# Eigenvalues of the Birman-Schwinger operator for singular measures: The noncritical case



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# Eigenvalues of the Birman-Schwinger operator for singular measures: The noncritical case [☆]

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A R T I C L E   I N F O

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In the memory of Mikhail Birman and Mikhail Solomyak, our teachers

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A B S T R A C T

For operators of the form $\mathbf{T} = \mathbf{T}_{\mathfrak{A},P} = \mathfrak{A}^* P \mathfrak{A}$ with pseudodifferential operator $\mathfrak{A}$ of negative order $-l$ in a domain in $\mathbb{R}^{\mathbf{N}}$, $2l \neq \mathbf{N}$, and a singular measure $P$, an estimate of eigenvalues is found with an order depending on the dimensional characteristics of the measure $P$ and the coefficient depending on an integral norm of the density of $P$ with respect to the Hausdorff measure of an appropriate dimension. These estimates are used to establish asymptotic formulas for the eigenvalues of $\mathbf{T}$ for the case when $P$ is supported on a Lipschitz surface of some codimension and on certain sets of a more complicated structure. In one of applications, a version of the CLR estimate for singular measures is proved.



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## 1. Introduction

In the recent papers [23], [25], Birman-Schwinger type operators in a domain $\Omega \subseteq \mathbb{R}^{\mathbf{N}}$ were considered, namely, the ones having the form $\mathbf{T} \equiv \mathbf{T}_P \equiv \mathbf{T}_{P,\mathfrak{A}} = \mathfrak{A}^* P \mathfrak{A}$. Here $\mathfrak{A}$ is a pseudodifferential operator in $\Omega$ of order $-l = -\mathbf{N}/2$ and $P = V\mu$ is a finite signed Borel measure containing a singular part. For this particular relation between the order of the operator $\mathfrak{A}$ and the dimension of the space, $-l = -\mathbf{N}/2$, the case we call '*critical*', it was found that the decay rate of the eigenvalues $\lambda_k^{\pm}(\mathbf{T})$ does not depend on the dimensional characteristics of measure $\mu$. If the measure $\mu$ possesses a certain regularity property, namely, it is $s$-regular in the sense of Ahlfors, $s > 0$, the eigenvalue counting function $n_{\pm}(\lambda, \mathbf{T}) = \#\{k : \pm\lambda_k^{\pm}(\mathbf{T}) > \lambda\}$ admits the estimate $n_{\pm}(\lambda, \mathbf{T}) \le C_{\pm}(V, \mu, \mathfrak{A})\lambda^{-1}$, with constants depending linearly on the $L \log L$-Orlicz norm of the function $V$ with respect to the measure $\mu$, thus the estimate has correct, quasi-classical, order in the coupling constant. This estimate enables us to prove for the eigenvalues of $\mathbf{T}$ asymptotic formulas of the Weyl type, for the case when $\mu$ is the Hausdorff measure on a certain class of sets, including compact Lipschitz surfaces. Moreover, even in the case when the asymptotics is unknown, the order sharpness of estimates was confirmed by the same order lower estimates.

The reasoning in [23], [25] was based upon the traditional variational approach to the study of eigenvalue distribution, in the version originating in papers by M.Sh. Birman and M.Z. Solomyak in the late 60-s - early 70-s. In the presence of more modern and more powerful methods, the variational one can still produce new results involving rather weak regularity requirements in the setting of spectral problems.

Taking into account the results in [23], [25], by our opinion, it is interesting to investigate spectral properties of operators of the form $\mathbf{T}_P$ with singular a measure $P = V\mu$ in the *noncritical* case, i.e., for $l \neq \mathbf{N}/2$. Our results in this paper show that the eigenvalue behavior in the *subcritical*, $l < \mathbf{N}/2$, and in the *supercritical*, $l > \mathbf{N}/2$, cases differ essentially from each other, and both from the critical case above.

To explain this difference, we compare the eigenvalue distribution for an operator with singular measure with the well understood case of an absolutely continuous one. In the latter case, $P = VdX$, with rather weak integrability conditions imposed on $V$, for an operator $\mathfrak{A}$ of order $-l$, the eigenvalue counting function has Weyl order (it is hinted at by the subscript $W$ in the notation), namely, asymptotically

$$n_{\pm}(\lambda, \mathbf{T}_P) \sim C_{\pm}(P, \mathfrak{A})\lambda^{-\theta_W},\ \lambda \to 0,\ \theta_W = \mathbf{N}/(2l).$$

Thus, $\theta_W = 1$ in the critical case. It was established in [25] that for a singular measure, in the critical case, the order of the eigenvalue distribution does not depend on certain (we call them dimensional) characteristics of measure $\mu$ and this order still equals 1.

We recall that a measure $\mu$ on $\mathbb{R}^{\mathbf{N}}$ with support (the smallest closed set of full measure) $\mathcal{M}$ is called *Ahlfors regular* of order $s$ if for some constants $\mathcal{A} = \mathcal{A}(\mu)$, $\mathcal{B} = \mathcal{B}(\mu) > 0$, the measure of the ball $B(X, r)$ with center $X \in \mathcal{M}$ and radius $r$ satisfies

$$\mathcal{B}r^s \le \mu(B(X,r)) \le \mathcal{A}r^s \tag{1.1}$$

for any $X \in \mathcal{M}$ and $r \le \operatorname{diam}(\mathcal{M})$. It is known that such a measure is equivalent to the Hausdorff measure $\mathcal{H}^s$ of dimension $s$ on $\mathcal{M}$ (see, e.g., [12], Lemma 1.2). In the critical case, estimates of the eigenvalues of operator $\mathbf{T}_P$ were obtained in [25] under the condition that (1.1) is satisfied for *some* $s > 0$; moreover, both inequalities in (1.1) were essentially used in the proof, while the decay order of eigenvalues does not depend on $s$.

In the noncritical cases, $2l \lesseqgtr \mathbf{N}$, as we find out in this paper, the situation is different. In the *subcritical* case, $2l < \mathbf{N}$, previously known results concern mostly order $-2$ operators ($l = 1$) in a domain $\Omega$ in $\mathbb{R}^{\mathbf{N}}$, $\mathbf{N} \ge 2$ and the singular part of $P$ being supported on a (more or less) regular codimension one surface $\Sigma \subset \Omega$; here the contribution of such measure to the spectral asymptotics of $n_\pm(\lambda, \mathbf{T}_P)$ is known to have order $\theta > \theta_W$, i.e., larger than the order of the Weyl term corresponding to the absolutely continuous part, see, e.g., [1].

In the present paper, we establish that in the subcritical case, such spectral property of a singular measure manifests itself under rather general conditions and the contribution of such a singular measure to the spectrum becomes stronger as the dimension of the support of the measure decreases but still satisfies $s > \mathbf{N} - 2l$. Moreover, this eigenvalue estimate is established *only* under the sole condition that the *upper* bound in (1.1) holds, namely

$$\mu(\mathcal{M} \cap B(X,r)) \le \mathcal{A}(\mu)r^s. \tag{1.2}$$

So, for measures $\mu$ satisfying (1.2), we obtain eigenvalue estimates for $n_\pm(\lambda, \mathbf{T})$ with order depending on $\mathbf{N}, l$, and the exponent $s \in (\mathbf{N} - 2l, \mathbf{N})$ in (1.2),

$$n_\pm(\lambda, \mathbf{T}) \le C\mathcal{A}(\mu)^{\theta-1} \int V_\pm(X)^\theta \mu(dX)\lambda^{-\theta},\ \theta = \frac{s}{2l - \mathbf{N} + s} > 1, \tag{1.3}$$

with a constant $C$ depending only on $\mathbf{N}, l, s$ and the operator $\mathfrak{A}$ but not on the density $V$. Thus, the order of the eigenvalues decay and the integrability class of $V$, involved in the eigenvalue estimate, depend on the exponent $s$.

In the usual way, via the Birman-Schwinger principle, estimate (1.3) leads to an estimate of the number $\mathbf{n}_-(\mathfrak{D}^*\mathfrak{D} - P)$ of negative eigenvalues of a, properly defined, Schrödinger-like operator $\mathfrak{D}^*\mathfrak{D} - P$ with a uniformly elliptic operator $\mathfrak{D}$ of order $l$: under some additional conditions,

$$\mathbf{n}_-(\mathfrak{D}^*\mathfrak{D} - P) \le C(\mathfrak{D}, \mu) \int V_+(X)^\theta \mu(dX).$$

This inequality can be treated as an analogue, for singular measures, of the well-known CLR estimate. The restriction $s > \mathbf{N} - 2l$ is quite natural; if $s \le \mathbf{N} - 2l$, i.e., the support of $\mu$ has too small dimension, the quadratic form, even defined first on nice (say, smooth) functions, is, generally, unbounded and not closable.

On the other hand, in the *supercritical* case, $2l > \mathbf{N}$, the singular part of the measure $\mu$ gives to the eigenvalue distribution a *weaker* contribution than the Weyl term. This kind of spectral behavior of singular measures was first discovered, probably, by M.G. Krein in 1951, see [19], where eigenvalues of the 'singular string', $-\lambda u'' = Pu$, on a finite interval with $P$ being a Borel measure, have been studied. The leading, Weyl order, term in the eigenvalue distribution, according to [19], is determined only by the absolutely continuous component of the measure $P$, while the singular component makes a weaker contribution. This spectral problem is equivalent to the one for an operator of the form $\mathbf{T}_{P,\mathfrak{A}}$ with $\mathbf{N} = 1, l = 1$, thus a supercritical one. Such relative weakness of the contribution by the singular part of the measure was established later in a rather general setting by M.Sh. Birman and V.V. Borzov ([3], [11], see also the exposition in [6]), however, without specifying the exact order of eigenvalues decay.

In this paper, we show that in the supercritical case the *lower* bound in (1.1) only, namely,

$$\mu(B(X,r)) \geq \mathcal{B}r^s,\ 0 < s < \mathbf{N},\ \mathcal{B} = \mathcal{B}(\mu), \tag{1.4}$$

is sufficient for the validity for the operator $\mathbf{T}_{P,\mathfrak{A}}$ of eigenvalue estimates for a compactly supported measure $\mu$,

$$n_{\pm}(\lambda, \mathbf{T}_{P,\mathfrak{A}}) \leq C\lambda^{-\theta}\mathcal{B}^{\theta-1}\left[\int V_{\pm}(X)\mu(dX)\right]^{\theta}\mu(\Omega)^{1-\theta}, \tag{1.5}$$

where (again) $\theta = \frac{s}{2l-\mathbf{N}+s}$, but $\theta < 1$ now.

More complicated – and less sharp – results are obtained when the compact support condition for $\mu$ is dropped. For absolutely continuous measures this problem has been studied earlier, see, e.g., [3], [4].

An essentially different approach to obtaining eigenvalue estimates for operators involving singular measures has been developed by H. Triebel, [34], based upon his earlier studies [33], as well as in his monographs, including the ones joint with D. Edmunds [14] and joint with D. Haroske [17]. These authors consider operators of the form $\mathbf{G} = P_2\mathfrak{B}(X,D)P_1$, where $P_j$ are compactly supported functions on a domain in $\mathbb{R}^{\mathbf{N}}$ (in [14], [17]) or, in the singular case, measures $V_j\mu$ supported on a compact set $\mathcal{M}$ whose Hausdorff measure $\mu = \mathcal{H}^s|_{\mathcal{M}}$ satisfies (1.1), see [34], and obtain eigenvalue estimates. We show that this kind of operators can be dealt with by our method as well, moreover, we obtain sharper results. We consider also non-selfadjoint operators of the form $\mathfrak{A}_2P\mathfrak{A}_1$ with pseudodifferential operators $\mathfrak{A}_1, \mathfrak{A}_2$ and $P$ being, as before, a singular measure. Additionally, in Appendix, we discuss the approach of [34] in more detail and compare the spectral results with ours.

The last part of the paper is devoted to obtaining eigenvalue asymptotics for operators of the form $\mathbf{T}_{P,\mathfrak{A}}$ with the measure $\mu$ being the Hausdorff measure on a Lipschitz surface and, further, on some more general sets of integer dimension, namely, on uniformly rectifiable sets. The general approach to the spectral asymptotics of non-smooth spectral problems, developed by M.Sh. Birman and M.Z. Solomyak more than 50 years ago, enables us to derive asymptotic formulas for the eigenvalues of this kind of compact self-adjoint operators under fairly weak regularity assumptions, as soon as correct order upper eigenvalue estimates involving the integral norm of the weight function are obtained. Here we follow essentially the pattern of [23–25], however certain modifications are needed. We use essentially our earlier results ([27], [28]) on the eigenvalue asymptotics of potential type integral operators on Lipschitz surfaces. In the presentation in this paper, we skip more standard points and concentrate on the essentials. We present also some examples demonstrating our general results.

As it has been established in the extensive literature devoted to the eigenvalue distribution, the variational method does not produce more or less sharp values for constants in eigenvalue estimates. The same shortcoming is present in our paper as well. With few exceptions, we do not care for values of constants in formulas, however we indicate which important characteristics of a particular problem these constants depend on. When needed, a constant is marked by the number of formula where it appears first, say $C_{3.15}$ denotes the constant met first in (3.15). Otherwise, constants are denoted by the symbol $C$ and may change value when passing to the next formula.

Our considerations have their roots in ideas of the paper [25], written by the first-named author jointly with Eugene Shargorodsky. We express deep gratitude to Eugene for kind attention and stimulating discussions on the topic.

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## 2. Estimates in Sobolev spaces and singular Birman-Schwinger operators

The starting point in our study is deriving eigenvalue estimates similar to (1.3), (1.5) for the operator $\mathbf{S}_{P,\Omega}$, which is defined in the Sobolev space $H^l(\Omega)$ or $H^l_0(\Omega)$, $\Omega \subset \mathbb{R}^\mathbf{N}$, or $\mathbf{L}^l$ (see Sect. 2.3 below) by the quadratic form $\mathbf{s}_P[v] = \int |v(X)|^2 P(dX)$. Under the conditions we impose on the measure $P$, the quadratic form $\mathbf{s}_P$, defined initially on continuous functions in the Sobolev space, admits a bounded extension to the whole space and thus defines a bounded self-adjoint operator which further on proves to be compact.

### *2.1. Sobolev spaces*

In our considerations, $\Omega$ is usually a bounded open set in $\mathbb{R}^\mathbf{N}$ with piecewise smooth boundary (of interest are domains with smooth boundary and cubes; such domains are called *nice*). The case $\Omega = \mathbb{R}^\mathbf{N}$ is considered as well. For the Sobolev space $H^l(\Omega)$, $l > 0$, the Sobolev norm and the homogeneous Sobolev semi-norm are defined by

$$\|u\|^2_{H^l(\Omega)} := \|u\|^2_{L_2(\Omega)} + |u|^2_{l,\mathrm{hom},\Omega}, \tag{2.1}$$

where

$$|u|^2_{l,\mathrm{hom},\Omega} := \sum_{|\nu|_1 = l} \|\partial^\nu u\|^2_{L_2(\Omega)}, \tag{2.2}$$

for an integer $l$,

$$|u|^2_{l,\mathrm{hom},\Omega} := \sum_{|\nu|_1=[l]} \int_\Omega \int_\Omega \frac{|(\partial^\nu u)(X) - (\partial^\nu u)(Y)|^2}{|X-Y|^{\mathbf{N}+2\tilde{l}}} dX dY, \tag{2.3}$$

for a non-integer $l = [l] + \tilde{l}$, $\tilde{l} \in (0,1)$ (here $|\cdot|_1$ denotes the standard norm in $\ell^1$).

Consider the case of a bounded domain $\Omega$ first. Here, the closure of the space of smooth compactly supported in $\Omega$ functions in $H^l(\Omega)$ is denoted $H^l_0(\Omega)$. The norm $|.|_{l,\mathrm{hom},\Omega}$ is equivalent to the standard norm in (2.1) on $H^l_0(\Omega)$. The important property of the space $H^l(\Omega)$, used for obtaining eigenvalue estimates, is the following one (called sometimes *the Poincaré inequality*). Let $\Omega$ be a cube and $H^l_\top(\Omega)$ denote the subspace in $H^l(\Omega)$ consisting of functions $L_2(\Omega)$-orthogonal to all polynomials of degree less than $l$.

**Proposition 2.1.** *On the subspace $H^l_\top(\Omega)$, the standard $H^l$-norm is equivalent to its homogeneous part,*

$$\|u\|^2_{H^l(\Omega)} \le C(l,\Omega) |u|^2_{l,\mathrm{hom},\Omega}, \ u \in H^l_\top(\Omega). \tag{2.4}$$

This fact was established in the initial publication by S.L. Sobolev [30], Sect. 1.9, for an integer $l$ and became folklore later (see, e.g., an elementary proof of (2.4) in [21] for $l \in (0,1)$).

The Sobolev spaces for $2l < \mathbf{N}$ are closely related to the Riesz potentials,

$$I_l(f)(X) = \int_\Omega |X-Y|^{l-\mathbf{N}} f(Y) dY, \ 2l < \mathbf{N};$$

this connection is discussed in detail in [20], Ch.12. For a nice domain $\Omega$, the space of Riesz potentials $v = I_l(f)$ with $f \in L_2(\Omega)$ coincides with the Sobolev space $H^l(\Omega)$ and the standard norm in the latter space and the $L_2$ norm of $f$ are equivalent.

### *2.2. The case $2l < \mathbf{N}$*

The basic result we use here is the following embedding theorem.

**Proposition 2.2.** *Let $\Omega \subset \mathbb{R}^\mathbf{N}$ be a nice bounded domain, $2l < \mathbf{N}$. Suppose that the Borel measure $\mu$ on $\Omega$ satisfies (1.2) with $s > \mathbf{N} - 2l$ and with constant $\mathcal{A} = \mathcal{A}(\mu)$. Then for any $v \in H^l(\Omega) \cap C(\Omega)$,*

$$\left(\int_{\Omega} |v(X)|^q \mu(dX)\right)^{2/q} \le C_{2.5}\mathcal{A}^{2/q}\|v\|^2_{H^l(\Omega)},\ q = \frac{2s}{\mathbf{N}-2l} = \frac{2}{1-\theta^{-1}} > 2, \tag{2.5}$$

*with constant $C_{2.5}$ not depending on measure $\mu$.*

For an integer $l$, this result is a special case of the generalized Sobolev embedding theorem, see Theorem 1.4.5 in [20], and for a general $l$, it is a special case $p = 2$ of the D. Adams theorem, see Theorem 1.4.1/2 and Theorem 11.8 in [20].

For a given complex Borel measure $P = V\mu$ on $\Omega$ with $\mu$ satisfying (1.2), we consider the quadratic form $\mathbf{s}[v] = \int |v(X)|^2 P(dX) = \int |v(X)|^2 V(X)\mu(dX)$ defined initially on the space $H^l(\Omega) \cap C(\Omega)$. We suppose that $V \in L_{\theta,\mu}$, $\theta = \frac{s}{2l+s-\mathbf{N}} > 1$. By the Hölder inequality and (2.5),

$$|\mathbf{s}[v]| \le \left[\int_{\Omega} |V(X)|^{\theta}\mu(dX)\right]^{\frac{1}{\theta}} \left[\int_{\Omega} |v(X)|^q \mu(dX)\right]^{\frac{2}{q}} \le \tag{2.6}$$
$$C_{2.5}\mathcal{A}^{2/q}\left[\int_{\Omega} |V(X)|^{\theta}\mu(dX)\right]^{\frac{1}{\theta}} \|v\|^2_{H^l(\Omega)}.$$

It follows that the quadratic form $\mathbf{s}[v]$, defined initially on $H^l(\Omega)\cap C(\Omega)$, admits a unique bounded extension by continuity to the whole of $H^l(\Omega)$ and thus defines a bounded selfadjoint operator, which we denote by $\mathbf{S}_{l,P,\Omega}$ (subscripts or some of them may be dropped if this does not cause confusion). Further on, we assume this extension already made.

Since on $H^l_0(\Omega)$ the Sobolev norm is equivalent to its homogeneous part, it follows from (2.6) that

$$|\mathbf{s}[v]| \le \mathcal{Z}|v|^2_{l,\mathrm{hom},\Omega},\ v \in H^l_0(\Omega),\ \mathcal{Z} = C_{2.7}\mathcal{A}^{2/q}\left[\int_{\Omega} |V(X)|^{\theta}\mu(dX)\right]^{\frac{1}{\theta}}, \tag{2.7}$$

with a constant $C_{2.7}$ depending only on the domain $\Omega, l, \mathbf{N}$.

We consider now the operator $\mathbf{S}^{\top}$ defined by the same quadratic form $\mathbf{s}[v]$, but on the subspace $H^l_{\top}(\Omega) \subset H^l(\Omega)$ with norm $|v|^2_{l,\mathrm{hom},\Omega}$, in the particular case of $\Omega$ being a cube $Q$ in $\mathbb{R}^{\mathbf{N}}$. We are interested in the dependence of the norm of this operator on the size of the cube $Q$. If $Q$ is the unit cube $Q_1$, then by (2.6) and Propositions 2.1,

$$\left(\int_{Q_1} |v(X)|^q \mu(dX)\right)^{2/q} \le C_{2.8}\mathcal{A}^{2/q}|v|^2_{l,\mathrm{hom},Q_1},\ v \in H^l_{\top}(Q_1) \tag{2.8}$$

and

$$|\mathbf{s}[v]| \le C_{2.8}\mathcal{A}^{2/q}\left[\int\limits_{Q_1}|V(X)|^\theta\mu(dX)\right]^{\frac{1}{\theta}}|v|^2_{l,\mathrm{hom},Q_1}, \tag{2.9}$$

with a certain constant $C_{2.8}$ not depending on the measure $\mu$.

Our aim now is to describe the behavior of the constant in (2.8) when the unit cube $Q_1$ is replaced by a cube $Q_t$ with side length $t$. We place the center of co-ordinates at some point in $\mathcal{M} = \mathrm{supp}\,(\mu)$. After the scaling $X \mapsto Y = t^{-1}X$, the cube $Q_t$ is mapped onto $Q_1$. Thus, for $v \in H^l(Q_1)$, $\tilde{v}(X) = v(t^{-1}X)$, we have $\tilde{v} \in H^l_\top(Q_t)$, and, by the dilation homogeneity of the norm,

$$|\tilde{v}|^2_{l,\mathrm{hom},Q_t} = t^{\mathbf{N}-2l}|v|^2_{l,\mathrm{hom},Q_1}.$$

Recall that the homogeneity order $\mathbf{N} - 2l$ is positive in the subcritical case under consideration. With a measure $\mu$ on $Q_1$ we associate the scaled measure $\tilde{\mu}$ on $Q_t$: $\tilde{\mu}(E) = t^s\mu(t^{-1}E)$, $t^{-1}E \subset Q_1$. The measure $\tilde{\mu}(E)$ satisfies $\tilde{\mu}(B(X,r)) = t^s\mu(B(t^{-1}X, t^{-1}r)) \le \mathcal{A}r^s$ with the same constant $\mathcal{A}$ as in (1.2). Therefore, since $s = q(\frac{\mathbf{N}}{2} - l)$, $q = \frac{2\theta}{\theta-1}$, after this change of variables, all terms in (2.8) acquire the same power of the scaling parameter $t$, therefore $|\int_{Q_t}|\tilde{v}(X)|^q\tilde{\mu}(dX)|^{2/q} \le \mathcal{A}^{2/q}C_{2.8}|\tilde{v}|^2_{l,\mathrm{hom},Q_t}$, $\tilde{v} \in H^l_\top(Q_t)$, with constant $C_{2.8}$ not depending on the size of the cube $Q_t$. As a result, by the Hölder inequality, we, finally, obtain

$$\left|\int\limits_{Q_t}|v(X)|^2V(X)\tilde{\mu}(dX)\right| \le C_{2.8}\mathcal{A}^{2/q}\left(\int\limits_{Q_t}|V(X)|^\theta\tilde{\mu}(dX)\right)^{1/\theta}|v|^2_{l,\mathrm{hom},Q_t}, \tag{2.10}$$

$v \in H^l_\top(Q_t)$. We denote $C_0 = \max(C^\theta_{2.7}, C^\theta_{2.8})$.

### 2.3. *An estimate in* $\mathbb{R}^\mathbf{N}$

For obtaining eigenvalue estimates, we also need a version of the inequality similar to (2.9), but without the condition of boundedness of $\Omega \subset \mathbb{R}^\mathbf{N}$, including the case of $\Omega = \mathbb{R}^\mathbf{N}$. It is sufficient to consider $\Omega = \mathbb{R}^\mathbf{N}$. There are equivalent definitions of the space $\mathbf{L}^l = \mathbf{L}^l(\mathbb{R}^\mathbf{N})$ (called usually the homogeneous Sobolev space or the space of Riesz potentials). On the one hand, it is the closure of $C_0^\infty(\mathbb{R}^\mathbf{N})$ in the metric $|v|^2_{l,\mathrm{hom},\mathbb{R}^\mathbf{N}} = \|(-\Delta)^{l/2}v\|^2_{L_2}$. Equivalently, $\mathbf{L}^l$ is the space of Riesz potentials $v(X) = (I_lf)(X) = \int|X-Y|^{l-\mathbf{N}}f(Y)dY$, $f \in L_2(\mathbb{R}^\mathbf{N})$, with norm equal to the $L_2(\mathbb{R}^\mathbf{N})$-norm of $f$. Finally, the space $\mathbf{L}^l$ can be described as the space of functions $v \in L_{2,loc}(\mathbb{R}^\mathbf{N})$ such that $(-\Delta)^{l/2}v \in L_2(\mathbb{R}^\mathbf{N})$ in the sense of distributions as well as $|X|^{-l}v(X) \in L_2(\mathbb{R}^\mathbf{N})$, with norm defined by the sum of these $L_2$-norms. Locally, the space $\mathbf{L}^l$ coincides with the Sobolev space $H^l(\mathbb{R}^\mathbf{N})$, however functions in $\mathbf{L}^l$ may have a slower decay rate at infinity.

**Lemma 2.3.** *Let $\mu$ be a locally finite Borel measure on $\mathbb{R}^{\mathbf{N}}$ satisfying (1.2) for $X \in \mathcal{M}$, $\mathcal{M} = \operatorname{supp}\mu$, and $0 < r \le \operatorname{diam}\mathcal{M}$ (for all $r > 0$ if $\mathcal{M}$ is unbounded). Let $V \in L_{\theta,\mu}$, $\theta = \frac{s}{2l+s-\mathbf{N}} > 1$. Then*

$$\left|\int V(X)|v(X)|^2\mu(dX)\right| \le \tag{2.11}$$

$$C_{2.7}\left(\int |V(X)|^{\theta}\mu(dX)\right)^{\frac{1}{\theta}} \mathcal{A}^{1-\theta^{-1}}|v|^2_{l,\mathrm{hom},\mathbb{R}^{\mathbf{N}}}$$

*for $v \in \mathbf{L}^l$.*

**Proof.** Since $C_0^\infty(\mathbb{R}^{\mathbf{N}})$ is dense in $\mathbf{L}^l$, it suffices to prove (2.11) for $v \in C_0^\infty$. Such embedding of the space of Riesz potentials into $L_{q,\mu}$, similar to inequality (2.5), is, again, contained in [20], see there Theorem 11.8; (2.11) follows then by the Hölder inequality. $\square$

Note that unlike the case of the Sobolev space, the inequality for $\mu(B(X,r))$ is required to hold for *all* $r > 0$ if $\mathcal{M}$ is not compact.

*2.4. The case $2l > \mathbf{N}$*

We consider now the supercritical case $2l > \mathbf{N}$. Let the measure $\mu$, $\operatorname{supp}\mu \subset Q_t$, satisfy the estimate (converse to (1.2)):

$$\mu(B(X,r)) \ge \mathcal{B}(\mu)r^s,\ r \le \operatorname{diam} Q_t,\ X \in \mathcal{M},$$

for some cube $Q_t$ with side length $t$. For such cube, the classical embedding $H^l(Q_t) \subset C(\overline{Q_t})$ is valid with estimate

$$|v(X)|^2 \le Ct^{2l-\mathbf{N}}|v|^2_{l,\mathrm{hom},Q_t} = C|Q_t|^{2l/\mathbf{N}-1}|v|^2_{l,\mathrm{hom},Q_t},\ X \in Q_t, \tag{2.12}$$

$v \in H^l_\top(Q_t)$, and, further,

$$\left|\int_{Q_t} V(X)|v(X)|^2\mu(dX)\right| \le \int_{Q_t} |V(X)|\mu(dX) \sup_{X\in Q_t}|v(X)|^2. \tag{2.13}$$

It follows from (1.4) that $t^{2l-\mathbf{N}} \le C\mathcal{B}^{-\frac{2l-\mathbf{N}}{s}}\mu(Q_t)^{\frac{2l-\mathbf{N}}{s}}$. Taking into account (2.12), (2.13), we obtain

$$\left|\int_{Q_t} V(X)|v(X)|^2\mu(dX)\right| \le \tag{2.14}$$

$$C_{2.14}\mathcal{B}^{-\frac{2l-\mathbf{N}}{s}}\mu(Q_t)^{\frac{2l-\mathbf{N}}{s}}\int\limits_{Q_t}|V(X)|\mu(dX)|v|^2_{l,\mathrm{hom},Q_t},$$

$v\in H^l_\top(Q_t)$, with constant $C_{2.14}$ not depending on $V,v,t,\mu$.

Inequality (2.12) is valid for $v\in H^l_0(Q_t)$ as well (probably, with a different constant which we still denote by $C_{2.14}$). Therefore estimate (2.14) is valid for $v\in H^l_0(Q_t)$.

For a nice bounded domain $\Omega$, a similar estimate holds for *all* functions in $H^l(\Omega)$,

$$|v(X)|^2\le C\|v\|^2_{H^l(\Omega)}, \tag{2.15}$$

and

$$\left|\int\limits_\Omega V(X)|v(X)|^2\mu(dX)\right|\le C\int\limits_\Omega |V(X)|\mu(dX)\|v\|^2_{H^l(\Omega)}, \tag{2.16}$$

but with *certain* dependence of the constant $C$ in (2.15), (2.16) on the domain $\Omega$. We, however, will use it only for $\Omega$ being a unit cube, and here the constant is absolute.

## 3. Eigenvalue estimates for $\mathbf{S}_{V\mu}$, and the CLR bound for singular measures

### *3.1. Preparation*

We will use the following geometrical property established in [25] (the two-dimensional version was proved in [18]). For a fixed $\mathbf{N}$-dimensional cube $\mathbf{Q}\subset\mathbb{R}^{\mathbf{N}}$ we call a cube $Q$ parallel to $\mathbf{Q}$ iff all (one-dimensional) edges of $Q$ are parallel to the ones of $\mathbf{Q}$.

**Lemma 3.1.** *Let $\mu$ be a locally finite Borel measure in $\mathbb{R}^{\mathbf{N}}$ not containing point masses. Then there exists a cube $\mathbf{Q}\subset\mathbb{R}^{\mathbf{N}}$ such that for any cube $Q$ parallel to $\mathbf{Q}$, $\mu(\partial Q)=0$.*

Further on in the paper, we suppose that such a cube $\mathbf{Q}$ is fixed and all cubes in the constructions are parallel to $\mathbf{Q}$. It follows in particular, that for any such *open* cube $Q$, we have $\mu(Q)=\mu(\bar{Q})$.

Another important ingredient is the Besicovitch covering lemma, see, e.g., Theorem 1.1 in [16].

**Lemma 3.2.** *Let $\mathcal{M}$ be a bounded set in $\mathbb{R}^{\mathbf{N}}$ and suppose that with each point $X\in\mathcal{M}$ a closed cube $Q_X$ centered at $X$ is associated, all cubes are parallel to each other, and the size of the cubes is uniformly upper bounded. Then there exists a covering $\Upsilon$ of $\mathcal{M}$ consisting of cubes $Q_X$ so that $\Upsilon$ can be split into no more than $\kappa=\kappa(\mathbf{N})$ families $\Upsilon_j$, $j=1,\dots,\kappa$, such that cubes in each of $\Upsilon_j$ are disjoint. The number $\kappa$ depends only on the dimension $\mathbf{N}$.*

*3.2. Estimates in the subcritical case $2l < \mathbf{N}$*

**Theorem 3.3.** *Let $\Omega \subset \mathbb{R}^{\mathbf{N}}$ be a bounded open set and $\mu$ be a measure on $\Omega$ satisfying (1.2) with some $s > \mathbf{N} - 2l$ (this automatically excludes measures containing point masses). Let $V$ be a real-valued function on $\mathcal{M} = \operatorname{supp} \mu$, $V \in L_\theta \equiv L_{\theta,\mu}$, $\theta = \frac{s}{s+2l-\mathbf{N}}$. Then for the operator $\mathbf{S} = \mathbf{S}_{l,V\mu,\Omega}$ defined by the quadratic form $\mathbf{s}[v] = \int V|v(X)|^2\mu(dX)$ on the Sobolev space $H_0^l(\Omega)$, the following estimate holds:*

$$n_\pm(\lambda, \mathbf{S}) \leq C_{3.1}\mathcal{A}^{\theta-1}\int V_\pm(X)^\theta \mu(dX)\lambda^{-\theta}, \tag{3.1}$$

*with constant $C_{3.1} = C_{3.1}(l, s, \mathbf{N})$.*

*If $V$ is a complex-valued function, $V \in L_\theta$, then the estimate, similar to (3.1), holds for the distribution function $n(\lambda, \mathbf{S})$ of the singular numbers of $\mathbf{S}$:*

$$n(\lambda, \mathbf{S}) \leq C_{3.2}\mathcal{A}^{\theta-1}\int |V(X)|^\theta \mu(dX)\lambda^{-\theta}. \tag{3.2}$$

**Proof.** Since any complex function is a linear combination of 4 nonnegative ones, $V = (\operatorname{Re} V)_+ - (\operatorname{Re} V)_- + i(\operatorname{Im} V)_+ - i(\operatorname{Im} V)_-$, it is sufficient to consider the case of a nonnegative density $V$ and prove the estimate for $n_+(\lambda, \mathbf{S})$. Denote $\mathbf{K} = \mathcal{Z}^\theta \equiv C_{2.7}^\theta \mathcal{A}^{\theta-1}\int V(X)^\theta\mu(dX)$. The structure of the proof is as follows. We find constants, $\nu_1$ and $\nu_2$ depending only on $l, \mathbf{N}, s$ such that for $\lambda > \nu_1\mathbf{K}^{\frac{1}{\theta}}$, we have $n_+(\lambda, \mathbf{S}) = 0$, while for $\lambda < \nu_2\mathbf{K}^{\frac{1}{\theta}}$ the required estimate holds, $n_+(\lambda, \mathbf{S}) \leq C\mathbf{K}\lambda^{-\theta}$. From these inequalities, it follows that (3.1) holds for *all* $\lambda > 0$, probably, with a different constant $C$. Indeed, if $\lambda \in [\nu_2\mathbf{K}^{\frac{1}{\theta}}, \nu_1\mathbf{K}^{\frac{1}{\theta}}]$, then the number $\lambda\nu_2\nu_1^{-1}$ is smaller than $\lambda$ and smaller than $\nu_2\mathbf{K}^{\frac{1}{\theta}}$, and therefore, for such $\lambda$,

$$n_+(\lambda, \mathbf{S}) \leq n_+(\lambda\nu_2\nu_1^{-1}, \mathbf{S}) \leq C(\lambda\nu_2\nu_1^{-1})^{-\theta}\mathbf{K} = C(\nu_2\nu_1^{-1})^{-\theta}\lambda^{-\theta}\mathbf{K},$$

exactly what we needed.

Due to the obvious homogeneity,

$$n_\pm(\lambda, \mathbf{S}) = n_\pm(\gamma\lambda, \gamma\mathbf{S}), \ \gamma > 0,$$

it is sufficient to consider the case $\mathbf{K} = 1$.

Following the way of reasoning described above, we consider large $\lambda$ first. By (2.7), the norm of operator $\mathbf{S}$ is not greater than $\mathbf{K} = 1$, therefore, $n_+(1, \mathbf{S}) = 0$, so we can take $\nu_1 = 1$.

Next, for sufficiently small $\lambda < \nu_2$, following the variational principle, we will construct a subspace $\mathcal{L}(\lambda)$ of codimension not greater than $C\lambda^{-\theta}$ such that $\mathbf{s}[v] < \lambda|v|^2_{l,\mathrm{hom},\Omega}$, $v \in \mathcal{L}(\lambda)$, $v \neq 0$.

We act by an adaptation of the original construction by M.Sh. Birman – M.Z. Solomyak, with contribution by G. Rozenblum, see e.g., [7]. Consider the function of $\mu$-measurable sets,

$$\mathbf{J}(E) = 2C_0\mathcal{A}^{\theta-1}\int\limits_E V(X)^\theta \mu(dX)$$

(the measure $\mu$ is supposed to be extended by zero to $\Omega \setminus \mathcal{M}$). Thus, $\mathbf{J}(\mathcal{M}) = 2$. We fix a cube $\mathbf{Q} \subset \mathbb{R}^\mathbf{N}$ according to Lemma 3.1.

Having $\lambda$ fixed, for each point $Y \in \mathcal{M}$, we consider the family $Q_\tau(Y)$ of closed cubes parallel to $\mathbf{Q}$, centered at $Y$, and with side length $\tau$. It follows from 3.1 that $\mathbf{J}(Q_\tau(Y))$ is a monotonous continuous function of the variable $\tau$; it equals zero for $\tau = 0$ and equals 2 for large $\tau$. Due to this continuity, for a given integer $N > 0$, to be determined later, there exists a value $\tau = \tau(Y)$ (not necessarily unique) such that $\mathbf{J}(Q_{\tau(Y)}(Y)) = N^{-1}$. Such cubes $Q_{\tau(Y)}(Y)$ form a covering of $\mathcal{M}$. Let $\Upsilon$ be the subcovering of $\mathcal{M}$ consisting of cubes $Q_{\tau(Y)}(Y)$, found according to the Besicovitch lemma, $\Upsilon = \cup_{1\le j\le\kappa}\Upsilon_j$. It follows, in particular, that the multiplicity of the covering $\Upsilon$ is not greater than $\kappa$. By the disjointness property and additivity of $\mathbf{J}$, for each fixed $j$, we have

$$|\Upsilon_j|N^{-1} = \sum_{Q\in\Upsilon_j}\mathbf{J}(Q) = \mathbf{J}(\cup_{Q\in\Upsilon_j}Q) \le \mathbf{J}(\mathcal{M}) = 2. \tag{3.3}$$

Therefore, the number $|\Upsilon_j|$ of cubes in $\Upsilon_j$ is not greater than $2N$, and altogether, the number of cubes in $\Upsilon$ is not greater than $2\kappa N$.

Using this covering, we construct the subspace $\mathcal{L}(\lambda)$. Consider the linear space $\mathcal{P}(l,\mathbf{N})$ of (real) polynomials in $\mathbb{R}^\mathbf{N}$ having degree less than $l$; we denote its dimension by $\mathfrak{p} \equiv \mathfrak{p}(l,\mathbf{N})$. Now we consider the collection of linear functionals $\Psi(l,\mathbf{N},\Upsilon)$ on $H^l_0(\Omega)$ of the form $\psi(v) = \psi_{p,Q}(v) = \int_Q v(X)p(X)dX$, where $Q$ are cubes in $\Upsilon$ and $p$ is in a fixed basis in $\mathcal{P}(l,\mathbf{N})$; these functionals are, obviously, continuous on $H^l(\Omega)$. Thus, there are altogether $\mathfrak{p}|\Upsilon| \le 2\kappa\mathfrak{p}N$ functionals in $\Psi(l,\mathbf{N},\Upsilon)$. We denote by $\mathcal{L}[N]$ the intersection of the null spaces of these functionals in $H^l_0(\Omega)$. It is a subspace of codimension not greater than $2\mathfrak{p}\kappa N$. This will be the space $\mathcal{L}(\lambda)$ we are looking for, with an appropriate relation of $\lambda$ and $N$ to be determined.

To find this relation we estimate the quantity $\int |v(X)|^2V(X)\mu(dX)$ for $v \in \mathcal{L}[N]$. Since the cubes $Q \in \Upsilon$ form a covering of $\mathcal{M} = \operatorname{supp}(\mu)$,

$$\mathbf{s}[v] \equiv \int\limits_\mathcal{M} |v(X)|^2V(X)\mu(dX) \le \sum_{Q\in\Upsilon}\int\limits_Q |v(X)|^2V(X)\mu(dX). \tag{3.4}$$

We recall that for each cube $Q \in \Upsilon$, the restriction of function $v \in \mathcal{L}[N]$ to $Q$ belongs to $H^l_\top(Q)$, and, therefore, (3.4) and (2.10) imply

$$\mathbf{s}[v] \le C\sum_{Q\in\Upsilon}|v|^2_{l,\mathrm{hom},Q}\mathbf{J}(Q)^{\frac{1}{\theta}} \le C\sum_{Q\in\Upsilon}|v|^2_{l,\mathrm{hom},Q}N^{-\frac{1}{\theta}}. \tag{3.5}$$

From the definition of the homogeneous norm $|.|^2_{l,\mathrm{hom}}$ and the finite multiplicity of the covering $\Upsilon$, it follows that $\sum_{Q\in\Upsilon} |v|^2_{l,\mathrm{hom},Q} \le \kappa |v|^2_{l,\mathrm{hom},\Omega}$. Therefore, by (3.5),

$$\mathbf{s}[v] \le C_{3.6}\kappa N^{-\frac{1}{\theta}} |v|^2_{l,\mathrm{hom},\Omega}. \tag{3.6}$$

Now, for a given $\lambda > 0$, we take an integer $N = N(\lambda)$ so that $\lambda > C_{3.6}\kappa N(\lambda)^{-\frac{1}{\theta}}$, i.e.,

$$N(\lambda) > (C_{3.6}\kappa)^{\theta}\lambda^{-\theta}. \tag{3.7}$$

Then (3.5) implies

$$\mathbf{s}[v] \le \lambda |v|^2_{l,\mathrm{hom},\Omega},\ v \in \mathcal{L}(\lambda),$$

as we wish. In order to estimate the codimension of the subspace $\mathcal{L}(\lambda)$, we need a converse inequality between $\lambda$ and $N$. Suppose that

$$(C_{3.6}\kappa)\lambda^{-1} > 1. \tag{3.8}$$

Then there exists an integer $N = N(\lambda)$ satisfying (3.7) and such that $N(\lambda) < 2C^{\theta}_{3.6}\kappa^{\theta}\lambda^{-\theta}$. Due to the last inequality, the codimension of the subspace $\mathcal{L}(\lambda)$ satisfies

$$\operatorname{codim}(\mathcal{L}(\lambda)) \le 2\mathfrak{p}\kappa N(\lambda) \le 4\mathfrak{p}\kappa C^{\theta}_{3.6}\kappa^{\theta}\lambda^{-\theta}.$$

Therefore, to find the required $N$, we need condition (3.8) be fulfilled, i.e., $\lambda < \nu_2$, where $\nu_2 = (4\mathfrak{p}C_{3.6})^{\frac{1}{\theta}}\kappa$. So, this number $\nu_2$ fits into the structure explained at the beginning of the proof. $\square$

A similar estimate, but without control of the dependence of constant on the domain $\Omega$, is valid for the operator $\mathbf{S}^{\mathcal{N}}$ defined by the same quadratic form $\mathbf{s}[v]$, but on the Sobolev space $H^l(\Omega)$, a generalization of the operator with the Neumann condition.

**Theorem 3.4.** *Let $\Omega \subset \mathbb{R}^{\mathbf{N}}$ be a nice bounded domain, $V \in L_{\theta,\mu}$. Then the eigenvalues of operator $\mathbf{S}^{\mathcal{N}}_{l,P,\Omega}$ satisfy*

$$n_{\pm}(\lambda, \mathbf{S}^{\mathcal{N}}) \le C\mathcal{A}^{\theta-1}\int V_{\pm}(X)^{\theta}\mu(dX)\lambda^{-\theta}.$$

**Proof.** We use the bounded extension operator $\boldsymbol{\mathcal{E}} : H^l(\Omega) \to H^l_0(\tilde{\Omega})$, $\|\boldsymbol{\mathcal{E}}\| = C(\Omega)$, for some bounded domain $\tilde{\Omega} \supset \Omega$. By the variation principle, $n_{\pm}(\lambda, \mathbf{S}^{\mathcal{N}}_{l,P,\Omega}) \le n_{\pm}(\lambda C(\Omega)^{-2}, \mathbf{S}_{l,P,\tilde{\Omega}})$ and (3.1) applies. $\square$

*3.3. The case $2l < \mathbf{N}$. The estimate in $\mathbb{R}^{\mathbf{N}}$ and the CLR-type estimate*

A standard trick (a form of the Birman-Schwinger principle) enables us to carry over the eigenvalue estimates in Theorem 3.3 to the case of operator acting in the whole space. The first stage consists in extending the results to operators in $\mathbb{R}^{\mathbf{N}}$ but with a measure having compact support.

**Proposition 3.5.** *Let $\Omega = \mathbb{R}^{\mathbf{N}}$, $2l < \mathbf{N}$; let $\mu$ be a measure on $\mathbb{R}^{\mathbf{N}}$ with compact support, satisfying (1.2) with $s > \mathbf{N} - 2l$, and $V \in L_{\theta,\mu}$, $\theta = \frac{s}{2l-\mathbf{N}+s} > 1$. Then for the operator $\mathbf{S} = \mathbf{S}_{P,\mathbb{R}^{\mathbf{N}}}$ in $\mathbf{L}^l$, the estimate*

$$n_{\pm}(\lambda, \mathbf{S}) \leq C_{3.9}\mathcal{A}^{\theta-1} \int\limits_{\mathbb{R}^{\mathbf{N}}} V_{\pm}(X)^{\theta}\mu(dX)\lambda^{-\theta} \tag{3.9}$$

*holds with a constant not depending on the size of support of measure $\mu$.*

**Proof.** As usual, by the min-max principle, it suffices to consider the case of a positive density $V$ and study $n_+(\lambda, \mathbf{S})$. Let, for some $\lambda > 0$, $n_+(\lambda, \mathbf{S}_{P,\mathbb{R}^{\mathbf{N}}}) > \mathbf{n}$ for a certain integer $\mathbf{n}$. Then, due to the variational principle and the density of $C_0^\infty(\mathbb{R}^{\mathbf{N}})$ in $\mathbf{L}^l$, there exists a subspace $\mathcal{L}_{\mathbf{n}} \subset C_0^\infty(\mathbb{R}^{\mathbf{N}})$ of dimension $\mathbf{n}$ such that $\mathbf{s}[v] > \lambda|v|^2_{l,\mathrm{hom},\mathbb{R}^{\mathbf{N}}}$ for $v \in \mathcal{L}_{\mathbf{n}} \setminus \{0\}$. However, since the space $\mathcal{L}_{\mathbf{n}}$ is finite-dimensional, there exists a common compact support, i.e., all functions in $\mathcal{L}_{\mathbf{n}}$ have support in a certain ball $\Omega'$ containing the support of measure $\mu$. Thus we obtain a subspace of dimension $\mathbf{n}$ in $C_0^\infty(\Omega')$, where $\mathbf{s}[v] > \lambda|v|^2_{l,\mathrm{hom},\Omega'}$. However, by Theorem 3.3, this subspace may not have dimension greater than $C\lambda^{-\theta}\mathcal{A}^{\frac{2\theta}{q}} \int V(X)^\theta\mu(dX)$. This last observation gives us the required inequality $\mathbf{n} \leq C\lambda^{-\theta}\mathcal{A}^{\frac{2\theta}{q}} \int V(X)^\theta\mu(dX)$. $\square$

Now we dispose of the condition of compactness of the support of measure $\mu$.

**Theorem 3.6.** *Let $\Omega = \mathbb{R}^{\mathbf{N}}$, $2l < \mathbf{N}$, $\mu$ be a measure on $\mathbb{R}^{\mathbf{N}}$ satisfying (1.2) with $s > \mathbf{N} - 2l$, and $V \in L_{\theta,\mu}$, $\theta = \frac{s}{2l-\mathbf{N}+s} > 1$. Then estimate (3.9) holds. Similarly to (3.2), an estimate of the same form holds for singular numbers of operator $\mathbf{S}$ for a complex-valued function $V$.*

**Proof.** As usual, we may suppose that $V \geq 0$. For a fixed $\lambda > 0$, we split $V$ into two parts, $V_\lambda$ and $V'_\lambda$, so that $V_\lambda$ has compact support and $V'_\lambda$ is small, $C_0\mathcal{A}^{\frac{2\theta}{q}} \int (V'_\lambda)^\theta\mu(dX) < (\lambda/2)^{2\theta}$. Correspondingly, the operator $\mathbf{S}$ splits into the sum of two operators, $\mathbf{S} = \mathbf{S}_\lambda + \mathbf{S}'_\lambda$. For $\mathbf{S}_\lambda$ we have an eigenvalue estimate of the form (3.9) with $V_\lambda$ replacing $V$, and for $\mathbf{S}'_\lambda$ we have the norm estimate, by Lemma 2.3, $\|\mathbf{S}'_\lambda\| < \lambda/2$, which is equivalent to:

$$n_+(\lambda/2, \mathbf{S}'_\lambda) = 0.$$

Now, by the Ky Fan inequality,

$$n_+(\lambda, \mathbf{S}) \le n_+(\lambda/2, \mathbf{S}_\lambda) + n_+(\lambda/2, \mathbf{S}'_\lambda). \tag{3.10}$$

The second term in (3.10) equals zero, and Proposition 3.5 applied to $\mathbf{S}_\lambda$ gives us the required estimate. □

**Remark 3.7.** Since $(-\Delta)^{-l/2}$ is an isometric isomorphism of $L_2(\mathbb{R}^{\mathbf{N}})$ onto $\mathbf{L}^l$, the last result can be expressed as an eigenvalue estimate for the operator

$$\mathbf{T}_{P,(-\Delta)^{-l/2}} = (-\Delta)^{-l/2} P (-\Delta)^{-l/2}$$

in $L_2(\mathbb{R}^{\mathbf{N}})$:

$$n_\pm(\lambda, \mathbf{T}_{P,(-\Delta)^{-l/2}}) \le C\mathcal{A}(\mu)^{\theta-1} \int_{\mathbb{R}^{\mathbf{N}}} V_\pm(X)^\theta \mu(dX) \lambda^{-\theta}.$$

As an automatic consequence of Theorem 3.6 we obtain a version of the CLR estimate for singular measures.

**Theorem 3.8.** *Let $2l < \mathbf{N}$, let the measure $\mu$ satisfy (1.2) with some $s > \mathbf{N} - 2l$, and the density $V \ge 0$ satisfy $\int V(X)^\theta \mu(dX) < \infty$, $\theta = \frac{s}{2l - \mathbf{N} + s} > 1$. Consider the Schrödinger-type operator $\mathfrak{H} = \mathfrak{H}(l, P) = (-\Delta)^l - P$ defined in $L_2(\mathbb{R}^{\mathbf{N}})$ by means of the quadratic form $\mathfrak{h}[v] = \|(-\Delta)^{l/2} v\|^2_{L_2} - \int |v(X)|^2 P(dX)$, $P = V\mu$. Then for $N_-(\mathfrak{H})$, the number of negative eigenvalues of $\mathfrak{H}$, the following estimate holds*

$$N_-(\mathfrak{H}) \le C(\mathbf{N}, l)\mathcal{A}(\mu)^{\theta-1} \int V(X)^\theta \mu(dX). \tag{3.11}$$

**Proof.** The proof, actually, the derivation of (3.11) from estimate (3.9), is a quite standard application of the Birman-Schwinger principle. We repeat it for the sake of completeness. The quantity $N_-(\mathfrak{H})$ equals the minimal codimension of subspaces $\mathcal{L} \subset C_0^\infty(\mathbb{R}_{\mathbf{N}})$, for which

$$\int |v(X)|^2 V(X) \mu(dX) < |v|^2_{l,\mathrm{hom},\mathbb{R}^{\mathbf{N}}},\ v \in \mathcal{L} \setminus \{0\}.$$

But this codimension is exactly $n_+(1, \mathbf{S}_{P,\mathbb{R}^{\mathbf{N}}})$, and for the latter quantity we already have the required estimate. □

Later, we obtain a more general version of (3.11).

As usual, CLR-type estimates lead, although not quite directly, to Lieb-Thirring (LT) type estimates for the sum of powers of absolute values of the negative eigenvalues of $\mathfrak{H}$. The corresponding reasoning as well as an alternative way of proving $LT$ estimates for a Schrödinger operator with singular measure as potential are presented in [26].

### *3.4. Eigenvalue estimates for* $\mathbf{S}$, $2l > \mathbf{N}$

The aim of this subsection is to prove eigenvalue estimates in the supercritical case $2l > \mathbf{N}$. The estimates have a somewhat different form, compared with the subcritical case, but the proof is quite analogous.

**Theorem 3.9.** *Let $\Omega \subset \mathbb{R}^{\mathbf{N}}$ be a nice bounded domain, $\mu$ be a compactly supported finite Borel measure without point masses on $\Omega$ satisfying (1.4) with $s > 0$ and $V$ be a real-valued function in $L_1 = L_{1,\mu}$. Then for the operator $\mathbf{S} = \mathbf{S}_{l,V\mu,\Omega}$ defined by the quadratic form $\mathbf{s}[v]$ in $H^l_0(\Omega)$ the eigenvalue estimate holds*

$$n_{\pm}(\lambda, \mathbf{S}) \le C\mathbf{K}\lambda^{-\theta}, \mathbf{K} = \mathcal{B}^{\theta-1}\left(\int V_{\pm}(X)\mu(dX)\right)^{\theta}\mu(\Omega)^{1-\theta} \tag{3.12}$$

*with a constant $C$ not depending on the measure $\mu$ and the weight function $V$. The same estimate holds for a nice bounded domain $\Omega$ for operator $\mathbf{S}^{\mathcal{N}}$ defined by the quadratic form $\mathbf{s}[v]$ in the Sobolev space $H^l(\Omega)$, now with a constant depending on $\Omega$.*

**Proof.** The statement about the operator in the space $H^l(\Omega)$ is reduced to the one for the space $H^l_0(\Omega)$, using the extension operator, as it was done for $2l < \mathbf{N}$ in Theorem 3.4. So, we consider the operator $\mathbf{S}_{l,P,\Omega}$ in the space $H^l_0(\Omega)$; the reasoning here is analogous to the one in Theorem 3.3, with the same separate treatment of larger and smaller $\lambda$. We explain the modifications in detail, when needed. Again, it suffices to consider only the case $V \ge 0$, and, by homogeneity, we can set $\mathbf{K} = 1$.

The estimate for large $\lambda$ follows from the inequality (2.14). For small $\lambda$, the reasoning, as in the case $2l < \mathbf{N}$, consists of constructing a subspace $\mathcal{L}(\lambda)$ of a controlled codimension so that $\mathbf{s}[v] \le \lambda\|v\|^2_{H_{l,\mathrm{hom},\Omega}}$ $v \in \mathcal{L}(\lambda)$. We introduce a function of $\mu$-measurable sets:

$$\mathbf{J}(E) = C_{2.14}\left[\mathcal{B}^{-\beta}\mu(E)^{\beta}\int\limits_E V(X)\mu(dX)\right]^{\frac{1}{\beta+1}} = \tag{3.13}$$
$$\left[C_{2.14}\mathcal{B}^{-\beta}\int\limits_E V(X)\mu(dX)\right]^{\frac{1}{\beta+1}}\mu(E)^{1-\theta},$$

where $\beta = \theta^{-1} - 1 = \frac{2l-\mathbf{N}}{s} > 0, \theta = \frac{s}{2l-Nb+s} < 1$.

By Lemma 3.1, for each point $Y \in \mathcal{M}$, for a family of concentric cubes $Q_\tau(Y)$ parallel to a certain fixed cube $\mathbf{Q}$ and centered at $Y \in \mathcal{M}$ with side length $\tau$, $\mathbf{J}(Q_\tau(Y))$ is a continuous nondecreasing function of the variable $\tau$. Therefore, for an integer $N$, to be determined later, there exists $\tau(Y)$ such that $\mathbf{J}(Q_{\tau(Y)}(Y)) = (2N)^{-1}$. From the covering $\{Q_{\tau(Y)}(Y), Y \in \mathcal{M}\}$, by the same Besicovitch covering lemma, we can extract a finite subcovering $\Upsilon$ which can be split into no more than $\kappa = \kappa(\mathbf{N})$ families $\Upsilon_j$, each consisting of disjoint cubes. Next, we evaluate the number of cubes in $\Upsilon$. For a fixed $j$, we have, by the Hölder inequality,

$$(2N)^{-1}|\Upsilon_j| = \sum_{Q\in\Upsilon_j} \mathbf{J}(Q) = \left[C_{2.14}\mathcal{B}^{-\beta}\right]^\theta \sum_{Q\in\Upsilon_j}\left[\int_Q V(X)\mu(dX)\right]^\theta \mu(Q)^{1-\theta}$$
$$\leq \left[C_{2.14}\mathcal{B}^{-\beta}\right]^\theta \left[\sum_{Q\in\Upsilon_j}\int_Q V(X)\mu(dX)\right]^\theta \left[\sum_{Q\in\Upsilon_j}\mu(Q)\right]^{1-\theta}$$
$$\leq \left[C_{2.14}\mathcal{B}^{-\beta}\right]^\theta \left[\int_\Omega V(X)\mu(dX)\right]^\theta \mu(\Omega)^{1-\theta} = \mathbf{J}(\Omega) = 1.$$

It follows that $|\Upsilon_j|$ is not greater than $2N$ and the number of cubes in the whole covering $\Upsilon$ satisfies

$$|\Upsilon| \leq 2\kappa N.$$

With each cube $Q \in \Upsilon$ we associate a collection of linearly independent functionals, scalar products in $L_2(Q)$ with polynomials of degree less than $l$; as before, there are $\mathfrak{p} = \mathfrak{p}(\mathbf{N}, l)$ of them. As in Theorem 3.3, we define $\mathcal{L}[N]$ as the common null-space of these functionals. Its codimension is not greater than $\mathfrak{p}|\Upsilon| \leq 2\mathfrak{p}\kappa N$. Next, similar to (3.6), we estimate $\mathbf{s}[v]$. Due to the finite multiplicity of the covering $\Upsilon$ and (2.14), we have

$$\mathbf{s}[v] \leq \kappa \sum_{Q\in\Upsilon}\int_Q |v(X)|^2 V(X)\mu(dX) \leq \tag{3.14}$$
$$C\mathcal{B}(\mu)^{-\beta}\sum_{Q\in\Upsilon}\mu(Q)^\beta \int_Q V(X)\mu(dX)|v|^2_{l,\mathrm{hom},Q} = C\sum_Q \mathbf{J}(Q)^{\beta+1}|v|^2_{l,\mathrm{hom},Q}$$
$$\leq C_{3.14}N^{-(\beta+1)}|v|^2_{l,\mathrm{hom},\Omega}.$$

Finally, we take $N > C^\theta_{3.14}\lambda^{-\theta}$, so that, by (3.14), $\mathbf{s}[v] \leq \lambda|v|^2_{l,\mathrm{hom},\Omega}$; $\theta = (\beta+1)^{-1}$, and we can see that for $\lambda$ small enough, such integer $N$ can be chosen, simultaneously, smaller than $2C^\theta_{3.14}\lambda^{-\theta}$, which gives for the subspace $\mathcal{L}(\lambda) = \mathcal{L}[N]$ the required estimate for its codimension: $\operatorname{codim}(\mathcal{L}(\lambda)) \leq C\lambda^{-\theta}$. □

### *3.5. The case $2l > \mathbf{N}$: the Birman-Borzov estimate in the whole space*

For a measure $\mu$ with noncompact support, it follows from (1.4) that $\mu(\mathcal{M}) = \infty$, and inequality (3.12) becomes trivial, i.e., does not give any estimate for the eigenvalues of the operator $\mathbf{S}$. Since the paper [3] by M.Sh. Birman and V.V. Borzov appeared, quite a lot of work has been done to obtain eigenvalue estimates for this operator as well as for the Schrödinger-like operator $\mathfrak{H} = (-\Delta)^l - P$ in $\mathbb{R}^{\mathbf{N}}$, $2l \ge \mathbf{N}$, for an absolutely continuous measure $P$, see, especially, [4] and [29]. We will not discuss all possible versions of these results for singular measures, and restrict ourselves to just several typical ones. An estimate in [3], probably, the first one obtained for the case $2l > \mathbf{N}$ in the whole space, bounds the eigenvalues of the problem $\lambda[(-\Delta)^l + 1]u = Vu$ by a sum of powers of $L_1$-norms of the function $V$ over the lattice of unit cubes. Later this result was considerably generalized in many directions, however a common feature remained: for operators with fast eigenvalue decay, equivalently, with slow growing $n_\pm(\lambda) = O(\lambda^{-\theta})$, $\theta < 1$, estimates should involve the sum of $L_1$-norms, taken to some power, of the weight function over an appropriate system of compact sets. To carry over all these results to the case of singular measures considered in this paper would be a huge task. We demonstrate such a generalization of the initial Birman-Borzov result.

**Theorem 3.10.** *Let $\mu$ be a measure on $\mathbb{R}^{\mathbf{N}}$ possessing no point masses, with noncompact support $\mathcal{M} = \operatorname{supp}(\mu)$, satisfying a local version of condition (1.4), namely*

$$\mu(B(X,r)) \ge \mathcal{B}(X) r^s,\ X \in \mathcal{M}, 0 < r < \sqrt{\mathbf{N}}, \mathcal{B}(X) > 0 \tag{3.15}$$

*(recall, $\sqrt{\mathbf{N}}$ is the diameter of the unit cube in $\mathbb{R}^{\mathbf{N}}$). For a cube $\mathbf{Q}$ whose existence is established in Lemma 3.1, consider the lattice $\mathfrak{L}$ of unit cubes in $\mathbb{R}^{\mathbf{N}}$ parallel to $\mathbf{Q}$. Denote by $\mathfrak{L}_\mu$ the set of those cubes $Q$ in $\mathfrak{L}$ which satisfy $\mu(Q) > 0$ and by $\mathcal{B}(Q), Q \in \mathfrak{L}_\mu$, we denote the quantity $\mathcal{B}(Q) = \inf_{X \in Q \cap \mathcal{M}} \mathcal{B}(X)$. Suppose that the real-valued density $V \in L_{1,\mathrm{loc},\mu}$ satisfies*

$$\mathbf{M}(\mu, V) := \sum_{Q \in \mathfrak{L}_\mu} \mathcal{B}(Q)^{\theta-1} \left[ \int_Q |V(X)| \mu(dX) \right]^\theta \mu(Q)^{1-\theta} < \infty \tag{3.16}$$

*for $\theta = \frac{s}{2l - \mathbf{N} + s}$. Consider the operator $\mathbf{S} = \mathbf{S}_{l,V\mu}$ defined by the quadratic form $\mathbf{s}[v]$ in the Sobolev space $H^l(\mathbb{R}^{\mathbf{N}})$, $2l > \mathbf{N}$. Then this operator is bounded, compact, and satisfies*

$$n_\pm(\lambda, \mathbf{S}) \le C \mathbf{M}(\mu, V_\pm) \lambda^{-\theta}. \tag{3.17}$$

**Proof.** Having our estimate (3.12), the proof of (3.17) can be constructed by repeating the reasoning in [3], where an eigenvalue estimate in the whole space was obtained by summing estimates in separate cubes. In the present understanding, this proof fits in just a few lines. Consider, for each $Q \in \mathfrak{L}_\mu$, operator $\mathbf{S}_Q$ which is defined by the quadratic form $\mathbf{s}_Q[v] = \int_Q |v(X)|^2 V(X) \mu(dX)$ in $H^l(Q)$. The eigenvalues can only grow if we replace the base space $H^l(\mathbb{R}^{\mathbf{N}})$ by $\oplus H^l(Q)$ with the sum over $Q$ in the lattice $\mathfrak{L}$. Operators $\mathbf{S}_Q$ act in orthogonal subspaces $H^l(Q)$, therefore, by the variational principle,

$n_\pm(\lambda, \mathbf{S}) \le \sum_{Q\in\mathfrak{L}_\mu} n_\pm(\lambda, \mathbf{S}_Q)$, and by summing estimate (3.12) over cubes $Q$, we arrive at (3.17). $\square$

Now, since the operator $(1-\Delta)^{-l/2}$ is an isometric isomorphism of $L_2(\mathbb{R}^{\mathbf{N}})$ onto $H^l(\mathbb{R}^{\mathbf{N}})$, Theorem 3.10 leads to an eigenvalue estimate for a singular Birman-Schwinger operator.

**Corollary 3.11.** *Let measure $\mu$ and density $V$ satisfy conditions of Theorem 3.10. Then for the operator $\mathbf{T} = (1-\Delta)^{-l/2}(V\mu)(1-\Delta)^{-l/2}$ the eigenvalue estimate (3.17) holds.*

## 4. Pseudodifferential Birman-Schwinger operators with singular weight. Spectral estimates

In this section we give a detailed definition of pseudodifferential operators with singular weight and study spectral estimates for the noncritical case(s). We mostly follow the presentation in [25], [23], where the critical case was considered, and emphasize only essential differences.

### *4.1. Operators in bounded domains*

Let $\Omega$ be a bounded open set in $\mathbb{R}^{\mathbf{N}}$ and $\mathfrak{A}$ be a classical pseudodifferential operator of order $-l \ne -\mathbf{N}/2$, with principal symbol $\mathbf{a}_{-l}(X, \Xi)$. We suppose that $\mathfrak{A}$ is an operator with compact support in $\Omega$ which means that it contains cut-offs to some proper subdomain $\Omega' \Subset \Omega$, $\mathfrak{A} = \chi\mathfrak{A}\chi$, $\chi \in C_0^\infty(\Omega)$.

Such an operator $\mathfrak{A}$ maps $L_2(\Omega)$ to $H_0^l(\Omega)$, its essential norm in these spaces is bounded by $C(\mathbf{N}, l)\sup_{(X,\Xi)\in S^*(\Omega)} |\mathbf{a}_{-l}(X, \Xi)|$, where $S^*(\Omega)$ is the cospheric bundle of $\Omega$.

Let $\mu$ be a singular measure supported in $\Omega$ and $V$ be a real $\mu$-measurable function, $P = V\mu$. For $u \in L_2(\Omega)$ we consider the quadratic form $\mathbf{t}[u] = \mathbf{t}_{P,\mathfrak{A}}[u] = \mathbf{t}_{P,\mathfrak{A},\Omega}[u]$ defined as

$$\mathbf{t}_{P,\mathfrak{A},\Omega}[u] = \int_\Omega |(\mathfrak{A}u)(X)|^2 P(dX). \tag{4.1}$$

Under the conditions we impose now on the measure $\mu$ and the density $V$, it will be shown that the quadratic form (4.1) is well-defined on $L_2(\Omega)$ and determines a compact operator $\mathbf{T} = \mathbf{T}_{P,\mathfrak{A},\Omega}$ whose eigenvalues satisfy estimates similar to the ones in Section 3.

The conditions we impose on $\mu, V$ are the following.

**Condition 4.1.** *If $2l < \mathbf{N}$, $\mu$ satisfies (1.2) with $s > \mathbf{N} - 2l$; if $2l > \mathbf{N}$, $\mu$ satisfies (1.4) with $0 < s < \mathbf{N}$; the density $V$ belongs to $L_{\vartheta,\mu}$, $\vartheta = \max(1, \theta), \theta = \frac{s}{2l-\mathbf{N}+s}$.*

The quadratic form is defined in the following way. By results of Sect. 2, see (2.13), under the above conditions, the quadratic form $\mathbf{s}[v] = \int_\Omega |v(X)|^2 P(dX)$, defined initially for $v \in H^l(\Omega) \cap C(\Omega)$, admits an extension by continuity to the whole of $H^l(\Omega)$ as a bounded quadratic form. In its turn, the operator $\mathfrak{A}$ maps $L_2(\Omega)$ to $H^l(\Omega)$, $\mathfrak{A}(L_2(\Omega)) \subset H^l(\Omega)$, thus the composition

$$\mathbf{t}[u] = \mathbf{s}[\mathfrak{A}u]$$

is well defined on $L_2(\Omega)$ and determines a bounded operator $\mathbf{T} = \mathbf{T}_{P,\mathfrak{A}}$.

The action of the operator $\mathbf{T}$ can be described explicitly, in a way similar to [23]. By the standard polarization, the sesquilinear form of the operator $\mathbf{T}$ is

$$(\mathbf{T}u, v)_{L_2(\Omega)} = \int (\mathfrak{A}u)(X)\overline{(\mathfrak{A}v)(X)}P(dX).$$

For a fixed $v \in L_2$, the linear functional $\psi_v(w) = \int w(X)\overline{(\mathfrak{A}v(X))}P(dX)$ is continuous on $H_0^l(\Omega)$ and its norm is majorized by $||v||_{L_2}$. This defines the product $\overline{(\mathfrak{A}v(X))}P$ as an element in the negative order Sobolev space of distributions $H^{-l}(\Omega)$, and $v \mapsto \overline{(\mathfrak{A}v)}P$ is a continuous mapping from $L_2(\Omega)$ to $H^{-l}(\Omega)$. In its turn, for $h \in H^{-l}(\Omega)$, the mapping $u \mapsto \langle \mathfrak{A}u, h\rangle$, defined initially on $u \in C_0^\infty(\Omega)$, acts as $\varphi_h : u \mapsto \langle u, \mathfrak{A}^*h\rangle$, where $\mathfrak{A}^* : H^{-l} \to L_2(\Omega)$ is the adjoint operator for $\mathfrak{A} : L_2(\Omega) \to H_0^l$, so $\mathfrak{A}^*h \in L_2(\Omega)$ and the mapping $\varphi_h$ extends by continuity to $u \in L_2(\Omega)$. As a result, we obtain $(\mathbf{T}u, v)_{L_2(\Omega)} = (\mathfrak{A}^*P\mathfrak{A}u, v)_{L_2(\Omega)}$ and the operator $\mathbf{T}$ factorizes as

$$\mathbf{T} = \mathfrak{A}^*P\mathfrak{A} : L_2(\Omega) \xrightarrow{\mathfrak{A}} H_0^l(\Omega) \xrightarrow{P} H^{-l}(\Omega) \xrightarrow{\mathfrak{A}^*} L_2(\Omega),$$

all factors being continuous mappings between the corresponding spaces. This justifies our writing $\mathbf{T} = \mathfrak{A}^*P\mathfrak{A}$.

Now we can formulate the eigenvalue estimates.

**Theorem 4.2.** *Let $\Omega$ be a bounded domain, $P = V\mu$ be a singular measure in $\Omega$, $\mathfrak{A}$ be a compactly supported pseudodifferential operator of order $-l$ and Condition 4.1 be satisfied. Denote by $\mathbf{K}_\pm$ the quantity*

$$\mathbf{K}_\pm = \mathcal{A}(\mu)^{\theta-1}\int_\Omega V_\pm(X)^\theta \mu(dX), \text{ if } 2l < \mathbf{N}, \tag{4.2}$$

$$\mathbf{K}_\pm = \mathcal{B}(\mu)^{\theta-1}\left(\int_\Omega V_\pm(X)\mu(dX)\right)^\theta \mu(\Omega)^{1-\theta}, \text{ if } 2l > \mathbf{N}.$$

*Then*

$$n_\pm(\lambda, \mathbf{T}) \leq C(\mathfrak{A}, \Omega)\mathbf{K}_\pm\lambda^{-\theta}, \lambda > 0, \tag{4.3}$$

*and*

$$\limsup_{\lambda\to 0}\lambda^{\theta}n_{\pm}(\lambda,\mathbf{T})\le C(l,\Omega)\sup_{(X,\Xi)\in S^*\Omega}|\mathbf{a}_{-l}(X,\Xi)|^2\mathbf{K}_{\pm}. \tag{4.4}$$

The rough estimate (4.3) in the first part of Theorem 4.2 is proved quite similarly to Theorem 2.3 in [25], where the critical case was considered. Namely, denote by $\mathfrak{A}_0$ the operator $(-\Delta_{\mathcal{D}}+1)^{-l/2}$ where $\Delta_{\mathcal{D}}$ is the Laplacian in a domain $\Omega''\supset\Omega'$ with Dirichlet boundary conditions; $\mathfrak{A}_0$ maps $L_2(\Omega'')$ to $H^l(\Omega'')$. Thus,

$$\mathbf{T}_{P,\mathfrak{A}}=\mathfrak{A}^*P\mathfrak{A}=(\mathfrak{A}_0^{-1}\mathfrak{A})^*(\mathfrak{A}_0^*P\mathfrak{A}_0)(\mathfrak{A}_0^{-1}\mathfrak{A}).$$

In this product, the middle term is exactly the operator $\mathbf{T}_{P,\mathfrak{A}_0}$ discussed in Sect. 3, for which the required eigenvalue estimates are already justified. The operator $\mathfrak{A}_0^{-1}\mathfrak{A}$ is, up to a compact additive term, the zero order pseudodifferential operator with principal symbol $|\Xi|^l\mathbf{a}_{-l}(X,\Xi)$, it is therefore bounded in $L_2(\Omega)$, together with its adjoint, and this gives the first estimate. The second, sharper estimate (4.4) follows from the fact that the essential norm of the zero order operator $\mathfrak{C}=\mathfrak{A}_0^{-1}\mathfrak{A}$ is majorated by $M(\mathfrak{A}):=\sup_{(X,\Xi)\in S^*(\Omega)}|\mathbf{a}_{-l}(X,\Xi)|$ (see, e.g., Theorem 6.3, Ch.II in [31]). According to this property, for any $\epsilon>0$, the operator $\mathfrak{C}$ can be represented as $\mathfrak{C}=\mathfrak{C}_\epsilon+\mathfrak{C}'_\epsilon$, where $\|\mathfrak{C}_\epsilon\|<M(\mathfrak{A})+\epsilon$ and $\mathfrak{C}'_\epsilon$ is compact. Therefore

$$\mathbf{T}_{P,\mathfrak{A}}\equiv\mathfrak{C}^*\mathbf{T}_{P,\mathfrak{A}_0}\mathfrak{C}=\mathfrak{C}_\epsilon^*\mathbf{T}_{P,\mathfrak{A}_0}\mathfrak{C}_\epsilon+\mathfrak{R}_\epsilon, \tag{4.5}$$

where the operator $\mathfrak{R}_\epsilon$ in (4.5) is the sum of three terms, each being the product of $\mathbf{T}_{P,\mathfrak{A}_0}$ and two more bounded operators, at least one of which is compact. By the Ky Fan inequality the multiplication by a compact operator leads to a faster decay of singular numbers, see [6], Sect. 11.1, and therefore,

$$\limsup_{\lambda\to 0}\lambda^{\theta}n_{\pm}(\lambda,\mathbf{T}_{P,\mathfrak{A}})\le(M(\mathfrak{A})+\epsilon)^2\limsup_{\lambda\to 0}\lambda^{\theta}n_{\pm}(\lambda,\mathbf{T}_{P,\mathfrak{A}_0}),$$

which, by the arbitrariness of $\epsilon$ proves (4.4).

More sharp asymptotic eigenvalue estimates follow from the eigenvalue asymptotics to be proved further on, in Sect. 6.

### 4.2. Operators in $\mathbb{R}^{\mathbf{N}}$ and pseudodifferential CLR estimates

For operators in the whole Euclidean space we express the conditions for eigenvalue estimates in terms of the mapping properties of the pseudodifferential operator $\mathfrak{A}$.

**Theorem 4.3.** *Let $2l<\mathbf{N}$ and let the order $-l$ pseudodifferential operator $\mathfrak{A}$ map $L_2(\mathbb{R}^{\mathbf{N}})$ to $\mathbf{L}^l$. Suppose that the measure $\mu$ satisfies (1.2) with some $s\in(\mathbf{N}-2l,\mathbf{N})$ and $V$ is a real-valued function, $V\in L_{\theta,\mu}$, $\theta=\frac{s}{2l-\mathbf{N}+s}$. Then for the operator $\mathbf{T}=\mathbf{T}_{P,\mathfrak{A}}$, $P=V\mu$, defined by the quadratic form $\mathbf{t}_{P,\mathfrak{A}}[u]=\int|(\mathfrak{A}u)(X)|^2P(dX)$ in $L_2(\mathbb{R}^{\mathbf{N}})$, the following estimate holds*

$$n_{\pm}(\lambda, \mathbf{T}) \leq C||\mathfrak{A}||^{2\theta}_{L_2\to\mathbf{L}^l}\mathcal{A}^{\theta-1}\int\limits_{\mathbb{R}^{\mathbf{N}}} V_{\pm}(X)^{\theta}\mu(dX)\lambda^{-\theta}. \tag{4.6}$$

*The estimate of the form (4.6) holds in the case of a complex-valued function $V$, with $V_{\pm}$ replaced by $|V|$, for the counting function $n(\lambda, \mathbf{T})$ of singular numbers of $\mathbf{T}$.*

**Proof.** The estimate follows immediately from Theorem 3.6 (see Remark 3.7) due to the identity

$$\mathbf{T}_{P,\mathfrak{A}} = \mathfrak{A}^* P\mathfrak{A} = ((-\Delta)^{l/2}\mathfrak{A})^*\mathbf{T}_{P,(-\Delta)^{-l/2}}((-\Delta)^{l/2}\mathfrak{A}),$$

with $(-\Delta)^{l/2}\mathfrak{A}$ bounded in $L_2(\mathbb{R}^{\mathbf{N}})$. $\square$

From Theorem 4.3 in the usual way follows the general CLR type estimate:

**Corollary 4.4.** *Let $\mathfrak{D}$ be an order $l < \mathbf{N}/2$ pseudodifferential operator in $\mathbb{R}^{\mathbf{N}}$ such that $\mathfrak{D}^{-1}$ maps $L_2(\mathbb{R}^{\mathbf{N}})$ to $\mathbf{L}^l$. Let the measure $\mu$ satisfy (1.2) with $s \in (\mathbf{N} - 2l, \mathbf{N})$ and $V \in L_{\theta,\mu}$, $\theta = \frac{s}{2l-\mathbf{N}+s}$. Then for the Schrödinger type operator $\mathfrak{H} = \mathfrak{D}^*\mathfrak{D} - P$ defined in $L_2(\mathbb{R}^{\mathbf{N}})$ by means of the quadratic form $\mathfrak{h}[u] = \|\mathfrak{D}u\|^2_{L_2} - \int V(X)|u(X)|^2\mu(dX)$ the following estimate for the number $N_-(\mathfrak{H})$ of negative eigenvalues holds*

$$N_-(\mathfrak{H}) \leq C(\mathfrak{D})\mathcal{A}^{\theta-1}\int\limits_{\mathbb{R}^{\mathbf{N}}} V_+(X)^{\theta}\mu(dX).$$

Results for $2l > \mathbf{N}$ follow in the same way from Theorem 3.10.

### *4.3. Singular numbers estimates for non-selfadjoint operators*

We extend here the class of operators for which spectral estimates are obtained. Let $\mathfrak{B}$ be an order $-\varkappa$ pseudodifferential operator in $\mathbb{R}^{\mathbf{N}}$, $\varkappa < \mathbf{N}$, and $\mathcal{M}$ be a closed set in $\mathbb{R}^{\mathbf{N}}$. At this stage, we suppose that $\mathcal{M}$ is compact. Let $\mu$ be a Borel measure on $\mathcal{M}$ which satisfies condition (1.2) with some $s \in (\mathbf{N} - \varkappa, \mathbf{N}]$.

We consider operators of the form

$$\mathbf{G} = \mathbf{G}_{P_1,P_2,\mathfrak{B}} = P_1\mathfrak{B}P_2,\ P_j = V_j\mu, \tag{4.7}$$

in $L_{2,\mu}(\mathcal{M})$. Here $V_j$ are complex-valued $\mu$–measurable functions on $\mathcal{M}$, subject to conditions $V_j \in L_{r_j,\mu}(\mathcal{M})$ with some conditions imposed on $r_j$, to be specified later.

Such operators have been systematically studied in [14], Chapter 5, for the case $s = \mathbf{N}$, i.e., for a measure $\mu$ absolutely continuous with respect to the Lebesgue measure, and later in [34], Chapter V, for an arbitrary $s$, under the additional condition that $\mu$ coincides with the Hausdorff measure $\mathcal{H}^s$ of dimension $s$ on $\mathcal{M}$, moreover the two-sided estimate (1.1) was required.

The conditions imposed on $r_j$ in [34] are the following (we present them in *our* notations, for the Hilbert space case, $p = 2$):

$$r_j > 2; 1/r_1 + 1/r_2 < \frac{\varkappa - \mathbf{N} + s}{s} = \theta^{-1}. \tag{4.8}$$

Under these conditions, an estimate for the eigenvalues $\lambda_k(\mathbf{G})$, $|\lambda_1| \geq |\lambda_2| \geq \dots$ of the operator $\mathbf{G}$ (counted with algebraic multiplicity), obtained in [34], see, e.g., Theorem 28.2 there, has the following form:

$$|\lambda_k(\mathbf{G})| \leq C_{4.9}(\mathcal{M}, r_1, r_2)\|V_1\|_{L_{r_1,\mu}}\|V_2\|_{L_{r_2,\mu}} k^{-\theta^{-1}}, k \geq 1. \tag{4.9}$$

For the case of a self-adjoint operator $\mathbf{G}$, estimate (4.9) is equivalent to a similar estimate, of the same order, for the singular numbers $s_k(\mathbf{G})$. Both inequalities in (4.8) are not sharp: the constant $C_{4.9}(\mathcal{M}, r_1, r_2)$ depends, according to [34], on the set $\mathcal{M}$ and grows uncontrollably as $\mathcal{M}$ expands or as $1/r_1+1/r_2$ approaches $\theta^{-1}$, or as one of $r_j$ approaches 2. In the case $1/r_1 + 1/r_2 = \theta^{-1} < 1$, called in [14], [34] the 'limiting' one, the conditions imposed upon one of the weight functions $V_j$ are strengthened, namely, say, for $V_1$, it is required that $V_1$ belongs to the Orlicz space, $L^{r_1} \log(1 + L)$.

We are going to demonstrate here that using our approach, these results, in the Hilbert space setting, can be improved in several aspects. We show that the estimates for singular numbers, with constant *not depending on* $\mathcal{M}$ for the 'limiting' case, $1/r_1 + 1/r_2 = \theta^{-1}$, follow from our general estimates, again, under the condition that only the upper estimate in (1.1), namely (1.2), holds for measure $\mu$. In the discussion to follow, for simplicity of formulations, we consider the most important case of $\mathfrak{B} = (-\Delta)^{-\varkappa/2}$ in $\mathbb{R}^{\mathbf{N}}$.

**Theorem 4.5.** *Let $\mathfrak{B} = (-\Delta)^{-\varkappa/2}$ be the pseudodifferential operator in $\mathbb{R}^{\mathbf{N}}$, $\varkappa < \mathbf{N}$, $\mathcal{M}$ be a compact set in $\mathbb{R}^{\mathbf{N}}$ supporting a measure satisfying (1.2) with $s \in (\mathbf{N} - \varkappa, \mathbf{N})$. Then $\mathbf{G}$ as an operator of the form (4.7), with $V_j \in L_{r_j,\mu}$, $r_j > 2$, $1/r_1 + 1/r_2 = \theta^{-1} = \frac{\varkappa - \mathbf{N} + s}{s}$, can be defined as a bounded operator in $L_{2,\mu}$ and for its singular numbers $s_k(\mathbf{G})$ the following estimate holds*

$$s_k(\mathbf{G}) \leq C\mathcal{A}^{1-\theta^{-1}} k^{-\theta^{-1}} \|V_1\|_{L_{r_1,\mu}}\|V_2\|_{L_{r_2,\mu}}, \tag{4.10}$$

**Remark 4.6.** In terms of the distribution function of singular numbers, (4.10) takes the form

$$n(\lambda, \mathbf{G}) \leq C\lambda^{-\theta}\mathcal{A}^{\theta-1} \left[\int |V_1(X)|^{r_1}\mu(dX)\right]^{\theta/r_1} \left[\int |V_2(X)|^{r_2}\mu(dX)\right]^{\theta/r_2}. \tag{4.11}$$

It is important to note that the constants in (4.10), (4.11) do not depend on the set $\mathcal{M}$. Therefore, similarly to Theorem 3.6, the above result extends to arbitrary measures satisfying (1.2).

**Proof.** Again, it suffices to consider the case of nonnegative densities $V$. We start by giving an exact definition of the operator $\mathbf{G}$. Set $\theta_j = \frac{s}{2l_j - \mathbf{N}+s} = \frac{r_j}{2}$, $l_j = \frac{\mathbf{N}-s}{2} + \frac{s}{r_j} < \mathbf{N}/2$.

Since $\varkappa = l_1 + l_2$, consider the composition

$$\mathbf{G} = \mathfrak{V}_1 \mathfrak{V}_2^*, \tag{4.12}$$

where $\mathfrak{V}_j = V_j \Gamma_{\mathcal{M}} (-\Delta)^{-l_1/2}$, and $\Gamma_{\mathcal{M}}$ is the operator of restriction of functions in the Sobolev space $H^{l_j}$ to $\mathcal{M}$. The operator $\mathfrak{V}_j$ is considered as acting from $L_2(\mathbb{R}^{\mathbf{N}})$ to $L_{2,\mu}(\mathcal{M})$. As it follows from Proposition 2.2 and the Hölder inequality, cf. Section 2.2, the operator $\mathfrak{V}_j : L_2(\mathbb{R}^{\mathbf{N}}) \to L_{2,\mu}(\mathcal{M})$, is bounded under the conditions of the Theorem. Therefore, similarly to Section 4.1, the operator $\mathfrak{V}_j^* = (-\Delta)^{-l_j} \Gamma_{\mathcal{M}}^* V_j : L_{2,\mu}(\mathcal{M}) \to L_2(\mathbb{R}^{\mathbf{N}})$ is bounded as well and it can be expressed by duality as the composition of continuous operators

$$\mathfrak{V}_j^* = L_{2,\mu}(\mathcal{M}) \xrightarrow{V_j} L_{\frac{2r_j}{2+r_j},\mu}(\mathcal{M}) \xrightarrow{\Gamma_{\mathcal{M}}^*} \mathbf{L}^{-l_2}(\mathbb{R}^{\mathbf{N}}) \xrightarrow{(-\Delta)^{-l_j/2}} L_2(\mathbb{R}^{\mathbf{N}}). \tag{4.13}$$

Here $\Gamma_{\mathcal{M}}^* : L_{\frac{2r_j}{2+r_j},\mu} \to \mathbf{L}^{-l_j}$ is the embedding operator into the negative order homogeneous Sobolev space $\mathbf{L}^{-l_j}$ of distributions (dual to $\mathbf{L}^{l_j}$). This operator is continuous since it is adjoint to the embedding of $\mathbf{L}^{l_j}$ into $L_{\frac{2r_j}{r_j-2},\mu}$, the latter being bounded by Proposition 2.2. Therefore, the operator $\mathbf{G} = \mathfrak{V}_1 \mathfrak{V}_2^*$, defined in $L_{2,\mu}$ by the sesquilinear form $\mathbf{g}[u,v] = (\mathfrak{V}_2^* u, \mathfrak{V}_1^* v)$, acts as the composition of continuous operators

$$\begin{aligned} \mathbf{G} : L_{2,\mu}(\mathcal{M}) &\to L_{\frac{2r_2}{2+r_2},\mu}(\mathcal{M}) \to \mathbf{L}^{-l_2}(\mathbb{R}^{\mathbf{N}}) \to L_2(\mathbb{R}^{\mathbf{N}}) \to \\ &\to \mathbf{L}^{l_1}(\mathbb{R}^{\mathbf{N}}) \to L_{\frac{2r_1}{r_1-2},\mu}(\mathcal{M}) \to L_{2,\mu}(\mathcal{M}). \end{aligned}$$

This reasoning justifies the representation (4.12).

By the Ky Fan inequality, for the singular numbers of operators in (4.12), $s_{2k-1}(\mathbf{G}) = s_{2k-1}(\mathfrak{V}_1 \mathfrak{V}_2^*) \le s_k(\mathfrak{V}_1) s_k(\mathfrak{V}_2)$. For the operator $\mathfrak{V}_2$, we have $s_k(\mathfrak{V}_2) = s_k(\mathfrak{V}_2^* \mathfrak{V}_2)^{\frac{1}{2}} = s_k(\mathbf{T}_2)^{\frac{1}{2}}$, where $\mathbf{T}_2 = \mathbf{T}_{U_2\mu, \mathfrak{A}_2}$ is the operator considered in Subsect. 4.1, with $V = U_2 = V_2^2$ and $\mathfrak{A} = \mathfrak{A}_2 = (-\Delta)^{-l_2/2}$. By Theorem 3.6,

$$\begin{aligned} n(\lambda, \mathbf{T}_2) \le C \mathcal{A}(\mu)^{\theta_2 - 1} \int |U_2(X)|^{\theta_2} \mu(dX) \lambda^{-\theta_2} = \\ C \mathcal{A}(\mu)^{\theta_2 - 1} \int |V_2(X)|^{2\theta_2} \mu(dX) \lambda^{-\theta_2}. \end{aligned}$$

The last estimate, expressed in terms of singular numbers, takes the form

$$s_k(\mathfrak{V}_2^*) = s_k(\mathfrak{V}_2) = s_k(\mathbf{T}_2)^{\frac{1}{2}} \le C \mathcal{A}(\mu)^{(1-\theta_2^{-1})/2} \|V_2\|_{2\theta_2,\mu} k^{-1/(2\theta_2)}. \tag{4.14}$$

Similarly, for $s_k(\mathfrak{V}_1)$,

$$s_k(\mathfrak{V}_1) \le C\mathcal{A}(\mu)^{(1-\theta_1^{-1})/2}\|V_1\|_{2\theta_1,\mu}k^{-1/\theta_2}. \tag{4.15}$$

We combine (4.14) and (4.15) by means of the Ky Fan inequality to get

$$s_{2k-1}(\mathbf{G}) \le s_k(\mathfrak{V}_1)s_k(\mathfrak{V}_2^*) \le$$
$$C\mathcal{A}(\mu)^{(2-\theta_1^{-1}-\theta_2^{-1})/2}\|V_1\|_{2\theta_1,\mu}\|V_2\|_{2\theta_2,\mu}k^{-\frac{1}{2}(\theta_1^{-1}+\theta_2^{-1})},$$

which, since $\theta_1^{-1}+\theta_2^{-1} = 2\theta^{-1}$, coincides with (4.10). □

We skip the discussion of spectral estimates for other relations between $r_1, r_2, \varkappa, \mathbf{N}$, which are established by means of a similar reasoning, using the results of our Subsection 4.1.

Another type of non-self-adjoint operators with singular weights, considered in [34], are

$$\mathbf{G} = \mathfrak{A}_2 P \mathfrak{A}_1,$$

where $\mathfrak{A}_1, \mathfrak{A}_2$ are pseudodifferential operators in $\mathbb{R}^{\mathbf{N}}$ of orders $-l_1, -l_2$ and $P$ is a singular measure of the form $P = V\mu$, with $\mu$ being, as before, the dimension $s$ Hausdorff measure $\mathcal{H}^s$ on a set $\mathcal{M} \subset \mathbb{R}^{\mathbf{N}}$, satisfying (1.1), and $V$ being a $\mu$-measurable function on $\mathcal{M}$ belonging to $L_{r,\mu}$. As in the previous case, the conditions imposed in [34] on the parameters of this operator are not sharp, which leads to non-sharpness of spectral estimates.

Spectral (singular numbers) estimates for such operators are obtained here in the same way as in Theorem 4.5. Namely, we perform a factorization, this time of the weight function $V$, which leads to the factorization of the operator $\mathbf{G}$. Again, we present the most aesthetic case.

**Theorem 4.7.** *Let $\mathfrak{A}_j = (-\Delta)^{-l_j/2}$, $j = 1, 2$, in $\mathbb{R}^{\mathbf{N}}$, $2l_j < \mathbf{N}$, $2l = l_1 + l_2$. Let, further, $\mu$ be a singular measure satisfying (1.2) with exponent $s$, $\mathbf{N} - 2l_j < s \le \mathbf{N}$, and $P = V\mu$, $V \in L_{\theta,\mu}$. We set $\theta_j = \frac{s}{2l_j - \mathbf{N} + s} > 1$, so that $\theta^{-1} = (2\theta_1)^{-1} + (2\theta_2)^{-1} = \frac{2l-\mathbf{N}+s}{s}$, $\theta > 1$. Consider the operator $\mathbf{G} = \mathbf{G}_{\mathfrak{A}_1,\mathfrak{A}_2,P} = \mathfrak{A}_2 P \mathfrak{A}_1$ in $L_2(\mathbb{R}^{\mathbf{N}})$ defined by the sesquilinear form*

$$\mathbf{g}[u, v] = \int_{\mathbb{R}_{\mathbf{N}}} (\mathfrak{A}_1 u)(X)\overline{(\mathfrak{A}_2 v)(X)}P(dX),\ u, v \in L_2(\mathbb{R}^{\mathbf{N}}).$$

*Then for the singular numbers of $\mathbf{G}$, the following estimate holds*

$$n(\lambda, \mathbf{G}) \le C \int |V(X)|^\theta \mu(dX)\lambda^{-\theta}. \tag{4.16}$$

**Proof.** As before, it suffices to consider the case of real, non-negative $V$. We represent $V$ as $V = V_1V_2$, $V_j = V^{\gamma_j}$, where $\gamma_j = \theta_{3-j}(\theta_1+\theta_2)^{-1} = \theta/(2\theta_j)$, $\theta_j = \frac{s}{2l_j - \mathbf{N}+s}$, so that $\gamma_1+\gamma_2 = 1$, $\theta_1^{-1}+\theta_2^{-1} = 2(\theta)^{-1}$. With this representation, the operator $\mathbf{G}$ factorizes as

$$\mathbf{G} = \mathfrak{U}_2^*\mathfrak{U}_1,\ \mathfrak{U}_j = V_j\Gamma_\mu\mathfrak{A}_j : L_2(\mathbb{R}^{\mathbf{N}}) \to L_{2,\mu}(\mathcal{M}).$$

As in Theorem 4.5, it follows from Lemma 2.3 that the operators $\mathfrak{U}_j$ are bounded as acting from $L_2(\mathbb{R}^{\mathbf{N}})$ to $L_{2,\mu}(\mathcal{M})$ and are factorized as compositions of continuous operators. By the Ky Fan inequality

$$n(\lambda_1\lambda_2, \mathfrak{U}_2^*\mathfrak{U}_1) \le n(\lambda_1, \mathfrak{U}_1) + n(\lambda_2, \mathfrak{U}_2), \tag{4.17}$$

where $\lambda_j$, $\lambda_1\lambda_2 = \lambda$, will be fixed later. For the terms on the right in (4.17), we have

$$n(\lambda_j, \mathfrak{U}_j) = n(\lambda_j^2, \mathfrak{U}_j^*\mathfrak{U}_j).$$

The operator $\mathfrak{U}_j^*\mathfrak{U}_j$ coincides with $\mathfrak{A}_j^*V_j^2\mathfrak{A}_j$, and by Theorem 4.3 the following estimate holds

$$\begin{aligned} n(\lambda_j, \mathfrak{U}_j) \le C\mathcal{A}^{\theta_j-1}\lambda_j^{-2\theta_j}\int V_j(X)^{2\theta_j}\mu(dX) = \\ C\mathcal{A}^{\theta_{3-j}-1}\lambda_j^{-2\theta_j}\int V(X)^{\theta}\mu(dX), \end{aligned} \tag{4.18}$$

since $V_j^{2\theta_j} = V^{2\gamma_j\theta_j} = V^\theta$. Finally, we set $\lambda_j = \lambda^{\frac{\theta_j}{\theta_1+\theta_2}}$, so that $\lambda_1\lambda_2 = \lambda$, and now (4.18), (4.17) give the required inequality (4.16). □

**Remark 4.8.** In a similar way, other results of Sections 3, 4 are carried over to non-selfadjoint operators.

## 5. Localization

In this section, we present some auxiliary results on perturbations and localization in asymptotic eigenvalue estimates for operators of the form $\mathbf{T}_{P,\mathfrak{A}}$, needed for obtaining eigenvalue asymptotics. Some of these results are analogous to the ones in Sect. 3 in [25]. We, however, give complete proofs.

It is convenient to describe perturbation and localization properties without supposing that asymptotic formulas hold. So, the asymptotic characteristics of eigenvalue distribution, the ones introduced in [6], are used.

**Definition 5.1.** Let $\mathbf{T}$ be a compact self-adjoint operator. For $\theta > 0$, we define the quantities

$$\mathbf{D}_{\pm}^{\theta}(\mathbf{T}) = \limsup_{\lambda\to 0} n_{\pm}(\lambda,\mathbf{T})\lambda^{\theta},\ \mathbf{d}_{\pm}^{\theta}(\mathbf{T}) = \liminf_{\lambda\to 0} n_{\pm}(\lambda,\mathbf{T})\lambda^{\theta}, \tag{5.1}$$

where, recall, $n_{\pm}(\lambda,\mathbf{T})$ is the distribution function of positive (negative) eigenvalues of $\mathbf{T}$.

Of course, these quantities equal zero or infinity for all values $\theta$ except, possibly, just one. However, if for some $\theta$, for a certain choice of $\pm$ sign, both $\mathbf{D}_{\pm}^{\theta}(\mathbf{T})$ and $\mathbf{d}_{\pm}^{\theta}(\mathbf{T})$ are finite, nonzero, and equal, this means that the eigenvalues of $\mathbf{T}$ of the corresponding sign are subject to an asymptotic formula of order $\theta$. Some of the statements to follow appear mostly in two versions each, the subcritical and supercritical ones. The proofs are identical, up to notations, therefore we present them only for one statement in the pair.

We will systematically refer to the following elementary observation.

**Proposition 5.2.** *Let $l < l'$ and a pseudodifferential operator $\mathfrak{A}'$ have order $-l'$; for $2l < \mathbf{N}$ it is supposed that $2l' < \mathbf{N}$. Suppose that the measure $\mu$ and the density $V$ satisfy Condition 4.1. Then $n_{\pm}(\lambda,\mathbf{T}_{V\mu,\mathfrak{A}'}) = o(\lambda^{-\theta})$, $\theta = \frac{s}{2l-\mathbf{N}+s}$.*

This property follows from the fact that for fixed $P, s$, the exponent $\theta$ decreases as the order $l$ grows.

**Lemma 5.3** *(Lower order perturbation). Let $\mathfrak{A}$ be a pseudodifferential operator in $\Omega$ of order $-l$ and $\mathfrak{B}$ be a pseudodifferential operator of order $-l' < -l$, $\mathfrak{A}_1 = \mathfrak{A} + \mathfrak{B}$. Let Condition 4.1 be satisfied. Then for $\theta = \frac{s}{s+2l-\mathbf{N}}$,*

$$\mathbf{D}_{\pm}^{\theta}(\mathbf{T}_{P,\mathfrak{A}_1}) = \mathbf{D}_{\pm}^{\theta}(\mathbf{T}_{P,\mathfrak{A}}),\ \mathbf{d}_{\pm}^{\theta}(\mathbf{T}_{P,\mathfrak{A}_1}) = \mathbf{d}_{\pm}^{\theta}(\mathbf{T}_{P,\mathfrak{A}}). \tag{5.2}$$

*In particular, if for the positive (negative) eigenvalues of operator $\mathbf{T}_{P,\mathfrak{A}}$ an asymptotic formula of order $\theta$ is valid, it is valid for eigenvalues of operator $\mathbf{T}_{P,\mathfrak{A}_1}$ as well, with the same coefficient.*

**Proof.** Consider, e.g., the subcritical case. The difference $\mathbf{T}_{P,\mathfrak{A}_1} - \mathbf{T}_{P,\mathfrak{A}}$ is defined by the quadratic form which is the sum of terms $\int |\mathfrak{B}u(X)|^2 P(dX)$ and $2\int \mathrm{Re}\Big[(\mathfrak{B}u)\overline{(\mathfrak{A}u)}\Big] P(dX)$. For the former term, $\mathfrak{B}$ is a lower order perturbation and Proposition 5.2 applies. As for the second form above, we have

$$\begin{aligned} 2\left|\int \mathrm{Re}\left[(\mathfrak{B}u)\overline{(\mathfrak{A}u)}\right] P(dX)\right| \le \epsilon \int |\mathfrak{A}u(X)|^2 |V(X)|\mu(dX) \\ +\epsilon^{-1}\int |\mathfrak{B}u(X)|^2 |V(X)|\mu(dX). \end{aligned} \tag{5.3}$$

Here, the first term on the right-hand side defines the operator $\epsilon\mathbf{T}_{|V|\mu,\mathfrak{A}}$ with an arbitrarily small $\epsilon$, so, for the corresponding operator an estimate $n(\lambda) \le C||V||^{\theta}_{L_{\theta,\mu}}\epsilon^{\theta}\lambda^{-\theta}$.

The second term defines the operator already considered, with eigenvalues subject to $n_\pm(\lambda) = o(\lambda^{-\theta})$; by the arbitrariness of $\epsilon$, we have $n_\pm(\lambda, \mathbf{T}_{P,\mathfrak{A}'} - \mathbf{T}_{P,\mathfrak{A}}) = o(\lambda^{-\theta})$. Then (5.2) follows from the Ky Fan inequality, compare with Theorem 11.6.8 in [10]. $\square$

**Lemma 5.4** *(Cf. Observation 4 in [25]). Suppose that the operator $\mathfrak{A}$ is of order $-l$ and the measure $P = V\mu$ satisfies Condition 4.1. Let $\chi$ be the characteristic function of a closed set $\mathbf{E} \subset \Omega$ such that $\mathbf{E} \cap \mathcal{M} = \varnothing$. Then for the operator $\mathbf{T}_{P,\mathfrak{A}\chi}$ defined by the quadratic form*

$$\int_\Omega |\mathfrak{A}(\chi u)(X)|^2 P(dX), \tag{5.4}$$

*the eigenvalues satisfy*

$$\mathbf{D}^\theta_\pm(\mathbf{T}_{P,\mathfrak{A}\chi}) = 0. \tag{5.5}$$

This result demonstrates the following spectral localization property: if the quadratic form is restricted to functions supported away from the support of the measure $\mu$, the eigenvalues decay faster than this is prescribed by the general estimate.

**Proof.** Let $\psi \in C_0^\infty(\Omega)$ be a function which is equal to 0 on $\mathbf{E}$ and to 1 in a neighborhood of $\mathcal{M}$. Then

$$\mathbf{t}_{P,\mathfrak{A}\chi}[u] = \int V(X)|\psi(X)\mathfrak{A}(u\chi)(X)|^2\mu(dX) =$$
$$\int V(X)|([\psi, \mathfrak{A}](u\chi))(X)|^2\mu(dX),$$

and since the commutator $[\psi, \mathfrak{A}]$ has order $-l-1$, Proposition 5.2 applies. $\square$

It follows, in particular, that the eigenvalue counting function gets a lower order perturbation if operator $\mathfrak{A}$ is perturbed outside a neighborhood of the set $\mathcal{M}$. This circumstance gives us freedom in choosing cut-off functions away from the $\mathcal{M}$ or adding operators that are smoothing near $\mathcal{M}$ – the possibility already mentioned.

A complication which is often encountered in the study of eigenvalue asymptotics is the non-additivity of asymptotic coefficients: if for some operators $T_1, T_2$, asymptotic formulas of the same order for $n_\pm(\lambda, T_1)$, $n_\pm(\lambda, T_2)$ are known, this does not automatically imply a similar formula for $n_\pm(\lambda, T_1 + T_2)$ (and does not generally imply any asymptotic formula for eigenvalues at all). In the following statement, important in the study of eigenvalue asymptotics, it is shown that if two measures have supports separated by a positive distance, then, up to a lower order term, the counting functions behave additively with respect to the measures: measures are, so to say, spectrally almost orthogonal.

**Lemma 5.5.** *Let Condition 4.1 be satisfied, $\mathfrak{A}$ be an operator of order $-l$. Suppose that $P = P_1 + P_2$, where $P_j = V_j\mu_j$ is a measure supported on a compact set $\mathcal{M}_j$ and* $\operatorname{dist}(\mathcal{M}_1, \mathcal{M}_2) > 0$. *Then*

$$n_\pm(\lambda, \mathbf{T}_{P_1+P_2}) = n_\pm(\lambda, \mathbf{T}_{P_1}) + n_\pm(\lambda, \mathbf{T}_{P_2}) + o(\lambda^{-\theta}) \tag{5.6}$$

*as $\lambda \to 0$; in particular,*

$$\mathbf{D}^\theta_\pm(\mathbf{T}_{P_1+P_2}) \le \mathbf{D}^\theta_\pm(\mathbf{T}_{P_1}) + \mathbf{D}^\theta_\pm(\mathbf{T}_{P_2}),\ \mathbf{d}^\theta_\pm(\mathbf{T}_{P_1+P_2}) \ge \mathbf{d}^\theta_\pm(\mathbf{T}_{P_1}) + \mathbf{d}^\theta_\pm(\mathbf{T}_{P_2}). \tag{5.7}$$

**Proof.** Consider two disjoint open sets $\Omega_1, \Omega_2 \subset \Omega$, such that $\mathcal{M}_j \subset \Omega_j$ and $\overline{\Omega_1} \cup \overline{\Omega_2} \supset \bar{\Omega}$. Every function $u \in L_2(\Omega)$ splits into the (orthogonal) sum $u = u_1 \oplus u_2$, $u_j \in L_2(\Omega_j)$. The quadratic form of the operator $\mathbf{T}_{P_1+P_2}$ splits as follows

$$\begin{gathered}
\mathbf{t}_{P_1+P_2}[u] := (\mathbf{T}_{P_1+P_2}u, u)_{L^2(\Omega)} = \\
\int_\Omega V_1(X)|\mathfrak{A}(u_1)(X) + \mathfrak{A}(u_2)(X)|^2\mu_1(dX) + \\
\int_\Omega V_2(X)|\mathfrak{A}(u_1)(X) + \mathfrak{A}(u_2)(X)|^2\mu_2(dX) \\
\equiv \mathbf{t}_1[u_1] + \mathbf{t}_2[u_2] + \mathbf{t}_R[u_1, u_2] := \\
\int_\Omega V_1(X)|\mathfrak{A}(u_1)(X)|^2\mu_1(dX) + \int_\Omega V_2(X)|\mathfrak{A}(u_2)(X)|^2\mu_2(dX) + \mathbf{t}_R[u_1, u_2].
\end{gathered} \tag{5.8}$$

The remainder term $\mathbf{t}_R[u_1, u_2]$ is a quadratic form of the function $u = u_1 \oplus u_2$, $u_j \in L_2(\Omega_j)$, with the following property: if a term in $\mathbf{t}_R$ contains the measure $P_j$, then it necessarily contains the function $u_{3-j}$, so it always contains a measure and a function with disjoint supports. If such a term has the form $\int_\Omega V_1|\mathfrak{A}u_2|^2\mu_1(dX)$, then the corresponding operator $\mathbf{T}$ satisfies $n_\pm(\lambda, \mathbf{T}) = o(\lambda^{-\theta})$ by Lemma 5.4. If, on the other hand, such term has the form $\int_\Omega V_1(\mathfrak{A}u_1)\overline{(\mathfrak{A}u_2)}\mu_1(dX)$, then by the Cauchy-Schwartz inequality,

$$\begin{gathered}
\left|\int_\Omega V_1(\mathfrak{A}u_1)\overline{(\mathfrak{A}u_2)}\mu_1(dX)\right| \le \\
\left(\int_\Omega |V_1||\mathfrak{A}u_1|^2\mu_1(dX)\right)^{1/2}\left(\int_\Omega |V_1||\mathfrak{A}u_2|^2\mu_1(dX)\right)^{1/2}.
\end{gathered} \tag{5.9}$$

The quadratic form in the last factor above, containing a function and a measure with disjoint supports, again by Lemma 5.4, defines an operator with faster decaying eigenvalues. We repeat now the reasoning in Lemma 5.3 which shows that the quadratic form on the left-hand side in (5.9) defines an operator, again, with singular numbers satisfying $n(\lambda) = o(\lambda^{-\theta})$.

Now we observe that the quadratic forms $\mathbf{t}_1[u_1]$, $\mathbf{t}_2[u_2]$ in (5.8) act on orthogonal subspaces $L_2(\Omega_1), L_2(\Omega_2)$. Therefore, the spectrum of the sum of the corresponding operators $\mathbf{T}_1, \mathbf{T}_2$ equals the union of the spectra of the summands, and hence $n_\pm(\lambda, \mathbf{T}_1 + \mathbf{T}_2) = n_\pm(\lambda, \mathbf{T}_1) + n_\pm(\lambda, \mathbf{T}_2)$. The term $\mathbf{t}_R$ in (5.8) makes a weaker contribution,

$$n_\pm(\lambda, \mathbf{T}_{P_1+P_2}) = n_\pm(\lambda, \mathbf{T}_1) + n_\pm(\lambda, \mathbf{T}_2) + o(\lambda^{-\theta}), \tag{5.10}$$

which gives a representation for the operator involved on the left-hand side in (5.6). Now we consider the operator $\mathbf{T}_{P_j}$, $j = 1, 2$, present on the right-hand side in (5.6). It is defined by the quadratic form

$$\mathbf{t}_{P_j}[u] = \int V_j(X)|\mathfrak{A}u|^2 \mu_j(dX) \equiv \int V_j(X)|\mathfrak{A}(u_1 \oplus u_2)(X)|\mu_1(dX).$$

Similarly to (5.8), we represent it as

$$\mathbf{t}_{P_j}[u] = \int\limits_{\Omega} V_j(X)|\mathfrak{A}(u_j)(X)|^2 \mu_j(dX) + \mathbf{t}_{R_j}[u_1, u_2], \tag{5.11}$$

with $\mathbf{t}_{R_j}$ having the same structure as $\mathbf{t}_R$ in (5.8). Again, the quadratic form $\mathbf{t}_{R_j}$ defines an operator $\mathbf{T}_{R_j}$ with eigenvalues satisfying $n_\pm(\lambda, \mathbf{T}_{R_j}) = o(\lambda^{-\theta})$, and we obtain

$$n_\pm(\lambda, \mathbf{T}_{P_j}) = n_\pm(\lambda, \mathbf{T}_j) + o(\lambda^{-\theta}). \tag{5.12}$$

Finally, we substitute (5.12) into (5.10), to obtain (5.6), and therefore, (5.7). (Different signs in two inequalities in (5.7) arise, of course, due to the fact that $\limsup(f(\lambda) + g(\lambda)) \le \limsup f(\lambda) + \limsup g(\lambda)$, while $\liminf(f(\lambda) + g(\lambda)) \ge \liminf f(\lambda) + \liminf g(\lambda)$.) $\square$

An important corollary of this lemma provides us with the additivity property in asymptotic formulas.

**Corollary 5.6.** *Under the conditions of Lemma 5.5, if, for some sign, the following asymptotic formula for the eigenvalues*

$$\lim_{\lambda \to 0} \lambda^\theta n_\pm(\lambda, \mathbf{T}_{P_j}) = A_j^\pm,\ j = 1, 2, \tag{5.13}$$

*holds, then a similar formula is valid for* $\mathbf{T}_{P_1+P_2} = \mathbf{T}_{P_1} + \mathbf{T}_{P_2}$*:*

$$\lim_{\lambda \to 0} \lambda^\theta n_\pm(\lambda, \mathbf{T}_{P_1} + \mathbf{T}_{P_2}) = A_1^\pm + A_2^\pm. \tag{5.14}$$

Relations (5.13) are equivalent to $\mathbf{D}_{\pm}^{\theta}(\mathbf{T}_{P_j}) = \mathbf{d}_{\pm}^{\theta}(\mathbf{T}_{P_j}) = A_j^{\pm}$, and now (5.14) follows from (5.7).

Another corollary of Lemma 5.5 allows us to separate the positive and negative parts of the function $V$ when studying the distribution of the positive and negative eigenvalues of $\mathbf{T}_P$ separately.

**Corollary 5.7.** *Suppose that Condition 4.1 is satisfied. Let $\mathcal{M}$ be a compact set, $\mu = \mu_+ + \mu_-$, $\operatorname{supp}\mu_\pm = \mathcal{M}_\pm$, $\operatorname{dist}(\mathcal{M}_+, \mathcal{M}_-) > 0$, and $V \in L_\vartheta(\mathcal{M})$, $V = V_+ - V_-$, $V_\pm \geq 0$ in $\mathcal{M}_\pm$, $V_\pm = 0$ outside $\mathcal{M}_\pm$, $P = V\mu$, $P_\pm = V_\pm \mu_\pm$. Then*

$$n_\pm(\lambda, \mathbf{T}_P) = n_+(\lambda, \mathbf{T}_{V_\pm \mu}) + o(\lambda^{-\theta}) \quad as \quad \lambda \to 0, \tag{5.15}$$

*in particular, $\mathbf{D}_{\pm}^{\theta}(\mathbf{T}_P) = \mathbf{D}_{+}^{\theta}(\mathbf{T}_{P_\pm})$, $\mathbf{d}_{\pm}^{\theta}(\mathbf{T}_P) = \mathbf{d}_{+}^{\theta}(\mathbf{T}_{P_\pm})$,*

In other words, up to a lower order remainder, the asymptotic behavior of the positive, resp., negative, eigenvalues of the operator $\mathbf{T}_{V\mu}$ with a density of variable sign is determined by the positive, resp., negative, part of the density $V$, as soon as these parts are separated. To prove this property, we can use (5.10), taking as $P_1$ the restriction of measure $P$ to the set $\mathcal{M}_+$, and as $P_2$ its restriction to $\mathcal{M}_-$, and recall that $n_-(\lambda, \mathbf{T}_{V\mu,\Omega_+}) = n_+(\lambda, \mathbf{T}_{V\mu,\Omega_-}) = 0$, where $\Omega_\pm$ are neighborhoods of $\mathcal{M}_\pm$ such that $\operatorname{dist}(\Omega_+, \Omega_-) > 0$.

Finally, we get rid of the condition for the above sets to be separated.

**Theorem 5.8.** *Let the measure $\mu$, the real-valued density $V$, and the operator $\mathfrak{A}$ satisfy Condition 4.1. Then (5.15) holds.*

**Proof.** We follow mostly the reasoning in [25], where a similar property was established in a more restricted form. For a given $\varepsilon > 0$ we approximate the density $V$ by a function $V_\varepsilon$, continuous on $\mathcal{M}$ so that $\int_{\mathcal{M}} |V - V_\varepsilon|^\vartheta \mu(dX) < \varepsilon$, $\vartheta = \max(\theta, 1)$. By Theorem 4.2

$$\mathbf{D}_\theta^{\pm}(\mathbf{T}_V - \mathbf{T}_{V_\varepsilon}) = \mathbf{D}_\theta^{\pm}(\mathbf{T}_{V-V_\varepsilon}) \leq C\varepsilon^{\theta/\vartheta}.$$

Consider the set $\mathcal{M}_0 = \{X \in \mathcal{M} : V_\varepsilon = 0\}$ and its $\delta$-neighborhood $\mathcal{M}_\delta$. For sufficiently small $\delta$, the quantity $\int_{\mathcal{M}_\delta} |V_\varepsilon|^\vartheta \mu(dX)$ is less than $\varepsilon$. So, for the weight $V'_\varepsilon$, the restriction of $V_\varepsilon$ to $\mathcal{M} \setminus \mathcal{M}_\delta$, by the same estimates as in Theorem 4.2,

$$\mathbf{D}_\theta^{\pm}(\mathbf{T}_V - \mathbf{T}_{V'_\varepsilon}) = \mathbf{D}_\theta^{\pm}(\mathbf{T}_{V-V'_\varepsilon}) \leq C\varepsilon^{\theta/\vartheta}.$$

Therefore,

$$\mathbf{D}_\theta^{\pm}(\mathbf{T}_V - \mathbf{T}_{V'_\varepsilon}) < C\varepsilon^{\theta/\vartheta}. \tag{5.16}$$

Now, the density $V'_\varepsilon$ satisfies the conditions of Corollary 5.7, and (5.15) follows from (5.16) and the arbitrariness of $\varepsilon$ by means of the Ky Fan inequality, see Corollary 11.6.5 in [10]. □

## 6. Eigenvalue asymptotics

Derivation of eigenvalue asymptotic formulas follows now the pattern of [25] and [23].

### *6.1. The asymptotic perturbation lemma*

For convenience of the readers, we reproduce here the asymptotic perturbation lemma by M.Sh.B̃irman and M.Z. Solomyak, see, e.g. [6], Lemma 1.5, which we use systematically.

**Lemma 6.1.** *Let $T$ be a compact self-adjoint operator. Suppose that for any $\varepsilon > 0$, the operator $T$ can be represented as the sum, $T = T_\varepsilon + T'_\varepsilon$, so that for $T_\varepsilon$ the asymptotic formula for positive, resp., negative, eigenvalues holds, $\lim_{\lambda\to 0} \lambda^\theta n_\pm(\lambda, T_\varepsilon) = \mathbf{C}^\pm_\varepsilon$, and for $T'_\varepsilon$ both relations*

$$\lim_{\varepsilon\to 0} \mathbf{D}^\pm_\theta(T'_\varepsilon) = 0 \tag{6.1}$$

*hold. Then the limit $\mathbf{C}^\pm = \lim_{\varepsilon\to 0} \mathbf{C}^\pm_\varepsilon$ exists and for the operator $T$, the asymptotic formula is valid, $\lim_{\lambda\to 0} \lambda^\theta n_\pm(\lambda, T) = \mathbf{C}^\pm$.*

### *6.2. Compact Lipschitz surfaces*

The first result on eigenvalues asymptotics concerns measures supported on Lipschitz surfaces. A compact Lipschitz surface is, locally, a graph of a Lipschitz vector-function.

Let $\Sigma \subset \mathbb{R}^{\mathbf{N}}$ be a compact Lipschitz surface of dimension $d : 0 < d < \mathbf{N}$ and codimension $\mathfrak{d} = \mathbf{N} - d$, in a domain $\Omega \subset \mathbb{R}^{\mathbf{N}}$, defined, locally, in appropriate local co-ordinates $X = (\mathbf{x}, \mathbf{y}), \mathbf{x} \in \omega \subset \mathbb{R}^d, \mathbf{y} \in \mathbb{R}^{\mathfrak{d}}$, by the equation $\mathbf{y} = \boldsymbol{\phi}(\mathbf{x})$ with a Lipschitz vector-function $\boldsymbol{\phi}$: $|\boldsymbol{\phi}(\mathbf{x}) - \boldsymbol{\phi}(\mathbf{x}')| \le C_{\boldsymbol{\phi}}|\mathbf{x} - \mathbf{x}'|$, $\mathbf{x}, \mathbf{x}' \in \omega \subset \mathbb{R}^d$. The measure $\mu$ on $\Sigma$, generated by the embedding of $\Sigma$ into $\mathbb{R}^{\mathbf{N}}$ coincides with the $d$-dimensional Hausdorff measure $\mathcal{H}^d$ on $\Sigma$. For a compact Lipschitz surface, in one co-ordinate neighborhood, and therefore, globally, condition (1.1) is satisfied, with $s = d$. For noncompact Lipschitz surfaces, the situation is somewhat more complicated, see Section 6.4.

By the Rademacher theorem, for $\mu$-almost every $X \in \Sigma$, there exists a tangent space $\mathtt{T}_X\Sigma$ to $\Sigma$ at the point $X$ and, correspondingly, the normal space $\mathtt{N}_X\Sigma$, which are identified naturally with the cotangent and the conormal spaces. By $\mathtt{S}_X\Sigma$ we denote the sphere $|\xi| = 1$ in $\mathtt{T}_X\Sigma$.

**Theorem 6.2.** *Let a real function $V$ on $\Sigma$ belong to $L_{\vartheta,\mu}(\Sigma)$, $\vartheta = \max(\theta, 1)$, $\theta = \frac{d}{2l-\mathbf{N}+d} = \frac{d}{2l-\mathfrak{d}}$, $\mathfrak{d} < 2l$ and $P = V\mu$. Let $\mathfrak{A}$ be an order $-l$ pseudodifferential operator compactly* *supported in* $\Omega \subset \mathbb{R}^{\mathbf{N}}$*, with principal symbol* $\mathbf{a}_{-l}(X, \Xi)$*. At those points* $X \in \Sigma$*, where the tangent plane exists, we define the auxiliary symbol* $\mathbf{r}_{-\sigma}(X, \xi)$*,* $\xi \in \mathrm{T}_X\Sigma$*, of order* $-\sigma = \mathfrak{d} - 2l < 0$*,*

$$\mathbf{r}_{-\sigma}(X, \xi) = (2\pi)^{-\mathfrak{d}} \int_{\mathrm{N}_X\Sigma} |\mathbf{a}_{-l}(X, \xi, \eta)|^2 d\eta, (X, \xi) \in \mathrm{T}^*\Sigma, \tag{6.2}$$

*and the density*

$$\rho_{\mathfrak{A}}(X) = \frac{1}{d(2\pi)^d} \int_{\mathrm{S}_X\Sigma} \mathbf{r}_{-\sigma}(X, \xi)^\theta d\xi,\ X \in \Sigma. \tag{6.3}$$

*Then for the eigenvalues of the operator* $\mathbf{T}_{V,\mu,\mathfrak{A}} = \mathfrak{A}^* P \mathfrak{A}$*,* $P = V\mu$*, the following asymptotic formulas are valid*

$$n_{\pm}(\lambda, \mathbf{T}_{V\mu,\mathfrak{A}}) \sim \lambda^{-\theta} A_{\pm}(V, \Sigma, \mathfrak{A}),\ \lambda \to 0, \tag{6.4}$$

*where*

$$A_{\pm}(V, \Sigma, \mathfrak{A}) = \frac{1}{d(2\pi)^d} \int_{\Sigma} \int_{S_X\Sigma} V_{\pm}(X)^\theta \mathbf{r}_{-\sigma}(X, \xi)^\theta d\xi \mu(dX) = \tag{6.5}$$

$$\int_{\Sigma} V_{\pm}(X)^\theta \rho_{\mathfrak{A}}(X) \mu(dX),$$

*with the density* $\rho_{\mathfrak{A}}(X)$ *defined in* (6.3).

In the particular case of $\mathfrak{A} = \mathfrak{A}_0 = (1 - \Delta)^{-l/2}$ in a neighborhood of $\Sigma$ in $\mathbb{R}^{\mathbf{N}}$, we have $\mathbf{a}_{-l}(X, \Xi) = |\Xi|^{-l}$ and

$$\mathbf{r}_{-\sigma}(X, \xi) = (2\pi)^{-\mathfrak{d}} \int_{\mathbb{R}^{\mathfrak{d}}} (|\xi|^2 + |\eta|^2)^{-l} d\eta =$$

$$|\xi|^{-\sigma} (2\pi)^{-\mathfrak{d}} \boldsymbol{\omega}_{\mathfrak{d}-1} \int_0^\infty \zeta^{\mathfrak{d}-1}(1 + \zeta^2)^{-l} d\zeta = \frac{\boldsymbol{\omega}_{\mathfrak{d}-1}}{2(2\pi)^{\mathfrak{d}}} \mathbf{B}\left(\frac{\mathfrak{d}}{2}, l - \frac{\mathfrak{d}}{2}\right) |\xi|^{-\sigma},$$

where $\boldsymbol{\omega}_{\mathfrak{d}-1}$ is the volume of the unit sphere in $\mathbb{R}^{\mathfrak{d}}$, $\mathbf{B}$ is Euler's Beta-function. Therefore,

$$\rho_{\mathfrak{A}_0} \equiv \boldsymbol{\rho}(\mathfrak{d}, l) = \frac{\omega_{d-1}}{d(2\pi)^d} \left[\frac{\omega_{\mathfrak{d}-1}}{2(2\pi)^{\mathfrak{d}}} \mathbf{B}\left(\frac{\mathfrak{d}}{2}, l - \frac{\mathfrak{d}}{2}\right)\right]^\theta \tag{6.6}$$

and, finally,

$$n_{\pm}(\lambda, \mathbf{T}_{V,\mathfrak{A}_0}) \sim \lambda^{-\theta} \boldsymbol{\rho}(\mathfrak{d}, l) \int_{\Sigma} V_{\pm}^\theta(X) \mu(dX), \tag{6.7}$$

**Proof.** The proof of Theorem 6.2 follows closely the one of Theorem 2.4 in [25], with modifications caused by a different order of the operators involved. We give detailed explanations of the corresponding changes, directing interested readers to [25] for more details.

First, by the usual localization (see how this procedure is performed, e.g., in [9] or [27]), it is sufficient to prove the asymptotic formula for $V$ supported in just one co-ordinate neighborhood of $\Sigma$. In such case, we can approximate $V$ by a density $V_\varepsilon$, defined and smooth in a neighborhood of $\Sigma$ (therefore, in $\mathbb{R}^{\mathbf{N}}$) such that the $L_{\vartheta,\mu}(\Sigma)$-norm of $V_\varepsilon - V$ is small, less than $\varepsilon^{\theta/\vartheta}$. By eigenvalue estimates in Sect. 4, see Theorem 2.4, the eigenvalue distribution functions for operators $\mathbf{T}_{V,\mu,\mathfrak{A}}$ and $\mathbf{T}_{V_\varepsilon,\mu,\mathfrak{A}}$ differ asymptotically by less than $C\varepsilon\lambda^{-\theta}$. By the asymptotic perturbation Lemma 6.1, such approximation enables us to restrict our task to proving asymptotic formulas for such nice densities $V_\varepsilon$ only, passing then to the limit as $V_\varepsilon$ approaches $V$ in the $L_{\vartheta,\mu}(\Sigma)$ norm. At the next step, we separate the positive and negative eigenvalues of our operator, using Theorem 5.8. In this way, the problem is reduced to the case of a non-negative density $V_\varepsilon$, which, again after adding a perturbation with arbitrarily small $L_{\theta,\mu}(\Sigma)$-norm and using Lemma 6.1, we may suppose being the restriction to $\Sigma$ of a smooth compactly supported non-negative function in $\mathbb{R}^{\mathbf{N}}$; we drop the subscript $\varepsilon$ further on. So, $V = U^2$, $U \in C_0^\infty(\mathbb{R}^{\mathbf{N}})$, see [25] for the detailed description of this construction.

Next, the spectral problem is reduced to the study of eigenvalues of an integral operator on $\Sigma$ with kernel having a weak singularity at the diagonal. This is done in the following way. The operator $\mathbf{T}_{V\mu,\mathfrak{A}}$ is defined by the quadratic form $\mathbf{t}_{V\mu,\mathfrak{A}}[u]$, see (4.1), which can be re-written as

$$(\mathbf{T}_{V\mu,\mathfrak{A}}u,u)_{L_2(\Omega)} = \mathbf{t}_{V\mu,\mathfrak{A}}[u] = (\Gamma_\Sigma U\mathfrak{A}u, \Gamma_\Sigma U\mathfrak{A}u)_{L_{2,\mu}} = \\ ((\Gamma_\Sigma U\mathfrak{A})^*(\Gamma_\Sigma U\mathfrak{A})u,u)_{L_2(\Omega)},$$

where $\Gamma_\Sigma : H^l(\Omega) \to L_{2,\mu}$ is the operator of restriction to $\Sigma$. In this way, the operator $\mathbf{T}_{V\mu,\mathfrak{A}}$ factorizes as

$$\mathbf{T}_{V\mu,\mathfrak{A}} = \mathfrak{K}^*\mathfrak{K},\ \mathfrak{K} = \Gamma_\Sigma U\mathfrak{A} : L_2(\Omega) \to L_{2,\mu}, \mathfrak{K}^* : L_{2,\mu} \to L_2(\Omega). \tag{6.8}$$

We know, however, that the nonzero eigenvalues of the operator $\mathfrak{K}^*\mathfrak{K}$ coincide with the nonzero eigenvalues of $\mathfrak{R} = \mathfrak{K}\mathfrak{K}^*$, counting multiplicities. By (6.8), the operator $\mathfrak{K}\mathfrak{K}^*$ acts in $L_{2,\mu}$ as

$$\mathfrak{R} = \mathfrak{K}\mathfrak{K}^* = \Gamma_\Sigma U\mathfrak{A}\mathfrak{A}^*U\Gamma_\Sigma^*. \tag{6.9}$$

The operator $U\mathfrak{A}\mathfrak{A}^*U$ is an order $-2l$ self-adjoint pseudodifferential operator in $\Omega$ with principal symbol $\mathcal{R}_{-2l}(X,\Xi) = V(X)|a_{-l}(X,\Xi)|^2$, or, equivalently, it is a self-adjoint *integral* operator with kernel $R(X,Y,X-Y)$, smooth for $X \neq Y$ and having a weak singularity as $X - Y \to 0$. The leading singularity of this kernel, being the Fourier transform of the symbol $\mathcal{R}_{-2l}(X, \Xi)$ in $\Xi$ variable, has the following structure. If $m = 2l - \mathbf{N}$ is *not* a positive even integer, the leading singularity in $X - Y$ of $R(X, Y, X - Y)$ has the form $R_m(X, X - Y)$, smooth for $X \neq Y$ and positively homogeneous of order $m$ in $X - Y$. In the case when $m = 2l - \mathbf{N}$ *is* an even positive integer, then, in addition to the above term, there may be present a term with leading singularity of the form $R_{\log}(X, X - Y) \log |X - Y|$, where $R_{\log}(X, X - Y)$ is a smooth function of all variables, being a homogeneous polynomial in $X - Y$ of degree $m$. The case $2l - \mathbf{N} = 0$, i.e., the critical one, was dealt with in [25], [23], and therefore we do not discuss it here. For a detailed explanation of such correspondence between symbols and kernels, see, e.g., [32], Ch. 2, especially, Proposition 2.6. It is possible that only the logarithmic term is present in the leading singularity of the kernel. This happens, e.g., when $\mathfrak{A}$ is, in the leading term, the fundamental solution of a power of the Laplacian, say, $\mathfrak{A} = (1 - \Delta)^{-l/2}$ with positive even integer $2l - \mathbf{N}$. The structure of the integral kernel of *this* fundamental solution has been known since long ago, see, e.g., [15], Ch.II.

After framing by $\Gamma_\Sigma$ and $\Gamma_\Sigma^*$, as in (6.9), we arrive at the representation of $\mathfrak{K}\mathfrak{K}^*$ as the integral operator $\mathfrak{R}$ in $L_{2,\mu} = L_{2,\mu}(\Sigma)$ with kernel $R(X, Y, X - Y)$. Exactly this kind of operators was considered in the paper [2] (where 'almost smooth' Lipschitz surfaces were studied) and (in the general Lipschitz case) in [27], for surfaces of codimension 1, and [28], for an arbitrary codimension. The result on the eigenvalue asymptotics, obtained for such integral operators in [27], [28], corresponds exactly to the formulas in Theorem 6.2 above. The relation of the symbol of pseudodifferential operator $\mathfrak{K}\mathfrak{K}^*$ and its integral kernel is used in [27], [28] systematically. Expression (6.2) for the auxiliary symbol $\mathbf{r}_{-m}$ is obtained by representing the restriction operator $\Gamma_\Sigma$ by means of the Fourier transform.

In [27], [28], the eigenvalue asymptotics formulas have been proved first for integral operators on a *smooth* surface $\Sigma$, passing again to a pseudodifferential representation. In this case, $\mathfrak{R}$ is a classical pseudodifferential operator on $\Sigma$, of order $-\sigma = -d - m$, with principal symbol $\mathbf{r}_{-\sigma}$ given by (6.2), expressed in the co-ordinate system generated by the orthogonal projection to the tangent plane at the point $X \in \Sigma$. The proof of eigenvalue asymptotics now follows immediately from the, now classical, result by Birman-Solomyak on eigenvalue asymptotics for negative order pseudodifferential operators, see [8] (a soft proof of this result appeared recently in [22]). Note that the coefficient in the asymptotic formula has the meaning of phase volume.

If the surface $\Sigma$ is not better than Lipschitz, $\mathfrak{R}$ is, generally, not a classical pseudodifferential operator, therefore, $\mathbf{r}_{-m}(X, \xi)$ is not a symbol of anything but it is just considered as an expression involved in calculating the density in the spectral asymptotics formula. The given Lipschitz surface $\Sigma : \mathbf{y} = \boldsymbol{\phi}(\mathbf{x})$ is approximated, locally, by smooth ones, $\Sigma_\epsilon$, so that in their local representation $\mathbf{y} = \boldsymbol{\phi}_\epsilon(\mathbf{x})$, the functions $\boldsymbol{\phi}_\epsilon$ converge to $\boldsymbol{\phi}$ in $L_\infty$ and their gradients $\nabla\boldsymbol{\phi}_\epsilon$ converge to $\nabla\boldsymbol{\phi}$ in all $L_p, p < \infty$ (one should not expect the convergence of gradients in $L_\infty$, of course). Expressed in the local variables $\mathbf{x} \mapsto (\mathbf{x}, \boldsymbol{\phi}(\mathbf{x}))$, resp., $\mathbf{x} \mapsto (\mathbf{x}, \boldsymbol{\phi}_\epsilon(\mathbf{x}))$, operators with kernel $R(X, Y, X - Y)$ on surfaces $\Sigma$ and $\Sigma_\epsilon$ are transformed to operators $\tilde{\mathfrak{R}}$, resp., $\tilde{\mathfrak{R}}_\epsilon$, on the same domain $\omega \subset \mathbb{R}^d$, while the eigenvalue asymptotics for $\tilde{\mathfrak{R}}_\epsilon$ is known. Now it is possible to consider the difference $\tilde{\mathfrak{R}} - \tilde{\mathfrak{R}}_\epsilon$ of these operators. The eigenvalues of this difference are estimated using the closeness of $\boldsymbol{\phi}$ and $\boldsymbol{\phi}_\epsilon$ as $\Sigma_\epsilon \to \Sigma$ (and this is the most technical part of the reasoning in [27], [28]), and this estimate implies that the eigenvalue asymptotic coefficients $\mathbf{D}_\theta^\pm$ of $\tilde{\mathfrak{R}} - \tilde{\mathfrak{R}}_\epsilon$ converge to zero. This property enables one to use again the asymptotic perturbation Lemma 6.1, to justify the eigenvalue asymptotics formula for $\mathfrak{R}$. □

### *6.3. Eigenvalue asymptotics on uniformly rectifiable sets*

(Uniformly) rectifiable sets are an important object of study in the geometric measure theory.

**Definition 6.3.** A set $\mathcal{E} \subset \mathbb{R}^{\mathbf{N}}$ is called *uniformly rectifiable* of dimension $d$ if $\mathcal{E}$ is the union of a countable collection of Lipschitz surfaces $\Sigma_j$ of dimension $d$, up to a set of Hausdorff measure 0, $\mathcal{H}^d(\mathcal{E} \vartriangle \cup\Sigma_j) = 0$.

A number of criteria for a set to be uniformly rectifiable are known, in particular, expressed in terms of the local density of the set, see, e.g., the review in [13]; a brief exposition with references is included in [23]. Note that any compact connected set of Hausdorff dimension 1 is uniformly rectifiable. In [23], in the critical case $2l = \mathbf{N}$, the result on eigenvalue asymptotics was proved for $\mu$ being the sum of the Hausdorff measures $\mathcal{H}^d$ on uniformly rectifiable sets of Hausdorff dimension $d$, $1 \le d < \mathbf{N}$.

In the noncritical case, the eigenvalue asymptotics is established in a similar way. We restrict ourselves to the case $\mathfrak{A} = \mathfrak{A}_0$.

**Theorem 6.4.** *Let $\mathcal{M}$ be a compact uniformly rectifiable set of Hausdorff dimension $d$, $\max(0, \mathbf{N} - 2l) < d < \mathbf{N}$, $\mathfrak{A}_0$ be a pseudodifferential operator with principal symbol $|\Xi|^{-l}$. Suppose that the Hausdorff measure $\mu = \mathcal{H}^d$ on $\mathcal{M}$ and the density $V$ on $\mathcal{M}$ satisfy Condition 4.1. Then for the operator $\mathbf{T}_{V\mu,\mathfrak{A}_0}$ the eigenvalue asymptotic formula (6.7) holds, with integration over $\mathcal{M}$.*

**Proof.** The proof repeats the reasoning in [23] with minor changes. We explain here the main steps. First of all, as before, it is sufficient to consider the case of a sign-definite $V$, e.g., $V \ge 0$. Let $\Sigma_j$, $j = 1, \dots$ be a sequence of Lipschitz surfaces exhausting $\mathcal{M}$ up to a set of zero Hausdorff measure. Namely, denote by $\mathcal{E}_k$ the finite union $\mathcal{E}_k = \cup_{j\le k}\Sigma_j$, $\mathcal{F}_k = \mathcal{M} \setminus \mathcal{E}_k$, so that $\mathcal{H}^d(\mathcal{F}_k) \to 0$ as $k \to \infty$. We split the operator $\mathbf{T}_{V\mu} = \mathbf{T}_{V\mu,\mathfrak{A}_0}$ into the sum

$$\mathbf{T}_{V\mu} = \mathbf{T}_{V_k\mu} + \mathbf{T}_{V_k'\mu} \equiv \mathbf{T}_k + \mathbf{T}_k', \tag{6.10}$$

where $V_k$ is the restriction of $V$ to the set $\mathcal{E}_k$ and $V_k'$ is the restriction of $V$ to $\mathcal{F}_k$. Since $\mathcal{H}^d(\mathcal{F}_k)$ tends to zero as $k \to \infty$, we have $\int (V_k')^\vartheta \mu(dX) \to 0$ as $k \to \infty$. By Theorem

4.2, the second term in (6.10) satisfies $\mathbf{D}_{+}^{\theta}(\mathbf{T}'_k) \to 0$ as $k \to \infty$. Therefore $\mathbf{T}'_k$ can serve as the second term in the decomposition in Lemma 6.1.

It remains to prove the required asymptotic formula for the first term in (6.10), i.e., for an operator with measure supported on a finite union of Lipschitz surfaces. This is performed by means of induction on $k$, this means on the number of Lipschitz surfaces $\Sigma_j$, $j \le k$. For $k = 1$, i.e., for one surface, the asymptotic formulas have been established in Theorem 6.2. Suppose now that the eigenvalue asymptotics has been already proved for the union of $k-1$ Lipschitz surfaces, i.e., for the compact set $\mathcal{E}_{k-1}$. Let us add one more Lipschitz surface $\Sigma_k$. Consider the set $\mathcal{N}_\delta$, the $\delta$-neighborhood of $\mathcal{E}_{k-1}$ and denote by $\mathcal{Q}_\delta$ the set $(\mathcal{N}_\delta \cap \Sigma_k) \setminus \mathcal{E}_{k-1}$, the latter set is the part of $\Sigma_k$ lying outside the $\delta-$ neighborhood of $\mathcal{E}_{k-1}$. The function $V$ splits into three parts $V = V_0 + V_\delta + V_k$, where $V_0$ is supported in $\mathcal{E}_{k-1}$, $V_k$ is supported in $\Sigma_k \setminus \mathcal{Q}_\delta$, i.e., in just one Lipschitz surface, and, finally, $V_\delta$ is supported in $\mathcal{Q}_\delta$. Correspondingly, the operator $T_{V,\mu,\mathcal{M}}$ splits into the sum

$$\mathbf{T}_{V\mu} = (\mathbf{T}_0 + \mathbf{T}_k) + \mathbf{T}_\delta \equiv (\mathbf{T}_{V_0\mu} + \mathbf{T}_{V_k\mu}) + \mathbf{T}_{V_\delta\mu}. \tag{6.11}$$

As $\delta \to 0$, the Hausdorff measure of $\mathcal{Q}_\delta$ tends to zero, and thus the $L_\vartheta$-norm of $V_\delta$ tends to zero. Therefore, by Theorem 4.2, $\mathbf{D}_{+}^{\theta}(\mathbf{T}_\delta)$ tends to zero as $\delta \to 0$. For operators $\mathbf{T}_{V_0\mu}$, $\mathbf{T}_{V_k\mu}$ the asymptotic formulas hold, by the inductive assumption. Moreover, the supports of functions $V_0, V_k$ are well separated, their distance is not less than $\delta$. This allows us to use Corollary 5.6 to obtain the eigenvalue asymptotics for $\mathbf{T}_0 + \mathbf{T}_k$. Finally, the decomposition (6.11) enables us to apply, again, Lemma 6.1. □

### *6.4. Eigenvalue asymptotics on noncompact locally Lipschitz sets*

It is interesting to extend the results on eigenvalue estimates and asymptotics to a reasonably large class of noncompact Lipschitz surfaces or sets composed of these. Generally, due to non-compactness, some additional conditions are needed to justify eigenvalue properties in question.

We call a set $\Sigma \subset \mathbb{R}^{\mathbf{N}}$ with Hausdorff measure $\mu = \mathcal{H}^d|_\Sigma$ a locally Lipschitz set if there exists a locally finite family of compact Lipschitz surfaces $\Sigma_j$ of dimension $d$ such that $\Sigma = \cup\Sigma_j$

**Remark 6.5.** Any finite collection of surfaces $\Sigma_j, j \le k$, has a common Lipschitz constant. For the infinite set of surfaces, $\Sigma_j, j < \infty$, such common constant does not necessarily exist, so the surfaces may become more and more curved as one goes to infinity.

**Theorem 6.6.** *Let $2l < \mathbf{N}$ and $\Sigma$ be a locally Lipschitz set of dimension $d$ in $\mathbb{R}^{\mathbf{N}}$, $\mathfrak{A}_0 = (1-\Delta)^{-l/2}$, $2l < \mathbf{N}$, $d > \mathbf{N} - 2l$. Suppose that the Hausdorff measure $\mu$ on $\Sigma$ satisfies (1.2) with $s = d$. Suppose that $V \in L_{\theta,\mu}(\Sigma)$. Then for the operator $\mathbf{T} = \mathbf{T}_{P,\mathfrak{A}_0}$, $P = V\mu$, the eigenvalue asymptotic formula (6.7) holds.*

**Proof.** The proof follows the pattern of the one of Theorem 6.4. For a given $\epsilon > 0$, we split $V$ into two parts, $V = V_\epsilon + V'_\epsilon$ so that $V_\epsilon$ is supported in the union of a finite set of surfaces $\Sigma_j$, while $V'_\epsilon$ has small $L_{\theta,\mu}$-norm, $\|V\|_{L_{\theta,\mu}} < \epsilon$. For the operator $\mathbf{T}_{P_\epsilon,\mathfrak{A}_0}$, $P_\epsilon = V_\epsilon\mu$, the eigenvalue asymptotic formula was established when proving Theorem 6.4. For the operator $\mathbf{T}_{P'_\epsilon,\mathfrak{A}_0}$, $P'_\epsilon = V'_\epsilon\mu$, we have, by Theorem 4.3 an eigenvalue estimate with small constant. Now, as usual, Lemma 6.1 finishes the job. □

For the supercritical case, we restrict ourselves to the Birman-Borzov type of results.

**Theorem 6.7.** *Let $\Sigma$ be a locally Lipschitz set of dimension $d$ in $\mathbb{R}^{\mathbf{N}}$, $\mathfrak{A}_0 = (1-\Delta)^{-l/2}$, $2l > \mathbf{N}$. Suppose that the Hausdorff measure $\mu = \mathcal{H}^d$ satisfies* (3.15)*. Let $V$ be a real function on $\Sigma$ satisfying* (3.16)*. Then for the eigenvalues of the operator $\mathbf{T} = \mathbf{T}_{P,\mathfrak{A}_0}$, $P = V\mu$, the eigenvalue asymptotic formula* (6.7) *holds.*

**Proof.** The reasoning is the same as in Theorem 6.6. We just apply Corollary 3.11 to estimate the eigenvalues outside a compact set. □

## Appendix A. The approach by D. Edmunds and H. Triebel

In this Appendix, we discuss relation of our results with the ones presented in books by D. Edmunds and H. Triebel, see [14], and further by H. Triebel, [33], [34], H. Triebel and D. Haroske, [17], and accompanying papers.

Starting from early papers by H. Triebel, an approach for obtaining quantitative characteristics of operators in Banach and quasi-Banach spaces was being developed, based upon the analysis of entropy numbers. We recall that for a *compact* linear operator $\mathbf{H} : \mathcal{X} \to \mathcal{Y}$ between (quasi-)Banach spaces, the *entropy number* $e_k(\mathbf{H})$ is defined as the smallest $\epsilon$ such that the image in $\mathcal{Y}$ of the unit ball in $\mathcal{X}$ can be covered by not more than $2^k$ balls of radius $\epsilon$ in the metric of $\mathcal{Y}$. The *approximation numbers* $a_k(\mathbf{H})$ of a compact operator $\mathbf{H}$ in a quasi-Banach space $\mathcal{X}$ are defined as

$$a_k(\mathbf{H}) = \inf\{\|\mathbf{H} - \mathbf{V}\|_{\mathcal{X}\to\mathcal{Y}} : \operatorname{rank}(\mathbf{V}) \le k.\}$$

When $\mathcal{X} = \mathcal{Y}$, the approximation numbers are closely related to the eigenvalues $\mu_k(\mathbf{H})$, and for the case of Hilbert spaces, $a_k(\mathbf{H})$ are the singular numbers of $\mathbf{H}$.

The approach we are discussing now consists of the following. Having an operator $\mathbf{H}$ in a Hilbert or Banach space, of a complicated structure, containing multiplications by weight functions, trace and cotrace operators, one factorizes $\mathbf{H}$ as a composition of several operators, using embedding, trace, and extension theorems for functional and distributional spaces and Hölder type inequalities, so that all but one operators in this composition, the ones containing weight functions, are known to be bounded, and there is just one *embedding* operator in some standard spaces, which is compact and for which estimates for entropy numbers are known. This composition implies estimates for entropy numbers for $\mathbf{H}$. Finally, estimates for approximation numbers or eigenvalues (singular numbers) follow from the wonderful theorem by B. Carl that relates these quantities.

The simplest example of this construction can be seen in [14], Sect. 5.2.4 (we use the notations of that book). The authors consider there a non-selfadjoint Birman-Schwinger type operator,

$$\mathbf{H} = b_2 C b_1 \tag{A.1}$$

in $L_2(\Omega)$, where $C = A^{-1}$, and $A$ is an order $2m$ elliptic operator in a bounded domain $\Omega \subset \mathbb{R}^{\mathbf{N}}$ with some elliptic boundary conditions, so that $C$ is defined and maps Sobolev spaces $H_q^l$ with gain of $2m$ derivatives, $C : H_q^l(\Omega) \to H_q^{l+2m}(\Omega)$, $q \in [1, \infty)$. The weight functions $b_1, b_2$ belong, respectively, to the spaces $L_{r_1}(\Omega), L_{r_2}(\Omega)$. By Proposition 5.2.4, (we present it in the Hilbert space setting, $p = 2$) if $r_1 > 2, r_2 > 2$, $\delta = \frac{2m}{\mathbf{N}} - r_1^{-1} - r_2^{-1} > 0$, the operator $b_2 A^{-1} b_1$ is compact and its eigenvalues in $L_2(\Omega)$ satisfy

$$|\lambda_k(\mathbf{H})| \le C(A, \Omega) ||b_1||_{L_{r_1}} ||b_2||_{L_{r_2}} k^{-\frac{2m}{\mathbf{N}}}. \tag{A.2}$$

The factorization mentioned above has the form

$$\mathbf{H} = b_2 \circ \mathrm{id}_{H_q^{2m} \to L_t} \circ A^{-1} \circ b_1, \tag{A.3}$$

where the factors are: $b_1 : L_2 \to L_q$, $q^{-1} = r_1^{-1} + 1/2$, $A^{-1} : L_q \to H_q^{2m}$, $\mathrm{id} : H_q^{2m} \to L_t$, $t^{-1} = 1/2 - r_2^{-1}$, $b_2 : L_t \to L_2$. Here, the operator id, the embedding of the Sobolev space $H_q^{2m}$ into $L_t$, is compact and an estimate of its entropy numbers was previously found, $e_k(\mathrm{id}) \le Ck^{-\frac{2m}{\mathbf{N}}}$. This implies a similar estimate for entropy numbers of $\mathbf{H}$, and, by the Carl theorem, produces the required estimate for singular values of $\mathbf{H}$.

The circumstance of great importance here is the fact that the entropy numbers $e_k(\mathrm{id})$ of the embedding $\mathrm{id} : H_q^{2m} \to L_t$ have decay rate not depending on the parameter $t$, in other words, on the space into which the Sobolev space is embedded, as long as this embedding is compact, $t^{-1} > q^{-1} - \frac{2m}{\mathbf{N}}$, equivalently, $\delta > 0$. This looks mysterious if one compares this property of entropy numbers with the corresponding properties of approximation numbers of the same embedding: the latter decay rate *does* depend on the parameter $t$. This property of entropy numbers was first discovered by M.Sh. Birman and M.Z. Solomyak in [5], and it is these results that formed the basis of considerations in [14]. Further, these estimates for entropy numbers have been carried over, mostly by H. Triebel, to many embedding operators in various spaces, using interpolation and other advanced constructions. About this property of entropy numbers, the authors of [17] write: “This somewhat surprising assertion is a consequence of the miraculous properties of entropy numbers...”, see p. 243.

What is inherent and unavoidable for this approach is that the integrability conditions for the weight functions, such as $b_1, b_2$ in the above example, are not sharp. Say, in the most classical case $b_1 = b_2$, $r_1 = r_2 = r$, $\mathbf{N} > 2m$ (the subcritical case in our terminology), these conditions require that $r > \mathbf{N}/m$ and $\delta > 0$. If $\delta$ in the above example equals zero, the embedding operator $\mathrm{id}_{H^{2m}_q \to L_t}$ is continuous but not compact and the entropy numbers estimates fail. On the contrary, the estimates of CLR type admit $r = \mathbf{N}/m$, equivalently, $\delta = 0$, see, e.g. estimates presented in [6] and further publications, up to this one. This, although seemingly minor, difference is, in fact, quite crucial: the constants in such estimates with sharp exponents do not depend on the size of the domain $\Omega$ and therefore these eigenvalue estimates can be carried over to operators in unbounded domains, in particular, to the whole space, say, as it was done in Theorem 4.3. It is this non-sharpness of results in [14] that leads to excessive conditions in estimates of the negative spectrum of Schrödinger like operators, namely, the compactness of the support of the potential, see, e.g., [14], Sect. 5.2.7.

A similar effect is observed for $\mathbf{N} < 2m$ (the supercritical case). The condition $r_j > 2$, caused, again, by the requirement that a certain embedding operator is compact, is not sharp. On the other hand, in the M.Sh. Birman–M.Z. Solomyak approach, in a similar setting, it is allowed that $r_j = 2$.

These limitations of their method for obtaining eigenvalue estimates were well understood by the authors of [14]: they write, see pp. 185, 186: "As would be expected, in these special situations, the deep Hilbert space techniques used by the authors listed above [M. Birman, M. Solomyak, B. Simon, G. Rozenblum, G. Tashchiyan] often give better results than those obtained by our simple arguments which are not confined to symmetric operators or Hilbert spaces."

Similar reservations are made in [34]. Say on p. 231, where complications arising in the 'limiting case' are discussed, in particular, as it concerns the passage to singular measures with unbounded support, it is emphatically stated: 'But we do not go into detail for a simple reason: Nothing has been done so far.'

The above approach has been applied to various types of spectral problems. In particular, operators of the form $B = b_2(\gamma)b(x, D)b_1(\gamma)$ have been considered in [34], with $b_j(\gamma)$ denoting functions on a set $\gamma \subset \mathbb{R}^{\mathbf{N}}$, under the assumption that they belong to $L_{r_j}(\gamma)$ with respect to the Hausdorff measure of dimension $d$, while the set $\gamma$ is a $d$-set (is Ahlfors regular of order $d$, $0 < d < \mathbf{N}$). Further, $b(x, D)$ is an order $-\varkappa$ pseudodifferential operator and a number of conditions on parameters of the problem are imposed, see Sect. 27, 28 in [34]. For obtaining eigenvalue estimates, again, the operator under study is factorized as a composition of several operators, including multiplication by weight functions, restriction to $\gamma$, extension from $\gamma$, and embedding operators for functional spaces on $\mathbb{R}^{\mathbf{N}}$ and on $\gamma$. Deep and complicated results of this book and previous ones are used to justify the boundedness of these operators. And for one of them, which turns out to be compact, namely, the embedding of Besov type spaces, the estimate of entropy numbers has been established.

As before, the compactness requirement for this particular embedding operator leads to non-sharp integrability conditions for the weight functions $b_1, b_2$, with the same kind of limitations of the resulting eigenvalue estimates for the operator $B$.

In Sect. 4 in our paper, we consider some of operators of the type studied in [34] and demonstrate that our approach produces spectral estimates, sharp both in order and integrability classes of weight functions.

So, when comparing the results by H. Triebel and his collaborators with the ones of the present paper, we can state the following.

(1) The results by H. Triebel and co-authors concern a wide class of operators, including the ones close to the operators in the present paper, and not necessarily in the Hilbert space setting. The integrability condition imposed upon the weight functions are not sharp, so the estimates lack the homogeneity property with respect to dilations for weight functions, needed for getting rid of the compact support conditions. Additionally, the two-sided Ahlfors condition (1.1) is required.
(2) The results in our paper deal with a more narrow class of operators, exclusively in the Hilbert space setting. However, the integrability conditions on the weight functions are sharp, and their homogeneity properties enable us to get rid of the support compactness conditions, in particular, producing CLR-type eigenvalue estimates. Only one-sided Ahlfors conditions (1.2) or (1.4) are imposed on the measure $\mu$.

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