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# SYNCHRONICITY FOR QUANTUM NON-LOCAL GAMES 

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#### Abstract

We introduce concurrent quantum non-local games, quantum output mirror games and concurrent classical-to-quantum non-local games, as quantum versions of synchronous non-local games, and provide tracial characterisations of their perfect strategies belonging to various correlation classes. We define *-algebras and $\mathrm{C}^{*}$-algebras of concurrent classical-to-quantum and concurrent quantum non-local games, and algebraic versions of the orthogonal rank of a graph. We show that quantum homomorphisms of quantum graphs can be viewed as entanglement assisted classical homomorphisms of the graphs, and give descriptions of the perfect quantum commuting and the perfect approximately quantum strategies for the quantum graph homomorphism game. We specialise the latter results to the case where the inputs of the game are based on a classical graph.


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## 1. Introduction

Over the past decade, the theory of non-local games has undergone a flurry of development and is now a fundamental branch of modern quantum information theory, with deep applications to many areas of mathematics, physics, and computer science, including operator algebras, noncommutative geometry, quantum non-locality, entanglement, and quantum complexity theory. Mathematically, a (two-player) non-local game consists of a tuple $\mathcal{G}=(X, Y, A, B, \lambda)$, where $X, Y, A, B$ are finite sets, and $\lambda: X \times Y \times A \times B \rightarrow\{0,1\}$ is a function. The game is played cooperatively by two spatially separated non-communicating players, Alice and Bob, against a referee. During each round of the game, the referee samples a pair of "questions" $(x, y) \in X \times Y$, and sends question $x$ to Alice, and quesiton $y$ to Bob. Alice is then required to supply an "answer" $a \in A$, and Bob - an answer $b \in B$, to the referee. Alice and Bob win the round of the game if and only if the rule function $\lambda$ evaluates to 1 on this question-answer combination, that is, if the condition $\lambda(x, y, a, b)=1$ is satisfied.

The fact that the players Alice and Bob are not allowed to communicate during play makes it difficult to win each round of a non-local game with high probability. On the other hand, it is precisely this nature of non-local games that makes them interesting as both theoretical and practical tools in quantum information. The idea here is that, in certain scenarios, Alice and Bob can utilise the phenomenon of quantum entanglement to help correlate their answers in a much stronger way than what the resources of classical physics allow.

A prototypical example of a non-local game is the graph homomorphism game: Given a pair of finite simple graphs $G$ and $H$ with vertex sets $V(G)$, $V(H)$ and edge sets $E(G), E(H)$, respectively, the $(G, H)$-homomorphism game is the non-local game $\mathcal{G}$ with $X=Y=V(G), A=B=V(H)$ and $\lambda(x, y, a, b)=0$ if either (i) $x=y$ and $a \neq b$ or (ii) $(x, y) \in E(G)$ and $(a, b) \notin$ $E(H)$. Clearly the graph homomorphism game captures, in the operational language of non-local games, the notion of a graph homomorphism $G \rightarrow H$ : Any winning strategy for this game would serve to convince an observer that there exists such a graph homomorphism $G \rightarrow H$.

Graph homomorphism games form an interesting class of non-local games for several reasons. First, they give rise to quantum analogues of graph parameters, including quantum chromatic numbers and quantum independence numbers [18, 24]. These parameters can be genuinely different than the corresponding classical versions, thus providing new manifestations of the fundamental Bell Theorem. Second, they provide some of the simplest examples of pseudo-telepathy games - ones which can be perfectly won only with the help of quantum entanglement as a resource [18, 11, 24]. Third, and perhaps most importantly, graph homomorphism games belong to the particularly important class of synchronous non-local games introduced in [24] (see also [12]). Recall that a non-local game $\mathcal{G}=(X, Y, A, B, \lambda)$ is
called sychronous if $X=Y, A=B$, and $\lambda(x, x, a, b)=0$ for all $x \in X$ and $a \neq b \in A$. This means that in order for Alice and Bob to win a round of $\mathcal{G}$, they must "sychronise" their answers whenever they both receive the same question from the referee. This seemingly innocuous constraint on a game $\mathcal{G}$ turns out to have very interesting quantum information theoretic and operator algebraic consequences. For example, the problem of finding perfect quantum strategies for a synchronous game $\mathcal{G}$ amounts to finding tracial states on a certain game *-algebra $\mathcal{A}(\mathcal{G})$ associated to $\mathcal{G}$ 12. The algebras $\mathcal{A}(\mathcal{G})$ play the role of a non-commutative analogue of the algebras of coordinate functions on spaces of perfect deterministic (classical) strategies for $\mathcal{G}$, and are therefore of significant interest from several perspectives in noncommutative geometry, quantum groups [28, 4], and von Neumann algebra theory [13]. It follows from the breakthrough work [13] that there exists a synchronous non-local game $\mathcal{G}$ whose game $*$-algebra $\mathcal{A}(\mathcal{G})$ admits a tracial state $\tau$ for which the generated von Neumann algebra $M=\pi_{\tau}(\mathcal{A}(\mathcal{G}))^{\prime \prime}$ fails to embed into an ultraproduct of the hyperfinite $\mathrm{II}_{1}$-factor - yielding a(n albeit non-constructive) counter-example to the Connes Embedding Problem in operator algebras and to the equivalent [14] strong Tsirelson Problem in quantum physics.

The purpose of the present paper is to introduce and study generalisations of synchronous non-local games within the framework of quantum non-local games - non-local games where the questions and answers are allowed to be quantum states, or possibly mixtures of classical and quantum states. In this paper, we use the language of quantum no-signalling (QNS) correlations and quantum non-local games recently introduced by two of the present authors 30. Classically, in the course of a non-local game $\mathcal{G}=(X, Y, A, B, \lambda)$, Alice and Bob's behaviour is described by a family $p=(p(a, b \mid x, y))_{(a, b, x, y) \in A \times B \times X \times Y}$ of conditional probability distributions, which can, in a canonical way, be viewed as a noisy information channel $\mathcal{N}: X \times Y \rightarrow A \times B$ with well-defined marginal channels. In the quantum setting, one replaces the classical state spaces $X, Y, A, B$ by their quantum analogues (i.e. the Hilbert spaces $\mathbb{C}^{|X|}, \mathbb{C}^{|Y|}$, etc.), and the classical channel $\mathcal{N}: X \times Y \rightarrow A \times B$ by a quantum channel $\Gamma: M_{X} \otimes M_{Y} \rightarrow M_{A} \otimes M_{B}$, where, for any finite set $Z$, we have let $M_{Z}=\mathcal{B}\left(\mathbb{C}^{|Z|}\right)$ be the matrix algebra of linear maps on $\mathbb{C}^{|Z|}$. In this framework, the rule function $\lambda$ can be generalized by replacing it with a zero-preserving, join-preserving mapping $\varphi$ from the projection lattice on $\mathcal{P}_{X Y}$ in $M_{X} \otimes M_{Y}$ to the projection lattice $\mathcal{P}_{A B}$ in $M_{A} \otimes M_{B}$. A winning strategy for a quantum non-local game $\varphi: \mathcal{P}_{X Y} \rightarrow \mathcal{P}_{A B}$ is then given by a QNS correlation $\Gamma$ satisfying the trace-orthogonality relation

$$
\left\langle\Gamma(P), \varphi(P)_{\perp}\right\rangle=0, \quad P \in \mathcal{P}_{X Y} ;
$$

the latter condition constrains the supports of the output states of $\Gamma$ according to the supports of its input states (see Section 3.1 for further motivation and details).

We note that non-local games with quantum inputs and/or outputs have been previously studied in [7] and [27]. The strategies used in the latter papers are the elements from the quantum QNS correlation class. Since our main interest lies in the characterisation of the perfect strategies of a game and their applications, we have adopted the present approach, where we only specify the rules of the non-local game, without fixing a probability distribution on the questions (or a quantum version thereof).

One of our main achievements in the present work is the introduction of quantum analogues of synchronous non-local games (called herein concurrent quantum games), as well as classical input-quantum output versions of the mirror games introduced in [17. Classically, synchronous games form a special class of mirror games, and both of these classes of games have the remarkable property that "Alice's quantum behaviour completely determine Bob's quantum behaviour" when considering perfect strategies for the games; moreover, such perfect strategies can always be described in terms of correlations coming from tracial states on a particular game algebra. We show that such a paradigm persists in the quantum case by associating *-algebras and $\mathrm{C}^{*}$-algebras to concurrent quantum and to concurrent classical-to-quantum games. Our main results in this direction (cf. Theorem [3.2, Corollary 3.7, Theorem 4.1, Corollary 4.4) provide an operational interpretation of the tracial QNS correlations introduced in [30] in terms of perfect strategies of concurrent and quantum mirror games, and their associated game algebras.

One of our long-term motivations for the present work is to develop tools that may eventually be useful for gaining a better understanding of the work [13], which, as mentioned above, implicitly constructs a game $\mathcal{G}$, whose game algebra is a witness to the failure of the Connes Embedding Problem. At present, the game constructed in [13] is not well understood, and involves very large input/output sets. There is some hope that quantum non-local games may provide additional flexibility in the construction of game algebras with pathological operator algebraic properties. A particularly interesting and tractable source of examples in this more general framework are the quantum graph homomorphism games. Quantum graphs have achieved a lot of attention in recent years, as objects that arise in a variety of areas (e.g. zero-error quantum information theory, quantum error correction, quantum groups, quantum teleportation schemes, and subfactor theory) [3, 4, 21, 29, 33]. In Section [5, we study the quantum graph homomorphism game in detail, extending previous work of the authors [5, 30] in the classical-quantum hybrid setting, and also making connections with the work of Stahlke [29] and the algebraic work of Musto-Reutter-Verdon [21] on quantum graph homomorphisms.

The paper is organised as follows. Section 2 introduces some necessary notation and background that will be used throughout the paper. Section 3 recalls the notions related to QNS correlations and their various subclasses (quantum commuting, approximately quantum, quantum, local), examines
in detail the case of classical to quantum non-local games, introducing the aforementioned semi-quantised mirror games and concurrent games, and studies them as operational realisations of tracial QNS correlations. In Section 4. we consider the fully quantum concurrent games, proving tracial characterisations of perfect strategies of these games. Finally, in Section 5 we focus on the quantum graph homomorphism game, and describe connections with the prior work of Stahlke [29] on entanglement assisted quantum graph homomorphisms, as well as with our prior works [5, 30]. We show that the perfect quantum strategies of the quantum graph homomorphism game can, in a rigorous sense, be thought of as entanglement assisted perfect classical strategies for this game, and extract characterisations of the corresponding quantum commuting and approximately quantum strategies in terms of natural inclusion relations relating the two quantum graphs. Our results are further specialised in the case where the inputs are based on a classical graph, leading to separation results on the algebraic and $\mathrm{C}^{*}$ algebraic versions of the orthogonal rank of a graph (cf. Propositions 5.16 and 5.17).

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Note on related work. After the first draft of this paper was completed, we learnt from Piotr Soltan that characterisations of concurrent correlations from the quantum commuting class, closely related to the ones described in Subsection 4.1, were independently obtained by Bochniak-Kasprzak-Sołtan in the recently posted preprint [2; more specifically, [2, Theorem 6.6] generalises the first statement within Theorem 4.1 in the present paper.

## 2. Preliminary notions and Results

For a finite set $X$, let $M_{X}$ be the algebra of all complex matrices indexed by $X \times X$; we identify $M_{X}$ with the algebra of all linear transformations on the Hilbert space $\mathbb{C}^{X}:=\oplus_{x \in X} \mathbb{C}$. If $T \in M_{A}$, we write $T^{\mathrm{t}}$ for the transpose matrix, and set $\bar{T}=\left(T^{*}\right)^{\mathrm{t}}$. We let $\mathcal{D}_{X}$ be the subalgebra of $M_{X}$ of all diagonal matrices, and $\Delta_{X}: M_{X} \rightarrow \mathcal{D}_{X}$ be the conditional expectation. We write $M_{X Y}=M_{X} \otimes M_{Y}, \mathcal{P}_{X Y}$ for the projection lattice of $M_{X Y}$, and $\mathcal{P}_{X Y}^{\mathrm{cl}}$ for the projection lattice of $\mathcal{D}_{X Y}$. We let $\xi \eta^{*}$ be the rank one operator given by $\xi \eta^{*}(\zeta)=\langle\zeta, \eta\rangle \xi$, and $\left(e_{x}\right)_{x \in X}$ be the canonical orthonormal basis of $\mathbb{C}^{X}$.

For a Hilbert space $H$ and vectors $\xi, \eta \in H$, we write $\xi \perp \eta$ if $\langle\xi, \eta\rangle=0$. Let $H^{\mathrm{d}}$ be the dual (Banach) space of $H$ and d: $H \rightarrow H^{\mathrm{d}}$ be the map, given by $\mathrm{d}(\xi)(\eta)=\langle\eta, \xi\rangle$; we write $\xi^{\mathrm{d}}=\mathrm{d}(\xi)$. Note that $(\lambda \xi)^{\mathrm{d}}=\bar{\lambda} \xi^{\mathrm{d}}, \lambda \in \mathbb{C}$, and that, if $T \in \mathcal{L}(H)$, then the dual operator $T^{\mathrm{d}}: H^{\mathrm{d}} \rightarrow H^{\mathrm{d}}$ satisfies the
relation

$$
\begin{equation*}
T^{\mathrm{d}} \xi^{\mathrm{d}}=\left(T^{*} \xi\right)^{\mathrm{d}}, \quad T \in \mathcal{L}(H) . \tag{1}
\end{equation*}
$$

Let $\omega \in M_{X}$. Writing $f_{\omega}$ for the functional on $M_{X}$ given by $f_{\omega}(\rho)=$ $\operatorname{Tr}\left(\rho \omega^{\mathrm{t}}\right)$, we have that the map $\omega \rightarrow f_{\omega}$ is a complete order isomorphism from $M_{X}$ onto the dual operator system $M_{X}^{\mathrm{d}}$ (see e.g. [26, Theorem 6.2]). On the other hand, the map $\omega^{\mathrm{d}} \mapsto \omega^{\mathrm{t}}$ is a *-isomorphism from $\mathcal{L}\left(\left(\mathbb{C}^{X}\right)^{\mathrm{d}}\right)$ onto $M_{X}$. The composition of these maps, $\omega^{\mathrm{d}} \mapsto f_{\omega^{\mathrm{t}}}$, is thus a complete order isomorphism from $\mathcal{L}\left(\left(\mathbb{C}^{X}\right)^{\mathrm{d}}\right)$ onto $M_{X}^{\mathrm{d}}$. In the sequel, we identify these two spaces; note that, via this identification,

$$
\begin{equation*}
\left\langle\rho, \omega^{\mathrm{d}}\right\rangle=\left\langle\rho, \omega^{\mathrm{t}}\right\rangle=\operatorname{Tr}(\rho \omega), \quad \rho, \omega \in M_{X} . \tag{2}
\end{equation*}
$$

If $P \in M_{X}$ is a projection, we write $P_{\perp}$ for the projection in $M_{X}^{\mathrm{d}}$ on the annihilator in $\left(\mathbb{C}^{X}\right)^{\mathrm{d}}$ of the range of $P$.

Write $\epsilon_{x, x^{\prime}}=e_{x} e_{x^{\prime}}^{*}$ for the matrix unit in $M_{X}$, corresponding to the pair $\left(x, x^{\prime}\right)$ of indices. Set

$$
J_{X}=\frac{1}{|X|} \sum_{x, x^{\prime} \in X} \epsilon_{x, x^{\prime}} \otimes \epsilon_{x, x^{\prime}}
$$

if $\mathfrak{m}_{X}=\frac{1}{\sqrt{|X|}} \sum_{x \in X} e_{x} \otimes e_{x}$ is the maximally entangled unit vector in $\mathbb{C}^{X} \otimes$ $\mathbb{C}^{X}$, then $J_{X}=\mathfrak{m}_{X} \mathfrak{m}_{X}^{*}$ is its corresponding rank one projection. Set also

$$
J_{X}^{\mathrm{cl}}=\sum_{x \in X} \epsilon_{x, x} \otimes \epsilon_{x, x},
$$

and note that $\Delta_{X X}\left(J_{X}\right)=\frac{1}{|X|} J_{X}^{\mathrm{cl}}$. Heuristically, $J_{X}^{\mathrm{cl}}$ is the (normalised) part of $J_{X}$ that can be seen by a classical observer.

Recall [30] that a quantum non-local game is a join-preserving map $\varphi$ : $\mathcal{P}_{X Y} \rightarrow \mathcal{P}_{A B}$ with $\varphi(0)=0$, while a classical-to-quantum (cq) non-local game is a join-preserving map $\varphi: \mathcal{P}_{X Y}^{\mathrm{cl}} \rightarrow \mathcal{P}_{A B}$ with $\varphi(0)=0$. Similarly, a classical non-local game is a join-preserving and zero-preserving map $\varphi$ : $\mathcal{P}_{X Y}^{\mathrm{cl}} \rightarrow \mathcal{P}_{A B}^{\mathrm{cl}}$.

Recall also that a non-local game on the quadruple $(X, Y, A, B)$ is a function $\lambda: X \times Y \times A \times B \rightarrow\{0,1\}$. In [30], we associated to such $\lambda$ the classical non-local game $\varphi_{\lambda}: \mathcal{P}_{X Y}^{\mathrm{cl}} \rightarrow \mathcal{P}_{A B}^{\mathrm{cl}}$ given by

$$
\varphi_{\lambda}\left(\sum_{x, y \in \kappa} \epsilon_{x, x} \otimes \epsilon_{y, y}\right)=\sum\left\{\epsilon_{a, a} \otimes \epsilon_{b, b}: \exists(x, y) \in \kappa \text { s.t. } \lambda(x, y, a, b)=1\right\}
$$

after recalling that projections in $\mathcal{P}_{X Y}^{\mathrm{cl}}$ correspond to subsets $\kappa \subseteq X \times Y$.
A non-local game ( $X, Y, A, B, \lambda$ ) is called

- a mirror game [17] if there exist functions $f: X \rightarrow Y$ and $g: Y \rightarrow X$ such that for every $x \in X$ (resp. $y \in Y$ ) the set

$$
\{(a, b) \in A \times B: \lambda(x, f(x), a, b)=1\}
$$

(resp.

$$
\{(a, b) \in A \times B: \lambda(g(y), y, a, b)=1\})
$$

is the graph of a bijection, and

- a synchronous game [24] (see also [12]) if $X=Y, A=B$ and

$$
a, b \in A, a \neq b \Longrightarrow \lambda(x, x, a, b)=0 .
$$

Mirror games include the subclass of unique games (that is, games for which the set $\{(a, b) \in A \times B: \lambda(x, y, a, b)=1\}$ is the graph of a bijection for every $(x, y) \in X \times Y$ [31]); in particular, they form a class, strictly larger than that of synchronous games.

Set $B=A$ and recall the standard (linear) identification of matrices in $M_{A}$ with vectors in $\mathbb{C}^{A} \otimes \mathbb{C}^{B}$, which associates to the matrix unit $\epsilon_{a, b}$ the vector $e_{a} \otimes e_{b}$ (see e.g. [32, Section 1.1.2]). Write $\tilde{\zeta}_{T} \in \mathbb{C}^{A} \otimes \mathbb{C}^{B}$ for the vector corresponding to $T \in M_{A}$ and set $\zeta_{T}=\frac{\tilde{\zeta}_{T}}{\left\|\tilde{\zeta}_{T}\right\|}$; we have that $\zeta_{I_{A}}=\mathfrak{m}_{A}$. We note the relations [32, Section 1.1.1]

$$
\begin{equation*}
(R \otimes S) \tilde{\zeta}_{T}=\tilde{\zeta}_{R T S^{t}}, \quad R, S, T \in M_{A} \tag{3}
\end{equation*}
$$

If $\alpha: A \rightarrow B$ is a bijection, let $P_{\alpha}=\sum_{a \in A} \epsilon_{a, a} \otimes \epsilon_{\alpha(a), \alpha(a)}$; clearly, $P_{\alpha} \in \mathcal{P}_{A B}^{\mathrm{cl}}$.

Remark 2.1. A non-local game $\lambda$ is
(i) synchronous if and only if $\varphi_{\lambda}\left(J_{X}^{\mathrm{cl}}\right) \leq J_{A}^{\mathrm{cl}}$;
(ii) mirror if and only if there exist functions $f: X \rightarrow Y, g: Y \rightarrow X$ and bijections $\alpha_{x}, \beta_{y}: A \rightarrow B, x \in X, y \in Y$, such that
$\varphi_{\lambda}\left(\epsilon_{x, x} \otimes \epsilon_{f(x), f(x)}\right)=P_{\alpha_{x}}$ and $\varphi_{\lambda}\left(\epsilon_{g(y), g(y)} \otimes \epsilon_{y, y}\right)=P_{\beta_{y}^{-1}}, x \in X, y \in Y$.
Proof. (i) If $\lambda$ is synchronous then, clearly,

$$
\varphi_{\lambda}\left(\epsilon_{x, x} \otimes \epsilon_{x, x}\right) \leq J_{A}^{\mathrm{cl}} \quad \text { for all } x \in X
$$

taking the span over all $x$, we get $\varphi_{\lambda}\left(J_{X}^{\mathrm{cl}}\right) \leq J_{A}^{\mathrm{cl}}$. Conversely, the condition $\varphi_{\lambda}\left(J_{X}^{\mathrm{cl}}\right) \leq J_{A}^{\mathrm{cl}}$ implies in particular $\varphi_{\lambda}\left(\epsilon_{x, x} \otimes \epsilon_{x, x}\right) \leq J_{A}^{\mathrm{cl}}$, which is equivalent to $\lambda(x, x, a, b)=0$ whenever $a \neq b$. Claim (ii) is equally straightforward.

Remark 2.1 motivates the following versions of mirror and synchronous games, where the inputs are still classical, while the outputs are allowed to be quantum. We assume that $|A|=|B|$ but continue to use different symbols to denote the sets $A$ and $B$ for clarity. If $\omega \in M_{A}$, let $L_{\omega}: M_{A B} \rightarrow M_{B}$ be the slice map, given by $L_{\omega}(S \otimes T)=\langle S, \omega\rangle T$ and write $\operatorname{Tr}_{A}=L_{I_{A}}$ for the partial trace; the slice map $L_{\rho}: M_{A B} \rightarrow M_{A}$, for $\rho \in M_{B}$, and the partial trace $\operatorname{Tr}_{B}$, are defined similarly. Call a rank one projection $P \in M_{A B}$ bijective if

$$
\begin{equation*}
e, f \in \mathbb{C}^{A}, e \perp f \Longrightarrow L_{e e^{*}}(P) \perp L_{f f^{*}}(P) \tag{4}
\end{equation*}
$$

(note that the orthogonality is understood in terms of the trace in $M_{B}$ ). Bijective projections can be thought of as quantum versions of bijections; in fact, if $\alpha: A \rightarrow B$ is a bijection then $P=P_{\alpha}$ satisfies (4) when $e$ and $f$ are taken to be elements of the standard basis.

Lemma 2.2. A rank one projection $P \in M_{A B}$ is bijective if and only if $P=\zeta_{U} \zeta_{U}^{*}$ for some unitary operator $U \in M_{A}$.

Proof. Let $P=\zeta \zeta^{*}$ for some $\zeta \in \mathbb{C}^{A}$, and $U \in M_{A}$ be an operator with $\zeta=\zeta_{U}$. Let $e=\sum_{a \in A} \lambda_{a} e_{a} \in \mathbb{C}^{A}$ and write $\bar{e}=\sum_{a \in A} \overline{\lambda_{a}} e_{a}$. Set $r=\left\|\tilde{\zeta}_{U}\right\|$. We have

$$
\begin{aligned}
\left\langle L_{e e^{*}}\left(\zeta \zeta^{*}\right), e_{a} e_{b}^{*}\right\rangle & =\left\langle\zeta \zeta^{*}, e e^{*} \otimes e_{a} e_{b}^{*}\right\rangle=\operatorname{Tr}\left(\left(\zeta \zeta^{*}\right)\left(e e^{*} \otimes e_{a} e_{b}^{*}\right)^{\mathrm{t}}\right) \\
& =\operatorname{Tr}\left(\left(\zeta \zeta^{*}\right)\left(e e^{*}\right)^{\mathrm{t}} \otimes\left(e_{a} e_{b}^{*}\right)^{\mathrm{t}}\right)=\operatorname{Tr}\left(\left(\zeta \zeta^{*}\right)\left(\overline{e e^{*}}\right) \otimes\left(e_{b} e_{a}^{*}\right)\right) \\
& =\operatorname{Tr}\left(\left(\zeta \zeta^{*}\right)\left(\bar{e} \otimes e_{b}\right)\left(\bar{e} \otimes e_{a}\right)^{*}\right)=\left\langle\bar{e} \otimes e_{b}, \zeta\right\rangle\left\langle\zeta, \bar{e} \otimes e_{a}\right\rangle \\
& =r^{-2} \sum_{c \in A} \lambda_{c}\left\langle U e_{a}, e_{c}\right\rangle \sum_{d \in A} \overline{\lambda_{d}}\left\langle U e_{b}, e_{d}\right\rangle \\
& =r^{-2}\left\langle U e_{a}, \bar{e}\right\rangle\left\langle\overline{\left\langle U e_{b}, \bar{e}\right\rangle}=r^{-2}\left\langle e_{a}, U^{*} \bar{e}\right\rangle\left\langle U^{*} \bar{e}, e_{b}\right\rangle\right. \\
& =r^{-2}\left\langle\left(U^{*} \bar{e}\right)\left(U^{*} \bar{e}\right)^{*} e_{a}, e_{b}\right\rangle=r^{-2}\left\langle\left(U^{*} \bar{e}\right)\left(U^{*} \bar{e}\right)^{*}, e_{b} e_{a}^{*}\right\rangle \\
& =r^{-2}\left\langle\left(\overline{U^{*} \bar{e}}\right)\left(\overline{U^{*} \bar{e}}\right)^{*}, e_{a} e_{b}^{*}\right\rangle=r^{-2}\left\langle\left(U^{t} e\right)\left(U^{\mathrm{t}} e\right)^{*}, e_{a} e_{b}^{*}\right\rangle
\end{aligned}
$$

thus, $L_{e e^{*}}\left(\zeta \zeta^{*}\right)=r^{-2}\left(U^{\mathrm{t}} e\right)\left(U^{\mathrm{t}} e\right)^{*}$. It follows that $P$ is bijective if and only if $U^{\mathrm{t}}$ is a multiple of a unitary operator, that is, if and only if $\mu U$ is unitary for some $\mu \in \mathbb{C}$. Clearly, $P=\zeta_{\mu U} \zeta_{\mu U}^{*}$.

A projection $P \in M_{A}$ of rank $r$ will be called bijective if there exist partial isometries $U_{i}, i=1, \ldots, r$, such that $\sum_{i=1}^{r} U_{i} U_{i}^{*}=\sum_{i=1}^{r} U_{i}^{*} U_{i}=I$ and $P=\sum_{i=1}^{r} \zeta_{U_{i}} \zeta_{U_{i}}^{*}$. Note that, if $\alpha: A \rightarrow B$ is a bijection and $P=P_{\alpha}$, then $P$ is bijective of rank $|A|$ with corresponding partial isometries $\epsilon_{\alpha(a), a}$, $a \in A$.

Definition 2.3. Let $\varphi: \mathcal{P}_{X Y}^{\mathrm{cl}} \rightarrow \mathcal{P}_{A B}$ be a classical-to-quantum non-local game and $\psi: \mathcal{P}_{X Y} \rightarrow \mathcal{P}_{A B}$ be a quantum non-local game.
(i) $\varphi$ is called $a$ quantum output mirror game if there exists functions $f: X \rightarrow Y, g: Y \rightarrow X$ such that the projections $\varphi\left(\epsilon_{x, x} \otimes \epsilon_{f(x), f(x)}\right)$ and $\varphi\left(\epsilon_{g(y), g(y)} \otimes \epsilon_{y, y}\right)$ are bijective, $x \in X, y \in Y$;
(ii) $\varphi$ is called concurrent if $\varphi\left(J_{X}^{\mathrm{cl}}\right)=J_{A}$;
(iii) $\psi$ is called concurrent if $\psi\left(J_{X}\right)=J_{A}$.

In view of Remark 2.1, we consider quantum output mirror games as a quantum version of mirror games, and concurrent games - as quantum versions of synchronous games.

## 3. Classical-to-Quantum games

This section contains characterisations of the prefect strategies of quantum output mirror games and classical-to-quantum concurrent games, and their applications to quantum orthogonal ranks of graphs. We start with recalling the main classes of quantum no-signalling correlations introduced in [30] that will be used subsequently.
3.1. Quantum no-signalling correlations. If $\mathcal{A}$ is a $\mathrm{C}^{*}$-algebra, we denote by $\mathcal{A}^{\mathrm{op}}$ its opposite $\mathrm{C}^{*}$-algebra. As a set, $\mathcal{A}^{\mathrm{op}}$ can be identified with $\mathcal{A}$ and we write $\mathcal{A}^{\mathrm{op}}=\left\{z^{\mathrm{op}}: z \in \mathcal{A}\right\} ;$ the $\mathrm{C}^{*}$-algebra $\mathcal{A}^{\mathrm{op}}$ has the same norm, additive and involutive structure as $\mathcal{A}$, and its multiplication is given by letting $z_{1}^{\mathrm{op}} z_{2}^{\mathrm{op}}=\left(z_{2} z_{1}\right)^{\mathrm{op}}, z_{1}, z_{2} \in \mathcal{A}$.

Let $\mathfrak{V}_{X, A}$ be the ternary ring, generated by elements $v_{a, x}, x \in X, a \in A$, such that the matrix $V=\left(v_{a, x}\right)_{a \in A, x \in X}$ satisfies the condition of an isometry, that is,

$$
\sum_{a \in A} v_{a^{\prime \prime}, x^{\prime \prime}} v_{a, x}^{*} v_{a, x^{\prime}}=\delta_{x, x^{\prime}} v_{a^{\prime \prime}, x^{\prime \prime}}, \quad x, x^{\prime}, x^{\prime \prime} \in X, a^{\prime \prime} \in A
$$

Let $\mathfrak{C}_{X, A}$ be the unital ${ }^{*}$-algebra, generated by the set $\left\{v_{a, x}^{*} v_{a^{\prime}, x^{\prime}}: x, x^{\prime} \in\right.$ $\left.X, a, a^{\prime} \in A\right\}$, and set $e_{x, x^{\prime}, a, a^{\prime}}=v_{a, x}^{*} v_{a^{\prime}, x^{\prime}}$ for brevity. Further, let $\mathcal{V}_{X, A}$ be the universal ternary ring of operators (TRO) of the isometry $V$, and let $\mathcal{C}_{X, A}$ be its right $\mathrm{C}^{*}$-algebra; thus, $\mathcal{C}_{X, A}$ is generated, as a $\mathrm{C}^{*}$-algebra, by $e_{x, x^{\prime}, a, a^{\prime}}, x, x^{\prime} \in X, a, a^{\prime} \in A$ (see [30]). We write

$$
E=\left(e_{x, x^{\prime}, a, a^{\prime}}\right)_{x, x^{\prime}, a, a^{\prime}} \text { and } E^{\mathrm{op}}=\left(e_{x^{\prime}, x, a^{\prime}, a}^{\mathrm{op}}\right)_{x, x^{\prime}, a, a^{\prime}} ;
$$

thus, $E \in M_{X A} \otimes \mathfrak{C}_{X, A}$ and $E^{\mathrm{op}} \in M_{X A} \otimes \mathfrak{C}_{X, A}^{\mathrm{op}}$.
A stochastic operator matrix acting on a Hilbert space $H$ is a positive block operator matrix $\tilde{E}=\left(E_{x, x^{\prime}, a, a^{\prime}}\right)_{x, x^{\prime}, a, a^{\prime}} \in M_{X A}(\mathcal{B}(H))$ such that $\operatorname{Tr}_{A} \tilde{E}=I$. Stochastic operator matrices $\tilde{E}$ acting on $H$ correspond to unital *-representations $\pi: \mathcal{C}_{X, A} \rightarrow \mathcal{B}(H)$ by via the assignment $\pi\left(e_{x, x^{\prime}, a, a^{\prime}}\right)=$ $E_{x, x^{\prime}, a, a^{\prime}}, x, x^{\prime} \in X, a, a^{\prime} \in A$ [30].

Let $X, Y, A$ and $B$ be finite sets. A quantum no-signalling (QNS) correlation 10 is a quantum channel (that is, a completely positive trace preserving map) $\Gamma: M_{X Y} \rightarrow M_{A B}$ such that
(5) $\operatorname{Tr}_{A} \Gamma\left(\rho_{X} \otimes \rho_{Y}\right)=0$ whenever $\rho_{X} \in M_{X}$ and $\operatorname{Tr}\left(\rho_{X}\right)=0$,
and
(6) $\quad \operatorname{Tr}_{B} \Gamma\left(\rho_{X} \otimes \rho_{Y}\right)=0$ whenever $\rho_{Y} \in M_{Y}$ and $\operatorname{Tr}\left(\rho_{Y}\right)=0$.

A QNS correlation $\Gamma: M_{X Y} \rightarrow M_{A B}$ is quantum commuting if there exist a Hilbert space $H$, a unit vector $\xi \in H$ and stochastic operator matrices $\tilde{E}=\left(E_{x, x^{\prime}, a, a^{\prime}}\right)_{x, x^{\prime}, a, a^{\prime}}$ and $\tilde{F}=\left(F_{y, y^{\prime}, b, b^{\prime}}\right)_{y, y^{\prime}, b, b^{\prime}}$ on $H$ such that

$$
E_{x, x^{\prime}, a, a^{\prime}} F_{y, y^{\prime}, b, b^{\prime}}=F_{y, y^{\prime}, b, b^{\prime}} E_{x, x^{\prime}, a, a^{\prime}}
$$

for all $x, x^{\prime} \in X, y, y^{\prime} \in Y, a, a^{\prime} \in A, b, b^{\prime} \in B$, and the Choi matrix of $\Gamma$ coincides with

$$
\begin{equation*}
\left(E_{x, x^{\prime}, a, a^{\prime}} F_{y, y^{\prime}, b, b^{\prime}}\right)_{x, x^{\prime}, a, a^{\prime}}^{y, y^{\prime}, b b^{\prime}} M_{X Y B}(\mathcal{B}(H)) . \tag{7}
\end{equation*}
$$

Quantum QNS correlations are defined as in (7), but using tensor products of stochastic operator matrices acting on finite dimensional Hilbert spaces (that is, ones having the form $E_{x, x^{\prime}, a, a^{\prime}} \otimes F_{y, y^{\prime}, b, b^{\prime}}$ ). Approximately quantum QNS correlations are limits of quantum QNS correlations, while local QNS
correlations are defined as in (17) by requiring that the entries of $\tilde{E}$ (resp. $\tilde{F}$ ) pairwise commute.

We write $\mathcal{Q}_{\mathrm{qc}}$ (resp. $\mathcal{Q}_{\mathrm{qa}}, \mathcal{Q}_{\mathrm{q}}, \mathcal{Q}_{\text {loc }}$ ) for the (convex) set of all quantum commuting (resp. approximately quantum, quantum, local) QNS correlations. It was shown in 30 that $\Gamma \in \mathcal{Q}_{\mathrm{qc}}$ precisely when there exists a state $s: \mathcal{C}_{X, A} \otimes_{\max } \mathcal{C}_{Y, B} \rightarrow \mathbb{C}$ such that $\Gamma=\Gamma_{s}$, where $\Gamma_{s}$ is given by

$$
\Gamma_{s}\left(\epsilon_{x, x^{\prime}} \otimes \epsilon_{y, y^{\prime}}\right)=\sum_{a, a^{\prime} \in A} \sum_{b, b^{\prime} \in B} s\left(e_{x, x^{\prime}, a, a^{\prime}} \otimes e_{y, y^{\prime}, b, b^{\prime}}\right) \epsilon_{a, a^{\prime}} \otimes \epsilon_{b, b^{\prime}},
$$

where $x, x^{\prime} \in X, y, y^{\prime} \in Y$. Similarly, $\Gamma \in \mathcal{Q}_{\text {qa }}$ precisely when $\Gamma=\Gamma_{s}$ for some state $s$ of $\mathcal{C}_{X, A} \otimes_{\min } \mathcal{C}_{Y, B}$, and $\Gamma \in \mathcal{Q}_{\mathrm{q}}$ (resp. $\Gamma \in \mathcal{Q}_{\text {loc }}$ ) if and only if $\Gamma=\Gamma_{s}$ for some state $s$ of $\mathcal{C}_{X, A} \otimes_{\min } \mathcal{C}_{Y, B}$ that factors through a finite dimensional (resp. abelian) representation of the latter $\mathrm{C}^{*}$-algebra. We point out that the elements of $\mathcal{Q}_{\text {loc }}$ are precisely the quantum channels of the form $\Gamma=\sum_{i=1}^{k} \lambda_{i} \Phi_{i} \otimes \Psi_{i}$ as a convex combination (where $\Phi_{i}: M_{X} \rightarrow M_{A}$ and $\Psi_{i}: M_{Y} \rightarrow M_{B}$ are quantum channels, $\left.i=1, \ldots, k\right)$.

Let $\mathfrak{B}_{X, A}$ (resp. $\mathcal{B}_{X, A}$ ) be the algebraic (resp. the $\mathrm{C}^{*}$-algebraic) free product $M_{A} *_{1} \cdots *_{1} M_{A}$, and $\mathfrak{A}_{X, A}$ (resp. $\mathcal{A}_{X, A}$ ) be the algebraic (resp. the $\mathrm{C}^{*}$-algebraic) free product $\mathcal{D}_{A} *_{1} \cdots *_{1} \mathcal{D}_{A}$, both having $|X|$ terms and amalgamated over the units. We denote by $e_{x, a, a^{\prime}}, a, a^{\prime} \in A$, the matrix units of the $x$-th copy of $M_{A}$ in $\mathfrak{B}_{X, A}$, and by $e_{x, a}, a \in A$, the canonical basis of the $x$-th copy of $\mathcal{D}_{A}$ in $\mathfrak{A}_{X, A}$. Set $E_{\mathrm{cq}}=\left(e_{x, a, a^{\prime}}\right)_{x, a, a^{\prime}} \in \mathcal{D}_{X} \otimes M_{A} \otimes \mathfrak{B}_{X, A}$ and $E_{\mathrm{cq}}^{\mathrm{op}}=\left(e_{x, a^{\prime}, a}^{\mathrm{op}}\right)_{x, a, a^{\prime}} \in \mathcal{D}_{X} \otimes M_{A} \otimes \mathfrak{B}_{X, A}^{\mathrm{op}}$; similarly, let $E_{\mathrm{cl}}=\left(e_{x, a}\right)_{x, a} \in$ $\mathcal{D}_{X A} \otimes \mathfrak{A}_{X, A}$ and $E_{\mathrm{cl}}^{\mathrm{op}}=\left(e_{x, a}^{\mathrm{op}}\right)_{x, a} \in \mathcal{D}_{X A} \otimes \mathfrak{A}_{X, A}^{\mathrm{op}}$.

A classical-to-quantum no-signalling (CQNS) correlation is a channel $\mathcal{E}$ : $\mathcal{D}_{X Y} \rightarrow M_{A B}$ such that (5) and (6) hold true for (traceless) elements $\rho_{X} \in \mathcal{D}_{X}$ and $\rho_{Y} \in \mathcal{D}_{Y}$. A semi-classical stochastic operator matrix acting on a Hilbert space $H$ is a positive block operator matrix $\tilde{E}=\left(E_{x, a, a^{\prime}}\right)_{x, a, a^{\prime}} \in$ $\mathcal{D}_{X} \otimes M_{A}(\mathcal{B}(H))$ with $\operatorname{Tr}_{A} \tilde{E}=I$. A CQNS correlation $\mathcal{E}$ is quantum commuting if its Choi matrix is given as in (7) but employing semi-classical stochastic operator matrices; this is equivalent to the requirement that its canonical extension to a QNS correlation $M_{X Y} \rightarrow M_{A B}$ is quantum commuting, as well as to the existence of a state $s$ of $\mathcal{B}_{X, A} \otimes_{\max } \mathcal{B}_{X, A}$ such that $\mathcal{E}=\Gamma_{s}$, where $\mathcal{E}_{s}$ is the CQNS correlation given by

$$
\Gamma_{s}\left(\epsilon_{x, x} \otimes \epsilon_{y, y}\right)=\sum_{a, a^{\prime} \in A} \sum_{b, b^{\prime} \in B} s\left(e_{x, a, a^{\prime}} \otimes e_{y, b, b^{\prime}}\right) \epsilon_{a, a^{\prime}} \otimes \epsilon_{b, b^{\prime}} .
$$

Similarly, approximately quantum (resp. quantum, local) CQNS correlations have the form $\mathcal{E}_{s}$, where $s$ is a state of $\mathcal{B}_{X, A} \otimes_{\min } \mathcal{B}_{X, A}$ (which in addition gives rise to a finite dimensional and abelain GNS representation, respectively). We denote by $\mathcal{C} \mathcal{Q}_{\text {qc }}$ (resp. $\mathcal{C} \mathcal{Q}_{\mathrm{qa}}, \mathcal{C} \mathcal{Q}_{\mathrm{q}}, \mathcal{C} \mathcal{Q}_{\text {loc }}$ ) the (convex) set of all quantum commuting (resp. approximately quantum, quantum, local) QNS correlations.

Let $\varphi: \mathcal{P}_{X Y} \rightarrow \mathcal{P}_{A B}$ be a quantum non-local game. A QNS correlation $\Gamma: M_{X Y} \rightarrow M_{A B}$ is called a perfect strategy for $\varphi$ if

$$
\left\langle\Gamma(P), \varphi(P)_{\perp}\right\rangle=0, \quad P \in \mathcal{P}_{X Y}
$$

Perfect strategies for classical-to-quantum non-local games are defined analogously [30].
3.2. Quantum output mirror games. We first describe the perfect strategies of quantum output mirror games that lie in the various correlation classes. In the sequel, we fix finite sets $X, Y, A$ and $B$, and for clarity denote the canonical generators of $\mathcal{B}_{Y, B}$ by $f_{y, b, b^{\prime}}, y \in Y, b, b^{\prime} \in B$. We fix a quantum output mirror game $\varphi: \mathcal{P}_{X Y}^{\mathrm{cl}} \rightarrow \mathcal{P}_{A B}$ and let $f: X \rightarrow Y$ and $g: Y \rightarrow X$ be as in Definition 2.3, We write

$$
\varphi\left(\epsilon_{x, x} \otimes \epsilon_{f(x), f(x)}\right)=\sum_{i=1}^{r(x)} \zeta_{U_{i}^{x}} \zeta_{U_{i}^{x}}^{*}, x \in X
$$

where $U_{i}^{x}, i=1, \ldots, r(x), x \in X$, are partial isometries satisfying the relations

$$
\begin{equation*}
\sum_{i=1}^{r(x)}\left(U_{i}^{x}\right)^{*} U_{i}^{x}=\sum_{i=1}^{r(x)} U_{i}^{x}\left(U_{i}^{x}\right)^{*}=I \tag{8}
\end{equation*}
$$

Let $D_{x}:=\sum_{i=1}^{r(x)} U_{i}^{x}$; the relations (8) imply that $D_{x}$ is unitary.
Lemma 3.1. Let $s$ be a state of $\mathcal{B}_{X, A} \otimes_{\max } \mathcal{B}_{Y, B}$ such that $\Gamma_{s}: \mathcal{D}_{X Y} \rightarrow M_{A B}$ is a perfect quantum commuting CQNS strategy for $\varphi$. Let $\pi_{1}: \mathcal{B}_{X, A} \rightarrow$ $\mathcal{B}(H)$ and $\pi_{2}: \mathcal{B}_{Y, B} \rightarrow \mathcal{B}(H)$ be ${ }^{*}$-representations with commuting ranges and $\xi \in H$ be a unit vector such that

$$
\begin{gathered}
s\left(u_{1} \otimes u_{2}\right)=\left\langle\pi_{1}\left(u_{1}\right) \pi_{2}\left(u_{2}\right) \xi, \xi\right\rangle, \quad u_{1} \in \mathcal{B}_{X, A}, u_{2} \in \mathcal{B}_{Y, B}, \\
E_{x}=\left(\pi_{1}\left(e_{x, a, a^{\prime}}\right)\right)_{a, a^{\prime} \in A} \text { and } F_{y}=\left(\pi_{2}\left(f_{y, b, b^{\prime}}\right)\right)_{b, b^{\prime} \in B} . \text { Then } \\
\left(U_{i}^{x} \otimes I\right)^{*} E_{x}\left(e_{a} \otimes \xi\right)=F_{f(x)}^{\mathrm{t}}\left(U_{i}^{x} \otimes I\right)^{*}\left(e_{a} \otimes \xi\right), \quad i=1, \ldots, r(x), a \in A .
\end{gathered}
$$

Proof. Set $P_{i, x}=U_{i}^{x}\left(U_{i}^{x}\right)^{*}$ and $Q_{i, x}=\left(U_{i}^{x}\right)^{*} U_{i}^{x}$; thus,

$$
\sum_{i=1}^{r(x)} P_{i, x}=\sum_{i=1}^{r(x)} Q_{i, x}=I,
$$

that is, $\left(P_{i, x}\right)_{i=1}^{r(x)}$ and $\left(Q_{i, x}\right)_{i=1}^{r(x)}$ are PVM's (in $\left.M_{A}\right)$ for every $x \in X$. We have that, if $\Gamma:=\Gamma_{s}$ then

$$
\begin{equation*}
\Gamma\left(\epsilon_{x, x} \otimes \epsilon_{f(x), f(x)}\right)=\left(\sum_{i=1}^{r(x)} \zeta_{U_{i}^{x}} \zeta_{U_{i}^{x}}^{*}\right) \Gamma\left(\epsilon_{x, x} \otimes \epsilon_{f(x), f(x)}\right)\left(\sum_{i=1}^{r(x)} \zeta_{U_{i}^{x}} \zeta_{U_{i}^{x}}^{*}\right) . \tag{9}
\end{equation*}
$$

Taking traces in (9), we obtain

$$
\begin{aligned}
1 & =\operatorname{Tr}\left(\Gamma\left(\epsilon_{x, x} \otimes \epsilon_{f(x), f(x)}\right)\right) \\
& =\sum_{i, j=1}^{r(x)} \sum_{a, b, a^{\prime}, b^{\prime}} s\left(e_{x, a, a^{\prime}} \otimes f_{f(x), b, b^{\prime}}\right) \operatorname{Tr}\left(\left(\zeta_{U_{j}^{x}} \zeta_{U_{j}^{x}}^{*}\right)\left(\epsilon_{a, a^{\prime}} \otimes \epsilon_{b, b^{\prime}}\right)\left(\zeta_{U_{i}^{x}} \zeta_{U_{i}^{x}}^{*}\right)\right) \\
& =\sum_{i, j=1}^{r(x)} \sum_{a, b, a^{\prime}, b^{\prime}} s\left(e_{x, a, a^{\prime}} \otimes f_{f(x), b, b^{\prime}}\right)\left\langle\zeta_{U_{i}^{x}}, e_{a^{\prime}} \otimes e_{b^{\prime}}\right\rangle\left\langle e_{a} \otimes e_{b}, \zeta_{U_{j}^{x}}\right\rangle\left\langle\zeta_{U_{j}^{x}}, \zeta_{U_{i}^{x}}\right\rangle \\
& =\sum_{i=1}^{r(x)} \sum_{a, b, a^{\prime}, b^{\prime}} s\left(e_{x, a, a^{\prime}} \otimes f_{f(x), b, b^{\prime}}\right)\left\langle\zeta_{U_{i}^{x}}, e_{a^{\prime}} \otimes e_{b^{\prime}}\right\rangle\left\langle e_{a} \otimes e_{b}, \zeta_{U_{i}^{x}}\right\rangle,
\end{aligned}
$$

where we have used the fact that $\left\langle\zeta_{U_{j}^{x}}, \zeta_{U_{i}^{x}}\right\rangle=0$ whenever $i \neq j$. Recall that $s\left(e_{x, a, a^{\prime}} \otimes e_{y, b, b^{\prime}}\right)=\left\langle E_{x, a, a^{\prime}} F_{y, b, b^{\prime}} \xi, \xi\right\rangle$ for all $x \in X, y \in Y, a, a^{\prime} \in A, b, b^{\prime} \in$ $B$. In the sequel, we denote by $T_{a, b}$ the $(a, b)$-entry of a (possibly block operator) matrix $T$. Noting that $\zeta_{U_{i}^{x}}=\left(\sum_{a, b}\left(U_{i}^{x}\right)_{a, b} e_{a} \otimes e_{b}\right) /\left\|U_{i}^{x}\right\|_{2}$ and $\left\|U_{i}^{x}\right\|_{2}^{2}=\operatorname{Tr}\left(U_{i}^{x}\left(U_{i}^{x}\right)^{*}\right)$ is the rank $r_{i}(x)$ of the projection $P_{i, x}$, we obtain

$$
\left\langle\zeta_{U_{i}^{x}}, e_{a^{\prime}} \otimes e_{b^{\prime}}\right\rangle=\frac{\left(U_{i}^{x}\right)_{a^{\prime}, b^{\prime}}}{r_{i}(x)^{1 / 2}} \text { and }\left\langle e_{a} \otimes e_{b}, \zeta_{U_{i}^{x}}\right\rangle=\frac{{\left.\overline{\left(U_{i}^{x}\right.}\right)}_{a, b}}{r_{i}(x)^{1 / 2}}
$$

Setting $\tilde{U}_{i}^{x}=U_{i}^{x} \otimes I, i=1, \ldots, r(x), x \in X$, we therefore have

$$
\begin{aligned}
1 & =\sum_{i=1}^{r(x)} \sum_{a, b, a^{\prime}, b^{\prime}}\left\langle E_{x, a, a^{\prime}} F_{f(x), b, b^{\prime}} \xi, \xi\right\rangle\left\langle\zeta_{U_{i}^{x}}, e_{a^{\prime}} \otimes e_{b^{\prime}}\right\rangle\left\langle e_{a} \otimes e_{b}, \zeta_{U_{i}^{x}}\right\rangle \\
& \left.=\sum_{i=1}^{r(x)} \frac{1}{r_{i}(x)} \sum_{a, b, a^{\prime}, b^{\prime}}\left\langle E_{x, a, a^{\prime}}\left(U_{i}^{x}\right)_{a^{\prime}, b^{\prime}} F_{f(x), b, b^{\prime}} \overline{\left(U_{i}^{x}\right.}\right)_{a, b} \xi, \xi\right\rangle \\
& =\sum_{i=1}^{r(x)} \frac{1}{r_{i}(x)} \sum_{a \in A}\left\langle\left(E_{x} \tilde{U}_{i}^{x} F_{f(x)}^{\mathrm{t}}\left(\tilde{U}_{i}^{x}\right)^{*}\right)_{a, a} \xi, \xi\right\rangle \\
& =\sum_{i=1}^{r(x)} \frac{1}{r_{i}(x)} \sum_{a \in A}\left\langle E_{x} \tilde{U}_{i}^{x} F_{f(x)}^{\mathrm{t}}\left(\tilde{U}_{i}^{x}\right)^{*}\left(e_{a} \otimes \xi\right), e_{a} \otimes \xi\right\rangle \\
& =\sum_{i=1}^{r(x)} \frac{1}{r_{i}(x)} \sum_{a \in A}\left\langle F_{f(x)}^{\mathrm{t}}\left(\tilde{U}_{i}^{x}\right)^{*}\left(e_{a} \otimes \xi\right),\left(\tilde{U}_{i}^{x}\right)^{*} E_{x}\left(e_{a} \otimes \xi\right)\right\rangle .
\end{aligned}
$$

By the Cauchy-Schwartz inequality,

$$
\begin{align*}
& 1 \leq \sum_{i=1}^{r(x)} \frac{1}{r_{i}(x)} \sum_{a \in A}\left|\left\langle F_{f(x)}^{\mathrm{t}}\left(\tilde{U}_{i}^{x}\right)^{*}\left(e_{a} \otimes \xi\right),\left(\tilde{U}_{i}^{x}\right)^{*} E_{x}\left(e_{a} \otimes \xi\right)\right\rangle\right|  \tag{10}\\
& \leq\left(\sum_{i=1}^{r(x)} \sum_{a \in A} \frac{1}{r_{i}(x)}\left\|F_{f(x)}^{\mathrm{t}}\left(\tilde{U}_{i}^{x}\right)^{*}\left(e_{a} \otimes \xi\right)\right\|\left\|\left(\tilde{U}_{i}^{x}\right)^{*} E_{x}\left(e_{a} \otimes \xi\right)\right\|\right)^{2} \\
& \leq\left(\sum_{i=1}^{r(x)} \sum_{a \in A} \frac{1}{r_{i}(x)}\left\|\left(\tilde{U}_{i}^{x}\right)^{*} E_{x}\left(e_{a} \otimes \xi\right)\right\|^{2}\right) \\
&= \quad\left(\sum_{a \in A}\left\langle\sum_{i=1}^{r(x)} \sum_{i=1}^{r(x)} \frac{1}{r_{i}(x)} E_{x}^{*} \tilde{U}_{i}^{x}\left(\tilde{U}_{i}^{x}\right)^{*} E_{x}\left(e_{a} \otimes \xi\right), e_{a} \otimes \xi\right\rangle\right) \\
& \times\left(\sum_{a \in A}^{\mathrm{t}}\left\langle\sum_{i=1}^{r(x)} \frac{1}{r_{i}(x)} \tilde{U}_{i}^{x}\left(F_{f(x)}^{\mathrm{t}}\right)^{*}\left(e_{a} \otimes \xi\right) \|^{2}\right)\right. \\
& \mathrm{r} \\
& \mathrm{t}(x) \\
&\left.\left.\left(\tilde{U}_{i}^{x}\right)^{*}\left(e_{a} \otimes \xi\right), e_{a} \otimes \xi\right\rangle\right) .
\end{align*}
$$

Since $\left(P_{i, x}\right)_{i=1}^{r(x)}$ is a PVM, there exist a partition $\left(S_{i}\right)_{i=1}^{r(x)}$ of $A$ with $\left|S_{i}\right|=$ $r_{i}(x)$ and a unitary $V_{x}$ in $M_{A}$ such that $V_{x}^{*} P_{i, x} V_{x}$ coincides with the projection $P_{S_{i}}$ onto $\operatorname{span}\left\{e_{a}: a \in S_{i}\right\}, i=1, \ldots, r(x)$. Let $\tilde{E}_{x}=V_{x}^{*} E_{x} V_{x}$, and write $\tilde{E}_{x}=\sum_{a, b} \epsilon_{a, b} \otimes \tilde{E}_{x, a, b}$. As $V_{x}$ is unitary and $E_{x, a, b} E_{x, a^{\prime}, b^{\prime}}=\delta_{b, a^{\prime}} E_{x, a, b^{\prime}}$, we also have $\tilde{E}_{x, a, b} \tilde{E}_{x, a^{\prime}, b^{\prime}}=\delta_{b, a^{\prime}} \tilde{E}_{x, a, b^{\prime}}$. Thus, we have

$$
\begin{aligned}
\tilde{E}_{x}^{*} P_{S_{i}} \tilde{E}_{x} & =\left(\sum_{a, b} \epsilon_{a, b} \otimes \tilde{E}_{x, a, b}\right)\left(\sum_{c \in S_{i}} \epsilon_{c, c} \otimes 1\right)\left(\sum_{a^{\prime}, b^{\prime}} \epsilon_{a^{\prime}, b^{\prime}} \otimes \tilde{E}_{x, a^{\prime}, b^{\prime}}\right) \\
& =r_{i}(x) \tilde{E}_{x}
\end{aligned}
$$

Let, similarly, $\left(R_{i}\right)_{i=1}^{r(x)}$ be a partition of $B$ with $\left|R_{i}\right|=r_{i}(x)$ and $W_{x}$ be a unitary such that $W_{x}^{*} Q_{x, i} W_{x}=P_{R_{i}}, i=1, \ldots, r(x)$. Setting $\tilde{F}_{f(x)}=$ $W_{x}^{*} F_{f(x)} W_{x}$, we have that $\left(\tilde{F}_{f(x)}^{\mathrm{t}}\right)^{*} P_{R_{i}} \tilde{F}_{f(x)}^{\mathrm{t}}=r_{i}(x) \tilde{F}_{f(x)}^{\mathrm{t}}$. This implies that the last product in (10) is equal to

$$
\begin{aligned}
& \left(\sum_{a \in A}\left\langle E_{x}\left(e_{a} \otimes \xi\right), e_{a} \otimes \xi\right\rangle\right)\left(\sum_{a \in A}\left\langle F_{f(x)}^{\mathrm{t}}\left(e_{a} \otimes \xi\right), e_{a} \otimes \xi\right\rangle\right) \\
& =\left\langle\sum_{a \in A} E_{x, a, a} \xi, \xi\right\rangle\left\langle\sum_{a \in A} F_{f(x), a, a} \xi, \xi\right\rangle=1 .
\end{aligned}
$$

Hence we have equalities in all chains of inequalities which implies that there exist scalars $\lambda_{x}$ such that

$$
F_{f(x)}^{\mathrm{t}}\left(\tilde{U}_{i}^{x}\right)^{*}\left(e_{a} \otimes \xi\right)=\lambda_{x}\left(\tilde{U}_{i}^{x}\right)^{*} E_{x}\left(e_{a} \otimes \xi\right), i=1, \ldots, r(x), a \in A .
$$

Summing up over $i$, we obtain that $\left(D_{x} \otimes I\right) F_{f(x)}^{\mathrm{t}}\left(D_{x}^{*} \otimes I\right)\left(e_{a} \otimes \xi\right)=\lambda_{x} E_{x}\left(e_{a} \otimes\right.$ $\xi$ ) for all $a \in A$. After applying $\operatorname{Tr}_{A}$, we conclude that $\lambda_{x}=1$, which yields the desired result.

Theorem 3.2. Let $\varphi: \mathcal{P}_{X Y}^{\mathrm{cl}} \rightarrow \mathcal{P}_{A B}$ be a quantum output mirror game and $\Gamma: \mathcal{D}_{X Y} \rightarrow M_{A B}$ be a perfect quantum commuting CQNS strategy for $\varphi$. Then there exists a tracial state $\tau: \mathcal{B}_{X, A} \rightarrow \mathbb{C}$ and a ${ }^{*}$-homomorphism $\rho: \mathcal{B}_{Y, B} \rightarrow \mathcal{B}_{X, A}$ such that

$$
\begin{equation*}
\Gamma\left(\epsilon_{x, x} \otimes \epsilon_{y, y}\right)=\left(\tau\left(e_{x, a, a^{\prime}} \rho\left(f_{y, b^{\prime}, b}\right)\right)\right)_{a, a^{\prime}, b, b^{\prime}}, \quad x, y \in X . \tag{11}
\end{equation*}
$$

Proof. We choose $f: X \rightarrow Y$ and $g: Y \rightarrow X$ as in Definition [2.3, and write

$$
\varphi\left(\epsilon_{x, x} \otimes \epsilon_{f(x), f(x)}\right)=\sum_{i=1}^{r(x)} \zeta_{U_{i}^{x}} \zeta_{U_{i}^{x}}^{*}, x \in X,
$$

for partial isometries $U_{i}^{x}, i=1, \ldots, r(x), x \in X$, such that

$$
\sum_{i=1}^{r(x)}\left(U_{i}^{x}\right)^{*} U_{i}^{x}=\sum_{i=1}^{r(x)} U_{i}^{x}\left(U_{i}^{x}\right)^{*}=I
$$

Keeping the notation from the proof of Lemma 3.1, we write $\Gamma\left(\epsilon_{x, x} \otimes \epsilon_{y, y}\right)=$ $\sum_{a, a^{\prime} \in A} \sum_{b, b^{\prime} \in B}\left\langle E_{x, a, a^{\prime}} F_{y, b, b^{\prime}} \xi, \xi\right\rangle$, where the assignments $e_{x, a, a^{\prime}} \mapsto E_{x, a, a^{\prime}} \in$ $\mathcal{B}(H)$ and $f_{y, b, b^{\prime}} \mapsto F_{y, b, b^{\prime}} \in \mathcal{B}(H)$ define ${ }^{*}$-representations of $\mathcal{B}_{X, A}$ and $\mathcal{B}_{Y, B}$, respectively, with commuting ranges. Set $E_{x}=\left(E_{x, a, a^{\prime}}\right)_{a, a^{\prime} \in A}$ and $F_{y}=\left(F_{y, b, b^{\prime}}\right)_{b, b^{\prime} \in B}$. By Lemma 3.1,

$$
F_{f(x)}^{\mathrm{t}}\left(\tilde{U}_{i}^{x}\right)^{*}\left(e_{a} \otimes \xi\right)=\left(\tilde{U}_{i}^{x}\right)^{*} E_{x}\left(e_{a} \otimes \xi\right), i=1, \ldots, r(x), a \in A .
$$

Let $\left.Q=\left(\left(D_{x} \otimes I\right)\left(f_{f(x), a, b}\right)_{(a, b)}^{\mathrm{t}}\right)\left(D_{x}^{*} \otimes I\right)\right)_{a, b}$ and write $Q=\left(q_{x, a, b}\right)_{b, a}$. Set

$$
h_{x, a, b}=e_{x, a, b} \otimes 1-1 \otimes q_{x, b, a}, \quad x \in X, a, b \in A .
$$

We have

$$
\begin{aligned}
h_{x, a, b}^{*} h_{x, a, b} & =\left(e_{x, b, a} \otimes 1-1 \otimes q_{x, a, b}\right)\left(e_{x, a, b} \otimes 1-1 \otimes q_{x, b, a}\right) \\
& =e_{x, b, b} \otimes 1-e_{x, b, a} \otimes q_{x, b, a}-e_{x, a, b} \otimes q_{x, a, b}+1 \otimes q_{x, a, a} .
\end{aligned}
$$

Let $s \in \mathcal{B}_{X, A} \otimes_{\max } \mathcal{B}_{Y, B}$ be such that $\Gamma=\Gamma_{s}$. As

$$
\begin{aligned}
& s\left(e_{x, b, a} \otimes q_{x, b, a}\right)=\left\langle E_{x, b, a}\left(\left(D_{x} \otimes I\right) F_{f(x)}^{t}\left(D_{x}^{*} \otimes I\right)\right)_{a, b} \xi, \xi\right\rangle \\
= & \left\langle\left(\left(D_{x} \otimes I\right) F_{f(x)}^{t}\left(D_{x}^{*} \otimes I\right)\right)_{a, b} \xi, E_{x, a, b} \xi\right\rangle=\left\langle E_{x, a, b} \xi, E_{x, a, b} \xi\right\rangle=\left\langle E_{x, b, b} \xi, \xi\right\rangle,
\end{aligned}
$$

we get

$$
\begin{equation*}
s\left(h_{x, a, b}^{*} h_{x, a, b}\right)=0, \quad x \in X, a, b \in A . \tag{12}
\end{equation*}
$$

For $u, v \in \mathcal{B}_{X, A} \otimes_{\max } \mathcal{B}_{Y, B}$, write $u \sim v$ if $s(u-v)=0$. Equations (12), combined with the Cauchy-Schwarz inequality, imply

$$
u h_{x, a, b} \sim 0 \text { and } h_{x, a, b}^{*} u \sim 0, \quad x \in X, a, b \in A, u \in \mathcal{B}_{X, A} \otimes_{\max } \mathcal{B}_{Y, B}
$$

Since $h_{x, a, b}^{*}=h_{x, b, a}$, we have

$$
\begin{equation*}
u h_{x, a, b} \sim 0 \text { and } h_{x, a, b} u \sim 0, \quad x \in X, a, b \in A, u \in \mathcal{B}_{X, A} \otimes_{\max } \mathcal{B}_{Y, B} \tag{13}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
z e_{x, a, b} \otimes 1 \sim z \otimes q_{x, b, a} \sim e_{x, a, b} z \otimes 1, \quad x \in X, a, b \in A, z \in \mathcal{B}_{X, A} \tag{14}
\end{equation*}
$$

Similarly, let $V_{i}^{y}, i=1, \ldots d(y)$, be partial isometries such that

$$
\sum_{i=1}^{d(y)} V_{i}^{y}\left(V_{i}^{y}\right)^{*}=\sum_{i=1}^{d(y)}\left(V_{i}^{y}\right)^{*} V_{i}^{y}=I
$$

and

$$
\varphi\left(\epsilon_{g(y), g(y)} \otimes \epsilon_{y, y}\right)=\sum_{i=1}^{d(y)} \zeta_{V_{i}^{y}} \zeta_{V_{i}^{y}}^{*}
$$

Similarly to the proof of Lemma 3.1, letting $G_{y}=\sum_{i=1}^{d(y)} V_{i}^{y}$, we obtain that $F_{y, a, b} \xi=\left(\left(G_{y} \otimes I\right)\left(E_{g(y), a, b}\right)_{a, b}^{\mathrm{t}}\left(G_{y}^{*} \otimes I\right)\right)_{a, b} \xi$.

Set $\left(p_{y, a, b}\right)_{b, a}=\left(\left(G_{y} \otimes I\right)\left(e_{g(y), a, b}\right)_{a, b}^{\mathrm{t}}\left(G_{y}^{*} \otimes I\right)\right)_{a, b}$ and note that $\left\{p_{y, a, b}: a, b\right\}$ is a matrix unit system, $y \in Y$. Letting $g_{y, b, b^{\prime}}=p_{y, b, b^{\prime}} \otimes 1-1 \otimes f_{y, b, b^{\prime}}$, where $y \in Y$ and $b, b^{\prime} \in B$, we obtain, similarly,

$$
\begin{equation*}
z p_{y, b, b^{\prime}} \otimes 1 \sim z \otimes f_{y, b, b^{\prime}} \sim p_{y, b, b^{\prime}} z \otimes 1, \quad y \in Y, b, b^{\prime} \in B, z \in \mathcal{B}_{X, A} \tag{15}
\end{equation*}
$$

Let $z$ and $w$ be (finite) words on the set $\mathcal{E}:=\left\{e_{x, a, b}: x \in X, a, b \in A\right\}$. We show by induction on the length $|w|$ of $w$ that

$$
\begin{equation*}
z w \otimes 1 \sim w z \otimes 1 \tag{16}
\end{equation*}
$$

In the case $|w|=1$, the claim reduces to (14). Suppose (16) holds if $|w| \leq$ $n-1$. Let $|w|=n$ and write $w=w^{\prime} e$, where $e \in \mathcal{E}$. Using (14), we have

$$
z w \otimes 1=z w^{\prime} e \otimes 1 \sim e z w^{\prime} \otimes 1 \sim w^{\prime} e z \otimes 1=w z \otimes 1
$$

Let $\tau: \mathcal{B}_{X, A} \rightarrow \mathbb{C}$ be given by $\tau(z)=s(z \otimes 1)$; it is clear that $\tau$ is a state on $\mathcal{B}_{X, A}$. From (16) and the fact that the set of all linear combinations of words on $\mathcal{E}$ is dense in $\mathcal{A}$, we conclude that $\tau$ is a trace on $\mathcal{B}_{X, A}$. Identity (15) implies that

$$
s\left(e_{x, a, a^{\prime}} \otimes f_{y, b, b^{\prime}}\right)=\tau\left(e_{x, a, a^{\prime}} p_{y, b, b^{\prime}}\right), \quad x \in X, y \in Y, a, a^{\prime}, b, b^{\prime} \in A
$$

Equality (11) is now immediate if we let $\rho: \mathcal{B}_{Y, B} \rightarrow \mathcal{B}_{X, A}$ be the ${ }^{*_{-}}$ homomorphism defined by letting $\rho\left(f_{y, b^{\prime}, b}\right)=p_{y, b, b^{\prime}}, y \in Y, b, b^{\prime} \in B$.

We will write $\Gamma=\Gamma_{\rho, \tau}$ if the CQNS correlation $\Gamma: \mathcal{D}_{X Y} \rightarrow M_{A B}$ is given as in (11). Keeping the notation from the proof of Theorem 3.2, let $\pi: \mathcal{B}_{X, A} \rightarrow \mathcal{B}_{Y, B}$ be the ${ }^{*}$-homomorphism given by $\pi\left(e_{x, a, a^{\prime}}\right)=q_{x, a, a^{\prime}}$. We will need the following lemma, which can be thought of as a dilation result for semi-classical stochastic operator matrices.

Lemma 3.3. Let $X$ and $A$ be finite sets and $\left(E_{x, a, a^{\prime}}\right)_{x, a, a^{\prime}}$, where $x \in X$ and $a, a^{\prime} \in A$, be a semi-classical stochastic operator matrix acting on a finite dimensional Hilbert space $H$. Then there exist matrix unit systems $\left(\tilde{E}_{x, a, a^{\prime}}\right)_{a, a^{\prime}}, x \in X$, on a finite dimensional Hilbert space $\tilde{H}$, and an isometry $V: H \rightarrow \tilde{H}$, such that $V^{*} \tilde{E}_{x, a, a^{\prime}} V=E_{x, a, a^{\prime}}$ for all $x \in X$ and all $a, a^{\prime} \in A$.

Proof. Write $X=[k]$ and use induction on $k$. If $k=1$, the result is a direct consequence of the Stinespring Theorem. Resorting to the inductive assumption, suppose that $H_{k-1}$ is a finite dimensional Hilbert space, $V_{k-1}$ : $H \rightarrow H_{k-1}$ is an isometry, and $\left(F_{x, a, a^{\prime}}\right)_{a, a^{\prime}}$ is a matrix unit system on $H_{k-1}$, such that

$$
V_{k-1}^{*} F_{x, a, a^{\prime}} V_{k-1}=E_{x, a, a^{\prime}}, \quad x \in[k-1], a, a^{\prime} \in A .
$$

Let $F_{k, a, a^{\prime}}^{\prime}=V_{k-1} E_{k, a, a^{\prime}} V_{k-1}^{*}, a, a^{\prime} \in A$. Note that $\left(F_{k, a, a^{\prime}}^{\prime}\right)_{a, a^{\prime}} \in\left(M_{A} \otimes\right.$ $\left.\mathcal{B}\left(H_{k-1}\right)\right)^{+}$and $\sum_{a \in A} F_{k, a, a}^{\prime}=P_{k-1}:=V_{k-1} V_{k-1}^{*}$. Fix $a_{0} \in A$ and define

$$
F_{k, a, a^{\prime}}= \begin{cases}F_{k, a_{0}, a_{0}}^{\prime}+P_{k-1}^{\perp} & \text { if } a=a^{\prime}=a_{0} \\ F_{k, a, a^{\prime}}^{\prime} & \text { otherwise } .\end{cases}
$$

Note that $\left(F_{k, a, a^{\prime}}\right)_{a, a^{\prime}}$ is a stochastic operator matrix acting on $H_{k-1}$. In addition,

$$
V_{k-1}^{*} F_{k, a_{0}, a_{0}} V_{k-1}=V_{k-1}^{*}\left(F_{k, a_{0}, a_{0}}^{\prime}+P_{k-1}^{\perp}\right) V_{k-1}=E_{k, a_{0}, a_{0}},
$$

and hence

$$
V_{k-1}^{*} F_{k, a, a^{\prime}} V_{k-1}=E_{k, a, a^{\prime}}, \quad a, a^{\prime} \in A .
$$

By [30, Theorem 3.1], there exists a Hilbert space $K$ and operators $V_{a}$ : $H_{k-1} \rightarrow K$ such that the column operator $V_{k}:=\left(V_{a}\right)_{a \in A}: H_{k-1} \rightarrow K \otimes \mathbb{C}^{A}$ is an isometry, and $\left(F_{k, a, a^{\prime}}\right)_{a, a^{\prime}}=V_{a}^{*} V_{a^{\prime}}, a, a^{\prime} \in A$. Let $\tilde{H}=K \otimes \mathbb{C}^{A}$ and $\tilde{E}_{k, a, a^{\prime}}=I_{K} \otimes \epsilon_{a, a^{\prime}}, a, a^{\prime} \in A$. Then $V_{k}^{*} \tilde{E}_{k, a, a^{\prime}} V_{k}=V_{a}^{*} V_{a^{\prime}}=F_{k, a, a^{\prime}}$ and hence, letting $V=V_{k} V_{k-1}$, we have that $V: H \rightarrow \tilde{H}$ is an isometry such that $V^{*} \tilde{E}_{k, a, a^{\prime}} V=E_{k, a, a^{\prime}}, a, a^{\prime} \in A$.

Let $P_{k}=V_{k} V_{k}^{*}$ and $\tilde{F}_{x, a, a^{\prime}}=V_{k} F_{x, a, a^{\prime}} V_{k}^{*}, x \in[k-1], a, a^{\prime} \in A$. Then

$$
\begin{equation*}
\tilde{F}_{x, a, a^{\prime}} \tilde{F}_{x, b, b^{\prime}}=V_{k} F_{x, a, a^{\prime}} V_{k}^{*} V_{k} F_{x, b, b^{\prime}} V_{k}^{*}=\delta_{a^{\prime}, b} \tilde{F}_{x, a, b^{\prime}}, \quad a, a^{\prime}, b, b^{\prime} \in A, \tag{17}
\end{equation*}
$$

and

$$
\sum_{a \in A} \tilde{F}_{x, a, a}=P_{k} .
$$

Note that, if $x_{0} \in[k-1], a_{0} \in A$ and $l=\operatorname{rank}\left(F_{x_{0}, a_{0}, a_{0}}\right)$, then $\operatorname{rank}\left(P_{k}\right)=$ $l|A|$. It follows that $l=\operatorname{rank}\left(F_{x, a, a}\right)$ for all $x \in[k-1]$ and all $a \in A$. Thus, $P_{k}^{\perp}\left(K \otimes \mathbb{C}^{A}\right)=K_{0} \otimes \mathbb{C}^{A}$ for some Hilbert space with $\operatorname{dim} K_{0}=\operatorname{dim} K-l$.

Let

$$
\tilde{F}_{x, a, a^{\prime}}^{\prime}=I_{K_{0}} \otimes \epsilon_{a, a^{\prime}}, \quad x \in[k-1], a, a^{\prime} \in A,
$$

considered as an operator on $P_{k}^{\perp}\left(K \otimes \mathbb{C}^{A}\right)$, and

$$
\tilde{E}_{x, a, a^{\prime}}:=\tilde{F}_{x, a, a^{\prime}}+\tilde{F}_{x, a, a^{\prime}}^{\prime}, \quad x \in[k-1], a, a^{\prime} \in A .
$$

For $a, a^{\prime}, b, b^{\prime} \in A$ and $x \in[k-1]$, using (17) we have

$$
\begin{aligned}
\tilde{E}_{x, a, a^{\prime}} \tilde{E}_{x, b, b^{\prime}} & =\left(\tilde{F}_{x, a, a^{\prime}}+\tilde{F}_{x, a, a^{\prime}}^{\prime}\right)\left(\tilde{F}_{x, b, b^{\prime}}+\tilde{F}_{x, b, b^{\prime}}^{\prime}\right) \\
& =\tilde{F}_{x, a, a a^{\prime}} \tilde{F}_{x, b, b^{\prime}}+\tilde{F}_{x, a, a^{\prime}}^{\prime} \tilde{F}_{x, b, b^{\prime}}^{\prime} \\
& =\delta_{a^{\prime}, b} \tilde{F}_{x, a, b^{\prime}}+\delta_{a^{\prime}, b} \tilde{F}_{x, a, b^{\prime}}^{\prime}=\delta_{a^{\prime}, b} \tilde{E}_{x, a, b^{\prime}}
\end{aligned}
$$

In addition, for $x \in[k-1]$ and $a, a^{\prime} \in A$ we have

$$
\begin{aligned}
V^{*} \tilde{E}_{x, a, a^{\prime}} V & =V^{*} \tilde{F}_{x, a, a^{\prime}} V+V^{*} \tilde{F}_{x, a, a^{\prime}}^{\prime} V=V^{*} \tilde{F}_{x, a, a^{\prime}} V \\
& =V_{k-1}^{*} V_{k}^{*}\left(V_{k} F_{x, a, a^{\prime}} V_{k}^{*}\right) V_{k} V_{k-1}=E_{x, a, a^{\prime}}
\end{aligned}
$$

Remark. In the notation of Lemma 3.3, if $E_{x, a, a^{\prime}}=\delta_{a, a^{\prime}} E_{x, a, a}$ for all $x, a, a^{\prime}$, the statement reduces to the simultaneous Naimark dilation of a finite family of POVM's exhibited in [23, Theorem 9.8]. We include the following consequence, which will be used later.
Corollary 3.4. Let $X, Y, A$ and $B$ be finite sets. $A C Q N S$ correlation $\Gamma: \mathcal{D}_{X Y} \rightarrow M_{A B}$ is quantum if and only if there exist finite dimensional Hilbert space $H_{X}$ and $H_{Y},{ }^{*}$-representations $\pi_{X}: \mathcal{B}_{X, A} \rightarrow \mathcal{B}\left(H_{X}\right)$ and $\pi_{Y}$ : $\mathcal{B}_{Y, B} \rightarrow \mathcal{B}\left(H_{Y}\right)$, and a unit vector $\xi \in H_{A} \otimes H_{B}$, such that

$$
\Gamma\left(\epsilon_{x, x} \otimes \epsilon_{y, y}\right)=\left(\left\langle\left(\pi_{X}\left(e_{x, a, a^{\prime}}\right) \otimes \pi_{Y}\left(f_{y, b, b^{\prime}}\right)\right) \xi, \xi\right\rangle\right)_{a, a^{\prime}, b, b^{\prime}}, \quad x \in X, y \in Y
$$

Proof. Let $\left(E_{x, a, a^{\prime}}\right)_{x, a, a^{\prime}}$ (resp. $\left.\left(F_{y, b, b^{\prime}}\right)_{y, b, b^{\prime}}\right)$ be a semi-classical stochastic operator matrix acting on finite dimensional Hilbert space $H_{A}$ (resp. $H_{B}$ ) and $\eta \in H_{A} \otimes H_{B}$ be a unit vector such that

$$
\Gamma\left(\epsilon_{x, x} \otimes \epsilon_{y, y}\right)=\left(\left\langle\left(E_{x, a, a^{\prime}} \otimes F_{y, b, b^{\prime}}\right) \eta, \eta\right\rangle\right)_{a, a^{\prime}, b, b^{\prime}}, \quad x \in X, y \in Y .
$$

Let $\left(\tilde{E}_{x, a, a^{\prime}}\right)_{a, a^{\prime}}$ and $V$ (resp. $\left(\tilde{F}_{x, a, a^{\prime}}\right)_{a, a^{\prime}}$ and $W$ ) be the matrix unit systems acting on a finite dimensional Hilbert space $H_{X}$ (resp. $H_{Y}$ ) and the corresponding isometry, obtained via Lemma 3.3, By the universal property of the $\mathrm{C}^{*}$-algebraic free product, there exists a ${ }^{*}$-representation $\pi_{X}: \mathcal{B}_{X, A} \rightarrow$ $\mathcal{B}\left(H_{X}\right)$ (resp. $\left.\pi_{Y}: \mathcal{B}_{Y, B} \rightarrow \mathcal{B}\left(H_{Y}\right)\right)$ such that $\pi_{X}\left(e_{x, a, a^{\prime}}\right)=\tilde{E}_{x, a, a^{\prime}}$ (resp. $\left.\pi_{Y}\left(f_{y, b, b^{\prime}}\right)=\tilde{F}_{y, b, b^{\prime}}\right), x \in X, a, a^{\prime} \in A$ (resp. $y \in Y, b, b^{\prime} \in B$ ). Letting $\xi=(V \otimes W) \eta$, we obtain the required representation of $\Gamma$.

Theorem 3.5. Let $\varphi: \mathcal{P}_{X Y}^{\mathrm{cl}} \rightarrow \mathcal{P}_{A B}$ be a quantum output mirror game, $\tau$ be a tracial state on $\mathcal{B}_{X, A}$ and $\rho: \mathcal{B}_{Y, B} \rightarrow \mathcal{B}_{X, A}$ be a unital ${ }^{*}$-homomorphism such that $\Gamma=\Gamma_{\rho, \tau}$ is a perfect quantum commuting CQNS strategy for $\varphi$. The following hold:
(i) $\Gamma \in \mathcal{C} \mathcal{Q}_{\mathrm{qa}}$ if and only if $\tau$ can be chosen to be amenable;
(ii) $\Gamma \in \mathcal{C} \mathcal{Q}_{\mathrm{q}}$ if and only if $\tau$ can be chosen to factor through a finitedimensional ${ }^{*}$-representation of $\mathcal{B}_{X, A}$.

Proof. (i) Assume that $\Gamma \in \mathcal{C} \mathcal{Q}_{\text {qa }}$. By the Remark after [30, Theorem 7.7], $s$ can be chosen to be a state of $\mathcal{B}_{X, A} \otimes_{\min } \mathcal{B}_{Y, B}$. Let $\partial: \mathcal{B}_{X, A} \rightarrow \mathcal{B}_{X, A}^{\mathrm{op}}$ be the
*-isomorphism given by $\partial\left(e_{x, a, a^{\prime}}\right)=e_{x, a^{\prime}, a}^{\mathrm{op}}$, whose existence is guaranteed by [30, Lemma 9.2]. Let $\phi: \mathcal{B}_{X, A} \otimes_{\min } \mathcal{B}_{X, A}^{\mathrm{op}} \rightarrow \mathbb{C}$ be the state defined by letting

$$
\phi=s \circ(\mathrm{id} \otimes \pi) \circ\left(\mathrm{id} \otimes \partial^{-1}\right)
$$

Let $z \in \mathcal{B}_{X, A}$ and $w=e_{x_{1}, a_{1}, a_{1}^{\prime}} \cdots e_{x_{k}, a_{k}, a_{k}^{\prime}}$, for some $x_{i} \in X, a_{i}, a_{i}^{\prime} \in A$, $i=1, \ldots, k$. Set $\bar{w}:=\partial^{-1}\left(w^{\mathrm{op}}\right)=e_{x_{k}, a_{k}^{\prime}, a_{k}} \cdots e_{x_{1}, a_{1}^{\prime}, a_{1}}$. Using (13), we have

$$
\begin{aligned}
\phi\left(z \otimes w^{\mathrm{op}}\right) & =s(z \otimes \pi(\bar{w}))=s\left(z \otimes q_{x_{k}, a_{k}^{\prime}, a_{k}} \cdots q_{x_{1}, a_{1}^{\prime}, a_{1}}\right) \\
& =\tau\left(z e_{x_{1}, a_{1}, a_{1}^{\prime}} \cdots e_{x_{k}, a_{k}, a_{k}^{\prime}}\right)=\tau(z w) .
\end{aligned}
$$

By linearity and continuity,

$$
\begin{equation*}
\phi\left(z \otimes w^{\mathrm{op}}\right)=\tau(z w), \quad z, w \in \mathcal{B}_{X, A} . \tag{18}
\end{equation*}
$$

By [6, Theorem 6.2.7], $\tau$ is amenable.
Conversely, if $\tau$ is an amenable trace that implements $\Gamma$ then the functional $\phi: \mathcal{B}_{X, A} \otimes_{\max } \mathcal{B}_{X, A} \rightarrow \mathbb{C}$ defined via the identity (18) factors through $\mathcal{B}_{X, A} \otimes_{\min } \mathcal{B}_{X, A} ;$ by the Remark after [30, Theorem 7.7], $\Gamma \in \mathcal{C} \mathcal{Q}_{\text {qa }}$.
(ii) Let $\Gamma: \mathcal{D}_{X Y} \rightarrow M_{A B}$ be a perfect strategy in $\mathcal{C} \mathcal{Q}_{\mathrm{q}}$. By Corollary 3.4, there exist finite dimensional spaces $H$ and $K$, representations $\pi^{\prime}: \mathcal{B}_{X, A} \rightarrow$ $\mathcal{B}(H)$ and $\rho^{\prime}: \mathcal{B}_{Y, B} \rightarrow \mathcal{B}(K)$, and a unit vector $\xi \in H \otimes K$ such that $\Gamma=\Gamma_{s}$, where $s: \mathcal{B}_{X, A} \otimes_{\min } \mathcal{B}_{Y, B}$ is a state such that

$$
\begin{equation*}
s\left(e_{x, a, a^{\prime}} \otimes f_{y, b, b^{\prime}}\right)=\left\langle\left(\pi^{\prime}\left(e_{x, a, a^{\prime}}\right) \otimes \rho^{\prime}\left(f_{y, b, b^{\prime}}\right)\right) \xi, \xi\right\rangle \tag{19}
\end{equation*}
$$

for all $x \in X, y \in Y, a, a^{\prime} \in A, b, b^{\prime} \in B$. The proof of Theorem 3.2 shows that the left marginal of $s$ is a trace on $\mathcal{B}_{X, A}$ that factors through the finite dimensional space $H \otimes K$ and satisfies (11). The converse direction follows from [30, Proposition 9.15].

Remark. In case the bijective projections $\varphi\left(\epsilon_{x, x} \otimes \epsilon_{f(x), f(x)}\right)$ and $\varphi\left(\epsilon_{g(y), g(y)} \otimes\right.$ $\epsilon_{y, y}$ ) from Definition 2.3 have full rank, the corresponding quantum output mirror games reduces to a classical one and has possesses non-trivial local perfect strategies. However, if $|A|>1$ and at least one of those projections has rank smaller than $|A|$, a local perfect strategy does not exist, since local CQNS correlations preserve separability of states.

The following is a partial converse of Theorem 3.2,
Proposition 3.6. Let $\tau: \mathcal{B}_{X, A} \rightarrow \mathbb{C}$ be a tracial state and let $\rho: \mathcal{B}_{Y, B} \rightarrow$ $\mathcal{B}_{X, A}$ be $a^{*}$-homomorphism for which there exist bijections $f: X \rightarrow Y$, $g: Y \rightarrow X$ and unitary operators $U_{x}, V_{y}: \mathbb{C}^{B} \rightarrow \mathbb{C}^{A}, x \in X, y \in Y$, such that $\left(\rho\left(f_{y, b, b^{\prime}}\right)\right)_{b, b^{\prime}}=\left(V_{y}^{*} \otimes I\right)\left(e_{g(y), a, a^{\prime}}\right)_{a, a^{\prime}}\left(V_{y} \otimes I\right)$ and $\left(\rho\left(f_{f(x), b, b^{\prime}}\right)\right)_{b, b^{\prime}}=$ $\left(U_{x}^{*} \otimes I\right)\left(e_{x, a, a^{\prime}}\right)_{a, a^{\prime}}\left(U_{x} \otimes I\right)$. Then $\Gamma_{\rho, \tau}$ is a perfect strategy for the game $\varphi$ given by

$$
\varphi\left(\epsilon_{x, x} \otimes \epsilon_{y, y}\right)= \begin{cases}\zeta_{\bar{U}_{x}} \zeta_{\bar{U}_{x}}^{*} & \text { if } y=f(x), \\ \zeta_{\bar{V}_{y}} \zeta_{\bar{V}_{y}}^{*} & \text { if } x=g(y), \\ I_{A A} & \text { otherwise } .\end{cases}
$$

Proof. We have that

$$
\tau\left(e_{x, a, a^{\prime}} \rho\left(f_{f(x), b^{\prime}, b}\right)\right)_{a, a^{\prime}, b, b^{\prime}}=\left(I \otimes U_{x}^{*}\right)\left(\tau\left(e_{x, a, a^{\prime}} e_{x, b^{\prime}, b}\right)\right)_{a, a^{\prime}, b, b^{\prime}}\left(I \otimes U_{x}\right) .
$$

As $\tau$ is a trace and $\left\{e_{x, a, a^{\prime}}\right\}_{a, a^{\prime} \in A}$ is a family of matrix units, $\tau\left(e_{x, a, a^{\prime}} e_{x, b^{\prime}, b}\right)=$ $\delta_{a^{\prime}, b^{\prime}} \delta_{a, b} \lambda_{x}$, where $\lambda_{x}=\tau\left(e_{x, a, a}\right)=\tau\left(e_{x, b, b}\right)$ for all $a, a^{\prime}, b, b^{\prime} \in A$ and $x \in X$. Hence

$$
\begin{aligned}
& \left\langle\Gamma\left(\epsilon_{x, x} \otimes \epsilon_{f(x), f(x)}\right)\left(e_{a^{\prime}} \otimes e_{b^{\prime}}\right), e_{a} \otimes e_{b}\right\rangle \\
& =\left\langle\left(\tau\left(e_{x, a, a^{\prime}} e_{x, b^{\prime}, b}\right)\right)_{a, a^{\prime}, b, b^{\prime}}\left(I \otimes U_{x}\right)\left(e_{a^{\prime}} \otimes e_{b^{\prime}}\right),\left(I \otimes U_{x}\right)\left(e_{a} \otimes e_{b}\right)\right\rangle \\
& =u_{a^{\prime}, b^{\prime}}^{x} \overline{u_{a, b}^{x}} \lambda_{x}
\end{aligned}
$$

where $U_{x}=\left(u_{a, b}^{x}\right)_{a, b}$. On the other hand,

$$
\left\langle\zeta_{\bar{U}_{x}} \zeta_{\bar{U}_{x}}^{*} e_{a^{\prime}} \otimes e_{b^{\prime}}, e_{a} \otimes e_{b}\right\rangle=\left\langle\zeta_{\bar{U}_{x}}, e_{a} \otimes e_{b}\right\rangle\left\langle e_{a^{\prime}} \otimes e_{b^{\prime}}, \zeta_{\bar{U}_{x}}\right\rangle=u_{a^{\prime}, b^{\prime}}^{x} \overline{u_{a, b}^{x}}
$$

showing that $\Gamma\left(\epsilon_{x, x} \otimes \epsilon_{f(x), f(x)}\right)=\lambda_{x} \zeta_{\bar{U}_{x}} \zeta_{\bar{U}_{x}}^{*}$ and hence for $P=\epsilon_{x, x} \otimes$ $\epsilon_{f(x), f(x)}$, we obtain

$$
\begin{equation*}
\left\langle\Gamma(P), \varphi(P)_{\perp}\right\rangle=0 . \tag{20}
\end{equation*}
$$

Similar arguments give (20) for $P=\epsilon_{g(y), g(y)} \otimes \epsilon_{y, y}$.
The classical-to-quantum concurrency game is the game $\varphi: \mathcal{P}_{X X}^{\mathrm{cl}} \rightarrow \mathcal{P}_{A A}$ defined as follows:

$$
\varphi\left(\epsilon_{x, x} \otimes \epsilon_{y, y}\right)= \begin{cases}J_{A} & \text { if } x=y \\ I_{A A} & \text { if } x \neq y\end{cases}
$$

A CQNS correlation $\Gamma$ will be called concurrent if $\Gamma$ is a perfect strategy for the concurrency game.

Corollary 3.7. Let $\Gamma: \mathcal{D}_{X X} \rightarrow M_{A A}$ be a quantum commuting CQNS correlation. The following are equivalent:
(i) $\Gamma$ is concurrent;
(ii) there exists a tracial state $\tau: \mathcal{B}_{X, A} \rightarrow \mathbb{C}$ such that

$$
\begin{equation*}
\Gamma\left(\epsilon_{x, x} \otimes \epsilon_{y, y}\right)=\left(\tau\left(e_{x, a, a^{\prime}} e_{y, b^{\prime}, b}\right)\right)_{a, a^{\prime}, b, b^{\prime}}, \quad x, y \in X \tag{21}
\end{equation*}
$$

Moreover,
(i') $\Gamma \in \mathcal{C} \mathcal{Q}_{\mathrm{qa}}$ if and only if the trace $\tau$ can be chosen to be amenable;
(ii') $\Gamma \in \mathcal{C} \mathcal{Q}_{\mathrm{q}}$ if and only if $\tau$ can be chosen to factor through a finite dimensional ${ }^{*}$-representation of $\mathcal{B}_{X, A}$.

Proof. (i) $\Rightarrow$ (ii) The concurrency game is a quantum output mirror game with $B=A, f$ and $g$ the identity maps, and $\varphi\left(\epsilon_{x, x} \otimes \epsilon_{x, x}\right)=J_{A}=\zeta_{I_{A}} \zeta_{I_{A}}^{*}$ for every $x \in X$. In this case the ${ }^{*}$-homomorphism $\rho: \mathcal{B}_{X, A} \rightarrow \mathcal{B}_{X, A}$ from the proof of Theorem 3.2 is given by $\rho\left(f_{y, b, b^{\prime}}\right)=e_{y, b, b^{\prime}}$. The statement now follows from Theorem 3.2.
(ii) $\Rightarrow$ (i) Fix $x \in X$ and note that, by the uniqueness of the trace on $M_{A}$, the restriction of $\tau$ to any of the free product terms in the definition of $\mathcal{B}_{X, A}$ coincides with the normalised trace tr; thus,

$$
\begin{equation*}
\tau\left(e_{x, a, a^{\prime}}\right)=\frac{1}{|A|} \delta_{a, a^{\prime}}, \quad a, a^{\prime} \in A . \tag{22}
\end{equation*}
$$

It follows that

$$
\begin{aligned}
\Gamma\left(\epsilon_{x, x} \otimes \epsilon_{x, x}\right) & =\sum_{a, a^{\prime}, b, b^{\prime} \in A} \tau\left(e_{x, a, a^{\prime}} e_{x, b^{\prime}, b}\right) \epsilon_{a, a^{\prime}} \otimes \epsilon_{b, b^{\prime}} \\
& =\sum_{a, b, b^{\prime} \in A} \tau\left(e_{x, a, b^{\prime}} e_{x, b^{\prime}, b}\right) \epsilon_{a, b^{\prime}} \otimes \epsilon_{b, b^{\prime}} \\
& =\sum_{a, b, b^{\prime} \in A} \tau\left(e_{x, a, b}\right) \epsilon_{a, b^{\prime}} \otimes \epsilon_{b, b^{\prime}}=\frac{1}{|A|} \sum_{a, b \in A} \epsilon_{a, b} \otimes \epsilon_{a, b}=J_{A}
\end{aligned}
$$

Statements (i') and (ii') are immediate from statements (i) and (ii) in Theorem 3.5,

Remark 3.8. Factorisable quantum channels were introduced in [1] and have been subsequently studied by a number of authors (see 19 and the references therein). It was shown in [19, Proposition 3.1] that a quantum channel $\Phi: M_{A} \rightarrow M_{A}$ is factorisable if and only if its Choi matrix has the form $\left(\tau\left(p_{a, a^{\prime}} q_{b^{\prime}, b}\right)\right)_{a, a^{\prime}, b, b^{\prime}}$, for some matrix unit systems $\left(p_{a, a^{\prime}}\right)_{a, a^{\prime}}$ and $\left(q_{b, b^{\prime}}\right)_{b, b^{\prime}}$ in $M_{A} *_{1} M_{A}$. It follows that the factorisable quantum channels on $M_{A}$ can be identified with the perfect quantum commuting CQNS strategies for concurrent games with two inputs. Note, in addition, that the perfect quantum commuting strategies of quantum output mirror games with a single input form a subclass of the factorisable quantum channels.

Remark 3.9. Let $\mathcal{A}$ be a unital C*-algebra, equipped with a tracial state $\tau$. Recall [30] that a semi-stochastic $\mathcal{A}$-matrix over $(X, A)$ is a positive matrix $\left(q_{x, a, a^{\prime}}\right)_{x, a, a^{\prime}} \in \mathcal{D}_{X} \otimes M_{A} \otimes \mathcal{A}$ such that $\sum_{a \in A} g_{x, a, a}=1$ for all $x \in X$. A CQNS correlation $\Gamma: \mathcal{D}_{X X} \rightarrow M_{A A}$ is called tracial [30] if there exists a semi-stochastic $\mathcal{A}$-matrix $\left(g_{x, a, a^{\prime}}\right)_{x, a, a^{\prime}}$ such that

$$
\Gamma\left(\epsilon_{x, x} \otimes \epsilon_{y, y}\right)=\sum_{a, a^{\prime}, b, b^{\prime}} \tau\left(g_{x, a, a^{\prime}} g_{y, b^{\prime}, b}\right) \epsilon_{a, a^{\prime}} \otimes \epsilon_{b, b^{\prime}}, \quad x, y \in X .
$$

It follows from Corollary 3.7 that every concurrent quantum commuting CQNS correlation is tracial.
3.3. Algebras of classical-to-quantum games. Let $P \in \mathcal{P}_{X X}^{\mathrm{cl}}$ and $Q \in$ $\mathcal{P}_{A A}$. We define a linear map

$$
\beta_{P, Q}: \mathcal{D}_{X X} \otimes M_{A A} \otimes \mathfrak{B}_{X, A} \otimes \mathfrak{B}_{X, A}^{\mathrm{op}} \rightarrow \mathfrak{B}_{X, A}
$$

by letting

$$
\beta_{P, Q}\left(\omega \otimes u \otimes v^{\mathrm{op}}\right)=\operatorname{Tr}(\omega(P \otimes Q)) u v, \quad \omega \in \mathcal{D}_{X X} \otimes M_{A A}, u, v \in \mathfrak{B}_{X, A} .
$$

When, in addition, $Q \in \mathcal{P}_{A A}^{\mathrm{cl}}$, define a corresponding map

$$
\alpha_{P, Q}: \mathcal{D}_{X X} \otimes \mathcal{D}_{A A} \otimes \mathfrak{A}_{X, A} \otimes \mathfrak{A}_{X, A}^{\mathrm{op}} \rightarrow \mathfrak{A}_{X, A} .
$$

Both $\beta_{P, Q}$ and $\alpha_{P, Q}$ will be considered as maps on the ampliations of the algebraic tensor products $\mathcal{B}_{X, A} \otimes \mathcal{B}_{X, A}^{\mathrm{op}}$ and $\mathcal{A}_{X, A} \otimes \mathcal{A}_{X, A}^{\mathrm{op}}$, with values in $\mathcal{B}_{X, A}$ and $\mathcal{A}_{X, A}$, respectively.

We use the notation $\langle S\rangle$ to refer to the *-ideal generated by a subset $S$ of a ${ }^{*}$-algebra. If $\varphi: \mathcal{P}_{X X}^{\mathrm{cl}} \rightarrow \mathcal{P}_{A A}$ is a classical-to-quantum game, set

$$
\mathfrak{I}(\varphi)=\left\langle\left\{\beta_{P, \varphi(P)^{\perp}}\left(E_{\mathrm{cq}} \otimes E_{\mathrm{cq}}^{\mathrm{op}}\right): P \in \mathcal{P}_{X X}^{\mathrm{cl}}\right\}\right\rangle \subseteq \mathfrak{B}_{X, A},
$$

and let $\mathcal{I}(\varphi)$ be the closure of $\mathfrak{I}(\varphi)$ in $\mathcal{B}_{X, A}$. Set $\mathfrak{B}(\varphi)=\mathfrak{B}_{X, A} / \mathfrak{I}(\varphi)$ and $\mathcal{B}(\varphi)=\mathcal{B}_{X, A} / \mathcal{I}(\varphi)$. Define $\mathfrak{A}(\varphi)$ and $\mathcal{A}(\varphi)$ similarly, using the ideal

$$
\left\langle\left\{\alpha_{P, \varphi(P)^{\perp}}\left(E_{\mathrm{cl}} \otimes E_{\mathrm{cl}}^{\mathrm{op}}\right): P \in \mathcal{P}_{X X}^{\mathrm{cl}}\right\}\right\rangle
$$

of $\mathfrak{A}_{X, A}$.
Given a synchronous non-local game $\lambda: X \times X \times A \times A \rightarrow\{0,1\}$, its *-algebra $\mathcal{A}(\lambda)$ was defined in 12 as the unital ${ }^{*}$-algebra with generators selfadjoint idempotents $e_{x, a}^{\prime}$, where $x \in X, a \in A$, subject to the relations

$$
\sum_{a \in A} e_{x, a}^{\prime}=1 \text { for all } x \in X \text {, and } e_{y, b}^{\prime} e_{z, c}^{\prime}=0 \text { if } \lambda(y, z, b, c)=0
$$

Proposition 3.10. Let $\lambda: X \times X \times A \times A \rightarrow\{0,1\}$ be a synchronous nonlocal game. Then $\mathfrak{A}\left(\varphi_{\lambda}\right)$ (resp. $\mathcal{A}\left(\varphi_{\lambda}\right)$ ) coincides with the ${ }^{*}$-algebra (resp. $C^{*}$-algebra) of the game $\lambda$.

Proof. Let $\mathfrak{A}(\lambda)$ be the ${ }^{*}$-algebra of the game $\lambda$ as defined in [12], and note that $\mathfrak{A}(\lambda)=\mathfrak{A}_{X, A} / \mathfrak{I}(\lambda)$, where

$$
\mathfrak{I}(\lambda)=\left\langle e_{x, a} e_{y, b}: \lambda(x, y, a, b)=0\right\rangle .
$$

We show that

$$
\begin{equation*}
\mathfrak{I}(\lambda)=\Im\left(\varphi_{\lambda}\right) . \tag{23}
\end{equation*}
$$

Note that $\varphi_{\lambda}\left(\epsilon_{x, x} \otimes \epsilon_{y, y}\right)^{\perp}=\sum_{(a, b): \lambda(x, y, a, b)=0} \epsilon_{a, a} \otimes \epsilon_{b, b} ;$ thus,

$$
\alpha_{\epsilon_{x, x} \otimes \epsilon_{y, y}, \varphi_{\lambda}\left(\epsilon_{x, x} \otimes \epsilon_{y, y}\right)^{\perp}}\left(E_{\mathrm{cl}} \otimes E_{\mathrm{cl}}^{\mathrm{op}}\right)=\sum_{(a, b): \lambda(x, y, a, b)=0} e_{x, a} e_{y, b} .
$$

Multipying from the left by $e_{x, a}$ and by $e_{y, b}$ from the right, we conclude that $e_{x, a} e_{y, b} \in \Im\left(\varphi_{\lambda}\right)$ whenever $\lambda(x, y, a, b)=0$; thus, $\Im(\lambda) \subseteq \Im\left(\varphi_{\lambda}\right)$.

Let $P=\sum_{k} \epsilon_{x_{k}, x_{k}} \otimes \epsilon_{y_{k}, y_{k}}$ as a finite sum. Then

$$
\varphi_{\lambda}(P)^{\perp}=\sum_{(a, b): \lambda\left(x_{k}, y_{k}, a, b\right)=0, \forall k} \epsilon_{a, a} \otimes \epsilon_{b, b}
$$

and hence

$$
\alpha_{P, \varphi_{\lambda}(P)^{\perp}}\left(E_{\mathrm{cl}} \otimes E_{\mathrm{cl}}^{\mathrm{op}}\right)=\sum_{(a, b): \lambda\left(x_{k}, y_{k}, a, b\right)=0, \forall k} \sum_{k} e_{x_{k}, a} e_{y_{k}, b}
$$

This shows that $\Im\left(\varphi_{\lambda}\right) \subseteq \Im(\lambda)$, establishing (23).
Remark 3.11. We have $\beta_{J_{X}^{\mathrm{c}}, J_{A}^{\perp}}\left(E_{\mathrm{cq}} \otimes E_{\mathrm{cq}}^{\mathrm{op}}\right)=0$.
Proof. The claim follows from the fact that

$$
\begin{aligned}
\beta_{J_{X}^{\mathrm{cl}, J_{A}}}\left(E_{\mathrm{cq}} \otimes E_{\mathrm{cq}}^{\mathrm{op}}\right) & =\frac{1}{|A|} \sum_{x \in X} \sum_{a, b \in A} e_{x, a, b} e_{x, b, a}=\sum_{x \in X} \sum_{a \in A} e_{x, a, a} \\
& =\sum_{x \in X} \sum_{a, b \in A} e_{x, a, a} e_{x, b, b}=\beta_{J_{X}^{\mathrm{c}}, I_{A A}}\left(E_{\mathrm{cq}} \otimes E_{\mathrm{cq}}^{\mathrm{op}}\right) .
\end{aligned}
$$

Corollary 3.12. Let $\varphi: \mathcal{P}_{X X}^{\mathrm{cl}} \rightarrow \mathcal{P}_{A A}$ be a classical-to-quantum concurrent game. The following are equivalent for a CQNS correlation $\Gamma: \mathcal{D}_{X X} \rightarrow$ $M_{A A}$ :
(i) $\Gamma$ is a perfect quantum commuting (resp. quantum) strategy for $\varphi$;
(ii) there exists trace $\tau$ (resp. a trace $\tau$ that factors through a finite dimensional *-representation) on $\mathcal{B}_{X, A}$ such that (21) holds and

$$
\tau\left(\beta_{P, \varphi(P)^{\perp}}\left(E_{\mathrm{cq}} \otimes E_{\mathrm{cq}}^{\mathrm{op}}\right)\right)=0,
$$

for all $P \in \mathcal{P}_{X X}^{\mathrm{cl}}$.
Proof. We only prove the statement in the case of quantum commuting strategies. Let $\tau$ be the trace of $\mathcal{B}_{X, A}$ that implements $\Gamma$, arising from Theorem [3.2, For any $P \in \mathcal{P}_{X X}^{\mathrm{cl}}$, taking into account the duality relations (2), we have

$$
\begin{aligned}
0 & =\left\langle\Gamma(P), \varphi(P)_{\perp}\right\rangle=\sum_{x, y \in X} \operatorname{Tr}\left(\left(\epsilon_{x, x} \otimes \epsilon_{y, y}\right) P\right)\left\langle\Gamma\left(\epsilon_{x, x} \otimes \epsilon_{y, y}\right), \varphi(P)_{\perp}\right\rangle \\
& =\sum_{x, y \in X} \operatorname{Tr}\left(\left(\epsilon_{x, x} \otimes \epsilon_{y, y}\right) P\right) \sum_{a, a^{\prime}, b, b^{\prime}} \tau\left(e_{x, a, a^{\prime}} e_{y, b^{\prime}, b}\right)\left\langle\epsilon_{a, a^{\prime}} \otimes \epsilon_{b, b^{\prime}}, \varphi(P)_{\perp}\right\rangle \\
& =\sum_{x, y \in X} \operatorname{Tr}\left(\left(\epsilon_{x, x} \otimes \epsilon_{y, y}\right) P\right) \sum_{a, a^{\prime}, b, b^{\prime}} \tau\left(e_{x, a, a^{\prime}} e_{y, b^{\prime}, b}\right) \operatorname{Tr}\left(\left(\epsilon_{a, a^{\prime}} \otimes \epsilon_{b, b^{\prime}}\right) \varphi(P)^{\perp}\right) \\
& =\tau\left(\beta_{P, \varphi(P)^{\perp}}\left(E_{\mathrm{cq}} \otimes E_{\mathrm{cq}}^{\mathrm{op}}\right)\right) .
\end{aligned}
$$

Remark 3.13. Clearly, any trace $\tau$ on $\mathcal{B}(\varphi)$ gives rise to a perfect quantum commuting strategy for $\varphi$. If, in particular, $\tau$ is amenable on $\mathcal{B}(\varphi)$, by [6. Proposition 6.3.5], the induced trace $\tilde{\tau}$ on $\mathcal{B}_{X, A}$ is amenable, and hence the CQNS correlation defined via (21) is approximately quantum. We do not know if any perfect quantum commuting strategy for a non-local game $\varphi$ arises from a trace of $\mathcal{B}(\varphi)$ in general. Similarly we are not aware if any the approximately quantum perfect strategies of a classical-to-quantum non-local game $\varphi$ all arise from amenable traces of $\mathcal{B}(\varphi)$.

## 4. Concurrent quantum games

In this section, we define the ${ }^{*}$-algebra and the $\mathrm{C}^{*}$-algebra of a quantum concurrent game and provide a characterisation of the prefect strategies for this type of games.
4.1. Tracial descriptions. Let $\tau: \mathcal{C}_{X, A} \rightarrow \mathbb{C}$ be a tracial state; then the linear $\operatorname{map} \Gamma_{\tau}: M_{X X} \rightarrow M_{A A}$, given by

$$
\Gamma_{\tau}\left(\epsilon_{x, x^{\prime}} \otimes \epsilon_{y, y^{\prime}}\right)=\sum_{a, a^{\prime}, b, b^{\prime}} \tau\left(e_{x, x^{\prime} a, a^{\prime}} e_{y^{\prime}, y, b^{\prime}, b}\right) \epsilon_{a, a^{\prime}} \otimes \epsilon_{b, b^{\prime}}
$$

is a QNS correlations; the QNS correlations arising in this way were called tracial in [30]. The classes of quantum tracial (resp. locally tracial) QNS correlations are defined by requiring that $\tau$ factors through a finite dimensional (resp. abelian) ${ }^{*}$-representation.

Theorem 4.1. Let $X$ and $A$ be finite sets, $\varphi: \mathcal{P}_{X X} \rightarrow \mathcal{P}_{A A}$ be a concurrent game and $\Gamma: M_{X X} \rightarrow M_{A A}$ be a perfect quantum commuting QNS strategy for $\varphi$. Then there exists a tracial state $\tilde{\tau}: \mathcal{C}_{X, A} \rightarrow \mathbb{C}$ such that $\Gamma=\Gamma_{\tilde{\tau}}$. Moreover,
(i) if $\Gamma \in \mathcal{Q}_{\mathrm{qa}}$ then $\tilde{\tau}$ can be chosen to be amenable;
(ii) if $\Gamma \in \mathcal{Q}_{\mathrm{q}}$ then $\tilde{\tau}$ can be chosen to factor through a finite dimensional *-representation of $\mathcal{C}_{X, A}$;
(iii) if $\Gamma \in \mathcal{Q}_{\text {loc }}$ then $\tilde{\tau}$ can be chosen to factor through an abelian ${ }^{*}$ representation of $\mathcal{C}_{X, A}$.

Proof. Let $\Gamma \in \mathcal{Q}_{\mathrm{qc}}$ be a perfect strategy for $\varphi$. By [30, Theorem 6.3], there exists a state $s: \mathcal{C}_{X, A} \otimes_{\max } \mathcal{C}_{X, A} \rightarrow \mathbb{C}$ such that

$$
\begin{equation*}
\Gamma\left(\epsilon_{x, x^{\prime}} \otimes \epsilon_{y, y^{\prime}}\right)=\sum_{a, a^{\prime}, b, b^{\prime}} s\left(e_{x, x^{\prime}, a, a^{\prime}} \otimes f_{y, y^{\prime}, b, b^{\prime}}\right) \epsilon_{a, a^{\prime}} \otimes \epsilon_{b, b^{\prime}} \tag{24}
\end{equation*}
$$

for all $x, x^{\prime}, y, y^{\prime} \in X$ and all $a, a^{\prime}, b, b^{\prime} \in A$ (for clarity, we use $f_{y, y^{\prime}, b, b^{\prime}}$ to denote the canonical generators of the second copy of $\mathcal{C}_{X, A}$ ). It follows that

$$
\frac{1}{|X|} \sum_{x, y} \sum_{a, a^{\prime}, b, b^{\prime}} s\left(e_{x, y, a, a^{\prime}} \otimes f_{x, y, b, b^{\prime}}\right) \epsilon_{a, a^{\prime}} \otimes \epsilon_{b, b^{\prime}}=J_{A}
$$

and hence

$$
\begin{equation*}
\sum_{x, y} s\left(e_{x, y, a, b} \otimes f_{x, y, a, b}\right)=\frac{|X|}{|A|}, \quad a, b \in A \tag{25}
\end{equation*}
$$

Let $V=\left(v_{a, x}\right)_{a, x}$ be the isometry such that $e_{x, x^{\prime}, a, a^{\prime}}=v_{a, x}^{*} v_{a^{\prime}, x^{\prime}}$. Then

$$
V V^{*}=\left(\sum_{x \in X} v_{a, x} v_{b, x}^{*}\right)_{a, b}
$$

is a projection, and hence

$$
\sum_{x \in X} v_{a, x} v_{a, x}^{*} \leq 1, \quad a \in A
$$

It follows that

$$
\begin{align*}
\sum_{x \in X} e_{y, x, b, a} e_{x, y, a, b} & =\sum_{x \in X} v_{b, y}^{*} v_{a, x} v_{a, x}^{*} v_{b, y}  \tag{26}\\
& \leq v_{b, y}^{*} v_{b, y}=e_{y, y, b, b}
\end{align*}
$$

for all $y \in X$ and all $a, b \in A$. Thus,

$$
\begin{align*}
\sum_{x, y \in X} \sum_{a, b \in A} e_{y, x, b, a} e_{x, y, a, b} & \leq \sum_{y \in X} \sum_{a, b \in A} e_{y, y, b, b} \\
& =|A| \sum_{y \in X} \sum_{b \in A} e_{y, y, b, b}=|X||A| 1 \tag{27}
\end{align*}
$$

Similarly,

$$
\begin{equation*}
\sum_{y \in X} f_{x, y, a, b} f_{y, x, b, a} \leq f_{x, x, a, a} \tag{28}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{x, y \in X} \sum_{a, b \in A} f_{x, y, a, b} f_{y, x, b, a} \leq|X||A| 1 \tag{29}
\end{equation*}
$$

Let

$$
h_{x, y, a, b}=e_{x, y, a, b} \otimes 1-1 \otimes f_{y, x, b, a}, \quad x, y \in X, a, b \in A
$$

Equation (25) and inequalities (27) and (29) imply

$$
\begin{aligned}
& \sum_{x, y, a, b} s\left(h_{x, y, a, b}^{*} h_{x, y, a, b}\right) \\
= & \sum_{x, y, a, b} s\left(\left(e_{y, x, b, a} \otimes 1-1 \otimes f_{x, y, a, b}\right)\left(e_{x, y, a, b} \otimes 1-1 \otimes f_{y, x, b, a}\right)\right) \\
= & \sum_{x, y, a, b} s\left(e_{y, x, b, a} e_{x, y, a, b} \otimes 1+1 \otimes f_{x, y, a, b} f_{y, x, b, a}\right) \\
- & \sum_{x, y, a, b} s\left(e_{y, x, b, a} \otimes f_{y, x, b, a}+e_{x, y, a, b} \otimes f_{x, y, a, b}\right) \\
\leq & 2|X||A| 1-2|X||A| 1=0 .
\end{aligned}
$$

It follows that

$$
\begin{equation*}
s\left(h_{x, y, a, b}^{*} h_{x, y, a, b}\right)=0, \quad x, y \in X, a, b \in A \tag{30}
\end{equation*}
$$

As in the proof of Theorem 3.2, write $u \sim v$ if $s(u-v)=0$ and note that, by (30),

$$
u h_{x, y, a, b} \sim 0 \text { and } h_{x, y, a, b} u \sim 0, \quad x, y \in X, a, b \in A, u \in \mathcal{C}_{X, A} \otimes_{\max } \mathcal{C}_{X, A}
$$

In particular,

$$
\begin{equation*}
z e_{x, y, a, b} \otimes 1 \sim z \otimes f_{y, x, b, a} \sim e_{x, y, a, b} z \otimes 1, \quad x, y \in X, a, b \in A, z \in \mathcal{C}_{X, A} . \tag{31}
\end{equation*}
$$

Using (31) and induction, as in the proof of Theorem 3.2. we conclude that the map $\tilde{\tau}: \mathcal{C}_{X, A} \rightarrow \mathbb{C}$, given by $\tilde{\tau}(z)=s(z \otimes 1)$, is a trace on $\mathcal{C}_{X, A}$. Identity (31) implies

$$
s\left(e_{x, x^{\prime}, a, a^{\prime}} \otimes f_{y, y^{\prime}, b, b^{\prime}}\right)=\tilde{\tau}\left(e_{x, x^{\prime}, a, a^{\prime}} e_{y^{\prime}, y, b^{\prime}, b}\right), x, x^{\prime}, y, y^{\prime} \in X, a, a^{\prime}, b, b^{\prime} \in A .
$$

Statements (i)-(ii) are proved similarly to Corollary 3.7. To see (iii), let $\Gamma$ be a perfect strategy of class $\mathcal{Q}_{\text {loc }}$. We have $\Gamma=\sum_{i=1}^{n} \lambda_{i} \Phi_{i} \otimes \Psi_{i}$ as a convex linear combination of quantum channels $\Phi_{i}, \Psi_{i}: M_{X} \rightarrow M_{A}, i=1, \ldots, n$. Let $\left(\lambda_{x, x^{\prime}, a, a^{\prime}}^{(j)}\right)_{a, a^{\prime}}=\Phi_{j}\left(\epsilon_{x, x^{\prime}}\right), x, x^{\prime} \in X,\left(\mu_{y, y^{\prime}, b, b^{\prime}}^{(j)}\right)_{b, b^{\prime}}=\Psi_{j}\left(\epsilon_{y, y^{\prime}}\right), y, y^{\prime} \in$ $X, \pi_{j}, \rho_{j}: \mathcal{C}_{X, A} \rightarrow \mathbb{C}$ be the ${ }^{*}$-representations given by $\pi_{j}\left(e_{x, x^{\prime}, a, a^{\prime}}\right)=$ $\lambda_{x, x^{\prime}, a, a^{\prime}}^{(j)}$ and $\rho_{j}\left(f_{y, y^{\prime}, b, b^{\prime}}\right)=\mu_{y, y^{\prime}, b, b^{\prime}}^{(j)}$, and $\pi^{\prime}, \rho^{\prime}: \mathcal{C}_{X, A} \rightarrow \mathcal{B}\left(\mathbb{C}^{n}\right)$ be the $*_{-}$ representations given by

$$
\pi^{\prime}(u)=\sum_{j=1}^{n} \pi_{j}(u) \epsilon_{j, j}, \quad \rho^{\prime}(v)=\sum_{j=1}^{n} \rho_{j}(v) \epsilon_{j, j} .
$$

The images of $\pi^{\prime}$ and $\rho^{\prime}$ are abelian. Set $\xi=\sum_{i=1}^{n} \sqrt{\lambda_{i}} e_{i} \otimes e_{i} \in \mathbb{C}^{n} \otimes \mathbb{C}^{n}$; then

$$
\Gamma\left(\epsilon_{x, x^{\prime}} \otimes \epsilon_{y, y^{\prime}}\right)=\left(\left\langle\left(\pi^{\prime}\left(e_{x, x^{\prime}, a, a^{\prime}}\right) \otimes \rho^{\prime}\left(f_{y, y^{\prime}, b, b^{\prime}}\right)\right) \xi, \xi\right\rangle\right)_{a, a^{\prime}, b, b^{\prime}}
$$

and the corresponding state $s$ is given by

$$
s\left(e_{x, x^{\prime}, a, a^{\prime}} \otimes f_{y, y^{\prime}, b^{\prime}, b}\right)=\left\langle\left(\pi^{\prime}\left(e_{x, x^{\prime}, a, a^{\prime}}\right) \otimes \rho^{\prime}\left(f_{y, y^{\prime}, b, b^{\prime}}\right)\right) \xi, \xi\right\rangle .
$$

It follows that the left marginal of $s$ is a trace on $\mathcal{C}_{X, A}$ that factors through the abelian representation $\pi^{\prime}$ of $\mathcal{C}_{X, A}$.

We now assume that $X=A$; we will see that in this case, we can obtain more precise conditions than the ones in Theorem 4.1 that are also sufficient. Let $\mathcal{B}_{X}$ be the universal $\mathrm{C}^{*}$-algebra (usually referred to as the Brown algebra), generated by the elements $u_{a, x}, x, a \in X$ such that the matrix $\left(u_{a, x}\right)_{a, x \in A}$ is unitary. Consider the $\mathrm{C}^{*}$-subalgebra $\mathcal{C}_{X}$ of $\mathcal{B}_{X}$ generated by $p_{x, x^{\prime}, a, a^{\prime}}=u_{a, x}^{*} u_{a^{\prime}, x^{\prime}}, x, x^{\prime}, a, a^{\prime} \in X$. Write $\mathcal{J}$ for the closed ideal of $\mathcal{C}_{X, A}$, generated by the elements

$$
\sum_{x \in X} e_{y, x, b, a} e_{x, y, a, b}-e_{y, y, b, b}, \quad y, a, b \in X
$$

Let $\mathcal{V}_{X, A}$ be the universal TRO of an isometry $\left(v_{a, x}\right)_{a, x}$, as defined in 30, Section 5]. In the sequel, we will consider products $v_{a_{1}, x_{1}}^{\varepsilon_{1}} v_{a_{2}, x_{2}}^{\varepsilon_{2}} \cdots v_{a_{k}, x_{k}}^{\varepsilon_{k}}$, where $\varepsilon_{i}$ is either the empty symbol or $*$, and $\varepsilon_{i} \neq \varepsilon_{i+1}$ for all $i$, as elements of either $\mathcal{V}_{X, A}, \mathcal{V}_{X, A}^{*}, \mathcal{C}_{X, A}$ or the left $\mathrm{C}^{*}$-algebra corresponding to the TRO $\mathcal{V}_{X, A}$.
Lemma 4.2. The map $\pi: e_{x, x^{\prime}, a, a^{\prime}} \mapsto p_{x, x^{\prime}, a, a^{\prime}}, x, x^{\prime}, a, a^{\prime} \in X$ extends to a surjective *-homomorphism $\pi: \mathcal{C}_{X, A} \rightarrow \mathcal{C}_{X}$ with $\operatorname{ker} \pi=\mathcal{J}$.

Proof. Since $U=\left(u_{a, x}\right)$ is unitary and hence an isometry, we have that $E=\left(p_{x, x^{\prime}, a, a^{\prime}}\right)_{x, x^{\prime}, a, a^{\prime}}$ is a stochastic operator matrix; thus, there exists a *-homomorphism $\pi: \mathcal{C}_{X, A} \rightarrow \mathcal{C}_{X}$ such that $\pi\left(e_{x, x^{\prime}, a, a^{\prime}}\right)=p_{x, x^{\prime}, a, a^{\prime}}$. We have

$$
\pi\left(\sum_{x \in X} e_{y, x, b, a} e_{x, y, a, b}-e_{y, y, b, b}\right)=\sum_{x \in X} u_{b, y}^{*} u_{a, x} u_{a, x}^{*} u_{b, y}-u_{b, y}^{*} u_{b, y}=0,
$$

showing that $\mathcal{J} \subseteq \operatorname{ker} \pi$.
For the reverse inclusion, let $\theta: \mathcal{C}_{X, A} \rightarrow \mathcal{B}(K)$ be a unital *-representation. By [30, Lemma 5.1], there exists a block operator matrix $V=\left(V_{a, x}\right)_{a, x}$ that is an isometry, such that $\theta\left(e_{x, x^{\prime}, a, a^{\prime}}\right)=V_{a, x}^{*} V_{a^{\prime}, x^{\prime}}, x, x^{\prime}, a, a^{\prime} \in X$; we write $\theta=\theta_{V}$. Note that $\theta$ annihilates $\mathcal{J}$ if and only if

$$
V_{b, y}^{*} V_{b, y}-\sum_{x \in X} V_{b, y}^{*} V_{a, x} V_{a, x}^{*} V_{b, y}=0, \quad a, b, y \in X .
$$

Letting $D_{a}=1-\sum_{x \in X} V_{a, x} V_{a, x}^{*}$, we have that $D_{a}$ is positive, $D_{a}^{1 / 2} V_{b, y}=0$ and hence

$$
\begin{equation*}
D_{a} V_{b, y}=\left(1-\sum_{x \in X} V_{a, x} V_{a, x}^{*}\right) V_{b, y}=0 . \tag{32}
\end{equation*}
$$

Since $\left(V_{b, y} \otimes I\right)^{*}\left(I-V V^{*}\right)\left(V_{b, y} \otimes I\right) \in M_{X}\left(\theta\left(\mathcal{C}_{X, A}\right)\right)^{+}$and has zeros on its main diagonal, it is the zero operator. In particular,

$$
\left(I-V V^{*}\right)^{1 / 2}\left(V_{b, y} \otimes I\right)=\left(I-V V^{*}\right)\left(V_{b, y} \otimes I\right)=0, \quad y, b \in A,
$$

implying that

$$
\begin{equation*}
\sum_{x \in X} V_{a, x} V_{a^{\prime}, x}^{*} V_{b, y}=0 \text { whenever } a \neq a^{\prime} . \tag{33}
\end{equation*}
$$

The block operator matrix

$$
U:=\left(\begin{array}{cc}
V & I-V V^{*} \\
0 & V^{*}
\end{array}\right)
$$

is unitary; let

$$
U_{a, x}=\left(\begin{array}{cc}
V_{a, x} & \delta_{a, x} I-\sum_{b \in X} V_{a, b} V_{x, b}^{*} \\
0 & V_{x, a}^{*}
\end{array}\right) .
$$

Since $\left(U_{a, x}\right)_{a, x}$ is unitary, it gives rise to a *-representation $\rho_{U}$ of $\mathcal{B}_{X}$ on the Hilbert space $K_{1} \oplus K_{2}$, where $K_{1}=K_{2}=K$. Using (32) and (33), we have

$$
\begin{aligned}
\rho_{U}\left(p_{x, x^{\prime}, a, a^{\prime}}\right)= & U_{a, x}^{*} U_{a^{\prime}, x^{\prime}}=\left(\begin{array}{cc}
V_{a, x}^{*} & 0 \\
\delta_{a, x} I-\sum_{b \in X} V_{x, b} V_{a, b}^{*} & V_{x, a}
\end{array}\right) \\
& \times\left(\begin{array}{cc}
V_{a^{\prime}, x^{\prime}} & \delta_{a^{\prime}, x^{\prime}} I-\sum_{b \in X} V_{a^{\prime}, b} V_{x^{\prime}, b}^{*} \\
0 & V_{x^{\prime}, a^{\prime}}^{*}
\end{array}\right) \\
& =\left(\begin{array}{cc}
\left(\delta_{a, x} I-\sum_{b \in X}^{*} V_{a^{\prime}, x^{\prime}}\right. & * \\
\left.V_{x, b} V_{a, b}^{*}\right) V_{a^{\prime}, x^{\prime}} & *
\end{array}\right) \\
= & \left(\begin{array}{cc}
V_{a, x}^{*} V_{a^{\prime}, x^{\prime}} & * \\
0 & *
\end{array}\right) .
\end{aligned}
$$

It follows that $K_{1}$ is an invariant subspace for $\rho_{U} \mid \mathcal{C}_{X}$, and $\theta_{V}\left(e_{x, x^{\prime}, a, a^{\prime}}\right)=$ $\left.\rho_{U}\left(p_{x, x^{\prime}, a, a^{\prime}}\right)\right|_{K_{1}}$ for all $x, x^{\prime}, a, a^{\prime} \in X$. This yields $\theta_{V}(T)=\left.\rho_{U}(\pi(T))\right|_{K_{1}}$, $T \in \mathcal{C}_{X, A}$. Thus, for a fixed $T \in \mathcal{C}_{X, A}$, we have

$$
\begin{aligned}
\|T+\mathcal{J}\| & =\sup \left\{\left\|\theta_{V}(T)\right\|: V=\left(V_{a, x}\right) \text { isometry with } \theta_{V}(\mathcal{J})=\{0\}\right\} \\
& \leq \sup \left\{\left\|\rho_{U}(\pi(T))\right\|: U=\left(U_{a, x}\right) \text { unitary }\right\}=\|\pi(T)\| .
\end{aligned}
$$

Therefore $T \in \operatorname{ker} \pi$ implies $T \in \mathcal{J}$.
Note that, according to Lemma 4.2, we have $\mathcal{C}_{X, A} / \mathcal{J} \cong \mathcal{C}_{X}$.
Theorem 4.3. Let $X$ be a finite set and $\varphi: \mathcal{P}_{X X} \rightarrow \mathcal{P}_{X X}$ be a concurrent game. A quantum commuting QNS correlation $\Gamma: M_{X X} \rightarrow M_{X X}$ is a perfect strategy for $\varphi$ if and only if there exists a tracial state $\tau: \mathcal{C}_{X} \rightarrow \mathbb{C}$ such that

$$
\Gamma\left(\epsilon_{x, x^{\prime}} \otimes \epsilon_{y, y^{\prime}}\right)=\left(\tau\left(p_{x, x^{\prime} a, a^{\prime}} p_{y^{\prime}, y, b^{\prime}, b}\right)\right)_{a, a^{\prime}, b, b^{\prime}}, \quad x, x^{\prime}, y, y^{\prime} \in X
$$

Moreover,
(i) $\Gamma \in \mathcal{Q}_{\mathrm{q}}$ if and only if $\tau$ can be chosen to factor through a finite dimensional ${ }^{*}$-representation of $\mathcal{C}_{X}$;
(ii) $\Gamma \in \mathcal{Q}_{\text {loc }}$ if and only if $\tau$ can be chosen to factor through an abelian *-representation of $\mathcal{C}_{X}$.

Proof. For clarity, we set $A=X$. Let $\Gamma \in \mathcal{Q}_{q c}$ be a perfect strategy for $\varphi$. Keeping the notation from the proofs of Theorem 4.1 and Lemma 4.2, we see that

$$
s\left(\sum_{x \in X} e_{y, x, b, a} e_{x, y, a, b} \otimes 1\right)=s\left(e_{y, y, b, b} \otimes 1\right)
$$

(for otherwise we would have $\sum_{x, y, a, b} s\left(h_{x, y, a, b}^{*} h_{x, y, a, b}\right)<0$ ). It follows that the trace $\tilde{\tau}$ on $\mathcal{C}_{X, A}$ annihilates the elements

$$
d_{y, a, b}:=e_{y, y, b, b}-\sum_{x \in X} e_{y, x, b, a} e_{x, y, a, b}, \quad y \in X, a, b \in A .
$$

As $d_{y, a, b} \geq 0$, we have that $\tilde{\tau}\left(d_{y, a, b}^{1 / 2} u\right)=0$ for every $u \in \mathcal{C}_{X, A}$ and so $\tilde{\tau}(\mathcal{J})=$ $\{0\}$. By Lemma 4.2, $\tau(\pi(u)):=\tilde{\tau}(u)$ is a well-defined trace on $\mathcal{C}_{X}$. Identity (31) implies that

$$
s\left(e_{x, x^{\prime}, a, a^{\prime}} \otimes f_{y, y^{\prime}, b, b^{\prime}}\right)=\tau\left(p_{x, x^{\prime}, a, a^{\prime}} p_{y^{\prime}, y, b^{\prime}, b}\right), x, x^{\prime}, y, y^{\prime} \in X, a, a^{\prime}, b, b^{\prime} \in A .
$$

Conversely, let $\tau$ be a trace on $\mathcal{C}_{X}$ and $\Gamma: M_{X X} \rightarrow M_{A A}$ be the QNS correlation, given by

$$
\Gamma\left(\epsilon_{x, x^{\prime}} \otimes \epsilon_{y, y^{\prime}}\right)=\sum_{a, a^{\prime}, b, b^{\prime} \in X} \tau\left(p_{x, x^{\prime}, a, a^{\prime}} p_{y^{\prime}, y, b^{\prime}, b}\right) \epsilon_{a, a^{\prime}} \otimes \epsilon_{b, b^{\prime}} .
$$

Write

$$
w_{a, b}=\sum_{x \in X} u_{a, x}^{*} u_{b, x}, \quad a, b \in X .
$$

We have that

$$
\begin{aligned}
\Gamma\left(J_{X}\right) & =\frac{1}{|X|} \sum_{x, y \in X} \sum_{a, a^{\prime}, b, b^{\prime} \in X} \tau\left(p_{x, y, a, a^{\prime}} p_{y, x, b^{\prime}, b}\right) \epsilon_{a, a^{\prime}} \otimes \epsilon_{b, b^{\prime}} \\
& =\frac{1}{|X|} \sum_{x, y \in X} \sum_{a, a^{\prime}, b, b^{\prime} \in X} \tau\left(u_{a, x}^{*} u_{a^{\prime}, y} u_{b^{\prime}, y}^{*} u_{b, x}\right) \epsilon_{a, a^{\prime}} \otimes \epsilon_{b, b^{\prime}} \\
& =\frac{1}{|X|} \sum_{x \in X} \sum_{a, a^{\prime}, b, b^{\prime} \in X} \delta_{a^{\prime}, b^{\prime}} \tau\left(u_{a, x}^{*} u_{b, x}\right) \epsilon_{a, a^{\prime}} \otimes \epsilon_{b, b^{\prime}} \\
& =\frac{1}{|X|} \sum_{x \in X} \sum_{a, b, c \in X} \tau\left(u_{a, x}^{*} u_{b, x}\right) \epsilon_{a, c} \otimes \epsilon_{b, c} \\
& =\frac{1}{|X|} \sum_{a, b, c \in X} \tau\left(w_{a, b}\right) \epsilon_{a, c} \otimes \epsilon_{b, c} \\
& =\frac{1}{|X|}\left(\sum_{a, b \in X} \tau\left(w_{a, b}\right) e_{a} \otimes e_{b}\right)\left(\sum_{c \in X} e_{c} \otimes e_{c}\right)^{*}
\end{aligned}
$$

Since $\Gamma$ is a quantum channel, $\Gamma\left(J_{X}\right)$ is a positive operator and hence

$$
\sum_{a, b \in X} \tau\left(w_{a, b}\right) e_{a} \otimes e_{b}=\sum_{c \in X} e_{c} \otimes e_{c},
$$

implying

$$
\tau\left(w_{a, b}\right)=\delta_{a, b}, \quad a, b \in X
$$

and $\Gamma\left(J_{X}\right)=J_{A}$.
(i)-(ii) If $\mathcal{A}$ is a unital $\mathrm{C}^{*}$-algebra, equipped with a $\operatorname{trace} \tau_{\mathcal{A}}$, and $\rho$ : $\mathcal{C}_{X, A} \rightarrow \mathcal{A}$ is a ${ }^{*}$-homomorphism such that $\tilde{\tau}=\tau_{\mathcal{A}} \circ \rho$, then $\tau_{\mathcal{A}}(\rho(\mathcal{J}))=0$. Let $\tilde{\rho}: \mathcal{C}_{X, A} / \mathcal{J} \rightarrow \mathcal{A} / \rho(\mathcal{J})$ be given by $\tilde{\rho}(u+\mathcal{J})=\rho(u)+\rho(\mathcal{J})$. Then $\tilde{\rho}$ is a ${ }^{*}$-homomorphism and the map $\tau_{\mathcal{A} / \rho(J)}: \mathcal{A} / \rho(\mathcal{J}) \rightarrow \mathbb{C}$, given by $\tau_{\mathcal{A} / \rho(J)}(a+\rho(\mathcal{J})):=\tau_{\mathcal{A}}(a)$, is a well-defined trace on $\mathcal{A} / \rho(\mathcal{J})$. We have $\tau_{\mathcal{A} / \rho(\mathcal{J})}(\tilde{\rho}(\pi(u)))=\tau_{\mathcal{A}}(\rho(u)), u \in \mathcal{C}_{X, A}$. Clearly, if $\mathcal{A}$ is finite-dimensional
(resp. abelian), so is $\mathcal{A} / \rho(\mathcal{J})$. The statements now follow after an inspection of the proof of Theorem 4.1.

We do not know if the approximately quantum perfect strategies for concurrent games admit a characterisation via amenable traces of $\mathcal{C}_{X}$ under the conditions of Theorem 4.3,
4.2. Algebras of quantum games. Similarly to concurrent classical-toquantum games, concurrent quantum games give rise to ${ }^{*}$ - and $\mathrm{C}^{*}$-algebras which we now describe. For $P \in \mathcal{P}_{X X}$ and $Q \in \mathcal{P}_{A A}$, define a linear map

$$
\gamma_{P, Q}: M_{X X} \otimes M_{A A} \otimes \mathfrak{C}_{X, A} \otimes \mathfrak{C}_{X, A}^{\mathrm{op}} \rightarrow \mathfrak{C}_{X, A}
$$

by letting

$$
\gamma_{P, Q}\left(\omega \otimes u \otimes v^{\mathrm{op}}\right)=\operatorname{Tr}(\omega(P \otimes Q)) u v, \quad \omega \in M_{X X} \otimes M_{A A}, u, v \in \mathfrak{C}_{X, A} .
$$

For a quantum game $\varphi: \mathcal{P}_{X X} \rightarrow \mathcal{P}_{A A}$, let

$$
\mathfrak{I}(\varphi)=\left\langle\gamma_{P, \varphi(P)^{\perp}}\left(E \otimes E^{\mathrm{op}}\right): P \in \mathcal{P}_{X X}\right\rangle
$$

be the ${ }^{*}$-ideal in $\mathfrak{C}_{X, A}$ generated by $\gamma_{P, \varphi(Q)^{\perp}}\left(E \otimes E^{\text {op }}\right), P \in \mathcal{P}_{X X}$, and $\mathcal{I}(\varphi)$ be the closed ideal in $\mathcal{C}_{X, A}$ generated by the same set. Write $\mathfrak{C}(\varphi)=$ $\mathfrak{C}_{X, A} / \mathfrak{I}(\varphi)$ (resp. $\left.\mathcal{C}(\varphi)=\mathcal{C}_{X, A} / \mathcal{I}(\varphi)\right)$ for the quotient *-algebra (resp. quotient $\mathrm{C}^{*}$-algebra). Similarly, we define an ideal $I(\varphi)$ in $\mathcal{C}_{X}$ and its quotient, where we write $E$ for $\left(p_{x, x^{\prime}, a, a^{\prime}}\right)_{x, x^{\prime}, a, a^{\prime}} \in M_{X X}\left(\mathcal{C}_{X}\right)$.

Similarly to Corollary 3.12, we obtain the following:
Corollary 4.4. Let $X$ be a finite set and $\varphi: \mathcal{P}_{X X} \rightarrow \mathcal{P}_{X X}$ be a concurrent quantum game. The following are equivalent for a QNS correlation $\Gamma$ : $M_{X X} \rightarrow M_{X X}:$
(i) $\Gamma$ is a perfect quantum commuting (resp. quantum/local) strategy for $\varphi$;
(ii) there exists a trace $\tau$ (resp. a trace $\tau$ that factors through a finite dimensional/abelian ${ }^{*}$-representation) of $\mathcal{C}_{X}$ such that
$\Gamma\left(\epsilon_{x, x^{\prime}} \otimes \epsilon_{y, y^{\prime}}\right)=\left(\tau\left(e_{x, x^{\prime}, a, a^{\prime}} e_{y, y^{\prime}, b^{\prime}, b}\right)\right)_{a, a^{\prime}, b, b^{\prime}}, \quad x, x^{\prime}, y, y^{\prime} \in X$,
and

$$
\tau\left(\gamma_{P, \varphi(P)^{\perp}}\left(E \otimes E^{\mathrm{op}}\right)\right)=0 .
$$

## 5. The quantum graph homomorphism game

In this section, we revisit the quantum graph homomorphism game as introduced in 30, and provide characterisations of its perfect QNS strategies of various classes.
5.1. Characterisations of the existence of perfect strategies. Let $Z$ be a finite set, $H=\mathbb{C}^{Z}$, and recall that $H^{\text {d }}$ stands for the dual (Banach) space of $H$. Let $\theta: H \otimes H \rightarrow \mathcal{L}\left(H^{\mathrm{d}}, H\right)$ be the linear map given by

$$
\theta(\xi \otimes \eta)\left(\zeta^{\mathrm{d}}\right)=\langle\xi, \zeta\rangle \eta, \quad \zeta \in H
$$

We have

$$
\begin{equation*}
\theta((S \otimes T) \zeta)=T \theta(\zeta) S^{\mathrm{d}}, \quad \zeta \in H \otimes H, S, T \in \mathcal{L}(H) \tag{34}
\end{equation*}
$$

We denote by $\mathrm{m}: H \otimes H \rightarrow \mathbb{C}$ the map, given by

$$
\mathrm{m}(\zeta)=\left\langle\zeta, \sum_{z \in Z} e_{z} \otimes e_{z}\right\rangle, \quad \zeta \in H \otimes H
$$

Let also $\mathfrak{f}: H \otimes H \rightarrow H \otimes H$ be the flip operator, given by $\mathfrak{f}(\xi \otimes \eta)=\eta \otimes \xi$.
Definition 5.1. A linear subspace $\mathcal{U} \subseteq H \otimes H$ is called skew if $\mathrm{m}(\mathcal{U})=\{0\}$ and symmetric if $\mathfrak{f}(\mathcal{U})=\mathcal{U}$.

If $\mathcal{U}$ is a symmetric skew subspace of $H \otimes H$ and $\mathcal{S}_{\mathcal{U}}=\theta(\mathcal{U})$ then the subspace $\mathcal{S}_{\mathcal{U}}$ of $\mathcal{L}\left(H^{\mathrm{d}}, H\right)$ has the following properties:

- $T \in \mathcal{S}_{\mathcal{U}} \Longrightarrow \mathrm{d}^{-1} \circ T^{*} \circ \mathrm{~d}^{-1} \in \mathcal{S}_{\mathcal{U}}$, and
- $T \in \mathcal{S}_{\mathcal{U}} \Longrightarrow \sum_{z \in Z}\left\langle(T \circ \mathrm{~d})\left(e_{z}\right), e_{z}\right\rangle=0$.

We call a subspace of $\mathcal{L}\left(H^{\mathrm{d}}, H\right)$ satisfying these properties a twisted operator anti-system, because of its resemblance to operator anti-systems (that is, selfadjoint subspaces of $M_{X}$ each of whose elements has trace zero [3]). Given a twisted operator anti-system $\mathcal{S} \subseteq \mathcal{L}\left(H^{\mathrm{d}}, H\right)$, one has that the subspace $\mathcal{U}_{\mathcal{S}}=\theta^{-1}(\mathcal{S})$ of $H \otimes H$ is symmetric and skew.

Given a graph $G$, let

$$
\mathcal{U}_{G}=\operatorname{span}\left\{e_{x} \otimes e_{y}: x \sim y\right\}
$$

then $\mathcal{U}_{G}$ is a symmetric skew subspace of $\mathbb{C}^{X} \otimes \mathbb{C}^{X}$. We thus consider symmetric skew subspaces of $\mathbb{C}^{X} \otimes \mathbb{C}^{X}$ as a non-commutative version of graphs. We note that a couple of other non-commutative incarnations of graphs were considered in the literature, namely, operator subsystems in $M_{X}$ in [9] - after noting that the subspace

$$
\mathcal{S}_{G}:=\operatorname{span}\left\{\epsilon_{x, x^{\prime}}: x \simeq x^{\prime}\right\}
$$

of $M_{X}$ is an operator system, and operator anti-systems in [29] - after noting that the subspace

$$
\mathcal{S}_{G}^{0}:=\operatorname{span}\left\{\epsilon_{x, x^{\prime}}: x \sim x^{\prime}\right\}
$$

of $M_{X}$ is an operator anti-system. Our use of symmetric skew subspaces, instead of some of these concepts, is dictated by the nature of the definition of QNS correlations, adopted in [10].

We write $P_{\mathcal{U}}$ for the orthogonal projection from $\mathbb{C}^{X} \otimes \mathbb{C}^{X}$ onto $\mathcal{U}$. Let $\mathcal{U}_{\perp} \subseteq\left(\mathbb{C}^{X} \otimes \mathbb{C}^{X}\right)^{\mathrm{d}}$ be the annihilator of $\mathcal{U}$ and write $P_{\mathcal{U}_{\perp}} \in \mathcal{L}\left(\left(\mathbb{C}^{X} \otimes \mathbb{C}^{X}\right)^{\mathrm{d}}\right)$
for the orthogonal projection onto $\mathcal{U}_{\perp}$. Observe that $\zeta^{\mathrm{d}} \in \mathcal{U}_{\perp}$ if and only if $\zeta$ belongs to the orthogonal complement $\mathcal{U}^{\perp}$ of $\mathcal{U}$ in $\mathbb{C}^{X} \otimes \mathbb{C}^{\bar{X}}$. In addition,

$$
P_{\mathcal{U}_{\perp}}=\left(P_{\mathcal{U}}^{\perp}\right)^{\mathrm{d}} .
$$

Let $\mathcal{U} \subseteq \mathbb{C}^{X}$ and $\mathcal{V} \subseteq \mathbb{C}^{A}$ be symmetric skew spaces. The quantum graph homomorphism game $\mathcal{U} \rightarrow \mathcal{V}$ is the quantum non-local game $\varphi_{\mathcal{U} \rightarrow \mathcal{V}}$ : $\mathcal{P}_{X X} \rightarrow \mathcal{P}_{A A}$ determined by

$$
\varphi_{\mathcal{U} \rightarrow \mathcal{V}}(P)= \begin{cases}0 & \text { if } P=0 \\ P_{\mathcal{V}} & \text { if } P \leq \mathcal{P}_{\mathcal{U}} \\ I_{A A} & \text { otherwise }\end{cases}
$$

Definition 5.2. Let $X$ and $A$ be finite sets and $\mathcal{U} \subseteq \mathbb{C}^{X} \otimes \mathbb{C}^{X}, \mathcal{V} \subseteq \mathbb{C}^{A} \otimes$ $\mathbb{C}^{A}$ be symmetric skew subspaces. We say that $\mathcal{U}$ is quantum commuting homomorphic (resp. quantum homomorphic, locally homomorphic) to $\mathcal{V}$, and write $\mathcal{U} \xrightarrow{\text { qc }} \mathcal{V}($ resp. $\mathcal{U} \xrightarrow{\text { q }} \mathcal{V}, \mathcal{U} \xrightarrow{\text { loc }} \mathcal{V})$, if $\varphi_{\mathcal{U} \rightarrow \mathcal{V}}$ has a perfect quantum commuting (resp. quantum, local) tracial strategy.

Given operator anti-systems $\mathcal{S} \subseteq M_{X}$ and $\mathcal{T} \subseteq M_{A}$, Stahlke [29] defines a non-commutative graph homomorphism from $\mathcal{S}$ to $\mathcal{T}$ to be a quantum channel $\Phi: M_{X} \rightarrow M_{A}$ whose family $\left\{M_{i}\right\}_{i=1}^{m}$ of Kraus operators satisfies the conditions

$$
M_{i} \mathcal{S} M_{j}^{*} \subseteq \mathcal{T}, \quad i, j=1, \ldots, m
$$

if such $\Phi$ exists, one writes $\mathcal{S} \rightarrow \mathcal{T}$. We recall the suitable version of this notion for twisted operator anti-systems, described in [30].

Definition 5.3. Let $X$ and $A$ be finite sets, and $\mathcal{S} \subseteq \mathcal{L}\left(\left(\mathbb{C}^{X}\right)^{\mathrm{d}}, \mathbb{C}^{X}\right)$ and $\mathcal{T} \subseteq \mathcal{L}\left(\left(\mathbb{C}^{A}\right)^{\mathrm{d}}, \mathbb{C}^{A}\right)$ be twisted operator anti-systems. $A$ homomorphism from $\mathcal{S}$ into $\mathcal{T}$ is a quantum channel

$$
\Phi: M_{X} \rightarrow M_{A}, \quad \Phi(T)=\sum_{i=1}^{m} M_{i} T M_{i}^{*}
$$

such that

$$
\bar{M}_{j} \mathcal{S} M_{i}^{\mathrm{d}} \subseteq \mathcal{T}, \quad i, j=1, \ldots, m
$$

If $\mathcal{S}$ and $\mathcal{T}$ are twisted operator anti-systems, we write $\mathcal{S} \rightarrow \mathcal{T}$ as in [29] to denote the existence of a homomorphism from $\mathcal{S}$ to $\mathcal{T}$. Further, if $G$ and $H$ are graphs, we write $G \rightarrow H$ if there exists a homomorphism from $G$ to $H$. The following was shown in 30.
Proposition 5.4. Let $X$ and $A$ be finite sets, $\mathcal{U} \subseteq \mathbb{C}^{X} \otimes \mathbb{C}^{X}, \mathcal{V} \subseteq \mathbb{C}^{A} \otimes \mathbb{C}^{A}$ be symmetric skew spaces, and $G, H$ be graphs. The following hold:
(i) $\mathcal{U} \xrightarrow{\text { loc }} \mathcal{V}$ if and only if $\mathcal{S}_{\mathcal{U}} \rightarrow \mathcal{S}_{\mathcal{V}}$;
(ii) $G \rightarrow H$ if and only if $\mathcal{U}_{G} \xrightarrow{\text { loc }} \mathcal{U}_{H}$.

Let $U_{X}:\left(\mathbb{C}^{X}\right)^{\mathrm{d}} \rightarrow \mathbb{C}^{X}$ be the unitary operator given on the standard basis by $U_{X} e_{x}^{\mathrm{d}}=e_{x}, x \in X$, and define $U_{A}:\left(\mathbb{C}^{A}\right)^{\mathrm{d}} \rightarrow \mathbb{C}^{A}$ similarly. Then $\mathcal{S} \subseteq \mathcal{L}\left(\left(\mathbb{C}^{X}\right)^{\mathrm{d}}, \mathbb{C}^{X}\right)$ is a twisted operator anti-system if and only if the space $\mathcal{S} U_{X}^{-1}$ of $M_{X}$ has the following properties:

- $T \in \mathcal{S} U_{X}^{-1} \Rightarrow T^{\mathrm{t}} \in \mathcal{S} U_{X}^{-1}$;
- $T \in \mathcal{S} U_{X}^{-1} \Rightarrow \operatorname{Tr}(T)=0$.

Indeed, the first property is a direct consequence of the fact that

$$
\begin{aligned}
\mathrm{d}^{-1} \circ\left(T U_{X}\right)^{*} \circ \mathrm{~d}^{-1} U_{X}^{-1} e_{x} & =\mathrm{d}^{-1} \circ U_{X}^{*}\left(T^{*} e_{x}\right)=\sum_{y \in X} \overline{\left\langle T^{*} e_{x}, e_{y}\right\rangle} e_{y} \\
& =\sum_{y \in X} t_{x, y} e_{y}=T^{\mathrm{t}} e_{x},
\end{aligned}
$$

while the second one follows directly from the definition of a twisted operator anti-system.

Recall from Section 2 that $\mathfrak{m}_{Z}$ denotes the (normalised) maximally entangled vector in $\mathbb{C}^{Z} \otimes \mathbb{C}^{Z}$. For a symmetric skew space $\mathcal{U} \subseteq \mathbb{C}^{X}$, set

$$
\mathcal{U} \otimes \mathfrak{m}_{Z}=\left\{\xi \otimes \mathfrak{m}_{Z}: \xi \in \mathcal{U}\right\} ;
$$

after applying the shuffle map, we view $\mathcal{U} \otimes \mathfrak{m}_{Z}$ as a symmetric skew subspace of $\left(\mathbb{C}^{X} \otimes \mathbb{C}^{Z}\right) \otimes\left(\mathbb{C}^{X} \otimes \mathbb{C}^{Z}\right)$.
Theorem 5.5. Let $X$ and $A$ be finite sets and $\mathcal{U} \subseteq \mathbb{C}^{X} \otimes \mathbb{C}^{X}, \mathcal{V} \subseteq \mathbb{C}^{A} \otimes \mathbb{C}^{A}$ be symmetric skew spaces. The following are equivalent:
(i) $\mathcal{U} \xrightarrow{q} \mathcal{V}$;
(ii) $\mathcal{U} \otimes \mathfrak{m}_{Z} \xrightarrow{\text { loc }} \mathcal{V}$ for some finite set $Z$.

Proof. (i) $\Rightarrow$ (ii) Let $\Gamma: M_{X X} \rightarrow M_{A A}$ be a tracial quantum QNS correlation such that

$$
\left\langle\Gamma\left(P_{\mathcal{U}}\right), P_{\mathcal{V}_{\perp}}\right\rangle=0,
$$

that is, such that

$$
\left\langle\Gamma\left(\xi \xi^{*}\right) \eta, \eta\right\rangle=\left\langle\Gamma\left(\xi \xi^{*}\right),\left(\eta \eta^{*}\right)^{\mathrm{d}}\right\rangle=0, \quad \xi \in \mathcal{U}, \eta \in \mathcal{V}^{\perp}
$$

By definition of tracial quantum QNS correlation, there exists a finite dimensional $C^{*}$-algebra $\mathcal{A}$, a tracial state $\tau_{\mathcal{A}}$ on $\mathcal{A}$ and a ${ }^{*}$-homomorphism $\pi: \mathcal{C}_{X, A} \rightarrow \mathcal{A}$ such that

$$
\Gamma\left(\epsilon_{x, x^{\prime}} \otimes \epsilon_{y, y^{\prime}}\right)=\left(\tau_{\mathcal{A}}\left(\pi\left(e_{x, x^{\prime}, a, a^{\prime}} e_{y^{\prime}, y, b^{\prime}, b}\right)\right)\right)_{a, a^{\prime}, b, b^{\prime}} .
$$

Writing $\xi=\sum_{x, y \in X} \alpha_{x, y} e_{x} \otimes e_{y}$ and $\eta=\sum_{a, b \in A} \eta_{a, b} e_{a} \otimes e_{b}$, we have

$$
\Gamma\left(\xi \xi^{*}\right)=\sum_{a, a^{\prime}, b, b^{\prime}} \sum_{x, x^{\prime}, y, y^{\prime}} \tau_{\mathcal{A}}\left(\pi\left(e_{x, x^{\prime}, a, a^{\prime}} e_{y^{\prime}, y, b^{\prime}, b}\right)\right) \alpha_{x, y} \overline{\alpha_{x^{\prime}, y^{\prime}}} \epsilon_{a, a^{\prime}} \otimes \epsilon_{b, b^{\prime}} .
$$

Let

$$
Y_{\bar{\xi}}:=\sum_{x^{\prime}, y^{\prime} \in X}{\overline{\alpha_{x^{\prime}, y^{\prime}}}}_{\epsilon_{x^{\prime}, y^{\prime}}, \quad Y_{\eta}=\sum_{a^{\prime}, b^{\prime} \in A} \eta_{a^{\prime}, b^{\prime}} \epsilon_{a^{\prime}, b^{\prime}}}
$$

and $E=\left(\pi\left(e_{x, x^{\prime}, a, a^{\prime}}\right)\right)_{x, x^{\prime}, a, a^{\prime}}$; then $E$ is a stochastic $\mathcal{A}$-matrix. Observe that

$$
\begin{equation*}
\theta(\xi) U_{X}^{-1}=\sum_{x, y \in X} \alpha_{x, y} \theta\left(e_{x} \otimes e_{y}\right) U_{X}^{-1}=\sum_{x, y \in X} \alpha_{x, y} \epsilon_{y, x}=Y_{\tilde{\xi}}^{*} \tag{35}
\end{equation*}
$$

We have

$$
\begin{aligned}
0= & \left\langle\Gamma\left(\xi \xi^{*}\right) \eta, \eta\right\rangle \\
= & \sum_{a, a^{\prime}, b, b^{\prime}} \sum_{x, x^{\prime}, y, y^{\prime}} \tau_{\mathcal{A}}\left(\pi\left(e_{x, x^{\prime}, a, a}\right) \pi\left(e_{y^{\prime}, y, b, b}\right)\right) \alpha_{x, y} \overline{\alpha_{x^{\prime}, y^{\prime}}} \eta_{a^{\prime}, b^{\prime}} \overline{\eta_{a, b}} \\
= & \left(\operatorname{Tr} \otimes \tau_{\mathcal{A}}\right)\left(\left(\sum_{x^{\prime}, y^{\prime}, a^{\prime}, b^{\prime}} \pi\left(e_{x, x^{\prime}, a, a^{\prime}} \overline{\alpha_{x^{\prime}, y^{\prime}}} \eta_{a^{\prime}, b^{\prime}} \pi\left(e_{y^{\prime}, y, b^{\prime}, b}\right)\right)_{x, y, a, b}\right.\right. \\
& \left.\times\left(\sum_{x, y, a, b} \overline{\alpha_{x, y}} \eta_{a, b} \epsilon_{x, y} \otimes \epsilon_{a, b} \otimes 1_{\mathcal{A}}\right)^{*}\right) \\
= & \left(\operatorname{Tr} \otimes \tau_{\mathcal{A}}\right)\left(E\left(Y_{\bar{\xi}} \otimes Y_{\eta} \otimes 1_{\mathcal{A}}\right) E\left(Y_{\bar{\xi}}^{*} \otimes Y_{\eta}^{*} \otimes 1_{\mathcal{A}}\right)\right) .
\end{aligned}
$$

After passing to a quotient, we may assume that $\mathcal{A}$ is faithfully represented on a Hilbert space $H$ and $\tau_{\mathcal{A}}$ is faithful. As $E$ is positive, we have

$$
E^{1 / 2}\left(Y_{\bar{\xi}} \otimes Y_{\eta} \otimes 1_{\mathcal{A}}\right) E\left(Y_{\bar{\xi}}^{*} \otimes Y_{\eta}^{*} \otimes 1_{\mathcal{A}}\right) E^{1 / 2}=0
$$

It follows that $E^{1 / 2}\left(Y_{\bar{\xi}} \otimes Y_{\eta} \otimes 1_{\mathcal{A}}\right) E^{1 / 2}=0$ and hence $E\left(Y_{\tilde{\xi}}^{*} \otimes Y_{\eta}^{*} \otimes 1_{\mathcal{A}}\right) E=0$.
Define a linear map $\psi: M_{A} \rightarrow M_{X} \otimes \mathcal{A}$ by letting

$$
\psi\left(\epsilon_{a, b}\right)=E_{a, b}:=\left(\pi\left(e_{x, x^{\prime}, a, b}\right)\right)_{x, x^{\prime}} ;
$$

by Choi's Theorem, $\psi$ is a unital completely positive map. Let $\psi(\omega)=$ $\sum_{i=1}^{m} M_{i} \omega M_{i}^{*}$ be a Kraus representation (here $M_{i}: \mathbb{C}^{A} \rightarrow \mathbb{C}^{X} \otimes H, i=$ $1, \ldots, m)$, and set

$$
X_{a, b, i, j}=\sum_{a^{\prime}, b^{\prime} \in A} \overline{\eta_{b^{\prime}, a^{\prime}}} \epsilon_{a, a^{\prime}} M_{j}^{*}\left(Y_{\tilde{\xi}}^{*} \otimes 1_{\mathcal{A}}\right) M_{i} \epsilon_{b^{\prime}, b}, \quad a, b \in A, i, j \in[m] .
$$

Let $\sigma^{1,2}: \mathbb{C}^{A} \otimes \mathbb{C}^{X} \otimes H \rightarrow \mathbb{C}^{X} \otimes \mathbb{C}^{A} \otimes H$ be the flip operator defined on the elementary tensors by $\sigma^{1,2}\left(\xi_{1} \otimes \xi_{2} \otimes \xi_{3}\right)=\xi_{2} \otimes \xi_{1} \otimes \xi_{3}$, and write $M_{i}^{1,3}: \mathbb{C}^{A} \otimes \mathbb{C}^{A} \rightarrow \mathbb{C}^{X} \otimes \mathbb{C}^{A} \otimes H$ for the operator $\sigma^{1,2}\left(1 \otimes M_{i}\right)$.

We have

$$
\begin{aligned}
& \operatorname{Tr}\left(X_{a, b, i, j} X_{a, b, i, j}^{*}\right) \\
& =\sum_{a^{\prime}, b^{\prime}, a^{\prime \prime}, b^{\prime \prime}} \overline{\eta_{b^{\prime}, a^{\prime}}} \eta_{b^{\prime \prime}, a^{\prime \prime}} \operatorname{Tr}\left(\epsilon_{a, a^{\prime}} M_{j}^{*}\left(Y_{\bar{\xi}}^{*} \otimes 1_{\mathcal{A}}\right) M_{i} \epsilon_{b^{\prime}, b} \epsilon_{b, b^{\prime \prime}} M_{i}^{*}\left(Y_{\bar{\xi}} \otimes 1_{\mathcal{A}}\right) M_{j} \epsilon_{a^{\prime \prime}, a}\right) .
\end{aligned}
$$

Letting $\epsilon_{a}=\left(\epsilon_{a, a^{\prime}}\right)_{a^{\prime} \in A}$, considered as a row operator over $M_{A}$, we have

$$
\begin{aligned}
& \sum_{i, j=1}^{m} \operatorname{Tr}\left(X_{a, b, i, j} X_{a, b, i, j}^{*}\right) \\
= & \sum_{j, a^{\prime}, b^{\prime}, a^{\prime \prime}, b^{\prime \prime}} \overline{\eta_{b^{\prime}, a^{\prime}}} \eta_{b^{\prime \prime}, a^{\prime \prime}} \operatorname{Tr}\left(\epsilon_{a, a^{\prime}} M_{j}^{*}\left(Y_{\bar{\xi}}^{*} \otimes 1_{\mathcal{A}}\right) E_{b^{\prime}, b^{\prime \prime}}\left(Y_{\bar{\xi}} \otimes 1_{\mathcal{A}}\right) M_{j} \epsilon_{a^{\prime \prime}, a}\right) \\
= & \sum_{j=1}^{m} \operatorname{Tr}\left(\epsilon_{a}\left(M_{j}^{1,3}\right)^{*}\left(Y_{\bar{\xi}}^{*} \otimes Y_{\eta}^{*} \otimes 1_{\mathcal{A}}\right) E\left(Y_{\bar{\xi}} \otimes Y_{\eta} \otimes 1_{\mathcal{A}}\right) M_{j}^{1,3} \epsilon_{a}^{*}\right) .
\end{aligned}
$$

Write $R_{a, j}=E^{1 / 2}\left(Y_{\bar{\xi}} \otimes Y_{\eta} \otimes 1_{\mathcal{A}}\right) M_{j}^{1,3} \epsilon_{a}^{*}$. Then

$$
\begin{aligned}
& \sum_{j=1}^{m}\left(\operatorname{Tr} \otimes \tau_{\mathcal{A}}\right)\left(R_{a, j} R_{a, j}^{*}\right) \\
= & \sum_{j=1}^{m} \operatorname{Tr}\left(E^{1 / 2}\left(Y_{\bar{\xi}} \otimes Y_{\eta} \otimes 1_{\mathcal{A}}\right) M_{j}^{1,3} \epsilon_{a}^{*} \epsilon_{a}\left(M_{j}^{1,3}\right)^{*}\left(Y_{\bar{\xi}}^{*} \otimes Y_{\eta}^{*} \otimes 1_{\mathcal{A}}\right) E^{1 / 2}\right) \\
= & \left(\operatorname{Tr} \otimes \tau_{\mathcal{A}}\right)\left(E^{1 / 2}\left(Y_{\bar{\xi}} \otimes Y_{\eta} \otimes 1_{\mathcal{A}}\right) E\left(Y_{\bar{\xi}}^{*} \otimes Y_{\eta}^{*} \otimes 1_{\mathcal{A}}\right) E^{1 / 2}\right)=0,
\end{aligned}
$$

giving $R_{a, j}=0$, as we assume that the trace is faithful and therefore $\sum_{i, j=1}^{m} \operatorname{Tr}\left(X_{a, b, i, j} X_{a, b, i, j}^{*}\right)=0$ implying

$$
\begin{equation*}
X_{a, b, i, j}=0, \quad a, b \in A, i, j=1, \ldots, m \tag{36}
\end{equation*}
$$

We may assume that $\mathcal{A}$ is faithfully represented on $\mathbb{C}^{Z}$ and so that $M_{i}$ is an operator from $\mathbb{C}^{A}$ into $\mathbb{C}^{X} \otimes \mathbb{C}^{Z}$. Let $R_{j}=\overline{M_{j}^{*}}, j=1, \ldots, m$. For $a \in A$ we have

$$
\begin{aligned}
\left(U_{X} \otimes U_{Z}\right) R_{i}^{\mathrm{d}} U_{A}^{-1}\left(e_{a}\right) & =\left(U_{X} \otimes U_{Z}\right) R_{i}^{\mathrm{d}}\left(e_{a}^{\mathrm{d}}\right)=\left(U_{X} \otimes U_{Z}\right)\left(R_{i}^{*} e_{a}\right)^{\mathrm{d}} \\
& =\left(U_{X} \otimes U_{Z}\right)\left(\bar{M}_{i} e_{a}\right)^{\mathrm{d}}=M_{i} e_{a} .
\end{aligned}
$$

Taking into account (35), we obtain

$$
\begin{aligned}
\overline{R_{j}} \theta\left(\xi \otimes \mathfrak{m}_{Z}\right) R_{i}^{\mathrm{d}} & =\overline{R_{j}}\left(\theta(\xi) U_{X}^{-1} \otimes \theta\left(\mathfrak{m}_{Z}\right) U_{Z}^{-1}\right)\left(\left(U_{X} \otimes U_{Z}\right) R_{i}^{\mathrm{d}} U_{A}^{-1} U_{A}\right. \\
& =M_{j}^{*}\left(Y_{\bar{\xi}}^{*} \otimes 1\right)\left(\left(U_{X} \otimes U_{Z}\right) R_{i}^{\mathrm{d}} U_{A}^{-1} U_{A}\right. \\
& =M_{j}^{*}\left(Y_{\bar{\xi}}^{*} \otimes 1\right) M_{i} U_{A} .
\end{aligned}
$$

Since $\psi$ is unital,

$$
\sum_{j=1}^{m} R_{j}^{*} R_{j}=\sum_{j=1}^{m} \overline{M_{j} M_{j}^{*}}=I
$$

and hence the map $\omega \mapsto \sum_{j=1}^{m} R_{j} \omega R_{j}^{*}$ from $M_{X Z}$ into $M_{A}$ is a quantum channel. We claim that

$$
\begin{equation*}
\overline{R_{j}} \theta\left(\xi \otimes \mathfrak{m}_{Z}\right) R_{i}^{\mathrm{d}} \subseteq \mathcal{S}_{\mathcal{V}} \tag{38}
\end{equation*}
$$

Indeed, fix $\eta \in \mathcal{V}^{\perp}$. Since $\theta(\eta) U_{A}^{-1}=Y_{\bar{\eta}}^{*}$, taking (36) and (37) into account, we have

$$
\begin{aligned}
\left\langle\overline{R_{j}} \theta\left(\xi \otimes \mathfrak{m}_{Z}\right) R_{i}^{\mathrm{d}}, \theta(\eta)\right\rangle & \left.=\operatorname{Tr}\left(M_{j}^{*}\left(Y_{\bar{\xi}}^{*} \otimes 1\right)\right) M_{i} U_{A} \theta(\eta)^{*}\right) \\
& =\operatorname{Tr}\left(M_{j}^{*}\left(Y_{\bar{\xi}}^{*} \otimes 1\right) M_{i} Y_{\bar{\eta}}\right) \\
& =\sum_{a^{\prime}, b^{\prime} \in A} \overline{\eta_{b^{\prime}, a^{\prime}}}\left\langle M_{j}^{*}\left(Y_{\bar{\xi}}^{*} \otimes 1\right) M_{i} e_{b^{\prime}}, e_{a^{\prime}}\right\rangle \\
& =\left\langle X_{a, b, i, j} e_{b}, e_{a}\right\rangle=0 ;
\end{aligned}
$$

(38) now follows.
(ii) $\Rightarrow$ (i) By Proposition 5.4.

$$
\begin{equation*}
\mathcal{S}_{\mathcal{U} \otimes \mathfrak{m}_{Z}} \rightarrow \mathcal{S}_{\mathcal{V}} . \tag{39}
\end{equation*}
$$

Let $\left(R_{i}\right)_{i=1}^{m} \subseteq \mathcal{L}\left(\mathbb{C}^{X} \otimes \mathbb{C}^{Z}, \mathbb{C}^{A}\right)$ be a family of Kraus operators of the quantum channel implementing (39). Keeping the notation from the previous paragraphs and reversing the arguments therein, we see that, if $M_{i}=\overline{R_{i}^{*}}$, $\xi \in \mathcal{U}$ and $\eta \in \mathcal{V}^{\perp}$, then

$$
\begin{equation*}
\sum_{a^{\prime}, b^{\prime} \in A} \overline{\eta_{b^{\prime}, a^{\prime}}}\left\langle M_{j}^{*}\left(Y_{\bar{\xi}}^{*} \otimes 1_{Z}\right) M_{i} e_{b^{\prime}}, e_{a^{\prime}}\right\rangle=0 . \tag{40}
\end{equation*}
$$

Thus, $X_{a, b, i, j}=0$ for all $a, b \in A$ and all $i, j=1, \ldots, m$.
Letting $\psi: M_{A} \rightarrow M_{X} \otimes M_{Z}$ be the unital completely positive map given by $\omega \mapsto \sum_{i} M_{i} \omega M_{i}^{*}$ and setting $E_{a, b}=\psi\left(\epsilon_{a, b}\right)$, we see that $E=\left(E_{a, b}\right)_{a, b}$ is a stochastic operator matrix acting on $\mathbb{C}^{Z}$. By [30, Theorem 5.2], there exists a *-representation $\pi: \mathcal{C}_{X, A} \rightarrow \mathcal{B}\left(\mathbb{C}^{Z}\right)$ such that $\left(\pi\left(e_{x, x^{\prime}, a, a^{\prime}}\right)\right)_{x, x^{\prime}, a, a^{\prime}}=E$. Let $\Gamma: M_{X X} \rightarrow M_{A A}$ be the linear map given by

$$
\Gamma\left(\epsilon_{x, x^{\prime}} \otimes \epsilon_{y, y^{\prime}}\right)=\left(\operatorname{Tr}\left(\pi\left(e_{x, x^{\prime}, a, a^{\prime}} e_{y^{\prime}, y, b^{\prime}, b}\right)\right)\right)_{a, a^{\prime}, b, b^{\prime}} ;
$$

thus, $\Gamma$ is a tracial quantum QNS correlation and, by (40) and the previous paragraphs,

$$
\left\langle\Gamma\left(\xi \xi^{*}\right),\left(\eta \eta^{*}\right)^{\mathrm{d}}\right\rangle=\operatorname{Tr}\left(E\left(Y_{\bar{\xi}} \otimes Y_{\eta} \otimes 1_{\mathcal{A}}\right) E\left(Y_{\bar{\xi}}^{*} \otimes Y_{\eta}^{*} \otimes 1_{Z}\right)\right)=0 .
$$

It follows that $\left\langle\Gamma\left(\xi \xi^{*}\right), P_{\mathcal{V}_{\perp}}\right\rangle=0$ for every $\xi \in \mathcal{U}$, giving $\left\langle\Gamma\left(P_{\mathcal{U}}\right), P_{\mathcal{V}_{\perp}}\right\rangle=$ 0.

Remark 5.6. It was shown as part of the proof of Theorem 5.5 that, for symmetric skew spaces $\mathcal{U} \subseteq \mathbb{C}^{X} \otimes \mathbb{C}^{X}$ and $\mathcal{V} \subseteq \mathbb{C}^{A} \otimes \mathbb{C}^{A}$, we have that $\mathcal{U} \xrightarrow{\mathrm{q}} \mathcal{V}$ if and only if there exist a finite-dimensional algebra $\mathcal{A}$, a unital completely positive map $\psi: M_{A} \rightarrow M_{X} \otimes \mathcal{A}$ with Kraus representation $\psi(T)=\sum_{i=1}^{m} M_{i} T M_{i}^{*}$, such that

$$
\begin{equation*}
M_{j}^{*}\left(\theta(\mathcal{U}) U_{X}^{-1} \otimes 1_{\mathcal{A}}\right) M_{i} \subseteq \theta(\mathcal{V}) U_{A}^{-1}, \quad i, j=1, \ldots, m \tag{41}
\end{equation*}
$$

The same arguments allow us to conclude the equivalence (i) $\Leftrightarrow$ (ii) in the following statement.

Theorem 5.7. Let $X$ and $A$ be finite sets and $\mathcal{U} \subseteq \mathbb{C}^{X} \otimes \mathbb{C}^{X}, \mathcal{V} \subseteq \mathbb{C}^{A} \otimes \mathbb{C}^{A}$ be symmetric skew spaces. The following are equivalent:
(i) $\mathcal{U} \xrightarrow{\text { qC }} \mathcal{V}$;
(ii) there exists a unital completely positive map $\psi: M_{A} \rightarrow M_{X} \otimes \mathcal{C}_{X, A}$ with Kraus representation $\psi(T)=\sum_{i=1}^{m} M_{i} T M_{i}^{*}$, for which inclusions (41) hold;
(iii) there exists a von Neumann algebra $\mathcal{N}$ with a faithful normal tracial state $\tau$ and a unital completely positive map $\psi: M_{A} \rightarrow M_{X} \otimes \mathcal{N}$ with Kraus representation $\psi(T)=\sum_{i=1}^{m} M_{i} T M_{i}^{*}$, for which inclusions (41) hold.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) was pointed out in Remark 5.6. The implication $(\mathrm{iii}) \Rightarrow(\mathrm{i})$ is similar to that of $(\mathrm{ii}) \Rightarrow$ (i) of Theorem [5.5. For $(\mathrm{ii}) \Rightarrow$ (iii), we take $\mathcal{N}=\pi_{\tau}\left(\mathcal{C}_{X, A}\right)^{\prime \prime}$, where $\pi_{\tau}$ is the GNS representation of $\tau$; if $\xi$ is the cyclic vector of $\pi_{\tau}$ then $\langle(\cdot) \xi, \xi\rangle$ is a faithful normal trace on $\mathcal{N}$.

Let $\mathcal{S} \subseteq M_{X}$ and $\mathcal{T} \subseteq M_{A}$ be operator anti-systems. Stalhke writes [29] $\mathcal{S} \xrightarrow{*} \mathcal{T}$ if there exists a finite set $B$ and a state $\Lambda \in M_{B}^{+}$such that $\mathcal{S} \otimes \Lambda \rightarrow \mathcal{T}$; in this case he says that there exists an entanglement assisted homomorphism from $\mathcal{S}$ to $\mathcal{T}$.

Corollary 5.8. Let $G, H$ be graphs. Then

$$
\mathcal{U}_{G} \xrightarrow{\mathrm{q}} \mathcal{U}_{H} \Longrightarrow \mathcal{S}_{G}^{0} \xrightarrow{*} \mathcal{S}_{H}^{0} .
$$

Proof. First observe that $\mathcal{S}_{G}^{0}=\theta\left(\mathcal{U}_{G}\right) U_{X}^{-1}$. The statement now follows from Remark 5.6.

In the next corollary, we partially improve [30, Proposition 10.5] by providing a lower bound on the relaxed orthogonal rank $\xi_{\mathrm{q}}(G)$.

Corollary 5.9. If $G$ is a graph then $\xi_{\mathrm{q}}(G) \geq \sqrt{\theta(\bar{G})}$.
Proof. We observe first that

$$
\xi_{\mathrm{q}}(G)=\min \left\{|A|: \mathcal{U}_{G} \xrightarrow{\mathrm{q}}\left\langle\mathfrak{m}_{A}\right\rangle^{\perp}\right\} .
$$

Moreover, $\theta\left(\left\langle\mathfrak{m}_{A}\right\rangle^{\perp}\right) U_{A}^{-1}=\left(\mathbb{C} I_{A}\right)^{\perp}$, and hence $\mathcal{U}_{G} \xrightarrow{\mathrm{q}}\left\langle\mathfrak{m}_{A}\right\rangle^{\perp}$ implies $\mathcal{S}_{G}^{0} \xrightarrow{*}$ $\left(\mathbb{C} I_{A}\right)^{\perp}$. It follows from [29, Corollary 20] that $\xi_{\mathrm{q}}(G) \geq \sqrt{\theta(\bar{G})}$.
5.2. Quantum colourings of graphs. Let $G$ be a (finite) simple graph with vertex set $X$. For $x, y \in X$, we write $x \sim y$ when $\{x, y\}$ is an edge of $G$, and $x \simeq y$ when $x \sim y$ or $x=y$. The classical-to-quantum colouring game $\varphi_{G}^{A}: \mathcal{P}_{X X}^{\mathrm{cl}} \rightarrow \mathcal{P}_{A A}$ is determined by the requirements

$$
\varphi_{G}^{A}\left(\epsilon_{x, x} \otimes \epsilon_{y, y}\right)= \begin{cases}J_{A} & \text { if } x=y \\ J_{A}^{\perp} & \text { if } x \sim y \\ I_{A A} & \text { otherwise }\end{cases}
$$

In this subsection, we apply the previous results to give a description of perfect quantum commuting and perfect quantum strategies for the classical-to-quantum colouring game in terms of quantum channels whose Kraus operators respect certain containment relations. These relations define a "pushforward" of the graph $G$ into $M_{A}$ or, in the terminology of Weaver [33], into the quantum graph $(\mathcal{S}, \mathcal{M})$ with $\mathcal{S}=\mathcal{M}=M_{A}$. Namely, for a von Neumann algebra $\mathcal{N}$, equipped with faithful tracial state $\tau$, and a unital completely positive map $\Psi: M_{A} \rightarrow \mathcal{D}_{X} \otimes \mathcal{N}$ with Kraus representation $\Psi(T)=\sum_{i=1}^{m} M_{i} T M_{i}^{*}$, we consider the inclusion relations

$$
\begin{equation*}
M_{i}^{*}\left(\mathcal{D}_{X} \otimes 1_{\mathcal{N}}\right) M_{j} \subseteq \mathbb{C} I_{A},, \quad i, j \in[m] \tag{42}
\end{equation*}
$$

and

$$
\begin{equation*}
M_{i}^{*}\left(\mathcal{S}_{G}^{0} \otimes 1_{\mathcal{N}}\right) M_{j} \perp \mathbb{C} I_{A}, \quad i, j \in[m] \tag{43}
\end{equation*}
$$

Definition 5.10. Let $X$ and $A$ be finite sets and $G$ be graph with vertex set $X$. A pair $(\mathcal{N}, \Psi)$, where $\mathcal{N}$ is a von Neumann algebra and $\Psi: M_{A} \rightarrow \mathcal{D}_{X} \otimes$ $\mathcal{N}$ is a unital completely positive map with Kraus representation $\Psi(T)=$ $\sum_{i=1}^{m} M_{i} T M_{i}^{*}$, is called a quantum colouring of $G$ if conditions 42) and (43) are satisfied.

Let $\mathcal{R}^{\mathfrak{u}}$ denote an ultrapower of the hyperfinite $\mathrm{II}_{1}$-factor $\mathcal{R}$ by a free ultrafilter $\mathfrak{u}$ on $\mathbb{N}$ and $\operatorname{tr}_{\mathcal{R}^{u}}$ be its trace.

Theorem 5.11. Let $G$ be a graph with vertex set $X$.
(1) The following are equivalent:
(i) the classical-to-quantum colouring game $\varphi_{G}^{A}$ has a perfect quantum commuting strategy;
(ii) there exists a quantum colouring $(\mathcal{N}, \Psi)$ of $G$, with $\mathcal{N}$ possessing a faithful tracial state.
(2) The following are equivalent:
(i) $\varphi_{G}^{A}$ has a perfect approximately quantum strategy;
(ii) there exist a quantum colouring of the form $\left(\mathcal{R}^{\mathfrak{u}}, \Psi\right)$.
(3) The following are equivalent:
(i) $\varphi_{G}^{A}$ has a perfect quantum strategy;
(ii) there exists a quantum colouring $(\mathcal{N}, \Psi)$ of $G$, where $\mathcal{N}$ is finite dimensional.

Proof. (1) (i) $\Rightarrow$ (ii) Let $\mathcal{E}: \mathcal{D}_{X X} \rightarrow M_{A A}$ be a CQNS correlation, which is a perfect quantum commuting strategy for $\varphi_{G}^{A}$. Let $\tilde{\tau}$ be a trace on $\mathcal{B}_{X, A}$ associated with $\mathcal{E}$ via Corollary 3.12, and $\mathcal{N}:=\pi_{\tilde{\tau}}\left(\mathcal{B}_{X, A}\right)^{\prime \prime}$, where $\pi_{\tilde{\tau}}$ is the GNS representation corresponding to $\tilde{\tau}$. If $\xi$ is the cyclic vector of $\pi_{\tilde{\tau}}$, then $\tau(T):=\langle T \xi, \xi\rangle$ is a faithful trace on $\mathcal{N}$. Let $\Gamma: M_{X X} \rightarrow M_{A A}$ be the canonical lift of $\mathcal{E}$ to a QNS correlation:

$$
\begin{aligned}
\Gamma\left(\epsilon_{x, x^{\prime}} \otimes \epsilon_{y, y^{\prime}}\right) & =\left(\delta_{x, x^{\prime}} \delta_{y, y^{\prime}} \tilde{\tau}\left(e_{x, a, b} e_{y, b^{\prime}, a^{\prime}}\right)\right)_{a, a^{\prime}, b, b^{\prime}} \\
& =\left(\delta_{x, x^{\prime}} \delta_{y, y^{\prime}} \tau\left(\pi_{\tilde{\tau}}\left(e_{x, a, b} e_{y, b^{\prime}, a^{\prime}}\right)\right)_{a, a^{\prime}, b, b^{\prime}}\right.
\end{aligned}
$$

As $\left(\delta_{x, x^{\prime}} \pi_{\tilde{\tau}}\left(e_{x, a, a^{\prime}}\right)_{x, x^{\prime}, a, a^{\prime}}\right)$ is a stochastic operator matrix, there exists a ${ }^{*}$-representation $\pi: \mathcal{C}_{X, A} \rightarrow \mathcal{N}$ such that $\pi\left(e_{x, x^{\prime}, a, a^{\prime}}\right)=\delta_{x, x^{\prime}} \pi_{\tilde{\tau}}\left(e_{x, a, a^{\prime}}\right)$, $x, x^{\prime} \in X, a, a^{\prime} \in A$. Therefore $\Gamma$ is a tracial QNS correlation with

$$
\left\langle\Gamma\left(P_{\left.\mathcal{U}_{G}\right)}\right), P_{\mathcal{V}_{\perp}}\right\rangle=0, \quad \text { where } \mathcal{V}=\left\langle\mathfrak{m}_{A}\right\rangle^{\perp}
$$

As $\theta\left(\mathcal{V}^{\perp}\right) \mathcal{U}_{A}^{-1}=\mathbb{C} I_{A}$ and $\theta\left(\mathcal{U}_{G}\right) U_{X}^{-1}=\mathcal{S}_{G}^{0}$, Theorem 5.7 shows that the unital completely positive map $\Psi: M_{A} \rightarrow \mathcal{D}_{X} \otimes \mathcal{N}$, given by $\Psi\left(\epsilon_{a, a^{\prime}}\right)=$ $\left(\pi\left(e_{x, x^{\prime}, a, a^{\prime}}\right)\right)_{x, x^{\prime}}$, has a Kraus representation $\Psi(T)=\sum_{i=1}^{m} M_{i} T M_{i}^{*}$ satisfying (43). As $\left\langle\Gamma\left(\epsilon_{x, x} \otimes \epsilon_{x, x}\right),\left(\eta \eta^{*}\right)^{\mathrm{d}}\right\rangle=0$ whenever $\eta \in \mathcal{V}$, similar arguments show that (42) is satisfied.
$($ ii $) \Rightarrow\left(\right.$ i) Let $E_{a, b}=\Psi\left(\epsilon_{a, b}\right), a, b \in A$. Then $E:=\left(E_{a, b}\right)_{a, b}$ is a semiclassical stochastic operator matrix; thus, there exists a *-representation $\pi: \mathcal{C}_{X, A} \rightarrow \mathcal{N}$ such that $\left(\pi\left(e_{x, x^{\prime}, a, a^{\prime}}\right)\right)_{x, x^{\prime}, a, a^{\prime}}=E$. Let $\Gamma: M_{X X} \rightarrow M_{A A}$ be the QNS correlation given by

$$
\Gamma\left(\epsilon_{x, x^{\prime}} \otimes \epsilon_{y, y^{\prime}}\right)=\left(\tau\left(\pi\left(e_{x, x^{\prime}, a, a^{\prime}} e_{y^{\prime}, y, b^{\prime}, b}\right)\right)\right)_{a, a^{\prime}, b, b^{\prime}}, \quad x, x^{\prime}, y, y^{\prime} \in X
$$

As $\pi\left(e_{x, x^{\prime}, a, a^{\prime}}\right)=0$ whenever $x \neq x^{\prime}$, we have that $\Gamma=\Gamma \circ \Delta_{X X}$. By Theorem 5.7. $\left\langle\Gamma\left(P_{\mathcal{U}_{G}}\right), P_{\mathcal{V}_{\perp}}\right\rangle=0$. It hence suffices to show that the CQNS correlation $\left.\Gamma\right|_{\mathcal{D}_{X X}}$ is concurrent.

Let $\eta=\sum_{a, b \in A} \eta_{a, b} e_{a} \otimes e_{b}$ be orthogonal to $\mathfrak{m}_{A}$; thus, $\sum_{a \in A} \eta_{a, a}=0$. Let $Y_{\eta}=\sum_{a^{\prime}, b^{\prime} \in A} \eta_{a^{\prime}, b^{\prime}} \epsilon_{a^{\prime}, b^{\prime}}$. We have to show that $\left\langle\Gamma\left(\epsilon_{x, x} \otimes \epsilon_{x, x}\right),\left(\eta \eta^{*}\right)^{\mathrm{d}}\right\rangle=0$. As $M_{j}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{i} \in \mathbb{C} I$, there exists $\lambda_{x} \in \mathbb{C}$ such that

$$
\begin{equation*}
\epsilon_{a, a^{\prime}} M_{j}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{i} \epsilon_{b^{\prime}, b}=\delta_{a^{\prime}, b^{\prime}} \lambda_{x} \epsilon_{a, b} \tag{44}
\end{equation*}
$$

for all $a, a^{\prime}, b, b^{\prime} \in A$. As in the proof of Theorem 5.5, let $\epsilon_{a}=\left(\epsilon_{a, a^{\prime}}\right)_{a^{\prime}}$, considered as a row operator over $M_{A}$, and $M_{i}^{1,3}: \mathbb{C}^{A} \otimes \mathbb{C}^{A} \rightarrow \mathbb{C}^{X} \otimes \mathbb{C}^{A} \otimes H$ be the operator $\sigma^{1,2}\left(1 \otimes M_{i}\right)$, where $\sigma^{1,2}: \mathbb{C}^{A} \otimes \mathbb{C}^{X} \otimes H \rightarrow \mathbb{C}^{X} \otimes \mathbb{C}^{A} \otimes H$ is the flip operator defined on the elementary tensors by $\sigma^{1,2}\left(\xi_{1} \otimes \xi_{2} \otimes \xi_{3}\right)=\xi_{2} \otimes \xi_{1} \otimes \xi_{3}$. Fix $a, b \in A$. We have

$$
\begin{aligned}
& \sum_{i, j=1}^{m} \sum_{a^{\prime}, b^{\prime}, a^{\prime \prime}, b^{\prime \prime} \in A} \overline{\eta_{b^{\prime}, a^{\prime}}} \eta_{b^{\prime \prime}, a^{\prime \prime}} \epsilon_{a, a^{\prime}} M_{j}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{i} \epsilon_{b^{\prime}, b} \epsilon_{b, b^{\prime \prime}} M_{i}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{j} \epsilon_{a^{\prime \prime}, a} \\
& =\sum_{j=1}^{m} \sum_{a^{\prime}, b^{\prime}, a^{\prime \prime}, b^{\prime \prime} \in A} \overline{\eta_{b^{\prime}, a^{\prime}}} \eta_{b^{\prime \prime}, a^{\prime \prime}} \epsilon_{a, a^{\prime}} M_{j}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) E_{b^{\prime}, b^{\prime \prime}}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{j} \epsilon_{a^{\prime \prime}, a} \\
& =\sum_{j=1}^{m} \epsilon_{a}\left(M_{j}^{1,3}\right)^{*}\left(\epsilon_{x, x} \otimes Y_{\eta}^{*} \otimes 1_{\mathcal{N}}\right) E\left(\epsilon_{x, x} \otimes Y_{\eta} \otimes 1_{\mathcal{N}}\right) M_{j}^{1,3} \epsilon_{a}^{*}
\end{aligned}
$$

On the other hand, by (44),

$$
\begin{aligned}
& \sum_{a^{\prime}, b^{\prime}, a^{\prime \prime}, b^{\prime \prime} \in A} \overline{\eta_{b^{\prime}, a^{\prime}}} \eta_{b^{\prime \prime}, a^{\prime \prime}} \epsilon_{a, a^{\prime}} M_{j}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{i} \epsilon_{b^{\prime}, b} \epsilon_{b, b^{\prime \prime}} M_{i}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{j} \epsilon_{a^{\prime \prime}, a} \\
= & \sum_{a^{\prime}, a^{\prime \prime} \in A} \overline{\eta_{a^{\prime}, a^{\prime}}} \eta_{a^{\prime \prime}, a^{\prime \prime}} \epsilon_{a, a^{\prime}} M_{j}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{i} \epsilon_{a^{\prime}, a^{\prime \prime}} M_{i}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{j} \epsilon_{a^{\prime \prime}, a} \\
= & \left(\sum_{a^{\prime} \in A} \overline{\eta_{a^{\prime}, a^{\prime}}} \epsilon_{a, a^{\prime}} M_{j}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{i} \epsilon_{a^{\prime}, a}\right) \\
& \times\left(\sum_{a^{\prime} \in A} \overline{\eta_{a^{\prime}, a^{\prime}}} \epsilon_{a, a^{\prime}} M_{j}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{i} \epsilon_{a^{\prime}, a}\right)^{*} \\
= & \left(\sum_{a^{\prime} \in A} \overline{\eta_{a^{\prime}, a^{\prime}}} \lambda_{x} \epsilon_{a, a}\right)\left(\sum_{a^{\prime} \in A} \overline{\eta_{a^{\prime}, a^{\prime}}} \lambda_{x} \epsilon_{a, a}\right)^{*}=0
\end{aligned}
$$

Hence

$$
\sum_{j=1}^{m} \epsilon_{a}\left(M_{j}^{1,3}\right)^{*}\left(\epsilon_{x, x} \otimes Y_{\eta}^{*} \otimes 1_{\mathcal{N}}\right) E\left(\epsilon_{x, x} \otimes Y_{\eta} \otimes 1_{\mathcal{N}}\right) M_{j}^{1,3} \epsilon_{a}^{*}=0
$$

this implies $E^{1 / 2}\left(\epsilon_{x, x} \otimes Y_{\eta} \otimes 1_{\mathcal{N}}\right) M_{j}^{1,3} \epsilon_{a}^{*}=0$ and therefore

$$
\begin{aligned}
& 0= \\
& (\operatorname{Tr} \otimes \tau)\left(\sum_{j=1}^{m} E^{1 / 2}\left(\epsilon_{x, x} \otimes Y_{\eta} \otimes 1_{\mathcal{N}}\right) M_{j}^{1,3} \epsilon_{a}^{*} \epsilon_{a}\left(M_{j}^{1,3}\right)^{*}\left(\epsilon_{x, x} \otimes Y_{\eta} \otimes 1_{\mathcal{N}}\right) E^{1 / 2}\right) \\
& =(\operatorname{Tr} \otimes \tau)\left(E^{1 / 2}\left(\epsilon_{x, x} \otimes Y_{\eta} \otimes 1_{\mathcal{N}}\right) E\left(\epsilon_{x, x} \otimes Y_{\eta}^{*} \otimes 1_{\mathcal{N}}\right) E^{1 / 2}\right) \\
& =\left\langle\Gamma\left(\epsilon_{x, x} \otimes \epsilon_{x, x}\right) \eta, \eta\right\rangle
\end{aligned}
$$

showing that $\Gamma$ is concurrent.
(2) (ii) $\Rightarrow$ (i) The arguments are similar to those in part (1): we first obtain a ${ }^{*}$-representation $\pi: \mathcal{C}_{X, A} \rightarrow \mathcal{R}^{\mathfrak{u}}$ by letting $\left(\pi\left(e_{x, x^{\prime}, a, a^{\prime}}\right)\right)_{x, x^{\prime}}=\Psi\left(\epsilon_{a, a^{\prime}}\right)$, and define

$$
\Gamma\left(\epsilon_{x, x^{\prime}} \otimes \epsilon_{y, y^{\prime}}\right)=\left(\operatorname{tr}_{\mathcal{R}^{u}}\left(\pi\left(e_{x, x^{\prime}, a, a^{\prime}} e_{y^{\prime}, y, b^{\prime}, b}\right)\right)\right)_{a, a^{\prime}, b, b^{\prime}}
$$

We have that $\tau:=\operatorname{tr}_{\mathcal{R}^{u}} \circ \pi$ is an amenable trace on $\mathcal{C}_{X, A}$ ([6], Proposition 6.3.5 (1),(2)]). Hence the assignment $s_{\tau}\left(x \otimes y^{\mathrm{op}}\right):=\tau(x y)$ determines a state on $\mathcal{C}_{X, A} \otimes_{\text {min }} \mathcal{C}_{X, A}^{\mathrm{op}}$ and if $\partial: \mathcal{C}_{X, A} \rightarrow \mathcal{C}_{X, A}^{\mathrm{op}}$ is the ${ }^{*}$-isomorphism given by $\partial\left(e_{x, x^{\prime}, a, a^{\prime}}\right)=e_{x^{\prime}, x, a^{\prime}, a}\left(\left[30\right.\right.$, Theorem 7.7]) then $s:=s_{\tau} \circ(\mathrm{id} \otimes \partial)$ is a state on $\mathcal{C}_{X, A} \otimes_{\min } \mathcal{C}_{X, A}$ such that $s\left(e_{x, x^{\prime}, a, a^{\prime}} \otimes e_{y, y^{\prime}, b, b^{\prime}}\right)=\tau\left(e_{x, x^{\prime}, a, a^{\prime}} e_{y^{\prime}, y, b^{\prime}, b}\right)$. Applying [30, Theorem 6.5], we obtain that $\Gamma$ is an approximately quantum QNS correlation. As $\Gamma=\Gamma \circ \Delta_{X X}$, by [30, Theorem 7.3], $\left.\Gamma\right|_{\mathcal{D}_{X X}}$ is an approximately quantum CQNS correlation. The above arguments also give that $\Gamma$ is concurrent and satisfies $\left\langle\Gamma\left(P_{\mathcal{U}_{G}}\right), P_{\mathcal{V}_{\perp}}\right\rangle=0$.
(i) $\Rightarrow$ (ii) By Corollary 3.7(i'), a perfect approximately quantum CQNS strategy $\Gamma$ is determined by an amenable trace $\tau$ on $\mathcal{B}_{X, A}$. Hence there
exists a ${ }^{*}$-homomorphism $\rho: \mathcal{B}_{X, A} \rightarrow \mathcal{R}^{\mathfrak{u}}$ such that $\tau=\operatorname{tr}_{\mathcal{R}^{u}} \circ \rho$ (see [15, Proposition 3.2]). The proof is completed similarly to (1)(i) $\Rightarrow(1)(\mathrm{ii})$.
(3) This part of the statement is similar to (1) and (2), and uses the representation of quantum strategies established in Theorem 3.5,

In the next two propositions, we clarify some useful properties of quantum colourings. The first one is an automatic homomorphism result.

Proposition 5.12. Let $X$ and $A$ be finite sets, $\mathcal{N}$ be a von Neumann algebra and $\Psi: M_{A} \rightarrow \mathcal{D}_{X} \otimes \mathcal{N}$ be a unital completely positive map with Kraus representation $\Psi(T)=\sum_{i=1}^{m} M_{i} T M_{i}^{*}$. The following are equivalent:
(i) $\Psi$ is $a^{*}$-homomorphism;
(ii) condition (42) holds.

Proof. (i) $\Rightarrow$ (ii) For each $a, b \in A$, we write $\Psi\left(\epsilon_{a, b}\right)=\sum_{x \in X} \epsilon_{x, x} \otimes r_{x, a, b}$. For $a \neq b$ in $A$ and $1 \leq i, j \leq m$, set

$$
\begin{equation*}
X_{a, b, i, j}=\epsilon_{a, a} M_{i}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{j} \epsilon_{b, b} \in M_{A} \tag{45}
\end{equation*}
$$

We have

$$
\begin{aligned}
& \sum_{j=1}^{m} X_{a, b, i, j} X_{a, b, i, j}^{*}=\sum_{j=1}^{m} \epsilon_{a, a} M_{i}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{j} \epsilon_{b, b} M_{j}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{i} \epsilon_{a, a} \\
& =\epsilon_{a, a} M_{i}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right)\left(\sum_{y \in X} \epsilon_{y, y} \otimes r_{y, b, b}\right)\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{i} \epsilon_{a, a} \\
& =\epsilon_{a, a} M_{i}^{*}\left(\epsilon_{x, x} \otimes r_{x, b, b}\right) M_{i} \epsilon_{a, a}
\end{aligned}
$$

$$
\begin{equation*}
=\left(\epsilon_{a, a} M_{i}^{*}\left(\epsilon_{x, x} \otimes r_{x, b, b}^{1 / 2}\right)\right)\left(\epsilon_{a, a} M_{i}^{*}\left(e_{x, x} \otimes r_{x, b, b}^{1 / 2}\right)\right)^{*} \tag{46}
\end{equation*}
$$

where we have used the fact that $e_{x, b, b}$ is positive. Let

$$
\begin{equation*}
Y_{a, b, i}=\epsilon_{a, a} M_{i}^{*}\left(\epsilon_{x, x} \otimes r_{x, b, b}^{1 / 2}\right), \quad x \in X, a, b \in A, i=1, \ldots, m \tag{47}
\end{equation*}
$$

By (46), $\sum_{j=1}^{m} X_{a, b, i, j} X_{a, b, i, j}^{*}=Y_{a, b, i} Y_{a, b, i}^{*}$. Furthermore,

$$
\begin{align*}
\sum_{i=1}^{m} Y_{a, b, i}^{*} Y_{a, b, i} & =\sum_{i=1}^{m}\left(\epsilon_{x, x} \otimes r_{x, b, b}^{1 / 2}\right) M_{i} \epsilon_{a, a} M_{i}^{*}\left(\epsilon_{x, x} \otimes r_{x, b, b}^{1 / 2}\right) \\
& =\left(\epsilon_{x, x} \otimes r_{x, b, b}^{1 / 2}\right)\left(\sum_{y \in X} \epsilon_{y, y} \otimes r_{y, a, a}\right)\left(\epsilon_{x, x} \otimes r_{x, b, b}^{1 / 2}\right) \\
& =\epsilon_{x, x} \otimes\left(r_{x, b, b}^{1 / 2} r_{x, a, a}^{1 / 2}\right)\left(r_{x, b, b}^{1 / 2} r_{x, a, a}^{1 / 2}\right)^{*} . \tag{48}
\end{align*}
$$

Since $\Psi$ is a homomorphism, $r_{x, b, b}^{1 / 2} r_{x, a, a}^{1 / 2}=r_{x, b, b} r_{x, a, a}=0$ if $a \neq b$. By (48), it follows that $Y_{a, b, i}=0$ for all $i$ and all $a \neq b$. Using (46), we have $X_{a, b, i, j}=0$ for all $i, j$ and $a \neq b$. By (45), this forces $M_{i}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{j} \in \mathcal{D}_{A}$.

Next, we show that $M_{i}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{j}$ lies in $\mathbb{C} I_{A}$. We set

$$
\lambda_{a, x, i, j}=\operatorname{Tr}\left(\epsilon_{a, a} M_{i}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{j} \epsilon_{a, a}\right) ;
$$

then $M_{i}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{j}=\sum_{a \in A} \lambda_{a, x, i, j} \epsilon_{a, a}$. To establish (iii), it suffices to show that $\lambda_{a, x, i, j}=\lambda_{b, x, i, j}$ for all $a, b \in A$.

Set

$$
\begin{equation*}
C_{a, b, x, i, j}=\epsilon_{a, a} M_{i}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{j} \epsilon_{a, a}-\epsilon_{a, b} M_{i}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{j} \epsilon_{b, a} \tag{49}
\end{equation*}
$$

and observe that

$$
C_{a, b, x, i, j}=\left(\lambda_{a, x, i, j}-\lambda_{b, x, i, j}\right) \epsilon_{a, a} .
$$

We note that $\epsilon_{a, a} M_{i}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{j} \epsilon_{b, a}=\epsilon_{a, b} M_{i}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{j} \epsilon_{a, a}=0$, since $M_{i}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{j} \in \mathcal{D}_{A}$. Therefore,

$$
\begin{equation*}
C_{a, b, x, i, j}=\left(\epsilon_{a, a}-\epsilon_{a, b}\right) M_{i}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{j}\left(\epsilon_{a, a}+\epsilon_{b, a}\right) . \tag{50}
\end{equation*}
$$

Since

$$
\left(\epsilon_{a, a}+\epsilon_{b, a}\right)\left(\epsilon_{a, a}+\epsilon_{a, b}\right)=\epsilon_{a, a}+\epsilon_{a, b}+\epsilon_{b, a}+\epsilon_{b, b},
$$

by summing over $j$ and setting $d_{x, a, b}=r_{x, a, a}+r_{x, a, b}+r_{x, b, a}+r_{x, b, b} \geq 0$, we obtain

$$
\begin{equation*}
\left.\sum_{j=1}^{m} C_{a, b, x, i, j} C_{a, b, x, i, j}^{*}=\left(\epsilon_{a, a}-\epsilon_{a, b}\right) M_{i}^{*}\left(\epsilon_{x, x} \otimes d_{x, a, b}\right)\right) M_{i}\left(\epsilon_{a, a}-\epsilon_{b, a}\right) \tag{51}
\end{equation*}
$$

Let $g_{x, a, b} \in \mathcal{N}$ satisfy $g_{x, a, b} g_{x, a, b}^{*}=d_{x, a, b}$ and define

$$
\begin{equation*}
D_{a, b, x, i}=\left(\epsilon_{a, a}-\epsilon_{a, b}\right) M_{i}^{*}\left(\epsilon_{x, x} \otimes g_{x, a, b}\right) \tag{52}
\end{equation*}
$$

By (51),

$$
\sum_{j=1}^{m} C_{a, b, x, i, j} C_{a, b, x, i, j}^{*}=D_{a, b, x, i} D_{a, b, x, i}^{*}
$$

Set $f_{x, a, b}=r_{x, a, a}-r_{x, a, b}-r_{x, b, a}+r_{x, b, b}$ and note that $f_{x, a, b} \geq 0$ and

$$
\begin{align*}
& \sum_{i=1}^{m} D_{a, b, x, i}^{*} D_{a, b, x, i}  \tag{53}\\
& =\left(\epsilon_{x, x} \otimes g_{x, a, b}^{*}\right) M_{i}\left(\epsilon_{a, a}-\epsilon_{b, a}\right)\left(\epsilon_{a, a}-\epsilon_{a, b}\right) M_{i}^{*}\left(\epsilon_{x, x} \otimes g_{x, a, b}\right) \\
& =\epsilon_{x, x} \otimes g_{x, a, b}^{*}\left(r_{x, a, a}-r_{x, a, b}-r_{x, b, a}+r_{x, b, b}\right) g_{x, a, b}^{*} \\
& =\epsilon_{x, x} \otimes\left(g_{x, a, b}^{*} f_{x, a, b} g_{x, a, b}\right)
\end{align*}
$$

Since $\Psi$ is a *-homomorphism, the element $g_{x, a, b}=r_{x, a, a}+r_{x, a, b}$ satisfies the relation $g_{x, a, b}^{*} g_{x, a, b}=d_{x, a, b}$. A calculation then shows that $g_{x, a, b}^{*} f_{x, a, b} g_{x, a, b}=$ 0 . By (53), $D_{a, b, x, i}=0$, and by (51), $C_{a, b, x, i, j}=0$. This forces $\lambda_{x, a, i, j}=$ $\lambda_{x, b, i, j}$ for all $a \neq b$. Hence, $M_{i}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{j} \in \mathbb{C} I_{A}$.
(ii) $\Rightarrow$ (i) The assumption implies that $M_{i}^{*}\left(\epsilon_{x, x} \otimes 1_{\mathcal{N}}\right) M_{j} \in \mathcal{D}_{A}$, so equations (45)-(48) show that $r_{x, a, a}^{1 / 2} r_{x, b, b}^{1 / 2}=0$, and hence $r_{x, a, a} r_{x, b, b}=0$, whenever $a \neq b$. Since each $e_{r, a, a} \geq 0$ and $\sum_{a \in A} r_{x, a, a}=1$, we have that $r_{x, a, a}^{2}=r_{x, a, a}$ for each $x \in X$ and $a \in A$. As $\Psi\left(\epsilon_{a, a}\right)=\sum_{x \in X} \epsilon_{x, x} \otimes r_{x, a, a}$, the diagonal
matrix unit $\epsilon_{a, a}$ belongs to the multiplicative domain of $\Psi$ for each $a$. In particular,

$$
\Psi\left(\epsilon_{a, a}\right) \Psi\left(\epsilon_{b, c}\right)=\delta_{a, b} \Psi\left(\epsilon_{a, c}\right)=\delta_{a, b} \sum_{x \in X} \epsilon_{x, x} \otimes r_{x, a, c}
$$

for all $b, c \in A$.
Now, choose $g_{x, a, b}$ with $g_{x, a, b}^{*} g_{a, x, b}=d_{x, a, b}$, and $h_{x, a, b}$ with $h_{x, a, b}^{*} h_{x, a, b}=$ $f_{x, a, b}$. Our assumption on implies that $C_{a, b, x, i, j}=0$ for all $a \neq b, x \in X$ and all $i$ and $j$. By (49)-(53), $g_{x, a, b}^{*} f_{x, a, b} g_{x, a, b}=0$, yielding $g_{x, a, b} h_{x, a, b}^{*}=0$. Multiplying on the left by $g_{x, a, b}^{*}$ and on the right by $h_{x, a, b}$, we get $d_{x, a, b} f_{x, a, b}=$ 0 . Using the fact that $\epsilon_{a, a}$ and $\epsilon_{b, b}$ are in the multiplicative domain of $\Psi$, a calculation shows that

$$
\begin{equation*}
0=d_{x, a, b} f_{x, a, b}=r_{x, a, a}+r_{x, b, b}-r_{x, a, b} r_{x, b, a}-r_{x, b, a} r_{x, b, a} \tag{54}
\end{equation*}
$$

Multiplying equation (54) on both sides by $r_{x, a, a}$, we get $0=r_{x, a, a}$ $r_{x, a, b} r_{x, b, a}$. Therefore, $r_{x, a, b} r_{x, b, a}=r_{x, a, a}$. Similarly, $r_{x, b, a} r_{x, a, b}=r_{x, b, b}$, so that $\epsilon_{a, b}$ belongs to the multiplicative domain of $\Psi$. Since $a, b \in A$ were arbitrary with $a \neq b, \Psi$ must be a homomorphism, completing the proof.

Remark. We note that an alternative proof of the implications (iii) $\Rightarrow$ (ii) in Theorem 5.11 can be given, using Proposition 5.12. We have decided to present the given argument instead as it shows that, it order to conclude that the game $\varphi_{G}^{A}$ has a perfect strategy (of the corresponding class) one does not need to necessarily resort to the fact that $\Psi$ has to be a homomorphism.

The next proposition shows the combinatorial meaning of (43).
Proposition 5.13. Let $X$ and $A$ be finite sets, $G$ be a graph with vertex set $X$, and $\mathcal{N}$ be a von Neumann algebra. Let $\pi: M_{A} \rightarrow \mathcal{D}_{X} \otimes \mathcal{N}$ be a unital *-homomorphism with Kraus representation $\pi(T)=\sum_{i=1}^{m} M_{i} T M_{i}^{*}$ and write $\pi\left(\epsilon_{a, b}\right)=\sum_{x \in X} \epsilon_{x, x} \otimes r_{x, a, b}$, where $r_{x, a, b} \in \mathcal{N}, x \in X, a, b \in A$. The following are equivalent:
(i) condition (43) holds;
(ii) if $v \sim w$ in $G$, then $\sum_{a, b \in A} r_{v, a, b} r_{w, b, a}=0$.

Proof. Let $v \sim w$ in $G$, and define

$$
R_{c, i, j}=\sum_{b \in A} \epsilon_{c, b} M_{i}^{*}\left(\epsilon_{v, w} \otimes 1_{\mathcal{N}}\right) M_{j} \epsilon_{b, c}, \quad c \in A, i, j \in[m]
$$

Set

$$
T_{c, i}=\sum_{a \in A} \epsilon_{c, a} F_{i}\left(\epsilon_{v, v} \otimes e_{w, a, c}\right)
$$

We have

$$
\begin{align*}
& \sum_{c \in A} \sum_{i, j=1}^{m} R_{c, i, j} R_{c, i, j}^{*} \\
& =\sum_{a, b, c \in A} \sum_{i, j=1}^{m} \epsilon_{c, a} M_{i}^{*}\left(\epsilon_{v, w} \otimes 1_{\mathcal{N}}\right) M_{j} \epsilon_{a, c} \epsilon_{c, b} M_{j}^{*}\left(\epsilon_{w, v} \otimes 1_{\mathcal{N}}\right) M_{i} \epsilon_{b, c} \\
& =\sum_{a, b, c \in A} \sum_{i, j=1}^{m} \epsilon_{c, a} M_{i}^{*}\left(\epsilon_{v, w} \otimes 1_{\mathcal{N}}\right) M_{j} \epsilon_{a, b} M_{j}^{*}\left(\epsilon_{w, v} \otimes 1_{\mathcal{N}}\right) M_{i} \epsilon_{b, c} \\
& =\sum_{a, b, c \in A} \sum_{i=1}^{m} \epsilon_{c, a} M_{i}^{*}\left(\epsilon_{v, w} \otimes 1_{\mathcal{N}}\right) \pi\left(\epsilon_{a, b}\right)\left(\epsilon_{w, v} \otimes 1_{\mathcal{N}}\right) M_{i} \epsilon_{b, c} \\
& =\sum_{a, b, c \in A} \sum_{i=1}^{m} \epsilon_{c, a} M_{i}^{*}\left(\epsilon_{v, v} \otimes r_{w, a, b}\right) M_{i} \epsilon_{b, c} \\
& =\sum_{a, b, c \in A} \sum_{i=1}^{m} \epsilon_{c, a} M_{i}^{*}\left(\epsilon_{v, v} \otimes r_{w, a, c}\right)\left(\epsilon_{v, v} \otimes r_{w, c, b}\right) M_{i} \epsilon_{b, c} \\
& =\sum_{c \in A} \sum_{i=1}^{m} T_{c, i} T_{c, i}^{*} \tag{55}
\end{align*}
$$

On the other hand,

$$
\begin{align*}
\sum_{c \in A} \sum_{i=1}^{m} T_{c, i}^{*} T_{c, i} & =\sum_{a, b, c \in A} \sum_{i=1}^{m}\left(\epsilon_{v, v} \otimes r_{w, c, a}\right) M_{i} \epsilon_{a, c} \epsilon_{c, b} M_{i}^{*}\left(\epsilon_{v, v} \otimes r_{w, b, c}\right) \\
& =\sum_{a, b, c \in A} \sum_{i=1}^{m}\left(\epsilon_{v, v} \otimes r_{w, c, a}\right) M_{i} \epsilon_{a, b} M_{i}^{*}\left(\epsilon_{v, v} \otimes r_{w, b, c}\right) \\
& =\sum_{a, b, c \in A}\left(\epsilon_{v, v} \otimes r_{w, c, a}\right)\left(\epsilon_{v, v} \otimes r_{v, a, b}\right)\left(\epsilon_{v, v} \otimes r_{w, b, c}\right) \\
& =\epsilon_{v, v} \otimes \sum_{a, b, c \in A} r_{w, c, a} r_{v, a, b} r_{w, b, c} . \tag{56}
\end{align*}
$$

Since $\pi$ is a ${ }^{*}$-homomorphism,

$$
\begin{equation*}
\sum_{a, b, c \in A} r_{w, c, a} r_{v, a, b} r_{w, b, c}=\left(\sum_{a, c \in A} r_{w, c, a} r_{v, a, c}\right)\left(\sum_{a, c \in A} r_{w, c, a} r_{v, a, c}\right)^{*} . \tag{57}
\end{equation*}
$$

Considering equations (55)-(57), it follows that condition (i) is equivalent to having $R_{c, i, j}=0$ for all $c \in A$ and all $i, j \in[m]$. The latter condition is in turn equivalent to the condition $\operatorname{Tr}\left(M_{i}^{*}\left(\epsilon_{v, w} \otimes 1_{\mathcal{N}}\right) M_{j}\right)=0$ for all $i, j \in[m]$. The proof is complete.
5.3. Algebraic versions of the orthogonal rank. Recall that the orthogonal $\operatorname{rank} \xi(G)$ of $G$ is the smallest $k \in \mathbb{N}$ for which there exists an orthogonal representation of $G$ in $\mathbb{C}^{k}$, that is, a collection $\left(\xi_{x}\right)_{x \in X}$ of unit vectors in $\mathbb{C}^{k}$ such that

$$
x \sim y \Longrightarrow\left\langle\xi_{x}, \xi_{y}\right\rangle=0 .
$$

In this subsection, we discuss algebraic and $\mathrm{C}^{*}$-algebraic versions of the parameter $\xi(G)$. To place this into context, we define the relaxed classical-to-quantum colouring game as the game $\psi_{G}^{A}: \mathcal{P}_{X X}^{\mathrm{cl}} \rightarrow \mathcal{P}_{A A}$ determined by the requirements

$$
\psi_{G}^{A}\left(\epsilon_{x, x} \otimes \epsilon_{y, y}\right)= \begin{cases}J_{A}^{\perp} & \text { if } x \sim y \\ I_{A A} & \text { otherwise }\end{cases}
$$

Let $\mathrm{x} \in\{\mathrm{loc}, \mathrm{q}, \mathrm{qa}, \mathrm{qc}\}$. We consider the following two parameters:

$$
\xi_{\mathrm{x}}(G)=\min \left\{|A|: \text { there exists a perfect tracial x-strategy for } \psi_{G}^{A}\right\}
$$

which we call the relaxed orthogonal x -rank of $G$, and

$$
\xi_{\mathrm{x}}^{\prime}(G)=\min \left\{|A|: \text { there exists a perfect x-strategy for } \varphi_{G}^{A}\right\},
$$

which we call the orthogonal x -rank of $G$ (we set $\xi_{\mathrm{x}}(G)=\infty$ if there is no perfect strategy for $\varphi_{G}^{A}$ for any $A$ ). These parameters were introduced in [30, Subsection 10.1] as quantum versions of the orthogonal rank. We note the following:
(i) Since $\varphi_{G}^{A}$ is more restrictive than $\psi_{G}^{A}$, we have that $\xi_{\mathrm{x}}(G) \leq \xi_{\mathrm{x}}^{\prime}(G)$;
(ii) By [30, Proposition 10.3], we have $\xi_{\text {loc }}(G)=\xi(G)$. On the other hand, if $|A|>1$ then $\xi_{\text {loc }}^{\prime}(G)=\infty$;
(iii) By [30, Proposition 10.5], $\xi_{\mathrm{q}}^{\prime}\left(K_{d^{2}}\right)=\xi_{\mathrm{qc}}^{\prime}\left(K_{d^{2}}\right)=d$, and hence $\xi_{\mathrm{q}}^{\prime}(G) \leq[\sqrt{|X|}]+1$. By Corollary [5.9, $\xi_{q}\left(K_{d^{2}}\right)=d$.
Taking into account Remark 3.11, we see that the ideal $\mathfrak{I}\left(\varphi_{G}^{A}\right)$ of $\mathfrak{B} X_{X, A}$ is given by

$$
\Im\left(\varphi_{G}^{A}\right)=\left\langle\left\{\sum_{a, b \in A} e_{x, a, b} e_{y, b, a}: x \sim y\right\}\right\rangle,
$$

that is, $\mathfrak{B}\left(\varphi_{G}^{A}\right)$ is the universal *-algebra generated by matrix unit systems $\left(e_{x, a, a^{\prime}}\right)_{a, a^{\prime} \in A}, x \in X$, subject to the relations $\sum_{a, b \in A} e_{x, a, b} e_{y, b, a}=0$ whenever $x \sim y$. Similarly, $\mathcal{B}\left(\varphi_{G}^{A}\right)$ is the universal C*-algebra generated by such matrix unit systems, subject to these relations.

Corollary 3.12 implies the following characterisations:
Corollary 5.14. Let $G$ be a graph with vertex set $X$. Then
(i) The quantum commuting colourings of $G$ correspond to traces of $\mathcal{B}\left(\varphi_{G}^{A}\right)$. In particular,

$$
\xi_{\mathrm{qc}}^{\prime}(G)=\min \left\{|A|: \mathcal{B}\left(\varphi_{G}^{A}\right) \text { possesses a tracial state }\right\},
$$

and
(ii) The quantum colourings of $G$ correspond to finite dimensional traces of $\mathcal{B}\left(\varphi_{G}^{A}\right)$. In particular,

$$
\xi_{\mathrm{q}}^{\prime}(G)=\min \left\{|A|: \mathcal{B}\left(\varphi_{G}^{A}\right) \text { possesses a finite dim. }{ }^{*} \text {-representation }\right\} .
$$

Proof. Suppose that $\tau$ is a tracial state on $\mathcal{C}_{X, A}$ that annihilates the generators $A_{x, y}=\sum_{a, b \in A} e_{x, a, b} e_{y, b, a}, x \sim y$, of $\Im\left(\varphi_{G}^{A}\right)$. Note that

$$
A_{x, y}^{*} A_{x, y}=\sum_{a, b, c, d \in A} e_{y, a, b} e_{x, b, a} e_{x, c, d} e_{y, d, c}=\sum_{b, c, d \in A} e_{y, c, b} e_{x, b, d} e_{y, d, c} ;
$$

it follows that

$$
\tau\left(A_{x, y}^{*} A_{x, y}\right)=|A| \tau\left(\sum_{b, d \in A} e_{x, b, d} e_{y, d, b}\right)=0 .
$$

Combining this with the Cauchy-Schwartz inequality we obtain the statements.

Definition 5.15. (i) The algebraic orthogonal rank $\xi_{\text {alg }}(G)$ is the smallest cardinality of a set $A$ for which $\mathfrak{B}\left(\varphi_{G}^{A}\right) \neq\{0\}$; if such $A$ does not exist, set $\xi_{\text {alg }}(G)=\infty$;
(ii) The $\mathrm{C}^{*}$-algebraic orthogonal rank $\xi_{C^{*}}(G)$ is the smallest cardinality of a set $A$ for which $\mathcal{B}\left(\varphi_{G}^{A}\right) \neq\{0\}$; if such $A$ does not exist, set $\xi_{C^{*}}(G)=\infty$.

Proposition 5.16. Let $G$ be a graph with vertex set $X$. Then $\xi_{C^{*}}(G) \geq$ $\sqrt{\frac{|X|}{\theta(G)}}$. Moreover, $\xi_{C^{*}}\left(K_{d^{2}}\right)=d$.

Proof. If $\xi_{C^{*}}(G)=\infty$ then the inequality is trivial; assume hence that $\mathcal{B}\left(\varphi_{G}^{A}\right) \neq\{0\}$. Since $\mathcal{B}\left(\varphi_{G}^{A}\right)$ is separable, it possesses a faithful state $s$. Let $\pi$ be the corresponding GNS representation and $\xi$ the corresponding cyclic vector. Set $E_{x, a, b}=\pi\left(e_{x, a, b}\right)$ and $\xi_{x, a, b}=E_{x, a, b} \xi, x \in X, a, b \in A$. The proof of the inequality is now concluded in the same way as the proof of 30, Proposition 10.5].

For the equality, realise $A=\mathbb{Z}_{d}=\{0,1, \ldots, d-1\}$ and let $X=A \times A$. Let $\zeta$ be a primitive $|A|$-th root of unity and, for $x=\left(a^{\prime}, b^{\prime}\right)$ and $y=\left(a^{\prime \prime}, b^{\prime \prime}\right) \in X$, set

$$
E_{x, z, z^{\prime}}=\zeta^{\left(z^{\prime}-z\right) b^{\prime}} e_{z-a^{\prime}} e_{z^{\prime}-a^{\prime}}^{*} \in M_{A}, \quad x=\left(a^{\prime}, b^{\prime}\right) \in X, z, z^{\prime} \in A
$$

For $x=\left(a^{\prime}, b^{\prime}\right)$ and $y=\left(a^{\prime \prime}, b^{\prime \prime}\right)$ with $x \neq y$, we have

$$
\begin{aligned}
\sum_{z, z^{\prime} \in A} E_{x, z, z^{\prime}} E_{y, z^{\prime}, z} & =\sum_{z, z^{\prime} \in A} \zeta^{\left(z^{\prime}-z\right) b^{\prime}} \zeta^{\left(z-z^{\prime}\right) b^{\prime \prime}}\left(e_{z-a^{\prime}} e_{z^{\prime}-a^{\prime}}^{*}\right)\left(e_{z^{\prime}-a^{\prime \prime}} e_{z-a^{\prime \prime}}^{*}\right) \\
& =\sum_{z, z^{\prime} \in A} \delta_{z^{\prime}-a^{\prime}, z^{\prime}-a^{\prime \prime}} \delta_{z-a^{\prime}, z-a^{\prime \prime}} \zeta^{\left(z^{\prime}-z\right)\left(b^{\prime}-b^{\prime \prime}\right)} I=0 .
\end{aligned}
$$

In addition,

$$
\begin{aligned}
E_{x, z, z^{\prime}} E_{x, z^{\prime}, z^{\prime \prime}} & =\zeta^{\left(z^{\prime}-z\right) b^{\prime}} \zeta^{\left(z^{\prime \prime}-z^{\prime}\right) b^{\prime}}\left(e_{z-a^{\prime}} e_{z^{\prime}-a^{\prime}}^{*}\right)\left(e_{z^{\prime}-a^{\prime}} e_{z^{\prime \prime}-a^{\prime}}^{*}\right) \\
& =\zeta^{\left(z^{\prime \prime}-z\right) b^{\prime}} e_{z-a^{\prime}} e_{z^{\prime \prime}-a^{\prime}}^{*} ;
\end{aligned}
$$

thus, $\mathcal{B}\left(\varphi_{K_{d}{ }^{2}}^{A}\right)$ is non-trivial.
As the next proposition shows, the algebraic orthogonal rank can be strictly smaller than the $\mathrm{C}^{*}$-algebraic one.

Proposition 5.17. $\xi_{\text {alg }}\left(K_{d^{2}}\right)=2$ for all $d \geq 2$.
Proof. We first show that $\xi_{\mathrm{alg}}\left(K_{d^{2}}\right) \leq 2$. The case of $d=2$ follows from Proposition 5.16, so we assume that $d \geq 3$. By Proposition 5.16, the algebra of the (classical) 4 -colouring game for $K_{d^{2}}$ is non-zero. Hence, there are selfadjoint idempotents $p_{v, w}$ in a non-zero, unital ${ }^{*}$-algebra $\mathcal{A}$, for $1 \leq v \leq d^{2}$ and $1 \leq w \leq 4$, such that $\sum_{w=1}^{4} p_{v, w}=1$ for all $v, p_{v, w} p_{v, z}=0$ if $w \neq z$, and

$$
\begin{equation*}
p_{u, w} p_{v, w}=0, \quad u \neq v \tag{58}
\end{equation*}
$$

By Proposition 5.16, the algebra $\mathcal{B}\left(\varphi_{K_{4}}^{A}\right)$ is non-zero when $|A|=2$. Hence, there are elements $e_{x, a, b}$ in a unital $*$-algebra $\mathcal{B}$, for $1 \leq x \leq 4$ and $1 \leq a, b \leq$ 2 , such that $e_{x, a, b} e_{x, c, d}=\delta_{b, c} e_{x, a, d}, e_{x, a, b}^{*}=e_{x, b, a}, \sum_{a=1}^{2} e_{x, a, a}=1$ and

$$
\begin{equation*}
\sum_{a, b=1}^{2} e_{x, a, b} e_{y, b, a}=0, \quad x \neq y \tag{59}
\end{equation*}
$$

For $1 \leq v \leq d^{2}$ and $1 \leq a, b \leq 2$, define

$$
f_{v, a, b}=\sum_{w=1}^{4} p_{v, w} \otimes e_{w, a, b} \in \mathcal{A} \otimes \mathcal{B}
$$

We will show that the elements $f_{v, a, b}$ satisfy the requirements of the generators for the classical-to-quantum colouring game for $K_{d^{2}}$ with $|A|=2$. Observe that

$$
\begin{aligned}
f_{v, a, b} f_{v, c, d} & =\sum_{w, z=1}^{4} p_{v, w} p_{v, z} \otimes e_{w, a, b} e_{z, c, d}=\sum_{w=1}^{4} p_{v, w} \otimes e_{w, a, b} e_{w, c, d} \\
& =\delta_{b, c} \sum_{w=1}^{4} p_{v, w} \otimes e_{w, a, d}=\delta_{b, c} f_{v, a, d} .
\end{aligned}
$$

Since $p_{v, w}^{*}=p_{v, w}$ and $e_{w, a, b}^{*}=e_{w, b, a}$, we have $f_{v, a, b}^{*}=f_{v, b, a}$. In addition,

$$
\sum_{a=1}^{2} f_{v, a, a}=\sum_{a=1}^{2} \sum_{w=1}^{4} p_{v, w} \otimes e_{w, a, a}=\sum_{w=1}^{4} p_{v, w} \otimes 1=1 \otimes 1
$$

Lastly, using (58) and (59), assuming that $u \neq v$, we have

$$
\begin{aligned}
\sum_{a, b=1}^{2} f_{u, a, b} f_{v, b, a} & =\sum_{a, b=1}^{2} \sum_{w, z=1}^{4} p_{u, w} p_{v, z} \otimes e_{w, a, b} e_{z, b, a} \\
& =\sum_{w=1}^{4} p_{u, w} p_{v, w} \otimes 1=0
\end{aligned}
$$

Thus, there is a unital ${ }^{*}$-homomorphism from $\mathcal{B}\left(\varphi_{K_{d^{2}}}^{A}\right)$ to $\mathcal{A} \otimes \mathcal{B}$ with $|A|=2$, so $\xi_{\text {alg }}\left(K_{d^{2}}\right) \leq 2$.

It remains to show that $\xi_{\text {alg }}\left(K_{d^{2}}\right) \geq 2$. To this end, we show that $\mathfrak{B}\left(\varphi_{K_{d^{2}}}^{A}\right)=\{0\}$ if $|A|=1$. When $|A|=1$, the relations defining $\mathfrak{B}\left(\varphi_{K_{d^{2}}}^{A}\right)$ reduce to having generators $q_{v, a, a}, a \in A, 1 \leq v \leq d^{2}$, such that $\sum_{a \in A} q_{v, a, a}=$ $1, q_{v, a, a}^{*}=q_{v, a, a}$ and $q_{v, a, a} q_{v, a, a}=0$ for $v \neq w$. Since $|A|=1$, the first relation implies that $q_{v, a, a}=1$ for all $v, a$. Then since $n \geq 2$, we may choose $1 \leq v, w \leq d^{2}$ with $v \neq w$. Since we must have $q_{v, a, a} q_{w, a, a}=0$, it follows that $1=1^{2}=0$ in $\mathfrak{B}\left(\varphi_{K_{d^{2}}}^{A}\right)$. Hence, $\xi_{\text {alg }}\left(K_{d^{2}}\right) \geq 2$.

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