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Chern currents of coherent sheaves

Richard Lärkäng and Elizabeth Wulcan

Abstract. Given a finite locally free resolution of a coherent analytic sheaf $\mathcal{F}$, equipped with Hermitian metrics and connections, we construct an explicit current, obtained as the limit of certain smooth Chern forms of $\mathcal{F}$, that represents the Chern class of $\mathcal{F}$ and has support on the support of $\mathcal{F}$. If the connections are $(1,0)$-connections and $\mathcal{F}$ has pure dimension, then the first nontrivial component of this Chern current coincides with (a constant times) the fundamental cycle of $\mathcal{F}$. The proof of this goes through a generalized Poincaré–Lelong formula, previously obtained by the authors, and a result that relates the Chern current to the residue current associated with the locally free resolution.

Keywords. Chern classes; coherent sheaves; residue currents

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Contents

1. Introduction

Let X be a complex manifold and let $\mathcal{F}$ be a coherent analytic sheaf on X of positive codimension, ***i.e.*** such that the support $\operatorname{supp}\mathcal{F}$ has positive codimension. Assume that $\mathcal{F}$ has a locally free resolution of the form

$$0 \to \mathcal{O}(E_N) \xrightarrow{\varphi_N} \cdots \xrightarrow{\varphi_1} \mathcal{O}(E_0) \to \mathcal{F} \to 0, \tag{1.1}$$

where E_k are holomorphic vector bundles on X, and $\mathcal{O}(E_k)$ denote the corresponding locally free sheaves. Then the (total) Chern class of $\mathcal{F}$ equals

$$c(\mathcal{F}) = \prod_{k=0}^{N} c(E_k)^{(-1)^k}, \tag{1.2}$$

where $c(E_k)$ is the (total) Chern class of E_k, see Section 3.1. In this paper we construct explicit representatives of the nontrivial part of $c(\mathcal{F})$ with support on $\operatorname{supp}\mathcal{F}$.

Let us briefly describe our construction; the representatives of $c(\mathcal{F})$ will be currents obtained as limits of certain Chern forms. Assume that (E,φ) is a locally free resolution of $\mathcal{F}$ of the form (1.1). Moreover assume that each vector bundle E_k is equipped with a Hermitian metric and a connection D_k (that is not necessarily the Chern connection of the metric). Let σ_k be the minimal inverse of φ_k, see Section 2.3, let $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a smooth cut-off function such that $\chi(t) \equiv 0$ for $t \ll 1$ and $\chi(t) \equiv 1$ for $t \gg 1$, let s be a (generically non-vanishing) holomorphic section of a vector bundle such that $\{s=0\} \supseteq \operatorname{supp}\mathcal{F}$, let $\chi_\epsilon = \chi(|s|^2/\epsilon)$, and let $\widehat{D}_k^\epsilon$ be the connection

$$\widehat{D}_k^\epsilon = -\chi_\epsilon \sigma_k D\varphi_k + D_k; \tag{1.3}$$

here D is a connection on $\operatorname{End} E$, where $E = \bigoplus E_k$, induced by the D_k, see Section 2.2. Then clearly the Chern form

$$c(E,\widehat{D}^\epsilon) := \prod_{k=0}^{N} c(E_k, \widehat{D}_k^\epsilon)^{(-1)^k} \tag{1.4}$$

is a representative for $c(\mathcal{F})$. Throughout this paper we consider Chern classes as de Rham cohomology classes of smooth forms or currents. Our first main result asserts that the limit of this form is a current with the desired properties.

Theorem 1.1. *Assume that $\mathcal{F}$ is a coherent analytic sheaf of positive codimension that admits a locally free resolution of the form (1.1). Moreover, assume that each E_k is equipped with a Hermitian metric and a connection D_k. Let $c(E,\widehat{D}^\epsilon)$ be the Chern form of $\mathcal{F}$ defined by (1.4), and let $c_\ell(E,\widehat{D}^\epsilon)$ denote the component of degree 2ℓ. Let $\ell_1,\dots,\ell_m \in \mathbb{N}_{>0}$. Then*

$$c^{\mathrm{Res}}_{\ell_1}(E,D)\wedge\cdots\wedge c^{\mathrm{Res}}_{\ell_m}(E,D) := \lim_{\epsilon\to 0} c_{\ell_1}(E,\widehat{D}^\epsilon)\wedge\cdots\wedge c_{\ell_m}(E,\widehat{D}^\epsilon) \tag{1.5}$$

is a well-defined closed current, independent of χ_ϵ, that represents $c_{\ell_1}(\mathcal{F})\wedge\cdots\wedge c_{\ell_m}(\mathcal{F})$ and has support on $\operatorname{supp}\mathcal{F}$.

The Chern currents (1.5) are pseudomeromorphic in the sense of [AW10], see Theorem 5.1, which means that they have a geometric nature similar to closed positive (or normal) currents, see Section 2.1. We let

$$c^{\mathrm{Res}}(E,D) = 1 + c^{\mathrm{Res}}_1(E,D) + c^{\mathrm{Res}}_2(E,D) + \cdots.$$

The first nontrivial component of $c^{\mathrm{Res}}(E,D)$ is (the current of integration along) a cycle, see Theorem 1.4 below. We do not know whether Chern currents of higher degree are of order 0 in general.

Remark 1.2. If all the connections D_k are $(1,0)$-connections, *i.e.* the $(0,1)$-part of each D_k equals $\bar\partial$, then so are the connections $\widehat{D}^\epsilon_k$. However, even if the D_k are Chern connections, the $\widehat{D}^\epsilon_k$ are not Chern connections in general. Thus, it might be the case that the involved forms and currents in (1.5) contain terms of bidegree $(\ell+m,\ell-m)$ with $m>0$ (but only when $\ell > \operatorname{codim}\mathcal{F}$ by Theorem 1.4 below).

Our construction of Chern currents is inspired by the paper [BB72] by Baum and Bott, where singular holomorphic foliations are studied by expressing characteristic classes associated to a foliation as certain cohomological residues, more precisely as push-forwards of cohomology classes living in the singular set of the foliation. A key point in the proofs in [BB72] are the concepts of connections compatible with and fitted to a complex of vector bundles. One may check that their constructions of fitted connections (with some minor adaptations) correspond to connections of the form (1.3). For the results in [BB72], it was sufficient to consider Chern forms associated to connections (1.3) for ϵ small enough, but fixed, while in the present paper, we study the limit of such forms when $\epsilon \to 0$.

Example 1.3. Let us compute $c^{\mathrm{Res}}(E,D)$ when $\mathcal{F}$ is the structure sheaf $\mathcal{O}_Z$ of a divisor $Z\subset X$, defined by a holomorphic section s of a holomorphic line bundle L over X, and (E,φ) is the locally free resolution

$$0 \to \mathcal{O}(L^*) \xrightarrow{s} \mathcal{O} \to \mathcal{O}_Z \to 0. \tag{1.6}$$

Assume that L is equipped with a connection D_L; equip $E_1 = L^*$ with the induced dual connection D_{L^*}, and E_0 with the trivial connection. The minimal inverse of s is $1/s$ and $Ds = D_L s$, so $\widehat{D}^\epsilon_1 = \chi_\epsilon(D_L s/s) + D_{L^*}$, and $\widehat{D}^\epsilon_0$ is the trivial connection. The curvature form of $\widehat{D}^\epsilon_1$ equals $\widehat{\Theta}^\epsilon_1 = d\chi_\epsilon\wedge(D_L s/s) - (1-\chi_\epsilon)\Theta_L$, where Θ_L is the curvature form of D_L (which equals minus the curvature form of D_{L^*}). By an appropriate formulation of the Poincaré-Lelong formula,

$$\lim_{\epsilon\to 0} d\chi_\epsilon \wedge \frac{D_L s}{s} = 2\pi i[Z], \tag{1.7}$$

where $[Z]$ is the current of integration along (the cycle of) Z. Note that

$$c(E,\widehat{D}^\epsilon) = c(E_1,\widehat{D}^\epsilon_1)^{-1} = 1 - c_1(E_1,\widehat{D}^\epsilon_1) + c_1(E_1,\widehat{D}^\epsilon_1)^2 - \cdots.$$

Thus, since Θ_L is smooth,

$$c^{\mathrm{Res}}_1(E,D) = \lim_{\epsilon\to 0} -c_1(E_1,\widehat{D}^\epsilon_1) = -\lim_{\epsilon\to 0}\frac{i}{2\pi}\widehat{\Theta}^\epsilon_1 = [Z]. \tag{1.8}$$

Thus, the first Chern current coincides with $[Z]$, which can be seen as a canonical representative of $c_1(\mathcal{O}_Z)$ with support on $\operatorname{supp}\mathcal{O}_Z$.

By further calculations of terms of all degrees, one can show that

$$c^{\mathrm{Res}}(E,D) = 1 + [Z](1 + c_1(L,D) + \cdots + c_1^{n-1}(L,D)).$$

Our next result is an explicit description of the first nontrivial Chern current $c_p^{\mathrm{Res}}(E,D)$ in the case when $\mathcal{F}$ has pure codimension p, *i.e.* $\operatorname{supp}\mathcal{F}$ has pure dimension $\dim X - p$, that generalizes (1.8). Recall that the *(fundamental) cycle* of $\mathcal{F}$ is the cycle

$$[\mathcal{F}] = \sum_i m_i[Z_i] \tag{1.9}$$

(considered as a current of integration), where Z_i are the irreducible components of $\operatorname{supp}\mathcal{F}$, and m_i is the *geometric multiplicity* of Z_i in $\mathcal{F}$, see *e.g.* [Ful98, Chapter 1.5].

Theorem 1.4. *Assume that $\mathcal{F}$ is a coherent analytic sheaf of pure codimension $p > 0$ that admits a locally free resolution (E,φ) of the form (1.1). Moreover, assume that each E_k is equipped with a Hermitian metric and a $(1,0)$-connection D_k. Then*

$$c_p^{\mathrm{Res}}(E,D) = (-1)^{p-1}(p-1)![\mathcal{F}].$$

Moreover

$$c_\ell^{\mathrm{Res}}(E,D) = 0 \quad \text{for } 0 < \ell < p \tag{1.10}$$

and

$$c_{\ell_1}^{\mathrm{Res}}(E,D) \wedge \cdots \wedge c_{\ell_m}^{\mathrm{Res}}(E,D) = 0 \quad \text{for } m \geq 2 \text{ and } 0 < \ell_1 + \cdots + \ell_m \leq p. \tag{1.11}$$

Here the Chern currents on the left hand sides are defined by (1.5).

In case $\mathcal{F}$ has codimension p, but not necessarily *pure* codimension p, then Theorem 1.4 still holds if we replace the first equation by

$$c_p^{\mathrm{Res}}(E,D) = (-1)^{p-1}(p-1)![\mathcal{F}]_p, \tag{1.12}$$

where $[\mathcal{F}]_p$ denotes the part of $[\mathcal{F}]$ of codimension p, *i.e.* in (1.9), one only sums over the components Z_i of codimension p.

In particular, $c_p^{\mathrm{Res}}(E,D)$ is independent of the choice of Hermitian metrics and $(1,0)$-connections on (E,φ). Moreover, it follows that on cohomology level

$$c_p(\mathcal{F}) = (-1)^{p-1}(p-1)![\mathcal{F}], \tag{1.13}$$

where now the right hand side should be interpreted as a de Rham class. When $\mathcal{F}$ is the pushforward of a vector bundle from a subvariety, that (1.13) holds is a well-known consequence of the Grothendieck-Riemann-Roch theorem, *cf.* [Ful98, Examples 15.2.16 and 15.1.2].

The proof of Theorem 1.4 relies on a generalization of the Poincaré-Lelong formula. Given a complex (1.1) equipped with Hermitian metrics, Andersson and the second author defined in [AW07] an associated so-called residue current $R^E = R = \sum R_k$ with support on $\operatorname{supp}\mathcal{F}$, where R_k is a $\operatorname{Hom}(E_0,E_k)$-valued $(0,k)$-current for $k = 0,\dots,N$, see Section 2.3. The construction involves the minimal inverses σ_k of φ_k. If (E,φ) is the complex (1.6), then R^E coincides with the residue current $\bar\partial(1/s) = \lim_{\epsilon\to 0}\bar\partial\chi_\epsilon(1/s)$; more generally if (E,φ) is the Koszul complex of a complete intersection, then R^E coincides with the classical Coleff-Herrera residue current, [CH78]. Using residue currents, we can write the Poincaré-Lelong formula (1.7) as

$$\bar\partial\frac{1}{s} \wedge D_L s = 2\pi i[Z];$$

indeed, the left hand side in (1.7) equals $\lim \bar\partial\chi_\epsilon \wedge (D_L s)/s$. Given a $\mathrm{Hom}(E_\ell, E_\ell)$-valued current α, let $\mathrm{tr}\,\alpha$ denote the trace of α. In [LW18, LW21] we proved the following generalization of the Poincaré-Lelong formula:
Assume that R^E is the residue current associated with a finite locally free resolution (E,φ) of a coherent analytic sheaf $\mathcal{F}$ of pure codimension p. Moreover assume that D is a connection on $\mathrm{End}\,E$ induced by arbitrary $(1,0)$-connections on E_k. Then

$$\frac{1}{(2\pi i)^p p!}\mathrm{tr}(D\varphi_1\cdots D\varphi_p R_p) = [\mathcal{F}]. \tag{1.14}$$

If $\mathcal{F}$ has codimension p, but not necessarily pure codimenion, (1.14) still holds if we replace $[\mathcal{F}]$ by $[\mathcal{F}]_p$, *cf.* [LW18, Theorem 1.5]. In view of this, Theorem 1.4, as well as (1.12), are direct consequences of the following explicit description of (the components of low degree of) $c^{\mathrm{Res}}(E,D)$ in terms of R^E.

Theorem 1.5. *Assume that $\mathcal{F}$ is a coherent analytic sheaf of codimension $p > 0$ that admits a locally free resolution (E,φ) of the form (1.1). Moreover, assume that each E_k is equipped with a Hermitian metric and a $(1,0)$-connection D_k. Let R be the associated residue current and D the connection on $\mathrm{End}\,E$ induced by the D_k. Then*

$$c_p^{\mathrm{Res}}(E,D) = \frac{(-1)^{p-1}}{(2\pi i)^p p}\mathrm{tr}(D\varphi_1\cdots D\varphi_p R_p).$$

Moreover (1.10) *and* (1.11) *hold.*

In fact, we formulate and prove our results in a slightly more general setting. We consider the Chern class $c(E)$ of a generically exact complex of vector bundles (E,φ) that is not necessarily a locally free resolution of a coherent sheaf. Theorem 5.1 below asserts that $c(E,\widehat{D}^\epsilon)$ as well as products of such currents have well-defined limits when $\epsilon \to 0$ and represent the corresponding (products of) Chern classes. In particular, Theorem 1.1 follows. In Theorem 6.1, if (E,φ) is exact outside a variety of codimension p, we give an explicit description of $c_p^{\mathrm{Res}}(E,D) := \lim_{\epsilon\to 0} c_p(E,\widehat{D}^\epsilon)$ terms of residue currents that generalizes Theorem 1.5. From this and a more general version of the Poincaré-Lelong formula (1.14) it follows that if the cohomology groups are of pure codimension p, then $c_p^{\mathrm{Res}}(E,D) = (-1)^{p-1}(p-1)![E]$, where $[E]$ is the cycle of (E,φ), see Corollary 6.7 and (2.19).

Our results could alternatively be formulated in term of the Chern character $\mathrm{ch}(E)$ of E. From Theorem 1.1, for $\ell > 0$, we obtain a current $\mathrm{ch}_\ell^{\mathrm{Res}}(E,D)$ that represents the ℓ^{th} graded piece $\mathrm{ch}_\ell(E)$ of the Chern character, see Section 6. Theorems 1.4 and 1.5 are then equivalent to

$$\begin{aligned}
\mathrm{ch}_p^{\mathrm{Res}}(E,D) &= \frac{1}{(2\pi i)^p p!}\mathrm{tr}(D\varphi_1\cdots D\varphi_p R_p) = [\mathcal{F}],\\
\mathrm{ch}_\ell^{\mathrm{Res}}(E,D) &= 0 \quad \text{for } \ell < p, \text{ and}\\
\mathrm{ch}_{\ell_1}^{\mathrm{Res}}(E,D)\wedge\cdots\wedge\mathrm{ch}_{\ell_m}^{\mathrm{Res}}(E,D) &= 0 \quad \text{for } m \geq 2 \text{ and } \ell_1+\cdots+\ell_m \leq p,
\end{aligned}$$

see Theorem 6.2 and Remark 6.3.

We refer to the currents in Theorem 1.1 as Chern currents, in analogy with the usual Chern forms representing Chern classes. In works of Bismut, Gillet, and Soulé [BGS90a, BGS90b] appears the similarly named concept of Bott-Chern currents, that are certain explicit dd^c-potentials in a transgresssion formula in a Grothendieck-Riemann-Roch theorem, and not directly related to our currents.

There are some similarities between our results and results by Harvey and Lawson. In [HL93] they study characteristic classes of morphisms $\varphi : E_0 \to E_1$ of vector bundles, and only in very special situations there is overlap between their results and ours. We remark that the connection (1.3) that plays a crucial role in our work essentially appears and is important in [HL93], see, in particular, [HL93, Section I.4].

Chern classes of coherent sheaves, without the assumption of the existence of a global locally free resolution, were studied in the thesis of Green, [Gre80], as well as in various recent papers, including

[Gri10, Hos20a, Hos20b, Qia16, BSW21, Wu20]. Several of these papers also concern classes in finer cohomology theories than de Rham cohomology, as for example (rational or complex) Bott-Chern or Deligne cohomology.

In the present paper, our focus has been to find explicit representatives of Chern classes of a coherent sheaf with support on the support of the sheaf, a type of result which as far as we can tell, none of the above mentioned works seems to consider. By incorporating the construction of residue currents associated with a twisted resolution from [JL21], it might be possible to extend our results to arbitrary coherent sheaves, without any assumptions about the existence of a global locally free resolution. We plan to explore this in future work. The currents we study provide representatives of the Chern classes in de Rham cohomology. Our methods unfortunately do not seem to yield representatives in the finer cohomology theories mentioned above, since for example Chern classes in complex Bott-Chern cohomology as in [Qia16, BSW21], are naturally obtained from Chern forms of the Chern connection of a hermitian metric, while our construction, building on the techniques in [BB72], involve Chern forms of connections that are not Chern connections of a hermitian metric.

The paper is organized as follows. In Section 2 we give some necessary background on (residue) currents. In Section 3 we describe Chern forms and Chern characters, and in Section 4 we discuss compatible connections. The proofs of (the generalized versions of) Theorems 1.1 and 1.5 occupy Sections 5 and 6, respectively. Finally in Section 7 we compute $c^{\mathrm{Res}}(E,D)$ for an explicit choice of a locally free resolution (E,φ) of a coherent sheaf $\mathcal{F}$. In particular, we compute $c_\ell^{\mathrm{Res}}(E,D)$ for $\ell > \operatorname{codim}\mathcal{F}$ in this case.

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2. Currents associated with complexes of vector bundles

We say that a function $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a *smooth approximand of the characteristic function* $\chi_{[1,\infty)}$ of the interval $[1,\infty)$ and write

$$\chi \sim \chi_{[1,\infty)}$$

if χ is smooth and $\chi(t) \equiv 0$ for $t \ll 1$ and $\chi(t) \equiv 1$ for $t \gg 1$. Note that if $\chi \sim \chi_{[1,\infty)}$ and $\hat{\chi} = \chi^\ell$, then $\hat{\chi} \sim \chi_{[1,\infty)}$ and

$$d\hat{\chi} = \ell \chi^{\ell-1} d\chi. \tag{2.1}$$

2.1. Pseudomeromorphic currents

Let f be a (generically nonvanishing) holomorphic function on a (connected) complex manifold X. Herrera and Lieberman [HL71], proved that the *principal value*

$$\lim_{\epsilon\to 0} \int_{|f|^2>\epsilon} \frac{\xi}{f}$$

exists for test forms ξ and defines a current $[1/f]$. It follows that $\bar\partial[1/f]$ is a current with support on the zero set $Z(f)$ of f; such a current is called a *residue current*. Assume that $\chi \sim \chi_{[1,\infty)}$ and that F is a (generically nonvanishing) section of a Hermitian vector bundle such that $Z(f) \subseteq \{F = 0\}$. Then

$$[1/f] = \lim_{\epsilon\to 0} \frac{\chi(|F|^2/\epsilon)}{f} \qquad \text{and} \qquad \bar\partial[1/f] = \lim_{\epsilon\to 0} \frac{\bar\partial\chi(|F|^2/\epsilon)}{f}, \tag{2.2}$$

see *e.g.* [AW18]. In particular, the limits are independent of χ and F.

In the literature there are various generalizations of residue currents and principal value currents. In particular, Coleff and Herrera [CH78] introduced products like

$$[1/f_1]\cdots[1/f_r]\bar\partial[1/f_{r+1}]\wedge\cdots\wedge\bar\partial[1/f_m]. \tag{2.3}$$

In order to obtain a coherent approach to questions about residue and principal value currents was introduced in [AW10] the sheaf $\mathcal{PM}_X$ of ***pseudomeromorphic currents*** on X, consisting of direct images under holomorphic mappings of products of test forms and currents like (2.3). See *e.g.* [AW18, Section 2.1] for a precise definition; in particular it follows from the definition that $\mathcal{PM}$ is closed under push-forwards of modifications. Also, we refer to [AW18] for the results mentioned in this subsection. The sheaf $\mathcal{PM}_X$ is closed under $\bar\partial$ and under multiplication by smooth forms. Pseudomeromorphic currents have a geometric nature, similar to closed positive (or normal) currents. For example, the ***dimension principle*** states that if the pseudomeromorphic current μ has bidegree $(*,p)$ and support on a variety of codimension strictly larger than p, then μ vanishes.

The sheaf $\mathcal{PM}_X$ admits natural restrictions to constructible subsets. In particular, if W is a subvariety of the open subset $\mathcal{U}\subseteq X$, and F is a section of a vector bundle such that $\{F=0\}=W$, then the restriction to $\mathcal{U}\setminus W$ of a pseudomeromorphic current μ on $\mathcal{U}$ is the pseudomeromorphic current

$$\mathbf{1}_{\mathcal{U}\setminus W}\mu := \lim_{\epsilon\to 0}\chi(|F|^2/\epsilon)\mu|_{\mathcal{U}},$$

where $\chi\sim\chi_{[1,\infty)}$ as above. This definition is independent of the choice of F and χ.

A pseudomeromorphic current μ on X is said to have the ***standard extension property*** (SEP) if $\mathbf{1}_{\mathcal{U}\setminus W}\mu=\mu|_{\mathcal{U}}$ for any subvariety $W\subseteq\mathcal{U}$ of positive codimension, where $\mathcal{U}\subseteq X$ is any open subset. By definition, it follows that if μ has the SEP and $F\not\equiv 0$ is any holomorphic section of a vector bundle, then

$$\lim_{\epsilon\to 0}\chi(|F|^2/\epsilon)\mu=\mu. \tag{2.4}$$

2.2. Superstructure and connections on a complex of vector bundles

Let (E,φ) be a complex

$$0\to E_N\xrightarrow{\varphi_N}E_{N-1}\xrightarrow{\varphi_{N-1}}\cdots\xrightarrow{\varphi_2}E_1\xrightarrow{\varphi_1}E_0\to 0, \tag{2.5}$$

of vector bundles over X. As in [AW07], see also [LW18], we will consider the complex (E,φ) to be equipped with a so-called superstructure, *i.e.* a $\mathbb{Z}_2$-grading, which splits $E:=\oplus E_k$ into odd and even parts E^+ and E^-, where $E^+=\oplus E_{2k}$ and $E^-=\oplus E_{2k+1}$. Also $\operatorname{End}E$ gets a superstructure by letting the even part be the endomorphisms preserving the degree, and the odd part the endomorphisms switching degrees.

This superstructure affects how form- and current-valued endomorphisms act. Assume that $\alpha=\omega\otimes\gamma$ is a section of $\mathcal{E}^\bullet(\operatorname{End}E)$, where γ is a holomorphic section of $\operatorname{Hom}(E_\ell,E_k)$, and ω is a smooth form of degree m. Then we let $\deg_f\alpha=m$ and $\deg_e\alpha=k-\ell$ denote the ***form*** and ***endomorphism*** degrees, respectively, of α. The ***total*** degree is $\deg\alpha=\deg_f\alpha+\deg_e\alpha$. If β is a form-valued section of E, *i.e.* $\beta=\eta\otimes\xi$, where η is a scalar form, and ξ is a section of E, both homogeneous in degree, then the action of α on β is defined by

$$\alpha(\beta):=(-1)^{(\deg_e\alpha)(\deg_f\beta)}\omega\wedge\eta\otimes\gamma(\xi). \tag{2.6}$$

If furthermore, $\alpha'=\omega'\otimes\gamma'$, where γ' is a holomorphic section of $\operatorname{End}E$, and ω' is a smooth form, both homogeneous in degree, then we define

$$\alpha\alpha':=(-1)^{(\deg_e\alpha)(\deg_f\alpha')}\omega\wedge\omega'\otimes\gamma\circ\gamma'.$$

For an $(m\times n)$-matrix A and an $(n\times m)$-matrix B, we have that $\operatorname{tr}(AB)=\operatorname{tr}(BA)$, while for the morphisms α and α' above, we get such an equality with a sign due to the superstructure,

$$\operatorname{tr}(\alpha\alpha')=(-1)^{(\deg\alpha)(\deg\alpha')-(\deg_e\alpha)(\deg_e\alpha')}\operatorname{tr}(\alpha'\alpha), \tag{2.7}$$

see [LW18, Equation (2.14)].

Note that $\bar\partial$ extends in a way that respects the superstructure to act on $\operatorname{End} E$-valued morphisms. In particular,

$$\bar\partial(\alpha\alpha') = \bar\partial\alpha\alpha' + (-1)^{\deg\alpha}\alpha\bar\partial\alpha'. \tag{2.8}$$

We will consider the situation when (E,φ) is equipped with a connection $D = D_E = (D_0,\dots,D_N)$, where D_k is a connection on E_k. Then there is an induced connection $\oplus D_k$ on E, that we also denote by D_E. This in turn induces a connection D_{End} on $\operatorname{End} E$ that takes the superstructure into account, defined by

$$D_{\mathrm{End}}\alpha := D_E \circ \alpha - (-1)^{\deg\alpha}\alpha \circ D_E, \tag{2.9}$$

if α is a $\operatorname{End} E$-valued form. It satisfies the following Leibniz rule, [LW18, Equation (2.4)], *cf.* (2.8)

$$D_{\mathrm{End}}(\alpha\alpha') = D_{\mathrm{End}}\alpha\alpha' + (-1)^{\deg\alpha}\alpha D_{\mathrm{End}}\alpha'. \tag{2.10}$$

To simplify notation, we will sometimes drop the subscript End and simply denote this connection by D. If Θ_k denotes the curvature form of D_k, and $\alpha : E_k \to E_\ell$, then, by (2.9),

$$DD\alpha = \Theta_\ell\alpha + (-1)^{\deg\alpha+\deg\alpha+1}\alpha\Theta_k = \Theta_\ell\alpha - \alpha\Theta_k. \tag{2.11}$$

The above formulas hold also when α and α' are current-valued instead of form-valued, as long as the involved products of currents are well-defined.

We let D_k' and D_k'' denote the $(1,0)$- and $(0,1)$-parts of D_k, respectively, and we let $D' = (D_k')$ and $D'' = (D_k'')$ denote the corresponding $(1,0)$- and $(0,1)$-parts of $D_E = (D_k)$. We say that D_E is a *$(1,0)$-connection* if each D_k is a $(1,0)$-connection, *i.e.* $D_k'' = \bar\partial$. We will use the following consequence of (2.11): assume that D_E is a $(1,0)$-connection, and $\alpha : E_k \to E_\ell$ is a holomorphic (or more generally a $\bar\partial$-closed form-valued) morphism. Then

$$\bar\partial D\alpha = (\Theta_\ell)_{(1,1)}\alpha - \alpha(\Theta_k)_{(1,1)}, \tag{2.12}$$

where $(\cdot)_{(1,1)}$ denotes the component of bidegree $(1,1)$.

Since (E,φ) is a complex and φ_k has odd degree, it follows from (2.10) that

$$\varphi_{k-1}D\varphi_k = D\varphi_{k-1}\varphi_k. \tag{2.13}$$

2.3. Residue currents associated to a complex

Let us briefly recall the construction in [AW07]. Assume that we have a generically exact complex (E,φ) of vector bundles over a complex manifold X of the form (2.5), and assume that each E_k is equipped with some Hermitian metric. If Z_k is the analytic set where φ_k has lower rank than its generic rank, then outside of Z_k the minimal (or Moore-Penrose) inverse $\sigma_k : E_{k-1} \to E_k$ of φ_k is determined by the following properties: $\varphi_k\sigma_k\varphi_k = \varphi_k$, $\operatorname{im}\sigma_k \perp \operatorname{im}\varphi_{k+1}$, and $\sigma_{k+1}\sigma_k = 0$. One can verify that σ_k is smooth outside of Z_k. Since $\sigma_k\sigma_{k-1} = 0$ and σ_k has odd degree, by (2.8),

$$\sigma_k\bar\partial\sigma_{k-1} = \bar\partial\sigma_k\sigma_{k-1}. \tag{2.14}$$

Let Z be the set where (E,φ) is not pointwise exact. It follows from the definition of σ_k that

$$\varphi_k\sigma_k + \sigma_{k-1}\varphi_{k-1} = \mathrm{Id}_{E_{k-1}} \tag{2.15}$$

outside Z, or more generally outside $Z_k \cup Z_{k-1}$. Applying (2.8) to (2.15), we obtain that outside Z

$$\varphi_k\bar\partial\sigma_k = \bar\partial\sigma_{k-1}\varphi_{k-1} \tag{2.16}$$

and furthermore applying (2.10) to this equality, we get that

$$D\varphi_k\bar\partial\sigma_k = D\bar\partial\sigma_{k-1}\varphi_{k-1} + \bar\partial\sigma_{k-1}D\varphi_{k-1} + \varphi_k D\bar\partial\sigma_k. \tag{2.17}$$

Lemma 2.1. *Let X, (E,φ), Z, and σ_k be as above. Assume that for each $j = 1,\dots,m$, s_j is an entry of σ_k, $\partial\sigma_k$, or $\bar\partial\sigma_k$ for some k in some local trivialization, and let $s = s_1\cdots s_m$. Assume that $\chi \sim \chi_{[1,\infty)}$ and that F is a (generically nonvanishing) holomorphic section of a vector bundle over X such that $Z \subset \{F = 0\}$. Then the limits*

$$\lim_{\epsilon\to 0}\chi(|F|^2/\epsilon)s \qquad \textit{and} \qquad \lim_{\epsilon\to 0}\bar\partial\chi(|F|^2/\epsilon)\wedge s$$

exist and define pseudomeromorphic currents on X that are independent of the choices of χ and F; the support of the second current is contained in Z. Furthermore,

$$\lim_{\epsilon\to 0}\partial\chi(|F|^2/\epsilon)\wedge s = 0.$$

Proof. By Hironaka's theorem there is a holomorphic modification $\pi : \tilde X \to X$, such that for each k, $\pi^*\sigma_k$ is locally of the form $(1/\gamma_k)\tilde\sigma_k$, where γ_k is holomorphic with $Z(\gamma_k) \subset \tilde Z := \pi^{-1}Z$, and $\tilde\sigma_k$ is smooth, see [AW07, Section 2]. Now, where $\chi(|\pi^*F|^2/\epsilon) \not\equiv 0$, $\bar\partial\pi^*\sigma_k = (1/\gamma_k)\bar\partial\tilde\sigma_k$ and $\partial\pi^*\sigma_k = \partial(1/\gamma_k)\tilde\sigma_k + (1/\gamma_k)\partial\tilde\sigma_k$. Since each holomorphic derivative $\partial/\partial z_i(1/\gamma_k)$ is a meromorphic function with poles contained in $\tilde Z$ it follows that π^*s_j equals (a sum of terms of the form) $(1/g_j)\tilde s_j$, where g_j is holomorphic with $Z(g_j) \subset \tilde Z$, and $\tilde s_j$ is smooth. Thus π^*s equals (a sum of terms of the form) $(1/g)\tilde s$, where g is holomorphic with $Z(g) \subset \tilde Z$, and $\tilde s$ is smooth. In view of (2.2),

$$\lim_{\epsilon\to 0}\chi(|\pi^*F|^2/\epsilon)\pi^*s \quad \text{and} \quad \lim_{\epsilon\to 0}\bar\partial\chi(|\pi^*F|^2/\epsilon)\wedge\pi^*s$$

are well-defined pseudomeromorphic currents on $\tilde X$ independent of χ and F; the second current has support on $\tilde Z$. Since $\mathcal{PM}$ is closed under push-forwards of modifications, *cf.* Section 2.1, this proves the first part of the lemma.

As proved above, the limit

$$\mu := \lim_{\epsilon\to 0}\chi(|F|^2/\epsilon)s$$

exists. This current is in fact a so-called almost semi-meromorphic current, *cf.* [AW18, Section 4], and in particular, it has the SEP. By [AW18, Theorem 3.7], $\partial\mu$ also has the SEP. Thus,

$$\lim_{\epsilon\to 0}\partial\chi(|F|^2/\epsilon)\wedge\mu = \lim_{\epsilon\to 0}\partial(\chi(|F|^2/\epsilon)\wedge\mu) - \lim_{\epsilon\to 0}\chi(|F|^2/\epsilon)\partial\mu = \partial\mu - \partial\mu = 0,$$

which proves the last part of the lemma. Here, in the second equality, we have used that the two limits exist and are both equal to $\partial\mu$ by (2.4). □

In particular

$$R_k^\ell := \lim_{\epsilon\to 0}\bar\partial\chi(|F|^2/\epsilon)\wedge\sigma_k\bar\partial\sigma_{k-1}\cdots\bar\partial\sigma_{\ell+1} \tag{2.18}$$

is a $\mathrm{Hom}(E_\ell, E_k)$-valued pseudomeromorphic current of bidegree $(0, k-\ell)$ with support on Z; in fact, it follows from the proof that the support is contained in $Z_{\ell+1}\cup\cdots\cup Z_k$. If $\ell = k-1$, then the right hand side of (2.18) should be interpreted as $\lim_{\epsilon\to 0}\bar\partial\chi(|F|^2/\epsilon)\wedge\sigma_k$. The residue current $R^E = R := \sum R_k^\ell$ associated with (E,φ) was introduced in [AW07], *cf.* the introduction. Assume that (E,φ) is a locally free resolution of a coherent analytic sheaf $\mathcal{F}$. Then R_k^ℓ vanishes for $\ell > 0$ by [AW07, Theorem 3.1]. In this case $R = \sum R_k$, where $R_k = R_k^0$.

Given a complex (E,φ) of vector bundles of the form (2.5), following [LW21], we define the *cycle*

$$[E] = \sum_{k=0}^{N}(-1)^k[\mathcal{H}_k(E)], \tag{2.19}$$

where $\mathcal{H}_k$ is the homology sheaf of (E,φ) at level k. Note that if (E,φ) is a locally free resolution of a coherent analytic sheaf $\mathcal{F}$, then $[E] = [\mathcal{F}]$. In [LW21] we prove the following generalization of (1.14).

Theorem 2.2. *Let (E, φ) be a complex of Hermitian vector bundles of the form (2.5) such that $\mathcal{H}_k(E)$ has pure codimension $p > 0$ or vanishes, for $k = 0, \dots, N$, and let D be an arbitrary $(1,0)$-connection on (E, φ). Then,*

$$\frac{1}{(2\pi i)^p p!} \sum_{k=0}^{N-p} (-1)^k \operatorname{tr}(D\varphi_{k+1} \cdots D\varphi_{k+p} R^k_{k+p}) = [E].$$

3. Chern forms and Chern characters

3.1. Chern classes and forms

Assume that E is a holomorphic vector bundle of rank r equipped with a connection D. Then recall that the (total) Chern form $c(E, D) = 1 + c_1(E, D) + \cdots + c_r(E, D)$ is defined by

$$\sum_{\ell=0}^{r} c_\ell(E, D) t^\ell = \det\left(I + \frac{i}{2\pi} \Theta t\right),$$

where Θ is the curvature matrix of D in a local trivialization; in particular, $c_\ell(E, D)$ is a form of degree 2ℓ. The de Rham cohomology class of $c(E, D)$ is the *(total) Chern class* $c(E) = \sum c_\ell(E)$ of the vector bundle E.

If (E, φ) is a complex of vector bundles of the form (2.5) that is not necessarily a locally free resolution of a coherent analytic sheaf, in line with the Chern theory of virtual bundles as in *e.g.* [BB72, Section 4] or [Suw98, Section II.8.C], we let

$$c(E) = \prod_{k=0}^{N} c(E_k)^{(-1)^k}.$$

Moreover, if (E, φ) is equipped with a connection $D = (D_k)$, *cf.* Section 2.2, we let

$$c(E, D) = \prod_{k=0}^{N} c(E_k, D_k)^{(-1)^k} \tag{3.1}$$

and we let $c_\ell(E, D) = c(E, D)_\ell$ be the component of degree 2ℓ.

Consider now a coherent analytic sheaf $\mathcal{F}$ with a locally free resolution (1.1). We define the Chern class of $\mathcal{F}$ by (1.2), *i.e.* $c(\mathcal{F}) = c(E)$, and if (E, φ) is equipped with a connection D, then this class may be represented by (3.1). This definition of Chern classes of coherent sheaves may be motivated in terms of K-theory. However, it is typically considered only on manifolds with the so-called resolution property. Recall that a complex manifold X is said to have the *resolution property* if any coherent analytic sheaf $\mathcal{F}$ on X has a finite locally free resolution (1.1). For such manifolds, the definition (1.2) is the unique extension of the definition of Chern classes from locally free sheaves to coherent analytic sheaves that satisfies the following *Whitney formula*: if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of sheaves, then $c(\mathcal{F}) = c(\mathcal{F}'')c(\mathcal{F}'')$, *cf.* [BS58, Théorème 2] or [EH16, Chapter 14.2].

In this paper, we define Chern classes of coherent sheaves by (1.2) also on manifolds which do not have the resolution property, but then necessarily only for coherent sheaves with a locally free resolution (1.1). Note that if we are on a manifold for which the resolution property does not hold, it is not immediate that the de Rham cohomology class of (1.2) is well-defined, *i.e.* independent of the resolution. However, that it is well-defined follows from a construction of Chern classes of arbitrary coherent analytic sheaves on arbitrary complex manifolds by Green, [Gre80], see also [TT86], since in case one has a global locally free resolution of finite length, the definition in [Gre80] coincides with the one in (1.2).

3.2. The Chern character (form) of a vector bundle

Assume that E is a holomorphic vector bundle of rank r. Then formally we can write

$$1 + c_1(E)t + \cdots + c_r(E)t^r = \prod_{i=1}^{r}(1 + \alpha_i t),$$

where α_i are the so-called *Chern roots* of E, see *e.g.* [Ful98, Remark 3.2.3]. In particular, this means that $c_\ell(E) = e_\ell(\alpha_1, \dots, \alpha_r)$, where e_ℓ is the ℓ^{th} *elementary symmetric polynomial*

$$e_\ell(x) = e_\ell(x_1, \dots, x_r) = \sum_{1 \le i_1 < \cdots < i_\ell \le r} x_{i_1} \cdots x_{i_\ell}.$$

The *Chern character* of E may formally be defined as the symmetric polynomial $\operatorname{ch}(E) = \sum_{i=1}^{r} e^{\alpha_i}$ in the Chern roots, see *e.g.* [Ful98, Example 3.2.3]. In particular, the ℓ^{th} graded piece is

$$\operatorname{ch}_\ell(E) = \frac{1}{\ell!} p_\ell(\alpha_1, \dots, \alpha_r), \tag{3.2}$$

where p_ℓ is the ℓ^{th} *power sum polynomial*

$$p_\ell(x) = p_\ell(x_1, \dots, x_r) = \sum_{i=1}^{r} x_i^\ell.$$

Since any symmetric polynomial in x_i may be expressed as a unique polynomial in $e_j(x)$, there are polynomials $Q_\ell(t_1, \dots, t_\ell)$, $\ell \ge 1$, such that $p_\ell(x) = Q_\ell(e_1(x), \dots, e_\ell(x))$; these are sometimes called *Hirzebruch–Newton polynomials*. If t_j is given weight j, then $Q_\ell(t_1, \dots, t_\ell)$ is a weighted homogenous polynomial of degree ℓ. Written out explicitly, Definition (3.2) should be read as

$$\operatorname{ch}_\ell(E) = \frac{1}{\ell!} Q_\ell\big(c_1(E), \dots, c_\ell(E)\big).$$

If E is equipped with a connection D, one can analogously define *Chern character forms*

$$\operatorname{ch}_\ell(E, D) = \frac{1}{\ell!} Q_\ell\big(c_1(E, D), \dots, c_\ell(E, D)\big) \tag{3.3}$$

and $\operatorname{ch}(E, D) = \sum \operatorname{ch}_\ell(E, D)$ representing the Chern character. If Θ is the curvature corresponding to D (in a local trivialization), then

$$\operatorname{ch}_\ell(E, D) = \frac{1}{\ell!} \operatorname{tr}\left(\frac{i}{2\pi}\Theta\right)^\ell, \tag{3.4}$$

cf. e.g. [Tu17, §B.4-6].

The polynomials Q_ℓ may be computed recursively through Newton's identities,

$$p_\ell(x) = (-1)^{\ell-1} \ell e_\ell(x) + \sum_{i=1}^{\ell-1} (-1)^{\ell-i-1} e_{\ell-i}(x) p_i(x), \quad \ell \ge 1. \tag{3.5}$$

In particular, it follows that the Q_ℓ are independent of r. Moreover, $\operatorname{ch}_\ell(E, D)$ is of the form

$$\operatorname{ch}_\ell(E, D) = \frac{(-1)^{\ell-1}}{(\ell-1)!} c_\ell(E, D) + \widetilde{Q}_\ell\big(c_1(E, D), \dots, c_{\ell-1}(E, D)\big), \tag{3.6}$$

where $\widetilde{Q}_\ell$ is a weighted homogeneous polynomial of degree ℓ, and conversely,

$$c_\ell(E, D) = (-1)^{\ell-1} (\ell-1)! \operatorname{ch}_\ell(E, D) + \widehat{Q}_\ell\big(\operatorname{ch}_1(E, D), \dots, \operatorname{ch}_{\ell-1}(E, D)\big), \tag{3.7}$$

where $\widehat{Q}_\ell$ is a weighted homogeneous polynomial of degree ℓ.

Example 3.1. We obtain from (3.5) that $p_1 = e_1$ and $p_2 = e_1^2 - 2e_2$. Thus, $Q_1(t_1) = t_1$ and $Q_2(t_1,t_2) = t_1^2 - 2t_2$, so

$$\mathrm{ch}_1(E,D) = c_1(E,D) \quad \text{and} \quad \mathrm{ch}_2(E,D) = \frac{1}{2}(c_1(E,D)^2 - 2c_2(E,D)). \tag{3.8}$$

We have the following (formal) relationship between $e_\ell(x)$ and $p_\ell(x)$, and thus $Q_\ell(e_1,\dots,e_\ell)$:

$$\ln\left(\sum_{\ell\geq 0} e_\ell(x)t^\ell\right) = \sum_{\ell\geq 1} \frac{(-1)^{\ell-1}}{\ell} p_\ell(x)t^\ell = \sum_{\ell\geq 1} \frac{(-1)^{\ell-1}}{\ell} Q_\ell(e_1,\dots,e_\ell)t^\ell. \tag{3.9}$$

This follows *e.g.* by integrating [Mac95, Chapter I, Equation (2.10')] with respect to t. Since $e_1,\dots,e_r$ are algebraically independent, (3.9) holds if we replace the e_ℓ by a_ℓ in any commutative ring. In particular, if we apply (3.9) to $e_\ell = c_\ell(E,D)$ and take the components of degree 2ℓ (the coefficents of t^ℓ) we get

$$\ln\big(c(E,D)\big)_\ell = (-1)^{\ell-1}(\ell-1)!\,\mathrm{ch}_\ell(E,D); \tag{3.10}$$

here $()_\ell$ denotes the part of form degree 2ℓ.

3.3. The Chern character of a complex of vector bundles

Let (E,φ) be a complex of vector bundles of the form (2.5). Then the Chern character can be defined as

$$\mathrm{ch}(E) = \sum_{k=0}^{N} (-1)^k \mathrm{ch}(E_k),$$

cf., *e.g.*, [EH16, Chapter 14.2.1] and [Kar08, Chapter V.3].

If (E,φ) is equipped with a connection $D = (D_k)$, for $\ell \geq 1$ we define a *Chern character form* $\mathrm{ch}_\ell(E,D)$ through (3.3). Then $\mathrm{ch}_\ell(E,D)$ inherits properties from the vector bundle case. In particular (3.6) and (3.7) hold. Also (3.10) holds and, using that

$$\ln\big(c(E,D)\big) = \ln\left(\prod_{k=0}^{N} c(E_k,D_k)^{(-1)^k}\right) = \sum_{k=0}^{N} (-1)^k \ln\big(c(E_k,D_k)\big),$$

we get that

$$\mathrm{ch}_\ell(E,D) = \sum_{k=0}^{N} (-1)^k \mathrm{ch}_\ell(E_k,D_k). \tag{3.11}$$

In particular, $\mathrm{ch}_\ell(E,D)$ represents $\mathrm{ch}_\ell(E)$.

Let Θ_k denote the curvature matrix of D_k (in some local trivialization) and define[1]

$$p_\ell(E,D) = \sum_{k=0}^{N} (-1)^k \operatorname{tr}\Theta_k^\ell. \tag{3.12}$$

In view of (3.4) and (3.11), for $\ell \geq 1$,

$$\mathrm{ch}_\ell(E,D) = \frac{i^\ell}{(2\pi)^\ell \ell!} p_\ell(E,D). \tag{3.13}$$

Assume that $D = (D_k)$ is a $(1,0)$-connection. Let $()_{(q,r)}$ denote the part of bidegree (q,r) of a form. Since the curvature matrices Θ_k (in local trivializations) consist of forms of bidegree $(2,0)$ and $(1,1)$, it follows that

$$p_{(\ell,\ell)}(E,D) = \sum_{k=0}^{N} (-1)^k \operatorname{tr}(\Theta_k)^\ell_{(1,1)} \tag{3.14}$$

[1]To be consistent with (3.2) we should have a factor $(i/2\pi)^\ell$ in the definition of $p_\ell(E,D)$, *cf.* (3.13). However, the normalization (3.12) is more convenient to work with.

and by (3.13) that

$$\mathrm{ch}_{(\ell,\ell)}(E,D) = \frac{i^\ell}{(2\pi)^\ell \ell!} p_{(\ell,\ell)}(E,D). \tag{3.15}$$

4. Connections compatible with a complex

Assume that (E,φ) is a complex of vector bundles of the form

$$0 \to E_N \xrightarrow{\varphi_N} \cdots \xrightarrow{\varphi_1} E_0 \xrightarrow{\varphi_0} E_{-1} \to 0. \tag{4.1}$$

Moreover assume that each E_k is equipped with a connection D_k. Then, following [BB72], we say that the connection $D = (D_{-1},\dots,D_N)$ on (E,φ) is *compatible* with (E,φ) if

$$D_{k-1} \circ \varphi_k = -\varphi_k \circ D_k \tag{4.2}$$

for $k = 0,\dots,N$. In terms of the induced connection $D = D_{\mathrm{End}}$ on $\mathrm{End}\, E$, this can succinctly be written as $D\varphi_k = 0$.

Note that in contrast to above, (4.1) starts at level -1. The typical situation we consider is when we start with a complex (2.5) that is pointwise exact outside an analytic variety Z and then restrict to $X \setminus Z$; then $E_{-1} = 0$.

Remark 4.1. By [BB72, Lemma 4.17], given a complex (E,φ) of vector bundles of the form (4.1) one can always extend a given connection D_{-1} on E_{-1} to a connection $D = (D_k)$ that is compatible with (E,φ) where it is pointwise exact. In fact, Lemma 4.4 below gives an explicit formula for such a connection, see Remark 4.5.

Remark 4.2. In [BB72], the condition of being compatible is stated without the minus sign in (4.2); our condition on D is actually the same, but we need to introduce the minus sign since we use the conventions of the superstructure. Indeed, if ξ is a section of E_k of form-degree 0, then $D_{k-1} \circ \varphi_k \xi$ is defined in the same way with or without the superstructure, while the action of φ_k on $D_k\xi$ changes sign depending on whether the superstructure is used or not since $D_k\xi$ has form-degree 1, *cf.* (2.6).

Compatible connections satisfy the following Whitney formula, [BB72, Lemma 4.22], *cf.* Section 3.1.

Lemma 4.3. *Assume that (E,φ) is an exact complex of vector bundles of the form (4.1) that is equipped with a connection $D = (D_k)$ that is compatible with (E,φ). Then*

$$c(E_{-1}, D_{-1}) = \prod_{k=0}^{N} c(E_k, D_k)^{(-1)^k}.$$

4.1. The connection $\widehat{D}^\epsilon$

We will consider a specific situation and choice of compatible connection. As in previous sections, let (E,φ) be a complex of vector bundles of the form (2.5) that is pointwise exact outside the analytic set Z. Moreover, let χ be a smooth approximand of $\chi_{[1,\infty)}$, let F be a (generically nonvanishing) section of a vector bundle such that $Z \subseteq \{F = 0\}$, and let $\chi_\epsilon = \chi(|F|^2/\epsilon)$. Then $\chi_\epsilon \equiv 0$ in a neighborhood of Z. Consider now a fixed choice of connection $D = (D_k)$ on (E,φ), and for $\epsilon > 0$, define a new connection $\widehat{D}^\epsilon = (\widehat{D}^\epsilon_k)$ on (E,φ) through

$$\widehat{D}^\epsilon_k = -\chi_\epsilon \sigma_k D\varphi_k + D_k. \tag{4.3}$$

Note that if D is a $(1,0)$-connection, then so is $\widehat{D}^\epsilon$.

Lemma 4.4. *The connection $\widehat{D}^\epsilon$ is compatible with (E,φ) where $\chi_\epsilon \equiv 1$.*

Proof. Using (2.9), (2.13) and (2.15) we obtain that

$$\begin{aligned}\widehat{D}^\epsilon \varphi_k &= \widehat{D}^\epsilon_{k-1} \circ \varphi_k + \varphi_k \circ \widehat{D}^\epsilon_k \\ &= \chi_\epsilon(-\sigma_{k-1} D\varphi_{k-1}\varphi_k - \varphi_k\sigma_k D\varphi_k) + D_{k-1}\circ\varphi_k + \varphi_k \circ D_k \\ &= -\chi_\epsilon(\sigma_{k-1}\varphi_{k-1} + \varphi_k\sigma_k) D\varphi_k + D_{k-1}\circ\varphi_k + \varphi_k\circ D_k \\ &= (1-\chi_\epsilon) D\varphi_k.\end{aligned}$$

In particular, $\widehat{D}^\epsilon$ is compatible with the complex where $\chi_\epsilon \equiv 1$. □

Remark 4.5. Assume that (E,φ) is a pointwise exact complex of vector bundles equipped with some connection $D = (D_k)$. Then, as in the proof above, it follows that the connection $\widetilde{D}$ defined by

$$\widetilde{D}_k = -\sigma_k D\varphi_k + D_k$$

is compatible with (E,φ). Moreover, $\widetilde{D}_{-1} = D_{-1}$, *cf.* Remark 4.1.

Assume that θ_k is a connection matrix for D_k in a local trivialization, *i.e.* $D_k\alpha = d\alpha + \theta_k \wedge \alpha$. Then the connection matrix for $\widehat{D}^\epsilon_k$ is

$$\hat{\theta}^\epsilon_k = -\chi_\epsilon \sigma_k D\varphi_k + \theta_k$$

and thus the curvature matrix of $\widehat{D}^\epsilon_k$ equals

$$\widehat{\Theta}^\epsilon_k = d\hat{\theta}^\epsilon_k + (\hat{\theta}^\epsilon_k)^2 = -d(\chi_\epsilon\sigma_k D\varphi_k) + \chi_\epsilon^2 \sigma_k D\varphi_k \sigma_k D\varphi_k - \chi_\epsilon(\theta_k\sigma_k D\varphi_k + \sigma_k D\varphi_k\theta_k) + \Theta_k, \tag{4.4}$$

where Θ_k is the curvature matrix of D_k.

5. The Chern current $c^{\mathrm{Res}}(E,D)$

In this section we prove that the limits as $\epsilon \to 0$ of products of Chern forms $c_\ell(E,\widehat{D}^\epsilon)$, where $\widehat{D}^\epsilon$ is the connection from the previous section, give the desired currents in (1.5). More generally, we prove the following generalization of Theorem 1.1.

Theorem 5.1. *Assume that (E,φ) is a complex of Hermitian vector bundles of the form (2.5) that is pointwise exact outside a subvariety Z of positive codimension. Moreover assume that $D = (D_k)$ is a connection on (E,φ) and let $\widehat{D}^\epsilon$ be the connection defined by (4.3). Then, for $\ell_1,\dots,\ell_m \in \mathbb{N}_{>0}$,*

$$c^{\mathrm{Res}}_{\ell_1}(E,D)\wedge\cdots\wedge c^{\mathrm{Res}}_{\ell_m}(E,D) = \lim_{\epsilon\to 0} c_{\ell_1}(E,\widehat{D}^\epsilon)\wedge\cdots\wedge c_{\ell_m}(E,\widehat{D}^\epsilon), \tag{5.1}$$

where the right side is defined by (3.1), is a well-defined closed pseudomeromorphic current that is independent of the choice of χ_ϵ, has support on Z, and represents $c_{\ell_1}(E)\wedge\cdots\wedge c_{\ell_m}(E)$.

Theorem 1.1 is an immediate consequence of Theorem 5.1.

Proof. Let

$$M_\epsilon = c_{\ell_1}(E,\widehat{D}^\epsilon)\wedge\cdots\wedge c_{\ell_m}(E,\widehat{D}^\epsilon). \tag{5.2}$$

We first prove that $\lim_{\epsilon\to 0} M_\epsilon$ exists and is a pseudomeromorphic current. This is a local statement and we may therefore work in a local trivialization where D_k is determined by the connection matrix θ_k. By (4.4), $\widehat{\Theta}^\epsilon_k$ is a (form-valued) matrix of the form

$$\widehat{\Theta}^\epsilon_k = \alpha_k + \chi_\epsilon\beta'_k + \chi_\epsilon^2\beta''_k + d\chi_\epsilon\wedge\beta'''_k,$$

where $\alpha_k = \Theta_k$ is smooth and β'_k, β''_k and β'''_k are polynomials in σ_k, $D\varphi_k$, θ_k and exterior derivatives of such factors. In particular α_k, β'_k, β''_k, and β'''_k are independent of ϵ.

Since M_ϵ is a polynomial in the entries of $\widehat{\Theta}_0^\epsilon, \dots, \widehat{\Theta}_N^\epsilon$, see Section 3.1, we can write

$$M_\epsilon = A + \sum_{j\geq 1} \chi_\epsilon^j B'_j + \sum_{j\geq 1} \chi_\epsilon^{j-1} d\chi_\epsilon \wedge B''_j,$$

where A, B'_j, and B''_j are independent of ϵ, A is smooth, and B'_j and B''_j are polynomials in entries of σ_k, $D\varphi_k$, θ_k and exterior derivatives of such factors. Let $\hat{\chi}_\epsilon = \hat{\chi}(|F|^2/\epsilon)$, where $\hat{\chi} = \chi^j \sim \chi_{[1,\infty)}$, *cf.* Section 2. Then by Lemma 2.1, the limits of

$$\chi_\epsilon^j B'_j = \hat{\chi}_\epsilon B'_j \qquad \text{and} \qquad \chi_\epsilon^{j-1} d\chi_\epsilon \wedge B''_j = d\hat{\chi}_\epsilon \wedge B''_j/j$$

as $\epsilon \to 0$ exist and are pseudomeromorphic currents that are independent of χ_ϵ. It follows that the limit (5.1) exists and is a pseudomeromorphic current that is independent of χ_ϵ.

By Lemma 4.4, $\widehat{D}^\epsilon$ is compatible with (E,φ) where $\chi_\epsilon \equiv 1$ and therefore, by Lemma 4.3, $c(E,\widehat{D}^\epsilon) = 0$ there. It follows that M_ϵ has support where $\chi_\epsilon \not\equiv 1$. Note that the σ_k are smooth outside of Z. By Lemma 2.1, the limit (5.1) is independent of the choice of χ_ϵ. In particular, we may assume that the section F defining $\chi_\epsilon = \chi(|F|^2/\epsilon)$ is locally defined such that $\{F=0\} = Z$. It then follows that the limit (5.1) has support on Z. That (5.1) represents $c_{\ell_1}(E) \wedge \cdots \wedge c_{\ell_m}(E)$ follows by Poincaré duality, since the forms on the right hand side of (5.1) represent this class for all $\epsilon > 0$. Also (5.1) is closed since the forms on the right hand side are for all $\epsilon > 0$. □

Remark 5.2. Assume that $D = (D_k)$ in Theorem 5.1 is a $(1,0)$-connection. Then $\widehat{\Theta}_k^\epsilon$ only has components of bidegree $(2,0)$ and $(1,1)$, *cf.* (4.4). It follows that (5.2) and consequently (5.1) consist of components of bidegree $(\ell+q, \ell-q)$ with $q \geq 0$, where $\ell = \ell_1 + \cdots + \ell_m$.

6. An explicit description of Chern currents of low degrees

In this section we study the Chern current $c^{\text{Res}}(E,D)$ of a complex (E,φ) that is equipped with a $(1,0)$-connection D. Our main result is the following generalization of Theorem 1.5 that is an explicit description of $c_p^{\text{Res}}(E,D)$ in terms of the residue current R associated with (E,φ).

Theorem 6.1. *Assume that (E,φ) is a complex of Hermitian vector bundles of the form (2.5) that is pointwise exact outside a subvariety Z of codimension p, and let R be the corresponding residue current. Moreover, assume that $D = (D_k)$ is a $(1,0)$-connection on (E,φ) and let $c^{\text{Res}}(E,D)$ be the corresponding Chern current. Then*

$$c_p^{\text{Res}}(E,D) = \frac{(-1)^{p-1}}{(2\pi i)^p p} \sum_{k=0}^{N-p} (-1)^k \operatorname{tr}(D\varphi_{k+1} \cdots D\varphi_{k+p} R_{k+p}^k). \tag{6.1}$$

Moreover

$$c_\ell^{\text{Res}}(E,D) = 0 \quad \textit{for } 0 < \ell < p \tag{6.2}$$

and

$$c_{\ell_1}^{\text{Res}}(E,D) \wedge \cdots \wedge c_{\ell_m}^{\text{Res}}(E,D) = 0 \quad \textit{for } m \geq 2 \textit{ and } 0 < \ell_1 + \cdots + \ell_m \leq p. \tag{6.3}$$

In fact, Theorem 6.1 follows from the following formulation in terms of the Chern character (forms). For (E,φ) and D as in the theorem and for $\ell_1, \dots, \ell_m \geq 1$ we let

$$\operatorname{ch}_{\ell_1}^{\text{Res}}(E,D) \wedge \cdots \wedge \operatorname{ch}_{\ell_m}^{\text{Res}}(E,D) := \lim_{\epsilon \to 0} \operatorname{ch}_{\ell_1}(E,\widehat{D}^\epsilon) \wedge \cdots \wedge \operatorname{ch}_{\ell_m}(E,\widehat{D}^\epsilon), \tag{6.4}$$

where $\widehat{D}^\epsilon$ is the connection defined by (4.3). By Theorem 5.1 this is a well-defined current with support on Z that represents $\operatorname{ch}_{\ell_1}(E) \wedge \cdots \wedge \operatorname{ch}_{\ell_m}(E)$.

Theorem 6.2. *Assume that* (E,φ), D, R, *and* p *are as in Theorem 6.1. For* $\ell \geq 1$, *let* $\mathrm{ch}^{\mathrm{Res}}_\ell(E,D)$ *be the corresponding Chern character current* (6.4). *Then*

$$\mathrm{ch}^{\mathrm{Res}}_p(E,D) = \frac{1}{(2\pi i)^p p!}\sum_{k=0}^{N-p}(-1)^k \operatorname{tr}(D\varphi_{k+1}\cdots D\varphi_{k+p}R^k_{k+p}). \tag{6.5}$$

Moreover

$$\mathrm{ch}^{\mathrm{Res}}_\ell(E,D) = 0 \quad \text{for } \ell < p \tag{6.6}$$

and

$$\mathrm{ch}^{\mathrm{Res}}_{\ell_1}(E,D)\wedge\cdots\wedge\mathrm{ch}^{\mathrm{Res}}_{\ell_m}(E,D) = 0 \quad \text{for } m \geq 2 \text{ and } \ell_1+\cdots+\ell_m \leq p. \tag{6.7}$$

Proof of Theorem 6.1. Since $\widehat{Q}_\ell$ in (3.7) is a polynomial of weighted degree ℓ we get that

$$\widehat{Q}_\ell\Big(\mathrm{ch}^{\mathrm{Res}}_1(E,D),\dots,\mathrm{ch}^{\mathrm{Res}}_{\ell-1}(E,D)\Big) \tag{6.8}$$

is a sum of terms

$$\mathrm{ch}^{\mathrm{Res}}_{\lambda_1}(E,D)\wedge\cdots\wedge\mathrm{ch}^{\mathrm{Res}}_{\lambda_s}(E,D), \text{ where } s\geq 2 \text{ and } \lambda_1+\cdots+\lambda_s=\ell. \tag{6.9}$$

Thus (6.8) vanishes by (6.7) for $\ell \leq p$. Now (6.1) and (6.2) follow by (3.7), (6.5), and (6.6). Also, the left hand side of (6.3) is a sum of terms of the form (6.9) and thus it vanishes. □

Remark 6.3. Taking Theorem 6.1 for granted, by similar arguments as in the proof above, using (3.6), we get Theorem 6.2. Thus Theorems 6.1 and 6.2 are equivalent.

Recall from Section 2.3 that if (E,φ) is a locally free resolution of a sheaf $\mathcal{F}$ of codimension p, then $R^\ell_k = 0$ for $\ell > 0$. Thus the only nonvanishing residue current in (6.1) is $R_p = R^0_p$, and hence Theorem 1.5 follows. It may be noted that our proof of Theorem 6.1 does not become simpler in the situation of Theorem 1.5.

To organize the proof of Theorem 6.2 we will introduce a certain class $\mathcal{O}_{Z,\ell,\epsilon}$ of forms depending on $\epsilon > 0$ that in the limit are pseudomeromorphic currents with support on Z that vanish if $\ell \leq \operatorname{codim} Z$. Throughout this section, let (E,φ) be fixed as the complex from Theorems 6.1 and 6.2 and let σ_k be the minimal inverse of φ_k as in Section 2.3. Let $\mathcal{E}_{q,\epsilon}$ denote smooth forms of bidegree $(*,q)$ that can be written as polynomials in χ_ϵ, $\bar\partial\chi_\epsilon$, entries of σ_k, $\partial\sigma_k$ or $\bar\partial\sigma_k$ in some local trivialization for $k=1,\dots,N$, and smooth forms independent of ϵ. Here $\chi_\epsilon = \chi(|F|^2/\epsilon)$, where χ is a smooth approximand of $\chi_{[1,\infty)}$ and F is a generically non-vanishing section of a holomorphic vector bundle such that $Z=\{F=0\}$. We say that $\psi_\epsilon \in \mathcal{E}_\epsilon := \oplus\mathcal{E}_{q,\epsilon}$ is in $\mathcal{O}_{Z,\ell,\epsilon}$ if ψ_ϵ is a sum of terms of the form $a\wedge b_\epsilon$, where a is a smooth form that is independent of ϵ, and b_ϵ is in $\mathcal{E}_{q,\epsilon}$, where $q<\ell$, and vanishes where $\chi_\epsilon \equiv 1$. In particular, if $\psi_\epsilon \in \mathcal{E}_{q,\epsilon}$ vanishes where $\chi_\epsilon\equiv 1$, then $\psi_\epsilon \in \mathcal{O}_{Z,\ell,\epsilon}$ for any $\ell > q$. Note that

$$\mathcal{E}_{q,\epsilon}\wedge\mathcal{O}_{Z,\ell,\epsilon}\subseteq\mathcal{O}_{Z,\ell+q,\epsilon}. \tag{6.10}$$

Lemma 6.4. *Assume that* Z *has codimension* p, *and let* ψ_ϵ *be a form in* $\mathcal{O}_{Z,\ell,\epsilon}$ *with* $\ell \leq p$. *Then* $\lim_{\epsilon\to 0}\psi_\epsilon = 0$.

Proof. Consider a term $a\wedge b_\epsilon$ of ψ_ϵ as above. Then $b_\epsilon \in \mathcal{E}_{q,\epsilon}$, where $q<\ell\leq p$. By Lemma 2.1, the limit $b := \lim_{\epsilon\to 0} b_\epsilon$ exists and is a pseudomeromorphic current of bidegree $(*,q)$. Since $b_\epsilon \equiv 0$ where $\chi_\epsilon \equiv 1$, b has support on Z and thus $b=0$ by the dimension principle, see Section 2.1. Since a is smooth and independent of ϵ, it follows that $\lim_{\epsilon\to 0}(a\wedge b_\epsilon) = a\wedge b = 0$. □

Throughout this section, let $D=(D_k)$ be a $(1,0)$-connection on (E,φ), let $\chi \sim \chi_{[1,\infty)}$, let χ_ϵ and $\widehat{D}^\epsilon$ be defined as in Section 4, and let $c(E,\widehat{D}^\epsilon) = \sum c_\ell(E,\widehat{D}^\epsilon)$ be the corresponding Chern form defined by (3.1). Since the limits in Theorem 5.1 are independent of the choice of χ_ϵ, and the results in this section are local statements, we may assume locally that the section F in the definition of $\chi_\epsilon = \chi(|F|^2/\epsilon)$ is such that $\{F=0\}=Z$.

Lemma 6.5. *For* $\ell \geq 1$ *and* $\epsilon > 0$, *we have*

$$\operatorname{ch}_\ell(E, \widehat{D}^\epsilon) = \frac{1}{(2\pi i)^\ell \ell!} \bar\partial \chi_\epsilon^\ell \wedge \sum_{k=1}^N (-1)^k \operatorname{tr}\Big(\sigma_k D\varphi_k (\bar\partial \sigma_k D\varphi_k)^{\ell-1}\Big) + O_{Z,\ell,\epsilon}. \tag{6.11}$$

Proof. We may work in a local trivialization; let $\widehat{\Theta}_k^\epsilon$ be the curvature matrix of $\widehat{D}_k^\epsilon$. By Remark 5.2, since the D_k are $(1,0)$-connections, $\operatorname{ch}_\ell(E, \widehat{D}^\epsilon)$ consists of components of bidegree $(\ell+q, \ell-q)$ with $q \geq 0$. From the proof of Theorem 5.1 it follows that $c(E, \widehat{D}^\epsilon)$ is in $\mathcal{E}_\epsilon$ and vanishes where $\chi_\epsilon \equiv 1$, and consequently, the same holds for $\operatorname{ch}(E, \widehat{D}^\epsilon)$. It follows that $\operatorname{ch}_{(\ell+q,\ell-q)}(E, \widehat{D}^\epsilon) \in O_{Z,\ell,\epsilon}$ for $q > 0$, so

$$\operatorname{ch}_\ell(E, \widehat{D}^\epsilon) = \operatorname{ch}_{(\ell,\ell)}(E, \widehat{D}^\epsilon) + O_{Z,\ell,\epsilon}. \tag{6.12}$$

Since $\widehat{D}^\epsilon$ is a $(1,0)$-connection, by (3.15),

$$\operatorname{ch}_{(\ell,\ell)}(E, \widehat{D}^\epsilon) = \frac{i^\ell}{(2\pi)^\ell \ell!} p_{(\ell,\ell)}(E, \widehat{D}^\epsilon), \tag{6.13}$$

where $p_{(\ell,\ell)}$ is given by (3.14). To prove the lemma it thus suffices to show that

$$p_{(\ell,\ell)}(E, \widehat{D}^\epsilon) = (-1)^\ell \bar\partial \chi_\epsilon^\ell \wedge \sum_{k=1}^N (-1)^k \operatorname{tr}\Big(\sigma_k D\varphi_k (\bar\partial \sigma_k D\varphi_k)^{\ell-1}\Big) + O_{Z,\ell,\epsilon}. \tag{6.14}$$

To prove (6.14), first note in view of (4.4) that since D is a $(1,0)$-connection,

$$(\widehat{\Theta}_k^\epsilon)_{(1,1)} = -\bar\partial(\chi_\epsilon \sigma_k D\varphi_k) + (\Theta_k)_{(1,1)}, \tag{6.15}$$

where Θ_k is the curvature matrix of D_k. We make the following decomposition:

$$\begin{aligned} -\bar\partial(\chi_\epsilon \sigma_k D\varphi_k) + (\Theta_k)_{(1,1)} &= -\bar\partial \chi_\epsilon \wedge \sigma_k D\varphi_k - \chi_\epsilon\Big(\bar\partial(\sigma_k D\varphi_k) - (\Theta_k)_{(1,1)}\Big) + (1-\chi_\epsilon)(\Theta_k)_{(1,1)} \\ &=: \alpha_k + \beta_k + \gamma_k. \end{aligned} \tag{6.16}$$

Let us consider

$$\operatorname{tr}(\widehat{\Theta}_k^\epsilon)_{(1,1)}^\ell = \operatorname{tr}(\alpha_k + \beta_k + \gamma_k)^\ell$$

and expand the product. Note that $\gamma_k \in O_{Z,1,\epsilon}$, and thus by (6.10) all terms with a factor γ_k are in $O_{Z,\ell,\epsilon}$. Next, note that since $(\bar\partial \chi_\epsilon)^2 = 0$, $\alpha_k^2 = 0$, and since α_k and β_k have total degree 4 and endomorphism degree 2, all terms containing one α_k and the remaining $\ell - 1$ factors being β_k are all equal to $\operatorname{tr}(\alpha_k \wedge \beta_k^{\ell-1})$ by (2.7). To conclude,

$$\operatorname{tr}(\widehat{\Theta}_k^\epsilon)_{(1,1)}^\ell = \ell \operatorname{tr}(\alpha_k \wedge \beta_k^{\ell-1}) + \operatorname{tr} \beta_k^\ell + O_{Z,\ell,\epsilon}. \tag{6.17}$$

We have that

$$\begin{aligned} \ell \operatorname{tr}(\alpha_k \wedge \beta_k^{\ell-1}) &= (-1)^\ell \bar\partial \chi_\epsilon^\ell \wedge \operatorname{tr}\Big(\sigma_k D\varphi_k \Big(\bar\partial(\sigma_k D\varphi_k) - (\Theta_k)_{(1,1)}\Big)^{\ell-1}\Big) \\ &= (-1)^\ell \bar\partial \chi_\epsilon^\ell \wedge \operatorname{tr}\Big(\sigma_k D\varphi_k \Big(\bar\partial(\sigma_k D\varphi_k)\Big)^{\ell-1}\Big) + O_{Z,\ell,\epsilon}, \end{aligned}$$

since $\ell \chi_\epsilon^{\ell-1} \bar\partial \chi_\epsilon = \bar\partial \chi_\epsilon^\ell$, *cf.* (2.1), and in the middle expression all terms having a factor $(\Theta_k)_{(1,1)}$ also contain a factor $\bar\partial \chi_\epsilon^\ell$, and thus are in $O_{Z,\ell,\epsilon}$. Moreover, by (2.8) and (2.12),

$$\bar\partial(\sigma_k D\varphi_k) = \bar\partial \sigma_k D\varphi_k - \sigma_k \bar\partial(D\varphi_k) = \bar\partial \sigma_k D\varphi_k - \sigma_k (\Theta_{k-1})_{(1,1)} \varphi_k + \sigma_k \varphi_k (\Theta_k)_{(1,1)},$$

and hence

$$\ell \operatorname{tr}(\alpha_k \wedge \beta_k^{\ell-1}) = (-1)^\ell \bar\partial \chi_\epsilon^\ell \wedge \operatorname{tr}\Big(\sigma_k D\varphi_k \Big(\bar\partial \sigma_k D\varphi_k\Big)^{\ell-1}\Big) + O_{Z,\ell,\epsilon}, \tag{6.18}$$

since all terms containing a factor $(\Theta_k)_{(1,1)}$ or $(\Theta_{k-1})_{(1,1)}$ also contain a factor $\bar\partial\chi_\epsilon^\ell$. Note that $\alpha_0 = 0$ since $\varphi_0 = 0$. It thus follows from (6.17) and (6.18) that

$$p_{(\ell,\ell)}(E,\widehat{D}^\epsilon) = \sum_{k=0}^{N}(-1)^k \operatorname{tr}(\widehat{\Theta}_k^\epsilon)^\ell_{(1,1)}$$
$$= (-1)^\ell\bar\partial\chi_\epsilon^\ell \wedge \sum_{k=1}^{N}(-1)^k \operatorname{tr}\Big(\sigma_k D\varphi_k\big(\bar\partial\sigma_k D\varphi_k\big)^{\ell-1}\Big) + \sum_{k=0}^{N}(-1)^k \operatorname{tr}\beta_k^\ell + O_{Z,\ell,\epsilon}.$$

Thus, to prove (6.14) it suffices to show that $\sum_{k=0}^N(-1)^k\operatorname{tr}\beta_k^\ell$ vanishes for $\ell\geq 1$. Outside Z, let $\widetilde{D}$ be the connection on (E,φ) defined by

$$\widetilde{D}_k := -\sigma_k D\varphi_k + D_k \tag{6.19}$$

and let $\tilde c = c(E,\widetilde{D})$ be the corresponding Chern form defined by (3.1). If follows from Lemma 4.4 that $\widetilde{D}$ is compatible with (E,φ) and thus by Lemma 4.3, $\tilde c_j$ vanishes for $j\geq 1$. For $\ell\geq 1$, let $\tilde p_\ell := p_{(\ell,\ell)}(E,\widetilde{D})$, where $p_{(\ell,\ell)}$ is given by (3.14). By (3.15) and (3.6), $\tilde p_\ell$ is a polynomial in $\tilde c_{(1,1)},\dots,\tilde c_{(\ell,\ell)}$. Since $\tilde c_{(j,j)}$ vanishes for any $j\geq 1$, $\tilde p_\ell = 0$. Note that $\beta_k = \chi_\epsilon(\widetilde{\Theta}_k)_{(1,1)}$, where $\widetilde{\Theta}_k$ is the curvature matrix corresponding to $\widetilde{D}_k$. Thus

$$\sum_{k=0}^{N}(-1)^k\operatorname{tr}\beta_k^\ell = \chi_\epsilon^\ell\sum_{k=0}^{N}(-1)^k\operatorname{tr}(\widetilde{\Theta}_k)^\ell_{(1,1)} = \chi_\epsilon^\ell\tilde p_\ell = 0$$

for $\epsilon > 0$. This concludes the proof of (6.14). □

Lemma 6.6. *For $\ell\geq 1$ and $\epsilon>0$, we have*

$$\bar\partial\chi_\epsilon\wedge\sum_{k=1}^{N}(-1)^k\operatorname{tr}\Big(\sigma_k D\varphi_k(\bar\partial\sigma_k D\varphi_k)^{\ell-1}\Big) =$$
$$(-1)^\ell\sum_{k=0}^{N-\ell}(-1)^k\bar\partial\chi_\epsilon\wedge\operatorname{tr}\Big(\sigma_{k+\ell}\bar\partial\sigma_{k+\ell-1}\cdots\bar\partial\sigma_{k+1}D\varphi_{k+1}\cdots D\varphi_{k+\ell}\Big) + O_{Z,\ell,\epsilon} + \bar\partial O_{Z,\ell,\epsilon}. \tag{6.20}$$

If $\ell > N$, the sum on the right hand side should be interpreted as 0.

Here $\bar\partial O_{Z,\ell,\epsilon}$ means forms of the form $\bar\partial\psi_\epsilon$, where $\psi_\epsilon\in O_{Z,\ell,\epsilon}$.

Proof. For $\ell = 1$ the sums differ only by a shift in the indices, so we may assume $\ell\geq 2$. For fixed $k\in\mathbb{Z}$ and $m,r,s\geq 0$, let

$$\rho_{k,m}^{r,s} = \bar\partial\chi_\epsilon\wedge\operatorname{tr}\Big(\sigma_{k+m+1}\bar\partial\sigma_{k+m}\cdots\bar\partial\sigma_{k+1}(\bar\partial\sigma_k D\varphi_k)^r D\bar\partial\sigma_k(D\varphi_k\bar\partial\sigma_k)^s D\varphi_k\cdots D\varphi_{k+m}\varphi_{k+m+1}\Big).$$

If $m = 0$, then the factor $\bar\partial\sigma_{k+m}\cdots\bar\partial\sigma_{k+1}$ should be interpreted as 1. Moreover since (E,φ) starts at level 0 and ends at level N, we interpret φ_j and σ_j as 0 if $j>N$ or $j<1$, and consequently we interpret $\rho_{k,m}^{r,s}$ as 0 if $k+m\geq N$ or $k\leq 0$.

We claim that

$$\bar\partial\chi_\epsilon\wedge\operatorname{tr}\Big(\sigma_{k+m}\bar\partial\sigma_{k+m-1}\cdots\bar\partial\sigma_k(\bar\partial\sigma_{k-1}D\varphi_{k-1})^r(D\varphi_k\bar\partial\sigma_k)^{s+1}D\varphi_k\cdots D\varphi_{k+m}\Big) =$$
$$\bar\partial\chi_\epsilon\wedge\operatorname{tr}\Big(\sigma_{k+m}\bar\partial\sigma_{k+m-1}\cdots\bar\partial\sigma_k(\bar\partial\sigma_{k-1}D\varphi_{k-1})^{r+1}(D\varphi_k\bar\partial\sigma_k)^s D\varphi_k\cdots D\varphi_{k+m}\Big)$$
$$+\rho_{k-1,m}^{r,s} + \rho_{k,m}^{r,s} + O_{Z,r+s+m+2,\epsilon} + \bar\partial O_{Z,r+s+m+2,\epsilon}. \tag{6.21}$$

Let us take (6.21) for granted and let $\rho^t_{k,m} = \sum_{r=0}^t \rho^{r,t-r}_{k,m}$. Then by inductively applying (6.21) to $r = 0, \dots, t$ with $s = t - r$, we get

$$(6.22)\quad \bar\partial\chi_\epsilon \wedge \operatorname{tr}\Big(\sigma_{k+m}\bar\partial\sigma_{k+m-1}\cdots\bar\partial\sigma_k(D\varphi_k\bar\partial\sigma_k)^{t+1}D\varphi_k\cdots D\varphi_{k+m}\Big) = \bar\partial\chi_\epsilon \wedge \operatorname{tr}\Big(\sigma_{k+m}\bar\partial\sigma_{k+m-1}\cdots\bar\partial\sigma_{k-1}(D\varphi_{k-1}\bar\partial\sigma_{k-1})^{t}D\varphi_{k-1}\cdots D\varphi_{k+m}\Big) + \rho^t_{k-1,m} + \rho^t_{k,m} + O_{Z,t+m+2,\epsilon} + \bar\partial O_{Z,t+m+2,\epsilon}.$$

It follows that, for fixed m and t,

$$\sum_{k=1}^{N-m}(-1)^k\bar\partial\chi_\epsilon \wedge \operatorname{tr}\Big(\sigma_{k+m}\bar\partial\sigma_{k+m-1}\cdots\bar\partial\sigma_k(D\varphi_k\bar\partial\sigma_k)^{t+1}D\varphi_k\cdots D\varphi_{k+m}\Big) = \sum_{k=1}^{N-m}(-1)^k\bar\partial\chi_\epsilon \wedge \operatorname{tr}\Big(\sigma_{k+m}\bar\partial\sigma_{k+m-1}\cdots\bar\partial\sigma_{k-1}(D\varphi_{k-1}\bar\partial\sigma_{k-1})^{t}D\varphi_{k-1}\cdots D\varphi_{k+m}\Big) - \rho^t_{0,m} + (-1)^{N-m}\rho^t_{N-m,m} + O_{Z,t+m+2,\epsilon} + \bar\partial O_{Z,t+m+2,\epsilon}.$$

Thus, since $\rho^{r,s}_{0,m}$ and $\rho^{r,s}_{N-m,m}$ vanish,

$$(6.23)\quad \sum_{k=1}^{N-m}(-1)^k\bar\partial\chi_\epsilon \wedge \operatorname{tr}\Big(\sigma_{k+m}\bar\partial\sigma_{k+m-1}\cdots\bar\partial\sigma_k(D\varphi_k\bar\partial\sigma_k)^{t+1}D\varphi_k\cdots D\varphi_{k+m}\Big) = -\sum_{k=1}^{N-m-1}(-1)^k\bar\partial\chi_\epsilon \wedge \operatorname{tr}\Big(\sigma_{k+m+1}\bar\partial\sigma_{k+m}\cdots\bar\partial\sigma_k(D\varphi_k\bar\partial\sigma_k)^{t}D\varphi_k\cdots D\varphi_{k+m+1}\Big) + O_{Z,t+m+2,\epsilon} + \bar\partial O_{Z,t+m+2,\epsilon}.$$

Assume that $2 \le \ell \le N$. By inductively applying (6.23) to $m = 0, \dots, \ell-2$ with $t = \ell - 2 - m$, we get

$$\begin{aligned}
&\sum_{k=1}^{N}(-1)^k\bar\partial\chi_\epsilon \wedge \operatorname{tr}\Big(\sigma_k(D\varphi_k\bar\partial\sigma_k)^{\ell-1}D\varphi_k\Big)\\
&\quad= -\sum_{k=1}^{N-1}(-1)^k\bar\partial\chi_\epsilon \wedge \operatorname{tr}\Big(\sigma_{k+1}\bar\partial\sigma_k(D\varphi_k\bar\partial\sigma_k)^{\ell-2}D\varphi_k D\varphi_{k+1}\Big) + O_{Z,\ell,\epsilon} + \bar\partial O_{Z,\ell,\epsilon}\\
&\quad= \cdots\\
&\quad= (-1)^{\ell-1}\sum_{k=1}^{N-\ell+1}(-1)^k\bar\partial\chi_\epsilon \wedge \operatorname{tr}\Big(\sigma_{k+\ell-1}\bar\partial\sigma_{k+\ell-2}\cdots\bar\partial\sigma_k D\varphi_k\cdots D\varphi_{k+\ell-1}\Big) + O_{Z,\ell,\epsilon} + \bar\partial O_{Z,\ell,\epsilon},
\end{aligned}$$

which after a shift in indices is exactly (6.20). If $\ell > N$ and we perform the same induction, after $N-1$ steps we end up with

$$(-1)^N\bar\partial\chi_\epsilon \wedge \operatorname{tr}\Big(\sigma_N\bar\partial\sigma_{N-1}\cdots\bar\partial\sigma_1(D\varphi_1\bar\partial\sigma_1)^{\ell-N}D\varphi_1\cdots D\varphi_N\Big) + O_{Z,\ell,\epsilon} + \bar\partial O_{Z,\ell,\epsilon},$$

which by (6.22) equals $\rho^{\ell-N-1}_{0,N-1} + \rho^{\ell-N-1}_{1,N-1} + O_{Z,\ell,\epsilon} = O_{Z,\ell,\epsilon}$; thus (6.20) holds also in this case.

It remains to prove (6.21). To do this let us replace the first factor $D\varphi_k\bar\partial\sigma_k$ in the left hand side of (6.21) by the right hand side of (2.17); we then get three terms. The term corresponding to the second term $\bar\partial\sigma_{k-1}D\varphi_{k-1}$ in (2.17) is precisely the first term in the right hand side of (6.21). Next, by (2.13) and (2.16),

$$\varphi_{k-1}D\varphi_k\bar\partial\sigma_k = D\varphi_{k-1}\varphi_k\bar\partial\sigma_k = D\varphi_{k-1}\bar\partial\sigma_{k-1}\varphi_{k-1}.$$

Applying this repeatedly we get

$$\varphi_{k-1}(D\varphi_k\bar\partial\sigma_k)^s = (D\varphi_{k-1}\bar\partial\sigma_{k-1})^s\varphi_{k-1}.$$

Using this and (2.13) (to "move" the D), we get

$$\sigma_{k+m}\bar\partial\sigma_{k+m-1}\cdots\bar\partial\sigma_k(\bar\partial\sigma_{k-1}D\varphi_{k-1})^r D\bar\partial\sigma_{k-1}\varphi_{k-1}(D\varphi_k\bar\partial\sigma_k)^s D\varphi_k\cdots D\varphi_{k+m} = \sigma_{k+m}\bar\partial\sigma_{k+m-1}\cdots\bar\partial\sigma_k(\bar\partial\sigma_{k-1}D\varphi_{k-1})^r D\bar\partial\sigma_{k-1}(D\varphi_{k-1}\bar\partial\sigma_{k-1})^s D\varphi_{k-1}\cdots D\varphi_{k+m-1}\varphi_{k+m}.$$

It follows that the term corresponding to the first term in (2.17) equals $\rho^{r,s}_{k-1,m}$.

Finally we consider the term corresponding to the last term in (2.17). As above, using (2.13) and (2.16), we get that

$$(\bar\partial\sigma_{k-1}D\varphi_{k-1})^r\varphi_k = \varphi_k(\bar\partial\sigma_k D\varphi_k)^r$$

and thus, using this and (2.14) (to "move" the $\bar\partial$),

$$\sigma_{k+m}\bar\partial\sigma_{k+m-1}\cdots\bar\partial\sigma_k(\bar\partial\sigma_{k-1}D\varphi_{k-1})^r\varphi_k D\bar\partial\sigma_k(D\varphi_k\bar\partial\sigma_k)^s D\varphi_k\cdots D\varphi_{k+m} = \bar\partial\sigma_{k+m}\cdots\bar\partial\sigma_{k+1}\sigma_k\varphi_k(\bar\partial\sigma_k D\varphi_k)^r D\bar\partial\sigma_k(D\varphi_k\bar\partial\sigma_k)^s D\varphi_k\cdots D\varphi_{k+m}. \tag{6.24}$$

In view of (2.15) we can replace the factor $\sigma_k\varphi_k$ by $\mathrm{Id}_{E_k} - \varphi_{k+1}\sigma_{k+1}$:

$$\begin{aligned}\bar\partial\sigma_{k+m}\cdots\bar\partial\sigma_{k+1}\sigma_k\varphi_k(\bar\partial\sigma_k D\varphi_k)^r D\bar\partial\sigma_k(D\varphi_k\bar\partial\sigma_k)^s D\varphi_k\cdots D\varphi_{k+m} = \\ \bar\partial\sigma_{k+m}\cdots\bar\partial\sigma_{k+1}(\bar\partial\sigma_k D\varphi_k)^r D\bar\partial\sigma_k(D\varphi_k\bar\partial\sigma_k)^s D\varphi_k\cdots D\varphi_{k+m} + \\ (-\bar\partial\sigma_{k+m}\cdots\bar\partial\sigma_{k+1}\varphi_{k+1}\sigma_{k+1}(\bar\partial\sigma_k D\varphi_k)^r D\bar\partial\sigma_k(D\varphi_k\bar\partial\sigma_k)^s D\varphi_k\cdots D\varphi_{k+m}) =: \xi + \delta.\end{aligned} \tag{6.25}$$

By repeatedly using (2.16) we get that

$$\bar\partial\sigma_{k+m}\cdots\bar\partial\sigma_{k+1}\varphi_{k+1}\sigma_{k+1} = \bar\partial\sigma_{k+m}\cdots\bar\partial\sigma_{k+2}\varphi_{k+2}\bar\partial\sigma_{k+2}\sigma_{k+1} = \cdots = \varphi_{k+m+1}\bar\partial\sigma_{k+m+1}\cdots\bar\partial\sigma_{k+2}\sigma_{k+1}.$$

It follows, using (2.14), that δ in (6.25) equals

$$\delta = -\varphi_{k+m+1}\sigma_{k+m+1}\bar\partial\sigma_{k+m}\cdots\bar\partial\sigma_{k+1}(\bar\partial\sigma_k D\varphi_k)^r D\bar\partial\sigma_k(D\varphi_k\bar\partial\sigma_k)^s D\varphi_k\cdots D\varphi_{k+m} =: -\varphi_{k+m+1}\beta.$$

Note that $\deg\beta = 4\kappa+2$ and $\deg_e\beta = 2\kappa+1$, where $\kappa = m+r+s+1$, and since

$$\deg\varphi_{k+m+1} = \deg_e\varphi_{k+m+1} = 1,$$

we get, in view of (2.7), that $\mathrm{tr}(\varphi_{k+m+1}\beta) = -\mathrm{tr}(\beta\varphi_{k+m+1})$ and it follows that this terms equals $\rho^{r,s}_{k,m}$.

It remains to consider ξ in (6.25). Let

$$\eta := \sigma_{k+m}\bar\partial\sigma_{k+m-1}\cdots\bar\partial\sigma_{k+1}(\bar\partial\sigma_k D\varphi_k)^r D\bar\partial\sigma_k(D\varphi_k\bar\partial\sigma_k)^s D\varphi_k\cdots D\varphi_{k+m}$$

Then, by (2.8), $\bar\partial\eta = \xi + \xi'$, where ξ' consists of a sum of terms with a factor $\bar\partial D\varphi_j$ or $\bar\partial D\bar\partial\sigma_j$. Let $q = m+r+s+2$. Then, note that $\bar\partial\chi_\epsilon\wedge\eta$ is in $O_{Z,q,\epsilon}$, and thus $\bar\partial\chi_\epsilon\wedge\bar\partial\eta = -\bar\partial(\bar\partial\chi_\epsilon\wedge\eta)\in\bar\partial O_{Z,q,\epsilon}$. Moreover, by (2.12), each term in ξ' has a factor that is a smooth $(1,1)$-form. Therefore $\bar\partial\chi_\epsilon\wedge\xi'\in O_{Z,q,\epsilon}$, and hence $\mathrm{tr}\,\bar\partial\chi_\epsilon\wedge\xi = -\mathrm{tr}(\bar\partial\chi_\epsilon\wedge\xi') + \mathrm{tr}(\bar\partial\chi_\epsilon\wedge\bar\partial\eta)\in O_{Z,q,\epsilon} + \bar\partial O_{Z,q,\epsilon}$. This concludes the proof of (6.21). □

Proof of Theorem 6.2. We first prove (6.5). Since Z has codimension p and $\chi^p\sim\chi_{[1,\infty)}$, by Lemmas 6.4, 6.5, and 6.6, and by (2.18), we have

$$\begin{aligned}\mathrm{ch}^{\mathrm{Res}}_p(E,D) &= \lim_{\epsilon\to 0}\mathrm{ch}_p(E,\widehat{D}^\epsilon) \\ &= \frac{1}{(2\pi i)^p p!}\lim_{\epsilon\to 0}\bar\partial\chi^p_\epsilon\wedge\sum_{k=1}^N(-1)^k\,\mathrm{tr}\Big(\sigma_k D\varphi_k(\bar\partial\sigma_k D\varphi_k)^{p-1}\Big) \\ &= \frac{(-1)^p}{(2\pi i)^p p!}\sum_{k=0}^{N-p}(-1)^k\lim_{\epsilon\to 0}\bar\partial\chi^p_\epsilon\wedge\mathrm{tr}\Big(\sigma_{k+p}\bar\partial\sigma_{k+p-1}\cdots\bar\partial\sigma_{k+1}D\varphi_{k+1}\cdots D\varphi_{k+p}\Big) \\ &= \frac{(-1)^p}{(2\pi i)^p p!}\sum_{k=0}^{N-p}(-1)^k\,\mathrm{tr}\Big(R^k_{k+p}D\varphi_{k+1}\cdots D\varphi_{k+p}\Big).\end{aligned}$$

Since $D\varphi_{k+1}\cdots D\varphi_{k+p}$ and R^k_{k+p} both have total degree $2p$ and endomorphism degree p, it follows from (2.7) that

$$\operatorname{tr}\Big(R^k_{k+p} D\varphi_{k+1}\cdots D\varphi_{k+p}\Big) = (-1)^p \operatorname{tr}\Big(D\varphi_{k+1}\cdots D\varphi_{k+p} R^k_{k+p}\Big),$$

and thus (6.5) follows.

Next, by Theorem 5.1 and Remark 5.2, $\mathrm{ch}^{\mathrm{Res}}_\ell(E,D)$ is a pseudomeromorphic current with support on Z and with components of bidegree $(\ell+q,\ell-q)$ where $q\geq 0$. Therefore it vanishes by the dimension principle when $\ell<p$, see Section 2.1. This proves (6.6).

It remains to prove (6.7). From the beginning of the proof of Lemma 6.5 and (6.10) it follows that

$$\mathrm{ch}_{\ell_1}(E,\widehat{D}^\epsilon)\wedge\cdots\wedge\mathrm{ch}_{\ell_m}(E,\widehat{D}^\epsilon) = C p_{(\ell_1,\ell_1)}(E,\widehat{D}^\epsilon)\wedge\cdots\wedge p_{(\ell_m,\ell_m)}(E,\widehat{D}^\epsilon) + O_{Z,\ell_1+\cdots+\ell_m,\epsilon}, \tag{6.26}$$

for some appropriate constant C. By (6.14), the fact that $(\bar\partial\chi_\epsilon)^2=0$, and (6.10), it follows that

$$p_{(\ell_1,\ell_1)}(E,\widehat{D}^\epsilon)\wedge\cdots\wedge p_{(\ell_m,\ell_m)}(E,\widehat{D}^\epsilon)\in O_{Z,p,\epsilon}$$

if $m\geq 2$ and $\ell_1+\cdots+\ell_m\leq p$. Thus the limit of (6.26) vanishes in this case, which proves (6.7). □

Assume that (E,φ) is a complex of Hermitian vector bundles of the form (2.5) such that $\mathcal{H}_k(E)$ has pure codimension p or vanishes for $k=0,\dots,N$, and let $Z=\cup\operatorname{supp}\mathcal{H}_k(E)$. Then (E,φ) is pointwise exact outside Z, which has codimension p. Now, by combining Theorem 2.2 and Theorem 6.1, we obtain the following generalization of Theorem 1.4.

Corollary 6.7. *Assume that (E,φ) is a complex of Hermitian vector bundles of the form (2.5) such that $\mathcal{H}_k(E)$ has pure codimension p or vanishes for $k=0,\dots,N$. Moreover, assume that $D=(D_k)$ is a $(1,0)$-connection on (E,φ). Then*

$$c_p^{\mathrm{Res}}(E,D) = (-1)^{p-1}(p-1)![E].$$

Moreover (1.10) and (1.11) hold.

Here $[E]$ is the cycle of (E,φ) defined by (2.19). In particular, it follows from Corollary 6.7 that $c_p^{\mathrm{Res}}(E,D)$ is independent of the choice of Hermitian metric and $(1,0)$-connection D on (E,φ).

By equipping a complex of vector bundles (E,φ) with Hermitian metrics and $(1,0)$-connections and taking cohomology we get the following generalization of (1.13).

Corollary 6.8. *Assume that (E,φ) is a complex of vector bundles of the form (2.5) such that $\mathcal{H}_k(E)$ has pure codimension p or vanishes for $k=0,\dots,N$. Then*

$$\Big[(-1)^{p-1}(p-1)![E]\Big] = c_p(E).$$

7. An example

We will compute (products of) Chern currents $c^{\mathrm{Res}}(E,D)$ for an explicit choice of (E,φ) and D. Let $\mathcal{J}\subseteq\mathcal{O}_{\mathbb{P}^2_{[t,x,y]}}$ be defined by $\mathcal{J}=\mathcal{J}(y^k,x^\ell y^m)$, where $m<k$, and let $\mathcal{O}_Z:=\mathcal{O}_{\mathbb{P}^2}/\mathcal{J}$. Then Z has pure dimension 1, since $Z_{\mathrm{red}}=\{y=0\}$, which is irreducible. However, note that $\mathcal{J}$ has an embedded prime $\mathcal{J}(x,y)$ of dimension 0. Now $\mathcal{F}=\mathcal{O}_Z$ has a locally free resolution of the form

$$0\to\mathcal{O}(-k-\ell)\xrightarrow{\varphi_2}\mathcal{O}(-k)\oplus\mathcal{O}(-\ell-m)\xrightarrow{\varphi_1}\mathcal{O}_{\mathbb{P}^2}\to\mathcal{F}\to 0, \tag{7.1}$$

where in the trivialization in the coordinate chart $\mathbb{C}^2=\mathbb{C}^2_{(x,y)}$

$$\varphi_2=\begin{bmatrix} -x^\ell \\ y^{k-m}\end{bmatrix} \text{ and } \varphi_1=\begin{bmatrix} y^k & x^\ell y^m\end{bmatrix}. \tag{7.2}$$

Let us start by computing the Chern class of $\mathcal{F}$. Let ω denote $c_1(\mathcal{O}(1))$. Then

$$c(E_0) = c(\mathcal{O}_{\mathbb{P}^2}) = 1$$
$$c(E_1) = c\Big(\mathcal{O}(-k) \oplus \mathcal{O}(-\ell - m)\Big) = 1 - (k + \ell + m)\omega + k(\ell + m)\omega^2$$
$$c(E_2) = c\Big(\mathcal{O}(-k - \ell)\Big) = 1 - (k + \ell)\omega,$$

see *e.g.* [Ful98, Chapter 3]. Moreover,

$$c(E_1)^{-1} = 1 - c_1(E_1) + c_1(E_1)^2 - c_2(E_1); \tag{7.3}$$

here the sum ends in degree 2, since we are in dimension 2. Thus, by (1.2),

$$c(\mathcal{F}) = c(E_0)c(E_1)^{-1}c(E_2) = 1 - c_1(E_1) + c_1(E_2) + c_1(E_1)^2 - c_2(E_1) - c_1(E_1)c_1(E_2)$$
$$= 1 + m\omega + \Big(m^2 + \ell(m - k)\Big)\omega^2.$$

In particular,

$$c_1(\mathcal{F}) = m\omega \quad \text{and} \quad c_2(\mathcal{F}) = \Big(m^2 + \ell(m - k)\Big)\omega^2. \tag{7.4}$$

7.1. Chern currents

Assume that each E_k in (7.1) is equipped with the metric induced by the standard metric on $\mathcal{O}(1) \to \mathbb{P}^2$, in turn induced by the standard metric on $\mathbb{C}^3$, and let D_k the corresponding Chern connection. Let $D = (D_k)$, let $\chi \sim \chi_{[1,\infty)}$, and let χ_ϵ and $\widehat{D}^\epsilon$ be as in Section 4. By Theorem 1.4,

$$c_1^{\text{Res}}(E, D) = [\mathcal{F}] = [Z] = m[y = 0],$$

which clearly is a representative of $c_1(\mathcal{F})$, see (7.4), with support on $\operatorname{supp} \mathcal{F} = Z_{\text{red}} = \{y = 0\}$. We want to compute $c_2^{\text{Res}}(E, D)$ and $(c_1^{\text{Res}})^2(E, D)$. Note that these currents are not covered by Theorem 1.5, since $p = 1$ in this case.

Let $\hat{p}_\ell = p_{(\ell,\ell)}(E, \widehat{D}^\epsilon)$, where $p_{(\ell,\ell)}$ is given by (3.14). For degree reasons, $\text{ch}_2(E, \widehat{D}^\epsilon) = \text{ch}_{(2,2)}(E, \widehat{D}^\epsilon)$ and $\text{ch}_1^2(E, \widehat{D}^\epsilon) = \text{ch}_{(1,1)}^2(E, \widehat{D}^\epsilon)$, *cf.* Remark 5.2. It follows in view of (3.8) and (3.15) that

$$c_2(E, \widehat{D}^\epsilon) = \left(\frac{i}{2\pi}\right)^2 \frac{1}{2}(\hat{p}_1^2 - \hat{p}_2) \quad \text{and} \quad c_1^2(E, \widehat{D}^\epsilon) = \left(\frac{i}{2\pi}\right)^2 \hat{p}_1^2. \tag{7.5}$$

Thus, to compute $c_2^{\text{Res}}(E, D)$ and $(c_1^{\text{Res}})^2(E, D)$, it suffices to calculate the limits of $\hat{p}_1^2$ and $\hat{p}_2$ as $\epsilon \to 0$.

Note first that only two of the standard coordinate charts of $\mathbb{P}^2$ intersect Z. In $\mathbb{C}^2_{(t,y)}$, we have that $\varphi_1 = y^m \begin{bmatrix} y^{k-m} & 1 \end{bmatrix}$, so $\sigma_1 = (1/y^m)\sigma_1'$, where σ_1' is smooth. By using (4.4) one can check that the limits of $\hat{p}_1^2$ and $\hat{p}_2$ put no mass at $\{t = y = 0\}$. Thus it is enough to compute the limits in the coordinate chart $\mathbb{C}^2_{(x,y)}$ where φ_j are given by (7.2). Note that $\varphi_1 = y^m\varphi_1'$, where $\varphi_1' = [y^{k-m} \quad x^\ell]$ has rank 1 outside of the origin. Then $\sigma_1 = (1/y^m)\sigma_1'$, where σ_1' is smooth outside the origin, and

$$\sigma_1'\varphi_1'|_{\{y=0\}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{7.6}$$

when $x \neq 0$ and $\varphi_1'\sigma_1' = 1$ outside the origin. Also note that σ_2 is smooth outside the origin, since φ_2 has constant rank there. Let $O'_{Z,\ell,\epsilon}$ be defined as in the beginning of Section 6 but with σ_1 replaced by σ_1', and let $O_\epsilon = O'_{Z,2,\epsilon}$. Then $\psi_\epsilon \in O_\epsilon$ is smooth outside the origin and, by arguments as in the proof of Lemma 6.4, $\lim_{\epsilon\to 0}\psi_\epsilon = 0$.

Next, let $\hat{\omega} = (2\pi/i)\omega$, where ω now denotes the Fubini-Study form. Then

$$\Theta_1 = \begin{bmatrix} -k & 0 \\ 0 & -(\ell + m) \end{bmatrix}\hat{\omega}, \quad \Theta_2 = -(k + \ell)\hat{\omega}. \tag{7.7}$$

In particular, Θ_k is of bidegree $(1,1)$. Let $\widetilde{D}=(\widetilde{D}_k)$ be the connection on $\mathbb{P}^2\setminus Z$ defined by (6.19) and let $\widetilde{\Theta}_k$ be the corresponding curvature forms. Then a computation, *cf.* (6.15) and (6.16), yields

$$(\widehat{\Theta}_k^\epsilon)_{(1,1)} = -\bar\partial\chi_\epsilon\wedge\sigma_k D\varphi_k + \chi_\epsilon(\widetilde{\Theta}_k)_{(1,1)} + (1-\chi_\epsilon)\Theta_k. \tag{7.8}$$

Let us start by computing $\hat{p}_1^2$. Recall from the proof of Lemma 6.5 that $\tilde{p}_j = -\operatorname{tr}(\widetilde{\Theta}_1)^j_{(1,1)} + \operatorname{tr}(\widetilde{\Theta}_2)^j_{(1,1)}$ vanishes where $\chi_\epsilon\not\equiv 0$ for $j=1,2$. Moreover, note in view of (7.7) that $-\operatorname{tr}\Theta_1+\operatorname{tr}\Theta_2 = m\hat\omega$. It follows that

$$\hat{p}_1 = -\operatorname{tr}(\widehat{\Theta}_1^\epsilon)_{(1,1)} + \operatorname{tr}(\widehat{\Theta}_2^\epsilon)_{(1,1)} = \bar\partial\chi_\epsilon\wedge\Big(\operatorname{tr}(\sigma_1 D\varphi_1) - \operatorname{tr}(\sigma_2 D\varphi_2)\Big) + (1-\chi_\epsilon)m\hat\omega.$$

Note that $(1-\chi_\epsilon)\bar\partial\chi_\epsilon = (1/2)\bar\partial\tilde\chi_\epsilon$, where $\tilde\chi = 2(\chi-\chi^2/2)\sim\chi_{[1,\infty)}$. Using this and that $(\bar\partial\chi_\epsilon)^2=0$, we get

$$\begin{aligned}\hat{p}_1^2 &= 2m\hat\omega\wedge(1-\chi_\epsilon)\bar\partial\chi_\epsilon\wedge\Big(\operatorname{tr}(\sigma_1 D\varphi_1)-\operatorname{tr}(\sigma_2 D\varphi_2)\Big) + (1-\chi_\epsilon)^2 m^2\hat\omega^2\\ &= m\hat\omega\wedge\bar\partial\tilde\chi_\epsilon\wedge\operatorname{tr}(\sigma_1 D\varphi_1) + O_\epsilon.\end{aligned}$$

Note that

$$\sigma_1 D\varphi_1 = -\frac{Dy^m}{y^m}\sigma_1'\varphi_1' + \theta_1\sigma_1'\varphi_1' + \sigma_1' D\varphi_1'. \tag{7.9}$$

Therefore, in view of the Poincaré-Lelong formula, *cf.* (1.7) and (7.6), since $\sigma_1'\varphi_1'$ is smooth outside the origin,

$$\bar\partial\tilde\chi_\epsilon\wedge\sigma_1 D\varphi_1 = -\bar\partial\tilde\chi_\epsilon\wedge\frac{Dy^m}{y^m}\sigma_1'\varphi_1' + O_\epsilon \xrightarrow[\epsilon\to 0]{} \frac{2\pi}{i}m[y=0]\begin{bmatrix}0&0\\0&1\end{bmatrix} \tag{7.10}$$

outside the origin. Since the limit is a pseudomeromorphic $(1,1)$-current, (7.10) holds everywhere by the dimension principle. It follows that

$$\lim_{\epsilon\to 0}\hat{p}_1^2 = \lim_{\epsilon\to 0} m\hat\omega\wedge\bar\partial\tilde\chi_\epsilon\wedge\operatorname{tr}(\sigma_1 D\varphi_1) = \frac{2\pi}{i}m^2\hat\omega\wedge[y=0]. \tag{7.11}$$

Let us next consider $\hat{p}_2 = -\operatorname{tr}(\widehat{\Theta}_1)^2_{(1,1)} + \operatorname{tr}(\widehat{\Theta}_2)^2_{(1,1)}$. A computation using (2.7), *cf.* (7.8), yields

$$\begin{aligned}\operatorname{tr}(\widehat{\Theta}_k^\epsilon)^2_{(1,1)} &= \operatorname{tr}\Big(-\bar\partial\chi_\epsilon\wedge\sigma_k D\varphi_k - \chi_\epsilon\bar\partial(\sigma_k D\varphi_k) + \Theta_k\Big)^2\\ &= \bar\partial\chi_\epsilon^2\wedge\operatorname{tr}\Big(\sigma_k D\varphi_k\bar\partial(\sigma_k D\varphi_k)\Big) - 2\bar\partial\chi_\epsilon\wedge\operatorname{tr}(\sigma_k D\varphi_k\Theta_k) + \operatorname{tr}\Big(\chi_\epsilon(\widetilde{\Theta}_k)_{(1,1)} + (1-\chi_\epsilon)\Theta_k\Big)^2.\end{aligned} \tag{7.12}$$

Again using that $\tilde{p}_j = -\operatorname{tr}(\widetilde{\Theta}_1)^j_{(1,1)} + \operatorname{tr}(\widetilde{\Theta}_2)^j_{(1,1)}$ vanishes where $\chi_\epsilon\not\equiv 0$, we get

$$-\Big(\chi_\epsilon(\widetilde{\Theta}_1)_{(1,1)} + (1-\chi_\epsilon)\Theta_1\Big)^2 + \Big(\chi_\epsilon(\widetilde{\Theta}_2)_{(1,1)} + (1-\chi_\epsilon)\Theta_2\Big)^2 = O_\epsilon\to 0. \tag{7.13}$$

Note that $\bar\partial\chi_\epsilon\wedge\sigma_2 D\varphi_2\Theta_2$ is in O_ϵ. Therefore, in view of (7.7) and (7.10),

$$\begin{aligned}&2\bar\partial\chi_\epsilon\wedge\operatorname{tr}(\sigma_1 D\varphi_1\Theta_1) - 2\bar\partial\chi_\epsilon\wedge\operatorname{tr}(\sigma_2 D\varphi_2\Theta_2) =\\ &\qquad -2\bar\partial\chi_\epsilon\wedge\frac{Dy^m}{y^m}\operatorname{tr}(\sigma_1'\varphi_1'\Theta_1) + O_\epsilon \xrightarrow[\epsilon\to 0]{} -\frac{2\pi}{i}2m(\ell+m)\hat\omega\wedge[y=0].\end{aligned} \tag{7.14}$$

Let us next consider the contribution from the first term

$$\bar\partial\chi_\epsilon^2\wedge\operatorname{tr}\Big(\sigma_k D\varphi_k\bar\partial(\sigma_k D\varphi_k)\Big) = \bar\partial\chi_\epsilon^2\wedge\operatorname{tr}\Big(\sigma_k D\varphi_k\bar\partial\sigma_k D\varphi_k\Big) - \bar\partial\chi_\epsilon^2\wedge\operatorname{tr}\Big(\sigma_k D\varphi_k\sigma_k\bar\partial(D\varphi_k)\Big) \tag{7.15}$$

in (7.12). We start by considering the contribution from the first term in (7.15). By arguments as in the proof of Lemma 6.6, we get that

$$\begin{aligned}&-\bar\partial\chi_\epsilon^2\wedge\operatorname{tr}\Big(\sigma_1 D\varphi_1\bar\partial\sigma_1 D\varphi_1\Big) + \bar\partial\chi_\epsilon^2\wedge\operatorname{tr}\Big(\sigma_2 D\varphi_2\bar\partial\sigma_2 D\varphi_2\Big) =\\ &\qquad \bar\partial\chi_\epsilon^2\wedge\operatorname{tr}\Big(\sigma_2\bar\partial\sigma_1 D\varphi_1 D\varphi_2\Big) - \bar\partial\chi_\epsilon^2\wedge\operatorname{tr}\Big(D\bar\partial\sigma_1 D\varphi_1\Big) + \bar\partial\chi_\epsilon^2\wedge\operatorname{tr}\Big(D\bar\partial\sigma_2 D\varphi_2\Big).\end{aligned} \tag{7.16}$$

Taking the limit of the first term in the right hand side of (7.16), we get

$$\begin{aligned}\lim_{\epsilon\to 0}\bar\partial\chi_\epsilon^2\wedge\operatorname{tr}(\sigma_2\bar\partial\sigma_1 D\varphi_1 D\varphi_2) &= \lim_{\epsilon\to 0}\bar\partial\chi_\epsilon^2\wedge\operatorname{tr}(D\varphi_1 D\varphi_2\sigma_2\bar\partial\sigma_1)\\ &= \operatorname{tr}(D\varphi_1 D\varphi_2 R_2^0)\\ &= \left(\frac{2\pi}{i}\right)^2\ell(2k-m)[0].\end{aligned}\tag{7.17}$$

Here, the first equality follows from (2.7), the second equality from (2.18), and the third equality is computed in [LW18, Example 5.2]. Next, by (2.7), $\operatorname{tr}(D\bar\partial\sigma_1 D\varphi_1) = -\operatorname{tr}(D\varphi_1 D\bar\partial\sigma_1)$. Using (2.10), (2.11) and the fact that $\varphi_1'\sigma_1' = 1$, so $\varphi_1'\bar\partial\sigma_1' = 0$, we have

$$\bar\partial\chi_\epsilon^2\wedge D\varphi_1 D\bar\partial\sigma_1 = \bar\partial\chi_\epsilon^2\wedge D(D\varphi_1\bar\partial\sigma_1) = \bar\partial\chi_\epsilon^2\wedge D(D\varphi_1'\bar\partial\sigma_1') = \bar\partial\chi_\epsilon^2\wedge D\varphi_1' D\bar\partial\sigma_1'.$$

If we let f be the section of $\mathcal{O}(k-m)\oplus\mathcal{O}(\ell)$ defined by $f = \left[\begin{array}{cc} y^{k-m} & x^\ell\end{array}\right]$, then φ_2, φ_1' are the morphisms in the Koszul complex defined by (contraction with) f. If we let σ be the minimal inverse of f, when f is viewed as a section of $\operatorname{Hom}(\mathcal{O}(-(k-m))\oplus\mathcal{O}(-\ell),\mathcal{O})$, then σ_2 and σ_1' are given by multiplication with σ. One may verify that $D\varphi_2$ and $D\varphi_1'$ are given by contraction with Df, and that $D\bar\partial\sigma_2$ and $D\bar\partial\sigma_1'$ are given by multiplication with $D\bar\partial\sigma$. A calculation then yields that

$$\operatorname{tr}\left(D\varphi_1' D\bar\partial\sigma_1'\right) = -\operatorname{tr}\left(D\bar\partial\sigma_2 D\varphi_2\right),$$

so by (7.16),

$$-\bar\partial\chi_\epsilon^2\wedge\operatorname{tr}\left(\sigma_1 D\varphi_1\bar\partial\sigma_1 D\varphi_1\right) + \bar\partial\chi_\epsilon^2\wedge\operatorname{tr}\left(\sigma_2 D\varphi_2\bar\partial\sigma_2 D\varphi_2\right) = \bar\partial\chi_\epsilon^2\wedge\operatorname{tr}\left(\sigma_2\bar\partial\sigma_1 D\varphi_1 D\varphi_2\right).\tag{7.18}$$

Thus, in view of (7.18) and (7.17),

$$-\bar\partial\chi_\epsilon^2\wedge\operatorname{tr}\left(\sigma_1 D\varphi_1\bar\partial\sigma_1 D\varphi_1\right) + \bar\partial\chi_\epsilon^2\wedge\operatorname{tr}\left(\sigma_2 D\varphi_2\bar\partial\sigma_2 D\varphi_2\right)\xrightarrow[\epsilon\to 0]{}\left(\frac{2\pi}{i}\right)^2\ell(2k-m)[0].\tag{7.19}$$

Next, let us consider the contribution from the second term in (7.15). As above, using (2.12), *cf.* (7.9),

$$\bar\partial\chi_\epsilon^2\wedge\left(\sigma_1 D\varphi_1\sigma_1\bar\partial(D\varphi_1)\right) = -\bar\partial\chi_\epsilon^2\wedge\left(\sigma_1 D\varphi_1\sigma_1\varphi_1\Theta_1\right) = \bar\partial\chi_\epsilon^2\wedge\left(\frac{Dy^m}{y^m}\sigma_1'\varphi_1'\Theta_1\right) + O_\epsilon.$$

Note that $\bar\partial\chi_\epsilon^2\wedge\sigma_2 D\varphi_2\sigma_2\bar\partial(D\varphi_2)$ is in O_ϵ. Thus, by (7.10) and (7.7),

$$\bar\partial\chi_\epsilon^2\wedge\operatorname{tr}\left(\sigma_1 D\varphi_1\sigma_1\bar\partial(D\varphi_1)\right) - \bar\partial\chi_\epsilon^2\wedge\operatorname{tr}\left(\sigma_2 D\varphi_2\sigma_2\bar\partial(D\varphi_2)\right)\xrightarrow[\epsilon\to 0]{}\frac{2\pi}{i}m(\ell+m)\hat\omega\wedge[y=0],\tag{7.20}$$

cf. (7.14).

From (7.15), (7.19), and (7.20), we conclude that

$$\begin{aligned}&-\bar\partial\chi_\epsilon^2\wedge\operatorname{tr}\left(\sigma_1 D\varphi_1\bar\partial(\sigma_1 D\varphi_1)\right) + \bar\partial\chi_\epsilon^2\wedge\operatorname{tr}\left(\sigma_2 D\varphi_2\bar\partial(\sigma_2 D\varphi_2)\right)\xrightarrow[\epsilon\to 0]{}\\ &\qquad\left(\frac{2\pi}{i}\right)^2\ell(2k-m)[0] + \frac{2\pi}{i}m(\ell+m)\hat\omega\wedge[y=0].\end{aligned}\tag{7.21}$$

Next, from (7.12), (7.13), (7.14), and (7.21), we conclude that

$$\hat p_2 = -\operatorname{tr}(\widehat{\Theta}_1)^2_{(1,1)} + \operatorname{tr}(\widehat{\Theta}_2)^2_{(1,1)}\xrightarrow[\epsilon\to 0]{} -(2\pi/i)m(\ell+m)\hat\omega\wedge[y=0] + (2\pi/i)^2\ell(2k-m)[0].\tag{7.22}$$

Finally from (7.5), (7.11), and (7.22) we conclude that

$$c_2^{\mathrm{Res}}(E,D) = \left(\frac{i}{2\pi}\right)^2\frac{1}{2}\lim_{\epsilon\to 0}(\hat p_1^2 - \hat p_2) = \frac{1}{2}\Big(m(2m+\ell)\omega\wedge[y=0] - \ell(2k-m)[0]\Big)$$

and that

$$(c_1^{\mathrm{Res}})^2(E,D) = \left(\frac{i}{2\pi}\right)^2\lim_{\epsilon\to 0}\hat p_1^2 = m^2\omega[y=0].$$

Taking cohomology, since $[[0]] = [[y = 0] \wedge \omega] = [\omega^2]$, we get

$$\left[c_2^{\mathrm{Res}}(E,D)\right] = \left(m^2 + \ell(m-k)\right)[\omega^2] = c_2(\mathcal{F}) \quad \text{and} \quad \left[(c_1^{\mathrm{Res}})^2(E,D)\right] = m^2[\omega^2] = c_1(\mathcal{F})^2,$$

see (7.4).

References

[AW07] M. Andersson and E. Wulcan, *Residue currents with prescribed annihilator ideals*, Ann. Sci. École Norm. Sup. **40** (2007), 985–1007.

[AW10] ———, *Decomposition of residue currents*, J. Reine Angew. Math. **638** (2010), 103–118.

[AW18] ———, *Direct images of semi-meromorphic currents*, Ann. Inst. Fourier **68** (2018), 875–900.

[BB72] P. Baum and R. Bott, *Singularities of holomorphic foliations*, J. Differential Geometry **7** (1972), 279–342.

[BGS90a] J.-M. Bismut, H. Gillet, and C. Soulé, *Bott-Chern currents and complex immersions*, Duke Math. J. **60** (1990), 255–284.

[BGS90b] ———, *Complex immersions and Arakelov geometry*. In: The Grothendieck Festschrift, Vol. I, pp. 249–331, Progr. Math., 86, Birkhäuser Boston, Boston, MA, 1990.

[BSW21] J.-M. Bismut, S. Shen, and Z. Wei, *Coherent sheaves, superconnections, and RRG*, preprint arXiv:2102.08129 (2021).

[BS58] A. Borel and J.-P. Serre, *Le théorème de Riemann-Roch*, Bull. Soc. Math. France **86** (1958), 97–136.

[CH78] N. R. Coleff and M. E. Herrera, *Les courants résiduels associés à une forme méromorphe*, Lecture Notes in Mathematics 633, Springer, Berlin, 1978.

[EH16] D. Eisenbud and J. Harris, *3264 and all that—a second course in algebraic geometry*, Cambridge University Press, Cambridge, 2016.

[Ful98] W. Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 2, Springer-Verlag, Berlin, 1998.

[Gre80] H. I. Green, *Chern classes for coherent sheaves*, Ph.D. Thesis, University of Warwick, (1980). Available from http://wrap.warwick.ac.uk/40592/1/WRAP_THESIS_Green_1980.pdf.

[Gri10] J. Grivaux, *Chern classes in Deligne cohomology for coherent analytic sheaves*, Math. Ann. **347** (2010), 249–284.

[Hos20a] T. Hosgood, *Simplicial Chern-Weil theory for coherent analytic sheaves, part I*, preprint arXiv:2003.10023 (2020).

[Hos20b] ———, *Simplicial Chern-Weil theory for coherent analytic sheaves, part II*, preprint arXiv:2003.10591 (2020).

[HL93] F. R. Harvey and H. B. Lawson Jr., *A theory of characteristic currents associated with a singular connection*, Astérisque **213**, Société Mathématique de France, 1993.

[HL71] M. Herrera and D. Lieberman, *Residues and principal values on complex spaces*, Math. Ann. **194** (1971), 259–294.

[JL21] J. Johansson and R. Lärkäng, *An explicit isomorphism of different representations of the Ext functor using residue currents*, preprint arXiv:2109.00480 (2021).

[Kar08] M. Karoubi, *K-theory*, Classics in Mathematics, Springer-Verlag, Berlin, 2008.

[LW18] R. Lärkäng, and E. Wulcan, *Residue currents and fundamental cycles*, Indiana Univ. Math. J. **67** (2018), 1085–1114.

[LW21] ______, *Residue currents and cycles of complexes of vector bundles*, Ann. Fac. Sci. Toulouse **30** (2021), no. 5, 961–984.

[Mac95] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995.

[Qia16] H. Qiang, *On the Bott-Chern characteristic classes for coherent sheaves*, preprint arXiv:1611.04238 (2016).

[Suw98] T. Suwa, *Indices of vector fields and residues of singular holomorphic foliations*, Actualités Mathématiques, Hermann, Paris, 1998.

[TT86] D. Toledo and Y. L. L. Tong, *Green's theory of Chern classes and the Riemann-Roch formula.* In: *The Lefschetz centennial conference, Part I* (Mexico City, 1984), pp. 261–275, Contemp. Math. 58, Amer. Math. Soc., Providence, RI, 1986.

[Tu17] L. W. Tu, *Differential geometry*, Graduate Texts in Mathematics, 275, Springer, Cham, 2017.

[Wu20] X. Wu, *Intersection theory and Chern classes in Bott-Chern cohomology*, preprint arXiv:2011.13759 (2020).