\big(\rho(s)\big) G_\ell(s)\, ds \wedge \xi = \int_{\mathbb{R}^p_{\geq 0}} \int_X T\big(\rho(s)\big) \wedge \xi G_\ell(s)\, ds \to \int_X T_\infty \wedge \xi \tag{5.6}$$

when $\ell \to \infty$.

Take $\epsilon > 0$. Then there is an $R > 0$ such that for $t > R$,

$$\left| \int_X \big(T(t) - T_\infty\big) \wedge \xi \right| < \epsilon.$$

Let $A_R = \{s_j > R : j = 1, \dots, p\}$. Then we have $\rho(s) > R$ on A_R and, since G_ℓ are probability measures, it follows that

$$\left| \int_{A_R} \int_X \big(T\big(\rho(s)\big) - T_\infty\big) \wedge \xi G_\ell(s)\, ds \right| < \epsilon \quad \text{for all } \ell. \tag{5.7}$$

Since $T(t)$ is continuous and convergent, there is an $M \in \mathbb{R}$ such that $|\int_X T(t) \wedge \xi| < M$ for all t. Now, by (5.5),

$$\left| \int_{\mathbb{R}^p_{\geq 0} \setminus A_R} \int_X \big(T\big(\rho(s)\big) - T_\infty\big) \wedge \xi G_\ell(s)\, ds \right| < 2M \int_{\mathbb{R}^p_{\geq 0} \setminus A_R} G_\ell(s) \to 0 \tag{5.8}$$

when $\ell \to \infty$. Since ϵ is arbitrary, (5.6) follows from (5.7) and (5.8). □

Lemma 5.4. *Assume that $u \in \mathcal{G}_k(X)$ and v is a smooth psh function on X. For $\ell \geq 0$, let $u_\ell = \max(u, v - \ell)$. Then for any $p \leq k+1$ and any $\ell_1, \dots, \ell_p$ such that $\ell_1 \geq \cdots \geq \ell_p$,*

$$\begin{aligned} &dd^c u_{\ell_p} \wedge \cdots \wedge dd^c u_{\ell_1} \\ &\quad = dd^c u_{\ell_p} \wedge \langle dd^c u \rangle^{p-1} \\ &\qquad + \sum_{j=1}^{p-1} \big(dd^c u_{\ell_j} - \langle dd^c u \rangle\big) \wedge \langle dd^c u \rangle^{j-1} \wedge (dd^c v)^{p-j}. \end{aligned} \tag{5.9}$$

Proof. We claim that for $j = 1, \dots, p-1$,

$$\begin{aligned}(5.10)\quad dd^c u_{\ell_{j+1}} \wedge \cdots \wedge dd^c u_{\ell_1} \wedge (dd^c v)^{p-j-1} \\ = dd^c u_{\ell_j} \wedge \cdots \wedge dd^c u_{\ell_1} \wedge (dd^c v)^{p-j} \\ + dd^c u_{\ell_{j+1}} \wedge \langle dd^c u\rangle^j \wedge (dd^c v)^{p-j-1} - \langle dd^c u\rangle^j \wedge (dd^c v)^{p-j}.\end{aligned}$$

Taking (5.10) for granted, by recursively applying it to $j = p-1, \dots, 1$, we obtain (5.9).

To prove (5.10), let ξ be a test form on X and consider

$$(5.11)\qquad \int_X u_{\ell_{j+1}}\big(dd^c u_{\ell_j} \wedge \cdots \wedge dd^c u_{\ell_1} \wedge (dd^c v)^{p-j-1} - \langle dd^c u\rangle^j \wedge (dd^c v)^{p-j-1}\big) \wedge dd^c\xi.$$

Since $u_{\ell_j} = \cdots = u_{\ell_1} = u$ in $O = \{u > v - \ell_{j+1}\}$, which is open in the plurifine topology, and since the non-pluripolar Monge–Ampère operator is local in the plurifine topology, we get

$$dd^c u_{\ell_j} \wedge \cdots \wedge dd^c u_{\ell_1} \wedge (dd^c v)^{p-j-1} = \langle dd^c u\rangle^j \wedge (dd^c v)^{p-j-1}$$

there. Using that $u_{\ell_{j+1}} = v - \ell_{j+1}$ in $X \setminus O$, we see that (5.11) equals

$$(5.12)\qquad \int_X (v - \ell_{j+1})\big(dd^c u_{\ell_j} \wedge \cdots \wedge dd^c u_{\ell_1} \wedge (dd^c v)^{p-j-1} - \langle dd^c u\rangle^j \wedge (dd^c v)^{p-j-1}\big) \wedge dd^c\xi.$$

Now (5.10) follows from (5.11) and (5.12) by Stokes' theorem, since $\langle dd^c u\rangle^j$ is closed. □

Proof of Theorem 5.1. Since (5.2) is a local statement, we may assume that $u - v$ is bounded from above. In fact, we may assume that $u - v < 0$. Otherwise, if $u - v < c$, let $\check{u} = u - c$ and $\check{\chi}_\ell(t) = \chi_\ell(t + c) - c$. Then $\check{u} - v < 0$ and $\check{\chi}_\ell$ is a sequence of functions as in the assumption of the theorem. Moreover,

$$[dd^c \check{u}]^k = [dd^c u]^k \quad\text{and}\quad \check{\chi}_\ell \circ (\check{u} - v) = \chi_\ell \circ (u - v) - c;$$

in particular, $dd^c(\check{\chi}_\ell \circ (\check{u} - v)) = dd^c(\chi_\ell \circ (u - v))$. Thus it suffices to prove (5.2) for $\check{u}$ and $\check{\chi}_\ell$.

Throughout this proof, let $u_\ell = \max(u, v - \ell)$ and $\widetilde{u}_\ell = \chi_\ell \circ (u - v) + v$. Let us first assume that $\chi_\ell(0) = 0$ and $\chi_\ell'(0) = 1$ so that χ_ℓ (restricted to $(-\infty, 0]$) is as in Lemma 5.2. Let $g_\ell(t) = \chi_\ell''(-t)$. Note that $u_\ell = \max(u - v, -\ell) + v$ and thus by Lemma 5.2

$$\begin{aligned}\int_{s=0}^\infty u_s g_\ell(s)\, ds &= \int_{s=0}^\infty \max(u - v, -s) g_\ell(s)\, ds + \int_{s=0}^\infty v g_\ell(s)\, ds \\ &= \chi_\ell \circ (u - v) + v = \widetilde{u}_\ell.\end{aligned}$$

It follows from Lemma 2.2 that $dd^c u_{s_p} \wedge \cdots \wedge dd^c u_{s_1}$ is continuous in s. Thus

$$(5.13)\qquad (dd^c \widetilde{u}_\ell)^p = \int_{s_p=0}^\infty \cdots \int_{s_1=0}^\infty dd^c u_{s_p} \wedge \cdots \wedge dd^c u_{s_1} g_\ell(s_1) \cdots g_\ell(s_p)\, ds.$$

Let $\rho_j : \mathbb{R}^p_{\geq 0} \to \mathbb{R}$ be the function that maps $s = (s_1, \dots, s_p)$ to the jth largest s_i; in particular, $\rho_p(s) = \min_i s_i$. Since $dd^c u_{s_p} \wedge \cdots \wedge dd^c u_{s_1}$ is commutative in the factors $dd^c u_{s_i}$, it follows from Lemma 5.4 that

$$dd^c u_{s_p} \wedge \cdots \wedge dd^c u_{s_1} = \sum_{j=1}^{p} dd^c u_{\rho_j(s)} \wedge \langle dd^c u\rangle^{j-1} \wedge (dd^c v)^{p-j} - \sum_{j=1}^{p-1} \langle dd^c u\rangle^{j} \wedge (dd^c v)^{p-j}.$$

For $j = 1, \dots, p$, let

$$T_j(t) = dd^c u_t \wedge \langle dd^c u\rangle^{j-1} \wedge (dd^c v)^{p-j}.$$

By Lemma 2.1, $T_j(t)$ is continuous in t. Moreover, since $u \in \mathcal{G}_p(X)$, it converges weakly to $[dd^c u]^j \wedge (dd^c v)^{p-j}$. By Lemma 5.2, $G_\ell(s) := g_\ell(s_1)\cdots g_\ell(s_p)\,ds$ is a probability measure. Since $\chi_\ell(t) \to t$ when $\ell \to \infty$, given $R > 0$, $\chi'_\ell(-R) \to 1$ and thus $\int_{-R}^{0} g_\ell(s)\,ds \to 0$. It follows that $G_\ell(s)$ satisfies (5.5). Since ρ_j is continuous and $\rho_j(s) \geq \min_i s_i$, Lemma 5.3 yields that

$$\lim_{\ell\to\infty} \int_{\mathbb{R}^p_{\geq 0}} dd^c u_{\rho_j(s)} \wedge \langle dd^c u\rangle^{j-1} \wedge (dd^c v)^{p-j} G_\ell(s)\,ds = [dd^c u]^j \wedge (dd^c v)^{p-j}.$$

Hence, since G_ℓ is a probability measure, the limit of (5.13) when $\ell \to \infty$ equals

$$\begin{aligned}\sum_{j=1}^{p} [dd^c u]^j \wedge (dd^c v)^{p-j} &- \sum_{j=1}^{p-1} \langle dd^c u\rangle^{j} \wedge (dd^c v)^{p-j} \\ &= [dd^c u]^p + \sum_{j=1}^{p-1} S_j(u) \wedge (dd^c v)^{p-j}.\end{aligned}$$

Finally, let us consider a sequence χ_ℓ where we drop the extra assumptions on $\chi_\ell(0)$ and $\chi'_\ell(0)$. Since $\chi_\ell(t)$ are convex functions converging to t, $\chi'_\ell(0) \to 1$; in particular, $\chi'_\ell(0) \neq 0$ for large enough ℓ. Let $\widehat{\chi}_\ell = (\chi_\ell - \chi_\ell(0))/\chi'_\ell(0)$. Then $\widehat{\chi}_\ell$ is a sequence of non-decreasing convex functions bounded from below such that $\widehat{\chi}_\ell(t) \to t$, when $\ell \to \infty$, and $\widehat{\chi}_\ell(0) = 0$ and $\widehat{\chi}'_\ell(0) = 1$. By the above arguments

$$(dd^c \hat{u}_\ell)^k \to [dd^c u]^k + \sum_{j=1}^{k-1} S_j(u) \wedge (dd^c v)^{k-j}$$

for $k \leq p$, where $\hat{u}_\ell = \widehat{\chi}_\ell \circ (u - v) + v$. Note that

$$dd^c \tilde{u}_\ell = \chi'_\ell(0) dd^c \hat{u}_\ell + (1 - \chi'_\ell(0)) dd^c v.$$

Since $\chi'_\ell(0) \to 1$, it follows that

$$\lim_{\ell\to\infty} (dd^c \tilde{u}_\ell)^p = \lim_{\ell\to\infty} (dd^c \hat{u}_\ell)^p = [dd^c u]^p + \sum_{j=1}^{p-1} S_j(u) \wedge (dd^c v)^{p-j}. \qquad \square$$

Remark 5.5. Note that the proof above only uses that $\chi_\ell(t) \to t$, when $\ell \to \infty$. Therefore we could, in fact, drop the assumption that χ_ℓ is a decreasing sequence in Theorem 5.1 (as well as in the theorems in the introduction).

Remark 5.6. It follows from the proof above that we can choose different sequences χ_ℓ in Theorem 5.1 (and the theorems in the introduction) and get the following generalization: *For $\lambda = 1, \dots, p$, let $\chi_\ell^\lambda : \mathbb{R} \to \mathbb{R}$ be a sequence of non-decreasing convex functions, bounded from below, that decreases to t as $\ell \to \infty$, and let $u_\ell^\lambda = \chi_\ell^\lambda \circ (u - v) + v$. Then*

$$dd^c u_\ell^p \wedge \cdots \wedge dd^c u_\ell^1 \to [dd^c u]^p + \sum_{j=1}^{p-1} S_j(u) \wedge (dd^c v)^{p-j}, \quad \ell \to \infty.$$

Indeed, the proof goes through verbatim if we let

$$g_\ell^\lambda(t) = (\chi_\ell^\lambda)''(-t) \quad \text{and} \quad G_\ell(s) = g_\ell^1(s_1) \cdots g_\ell^p(s_p)\, ds.$$

Remark 5.7. Let us relate Theorem 5.1 to [11, Theorem 1]. Note that

$$u_\ell = \chi_\ell \circ (u - v) + v =: \varphi_\ell + v,$$

where φ_ℓ is a sequence converging to the quasiplurisubharmonic (qpsh) function $\varphi := u - v$. Theorem 1 in [11] asserts that if φ has analytic singularities, then

$$(dd^c \varphi_\ell)^p \to [dd^c \varphi]^p, \tag{5.14}$$

where $[dd^c \varphi]^n$ is an extension of (1.2) to qpsh functions, see, e.g., [11, 21] for details. Using (5.14), we see that

$$(dd^c u_\ell)^p = (dd^c \varphi_\ell + dd^c v)^p \to ([dd^c \varphi] + dd^c v)^p.$$

It follows from the definition of $[dd^c \varphi]^p$, cf. (1.2) that, in fact, $([dd^c \varphi] + dd^c v)^p$ equals the right-hand side of (5.2) (e.g., by arguments as in [21]). Thus if u has analytic singularities, Theorem 5.1 follows from (5.14).

Example 5.8. Let $u = \log |z_1|^2 + |z_2|^2$ in the unit ball $\mathbf{B}$ in $\mathbb{C}^2$. Then u has analytic singularities and thus $u \in \mathcal{G}(\mathbf{B})$. By the Poincaré–Lelong formula

$$[dd^c u] = [z_1 = 0] + dd^c |z_2|^2$$

and one easily checks that

$$[dd^c u]^2 = [z_1 = 0] \wedge dd^c |z_2|^2 \neq 0.$$

Since $S_1(u) = [z_1 = 0] \neq 0$, we know from Theorem 5.1 that there are sequences of bounded psh functions u_ℓ decreasing to u such that the limits of the Monge–Ampère currents $(dd^c u_\ell)^2$ converge to different measures. In fact, it follows that we can find u_ℓ so that the mass of the measures are arbitrarily large: Let $v_\ell = \ell |z_2|^2$ and let $u_{\ell,\lambda} = \max(u, v_\ell - \lambda)$. By Theorem 5.1,

$$\lim_{\lambda \to \infty} (dd^c u_{\ell,\lambda})^2 = [dd^c u]^2 + \ell [z_1 = 0] \wedge dd^c |z_2|^2. \tag{5.15}$$

If we choose $\lambda_1 \ll \lambda_2 \ll \lambda_3 \ll \cdots$, then $u_\ell := u_{\ell,\lambda_\ell}$ is a sequence of bounded psh functions decreasing to u and in view of (5.15) $(dd^c u_\ell)^2$ do not have locally uniformly bounded mass.

In fact, the function u is a maximal psh function and therefore in this case it is possible to find a sequence of smooth psh functions u_ℓ decreasing to u such that $(dd^c u_\ell)^2$ converges weakly to 0, see [2, Example 3.4].

The following example shows that one needs some condition on a psh function u for the Monge–Ampère currents of the natural regularization $u_\ell = \max(u, -\ell)$ to converge. In particular, u below is an example of a psh function that does not have locally finite non-pluripolar energy.

Example 5.9. Consider the plurisubharmonic function

$$u(z,w) = \sup_{k\geq 1}{}^{*}\left\{\left(1+\frac{1}{k}\right)\log|z|^2 - a_k + (1-(-1)^k)|w|^2\right\}$$

in the bidisc $\mathbf{D}\times\mathbf{D}$ for some choice of $a_k > 0$, $k = 1, 2, \ldots$; here $*$ denotes the usc regularization. It is not hard to see that if we choose $0 \ll a_0 \ll a_1 \ll a_2 \ll \cdots$, then there is an increasing sequence $\ell_k \in \mathbb{N}$ such that

$$u(z,w) = \left(1+\frac{1}{2k}\right)\log|z|^2 - a_{2k}$$

on the set $\{|u(z,w)+\ell_{2k}| < 1\}$, whereas

$$u(z,w) = \left(1+\frac{1}{2k+1}\right)\log|z|^2 - a_{2k+1} + 2|w|^2$$

on the set $\{|u(z,w)+\ell_{2k+1}| < 1\}$. Note that

$$\left(dd^c \max\left(\left(1+\frac{1}{k}\right)\log|z|^2 - a_k, -\ell\right)\right)^2 = 0$$

while

$$\left(dd^c \max\left(\left(1+\frac{1}{k}\right)\log|z|^2 - a_k + 2|w|^2, -\ell\right)\right)^2 \to 2\left(1+\frac{1}{k}\right)[z=0]\wedge dd^c|w|^2$$

as $\ell\to\infty$. It follows that

$$\lim_{k\to\infty}\left(dd^c \max(u, -\ell_{2k})\right)^2 = \langle dd^c u\rangle^2,$$

whereas

$$\lim_{k\to\infty}\left(dd^c \max(u, -\ell_{2k+1})\right)^2 = \langle dd^c u\rangle^2 + 2[z=0]\wedge dd^c|w|^2.$$

6. Global Monge–Ampère products

Let (X,ω) be a compact Kähler manifold. To define global analogues of the classes $\mathcal{G}_k(\Omega)$, let us first recall some results on the non-pluripolar Monge–Ampère operator.

Assume that for $j = 1, \ldots, n$, ω_j is a Kähler form on X and φ_j is ω_j-psh. Since (2.5) only depends on the currents $dd^c u_j$,

$$\langle dd^c\varphi_p + \omega_p\rangle \wedge\cdots\wedge \langle dd^c\varphi_1 + \omega_1\rangle, \tag{6.1}$$

locally defined as $\langle dd^c(\varphi_p + h_p)\rangle\wedge\cdots\wedge\langle dd^c(\varphi_1+h_1)\rangle$, where h_j are local dd^c-potentials of the ω_j, is a global closed positive current on X, see Section 2.1.

Assume that $\varphi_1,\ldots,\varphi_n \in \mathrm{PSH}(X,\omega)$. Then, by Stokes' theorem,

$$\int_X (dd^c\varphi_n + \omega)\wedge\cdots\wedge(dd^c\varphi_1+\omega) = \int_X \omega^n, \tag{6.2}$$

cf. (1.3). For the non-pluripolar products we have the following monotonicity property. Recall that a function φ on X is quasiplurisubharmonic (qpsh) if it is locally the sum of a psh function and a smooth function.

Proposition 6.1. *Assume that $\varphi_1, \dots, \varphi_n$ and $\psi_1, \dots, \psi_n$ are ω-psh functions such that $\varphi_j \succeq \psi_j$. Then*

$$\int_X \langle dd^c\varphi_n + \omega\rangle \wedge \cdots \wedge \langle dd^c\varphi_1 + \omega\rangle \geq \int_X \langle dd^c\psi_n + \omega\rangle \wedge \cdots \wedge \langle dd^c\psi_1 + \omega\rangle.$$

As a consequence, the integral of (6.1) only depends on the singularity types of the φ_j. For ω-psh functions with sul Proposition 6.1 was proved in [12, Theorem 1.16], in the case when $\varphi_j = \varphi$ and $\psi_j = \psi$ in [25, Theorem 1.2], and in the general case in [15, Theorem 1.1]. Also, see [23, Theorem 1.1] for an even stronger monotonicity result.

We will use the following integration by parts result.

Proposition 6.2 ([12, Theorem 1.14]). *Let $A \subset X$ be a closed complete pluripolar set, and let T be a closed positive $(n-1, n-1)$-current on X. Let φ_i and ψ_i, $i = 1, 2$, be qpsh functions on X that are locally bounded on $X \setminus A$. If $u := \varphi_1 - \varphi_2$ and $v := \psi_1 - \psi_2$ are globally bounded on X, then*

$$\int_{X\setminus A} u\, dd^c v \wedge T = \int_{X\setminus A} v\, dd^c u \wedge T = -\int_{X\setminus A} dv \wedge d^c u \wedge T. \tag{6.3}$$

Remark 6.3. In particular, the integrals in (6.3) are well-defined, cf. [12, Lemma 1.15 and the discussion after Theorem 1.14]. Note that if $v = u$, then (6.3) is non-positive.

Remark 6.4. Note that Proposition 6.2 recently has been generalized to the case when φ_i and ψ_i do not necessarily have sul, see [26, Theorem 1.1] and [24, Theorem 2.6].

6.1. The classes $\mathscr{G}_k(X, \omega)$. The classes $\mathscr{G}_k(X)$ in Definition 3.1, are naturally carried over to the global setting. Recall from the introduction that on (X, ω), the non-pluripolar Monge–Ampère currents $\langle dd^c\varphi + \omega\rangle^j$ are always finite.

Definition 6.5. Let (X, ω) be a compact Kähler manifold of dimension n. For k with $1 \leq k \leq n-1$, we say that a function $\varphi \in \mathrm{PSH}(X, \omega)$ has *finite non-pluripolar energy of order k*, $\varphi \in \mathscr{G}_k(X, \omega)$, if, for each $1 \leq j \leq k$, φ is integrable with respect to $\langle dd^c\varphi + \omega\rangle^j$.

Note that $\varphi \in \mathrm{PSH}(X, \omega)$ is in $\mathscr{G}_k(X, \omega)$ if and only if φ is integrable with respect to

$$\sum_{j=0}^{k} \langle dd^c\varphi + \omega\rangle^j \wedge \omega^{n-j},$$

cf. (3.1). Clearly

$$\mathscr{G}_1(X, \omega) \supset \mathscr{G}_2(X, \omega) \supset \cdots \supset \mathscr{G}_{n-1}(X, \omega) = \mathscr{G}(X, \omega),$$

where $\mathscr{G}(X, \omega)$ is as in Definition 1.6.

If $\varphi \in \mathcal{G}_k(X, \omega)$, for $p = 1, \dots, k+1$, we can define currents

$$[dd^c\varphi + \omega]^p = [dd^c(\varphi + h)]^p,$$

where h is a local potential for ω, and

$$S_p^\omega(\varphi) = [dd^c\varphi + \omega]^p - \langle dd^c\varphi + \omega\rangle^p$$

as in Definition 1.7. By Propositions 3.2 and 3.3, they are well-defined global closed positive currents on X that only depend on the current $dd^c\varphi + \omega$ and not on the choice of ω as a Kähler representative in the class $[\omega]$.

Remark 6.6. Assume that φ has sul and that A is a closed complete pluripolar set such that φ is locally bounded in $X \setminus A$. Then $\mathbf{1}_{X\setminus A}[dd^c\varphi + \omega]^p = \langle dd^c\varphi + \omega\rangle^p$ so that $S_p^\omega(\varphi) = \mathbf{1}_A[dd^c\varphi + \omega]^p$.

From the definitions above and the basic properties of the Monge–Ampère currents we get an immediate proof of the mass formula (Theorem 1.10). In fact, we prove the following slightly more general version.

Theorem 6.7. *Assume that $\varphi \in \mathcal{G}_k(X, \omega)$. Then for $p \le k+1$,*

$$\int_X \langle dd^c\varphi + \omega\rangle^p \wedge \omega^{n-p} + \sum_{j=1}^{p} \int_X S_j^\omega(\varphi) \wedge \omega^{n-j} = \int_X \omega^n. \tag{6.4}$$

Proof. First, note in view of Proposition 3.3 that, for $1 \le j \le k$,

$$dd^c\big(\varphi\langle dd^c\varphi + \omega\rangle^{j-1}\big) := [dd^c\varphi + \omega]^j - \langle dd^c\varphi + \omega\rangle^{j-1} \wedge \omega \tag{6.5}$$

is a well-defined exact current on X. We claim that for $1 \le j \le k$ we have

$$\int_X \langle dd^c\varphi+\omega\rangle^j \wedge\omega^{n-j} - \int_X \langle dd^c\varphi+\omega\rangle^{j-1} \wedge\omega^{n-j+1} = -\int_X S_j^\omega(\varphi)\wedge\omega^{n-j}. \tag{6.6}$$

In fact,

$$\begin{aligned}
&\int_X \langle dd^c\varphi + \omega\rangle^j \wedge \omega^{n-j} + \int_X S_j^\omega(\varphi) \wedge \omega^{n-j} \\
&\quad = \int_X [dd^c\varphi + \omega]^j \wedge \omega^{n-j} \\
&\quad = \int_X dd^c(\varphi\langle dd^c\varphi + \omega\rangle^{j-1}) \wedge \omega^{n-j} + \int_X \langle dd^c\varphi + \omega\rangle^{j-1} \wedge \omega^{n-j+1} \\
&\quad = \int_X \langle dd^c\varphi + \omega\rangle^{j-1} \wedge \omega^{n-j+1},
\end{aligned}$$

where we have used (6.5) for the second equality and the last equality follows from Stokes' theorem. Thus (6.6) holds, and summing from 1 to k we get (6.4). □

Theorem 6.7 also follows immediately from (6.2) and the following slightly generalized version of Theorem 1.9.

Theorem 6.8. *Assume that $\varphi \in \mathcal{G}_k(X,\omega)$ and that η is a Kähler form in $[\omega]$ so that $\eta = \omega + dd^c g$, where g is a smooth function on X. Let $\varphi_\ell = \max(\varphi - g, -\ell) + g$. Then, for $1 \le p \le k+1$,*

$$(6.7)\qquad (dd^c\varphi_\ell + \omega)^p \to [dd^c\varphi + \omega]^p + \sum_{j=1}^{p-1} S_j^\omega(\varphi) \wedge \eta^{p-j}, \quad \ell \to \infty.$$

More generally, let $\chi_\ell : \mathbb{R} \to \mathbb{R}$ be a sequence of non-decreasing convex functions, bounded from below, that decreases to t as $\ell \to \infty$, and let $\varphi_\ell = \chi_\ell \circ (\varphi - g) + g$. Then (6.7) *holds for* $1 \le p \le k+1$.

Proof. It is enough to prove the statement locally. We may therefore write $\varphi = u - h$, where u is psh and h is a smooth dd^c-potential for ω. Let

$$u_\ell = \chi_\ell \circ (u - h - g) + h + g.$$

Now Theorem 5.1 asserts that

$$(6.8)\qquad (dd^c u_\ell)^p \to [dd^c u]^p + \sum_{j=1}^{p-1} S_j(u) \wedge \big(dd^c(h+g)\big)^{p-j}.$$

Note that $u_\ell = \varphi_\ell + h$. Thus the left-hand side of (6.8) equals $(dd^c\varphi_\ell + \omega)^p$ and the right-hand side equals $[dd^c\varphi + \omega]^p + \sum_{j=1}^{p-1} S_j^\omega(\varphi) \wedge \eta^{p-j}$. □

Remark 6.9. If φ has analytic singularities, then Theorem 6.8 follows from [11, Theorem 1] as in Remark 5.7.

7. The Monge–Ampère energy

We want to describe $\mathcal{G}_k(X,\omega)$ as finite energy classes. To do this, let us start by recalling the classical setting. If $\varphi \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$, then its Monge–Ampère energy is defined as (1.5). More generally, for $0 \le k \le n$ and $\varphi \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ one can define the *Monge–Ampère energy of order k* as

$$E_k(\varphi) := \frac{1}{k+1} \sum_{j=0}^{k} \int_X \varphi (dd^c\varphi + \omega)^j \wedge \omega^{n-j}.$$

These functionals can be extended to the entire class $\mathrm{PSH}(X,\omega)$ by letting

$$E_k(\varphi) := \inf\big\{E_k(\psi) : \psi \ge \varphi,\ \psi \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)\big\},$$

cf. (1.6). We let

$$\mathcal{E}_k(X,\omega) := \{\varphi \in \mathrm{PSH}(X,\omega) : E_k(\varphi) > -\infty\}$$

be the corresponding *finite energy classes*. Moreover, we consider the *full mass classes*

$$\mathcal{F}_k(X,\omega) := \Big\{\varphi \in \mathrm{PSH}(X,\omega) : \int_X \langle dd^c\varphi + \omega\rangle^k \wedge \omega^{n-k} = \int_X \omega^n\Big\}.$$

Note that

$$\mathcal{E}(X,\omega) = \mathcal{E}_n(X,\omega) \subset \cdots \subset \mathcal{E}_1(X,\omega),$$

where $\mathcal{E}(X,\omega)$ is the standard finite energy class (1.7) corresponding to the energy functional $E = E_n$. Similarly, $\mathcal{F}(X,\omega) = \mathcal{F}_n(X,\omega)$ is the standard full mass class (1.8).

Proposition 7.1. *We have* $\mathcal{F}_n(X,\omega) \subset \cdots \subset \mathcal{F}_1(X,\omega)$.

Proof. Assume that $\varphi \in \mathcal{F}_k(X,\omega)$. Then

$$\begin{aligned} 0 &= \int_X \langle dd^c\varphi + \omega\rangle^k \wedge \omega^{n-k} - \int_X \omega^n \\ &= \Big(\int_X \langle dd^c\varphi + \omega\rangle^k \wedge \omega^{n-k} - \int_X \langle dd^c\varphi + \omega\rangle^{k-1} \wedge \omega^{n-k+1} \Big) \\ &\quad + \Big(\int_X \langle dd^c\varphi + \omega\rangle^{k-1} \wedge \omega^{n-k+1} - \int_X \omega^n \Big). \end{aligned}$$

By Proposition 6.1 both terms on the right-hand side are ≤ 0 and thus they must both vanish. In particular,

$$\int_X \langle dd^c\varphi + \omega\rangle^{k-1} \wedge \omega^{n-k+1} = \int_X \omega^n$$

and thus $\varphi \in \mathcal{F}_{k-1}(X,\omega)$. □

Remark 7.2. Note that $\varphi \in \mathcal{G}_{k-1}(X,\omega)$ is in $\mathcal{F}_k(X,\omega)$ if and only if $S_j^\omega(\varphi)$ vanishes for $j = 1,\dots,k$.

The finite energy classes $\mathcal{E}_k(X,\omega)$ have the following fundamental properties.

Theorem 7.3. *Let (X,ω) be a compact Kähler manifold. Then:*

(1) *if $\varphi \in \mathcal{E}_k(X,\omega)$ and $\psi \sim \varphi$ then $\psi \in \mathcal{E}_k(X,\omega)$,*

(2) *$\mathcal{E}_k(X,\omega)$ is convex,*

(3) *$\mathcal{E}_k(X,\omega) \subseteq \mathcal{F}_k(X,\omega)$.*

The last part is a consequence of the second part of Theorem 8.3 below, but it also follows from [12, Proposition 2.11] (for $k = n$). The first two parts follow from the following result.

Proposition 7.4. *The functional E_k is non-decreasing and concave on* $\mathrm{PSH}(X,\omega)$.

It is not hard to see that one can reduce the proof of Proposition 7.4 to prove that E_k is non-decreasing and concave on $\mathrm{PSH}(X,\omega) \cap L^\infty(X)$. This, in turn, is an immediate consequence of the following result.

Proposition 7.5. *If φ and $\varphi + u$ are ω-psh and bounded, then*

$$\frac{d}{dt}\Big|_{t=0^+} E_k(\varphi + tu) = \int_X u (dd^c\varphi + \omega)^k \wedge \omega^{n-k} \tag{7.1}$$

and

$$\frac{d^2}{dt^2}\Big|_{t=0^+} E_k(\varphi + tu) = -k \int_X du \wedge d^c u \wedge (dd^c\varphi + \omega)^{k-1} \wedge \omega^{n-k}. \tag{7.2}$$

For $k = n$ this was proved in [9, Propositions 4.1 and 4.4] and the general case can be proved by the same arguments.

8. The non-pluripolar energy

Let us now introduce an alternative way of extending the energies $E_k(\varphi)$ to the entire class $\mathrm{PSH}(X, \omega)$.

Definition 8.1. Let (X, ω) be a compact Kähler manifold of dimension n. For k with $1 \le k \le n-1$ we define the *non-pluripolar energy of order* k of $\varphi \in \mathrm{PSH}(X, \omega)$ as

$$E_k^{np}(\varphi) = \frac{1}{k+1} \sum_{j=0}^{k} \int_X \varphi \langle dd^c\varphi + \omega\rangle^j \wedge \omega^{n-j}.$$

Note that the *non-pluripolar energy* $E^{np}(\varphi)$, as defined in Definition 1.12, is equal to $E_{n-1}^{np}(\varphi)$. Note that if $\varphi \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$, then $E_k^{np}(\varphi) = E_k(\varphi)$. Moreover, in view of Definition 6.5,

$$\mathcal{G}_k(X, \omega) = \{\varphi \in \mathrm{PSH}(X, \omega) : E_k^{np}(\varphi) > -\infty\}.$$

Remark 8.2. Since $0 \le \int_X \langle dd^c\varphi + \omega\rangle^j \wedge \omega^{n-j} \le \int_X \omega^n$, if $C \ge 0$,

$$E_k^{np}(\varphi) \le E_k^{np}(\varphi + C) \le E_k^{np}(\varphi) + C \int_X \omega^n.$$

The functional E_k^{np} is neither monotone nor concave on $\mathrm{PSH}(X, \omega)$ in general, see Examples 11.4 and 11.5 below. In particular, $\mathcal{G}_k(X, \omega)$ is not convex in general. However, we have the following partial generalization of Theorem 7.3.

Theorem 8.3. *Let (X, ω) be a compact Kähler manifold.*

(1) *Assume that $\varphi, \psi \in \mathrm{PSH}(X, \omega)$. If $\varphi \in \mathcal{G}_k(X, \omega)$ and $\psi \sim \varphi$, then $\psi \in \mathcal{G}_k(X, \omega)$.*

(2) *We have*

$$\mathcal{E}_k(X, \omega) = \mathcal{G}_k(X, \omega) \cap \mathcal{F}_k(X, \omega).$$

Moreover, if $\varphi \in \mathcal{F}_k(X, \omega)$, then $E_k^{np}(\varphi) = E_k(\varphi)$.

In particular, $\mathcal{G}_k(X, \omega)$ contains the convex subclass $\mathcal{E}_k(X, \omega)$. Note that Theorem 1.13 follows from Theorem 8.3.

The proof relies on the following description of the non-pluripolar energy as a limit of energies of bounded ω-psh functions and Monge–Ampère masses.

Proposition 8.4. *If $\varphi \in \mathrm{PSH}(X, \omega)$, then*

$$E_k\big(\max(\varphi, -\ell)\big) + \frac{\ell}{k+1} \sum_{j=0}^{k} \int_X \big(\omega^j - \langle dd^c\varphi + \omega\rangle^j\big) \wedge \omega^{n-j} \searrow E_k^{np}(\varphi). \tag{8.1}$$

Proof. Let $\varphi_\ell = \max(\varphi, -\ell)$. We claim that

$$(8.2)\qquad \int_X \varphi_\ell (dd^c\varphi_\ell + \omega)^j \wedge \omega^{n-j} + \ell \int_X \big(\omega^j - \langle dd^c\varphi + \omega\rangle^j\big) \wedge \omega^{n-j} = \int_X \varphi_\ell \langle dd^c\varphi + \omega\rangle^j \wedge \omega^{n-j}.$$

Taking this for granted and noting that the right side decreases to $\int_X \varphi\langle dd^c\varphi + \omega\rangle^j \wedge \omega^{n-j}$, the proposition follows by summing over j.

It remains to prove the claim. First, since $\varphi_\ell = \varphi$ in $O := \{\varphi > -\ell\}$, which is open in the plurifine topology, since $\varphi_\ell = -\ell$ in $X \setminus O$, and since the non-pluripolar Monge–Ampère operator is local in the plurifine topology,

$$(8.3)\quad \varphi_\ell\big((dd^c\varphi_\ell + \omega)^j - \langle dd^c\varphi + \omega\rangle^j\big) = -\ell\big((dd^c\varphi_\ell + \omega)^j - \langle dd^c\varphi + \omega\rangle^j\big).$$

Next, since φ_ℓ is bounded, cf. (6.2),

$$(8.4)\qquad \int_X (dd^c\varphi_\ell + \omega)^j \wedge \omega^{n-j} = \int_X \omega^n.$$

Combining (8.3) and (8.4), we get (8.2). □

We have the following partial generalization of Proposition 7.4. Although E_k^{np} is not monotone on $\mathrm{PSH}(X,\omega)$, it is monotone on functions of the same singularity type.

Proposition 8.5. *Assume that $\varphi, \psi \in \mathrm{PSH}(X,\omega)$. If $\varphi \sim \psi$ and $\psi \geq \varphi$, then*

$$E_k^{np}(\psi) \geq E_k^{np}(\varphi).$$

Proof. Since E_k is non-decreasing, see Proposition 7.4,

$$E_k\big(\max(\psi, -\ell)\big) \geq E_k\big(\max(\varphi, -\ell)\big).$$

Moreover, since $\varphi \sim \psi$, by Proposition 6.1,

$$\int_X \langle dd^c\psi + \omega\rangle^j \wedge \omega^{n-j} = \int_X \langle dd^c\varphi + \omega\rangle^j \wedge \omega^{n-j}.$$

Now, the proposition follows from Proposition 8.4. □

Proof of Theorem 8.3. Part (1) follows from Proposition 8.5 in view of Remark 8.2.

It remains to prove part (2). Since $E_k(\varphi) = \lim_{\ell\to\infty} E_k(\varphi_\ell)$ and

$$T := \sum_{j=0}^{k} \int_X (\omega^j - \langle dd^c\varphi + \omega\rangle^j) \wedge \omega^{n-j} \geq 0,$$

it follows from (8.1) that

$$E_k(\varphi) \leq E_k^{np}(\varphi),$$

and thus $\mathcal{E}_k(X,\omega) \subset \mathcal{G}_k(X,\omega)$. Moreover, if $E_k(\varphi) > -\infty$, then $T = 0$, since clearly $E^{np}(\varphi)$ is bounded from above. It follows that $\mathcal{E}_k(X,\omega) \subset \mathcal{F}_k(X,\omega)$. If $\varphi \in \mathcal{F}_k(X,\omega)$, then $T = 0$ by Proposition 7.1, and thus $E_k(\varphi) = E_k^{np}(\varphi)$. Hence $\mathcal{E}_k(X,\omega) = \mathcal{G}_k(X,\omega) \cap \mathcal{F}_k(X,\omega)$. □

8.1. Concavity of E_k^{np}. We have the following generalization of (the second part of) Proposition 7.4.

Proposition 8.6. *Assume that $\varphi \in \mathcal{G}_k(X,\omega)$ has sul. Then E_k^{np} is concave on the set of $\psi \in \mathrm{PSH}(X,\omega)$ such that $\psi \sim \varphi$.*

This is a consequence of the following generalization of (7.2).

Proposition 8.7. *Assume that $\varphi, \psi \in \mathcal{G}_k(X,\omega)$ have sul and $\psi \sim \varphi$. Let A be a closed complete pluripolar set such that φ and thus ψ are locally bounded outside A. Moreover, let $u = \psi - \varphi$. Then*

$$\begin{aligned}(8.5)\qquad &\frac{d^2}{dt^2}\Big|_{t=0^+} E_k^{np}(\varphi + tu)\\ &= -k\int_{X\setminus A} du\wedge d^c u \wedge \langle dd^c\varphi+\omega\rangle^{k-1}\wedge\omega^{n-k}\\ &\quad - \frac{1}{k+1}\sum_{j=2}^{k} j(j-1)\lim_{\ell\to\infty}\int_{X\setminus(O_\ell\cup A)} du\wedge du^c\wedge(dd^c\varphi_\ell+\omega)\\ &\qquad\qquad \wedge\langle dd^c\varphi+\omega\rangle^{j-2}\wedge\omega^{n-j},\end{aligned}$$

where $O_\ell = \{\varphi > -\ell\}\cap\{\psi > -\ell\}$ and $\varphi_\ell = \max(\varphi, -\ell)$.

The right-hand side of (8.5) is non-positive, cf. Remark 6.3. If $\varphi + tu$ is ω-psh also for $t > -\epsilon$ for some $\epsilon > 0$ so that

$$g(t) := E_k^{np}(\varphi + tu) = E_k^{np}((1-t)\varphi + t\psi)$$

is defined in a neighborhood of $t = 0$, then it follows from the proof below that (8.5) is indeed the two-sided second derivative of $g(t)$ at $t = 0$. It follows that $g(t)$ is concave on the interval $(0, 1)$ (or more generally where it is defined). Thus Proposition 8.6 follows.

For the proof of Proposition 8.7 we need the following lemma, cf. Lemma 4.5.

Lemma 8.8. *Assume that $\varphi, \psi \in \mathcal{G}_k(X,\omega)$ and $\psi \sim \varphi$. Then, for any $i \le j \le k$,*

$$\int_X \varphi\langle dd^c\varphi+\omega\rangle^i\wedge\langle dd^c\psi+\omega\rangle^{j-i}\wedge\omega^{n-j} > -\infty.$$

Proof. We may assume that $\varphi, \psi \le 0$. Since $\varphi \sim \psi$, it follows from Theorem 8.3 (1) that $\varphi + \psi \in \mathcal{G}_k(X, 2\omega)$. Thus

$$(8.6)\qquad \int_X (\varphi+\psi)\big\langle dd^c(\varphi+\psi)+2\omega\big\rangle^j\wedge(2\omega)^{n-j} > -\infty$$

for $j \le k$. Since the non-pluripolar Monge–Ampère product is multilinear, (8.6) is a sum of terms

$$(8.7)\qquad \int_X \phi\langle dd^c\varphi+\omega\rangle^i\wedge\langle dd^c\psi+\omega\rangle^{j-i}\wedge\omega^{n-j},$$

where ϕ is φ or ψ. Since they are all non-positive, the lemma follows. □

Proof of Proposition 8.7. Since the non-pluripolar Monge–Ampère product is multilinear, it follows that

$$E_k^{np}(\varphi + tu) = \frac{1}{k+1}\sum_{j=0}^{k}\int_X (\varphi + tu)\big\langle dd^c(\varphi + tu) + \omega\big\rangle^j \wedge \omega^{n-j}$$

is a polynomial in t with coefficients that are sums of terms of the form (8.7), where ϕ is φ or ψ. Since each such integral is finite by Lemma 8.8, we may differentiate $E_k^{np}(\varphi + tu)$ formally. Thus

$$(k+1)\frac{d^2}{dt^2}\bigg|_{t=0^+} E_k^{np}(\varphi + tu) = \sum_{j=1}^{k} 2j\int_X u\,dd^c u \wedge \big\langle dd^c\varphi + \omega\big\rangle^{j-1} \wedge \omega^{n-j} + \sum_{j=2}^{k}(j-1)j\int_X \varphi(dd^c u)^2 \wedge \big\langle dd^c\varphi + \omega\big\rangle^{j-2} \wedge \omega^{n-j}, \tag{8.8}$$

where

$$dd^c u = \langle dd^c\psi + \omega\rangle - \langle dd^c\varphi + \omega\rangle.$$

Since currents of the form $(dd^c u)^\ell \wedge \langle dd^c\varphi + \omega\rangle^i$ do not charge A, we may replace X by $X \setminus A$ in (8.8).

Let $T = \langle dd^c\varphi + \omega\rangle^{j-2} \wedge \omega^{n-j}$. Since $\varphi_\ell := \max(\varphi, -\ell)$ decreases to φ, the integral in the jth term in the second sum in (8.8) equals

$$\int_{X\setminus A} \varphi(dd^c u)^2 \wedge T = \lim_{\ell\to\infty}\int_{X\setminus A} \varphi_\ell (dd^c u)^2 \wedge T. \tag{8.9}$$

Since $dd^c u \wedge T$ is the difference of two closed positive currents, we can apply Proposition 6.2 to this. It follows that the right-hand side of (8.9) equals

$$\lim_{\ell\to\infty}\int_{X\setminus A} u\,dd^c u \wedge (dd^c\varphi_\ell + \omega) \wedge T - \int_{X\setminus A} u\,dd^c u \wedge \omega \wedge T. \tag{8.10}$$

The first term in (8.10) equals

$$\lim_{\ell\to\infty}\int_{O_\ell\setminus A} u\,dd^c u \wedge (dd^c\varphi_\ell + \omega) \wedge T + \lim_{\ell\to\infty}\int_{X\setminus(O_\ell\cup A)} u\,dd^c u \wedge (dd^c\varphi_\ell + \omega) \wedge T.$$

In view of (2.5) we conclude that

$$\begin{aligned}
\int_{X\setminus A} &\varphi(dd^c u)^2 \wedge \big\langle dd^c\varphi + \omega\big\rangle^{j-2} \wedge \omega^{n-j} \\
&= \int_{X\setminus A} u\,dd^c u \wedge \big\langle dd^c\varphi + \omega\big\rangle^{j-1} \wedge \omega^{n-j} \\
&\quad - \int_{X\setminus A} u\,dd^c u \wedge \big\langle dd^c\varphi + \omega\big\rangle^{j-2} \wedge \omega^{n-j+1} \\
&\quad + \lim_{\ell\to\infty}\int_{X\setminus(O_\ell\cup A)} u\,dd^c u \wedge (dd^c\varphi_\ell + \omega) \wedge \langle dd^c\varphi + \omega\rangle^{j-2} \wedge \omega^{n-j}.
\end{aligned}$$

Plugging this into (8.8) and, as above, replacing integrals over X by integrals over $X \setminus A$, we get

$$\begin{aligned}\frac{d^2}{dt^2}\Big|_{t=0^+} & E_k^{np}(\varphi + tu) \\ &= k \int_{X\setminus A} u\,dd^c u \wedge \langle dd^c\varphi + \omega\rangle^{k-1} \wedge \omega^{n-k} \\ &\quad + \frac{1}{k+1}\sum_{j=0}^{k}(j-1)j \lim_{\ell\to\infty}\int_{X\setminus(O_\ell\cup A)} u\,dd^c u \wedge (dd^c\varphi_\ell + \omega) \\ &\qquad\qquad \wedge \langle dd^c\varphi + \omega\rangle^{j-2} \wedge \omega^{n-j}.\end{aligned}$$

By Proposition 6.2,

$$\begin{aligned}k \int_{X\setminus A} & u\,dd^c u \wedge \langle dd^c\varphi + \omega\rangle^{k-1} \wedge \omega^{n-k} \\ &= -k \int_{X\setminus A} du \wedge d^c u \wedge \langle dd^c\varphi + \omega\rangle^{k-1} \wedge \omega^{n-k},\end{aligned}$$

which is precisely the first term on the right-hand side of (8.5).

Next, we claim that

$$\begin{aligned}\lim_{\ell\to\infty} & \int_{X\setminus(O_\ell\cup A)} u\,dd^c u \wedge (dd^c\varphi_\ell + \omega) \wedge \langle dd^c\varphi + \omega\rangle^{j-2} \wedge \omega^{n-j} \\ &= -\lim_{\ell\to\infty}\int_{X\setminus(O_\ell\cup A)} du \wedge du^c \wedge (dd^c\varphi_\ell + \omega) \wedge \langle dd^c\varphi + \omega\rangle^{j-2} \wedge \omega^{n-j}.\end{aligned} \tag{8.11}$$

Taking this for granted, the last sum in (8.10) equals the sum in (8.5), and thus the proposition follows.

It remains to prove (8.11). To do this, let $T = \langle dd^c\varphi + \omega\rangle^{j-2} \wedge \omega^{n-j}$, and write

$$\begin{aligned}\int_{X\setminus(O_\ell\cup A)} & u\,dd^c u \wedge (dd^c\varphi_\ell + \omega) \wedge T \\ &= \int_{X\setminus A} u\,dd^c u \wedge (dd^c\varphi_\ell + \omega) \wedge T - \int_{O_\ell\setminus A} u\,dd^c u \wedge (dd^c\varphi_\ell + \omega) \wedge T.\end{aligned} \tag{8.12}$$

By Proposition 6.2,

$$\int_{X\setminus A} u\,dd^c u \wedge (dd^c\varphi_\ell + \omega) \wedge T = -\int_{X\setminus A} du \wedge d^c u \wedge (dd^c\varphi_\ell + \omega) \wedge T. \tag{8.13}$$

Next, in view of (2.5),

$$-\lim_{\ell\to\infty}\int_{O_\ell\setminus A} u\,dd^c u \wedge (dd^c\varphi_\ell + \omega) \wedge T = -\int_{X\setminus A} u\,dd^c u \wedge \langle dd^c\varphi + \omega\rangle \wedge T. \tag{8.14}$$

By Proposition 6.2, using that the non-pluripolar Monge–Ampère operator is local in the plurifine topology, this equals

$$\begin{aligned}\int_{X\setminus A} & du \wedge d^c u \wedge \langle dd^c\varphi + \omega\rangle \wedge T \\ &= \int_{O_\ell\setminus A} du \wedge d^c u \wedge (dd^c\varphi_\ell + \omega) \wedge T \\ &\quad + \int_{X\setminus(O_\ell\cup A)} du \wedge d^c u \wedge \langle dd^c\varphi + \omega\rangle \wedge T.\end{aligned} \tag{8.15}$$

Since u is bounded, (8.14) is finite and thus the second term in (8.15) tends to 0 when $\ell \to \infty$. We conclude that

$$-\lim_{\ell\to\infty}\int_{O_\ell\setminus A} u\,dd^c u \wedge (dd^c\varphi_\ell + \omega)\wedge T = \lim_{\ell\to\infty}\int_{O_\ell\setminus A} du\wedge d^c u \wedge (dd^c\varphi_\ell + \omega)\wedge T. \tag{8.16}$$

Now combining (8.12), (8.13), and (8.16), we get (8.11). □

Remark 8.9. By arguments as in the above proof we also get a formula for the first derivative of $E(\varphi + tu)$ in the situation of Proposition 8.7, cf. (7.1):

$$\frac{d}{dt}E_k^{np}(\varphi+tu) = \int_X u\langle dd^c(\varphi+tu)+\omega\rangle^k \wedge \omega^{n-k} + \frac{1}{1+k}\sum_{j=0}^{k} j \lim_{\ell\to\infty}\int_{X\setminus O_\ell} u\big(dd^c(\varphi_\ell + tu_\ell)+\omega\big) \wedge \big\langle dd^c(\varphi+tu)+\omega\big\rangle^{j-1}\wedge\omega^{n-j}. \tag{8.17}$$

In particular, (8.17) is non-negative if u is; thus we get an alternative proof of Proposition 8.5 in this case.

9. Relative energy

We slightly extend the notion of relative energy from the introduction.

Definition 9.1. Let $\psi \in \mathcal{G}_k(X,\omega)$. For $\varphi \in \mathrm{PSH}(X,\omega)$ such that $\varphi \preceq \psi$ and for k with $1 \le k \le n-1$, we define the *energy relative to ψ of order k* as

$$E_k^\psi(\varphi) = \inf\{E_k^{np}(\varphi') : \varphi' \ge \varphi,\ \varphi' \sim \psi\}.$$

We define the corresponding *finite relative energy classes*

$$\mathcal{E}_k^\psi(X,\omega) = \{\varphi \preceq \psi : E_k^\psi(\varphi) > -\infty\}$$

and the *relative full mass classes*

$$\mathcal{F}_k^\psi(X,\omega) = \left\{\varphi \in \mathrm{PSH}(X,\omega) : \varphi \preceq \psi,\ \sum_{j=0}^{k}\int_X \langle dd^c\varphi+\omega\rangle^j \wedge \omega^{n-j} = \sum_{j=0}^{k}\int_X \langle dd^c\psi+\omega\rangle^j\wedge\omega^{n-j}\right\}.$$

Note that if $\psi = 0$, or more generally $\psi \in \mathrm{PSH}(X,\omega)\cap L^\infty(X)$, then

$$E_k^\psi = E_k, \quad \mathcal{E}_k^\psi(X,\omega) = \mathcal{E}_k(X,\omega), \quad \mathcal{F}_k^\psi(X,\omega) = \mathcal{F}_k(X,\omega),$$

cf. Proposition 7.1. Also, note that $E^{\psi} = E^{\psi}_{n-1}$,

$$\mathcal{E}^{\psi}(X,\omega) = \mathcal{E}^{\psi}_{n-1}(X,\omega) \subset \cdots \subset \mathcal{E}^{\psi}_{1}(X,\omega)$$

and

$$\mathcal{F}^{\psi}(X,\omega) = \mathcal{F}^{\psi}_{n-1}(X,\omega) \subset \cdots \subset \mathcal{F}^{\psi}_{1}(X,\omega),$$

where E^{ψ}, $\mathcal{E}^{\psi}(X,\omega)$ and $\mathcal{F}^{\psi}(X,\omega)$ are as in Definition 1.14.

We have the following generalization of Theorems 7.3 and 8.3.

Theorem 9.2. *Let (X,ω) be a compact Kähler manifold. Then:*

(1) *if $\varphi \in \mathcal{E}^{\psi}_{k}(X,\omega)$ and $\varphi' \sim \varphi$, then $\varphi' \in \mathcal{E}^{\psi}_{k}(X,\omega)$,*

(2) *if ψ has sul, then $\mathcal{E}^{\psi}_{k}(X,\omega)$ is convex,*

(3) *we have*

$$\mathcal{E}^{\psi}_{k}(X,\omega) = \mathcal{G}_k(X,\omega) \cap \mathcal{F}^{\psi}_{k}(X,\omega),$$

and moreover if $\varphi \in \mathcal{F}^{\psi}_{k}(X,\omega)$, then $E^{\psi}_{k}(\varphi) = E^{np}_{k}(\varphi)$.

Note that Theorem 1.15 corresponds to $k = n-1$. For the proof we will use the following observation.

Lemma 9.3. *We have*

$$E^{\psi}_{k}(\varphi) = \lim_{\ell\to\infty} E^{np}_{k}\big(\max(\varphi, \psi - \ell)\big).$$

Proof. First, note that $\varphi_\ell := \max(\varphi, \psi-\ell) \sim \psi$ and φ_ℓ decreases to φ. Thus, by Proposition 8.5, $\lim_{\ell\to\infty} E^{np}_{k}(\varphi_\ell) \geq E^{\psi}_{k}(\varphi)$. Next, assume that $\phi^j \sim \psi$ is a sequence decreasing to φ. Since $\phi^j \sim \psi$ we can take $\ell_j \to \infty$ such that $\phi^j \geq \psi - \ell_j$ and thus $\phi^j \geq \varphi_{\ell_j}$. Now $\lim_{j\to\infty} E^{np}_{k}(\phi^j) \geq \lim_{j\to\infty} E^{np}_{k}(\varphi_{\ell_j}) = \lim_{\ell\to\infty} E^{np}_{k}(\varphi_\ell)$ by Proposition 8.5. □

We get the following partial generalization of Proposition 7.4.

Proposition 9.4. *Take $\psi \in \mathcal{G}_k(X,\omega)$. Then E^{ψ}_{k} is non-decreasing. If ψ has sul, then E^{ψ}_{k} is concave.*

Proof. As above, let $\varphi_\ell = \max(\varphi, \psi - \ell)$. Assume that $\varphi, \varphi' \preceq \psi$ are such that $\varphi \geq \varphi'$. Then $\varphi_\ell \sim \varphi'_\ell \sim \psi$ and $\varphi_\ell \geq \varphi'_\ell$ and thus, by Proposition 8.5, $E^{np}_{k}(\varphi_\ell) \geq E^{np}_{k}(\varphi'_\ell)$. Taking limits over ℓ, in view of Lemma 9.3, we get $E^{\psi}_{k}(\varphi) \geq E^{\psi}_{k}(\varphi')$.

Assume that ψ has sul. It remains to prove that then E^{ψ}_{k} is concave. Take $\varphi, \varphi' \preceq \psi$, $t \in (0,1)$, and a sequence $\phi^j \sim \psi$ decreasing to $(1-t)\varphi + t\varphi'$. Note that we can choose $\ell_j \to \infty$ such that for each j, $\phi^j \geq (1-t)\varphi_{\ell_j} + t\varphi'_{\ell_j}$. Now

$$E^{np}_{k}(\phi^j) \geq E^{np}_{k}\big((1-t)\varphi_{\ell_j} + t\varphi'_{\ell_j}\big) \geq (1-t)E^{np}_{k}(\varphi_{\ell_j}) + tE^{np}_{k}(\varphi'_{\ell_j}).$$

The first inequality follows since E^{np}_{k} is non-decreasing on ω-psh functions of the same singularity type, Proposition 8.5. The second inequality follows since E^{np}_{k} is concave on ω-psh functions with sul of the same singularity type, Proposition 8.6. Indeed, note that

$$\varphi_\ell \sim \varphi'_\ell \sim (1-t)\varphi_\ell + t\varphi'_\ell \sim \psi.$$

Now, taking limits over j, and inf over all sequences ϕ^j, we get

$$E_k^{\psi}((1-t)\varphi + t\varphi') \geq (1-t)E_k^{\psi}(\varphi) + tE_k^{\psi}(\varphi').$$

Thus E_k^{ψ} is concave. □

Proof of Theorem 9.2. Note, in view of Remark 8.2 and Lemma 9.3, that

$$E_k^{\psi}(\varphi) \leq E_k^{\psi}(\varphi + C) \leq E_k^{\psi}(\varphi) + C\int_X \omega^n$$

if $C \geq 0$. Hence part (1) follows from (the first part of) Proposition 9.4. Moreover, part (2) follows immediately from (the second part of) Proposition 9.4.

It remains to prove part (3). We start by proving

$$\mathcal{G}_k(X,\omega) \cap \mathcal{F}_k^{\psi}(X,\omega) \subset \mathcal{E}_k^{\psi}(X,\omega). \tag{9.1}$$

Let

$$T(\varphi) = \frac{1}{k+1}\sum_{j=0}^{k}\int_X \big(\omega^j - \langle dd^c\varphi + \omega\rangle^j\big) \wedge \omega^{n-j}.$$

Moreover, let $\varphi_\ell = \max(\varphi, \psi - \ell)$. Note that $\varphi_\ell \geq \varphi$. Since $\varphi_\ell \sim \psi$, it follows from Proposition 6.1 that

$$T(\varphi_\ell) = T(\psi). \tag{9.2}$$

Now

$$\begin{aligned} E_k^{\psi}(\varphi) &= \lim_{\ell\to\infty} E_k^{np}(\varphi_\ell) \\ &= \lim_{\ell\to\infty}\lim_{\lambda\to\infty}\big(E_k\big(\max(\varphi_\ell, -\lambda)\big) + \lambda T(\varphi_\ell)\big) \\ &\geq \lim_{\lambda\to\infty}\big(E_k\big(\max(\varphi, -\lambda)\big) + \lambda T(\psi)\big) \\ &= E_k^{np}(\varphi) + \lim_{\lambda\to\infty}\lambda\big(T(\psi) - T(\varphi)\big); \end{aligned}$$

here we have used Lemma 9.3 for the first equality, Proposition 8.4 for the second and last equality, and the monotonicity of E_k for the inequality. Note that $T(\varphi) = T(\psi)$ if and only if $\varphi \in \mathcal{F}_k^{\psi}(X,\omega)$. Hence (9.1) follows. Also if $\varphi \in \mathcal{F}_k^{\psi}(X,\omega)$, then

$$E_k^{\psi}(\varphi) = E_k^{np}(\varphi).$$

To prove the reverse inclusion, consider

$$\begin{aligned} \sum_{j=0}^{k}\int_X \varphi_\ell\langle dd^c\varphi_\ell\rangle^j \wedge \omega^{n-j} = {} & \sum_{j=0}^{k}\int_{O_\ell} \varphi_\ell\langle dd^c\varphi\rangle^j \wedge \omega^{n-j} \\ & + \sum_{j=0}^{k}\int_{X\setminus O_\ell} \psi\langle dd^c\varphi_\ell\rangle^j \wedge \omega^{n-j} \\ & - \ell\sum_{j=0}^{k}\int_{X\setminus O_\ell} \langle dd^c\varphi_\ell\rangle^j \wedge \omega^{n-j}; \end{aligned} \tag{9.3}$$

here the equality follows since (2.5) is local in the plurifine topology. By Lemma 9.3, the left-hand side converges to $(k+1)E_k^{\psi}(\varphi)$. Assume that $\varphi \in \mathcal{E}_k^{\psi}(X,\omega)$ so that $E_k^{\psi}(\varphi) > -\infty$. Note that the three terms on the right-hand side are bounded from above. Thus each of them is $> -\infty$. In particular,

$$\sum_{j=0}^{k} \int_{O_\ell} \varphi_\ell \langle dd^c\varphi\rangle^j \wedge \omega^{n-j} \to (k+1)E_k^{np}(\varphi) > -\infty,$$

which means that $\varphi \in \mathcal{G}_k(X,\omega)$. Moreover, the finiteness of the third term on the right-hand side of (9.3) implies that

$$\begin{aligned} 0 &= \lim_{\ell\to\infty} \sum_{j=0}^{k} \int_{X\setminus O_\ell} \langle dd^c\varphi_\ell\rangle^j \wedge \omega^{n-j} \\ &= \lim_{\ell\to\infty} \sum_{j=0}^{k} \int_{X} \langle dd^c\varphi_\ell\rangle^j \wedge \omega^{n-j} - \lim_{\ell\to\infty} \sum_{j=0}^{k} \int_{O_\ell} \langle dd^c\varphi_\ell\rangle^j \wedge \omega^{n-j} \\ &= \sum_{j=0}^{k} \int_{X} \langle dd^c\psi\rangle^j \wedge \omega^{n-j} - \sum_{j=0}^{k} \int_{X} \langle dd^c\varphi\rangle^j \wedge \omega^{n-j}, \end{aligned}$$

i.e., $\varphi \in \mathcal{F}_k^{\psi}(X,\omega)$. Here we have used (9.2) for the last equality. □

It is quite possible that the more general integration by parts results [26, Theorem 1.1] and [24, Theorem 2.6], cf. Remark 6.4, can be used remove the sul assumption from Theorem 9.2 (2). In fact, Vu has already proved the convexity of certain related finite relative energy classes [24, Theorem 1.1].

10. Relations to the Błocki–Cegrell class

In [10] the domain of the Monge–Ampère operator was characterized in several equivalent ways. In order to prove Theorem 1.11 we will use the following characterization from [10, Theorem 1.1].

Proposition 10.1. *Let Ω be an open subset of $\mathbb{C}^n$. Then $u \in \mathrm{PSH}(\Omega)$ is in $\mathcal{D}(\Omega)$ if and only if for each open $\mathcal{U} \subset \Omega$ and any sequence of smooth $u_\lambda \in \mathrm{PSH}(\mathcal{U})$ decreasing to u in $\mathcal{U}$, the sequences*

$$|u_\lambda|^{n-j-2} du_\lambda \wedge d^c u_\lambda \wedge (dd^c u_\lambda)^j \wedge \omega^{n-j-1}, \quad j = 0, 1, \dots, n-2,$$

where ω is a smooth strictly positive $(1,1)$-form, are locally weakly bounded in $\mathcal{U}$.

We will also use the following lemma.

Lemma 10.2. *Assume that $\{\mathcal{U}_i\}$ is a finite open covering of X and that, for each i, g_i is a local dd^c-potential of ω in $\mathcal{U}_i$. Moreover, assume that χ_i is partition of unity subordinate $\{\mathcal{U}_i\}$. Then there is a $C > 0$, such that, if $\varphi \in \mathrm{PSH}(X,\omega)$ is smooth and $\varphi < 0$, then,* *for* $1 \leq j \leq n-1$,

$$\int_X -\varphi(dd^c\varphi+\omega)^j \wedge \omega^{n-j} \tag{10.1}$$
$$\leq 2\sum_i \int_X \chi_i d(\varphi+g_i)\wedge d^c(\varphi+g_i)\wedge (dd^c\varphi+\omega)^{j-1}\wedge\omega^{n-j}$$
$$+\int_X -\varphi(dd^c\varphi+\omega)^{j-1}\wedge\omega^{n-j+1}+C.$$

Proof. We will use the following statement. Assume that A, B, a, b are real smooth functions. Then

$$\begin{aligned} AB\,da\wedge d^cb + AB\,db\wedge d^ca &= 2AB(da\wedge d^cb)_{(1,1)} \\ &\leq A^2 da\wedge d^ca + B^2 db\wedge d^cb \end{aligned} \tag{10.2}$$

as forms, where $(1,1)$ denotes the component of bidegree $(1,1)$. This is an immediate consequence of the inequality $i\alpha\wedge\bar\beta + i\beta\wedge\bar\alpha \leq i\alpha\wedge\bar\alpha + i\beta\wedge\bar\beta$ for $(1,0)$-forms α,β, applied to $\alpha = A\partial a$ and $\beta = B\partial b$.

Take $1\leq j\leq n-1$, let $T=(dd^c\varphi+\omega)^{j-1}\wedge\omega^{n-j}$, and let

$$I=\sum_i\int_X \chi_i d(\varphi+g_i)\wedge d^c(\varphi+g_i)\wedge T.$$

Then, by Stokes' theorem, the left-hand side of (10.1) equals

$$\int_X -\varphi(dd^c\varphi+\omega)\wedge T = \int_X d\varphi\wedge d^c\varphi\wedge T + \int_X -\varphi\wedge\omega T. \tag{10.3}$$

Moreover,

$$\begin{aligned} \int_X d\varphi\wedge d^c\varphi\wedge T &= \sum_i\int_X \chi_i d\varphi\wedge d^c\varphi\wedge T \\ &= I - \sum_i\int_X \chi_i d\varphi\wedge d^c g_i\wedge T - \sum_i\int_X \chi_i dg_i\wedge d^c\varphi\wedge T \\ &\quad - \sum_i\int_X \chi_i dg_i\wedge d^cg_i\wedge T \\ &\leq I + \frac12\sum_i\int_X \chi_i d\varphi\wedge d^c\varphi\wedge T + \sum_i\int_X \chi_i dg_i\wedge d^cg_i\wedge T \\ &= I + \frac12\int_X d\varphi\wedge d^c\varphi\wedge T + \sum_i\int_X \chi_i dg_i\wedge d^cg_i\wedge T. \end{aligned} \tag{10.4}$$

Here, the inequality follows by (10.2) applied to $A=-\frac{1}{\sqrt2}$, $B=\sqrt2$, $a=\varphi$, and $b=g_i$. Note that there is a $D>0$, such that for all i, $dg_i\wedge d^cg_i\wedge T\leq D\omega\wedge T$ as forms. Thus, from (10.4) we conclude that

$$\frac12\int_X d\varphi\wedge d^c\varphi\wedge T \leq I + D\int_X \omega\wedge T = I + D\int_X \omega^n. \tag{10.5}$$

Combining (10.3) and (10.5), we get (10.1) (with $C = 2D\int_X \omega^n$). □

Proof of Theorem 1.11. Assume that $\varphi \in \mathcal{D}(X,\omega)$. Clearly, we may assume that $\varphi < 0$. Let $\varphi_\ell = \max(\varphi, -\ell)$. Moreover, choose a sequence $\epsilon_\lambda \to 0$, $\lambda \in \mathbb{N}$. By [17, Theorem 1.1], for each ℓ, there is a sequence $\varphi_{\ell,\lambda}$ of smooth negative $(1+\epsilon_\lambda)\omega$-psh functions decreasing to φ_ℓ. We may assume that $\varphi_{\ell,\lambda} > \varphi$; otherwise replace $\varphi_{\ell,\lambda}$ by $\varphi_{\ell,\lambda} + \delta_{\ell,\lambda}$, where $\delta_{\ell,\lambda} \to 0$. Since X is compact, we may inductively choose $\lambda_\ell \geq \ell$ so that $\varphi_{\ell,\lambda_\ell} < \varphi_{\kappa,\lambda_{\ell-1}}$ for all $\kappa < \ell$, and

$$\int_X -\varphi_\ell (dd^c\varphi_\ell + \omega)^j \wedge \omega^{n-j} < \int_X -\varphi_{\ell,\lambda_\ell}(dd^c\varphi_{\ell,\lambda_\ell} + \omega)^j \wedge \omega^{n-j} + \frac{1}{\ell}. \tag{10.6}$$

Set $\psi_\ell = \varphi_{\ell,\lambda_\ell}$. Then ψ_ℓ is a sequence of smooth negative $(1+\epsilon_{\lambda_\ell})$-psh functions decreasing to φ. Assume that $\{\mathcal{U}_i\}$, g_i, and χ_i are as in Lemma 10.2. Then $\varphi + g_i \in \mathcal{D}(\mathcal{U}_i)$ and $\psi_\ell + (1+\epsilon_{\lambda_\ell})g_i$ is a sequence of smooth psh functions decreasing to $\varphi + g_i$. It follows by Proposition 10.1 that there is a $K > 0$, such that, for each ℓ and $j = 0, \dots, n-2$,

$$\sum_i \int_X \chi_i d\big(\psi_\ell + (1+\epsilon_{\lambda_\ell})g_i\big) \wedge d^c\big(\psi_\ell + (1+\epsilon_{\lambda_\ell})g_i\big) \\ \wedge \big(dd^c\psi_\ell + (1+\epsilon_{\lambda_\ell})\omega\big)^j \wedge \big((1+\epsilon_{\lambda_\ell})\omega\big)^{n-j-1} \leq K.$$

By Lemma 10.2, inductively applied to $j = 1, \dots, n-1$, we get that there is an $M > 0$ such that, for $1 \leq j \leq n-1$,

$$\int_X -\psi_\ell (dd^c\psi_\ell + \omega)^j \wedge \omega^{n-j} \leq M.$$

By (10.6),

$$\int_{O_\ell} -\varphi_\ell (dd^c\varphi_\ell + \omega)^j \wedge \omega^{n-j} \leq M + \frac{1}{\ell}.$$

Thus, in view of (2.5), as desired,

$$\int_X -\varphi \langle dd^c\varphi + \omega\rangle^j \wedge \omega^{n-j} \leq M. \qquad \square$$

11. Examples of functions with finite non-pluripolar energy

Let us present some examples of functions with finite non-pluripolar energy. As in the local situation ω-psh functions with analytic singularities are in $\mathcal{G}_k(X,\omega)$.

Example 11.1. Assume that $\varphi \in \mathrm{PSH}(X,\omega)$ has analytic singularities, i.e., locally φ is of the form $\varphi = c\log|f|^2 + b$, where $c > 0$, f is a tuple of holomorphic functions, and b is bounded. Then $\varphi \in \mathcal{G}(X,\omega)$ by Example 4.1. Note that $\varphi \in \mathcal{F}(X,\omega)$ if and only if φ is locally bounded, cf. Remark 7.2.

Moreover, note that if $\varphi, \psi \in \mathrm{PSH}(X,\omega)$ have analytic singularities, then the function $(1-t)\varphi + t\psi \in \mathrm{PSH}(X,\omega)$ has analytic singularities. Thus the set of functions in $\mathcal{G}_k(X,\omega)$ with analytic singularities is a convex subclass.

We will now consider some examples on the space $\mathbb{P}^n = \mathbb{P}^n_x$ with homogeneous coordinates $[x] = [x_0, \dots, x_n]$ and equipped with the Fubini–Study form

$$\omega_{\mathrm{FS}} = dd^c \log|x|^2 = \log(|x_0|^2 + \dots + |x_n|^2). \tag{11.1}$$

Example 11.2. Let f be a λ-homogeneous polynomial on $\mathbb{C}^{n+1}$ that we consider as a holomorphic section of $\mathcal{O}(\lambda) \to \mathbb{P}^n$. Now $\varphi := \log(|f|^2/|x|^2) = \log|f|^2_{\mathrm{FS}}$, where $|\cdot|_{\mathrm{FS}}$ denotes the Fubini–Study metric, is in $\mathrm{PSH}(\mathbb{P}^n, \lambda\omega_{\mathrm{FS}})$ and has analytic singularities. In particular, $\varphi \in \mathcal{G}(\mathbb{P}^n, \lambda\omega_{\mathrm{FS}})$ by Example 11.1.

Next, let us consider examples of functions with finite non-pluripolar energy that do not have analytic singularities. First, we present a global version of Example 4.3 that shows that $\mathcal{G}_k(X,\omega)$ is not convex in general. For this we need the following lemma.

Lemma 11.3. *Assume that $\varphi \in \mathrm{PSH}(X,\omega)$, where (X,ω) is a compact Kähler manifold. Moreover, assume that $g : \mathbb{R} \to \mathbb{R}$ is smooth, non-decreasing and convex, and $g'(\varphi) \le 1$. Then $g(\varphi) \in \mathrm{PSH}(X,\omega)$.*

Proof. Note that

$$dd^c\big(g(\varphi)\big) + \omega = g''(\varphi)d\varphi \wedge d^c\varphi + g'(\varphi)(dd^c\varphi + \omega) + \big(1 - g'(\varphi)\big)\omega,$$

and that each term on the right-hand side is ≥ 0 by the assumptions on φ and g. □

Example 11.4. Let X be a projective manifold and let $L \to X$ be an ample line bundle equipped with a positive metric h with corresponding Kähler form ω. Let $f = (f_1, \dots, f_m)$ be a holomorphic section of $L^{\oplus m}$, take $C > 0$ such that $|f|^2_h = |f_1|^2_h + \dots + |f_m|^2_h < C$, and let $\varphi = \log|f|^2_h - C$. For $\epsilon \in (0,1)$, let $g(t) = -(-t)^\epsilon$ and

$$\psi = g(\varphi) = -(-\log|f|^2_h + C)^\epsilon.$$

Then we have $\psi \in \mathrm{PSH}(X,\omega)$ by Lemma 11.3. As in Example 4.3 one sees that, for each k, $\psi \in \mathcal{G}_k(X,\omega)$ if and only if $\epsilon < \frac{1}{2}$, and that $(1-t)\varphi + t\psi \in \mathrm{PSH}(X,\omega) \setminus \mathcal{G}_k(X,\omega)$ for all $t \in (0,1)$ and all $\epsilon \in (0,1)$.

The previous example also shows that E_k^{np} is not monotone in general.

Example 11.5. Let us use the notation from Example 11.4. Note that $g(t) > t$ for $t \le -1$. It follows that $\psi > \varphi$, and thus $\varphi' := (1-t)\varphi + t\psi > \varphi$ if $t \in (0,1)$. By Example 11.4, $E_1^{np}(\varphi') = -\infty$, whereas $E_1^{np}(\varphi) > -\infty$. Thus E_1^{np} is not non-decreasing.

Next, let us consider some examples of functions with finite non-pluripolar energy that do not have sul. The following is a global version of Example 4.6.

Example 11.6. Let (X,ω) be a compact Kähler manifold of dimension n. For each $i \ge 1$, let ψ_i be an ω-psh function with analytic singularities and let $b_i > 0$. If $B := \sum b_i < \infty$, then

$$\varphi := \sum_1^\infty b_i\psi_i \in \mathrm{PSH}(X, B\omega).$$

Given $C > 0$, by the same arguments as in Example 4.6 we can choose b_i inductively so that

$$\int_X \varphi\langle dd^c\varphi + B\omega\rangle^j \wedge (B\omega)^{n-j} \ge -C, \quad j \le n-1,$$

and thus $\varphi \in \mathcal{G}(X, B\omega)$.

The following global variant of Example 4.8 gives an example of a function with finite non-pluripolar energy that neither has sul nor full mass.

Example 11.7. Let $\sigma = (\sigma_1, \dots, \sigma_k)$ be a holomorphic section of $\mathcal{O}(1)^{\oplus k} \to \mathbb{P}^n_x$, such that the zero set is a codimension k-plane P_σ, and let $\psi_\sigma = \log(|\sigma|^2/|x|^2)$. Then the function $\psi_\sigma \in \mathrm{PSH}(\mathbb{P}^n, \omega_{\mathrm{FS}})$ has analytic singularities and unbounded locus P_σ. Next, let $\sigma^1, \sigma^2, \dots$ be holomorphic sections of $\mathcal{O}(1)^{\oplus k}$ that define codimension k-planes $P_{\sigma^1}, P_{\sigma^2}, \dots$, respectively, such that $\bigcup P_{\sigma^i}$ is dense in $\mathbb{P}^n$. Let $\psi_i = \psi_{\sigma^i}$, and let $\psi = \sum b_i \psi_i$ be constructed as in Example 11.6. Then $\varphi := \psi/B \in \mathcal{G}(\mathbb{P}^n, \omega_{\mathrm{FS}})$, where $B = \sum_i b_i$, but φ is not locally bounded anywhere; in particular, φ does not have sul. Moreover, since $\psi_i \notin \mathcal{F}_k(\mathbb{P}^n, \omega_{\mathrm{FS}})$, cf. Example 11.1, it follows that $\varphi \notin \mathcal{F}_k(\mathbb{P}^n, \omega_{\mathrm{FS}})$ and thus $\varphi \notin \mathcal{E}_k(\mathbb{P}^n, \omega_{\mathrm{FS}})$.

Next, let us consider a variant of Example 11.7 in $\mathbb{P}^1_x$.

Example 11.8. Let $\psi_i = \max(\log|x_1 - a_i x_0|^2_{\mathrm{FS}}, -c_i)$, where $(1, a_i) \in \mathbb{P}^1$, $c_i \in (0, \infty]$. Then $\psi_i \in \mathrm{PSH}(\mathbb{P}^1, \omega_{\mathrm{FS}})$ has analytic singularities and as in Example 11.6 we can choose b_i so that

$$\varphi = \sum_{i=1}^{\infty} b_i \psi_i = \sum_{i=1}^{\infty} b_i \max(\log|x_1 - a_i x_0|^2_{\mathrm{FS}}, -c_i)$$

is in $\mathcal{G}(\mathbb{P}^n, B\omega_{\mathrm{FS}})$, where $B = \sum_i b_i$. If $c_i = \infty$, we are in the situation in Example 11.7.

Moreover, if the points $(1, a_i)$ are dense in $\mathbb{P}^1$ and $b_i c_i \to \infty$, then φ is not locally bounded anywhere; in particular, it does not have sul. By choosing b_i and c_i inductively it is possible to arrange this so that in addition $\varphi \in \mathcal{G}(\mathbb{P}^n, B\omega_{\mathrm{FS}})$. One can check that if $b_i \to 0$ fast enough and $b_i^2 c_i \to 0$, then, in fact, $\varphi \in \mathcal{E}(\mathbb{P}^n, B\omega_{\mathrm{FS}})$.

11.1. Product spaces. Let (X_1, ω_1) and (X_2, ω_2) be compact Kähler manifolds. Let $X = X_1 \times X_2$ and $\omega = \pi_1^*\omega + \pi_2^*\omega_2$, where $\pi_i : X \to X_i$ are the natural projections. Then it is readily verified that (X, ω) is a compact Kähler manifold. Moreover, if $\varphi^i \in \mathrm{PSH}(X_i, \omega_i)$, then $\varphi := \pi_1^*\varphi^1 + \pi_2^*\varphi^2 \in \mathrm{PSH}(X, \omega)$. We have the following global version of Proposition 4.9 that can be proved in the same way.

Proposition 11.9. *Let (X_i, ω_i), (X, ω), φ^i, and φ be as above. Assume $\varphi^i \in \mathcal{G}_k(X_i, \omega_i)$, $i = 1, 2$. Then $\varphi \in G_k(X, \omega)$.*

We can use Proposition 11.9 to find new non-trivial examples of ω-psh functions with finite non-pluripolar energy.

Example 11.10. Take $\varphi^1 \in \mathrm{PSH}(\mathbb{P}^n, \omega_{\mathrm{FS}})$ that has analytic singularities and is not locally bounded. Then $\varphi^1 \in \mathcal{G}(\mathbb{P}^n, \omega_{\mathrm{FS}})$ but $\varphi^1 \notin \mathcal{F}(\mathbb{P}^n, \omega_{\mathrm{FS}})$. Next, take $\varphi^2 \in \mathcal{E}(\mathbb{P}^1, \omega_{\mathrm{FS}})$ that does not have sul, e.g., as in Example 11.8. Then, by Proposition 11.9,

$$\varphi = \pi_1^*\varphi^1 + \pi_2^*\varphi^2 \in \mathcal{G}_k(\mathbb{P}^n \times \mathbb{P}^1, \pi_1^*\omega_{\mathrm{FS}} + \pi_2^*\omega_{\mathrm{FS}}),$$

but φ neither has sul nor full mass.

11.2. Convex combinations in projective space. Throughout this subsection we assume that $\mathbb{P}^n = \mathbb{P}^n_x$ is equipped with the Fubini–Study form (11.1) that we denote by ω or $\omega_{\mathbb{P}^n}$.

We know from Example 11.4 that convex combinations of functions in $\mathcal{G}_k(\mathbb{P}^n,\omega)$ are not in $\mathcal{G}_k(\mathbb{P}^n,\omega)$ in general. It turns out, however, that if $\varphi^i \in \mathcal{G}_k(\mathbb{P}^{n_i},\omega)$ for $i=1,2$, then there are natural associated functions $\widetilde{\varphi}^i \in \mathcal{G}_k(\mathbb{P}^N,\omega)$, where $N=n_1+n_2+1$, such that any convex combination of $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ is in $\mathcal{G}_k(\mathbb{P}^N,\omega)$.

To make this more precise, let us first settle some notation. Recall that $\mathbb{P}^n$ is covered by coordinate charts $\mathcal{U}_i = \mathcal{U}_i^{\mathbb{P}^n} = \{x_i \neq 0\}$. In $\mathcal{U}_0 = \{x_0 \neq 0\} = \{[1,x_1,\dots,x_n]\}$ we have local coordinates $x'=(x_1,\dots,x_n)$ and $\gamma_{x'} := dd^c \log(1+|x'|^2)$ is a local potential for ω.

Note that the maps $p_1 : \mathbb{P}^N_{x,y} \dashrightarrow \mathbb{P}^{n_1}_x$, $[x,y]\mapsto [x]$ and $p_2 : \mathbb{P}^N_{x,y} \dashrightarrow \mathbb{P}^{n_2}_x$, $[x,y]\mapsto [y]$ are well-defined outside the planes $\{x=0\}$ and $\{y=0\}$, respectively. If we let $\pi : Y \to \mathbb{P}^N$ be the blowup of $\mathbb{P}^N$ along these planes, then there are maps $\widehat{p}_i : Y \to \mathbb{P}^{n_i}$, $i=1,2$, so that the diagrams

$$\begin{tikzcd} Y \arrow[d, "\pi"'] \arrow[dr, "\widehat{p}_i"] & \\ \mathbb{P}^N \arrow[r, dashed, "p_i"'] & \mathbb{P}^{n_i} \end{tikzcd}$$

commute. Moreover, if $\varphi \in \mathrm{PSH}(\mathbb{P}^{n_i},\omega)$, then $p_i^*\varphi := \pi_*\widehat{p}_i^*\varphi$ is an upper semicontinuous function on $\mathbb{P}^N$ with possible singularities along $\{x=0\}$ or $\{y=0\}$. If we understand $p_i^*\varphi$ as the upper semicontinuous regularization, we get a well-defined upper semicontinuous function on $\mathbb{P}^N$.

Now, assume that $\varphi \in \mathrm{PSH}(\mathbb{P}^{n_1},\omega)$. We want to show that there is a natural associated $\widetilde{\varphi} \in \mathrm{PSH}(\mathbb{P}^N,\omega)$. Consider the functions $\Gamma_x = \log|x|^2$ and $\Gamma_{x,y} = \log|x,y|^2$ on $\mathbb{C}^{N+1}$. Note that $\Gamma_x - \Gamma_{x,y}$ is a 0-homogeneous function on $\mathbb{C}^{N+1}$; thus it defines a global function on $\mathbb{P}^N$. Now, let

$$\widetilde{\varphi} = p_1^*\varphi + \Gamma_x - \Gamma_{x,y}.$$

We claim that $\widetilde{\varphi} \in \mathrm{PSH}(\mathbb{P}^N,\omega)$. To prove this we need to show that $dd^c\widetilde{\varphi} + \omega_{\mathbb{P}^N} \geq 0$. In fact, it suffices to do this outside $\{x=0\}$, since $\{x=0\}$ has codimension at least 2 and $dd^c\widetilde{\varphi}$ is a normal $(1,1)$-current and thus we may assume that p_1 is holomorphic. Note that

$$dd^c(\Gamma_x - \Gamma) = p_1^*\omega_{\mathbb{P}^{n_1}} - \omega_{\mathbb{P}^N}.$$

Thus

$$dd^c\widetilde{\varphi} + \omega_{\mathbb{P}^N} = p_1^*(dd^c\varphi + \omega_{\mathbb{P}^{n_1}}) \geq 0,$$

since φ is $\omega_{\mathbb{P}^{n_1}}$-psh.

Analogously, if $\varphi \in \mathrm{PSH}(\mathbb{P}^{n_2},\omega)$, then $\widetilde{\varphi} := p_2^*\varphi + \Gamma_y - \Gamma_{x,y}$, where $\Gamma_y = \log|y|^2$, is a well-defined function in $\mathcal{G}_k(\mathbb{P}^N,\omega)$.

Proposition 11.11. *Assume that $\varphi^i \in \mathcal{G}_k(\mathbb{P}^{n_i},\omega)$, $i=1,2$. Then for $0\leq t\leq 1$,*

$$(1-t)\widetilde{\varphi}^1 + t\widetilde{\varphi}^2 \in \mathcal{G}_k(\mathbb{P}^{n_1+n_2+1},\omega),$$

where $\widetilde{\varphi}^i$ are defined as above.

Proof. Let $\varphi = (1-t)\widetilde{\varphi}^1 + t\widetilde{\varphi}^2$ and, as above, let $N=n_1+n_2+1$. We need to prove that for $0\leq j\leq k$,

$$\int_{\mathcal{U}} \varphi\langle dd^c\varphi + \omega\rangle^j \wedge \omega^{N-j} > -\infty, \tag{11.2}$$

where $\mathcal{U} \subset \mathbb{P}^N$ is open. We may assume that $\mathcal{U}$ has compact support in one of the coordinate charts $\mathcal{U}_i^{\mathbb{P}^N}$, say in $\mathcal{U}_0^{\mathbb{P}^N}$ with homogeneous coordinates $[1, x', \lambda, \lambda y']$.

Let us consider the integrand in (11.2) in these coordinates. First note that

$$\Gamma_x = \log(1 + |x'|) = \gamma_{x'} \quad \text{and} \quad \Gamma_y = \log\big(|\lambda|^2(1 + |y'|)\big) = \log|\lambda|^2 + \gamma_{y'}.$$

We write γ for $\Gamma_{x,y}$. Since φ^i are 0-homogeneous, it follows that $p_1^*\varphi^1$ and $p_2^*\varphi^2$ only depend on x' and y', respectively; we write $p_1^*\varphi^1 = \varphi^1(x')$ and $p_2^*\varphi^2 = \varphi^2(y')$.

We start with the factor $\langle dd^c\varphi + \omega\rangle^j = \langle dd^c(\varphi + \gamma)\rangle^j$. Now

$$\begin{aligned}\varphi + \gamma &= (1-t)\widetilde{\varphi}^1 + t\widetilde{\varphi}^2 + \gamma \\ &= (1-t)\big(\varphi^1(x') + \gamma_{x'}\big) + t\big(\varphi^2(y') + \log|\lambda|^2 + \gamma_{y'}\big).\end{aligned}$$

Note that $dd^c \log|\lambda|^2 = [\lambda = 0]$; in particular, it has support where $\log|\lambda|^2 = -\infty$. It follows that

$$\langle dd^c\varphi + \omega\rangle^j = \big\langle(1-t)(dd^c\varphi^1(x') + \omega_x) + t(dd^c\varphi^2(y') + \omega_y)\big\rangle^j,$$

where $\omega_x = dd^c\gamma_{x'}$ and $\omega_y = dd^c\gamma_{y'}$.

Next, let $\theta = \omega_x + \omega_y + \eta_\lambda$, where $\eta_\lambda = i\,d\lambda \wedge d\bar{\lambda}$. Then θ is a smooth strictly positive $(1,1)$-form on $\mathbb{P}^N$. Thus we can replace ω^{N-j} in (11.2) by θ^{N-j}. Note that for degree reasons (11.2) is a sum of terms of the form

$$\text{(11.3)} \quad \int_{\mathcal{U}} \varphi\big\langle(1-t)(dd^c\varphi^1(x')+\omega_x)+t(dd^c\varphi^2(y')+\omega_y)\big\rangle^j(\omega_x+\omega_y)^{n_1+n_2-j}\wedge\eta_\lambda.$$

Finally, consider

$$\begin{aligned}\varphi &= (1-t)\widetilde{\varphi}^1 + t\widetilde{\varphi}^2 \\ &= (1-t)\varphi^1(x') + t\varphi^2(y') + t\log|\lambda|^2 + (1-t)\gamma_x + t\gamma_y - \gamma.\end{aligned}$$

Since the last three terms are smooth, the contribution from these terms in (11.3) is finite. Next, the contribution from $\log|\lambda|^2$ is finite since $\log|\lambda|^2$ is (locally) integrable. Finally, if we replace φ by $\varphi^1(x')$ or $\varphi^2(y')$, then we are in the situation in Section 11.1. Indeed, we can regard $\mathcal{U}$ as a relatively compact subset of $\mathcal{U}_0^{\mathbb{P}^{n_1}} \times \mathcal{U}_0^{\mathbb{P}^{n_2}} \times \mathbb{C}_\lambda \subset \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{C}_\lambda$. Then $\varphi^1(x')$, ω_x, $\varphi^2(y')$, and ω_y are just the pullbacks of φ^1, $\omega_{\mathbb{P}^{n_1}}$, φ^2, and $\omega_{\mathbb{P}^{n_2}}$, respectively, under the natural projections $\mathcal{U} \to \mathcal{U}_0^{\mathbb{P}^{n_i}}$. Thus by Proposition 11.9 this contribution is also finite. We conclude that (11.2) holds and hence $\varphi \in \mathcal{G}_k(\mathbb{P}^N, \omega)$. □

Example 11.12. Let φ^i, $i = 1, 2$, be as in Example 11.10. Then

$$\varphi := (1-t)\widetilde{\varphi}^1 + t\widetilde{\varphi}^2 \in \mathcal{G}_k(\mathbb{P}^{n+1}, \omega)$$

and φ neither has analytic singularities nor full mass.

Remark 11.13. Let us consider the complex Monge–Ampère equation

$$\text{(11.4)} \qquad \langle dd^c\varphi + \omega\rangle^n = f\omega^n$$

on the compact Kähler manifold (X, ω). By Yau's solution to the famous Calabi conjecture, (11.4) has a smooth solution if f is smooth. This result has been generalized by several authors in many different directions allowing less regular f. For instance, in [15, Theorem 1.4] it was proved that under certain conditions on f there is a unique solution $\varphi \in \mathrm{PSH}(X, \omega)$ with prescribed singularity type.

One could hope that it would be possible to use our Monge–Ampère currents to solve a Monge–Ampère equation of the form

$$[dd^c\varphi + \omega]^n = \mu, \tag{11.5}$$

where μ is allowed to be a more general current. For instance, assume that $\mu = \mu_1 + \mu_2$, where $\mu_1 = f\omega^n$ and μ_2 has support on a subvariety $Z \subset X$, defined by a holomorphic section s of a Hermitian vector bundle over X. Then it might be natural to look for a solution $\varphi \sim \log|s|^2$ to (11.5). Note that if φ solves (11.5), then, in particular, $\langle dd^c\varphi + \omega\rangle^n = \mu_1$. Now, by [15, Theorem 1.4] this completely determines φ. Thus one can only hope to solve (11.5) for very special choices of μ_1 and μ_2.

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