#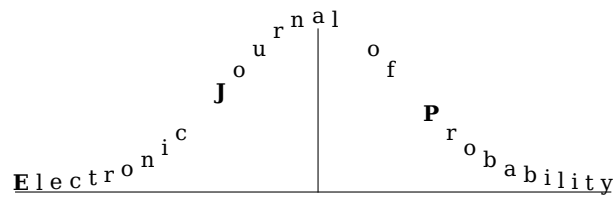 Priors leading to well-behaved Coulomb and Riesz gases versus zeroth-order phase transitions - a potential-theoretic characterization

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Priors leading to well-behaved Coulomb and Riesz gases versus zeroth-order phase transitions – a potential-theoretic characterization

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Abstract

We give a potential-theoretic characterization of measures μ_0 which have the property that the Coulomb gas, defined with respect to the prior μ_0, is "well-behaved" and similarly for more general Riesz gases. This means that the laws of the empirical measures of the corresponding random point process satisfy a Large Deviation Principle with a rate functional which depends continuously on the temperature, in the sense of Gamma-convergence. Equivalently, there is no zeroth-order phase transition at zero temperature, in the mean field regime. This is shown to be the case for the Hausdorff measure on a compact Lipschitz hypersurface, as well as Lesbesgue measure on a bounded Lipschitz domain. We also provide constructions of priors μ_0, absolutely continuous with respect to Lebesgue measure on a smoothly bounded domain, such that the corresponding 2d Coulomb exhibits a zeroth-order phase transition. This is based on relations to Ullman's criterion in the theory of orthogonal polynomials and Bernstein-Markov inequalities.

1 Introduction

In broad terms, the main aim of the present work is to study the interplay between fine potential-theoretic properties of a measure μ_0 in $\mathbb{R}^d$ (the "prior") and properties of the corresponding Coulomb gas in $\mathbb{R}^d$, in the mean-field regime. The pair-interaction potential of this gas (also known as the one-component plasma in the physics literature) is the fundemental solution of the Laplacian Δ in $\mathbb{R}^d$ and yields a family of random point processes on the support of μ_0 in $\mathbb{R}^d$, parametrized by temperatures $T_N \in]0, \infty]$. We will, in particular, give a potential-theoretic characterization of measures μ_0 for which the

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corresponding Coulomb gas is "well-behaved at zero-temperature", which is equivalent to the absence of a zeroth-order phase transition. In fact, the main results will be shown to hold in a more general setting involving a Riesz gas in $\mathbb{R}^d$, where the role of the Laplacian is played by the fractional Laplacian $-(-\Delta)^{\alpha/2}$, assuming that $\alpha \in]0, 2]$. Complex-geometric analogs of the results, where the role of the Laplacian is played by the complex Monge-Ampère operator, are described in [9]. Our results yield new probabilistic constructions of capacities, equilibrium measures etc, using random point processes, in contrast to the usual probabilistic approach based on Markov processes (and their hitting probabilities [33, 22]). More precisely, the setting of a Riesz gas with $\alpha \in]0, 2]$ corresponds precisely to the class of symmetric stable Levy processes in $\mathbb{R}^d$ [20] (i.e. Brownian motion in the "Coulomb case" $\alpha = 2$).

For concreteness we will introduce the main results in the Coulomb case ($\alpha = 2$), where the *energy* $E(\mu)$ of a compactly supported measure μ in $\mathbb{R}^d$, with $d \geq 2$, is defined by

$$E(\mu) = \frac{1}{2} \int_{\mathbb{R}^d} W \mu \otimes \mu,$$

where $W(x, y)$ denotes the standard Green function of the Laplacian Δ, i.e. $W(x, y)$ is proportional to $|x - y|^{2-d}$ when $d \geq 3$ and to $-\log|x - y|$ when $d = 2$. The *potential* ψ_μ of μ is the subharmonic function on $\mathbb{R}^d$ defined by

$$\psi_\mu(x) := -\int_{\mathbb{R}^d} W(x, y)\mu(y)$$

(using the opposite sign convention compared to the standard convention in physics). A bounded subset S of is said to be *polar* if there exists a potential ψ_μ such that $S \Subset \{\psi_\mu = -\infty\}$. We will be particularly interested in measures μ_0 not charging polar subsets (for example, this is the case if μ_0 has finite energy or if μ_0 is absolutely continuous wrt Lebesgue measure). We will denote by $\mathcal{P}(S)$ the space of all probability measures on a closed subset $S \subset \mathbb{R}^d$, endowed with the weak topology.

1.1 Energy approximation and determining measures

The main analytical result may be formulated in terms of potential theory and approximation theory as follows. Assume given a measure μ_0 on $\mathbb{R}^d$ and denote by S_0 its support. We will say that μ_0 has the *Energy Approximation Property* if for any measure μ supported on S_0 there exists a sequence μ_j converging weakly towards μ such that

- μ_j is absolutely continuous with respect to μ_0
- $\lim_{j\to\infty} E(\mu_j) = E(\mu)$

Note that, by the lower semi-continuity of W, the second point is equivalent to

$$\limsup_{j\to\infty} E(\mu_j) \leq E(\mu) \tag{1.1}$$

Theorem 1.1 below relates the Energy Approximation Property to the potential-theoretic notion of determining measures. Given a weighted set (S, ϕ) consisting of a subset S of $\mathbb{R}^d$ and a continuous function ϕ on S, a measure ν on $\mathbb{R}^d$ is said to be *determining for* (S, ϕ) if for all potentials ψ on $\mathbb{R}^d$

$$\psi \leq \phi \text{ almost everywhere wrt } \nu \implies \psi \leq \phi \text{ on } S \tag{1.2}$$

We will say that ν is *determining for* S if ν is determining for $(S, 0)$ and *strongly determining for* S if ν is determining for (S, ϕ) for all $\phi \in C(S)$. Similarly we will say that ν is (strongly) determining if it is (strongly) determining for its support. For example,

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Lebesgue measure $1_\Omega dx$ on a bounded domain Ω in $\mathbb{R}^d$ is strongly determining if Ω is *non-thin* at all boundary points, in the classical sense (Prop 2.21). In general, if ν is strongly determining and does not charge polar subsets then the support of ν is automatically *locally regular* (see Section 2.7 for the potential-theoretic notions of regularity).

Theorem 1.1. *Let μ_0 be a measure on $\mathbb{R}^d$ which does not charge polar subsets and assume that the support S_0 of μ_0 is compact and locally regular. Then μ_0 has the Energy Approximation Property iff μ_0 is strongly determining.*

The main virtue of the property of beeing strongly determining is that it can often be verified using maximum (/domination) principle type arguments. For example, using [27], we will show that the $(d-1)$-dimensional Hausdorff measure on a Lipschitz hypersurface is strongly determining (Theorem 2.22).

Theorem 1.1 appears to be new even in the simplest case when S_0 is an interval in $\mathbb{R} \subset \mathbb{R}^2$, which is the classical setting where the notion of determining measures was first introduced by Ullman (as discussed in Section 1.3 below). In this special case a measure μ_0 is determining iff it is strongly determining.

1.2 The zero-temperature limit of the Coulomb gas in the mean-field regime

The main motivation for Theorem 1.1 above comes from the study of the large deviations of the Coulomb gas on a measure μ_0 in $\mathbb{R}^d$, where the Energy Approximation Property has previously appeared as a technical hypothesis [26, 29, 7, 32]. To give some background, assume given a continuous function ϕ on $\mathbb{R}^d$ and consider the corresponding mean field N-particle Hamiltonian

$$H_\phi^{(N)}(x_1, ...x_N) := \frac{1}{(N-1)} \frac{1}{2} \sum_{i \neq j} W(x_i, x_j) + \sum_{i=1}^N \phi(x_i)$$

describing the Coulomb energy of N-particles in the exterior potential ϕ (with the divergent self-energies removed) with a mean field scaling. In physical terms this means that each particle is subject to the average of the Coulomb potentials created by the other particles, plus the exterior potential ϕ. We recall that the mean-field scaling is often used for systems with long-range interactions, i.e. such that $W(x,y) \leq C1/|x-y|^d$ (which includes general Riesz gases) [3, 23, 40]. Given a measure μ_0 on $\mathbb{R}^d$ (the "prior") with compact support S_0 and a sequence of numbers $T_N \in]0, \infty[$ the corresponding *(mean field) Coulomb gas at temperature T_N on μ_0* is defined as the probability space (canonical ensemble) $\left((\mathbb{R}^d)^N, \mu_{T_N}^{(N)}\right)$, where

$$\mu_{\phi, T_N}^{(N)} := \frac{1}{Z_{N,\phi,T_N}} e^{-T_N^{-1} H_\phi^{(N)}} \mu_0^{\otimes N}, \quad Z_{N,\phi,T_N} := \int_{(\mathbb{R}^d)^N} e^{-T_N^{-1} H_\phi^{(N)}} \mu_0^{\otimes N} \quad (1.3)$$

The normalizing constant Z_{N,ϕ,T_N} is called the *partition function* and

$$F_{N,\phi,T_N} := -\frac{T_N}{N} \log Z_{N,\phi,T_N}$$

is called the *N-particle free energy at temperature T_N*. It extends continuously to $T_N = 0$ by setting

$$F_{N,\phi,0} := \frac{1}{N} \inf_{S_0^N} H_\phi^{(N)},$$

where S_0 is the support of μ_0 (using that $e^{-H_\phi^{(N)}}$ is continuous; see the beginning of the proof of Lemma 5.3). In the case when $\phi = 0$ we will simply drop the subscripts

ϕ. The mean field scaling $1/(N-1)$ appearing in the definition of $H_\phi^{(N)}(x_1, ...x_N)$ could alternatively have been absorbed by the temperature T_N. Anyhow, throughout the paper we will employ the mean field scaling above and assume that the corresponding temperatures T_N have a limit T as $N \to \infty$:

$$T := \lim_{N \to \infty} T_N \in [0, \infty[.$$

For example, if, with the present mean field scaling, T_N is taken to be proportional to $1/(N-1)$, so that $\mu_{T_N}^{(N)}$ is the Gibbs measure for a Coulomb gas without a mean field scaling at a fixed positive temperature [52], then the corresponding limiting temperature T above vanishes (compare Section 1.3). But here it will be important to allow non-vanishing limiting temperatures T, where entropy enters the picture.

As first shown in [23, 41], in the case when $T > 0$ and μ_0 is equal to Lebesgue measure on a compact domain S_0, the empirical measure

$$\delta_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}, \tag{1.4}$$

viewed as a random measure on $\left((\mathbb{R}^d)^N, \mu_{T_N}^{(N)} \right)$, converges in probability, as $N \to \infty$, towards a deterministic measure $\mu_{\phi, \beta}$

$$\lim_{N \to \infty} \delta_N = \mu_{\phi, T},$$

where $\mu_{\phi, T}$ is the unique minimizer of the following *free energy functional* $F_{\phi, T}$ on the space $\mathcal{P}(S_0)$:

$$F_{\phi, T}(\mu) = E_\phi(\mu) + T D_{\mu_0}(\mu), \quad E_\phi(\mu) := E(\mu) + \int \phi \mu$$

where D_{μ_0} denotes the entropy of μ relative to μ_0, using the sign convention making D_μ non-negative (see formula (3.2)). In particular,

$$F_{\phi, 0} := E_\phi$$

and we denote by $\mu_{(S, \phi)}$ the *equilibrium measure* of a non-polar compact weighted set (S, ϕ), i.e. the unique minimizer of E_ϕ on $\mathcal{P}(S)$. It should be stressed that the presence of the entropy term in the free energy $F_{\phi, T}(\mu)$ entails that the minimizer μ_T is absolutely continuous wrt μ_0 for any $T > 0$, while this is often not the case for the equilibrium measure $\mu_{(S_0, \phi)}$. For example, if μ_0 is Lesbesgue measure on a compact domain S_0 with smooth boundary and $\phi = 0$, then $\mu_{(S_0, \phi)}$ is supported in the boundary of S_0 (in the Coulomb case).

Under appropriate regularity assumptions on μ_0 it is shown in [26, 29, 7, 32] that the convergence of δ_N towards $\mu_{\phi, T}$ is, in fact, exponential in the sense of large deviation theory. More precisely, the laws of the empirical measures δ_N satisfy a *Large Deviation Principle (LDP)* at speed $T_N^{-1} N$, whose rate functional $I_{\phi, T}$ coincides with $F_{\phi, T}$, up to an additive constant. In symbolic notation this may be expressed as

$$(\delta_N)_* \left(e^{-T_N^{-1} H_\phi^{(N)}} \mu_0^{\otimes N} \right) \sim e^{-T_N^{-1} N F_{\phi, T}}, \quad N \to \infty$$

as measures on $\mathcal{P}(S_0)$. We recall that, in general, the rate functional I for an LDP is a proper lower-semicontinuous function (the precise meaning of the LDP is recalled in Section 4).

1.2.1 The LDP in the zero-temperature limit

When $T > 0$ the LDP for the Coulomb gas holds for any measure μ_0 not charging polar subsets (Theorem 4.2). A natural question is thus what further conditions on μ_0 need to be imposed in order to ensure that the LDP also holds for $T = 0$? As shown in [26, 29, 7, 32], the Energy Approximation Property is a sufficient condition. However, as will be shown below, this condition is not necessary, but rather equivalent to a "well-behaved" LDP. The starting point is the basic observation that the Energy Approximation Property is equivalent to a certain continuity property of the free energy functional F_T, namely that F_T be continuous with respect to Gamma-convergence of functionals, as $T \to 0$. Recall that the notion of Gamma-convergence plays a prominent role in variational calculus (see definition 3.1) and corresponds to the Fell topology on the space of lower-semicontinuous functions [24]. It implies, in particular, that the minimizer μ_T of F_T converges towards the minimizer of F_0. In the present setting the Gamma-convergence of the functional F_T is, by a duality argument, equivalent to the continuity of

$$f_\phi(T) := \inf_{\mu \in \mathcal{M}(S_0)} F_{\phi,T}(\mu)$$

as $T \to 0$ for all exterior potentials ϕ. The number $f_\phi(T)$ is usually called the *free energy at temperature T* (wrt the exterior potential ϕ). The main result may now be formulated as the following

Theorem 1.2. *Let μ_0 be a measure on $\mathbb{R}^d$ which does not charge polar subsets and assume that the support S_0 of μ_0 is compact and locally regular. Then the following is equivalent:*

- *μ_0 is strongly determining*
- *For any given continuous function ϕ, the corresponding free energy $f_\phi(T)$ at temperature T,*

$$f_\phi(T) := \inf_{\mu \in \mathcal{M}(S_0)} F_{\phi,T}(\mu) \tag{1.5}$$

  is continuous wrt $T \in [0, \infty]$

- *For any given potential ϕ the minimizer $\mu_{\phi,T}$ of the functional $F_{T,\phi}$ on $\mathcal{P}(S_0)$ converges, as $T \to 0$, towards the equilibrium measure of (S_0, ϕ), i.e. towards the minimizer $\mu_{(S_0,\phi)}$ of $F_{\phi,0}$:*

$$\lim_{T \to 0} \mu_{\phi,T} = \mu_{(S_0,\phi)} \tag{1.6}$$

  in the weak topology on $\mathcal{P}(S_0)$.

- *The LDP for the Coulomb gas on μ_0 holds for all exterior potentials ϕ and all $T \in [0, \infty[$ with a rate functional which is continuous wrt $T \in [0, \infty[$ in the sense of Gamma-convergence*

In fact, in the present setting the LDP in the previous theorem is equivalent to the free energy asymptotics

$$\lim_{N \to \infty} F_{N,\phi,T_N} = f_\phi(T), \;\; T_N \to T, \tag{1.7}$$

for any potential ϕ and sequence $T_N \in [0, \infty[$ such that $T_N \to T \in [0, \infty[$ for a function $f_\phi(T)$ which is continuous on $[0, \infty[$ (and a posteriori of the form (1.5)). The continuity of $f_\phi(T)$ when $T > 0$ is automatic and, as discussed in Section 5, a discontinuity at $T = 0$ can be interpreted as a *zeroth-order phase transition*. Another equivalent formulation is obtained by taking $T_N = T \in]0, \infty[$ and demanding that the limits $T \to 0$ and $N \to \infty$ of

$F_{N,\phi,T}$ commute (see Lemma 5.3). The existence of discontinuities at $T = 0$ should be contrasted with the fact that

$$\lim_{T \to \infty} \mu_{\phi,T} = \mu_0$$

always holds (see Section 3.4). This means that if μ_0 is strongly determining, then the curve $\mu_{T,\phi} \in \mathcal{P}(S_0)$ interpolates between μ_0 at $T = \infty$ and the weighted equilibrium measure $\mu_{(S_0,\phi_0)}$ at $T = 0$.

The previous theorem will be deduced from the following result concerning the case when the potential ϕ is fixed:

Theorem 1.3. *Let μ_0 be a measure on $\mathbb{R}^d$ which does not charge polar subsets and assume that the support S_0 of μ_0 is compact and locally regular. For a given continuous function ϕ on S_0 the following is equivalent*

- *μ_0 is determining for (S_0, ϕ)*
- *The following convergence of free energies holds:*

$$\lim_{T \to 0} \lim_{N \to \infty} F_{N,\phi,T_N} = \inf_{\mathcal{P}(S_0)} E_\phi \tag{1.8}$$

- *The following weak convergence of the expectations $\mathbb{E}_{T,\phi}(\delta_N)$ of the empirical measure δ_N holds:*

$$\lim_{T \to 0} \lim_{N \to \infty} \mathbb{E}_{T,\phi}(\delta_N) = \mu_{(S_0,\phi)} \tag{1.9}$$

We recall that the inverse of the infimum of the functional E on $\mathcal{P}(K)$, for a given compact set K in $\mathbb{R}^d$, is usually called the *(Wiener) capacity* of K (see Section 6.1 for the general weighted setting).

Coming back to the energy approximation property in Section 1.1 we will also show, in Section 3.3, that the approximating sequence in question can be constructed quasi-explicitly.

1.3 Relations to Bernstein-Markov measures, orthogonal polynomials and Ullman's criterion

Now specialize to the two-dimensional case and identify $\mathbb{R}^2$ with $\mathbb{C}$. Then the Gibbs measure (1.3) may, for $\phi = 0$, be expressed as

$$\mu_{\beta_N}^{(N)} := \frac{1}{Z_{N,\beta_N}} \left| D^{(N)} \right|^{p_N} \mu_0^{\otimes N}, \;\; p_N := 2 \frac{1}{T_N(N-1)} \tag{1.10}$$

where $D^{(N)}(z_1, ..., z_N)$ denotes the *Vandermonde determinant,* i.e. the polynomial on $\mathbb{C}^N$ defined by

$$D^{(N)}(z_1, ..., z_{N_k}) = \prod_{1 \le i < j \le N} (z_i - z_j) = \det_{1 \le i,j \le N}(z_i^{j-1})$$

Accordingly, the corresponding partition function Z_{N,T_N} is equal to the L^{p_N}-norm of $D^{(N)}$. More generally, introducing an exterior potential ϕ corresponds to replacing the $L^{p_N}-$ norms with weighted norms, i.e replacing μ_0 with $e^{-T_N^{-1}\phi}\mu_0$. Note that for a fixed p, the corresponding T_N, tend to zero as $N \to \infty$, i.e. a fixed p induces a vanishing limiting temperature T (in terms of the mean field setup in the previous section). A notion of a phase transition for the Coulomb gas with respect to the parameter p – from a liquid to a crystalline phase, as p is increased towards a critical value – has been discussed extensively in the physics literature, supported by numerical studies [54] (and related to microscopic large deviation principles in [52, Section 4.6]). But this notion is different than the zeroth-order phase transitions discussed here, where the parameter T is decreased towards zero.

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The Coulomb gas has been studied extensively in connection to Random Matrix Theory, in particular in the case when μ_0 is Lesbegue measure on $\mathbb{R} \subset \mathbb{R}^2$ with ϕ of suffcient growth at infinity and $p = 2$. The corresponding Gibbs measure $\mu_{\beta_N}^{(N)}$ then arises as the eigenvalue distribution of a Hermitian matrix [46, 41, 31] (see also [28, 1] for the case of general p). The asymptotics of the corresponding free energies, as $T_N \to 0$ was established in [38, Thm 2.1] and the LDP in [4].

We recall that a measure μ_0 in $\mathbb{C}$ is said to satisfy a *Bernstein-Markov inequality with weight ϕ* if, for any given $\epsilon > 0$ there exists a constant C such that

$$\sup_{S_0} |p_k|^2 e^{-k\phi} \le Ce^{\epsilon k} \int_{\mathbb{C}} |p_k|^2 e^{-k\phi} \mu_0 \tag{1.11}$$

for all polynomials p_k on $\mathbb{C}$, where k denotes the degree of p_k (see the survey [17]). For such a measure the existence of the limit $f_\phi(0)$ of the corresponding N-particle free energies at $T = 0$ was established in [15]. In fact, if the Bernstein-Markov inequality holds for all weights then the LDP holds at zero temperature, by the results in [5] (see also [16] for a different approach and [57] for relations to random polynomials). In view of Theorem 1.3 this means that the Coulomb gas on a measure μ_0 with compact support S_0, which satisfies the weighted Bernstein-Markov inequality for (S_0, ϕ) – but which is not determining for (S_0, ϕ) – exhibits a zeroth-order phase transition at $T = 0$. A general procedure for constructing such measures is explained in [17], where a concrete example of Totik on the interval is reported (see the appendix). We thus arrive at the following corollary, exhibiting a zeroth-order phase transition:

Corollary 1.4. *Let K be a compact domain in $\mathbb{C}$ with smooth boundary or equal to a disjoint finite union of intervals in $\mathbb{R}$. For any given continuous function ϕ on K there exists a measure μ_0 with support K such that μ_0 is absolutely continuous wrt dx and such that the corresponding 2d Coulomb gas satisfies a LDP for any $T \in [0, \infty[$ with a rate functional which is discontinuous at $T = 0$ in the sense of Gamma-convergence. More precisely, in the case $\phi = 0$, the function*

$$f(T) := -\lim_{N\to\infty} \frac{T_N}{N} \log \int_{\mathbb{R}^N} \left| D^{(N)} \right|^{\frac{2}{T_N N}} \mu_0^{\otimes N} \quad T = \lim_{N\to\infty} T_N$$

is well-defined on $[0, \infty[$, continuous on $]0, \infty[$, but discontinuous at $T = 0$ (and similarly for a general ϕ).

The property of being determining can, for a non-polar measure μ_0, be viewed as a potential-theoretic refinement of the Bernstein-Markov inequality, where a polynomial p_k of degree k is replaced by $e^{T^{-1}\psi_\mu}$ for a general measure μ and positive number T^{-1}, playing the role of k. This was first shown in [12] in a general complex geometric setting. In Section 4 we will extend these notions to the case of general pair interaction potentials (the notion of Bernstein-Markov measures for Riesz interaction was introduced in [18]).

1.3.1 Orthogonal polynomials and Ullman's criterion

We recall that the Bernstein-Markov-inequality has it roots in the theory of orthogonal polynomials on $\mathbb{R}$. In fact, for a measure μ_0 on $\mathbb{R}$ with compact and regular support the Bernstein-Markov inequality is equivalent to the notion of *regular measures* on $\mathbb{R}$ introduced in [55], whose definition involves the asymptotics of the degree N orthogonal polynomials p_N associated to μ_0 (see the proof of Prop 4.13). In the case when the support of μ_0 is $[-1, 1]$ this notion goes back to Ullman. He also introduced the notion of determining measures on $[-1, 1]$ to get a sufficient condition for regularity, known as *Ullman's criterion* in the general setting of measures on $\mathbb{R}$ [55]. In view of Theorem 1.2

Ullman's criterion naturally fits into the probabilistic setting of the Coulomb gas, since it is equivalent to the continuity properties discussed above. It should be pointed out that Ullman originally used a different, but equivalent, capacity formulation of determining measures on $[-1, 1]$ (see Prop 6.2 for the relation to the present setting). The definition of determining measures in the present general potential-theoretic setting on $\mathbb{R}^d$ mimics the definition used in the complex-geometric setting of [12], which goes back to [43].

1.4 Towards the case of general pair interactions

Finally, let us make some remarks about the case when the Gibbs measure (1.3) is defined by a general proper lsc function W. Then it essentially follows from [29, 7, 32] that the corresponding LDP holds for $T > 0$ if and only if the corresponding free energy functional F_T is a proper lsc functional (see Theorem 4.2). However, it seems challenging to find a general potential-theoretic characterization of measures μ_0 such that the corresponding LDP also holds at $T = 0$. In view of Theorem 1.2 (and its generalization to Riesz gases 4.8) the problem of characterizing measures μ_0 such that the LDP is "well-behaved at $T = 0$" should be more accessible also in the general case. It seems likely that the answer should be given by determining measures for rather general interactions W, but we shall not pursue this here and only point out that in the case when $W(x, y) = W_\alpha(x, y) + K(x, y)$, where W_α denote the Riesz kernel and the perturbation $K(x, y)$ is continuous on S_0 the present results (concerning the case $K = 0$) directly generalize. Indeed, since $K(x, y)$ is continuous it influences neither the energy approximation property, nor the determining property appearing in Theorem 1.1. For example, this situation naturally appears in electrostatics in $\mathbb{R}^3$ with the screened Coulomb interaction (Yukawa potential) $W(x, y) = e^{-m|x-y|}/|x - y|$ for a given positive number m.

1.5 Further relations to previous results

The idea of studying the Gibbs measures corresponding to a general lsc pair interaction potential W, in the case $T = 0$, by letting $T \to 0$ goes back to [41], which builds on the variational approach introduced in [47] (where the case of a continuous W was considered). In the main result of [41] it is claimed that, in general, any limit point μ in $\mathcal{P}(\mathbb{R}^d)$ of the empirical measure δ_N minimizes the corresponding energy functional E. However, in the case of the Coulomb gas this is contradicted by the example in Theorem 4.12, where μ_0 has support $[-1, 1]$ and is absolutely continuous wrt dx (see also Example 4.4 for a simple counter-example in the weighted setting). The mistake in [41] appears to be the claimed inequality [41, 3.12], which, in general, requires assumptions on μ_0.

The "only if" direction in Theorem 1.3 was first shown in the complex-geometric setting of compact Kähler manifolds X in [9] and generalized to the non-compact setting pluripotential setting of $\mathbb{C}^n$ in [9], using a compactification argument. See also [35] for a far-reaching generalization of [8, Theorem 2.1] to general measures not charging pluripolar subsets and bounded weights.

It should also be pointed out that there are particular situations where the *rate* of the convergence of $\mu_{T,\phi}$ towards the equilibrium measure $\mu_{(S_0,\phi)}$ can be quantified. In the setting of a compact Kähler manifold X, studied in [8], the rate of convergence of the L^∞-norms of the corresponding potentials $\psi_{\mu_{T,\phi}}$ was shown to be of the order $O(T) \log T^{-1}$. In particular, the case when X is the Riemann sphere implies (by a compactification argument, as in [9]) that such a rate holds for the Coulomb gas in $\mathbb{R}^2$ with $\mu_0 = dx$ if ϕ has strictly super logarithmic growth. More general and precise quantitative convergence results for Coulomb gases in any dimension have very recently been established in

[2] (see also [1, formula 18] for an explicit integral formula for $\mu_{T,\phi}$ in the case of the Coulomb gas on $\mathbb{R}$ with $\phi(x) = x^2$ and μ_0 the Gaussian measure, so that $\mu_{T,\phi}$ converges, as $T \to 0$, to Wigner's semi-circle law). However, it should be stressed that in the general setting of determining measures μ_0, studied in the present paper, one can not hope for quantitative rates, unless regularity assumptions on μ_0 are imposed.

1.6 On the proofs

The core analytic result is Theorem 1.3 and its general form 3.10 (which applies to any Riesz interaction with $\alpha \leq 2$), saying, in particular, that the measure μ_0 *is determining iff the corresponding free energy $f(T)$ is continuous as $T \to 0$.* The proof of the "if" direction" mimics the variational proof of a similar result in the complex geometric setting on a compact Kähler manifold X [8, Theorem 2.1] (which applies, in particular, to the case $d = \alpha = 2$ by taking X to be the Riemann sphere). An important ingredient is the potential-theoretic analog of the Bernstein-Markov property for determining measures in Prop 3.6 (proved in [12] in the complex geometric setting). In the present setting we also have to deal with the non-compactness of $\mathbb{R}^d$ and the non-local properties of the fractional Laplacian $-(-\Delta)^{\alpha/2}$ (for $\alpha < 2$). In the case when $d = \alpha = 2$ the "only if" direction could alternatively be deduced from the generalization in [35] of [8, Theorem 2.1] to arbitrary measures μ_0, not charging pluripolar subsets. However, the proof in [35] (which is not variational) appears to exploit some special local features of the complex geometric setting, which do not seem to apply when $\alpha \neq 2$. Here we instead use a variational approach, which has the virtue of only demanding some rather general axioms of potential theory (compare Remark 2.1).

1.7 Acknowledgments

1.8 Organization

In Section 2 we introduce the weighted potential theory needed for the proofs of the main analytic results. In particular, a dual representation of the energy $E(\mu)$ as a Legendre transform is given. In Section 1.1 we reformulate the Energy-Approximation Property in terms of Gamma-convergence of the free energy functional F_T, which in turn is a given a dual formulation using Legendre transforms. Then in Section 3.2 the proofs of the main analytic results for Riesz interactions are given, by relating Gamma-convergence of F_T to determining measures. The connections to large deviation principles is studied in Section 4 and connections to Bernstein-Markov inequalities are explored. The results are then reformulated in terms of phase transitions in Section 5. In the appendix some construction of measures μ_0 are provided, which illustrate the sharpness of the main results.

1.9 General notation

We will denote by $\mathcal{P}_c(\mathbb{R}^d)$ the space of all compactly supported probability measures on $\mathbb{R}^d$ and by $\mathcal{P}(K)$ the subset consisting of measures supported on a compact subset K of $\mathbb{R}^d$. We endow the space $\mathcal{P}(K)$ with the weak topology. Throughout the paper we fix a probability measure μ_0 with compact support, denoted by S_0. Given a compact subset K

of $\mathbb{R}^d$ we will denote by $\mathcal{C}(K)$ the space of all continuous functions on K.

We recall that a function f on a topological space X, taking values in $]\infty,\infty]$ is lower semi-continuous (lsc) if $\{f \leq \alpha\}$ is closed for any $\alpha \in]\infty,\infty]$. We will say that f is *proper lower-semicontinuous* under the further assumption that f is not identically equal to ∞ (following standard terminology in convex analysis). If X is compact and f is lsc the latter condition equivalently means that $\inf_X f$ is finite. Hence, under the map $f \mapsto f - \inf_X f$ the space of proper lsc functions on X corresponds to the space of *rate functionals,* in the sense of large deviation theory. Finally, it will be convenient to work with inverse temperatures $\beta_N := T_N^{-1}$ and $\beta := T^{-1}$ rather than temperatures.

2 Weighted potential theory and Legendre transforms

In this section we develop the weighted potential theory needed for the proofs of the main results. The key result is the Legendre transform representation of the energy in Theorem 2.12. The presentation is inspired by the complex-geometric framework in [11, 12, 13], which covers in particular the Coulomb case in $\mathbb{R}^2$ (see also [51, 26] for different points of view).

2.1 Potential-theoretic preliminaries

We start by recalling some basic potential-theoretic results. We follow the classical reference [42], but with a different sign convention for the kernels and the potentials (ensuring that the potentials are subharmonic in the Coulomb case).

We will denote by $W_\alpha(x,y)$ the Riesz kernel with parameter $\alpha \in]0,d[$, i.e. the lsc function on $\mathbb{R}^d \times \mathbb{R}^d$ defined by

$$W_\alpha(x,y) := \frac{1}{|x-y|^{d-\alpha}}.$$

When $d = 2$ we will allow the case $\alpha = d = 2$, by setting

$$W_2(x,y) = -2\log|x-y|,$$

The definition ensures that when $d \geq 2$ the function $W_2(x,y)$ is a (up to multiplication by a negative constant) a Green's kernel for the Laplacian Δ on $\mathbb{R}^d$. Accordingly, we will refer to the case $\alpha = 2$ as the "Coulomb case" and the special case $\alpha = 2 = d$ the "logarithmic case".

The *energy* $E(\mu)$ of a measure $\mu \in \mathcal{P}_c(\mathbb{R}^d)$ is defined by

$$E(\mu) = \frac{1}{2}\int_{\mathbb{R}^d} W_\alpha \mu \otimes \mu \in]-\infty,\infty] \tag{2.1}$$

Given a measure μ on $\mathbb{R}^d$ we will denote by ψ_μ its *potential:*

$$\psi_\mu(x) := -\int_{\mathbb{R}^d} W_\alpha(x,y)\mu(y),$$

Since W_α is symmetric the following *symmetry property* holds:

$$\int \psi_\nu \mu = \int \nu \psi_\mu \tag{2.2}$$

if μ and ν are in $\mathcal{P}_c(\mathbb{R}^d)$ and of finite energy. Moreover, W_α defines a *strictly positive definite bilinear form* in the following sense:

$$-\int (\psi_\mu - \psi_\nu)(\mu - \nu) \geq 0 \tag{2.3}$$

with equality iff $\nu = \mu$. This implies that the map $\mu \mapsto \psi_\mu$ is injective and we will denote the inverse operator by Δ_α, which coincides with the ordinary Laplacian when $\alpha = 2$. [1] A bounded set S is said to be *polar* if $S \subset \{\psi_\mu = -\infty\}$ for some measure μ (equivalently, S has vanishing outer capacity) [2]. If S is compact then S is polar iff $E(\mu) = \infty$ for any measure $\mu \in \mathcal{P}(S)$. A property is said to hold *quasi-everywhere (q.e)* if it holds on the complement of a polar set.

If $\mu_j \in \mathcal{P}(K)$ for a compact subset K and $\mu_j \to \mu$ weakly, then, for any given $x \in \mathbb{R}^d$

$$\limsup_{j \to \infty} \psi_{\mu_j}(x) \leq \psi_\mu(x),$$

as follows directly from the lower semi-continuity of W. Moreover, for q.e. x in $\mathbb{R}^d$,

$$\limsup_{j \to \infty} \psi_{\mu_j}(x) = \psi_\mu(x) \tag{2.4}$$

(see [42, Therem 3.8, page 190]). As a consequence [42, page 191, Remark 2],

$$\left(\limsup_{j \to \infty} \psi_{\mu_j} \right)^* = \psi_\mu \tag{2.5}$$

on all of $\mathbb{R}^d$, where the limsup is defined point-wise and f^* denotes the upper semi-continuous regularization of a function f on $\mathbb{R}^d$:

$$f^*(x) = \sup \{ f(x_j) : \ x_j \to x \},$$

where the sup runs over all sequences x_j converging to x. We will be mainly interested in the case when $\alpha \leq 2$, since the following *domination principle* then applies [42, Thm 1.29]: for a given constant C

$$\psi_\nu \leq \psi_\mu + C \ \ \mu - \text{a.e} \ \ \implies \ \psi_\nu \leq \psi_\mu + C, \tag{2.6}$$

assuming that μ has finite energy.

Remark 2.1. When $\alpha \leq 2$ it is also known that the space of potentials is preserved under the max operation [42, Thm 1.31] (the case $\alpha = 2$ follows directly from subharmonicity). But for our purposes it will be enough to use the domination principle. This should be useful in order to extend Theorem 3.11 to more general kernels $W(x,y)$ appearing in axiomatic potential theory, where the domination principle (aka the second maximum principle) is often is taken as an axiom [42, Page 364]. For example, the domination principle holds when $W(x,y)$ is the potential kernel of a Markov process satisfying Hunt's hypothesis (H) [30] (then $-\psi_\mu$ is called the excessive function associated to μ). But we shall not go further into this here.

In order to simplify the notation we we will omit the dependence on α of the potential-theoretic objects associated to the Riesz kernel $W_\alpha(x,y)$, such as the energy $E(\mu)$, potentials ψ_μ and the corresponding inverse operator Δ (which coincides with the Laplacian when $\alpha = 2$). As α will be fixed this should not cause any confusion.

---

[1] In general, Δ_α is a fractional Laplacian $\Delta_\alpha := -(-\Delta)^{\frac{\alpha}{2}}$ in the sense of functional calculus, which in the case $\alpha \in]0, 2]$ corresponds precisely to the generator of a symmetric stable Levy process [19, 20].

[2] This terminology is standard, but different from the one in [42], where the terminology polar is used for the sets which are precisely equal to some $\{\psi_\mu = -\infty\}$ (such sets are called completely polar in modern terminology).

2.2 Function spaces

We will use the notation

$$\mathcal{L}_c(\mathbb{R}^d) := \left\{ \psi : \ \psi = \psi_\mu + C, \ \mu \in \mathcal{P}_c(\mathbb{R}^d), \ C \in \mathbb{R} \right\}.$$

We endow the space $\mathcal{L}_c(\mathbb{R}^d)$ with the L^1_{loc}-topology induced from the inclusion $\mathcal{L}_c(\mathbb{R}^d) \Subset L^1_{loc}(\mathbb{R}^d)$. Denote by $\mathcal{E}_c(\mathbb{R}^d)$ the subspace of $\mathcal{L}_c(\mathbb{R}^d)$ satisfying $E(\Delta\psi) < \infty$. Given a compact subset K of $\mathbb{R}^d$ we will write $\mathcal{L}_K(\mathbb{R}^d)$ and $\mathcal{E}_K(\mathbb{R}^d)$ for the subspaces of $\mathcal{L}_c(\mathbb{R}^d)$ and $\mathcal{E}_c(\mathbb{R}^d)$, respectively, obtained by demanding that $\Delta\psi$ be a probability measure supported in K. The definitions are made so that, for any compact S, we have a bijection (whose inverse is Δ) :

$$\mu \mapsto \psi_\mu \quad \mathcal{P}(S) \longleftrightarrow \mathcal{L}_S(\mathbb{R}^d)/\mathbb{R} \tag{2.7}$$

Proposition 2.2. *(Compactness) Let S be a compact subset of $\mathbb{R}^d$ and fix a closed ball B containing S. Then the subspace of $\mathcal{L}_S(\mathbb{R}^d)$ consisting of all ψ which are "mean normalized", i.e.*

$$\int_B \psi dx = 0,$$

is compact wrt the L^1_{loc}-topology. Moreover, given a compact subset K there exists a constant C such that

$$\sup_K \psi \leq \frac{\int_B \psi dx}{\int_B dx} + C \tag{2.8}$$

on $\mathcal{L}_S(\mathbb{R}^d)$.

Proof. Step 1: Compactness for "mean-normalized functions"

The compactness is a consequence of the following general properties of the kernel $-W_\alpha$: it is symmetric, usc, continuous on the complement of the diagonal, in L^1_{loc} and the corresponding integral operator yields a bijection, as in formula (2.7). To see this take any $\mu \in \mathcal{P}(S)$ and first observe that

$$\int_B \psi_\mu dx = \int f\mu$$

for the continuous function $f := \psi_\nu$ with $\nu := 1_B dx / \int_B dx$ (using the symmetry property (2.2)). Now decompose any mean-normalized ψ as

$$\psi = \left(\psi + \int f\mu \right) - \int f\mu$$

Since ψ is normalized the bijection (2.7) shows that the first term equals ψ_μ, where $\mu := \Delta\psi$. Let now ψ_j be a sequence in $\mathcal{L}_S(\mathbb{R}^d)$ and set $\mu_j := \Delta\psi_j$. Since S is compact the space $\mathcal{P}(S)$ is also compact and hence there exists $\mu \in \mathcal{P}(S)$ such that $\mu_j \to \mu$ in $\mathcal{P}(S)$, after perhaps passing to a subsequence. All that remains is thus to verify that $\psi_{\mu_j} \to \psi_\mu$ in L^1_{loc}. To this end first note that, since $-W_\alpha$ is usc, we have

$$\limsup_{j\to\infty} \psi_{\mu_j} \leq \psi$$

Moreover, by symmetry,

$$\int_B \psi_{\mu_j} dx = \int f\mu_j \to \int f\mu := \int_B \psi dx$$

since f is continuous. But then it follows from general integration theory that $\psi_{\mu_j} \to \psi$ in $L^1(B, dx)$. Finally, the L^1_{loc} convergence on the complement of S follows directly from the fact that W_α is continuous on the complement of the diagonal.

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Step 2: the upper bound on $\sup_K \psi$

By the previous step it will be enough to prove that the functional $\psi \mapsto \sup_K \psi$ is usc on $\mathcal{L}_S(\mathbb{R}^d)$. To this end first observe that, if $\mu_j \to \mu$ in $\mathcal{P}(S)$ and $x_j \to x$ then

$$\limsup \psi_{\mu_j}(x_j) \leq \psi_\mu(x),$$

using that the kernel $-W_\alpha$ is usc (see [42, Thm 1.3]). As consequence, if $\psi_j \to \psi_\infty$ in $\mathcal{L}_S(\mathbb{R}^d)$ then the previous inequality still holds if ψ_{μ_j} and ψ_μ are replaced by ψ_j and ψ_∞, respectively (using the decomposition argument in the previous step). Now, taking x_j so that $\psi_j(x_j) = \sup_K \psi_j$ concludes the proof of Step 2. $\qquad\square$

2.3 Energy functionals on $\mathcal{P}(S)$ and the equilibrium measure of a weighted set (S, ϕ)

If μ has compact support and ϕ is a continuous function on $\mathbb{R}^d$ we set

$$E_\phi(\mu) := -\frac{1}{2} \int \psi_\mu \mu + \int \phi \mu \in]-\infty, \infty] \tag{2.9}$$

The definition is made so that $E_0(\mu) := E(\mu)$ is the classical energy of μ (formula (2.1)).

Proposition 2.3. *Let S be a non-polar subset. Then the restriction of E to $\mathcal{P}(S)$ is lsc and strictly convex. Hence, so is E_ϕ for any given continuous function ϕ.*

Proof. This follows from (and is equivalent to) the positivity (2.3). $\qquad\square$

Thus, if S is a non-polar compact subset, then E_ϕ admits a unique minimizer $\mu_{(S,\phi)}$ on $\mathcal{P}(S)$, which is called the *equilibrium measure* of the *weighted set* (S, ϕ).

Given a non-polar compact weighted set (S, ϕ) it will also be convenient to consider a normalized version of the functional E_ϕ on $\mathcal{P}(S)$ defined by

$$E_{\omega_\phi} := E_\phi - \inf_{\mathcal{P}(S)} E_\phi, \tag{2.10}$$

which is invariant under $\phi \mapsto \phi + C$ for any constant C.

Remark 2.4. The notation is inspired by the corresponding complex-geometric setup, using the notation in [5, 9], applied to the special case when $d = \alpha = 2$. In this case, when ϕ is expressed as the restriction to S of a function with logarithmic growth in $\mathbb{C}$, the symbol ω_ϕ denotes the signed measure $\omega_\phi := \Delta\phi$ (extended by zero from $\mathbb{C}$ to the Riemann sphere) and E_{ω_ϕ} is the corresponding pluricomplex energy. Since the normalized energy, defined by formula (2.10), is invariant under $\phi \mapsto \phi + C$ it depends, indeed, only on $\Delta\phi$.

Lemma 2.5. *Given a weighted non-polar compact subset (S, ϕ) the potential $\psi_\mu \in \mathcal{L}_c(\mathbb{R}^d)$ of the equilibrium measure $\mu := \mu_{(S,\phi)}$ has the following property: there exists a constant C such that*

$$(i)\, \psi_\mu \leq \phi + C \quad q.e.\, on\, S$$
$$(ii)\, \psi_\mu \geq \phi + C \quad on\, S_\mu,$$

where S_μ denotes the support of μ.

Proof. This goes back to Frostman and is proved in [26, Theorem 1.2] when $S = \mathbb{R}^n$ and in [51] in the logarithmic case. The proof in the general case is essentially the same and follows from rather general variational considerations. Indeed, one first observes that $u_\mu := -(\psi_\mu - \phi)$ is a sub-gradient for the functional $E_\phi(\mu)$. Hence, if μ minimizes E_ϕ on $\mathcal{P}(S)$, then $\langle u_\mu, \nu - \mu \rangle \geq 0$ for any $\nu \in \mathcal{P}(S)$ of finite energy. Applying this inequality to

$\nu = \mu(1 + f)$ for any $f \in L^\infty(\mu)$ such that $\int f\mu = 0$ and $\|f\|_{L^\infty(\mu)} < 1$ gives $\langle u_\mu, f\mu \rangle \geq 0$. Replacing f with $-f$ yields the reversed inequality, showing that $\langle u_\mu, f\mu \rangle = 0$ for any f such that $\int f\mu = 0$. But this implies that $u_\mu = -C$ μ-a.e. for some constant C. Since, u is lsc it follows that $u_\mu \leq -C$ on S_μ. As a consequence, $\langle u_\mu, \nu \rangle \geq -C$ for any $\nu \in \mathcal{P}(S)$ of finite energy, which implies $u_\mu \geq -C$ q.e. on S (using that u_μ is lsc and that any subset of positive capacity has a compact subset of positive capacity). $\qquad\square$

To a compact weighted set (S, ϕ) we now attach the following function in $\mathcal{L}_S(\mathbb{R}^d)$:

$$\psi_{(S,\phi)} := \psi_{\mu_{(S,\phi)}} - C \tag{2.11}$$

where C is the constant appearing in the previous lemma.

2.4 The projection operator P_S

In this section we assume that $\alpha \leq 2$ (so that the domination principle (2.6) applies).

Now assume given a *weighted set* (S, ϕ), i.e. a subset S of $\mathbb{R}^d$ and a continuous function $\phi \in C(S)$. Consider the function

$$(\Pi_S \phi)(x) := \sup_{\mathcal{L}_c(\mathbb{R}^d)} \{\psi(x) : \ \psi \leq \phi \text{ on } S\}$$

Its upper semi-continuous regularization will be denoted by

$$P_S \phi := (\Pi_S \phi)^* \tag{2.12}$$

Proposition 2.6. *Let (S, ϕ) be a weighted subset and assume that S is compact and non-polar. Then*

$$\psi_{(S,\phi)} = P_S \phi \tag{2.13}$$

In particular, $\Delta(P_S \phi)$ is the unique minimizer of the functional E_ϕ on $\mathcal{P}(S)$. Moreover, as a consequence of (2.13),

$$P_S \phi \in \mathcal{E}_S(\mathbb{R}^d)$$

and the operator P_S satisfies the following "orthogonality relation"

$$\int (\phi - P_S \phi) \, \Delta(P_S \phi) = 0. \tag{2.14}$$

More generally, if S is bounded and equal to the union of increasing compact subsets, then $P_S \phi \in \mathcal{E}_{\overline{S}}(\mathbb{R}^d)$ for any given continuous function ϕ on $\mathbb{R}^d$.

Proof. Combining Lemma 2.5 with the domination principle (2.6) gives $\Pi_S \phi \leq \psi_{(S,\phi)}$ and hence $P_S \phi \leq \psi_{(S,\phi)}$. Indeed, let ψ be a candidate for the sup defining $\Pi_S \phi$. By Lemma (2.19) $\psi \leq \phi \leq \psi_{(S,\phi)}$ on the support of $\Delta(\psi_{(S,\phi)})$. Hence, the domination principle implies that $\psi \leq \psi_{(S,\phi)}$ everywhere. Moreover, since, by Lemma 2.5, $\psi_{(S,\phi)} \leq \phi$ on $S - N$ where N is polar we also have $\psi_{(S,\phi)} \leq P_{S-N}\phi$. The proof is thus concluded by invoking the fact that $P_{T \cup N}\phi = P_T \phi$ for any bounded Borel set T and polar subset N (applied to $T := S - N$). To see this first note that, trivially, $P_{T \cup N} \leq P_T$. To prove the converse fix $\psi_N \in \mathcal{L}_c(\mathbb{R}^d)$ such that $\psi_N = -\infty$ on N and $\psi_N \leq \phi$ on T. The existence of ψ_N follows form the fact that, by definition, any polar subset N is contained in the $-\infty$-locus of some potential ψ_N. Using the compactness of S we can then arrange that $\psi_N \in \mathcal{L}_c(\mathbb{R}^d)$ and $\psi_N \leq \phi$ on S and hence also $\psi_N \leq \phi$ on T. Thus, for any $\psi \in \mathcal{L}_c(\mathbb{R}^d)$ such that $\psi \leq 0$ on T we get $\psi_\epsilon := (1 - \epsilon)\psi + \epsilon\psi_N \leq \phi$ on $T \cup N$. Hence, $\psi_\epsilon \leq \Pi_{T \cup N}\phi$. Letting $\epsilon \to 0$ gives $\Pi_T \phi \leq \Pi_{T \cup N}\phi$ on the complement of the polar subset $\{\psi_N = -\infty\}$ and hence $P_T \phi \leq P_{T \cup N}\phi$ everywhere, as desired (using that if $\psi_\mu \leq \psi_\nu + C$ q.e. for a

measure ν of finite energy and a constant C, then $\psi_\mu \leq \psi_\nu + C$ everywhere, as a special case of the domination principle, since ν does not charge polar subsets).

Finally, if S it the union of increasing compact subsets K_i then $P_S \phi \leq \psi_i := P_{K_i} \phi$ for any i. By the previous step ψ_i is a decreasing sequence in $\mathcal{E}_{\overline{S}}(\mathbb{R}^d)$ and $\psi_i \leq \phi$ on $K_i - N_i$, where N_i is polar. Hence, by Prop 2.2, ψ_i converges in L^1_{loc} to $\psi_\infty \in \mathcal{E}_{\overline{S}}(\mathbb{R}^d)$, where $\psi_\infty \leq \phi$ on $S - N$, where N is the union of the polar sets N_i and hence polar (since the outer capacity is sub-additive). This means that $P_S \phi \leq \psi_\infty$ and $\psi_\infty \leq P_{S-N} \phi$. Since, as explained in the proof of the previous step, $P_{S-N} \phi = P_S \phi$ this concludes the proof. $\qquad\square$

Lemma 2.7. *Let S be a compact subset. The operator P_S defines a (non-linear) increasing concave operator from $\mathcal{C}(S)$ onto $\mathcal{L}_S(\mathbb{R}^d)$. Moreover, for any $\psi \in \mathcal{L}_c(\mathbb{R}^d)$*

$$P_S \psi \geq \psi$$

with equality if $\psi \in \mathcal{L}_S(\mathbb{R}^d)$. Hence, P_S is a projection operator, i.e. $P_S^2 = P_S$.

Proof. It follows directly from the definitions that $\phi_0 \leq \phi_1$ implies that $\Pi_S \phi_0 \leq \Pi_S \phi_1$ and hence also $P_S \phi_0 \leq P_S \phi_1$, i.e. P_S is increasing. Concavity of Π_S follows directly from the definition as a sup of linear functionals (defined by evaluation) and this implies the concavity of P_S, as well. Next, if $\psi \in \mathcal{L}_c(\mathbb{R}^d)$ then $\Pi_S \psi \geq \psi$ (since ψ is a candidate for the sup defining $\Pi_S \psi$). Since $\psi^* = \psi$ it follows that $P_S \psi \geq \psi$. The fact that $P_S \psi \in \mathcal{L}_S(\mathbb{R}^d)$ was proved in the previous proposition. Conversely, if $\psi \in \mathcal{L}_S(\mathbb{R}^d)$ then it follows, directly from the domination principle, that $\Pi_S \psi \leq \psi$ and hence also $P_S \psi \leq \psi \leq P_S \psi$, which proves the projection property in question. $\qquad\square$

2.5 The primitive functional $\mathcal{E}$ on $\mathcal{L}_c(\mathbb{R}^d)$ and its projection $\mathcal{F}$ to $C(S)$.

The operator Δ can be naturally identified with a one-form on the convex space $\mathcal{L}_c(\mathbb{R}^d) \cap C(\mathbb{R}^d)$:

$$\langle \Delta \psi, v \rangle := \int \Delta \psi v.$$

According to the next proposition this one-form admits a primitive that we shall denote by $\mathcal{E}$, i.e. a functional on $\mathcal{L}_c(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ whose differential is the operator Δ :

$$(d\mathcal{E})(\psi) = \Delta \psi$$

Since $\mathcal{L}_c(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ is convex the primitive $\mathcal{E}$ is uniquely determined up to an overall constant, which may be fixed by imposing the normalization condition

$$\mathcal{E}(\psi_0) = 0$$

for a fixed reference element $\psi_0 \in \mathcal{L}_c(\mathbb{R}^d) \cap C(\mathbb{R}^d)$. We will sometimes use a subscript $\mathcal{E}_{\psi_0}$ to indicate the dependence on the choice of ψ_0. Integrating along an affine line in $\mathcal{L}_c(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ suggests the following explicit formula, that we shall take as the definition of $\mathcal{E}_{\psi_0}$ on the whole space $\mathcal{L}_c(\mathbb{R}^d)$:

$$\mathcal{E}_{\psi_0}(\psi) := \frac{1}{2} \int (\psi - \psi_0)(\Delta \psi + \Delta \psi_0) \in [-\infty, \infty[$$

Moreover, it will be convenient to allow the reference ψ_0 to be in $\mathcal{E}_c(\mathbb{R}^d)$.

Proposition 2.8. *The functional $\mathcal{E}_{\psi_0}$ on $\mathcal{L}_c(\mathbb{R}^d)$ has the following properties*

(1) *$\mathcal{E}_{\psi_0}(\psi) > -\infty$ iff $\psi \in \mathcal{E}_c(\mathbb{R}^d)$. Moreover, $\Delta \psi_j \to \Delta \psi$ in $\mathcal{P}(S)$ and $E(\Delta \psi_j) \to E(\Delta \psi_j)$ for S compact iff $\psi_j \to \psi$ in L^1_{loc} and $\mathcal{E}_{\psi_0}(\psi_j) \to \mathcal{E}_{\psi_0}(\psi)$*

(2) *$\mathcal{E}_{\psi_0}$ is usc on $\mathcal{L}_c(\mathbb{R}^d)$*

(3) *Given $\psi_1, \psi_2 \in \mathcal{E}_c(\mathbb{R}^d)$ we have*

$$\frac{d\mathcal{E}_{\psi_0}(\psi_1 + t(\psi_2 - \psi_1))}{dt}\Big|_{t=0} = \int \Delta\psi_1(\psi_2 - \psi_1) \qquad (2.15)$$

(4) $\mathcal{E}_{\psi_0}$ *is concave on $\mathcal{L}_c(\mathbb{R}^d)$*

(5) $\mathcal{E}_{\psi_0}$ *is strictly increasing on $\mathcal{E}_c(\mathbb{R}^d)$: if $\psi \leq \Psi$, then $\mathcal{E}_{\psi_0}(\psi) \leq \mathcal{E}_{\psi_0}(\Psi)$ with equality iff $\psi = \Psi$.*

(6) *The following cocycle property holds: for any triple $\psi_i \in \mathcal{E}_c(\mathbb{R}^d)$ the difference*

$$\mathcal{E}_{\psi_0}(\psi_2) - \mathcal{E}_{\psi_0}(\psi_1)$$

is independent of ψ_0.

(7) *For any $c \in \mathbb{R}$ we have $\mathcal{E}_{\psi_0}(\psi + c) = \mathcal{E}_{\psi_0}(\psi) + c$*

Proof. To prove item 1 we may, by the cocycle property 6 proved below, assume that $\Delta\psi_0 = \rho dx$ for a continuous function ρ with compact support. Now decompose

$$\mathcal{E}_{\psi_0}(\psi) - \frac{1}{2}\int \psi\Delta\psi = -\frac{1}{2}\int \psi_0\Delta\psi + \frac{1}{2}\int \psi\Delta\psi_0 + C_0$$

Since ψ_0 is bounded on the support of $\Delta\psi$ the first term in the rhs above is finite and so is the second one since $\psi \in L^1_{loc}$. The same argument proves the convergence statement. Item 2 also follows from the previous decomposition, using that W is usc, just as in the proof of Prop 2.2. As for the formula in item 3 it follows directly from the symmetry (2.2). Similarly, the concavity of $\mathcal{E}_{\psi_0}$ follows from the fact that $\langle \Delta u, u \rangle \geq 0$ if $u = \psi_2 - \psi_1$ for $\psi_i \in \mathcal{E}_c(\mathbb{R}^d)$ (by the positivity (2.3)). That $\mathcal{E}_{\psi_0}$ is increasing follows directly from item 3, since $\Delta\psi \geq 0$ for any $\psi \in \mathcal{L}_c(\mathbb{R}^d)$ and strictly increasing follows from the domination principle when $\alpha \leq 2$. In the general case it follows from the strict concavity of $\mathcal{E}_{\psi_0}$, which in turn follows from the strict positivity in (2.3)). The cocycle property in item 6 follows directly from expressing $\mathcal{E}_{\psi_0}(\psi_2) - \mathcal{E}_{\psi_0}(\psi_1)$ as an integral on $[0,1]$ of the derivative $d\mathcal{E}_{\psi_0}(\psi_1 + t(\psi_2 - \psi_1))/dt$ and noting that, by item 3, the derivative is independent of ψ_0. Finally, item 7 also follows directly from item 3. □

Remark 2.9. If ψ_0 is normalized so that $\psi_0 = \psi_{\Delta\psi_0}$, then it follows directly from the symmetry (2.2) that $-\mathcal{E}_{\psi_0}(\psi_\mu) = E(\mu) + C_0$. However, it will be important to consider the functional $\mathcal{E}$ defined on all of $\mathcal{E}_c(\mathbb{R}^d)$.

Next, assume that $\alpha \leq 2$ (so that the domination principle (2.6) applies). Given a weighted compact and non-polar set (S, ϕ), consider the following functional defined on $C(S)$:

$$-\mathcal{F}_{(S,\phi)}(u) := \mathcal{E}_{\psi_0} \circ P_S(\phi - u) - \mathcal{E}_{\psi_0}(P_S(\phi))$$

Equivalently, this means, by the cocycle property in Prop 2.8, that, if choose the particular *canonical reference weight*

$$\psi_0 := P_S(\phi), \qquad (2.16)$$

(canonically attached to the weighted set (S, ϕ)), then

$$\mathcal{F}_{(S,\phi)}(u) = -\mathcal{E}_{\psi_0} \circ P_S(\phi - u). \qquad (2.17)$$

This choice of reference ψ_0 ensures the normalization $\mathcal{F}_{(S,\phi)}(0) = 0$.

Proposition 2.10. *The functional $\mathcal{F}_{(S,\phi)}$ is convex and Gateaux differentiable on $\mathcal{C}(S)$ and its differential at u is represented by the measure $\Delta\left(P_S(\phi - u)\right)$, i.e.*

$$\frac{d\mathcal{F}_{(S,\phi)}(u + tv)}{dt}\bigg|_{t=0} = \int v\Delta\left(P_S(\phi - u)\right)$$

for any $v \in C(S)$.

Proof. This can be shown directly using the orthogonality relation (2.14) (as in the complex geometric setting in [11], which covers the logarithmic case $d = \alpha = 2$). Alternatively, by Theorem 2.12 below $\mathcal{F}_{(S,\phi)}$ is the Legendre-Fenchel transform of the strictly convex functional E_{ω_ϕ}. Hence, by basic convex duality theory $\mathcal{F}_{(S,\phi)}$ is Gateaux differentiable and the differential at u is the minimizer of $E_{\omega_{\phi-u}}$, i.e. $\mu_{\phi-u}$, which is equal to $\Delta P_S(\phi - u)$, by Prop 2.6. $\qquad\square$

2.6 Energy and Legendre transforms

In this section we assume that $\alpha \leq 2$ or (so that the domination principle (2.6) applies) and show that the functional $\mathcal{F}_{(S,\phi)}$ can be viewed as a Legendre-Fenchel transform of E_{ω_ϕ}.

Lemma 2.11. *Assume that, on S, $\phi = \psi_\nu$, for a probability measure ν on S (which implies that the canonical reference weight $\psi_0 := P_S\phi$ coincides with ψ_ν). Then, for any $\mu \in \mathcal{P}_S(\mathbb{R}^d)$ such that $E(\mu) < \infty$, the corresponding normalized energy functional is given by*

$$E_{\omega_\phi}(\mu) = \mathcal{E}_{\psi_0}(\psi_\mu) - \int (\psi_\mu - \psi_0)\mu. \tag{2.18}$$

Proof. First rewrite

$$\mathcal{E}_{\psi_0}(\psi) = \frac{1}{2}\int (\psi - \psi_0)\Delta(\psi - \psi_0) + \int (\psi - \psi_0)\Delta\psi_0$$

Next, note that, in general, if μ and ν have finite energy, then there exists a constant C_1, depending on ν, such that

$$\frac{1}{2}\int (\psi_\mu - \psi_\nu)\Delta(\psi_\mu - \psi_\nu) = \frac{1}{2}\int \psi_\mu\Delta\psi_\mu - \int \psi_\mu\Delta\psi_\nu + C_1 =: -E_\phi(\mu) + C_1$$

using the symmetry (2.2) in the first equality. This shows that formula (2.18) holds up to an over all constant C_2. But the rhs in the formula vanishes for $\psi = \psi_\nu$ and so does $E_{\omega_\phi}(\nu)$, i.e. the minimum of $E_{\omega_\phi}(\mu)$ is realized for $\mu = \nu$, as follows from "completing the square". Hence, $C_2 = 0$, as desired. $\qquad\square$

Before stating the next theorem we recall the general definition of the Legendre-Fenchel transform. Let f be a function on a topological vector space V. The *Legendre-Fenchel transform* $\widehat{f}$ of f is defined as following convex lower semi-continuous function $\widehat{f}$ on the topological dual V^*

$$\widehat{f}(w) := \sup_{v \in V} \langle v, w \rangle - f(v) \tag{2.19}$$

in terms of the canonical pairing between V and V^*. In the present setting we will take $V = \mathcal{C}(S)$ and $V^* = \mathcal{M}(S)$, the space of all continuous functions and the space of all signed Borel measures, respectively, on a compact topological space S. Then the Legendre-Fenchel transform is involutive [25]. Given a compact subset S of $\mathbb{R}^d$ we will denote by $\chi_{\mathcal{P}(S)}$ the lsc functional on the space of all signed measures $\mathcal{M}(S)$ on $S \Subset \mathbb{R}^d$ which is equal to 0 on $\mathcal{P}(S)$ and equal to ∞ on the complement of $\mathcal{P}(S)$ in $\mathcal{M}(S)$. The definition is made so that the functional $\chi_{\mathcal{P}(S)} + E_{(S,\omega_\phi)}$ on $\mathcal{M}(S)$ is equal to $E_{(S,\omega_\phi)}$ if $\mu \in \mathcal{P}(S)$ and otherwise equal to ∞.

Theorem 2.12. *Let S be a compact subset of $\mathbb{R}^d$ and ϕ a continuous function on S. Consider the functional $\chi_{\mathcal{P}(S)} + E_{\omega_\phi}$ on the space $\mathcal{M}(S)$ of all signed measures on S. Its Legendre-Fenchel transform is given by*

$$\widehat{\chi_{\mathcal{P}(S)} + E_{\omega_\phi}} = \mathcal{F}_{(S,\phi)}, \tag{2.20}$$

where $\mathcal{F}_{(S,\phi)}$ is the functional defined in formula (2.17). Conversely,

$$\chi_{\mathcal{P}(S)} + E_{\omega_\phi} = \widehat{\mathcal{F}_{(S,\phi)}} \tag{2.21}$$

Moreover, for any $\mu \in \mathcal{P}(S)$ such that $E(\mu) < \infty$

$$E_{\omega_\phi}(\mu) = \mathcal{E}_{P_S\phi}(\psi_\mu) - \int (\psi_\mu - \phi)\mu \tag{2.22}$$

Proof. In order to prove (2.20) and (2.21) it is, since the Legendre-Fenchel transform is involutive on $\mathcal{M}(S)$, enough to prove (2.20), or equivalently that, for any given $u \in \mathcal{C}(S)$,

$$\inf_{\mathcal{P}(S)} \left(E_{\omega_\phi}(\mu) + \langle u, \mu \rangle \right) = \mathcal{E}_{P_S\phi}(P_S(\phi + u)). \tag{2.23}$$

Setting $\Phi = \phi + u$ and using that the inf above is attained at $\mu = \Delta(P_S\Phi)$ (by Prop 2.6) gives

$$\inf_{\mathcal{P}(S)} \left(E_\phi(\mu) + \langle u, \mu \rangle \right) = E_\phi(\Delta(P_S\Phi)) + \langle \Phi - \phi, \Delta(P_S\Phi) \rangle.$$

Rewriting the first term in the rhs as

$$E_\phi(\Delta(P_S\Phi)) = E_{P\phi}(\Delta(P_S\Phi)) + \langle (\phi - P_S\phi), \Delta(P_S\Phi) \rangle,$$

yields

$$\inf_{\mathcal{P}(S)} \left(E_{\omega_\phi}(\mu) + \langle u, \mu \rangle \right) = (E_{P_S\phi}(\Delta(P_S\Phi)) - E_\phi((\Delta(P_S\phi)))) + \langle \Phi - P_S\phi, \Delta(P_S\Phi) \rangle.$$

Using the orthogonality relation in Prop 2.6 reveals that $E_\phi((\Delta(P_S\phi)) = E_{P\phi}((\Delta(P_S\phi))$. Hence, by the previous lemma the first term in the rhs of the previous equation may be expressed as

$$E_{P_S\phi}(\Delta(P_S\Phi)) = \mathcal{E}_{P_S\phi}(\Phi) - \langle (\Phi - P_S\phi), \Delta(P_S\Phi) \rangle$$

Since the second term in the latter equation cancels the second term in former equation we thus get

$$\inf_{\mathcal{P}(S)} \left(E_{\omega_\phi}(\mu) + \langle u, \mu \rangle \right) = \mathcal{E}_{P_S\phi}(\Phi),$$

which coincides the right hand side in formula (2.23).

To prove the final statement we proceed essentially as above. First observe that, by definition,

$$E_\phi(\mu) = E_{P_S\phi}(\mu) + \langle \phi - P_S\phi, \mu \rangle.$$

Hence, applying the previous lemma to the weight $P_S\phi$, shows that there exists a constant C (only depending on ϕ) such that

$$E_{\omega_\phi}(\mu) = \mathcal{E}_{P_S\phi}(\psi_\mu) - \langle \psi_\mu - P_S\phi, \mu \rangle + \langle \phi - P_S\phi, \mu \rangle + C =$$
$$= \mathcal{E}_{P_S\phi}(\psi_\mu) - \langle \psi_\mu - \phi, \mu \rangle + C.$$

Now, evaluating the previous equality for $\mu = \Delta(P_S\phi)$ and using that $\psi_\mu - P_S\phi$ is constant, by Prop 2.6 (so that we can replace ψ_μ with $P_S\phi$ in the rhs of the previous equation, using item 7 in Prop 2.8) gives

$$0 = 0 - \langle P_S\phi - \phi, \Delta(P_S\phi) \rangle + C.$$

Finally, by the orthogonality relation in Prop 2.6, it follows that $C = 0$, which concludes the proof of the theorem. □

2.7 Regularity

In this section we assume that $\alpha \leq 2$. A weighted set (S, ϕ) will be said to be *regular* if $P_S\phi \leq \phi$ and a set S is said to be regular if $(S, 0)$ is regular. A compact set K is said to be *locally regular* if it is regular at any point $x \in K$, i.e. if $(P_{K \cap U}0)(x) \leq 0$ for any open ball U centered at x. In general, $P_S\phi \leq \phi$ always holds in the interior of S (see the beginning of the proof of Lemma 2.15).

Lemma 2.13. *A non-polar weighted compact set (S, ϕ) is regular iff $\sup_S(P_S\phi - \phi) = 0$ iff $P_S\phi = \Pi_S\phi$ iff $P_S\phi$ is continuous.*

Proof. The first equivalence follows from the extremal definition of $P_S\phi$, combined with Lemma 2.5 and Prop 2.6. To prove the second equivalence we note that if (S, ϕ) is regular, then $P_S\phi$ is a candidate for the sup defining $\Pi_S\phi$ and hence $P_S\phi \leq \Pi_S\phi$. Since the reverse inequality always holds we conclude that $P_S\phi = \Pi_S\phi$. Finally, let us show (S, ϕ) is regular iff $P_S\phi$ is continuous. First assume that (S, ϕ) is regular. By the previous step $P_S\phi = \Pi_S\phi$. Now, $P_S\phi$ is, by construction, usc. Hence, $\Pi_S\phi$ is continuous iff it is lsc. Accordingly, to prove that $\Pi_S\phi$ is continuous it is enough to show the following claim: the sup defining $\Pi_S\phi$ can be taken over all *continuous* $\psi \in \mathcal{L}_c(\mathbb{R}^d)$ satisfying $\psi \leq \phi$. To this end first note that there exists a sequence $\psi_j \in \mathcal{L}_c(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ such that $\psi_j \to P_S\phi$ in L^1_{loc} and such that $\psi_j(x) \to P_S\phi(x)$ for any x (as follows from [42, Thm 1.11 or Thm 3.7]). Moreover, ψ_j may be taken to in $\mathcal{L}_K(\mathbb{R}^d)$ for some compact set K containing S. Now, since (S, ϕ) is regular Prop 3.6 gives that the functional $\psi \mapsto \sup_S(\psi - \phi)$ is continuous on $\mathcal{L}_K(\mathbb{R}^d)$. Hence, replacing ψ_j with $\tilde{\psi}_j := \psi_j - \sup_S(\psi_j - \phi)$ and using that $\sup_S(P_S\phi - \phi) = 0$ we may as well assume that $\psi_j \leq \phi$. But then the point-wise convergence of ψ_j towards $P_S\phi$ proves the claim. Hence, $P_S\phi$ is continuous. Conversely, if $P_S\phi$ is continuous, then $P_S\phi \leq \phi$ q.e on S implies that $P_S\phi \leq \phi$ everywhere on S, which means that (S, ϕ) is regular. $\square$

Lemma 2.14. *Let K be a non-polar compact set K. Then (K, ϕ) is regular for any $\phi \in C(K)$ iff K is locally regular.*

Proof. This is shown as in the complex setting [48, Prop 6.1]. First assume that K is locally regular. Since ϕ is continuous we have that $\phi \leq \phi(x) + \delta(\epsilon)$ on an open ball $B_\epsilon(x)$ of radius ϵ centered at x, where $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. Hence,

$$P_K\phi \leq P_{K \cap B_\epsilon(x)}\phi \leq P_{K \cap B_\epsilon(x)}(\phi(x) + \delta(\epsilon)) \leq P_{K \cap B_\epsilon(x)}0 + \phi(x) + \delta(\epsilon).$$

Letting $\epsilon \to 0$ thus gives $(P_K\phi)(x) \leq \phi(x)$, showing that (K, ϕ) is regular. Conversely, assume that (K, ϕ) is regular for all $\phi \in \mathcal{C}(K)$. Take a point $x \in X$ and an open ball B centered at x. Define a function ϕ on K by setting $\phi = 0$ on $K \cap B$ and $\phi = P_{K \cap B}0$ on $K - B$. The function ϕ is clearly usc and hence there exists a sequence $\phi_j \in \mathcal{C}(K)$ decreasing to ϕ. Now, if ψ is candidate for the sup defining $\Pi_{K \cap B}0$, then $\psi \leq 0$ on $K \cap B$ and hence $\psi \leq \Pi_{K \cap B}0$ everywhere. As a consequence, $\psi \leq \phi$ on K, which, in turn, implies $\psi \leq P_K\phi_j \leq \phi_j$ for any j, using in the last equality that (K, ϕ_j) is assumed regular. Hence, taking the sup over all such ψ and using that ϕ_j is continuous on K gives $P_{K \cap B}0 \leq \phi_j$. Finally, letting $j \to \infty$ we conclude that $P_{K \cap B}0 \leq 0$ on $K \cap B$, as desired. $\square$

2.7.1 Compact domains

Now consider the case when K is a compact domain, i.e. K is the closure of an open bounded set Ω. Following standard classical terminology Ω is said to be *thin at* $x_0 \in \partial K := K - \Omega$ if there exists some potential ψ_μ such that $\limsup_{x \to x_0} \psi_\mu(x) < \psi_\mu(x_0)$, assuming $x \in \Omega$ (see the definition and discussion in [42, page 307]).

Lemma 2.15. *Let K be a compact domain, i.e. K is the closure of an open bounded set Ω. If Ω is non-thin at all boundary points, then (K, ϕ) is regular for any continuous ϕ.*

Proof. Set $\psi := P_K \phi$. Then it follows from the continuity of ϕ that $\psi \leq \phi$ in Ω. Indeed, by definition ψ is the upper semi-continuous regularization $(\Pi_K \phi)^*$ of the function $\Pi_K \phi$, which satisfies $\Pi_K \phi \leq \phi$ on K. Thus, given $x \in \Omega$ and using that any ball $B_\epsilon(x)$ centered at x of sufficiently small radius ϵ is contained on Ω, we deduce that

$$\psi(x) := \lim_{\epsilon \to 0} \sup_{B_\epsilon(x)} (\Pi_K \phi) \leq \lim_{\epsilon \to 0} \sup_{B_\epsilon(x)} \phi = \phi(x),$$

since ϕ is continuous. Finally, given $x_0 \in K - \Omega$, the assumption of non-thinness implies that $\psi(x_0) \leq \limsup_{x \to x_0} \psi(x) \leq \phi(x_0)$ since $x \in \Omega$ and ϕ is continuous. $\quad\square$

The notion of thinness of a set E at a point x_0 can equivalently be formulated in terms of Wiener's capacity criterion (see [42, Thm 5.2] and [42, Thm 5.10]), which, in turn, is equivalent to the following capacity criterion ([42, formula 5.1.7]): for a given number $q \in]0, 1[$,

$$\sum_{m=1}^{\infty} \frac{\mathcal{C}_\alpha(E^{(m)})}{q^{m(d-\alpha)}} < \infty, \quad E^{(m)} := E \cap \left\{ x \in \mathbb{R}^d : |x - x_0| < q^m \right\}, \tag{2.24}$$

where $\mathcal{C}_\alpha$ denotes the capacity corresponding to α (recalled in Section 6.1 in the appendix). Here we have assumed that we are not in the logarithmic case $d = \alpha = 2$, where a similar capicity criterion applies (see [42, Thm 5.6]).

Example 2.16. In the three-dimensional Coulomb case, $d = 3$ and $\alpha = 2$, algebraic cusps are non-thin at the vertex, while Lebesgue cusps are thin at the vertex [42, page 287]. These cusp as defined for a given $m > 0$, by the surfaces of revolution in $\mathbb{R}^3$ where $\theta \leq r^m$ and $\theta \leq e^{-m/r}$ respectively (using planar polar coordinates). See also [37] for more general results in the Coulomb case.

For general $\alpha \in]0, 2]$ the capacity criterion above yields the following

Proposition 2.17. *Let K be a compact domain in $\mathbb{R}^d$ satisfying the* interior cone condition, *i.e. for any point $x_0 \in \partial K$ there exists a cone contained in K with a vertex at x_0. Then K is locally regular.*

Proof. By the previous lemma it is equivalent to show that $(P_K \phi)(x) \leq \phi(x)$ for any $\phi \in C(K)$ and $x \in K$. Given $x_0 \in \partial K$ denote by C_{x_0} a cone in K with vertex at x_0. Note that it is enough to verify that interior of a cone C_{x_0} is non-thin at the vertex. Indeed, since ϕ is continuous $P_K \phi \leq 0$ in the interior of K and hence in the interior of C_{x_0}. Taking a sequence of points x_i in the interior of C_{x_0} converging towards x_0 and exploiting that C_{x_0} is non-thin at x_0 it thus follows that $P_S \phi(x_0) \leq \limsup_{x \to x_0} P_S \phi(x) \leq \phi(x_0)$, using in the last inequality that ϕ is continuous. Finally, to verify that the interior of a compact cone C_{x_0} is non-thin at x_0 first observe that, since W_α is translationally invariant, we may as well assume that x_0 is the origin. But then $E := C_{x_0}$ has the scaling property that $E^{(m)} = q^m \cdot E^{(1)}$. It thus follows from the scaling properties of the kernel W_α that $\mathcal{C}_\alpha(E^{(m)}) = q^{m(d-\alpha)} \mathcal{C}_\alpha(E^{(1)})$ and hence the capacity criterion (2.24) is trivially satisfied (for notational simplicity we have assued that we are not in the logarithmic case $d = 2 = \alpha$, where essentially the same argument applies). $\quad\square$

2.8 Determining measures

The definition of (strongly) determining measures was given in Section 1.1. It may be equivalently formulated as follows. A measure ν on $\mathbb{R}^d$ is said to be *determining for a*

weighted set (S, ϕ) if for all $\psi \in \mathcal{L}_c(\mathbb{R}^d)$

$$\sup_S e^{\psi - \phi} = \left\| e^{\psi - \phi} \right\|_{L^\infty(S, \nu)}$$

A measure ν is said to be *determining for S* if ν is determining for $(S, 0)$ and *strongly determining* if ν is determining for (S, ϕ) for all $\phi \in C(S)$. Similarly we will say that ν is (strongly) determining if it is (strongly determining) for its support.

Proposition 2.18. *If μ_0 does not charge polar subsets, has compact support S_0 and is (strongly) determining, then S_0 is (locally) regular.*

Proof. Since $\psi_\phi := P_{S_0} \phi \le \phi$ q.e. and μ_0 does not charge polar subsets it follows that $\psi_\phi \le \phi$ a.e. wrt μ_0. By assumption this means that $\psi_\phi \le \phi$ on S_0, i.e. (S_0, ϕ) is regular, as desired. □

Any compact weighted regular compact subset carries determining measures:

Proposition 2.19. *Let (K, ϕ) be a regular weighted compact set. Then the corresponding equilibrium measure $\mu_{(K, \phi)}$ is determining for (K, ϕ). As a consequence, if μ_0 has the property that $\mu_{(K, \phi)}$ is absolutely continuous with respect to μ_0, then μ_0 is also determining for (K, ϕ).*

Proof. Assume that $\psi \le \phi$ a.e. wrt $\mu_{(K, \phi)}$. Recall that, by Prop 2.6, $\mu_{(K, \phi)} = \Delta(P_K \phi)$ and $\phi = P_K \phi$ a.e. wrt to $\Delta(P_K \phi)$. Hence, by the domination principle (2.6), $\psi \le P_K \phi$ everywhere. Finally, since (K, ϕ) is assumed regular it thus follows that $\psi \le \phi$ everywhere, as desired.

This follows directly from the domination principle (2.6). □

2.8.1 Compact domains

We will next consider the special case when S_0 is a compact domain (i.e. S_0 is the closure of an open bounded set), using the following lemma:

Lemma 2.20. *Lebesgue measure dx is strongly determining for any open subset $U \subset \mathbb{R}^d$. In other words, the measure $1_U dx$ is strongly determining.*

Proof. Fix a smooth compactly supported function ρ such that $\rho dx \in \mathcal{P}(\mathbb{R}^n)$ and set $\rho_\delta := \delta^n \rho(x/\delta)$. Now, if $\psi_\mu \le \phi$ a.e. on U, then, for any given compact subset K of U, there exists a sequence $\epsilon_j \to 0$ such that $\psi_j := \psi_\mu * \rho_{j^{-1}} \le \phi + \epsilon_j$ on K. But $\psi_j = \psi_{\mu * \rho_{j^{-1}}}$ and hence, by (2.5),

$$\psi(x) \le \limsup_{j \to \infty} (\phi(x) + \epsilon_j) = \phi(x)$$

for any $x \in K$ and hence for any $x \in U$. □

Note that in the Coulomb case, $\alpha = 2$, the previous lemma follows directly from the submean property of subharmonic functions.

Proposition 2.21. *Let S_0 be a compact domain, i.e. $S_0 = \bar{\Omega}$, where Ω is an open bounded set. If Ω is non-thin at all boundary points, i.e. at all points in $X - \Omega$, then $\mu_0 := 1_\Omega dx$ is strongly determining for S_0. In particular, if S_0 satisfies the interior cone condition (appearing in Prop 2.17), then $1_\Omega dx$ is strongly determining.*

Proof. Assume that $\psi_\mu \le \phi$ a.e. wrt $1_\Omega dx$. Then, by the previous lemma, $\psi_\mu \le \phi$ on Ω. Now take $x_0 \in S_0 - \Omega$. By the non-thinness assumption $\psi_\mu(x_0) \le \limsup_{x \to x_0} \psi_\mu(x)$ for any sequence of points $x \in \Omega$ converging towards x_0. Since ϕ is continuous we deduce that $\psi_\mu(x_0) \le \phi(x_0)$, showing that $\mu_0 := 1_\Omega dx$ is strongly determining. □

In particular, $1_\Omega dx$ is strongly determining if Ω is a bounded Lipschitz domain. Moreover, as shown next, the Hausdorff measure on the boundary $\partial\Omega$ of a bounded Lipschitz domain is strongly determining for $\partial\Omega$, in the Coulomb case.

2.8.2 Lipschitz hypersurfaces

Theorem 2.22. *Consider the Coulomb case $\alpha = 2$. The $(d-1)$-dimensional Hausdorff measure μ_0 on a Lipschitz hypersurface K in $\mathbb{R}^d$ without boundary is strongly determining.*

Proof. Denote by T be the closure of a bounded tubular neighborhood of K and decompose it into two closed domains $T_\pm$ intersecting along K :

$$T = T_- \bigcup T_+.$$

By assumption, the domains $T_\pm$ may be taken to be compact Lipschitz domains. Given a potential ψ in $\mathbb{R}^d$ we fix a constant C such that $\psi \le C$ on T. Denote by $f_\pm$ the continuous function on $\partial T_\pm$ which is equal to a given continuous function ϕ on K and equal to C on ∂T. We denote by $h_\pm$ the harmonic extension of $f_\pm$ to $T_\pm$. The function $h_\pm$ is in $\mathcal{C}(T_\pm)$, as follows from the fact that $T_\pm$ satisfies the interior cone condition (see 2.17). Now, by assumption, $\psi \le f_\pm$ almost everywhere with respect to the Hausdorff measure $\sigma_\pm$ on $\partial T_\pm$. But then it follows from [27] that

$$\psi \le h_\pm, \text{ in the interior of} T_\pm. \tag{2.25}$$

Accepting, this for the moment and denoting by h the continuous function on T which is equal to $h_\pm$ on $T_\pm$ we get $\psi \le h$ on T. Hence, since ψ is subharmonic, for any $x \in K$ we have

$$\psi(x) \le \frac{1}{|B_\delta(x)|} \int_{B_\delta(x)} h dx$$

Letting $\delta \to 0$ and using that h is continuous and equal to $\phi(x)$ at x we conclude that $\psi(x) \le \phi(x)$, as desired.

Finally, we note that the inequality (2.25) is a standard consequence of the result in [27], saying, in particular, that for a Lipschitz domain D the harmonic measure ν_x on ∂D is absolutely continuous wrt the $(d-1)$-dimensional Hausdorff measure σ on ∂D, for any $x \in \partial D$. Indeed, by the standard maximum principle for subharmonic functions

$$\psi(x) \le \int \nu_x \psi \tag{2.26}$$

if x is in the interior of D (this is immediate in the case when ψ is continuous in a neighborhood of D and then general case then follows writing ψ as a decreasing limit of such functions). Hence, applying (2.26) to $D = T_\pm$ and using that $\psi \le f_\pm$ almost everywhere with respect $\sigma_\pm$ gives

$$\psi(x) \le \int \nu_x f_\pm = h_\pm,$$

proving (2.25). □

Remark 2.23. The method of proof can be adapted to many other situations. Indeed, it only requires the existence of a neighborhood T of K such that the harmonic measures on the corresponding boundaries of $T_\pm$ are absolutely continuous wrt the corresponding Hausdorff measures.

3   Determining measures, Energy approximation and Gamma-convergence

Given a probability measure μ_0 with compact support S_0 and a continuous function ϕ on $\mathbb{R}^d$ the corresponding *free energy functional $F_{\phi,\beta}$ at inverse temperature $\beta \in]0,\infty]$* is defined by the following functional on $\mathcal{P}_c(\mathbb{R}^d)$:

$$F_{\phi,\beta}(\mu) = E_\phi(\mu) + \frac{1}{\beta} D_{\mu_0}(\mu), \ \ E_\phi(\mu) := E(\mu) + \int \phi \mu \tag{3.1}$$

where D_{μ_0} denotes the *entropy of μ relative to μ_0* (also known as the *Kullback–Leibler Divergence*), i.e.

$$D_{\mu_0}(\mu) := \int_{\mathbb{R}^d} \log \frac{\mu}{\mu_0} \mu, \tag{3.2}$$

when μ is absolutely continuous wrt to μ_0 and otherwise $D_{\mu_0}(\mu) := \infty$. We define $F_{\phi,\infty} := E_\phi$. Note that

$$F_{\phi,\beta} \geq E_\phi,$$

since $D_{\mu_0} \geq 0$. We also recall that the functional D_{μ_0} is lower semi-continuous (lsc) on $\mathcal{P}(K)$, for any given compact subset K [25].

Similarly, when replacing the energy E_ϕ with its normalized version E_{ω_ϕ} (2.10) we will write

$$F_{\omega_\phi,\beta} := E_{\omega_\phi}(\mu) + \frac{1}{\beta} D_{\mu_0}(\mu) = F_{\phi,\beta} - \inf_{\mathcal{P}(S_0)} E_\phi,$$

where S_0 denotes the support of μ_0. When $\phi = 0$ we will simply use the notation $F_\beta := F_{\phi,\beta} = F_{0,\beta}$.

3.1   The Energy Approximation property vs Gamma-convergence of free energies

We recall the definition of *Gamma-convergence,* introduced by De Georgi (see the book [21] for background on Gamma-convergence):

Definition 3.1. A family of functions F_β on a topological space $\mathcal{M}$ *is said to Gamma-converge* to a function F on $\mathcal{M}$, as $\beta \to \infty$, if

$$\begin{array}{ccc} \mu_\beta \to \mu \text{ in } \mathcal{M} & \implies & \liminf_{\beta \to \infty} F_\beta(\mu_\beta) \geq F(\mu) \\ \forall \mu & \exists \mu_\beta \to \mu \text{ in } \mathcal{M} : & \lim_{\beta \to \infty} F_\beta(\mu_\beta) = F(\mu) \end{array} \tag{3.3}$$

A sequence (family) μ_β as in the last point above is called a *recovery sequence (family)* for μ. The limiting functional F_∞ is automatically lower semi-continuous on $\mathcal{M}$.

We first make the following simple observation:

Lemma 3.2. *A measure μ_0 satisfies the Energy Approximation Property (section 1.1) iff the free energy F_β Gamma-converges towards the energy E on $\mathcal{P}(K)$.*

Proof. First suppose that the Gamma-convergence holds. Given $\mu \in E(\mu)$ such that $E(\mu) < \infty$ we take a recovery family μ_β, i.e.

$$E(\mu) \geq \limsup_{\beta \to \infty} F_\beta(\mu_\beta).$$

Since $F_\beta \geq E$ this directly implies the inequality (1.1) and hence the Energy Approximation Property. To prove the converse first observe that, since E and D_{μ_0} are lsc on $\mathcal{P}(K)$ it is enough to show that for any $E(\mu) < \infty$ there exists a recovery family, which, in turn, is equivalent to finding a family μ_β such that $(i) \mu_\beta \to \mu$ in $\mathcal{P}(K)$, $(ii) E(\mu_\beta) \to E(\mu)$,

and $\beta^{-1} D_{\mu_0}(\mu_\beta) \to 0$. Moreover, the latter condition may be replaced by the condition that $(iii) \, D_{\mu_0}(\mu_\beta) < \infty$. Indeed, by relabeling the family μ_β we can then arrange that $D_{\mu_0}(\mu_\beta) \le \beta^{1/2}$, say. Now, assuming that the energy approximation property holds, there exists a family μ_β satisfying the conditions (i) and (ii) and such that $\mu_\beta = \rho_\beta \mu_0$ for some $\rho_\beta \in L^1(\mu_0)$. For any positive integer j we set

$$\mu_{\beta,j} := \max(\rho_\beta, j)\mu_0 / \int \max(\rho_\beta, j)\mu_0 \in \mathcal{P}(K),$$

which satisfies $D_{\mu_0}(\mu_{\beta,j}) < \infty$. Moreover, by the monotone convergence theorem, $E(\mu_{\beta,j}) \to E(\mu_\beta)$ and $\mu_{\beta,j} \to \mu_\beta$, as $j \to \infty$. Hence, we can conclude using a standard diagonal argument. $\qquad\square$

Gamma-convergence is stable under addition by continuous functionals, as follows directly from the definition. We will make use of the following criterion for Gamma-convergence on $\mathcal{P}(K)$, formulated in terms of the Legendre-Fenchel transform (Definition (2.19)):

Proposition 3.3. *Let F_β be a family of functions on the space $\mathcal{P}(K)$ of all probability measures on a compact space K and extend F_β by infinity to all of $\mathcal{M}(K)$. Assume that*

$$\lim_{\beta \to \infty} \widehat{F_\beta}(\phi) = f(\phi)$$

for any $\phi \in C(X)$ and that f defines a Gateaux differentiable function on $\mathcal{C}(K)$. Then F_β Gamma-converges to $\widehat{f}$ on $\mathcal{M}(K)$ (the converse holds without any differentiability assumption).

See [6] for the proof of the previous proposition. Unraveling definitions reveals that, in the present setting, where F_β is the free energy functional we have

$$\lim_{\beta \to \infty} \widehat{F_\beta}(\phi) = \widehat{E}(\phi)$$

iff

$$\lim_{\beta \to \infty} \inf_{\mathcal{P}(S_0)} F_{\phi,\beta} = \inf_{\mathcal{P}(S_0)} E_\phi$$

Thus in order to establish the Energy Approximation property, or equivalently, the Gamma-convergence of F_β towards E, it is equivalent to establish the asymptotics above for the infima of $F_{\phi,\beta}$. for all continuous weights ϕ.

Remark 3.4. Lemma 3.2 still holds if the entropy $D_{\mu_0}(\mu)$ is replaced by any lsc functional $\tilde{D}$ on $\mathcal{P}(K)$ with the property that $\tilde{D}(\mu) < \infty$ implies that μ is absolutely continuous with respect to μ_0 and such that $\tilde{D}$ is finite on $L^\infty(K)\mu_0$ (using the same proof). But in the proof of Theorem 3.10 we will (implicitly) exploit that the Legendre-Fenchel transform of $\beta^{-1} D_{\mu_0}$ has good monotonicity and convergence properties with respect to β. Indeed, as is well-known, the Legendre-Fenchel transform of $\beta^{-1} D_{\mu_0}$ (extended by ∞ to the space $\mathcal{M}(S_0)$ of all signed measures on S_0) is the functional on $\mathcal{C}(S_0)$ defined by $u \mapsto \beta^{-1} \log \int e^{\beta u} \mu_0$, which increases to $\sup_X u$ as $\beta \to \infty$. But the actual proof of Theorem 3.10 does not explicitly invoke the Legendre-Fenchel transform, since we will need to allow u to be non-continuous, namely of the form $\psi - \phi$, where ψ is a potential.

3.2 Determining measures vs Gamma-convergence of the free energies

In this section we will assume that $\alpha \le 2$ (so that the domination principle (2.6) applies).

Lemma 3.5. *Assume given μ_0 in $\mathcal{P}(\mathbb{R}^d)$ not charging polar subsets and of compact support S_0. Then, for any $\phi \in C(\mathbb{R}^d)$ and $\beta \in]0, \infty[$, the corresponding free energy functional $F_{\phi,\beta}$ on $\mathcal{P}(S_0)$ admits a unique minimizer $\mu_{\phi,\beta}$. Moreover, $\mu_{\phi,\beta} = \Delta \psi_{\phi,\beta}$, where $\psi_{\phi,\beta}$ is the unique solution in $\mathcal{E}_{S_0}(\mathbb{R}^d)$ of the following equation:*

$$\Delta \psi = e^{\beta(\psi - \phi)} \mu_0 \tag{3.4}$$

and we have

$$\inf_{\mathcal{M}(S_0)} F_{\omega_\phi, \beta} = \sup_{\mathcal{E}_c(\mathbb{R}^d)} \mathcal{G}_{\phi,\beta} = \sup_{\mathcal{E}_{S_0}(\mathbb{R}^d)} \mathcal{G}_{\phi,\beta} = \mathcal{G}_{\phi,\beta}(\psi_{\phi,\beta}) \tag{3.5}$$

where $\mathcal{G}_{\phi,\beta}$ is the following functional on $\mathcal{L}_S(\mathbb{R}^d)$, taking values in $[-\infty, \infty[$:

$$\mathcal{G}_{\phi,\beta}(\psi) := \mathcal{E}_{\psi_0}(\psi) - \mathcal{I}_\beta(\psi), \;\; \psi_0 := P_{S_0}(\phi) \tag{3.6}$$

where

$$\mathcal{I}_\beta(\psi) := \beta^{-1} \log \int e^{\beta(\psi - \phi)} \mu_0 \tag{3.7}$$

(where we have suppressed the dependence on ϕ in the notation $\mathcal{I}_\beta$).

Proof. Step 1: $\sup_{\mathcal{E}_c(\mathbb{R}^d)} \mathcal{G}_{\phi,\beta} = \sup_{\mathcal{E}_{S_0}(\mathbb{R}^d)} \mathcal{G}_{\phi,\beta} = \mathcal{G}_{\phi,\beta}(\psi_{\phi,\beta})$

To simplify the notation we will write $\mathcal{E}_{\psi_0} = \mathcal{E}$. First observe that $\mathcal{I}_\beta(\psi) > -\infty$ on $\mathcal{L}_c(\mathbb{R}^d)$. Indeed, if $\mathcal{I}_\beta(\psi) = -\infty$ then μ_0 charges the polar set $\{\psi = -\infty\}$, which contradicts the assumption on μ_0. Now fix any compact set S containing S_0 and consider the functional $\mathcal{G}_{\phi,\beta}$ on $\mathcal{L}_S(\mathbb{R}^d)$. We note that $\mathcal{G}_\beta$ is usc. Indeed, by Prop 2.8 $\mathcal{E}$ is usc and so is $-\mathcal{I}_\beta$, by Fatou's lemma. Moreover, $\mathcal{G}_{\phi,\beta}(\psi + c) = \mathcal{G}_{\phi,\beta}(\psi)$ and hence it follows from the compactness in Prop 2.2 that $\mathcal{G}_\beta$ admits a maximizer ψ_β. Since $\mathcal{I}_\beta(\psi) > -\infty$ we have $\psi_\beta \in \mathcal{E}_S(\mathbb{R}^d)$. All that remains is to verify that ψ_β satisfies the equation (3.4) (after perhaps shifting ψ_β by a constant). To this end fix a continuous bounded function u, $u \in \mathcal{C}_b(\mathbb{R}^d)$, and set

$$g(t) := \mathcal{E}(P_S(\psi_\beta + tu)) - \mathcal{I}_\beta(\psi_\beta + tu).$$

The maximum of the function g is attained at $t = 0$. Indeed, for any $\psi \in \mathcal{E}_S(\mathbb{R}^d) + \mathcal{C}(S_0)$ we have that $\mathcal{I}_\beta(P_S\psi) \leq \mathcal{I}_\beta(\psi)$, using that $P_S\psi \leq \psi$ q.e. on S and hence a.e. with respect to μ_0 (since μ_0 does not charge polar subsets). As a consequence,

$$\mathcal{E}(P_S(\psi)) - \mathcal{I}_\beta(\psi) \leq \mathcal{E}(P_S(\psi)) - \mathcal{I}_\beta(P_S(\psi)) := \mathcal{G}_\beta(P_S\psi)$$

In particular, since $P_S\psi \in \mathcal{E}_S(\mathbb{R}^d)$,

$$\sup_{\psi \in \mathcal{E}_S(\mathbb{R}^d) + C(S_0)} (\mathcal{E}(P_S(\psi)) - \mathcal{I}_\beta(\psi)) \leq \sup_{\psi \in \mathcal{E}_S(\mathbb{R}^d)} \mathcal{G}_{\phi,\beta} = \mathcal{G}_{\phi,\beta}(\psi_{\phi,\beta}).$$

Thus, restricting ψ in the lhs above to be in $\psi_\beta + \mathbb{R}u$ shows that the maximum of the function g is, indeed, attained at $t = 0$. Moreover, by Prop 2.10 $g(t)$ is differentiable and hence $g'(0) = 0$ shows, using that $P_S\psi = \psi$, that the equation (3.4) holds when integrated against any $u \in \mathcal{C}_b(\mathbb{R}^d)$. Since a probability measure is uniquely determined by its action on $C_b(\mathbb{R}^d)$ this conclude the proof of Step 1.

Step 2: $\inf_{\mathcal{M}(S_0)} F_{\omega_\phi, \beta} = \sup_{\mathcal{E}_{S_0}(\mathbb{R}^d)} \mathcal{G}_{\phi,\beta}$, where the infimum is realized precisely at $\Delta \psi_{\phi,\beta}$

Since E_{ω_ϕ} is convex and D_{μ_0} is strictly convex on the subset $\{D_{\mu_0} < \infty\} \subset \mathcal{P}(S_0)$ (by Jensen's inequality) the functional $F_{\omega_\phi, \beta}$ has at most one minimizer. By the previous step it will thus be enough to show that $\mu_{\phi,\beta} := \Delta \psi_{\phi,\beta}$ minimizes $F_{\omega_\phi, \beta}$. But this follows

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directly from the fact that $-(\psi_\mu - \phi)$ is a subgradient for E_ϕ and $\log(\mu/\mu_0)$ is a subgradient for D_{μ_0} (by convexity). Finally, evaluating $F_{\omega_\phi,\beta}$ at $\mu_{\phi,\beta}$ and using formula (2.22) gives

$$F_{\omega_\phi,\beta}(\mu_{\phi,\beta}) = \mathcal{E}_{P_S\phi}(\psi_{\phi,\beta}) - \int (\psi_{\phi,\beta} - \phi)\mu + \frac{1}{\beta} \int \log e^{\beta(\psi_{\phi,\beta} - \phi)} = \mathcal{E}_{P_S\phi}(\psi_{\phi,\beta}).$$

Finally, since $e^{\beta(\psi_{\phi,\beta} - \phi)} (= \Delta\psi_{\phi,\beta})$ is a probability measure we have that $\mathcal{I}_\beta(\psi_{\beta,\phi}) = 0$ and hence $\mathcal{E}_{P_S\phi}(\psi_{\phi,\beta}) = \mathcal{G}_{\phi,\beta}(\psi_{\beta,\phi})$, which concludes the proof of Step 2.

Step 3: The solution $\psi_{\phi,\beta}$ of equation (3.4) is uniquely determined.

By the previous step $\Delta(\psi_{\phi,\beta})$ is the unique minimizer of $F_{\omega_\phi,\beta}$ on $\mathcal{P}(S_0)$. Hence, $\psi_{\phi,\beta}$ is uniquely determined up to an additive constant. But, since, as explained in the previous step, $\mathcal{I}_\beta(\psi_{\beta,\phi}) = 0$ the constant in question vanishes. $\qquad\square$

We next establish an approximate reversed Hölder type inequality for measures μ_0 not charging polar subsets. The result mimics the logarithmic case, which is covered by the complex-geometric setting in [12, Thm 1.14] and shows that μ_0 is determining iff μ_0 satisfies a potential-theoretic analog of the Bernstein-Markov inequality for polynomials:

Proposition 3.6. *Assume that μ_0 has compact support S_0 and does not charge polar subsets. Then the following is equivalent for a given continuous function ϕ :*

- *μ_0 is determining for (S_0, ϕ)*
- *For all $\epsilon > 0$ there exist a constant C such that*

$$\sup_{S_0} e^{\psi - \phi} \leq C^{1/p} e^\epsilon \left\| e^{\psi - \phi} \right\|_{L^p(S_0, \mu_0)} \tag{3.8}$$

for any $\psi \in \mathcal{L}_K(\mathbb{R}^d)$ and $p > 0$.

As a consequence, if K is compact and (K, ϕ) is regular then the functional

$$\mathcal{L}_K(\psi) := \sup_K(\psi - \phi)$$

is continuous on $\mathcal{L}_S(\mathbb{R}^d)$ for any given compact set S.

Proof. Given the general properties recalled in Section 2.1 and the compactness result in Prop 2.2 the proof follows, more or less verbatim, from the proof of the corresponding result in [12, Thm 1.14]. For completeness we provide the argument here.

Step 1: The functional $\mathcal{L}_K$ is usc on $\mathcal{L}_K(\mathbb{R}^n)$ for any compact set K.

This is shown exactly as in the case $\phi = 0$ appearing in the proof of Step 2 in Prop 2.2.

Step 2: If μ_0 does not charge polar sets, then the functional $\mathcal{I}_p$ (formula (3.7)) is continuous on $\mathcal{L}_K(\mathbb{R}^n)$ for any $p > 0$.

If ψ_j is a sequence of functions in $\mathcal{L}_K(\mathbb{R}^n)$ converging in L^1_{loc} towards ψ, then, by (2.8) and (2.4)

$$(i) \sup_K \psi_j \leq C, \quad \limsup \psi_j = \psi \ \ \mu_0 - \text{a.e}$$

since μ_0 does not charge polar sets. The continuity of the functional $\mathcal{I}_p$ now follows from a Hilbert space argument using convex combinations of $f_j := e^{\psi_j - \phi}$, by repeating the argument in the proof of [12, Thm 1.14] word by word.

Step 3: In general, $\mathcal{I}_p$ is increasing in p and

$$\lim_{p \to \infty} \mathcal{I}_p(\psi) = \mathcal{L}_\infty(\psi) := \log \left\| e^{\psi - \phi} \right\|_{L^\infty(S, \mu_0)}$$

Indeed, this follows from Hölder's inequality and standard integration theory.

Now, *if μ_0 does not charge polar sets*, then, combining Step 2 and Step 3, reveals that the functional $\mathcal{L}_\infty$ is lsc. If μ_0 is moreover determining then $\mathcal{L}_\infty = \mathcal{L}_{S_0}$ and hence $\mathcal{L}_\infty$ is also usc continuous by Step 1 and hence continuous. To conclude the proof of the inequality (3.8) it will be enough to show that $f_p := \mathcal{I}_p - \mathcal{L}_\infty$ converges uniformly to 0 on $\mathcal{L}_{S_0}(\mathbb{R}^d)$. Since $f_p(\psi + c) = f_p(\psi)$ it is enough to prove this on the subspace of all mean-normalized ψ. But since the latter space if compact (Prop 2.2) the uniform convergence in question follows from Step 3, using Dini's lemma. Conversely, if the inequality (3.8) holds, then letting $p \to \infty$ gives $\mathcal{L}_{S_0} \leq \mathcal{L}_\infty$ on $\mathcal{L}_{S_0}(\mathbb{R}^d)$, i.e. μ_0 is determining (since trivially $\mathcal{L}_\infty \leq \mathcal{L}_{S_0}$).

Finally, the last statement in the proposition is obtained by taking μ_0 to be the equilibrium measure of (K, ϕ) and using Prop 2.19. $\qquad\qquad\qquad\qquad\square$

We note that for any measurable function u on a measure space (S, μ_0)

$$\sup_{\mu_0} u := \log \|e^u\|_{L^\infty(S, \mu_0)},$$

is called the *essential sup* of u on (S, μ_0). Given a measure μ_0 we now define the following function on $\mathbb{R}^d$, taking values in $]0, \infty]$:

$$(\Pi_{\mu_0}\phi)(x) := \sup_{\mathcal{L}_{S_0}(\mathbb{R}^d)} \{\psi(x) : \sup_{\mu_0}(\psi - \phi) \leq 0\}, \qquad (3.9)$$

where S_0 denotes the support of μ_0. Its upper semi-continuous regularization is denoted by

$$P_{\mu_0}\phi := (\Pi_{\mu_0}\phi)^*$$

This definition should be compared with definition of $P_S\phi$, in formula (2.12). However, in general, $P_{\mu_0}\phi$ can be different from $P_{S_0}\phi$, for the support S_0 of μ_0 (unless μ_0 is determining).

Remark 3.7. In the logarithmic case $d = \alpha = 2$ the function $P_{\mu_0}\phi$ coincides with the minimal carrier Green function [55] when $\phi = 0$. For a general ϕ it coincides with the quasi-plurisubharmonic envelope on Kähler manifolds X introduced in [35] (specialized to the case when X is the Riemann sphere).

Lemma 3.8. *Let μ_0 be a measure on $\mathbb{R}^d$ which does not charge polar subsets and with compact support S_0 and ϕ a continuous function on $\mathbb{R}^d$. Then*

$$P_{\mu_0}\phi \in \mathcal{E}_{S_0}(\mathbb{R}^d)$$

and

$$\sup_{\mu_0}(P_{\mu_0}\phi - \phi) = 0$$

Proof. Step1: $\Pi_\mu\psi$ is locally bounded from above

Given a large ball B it is enough to show the existence of a constant C such that

$$\delta(\psi) := \sup_B(\psi - \phi) - \sup_\mu(\psi - \phi) \leq C.$$

By Step 1 in the proof of Prop 3.6 the first functional in the lhs above is usc on $\mathcal{L}_{S_0}(\mathbb{R}^d)$ for any compact set B. Moreover, as explained in the proof of Prop 3.6 the second functional is lsc for any measure μ_0 not charging polar subsets. This means that the functional $\delta(\psi)$ is usc on $\mathcal{L}_{S_0}(\mathbb{R}^d)$ and satisfies $\delta(\psi + c) = \delta(\psi)$ for any $c \in \mathbb{R}$. By the compactness of the subspace of $\mathcal{L}_{S_0}(\mathbb{R}^d)$ consisting of mean-normalized functions this yields the existence of a constant C as above.

Step 2: $P_{\mu_0}\phi \in \mathcal{E}_{S_0}(\mathbb{R}^d)$ and $\sup_{\mu_0}(P_{\mu_0}\phi - \phi) = 0$

First we recall "Choquet's lemma": let $\{u_\alpha\}_{\alpha \in A}$ be a family of real valued functions on a metric separable space X (that we shall take to be $\mathbb{R}^d$). Suppose furthermore that this family is locally bounded from above. Then there exists a countable subset B of A such that

$$(\sup\{u_\beta\}_{\beta \in B})^* = (\sup\{u_\alpha\}_{\alpha \in A})^*,$$

where $\sup\{u_\beta\}$ denotes the function on $\mathbb{R}^d$ defined as the point-wise sup. Thus, by Choquet's lemma, there exists a countable family of functions ψ_i, which are candidates for the sup defining $P_\mu \phi$, satisfying

$$(\sup\{\psi_i\})^* = (P_\mu \phi).$$

Recall that, in general, a Borel subset $S \subset \mathbb{R}^d$ is called a μ_0-*carrier* of a measure μ_0 on $\mathbb{R}^d$ if $\mu(S) = \mu(\mathbb{R}^d)$, i.e. if $\mu_0(\mathbb{R}^d - S) = 0$. Since $\sup_{\mu_0}(P_{\mu_0}\psi_i - \psi_i) = 0$ we have that $\psi_i \leq \phi$ on a μ_0-carrier S_i. Denote by S the intersection of all S_i. Then S is also a μ_0-carrier. Take a subset $K_\sigma \Subset S$ which is a union of increasing compact subsets of S such that $\mu_0(K_\sigma) = \mu_0(S)$ (the existence of K_σ follows from the fact that a Borel measure μ_0 is, in particular, interior regular). Since $\psi_i \leq \phi$ on K_σ we have $\psi_i \leq P_{K_\sigma}\phi$. Moreover, by Prop 2.6, $P_{K_\sigma}\phi \in \mathcal{L}_{S_0}(\mathbb{R}^d)$ and $P_{K_\sigma}\phi \leq \phi$ q.e. on the μ_0-carrier K_σ. Hence, $P_\mu \phi \leq P_{K_\sigma}\phi \leq P_\mu \phi$, using in the last inequality that $P_{K_\sigma}\phi \leq \phi$ μ_0-almost everywhere, since μ_0 does not charge polar sets. This shows that $P_\mu \phi \in \mathcal{E}_{S_0}(\mathbb{R}^d)$ and $\sup_{\mu_0}(P_{\mu_0}\phi - \phi) \leq 0$. But then the extremal definition of Π_{μ_0} forces $\sup_{\mu_0}(P_{\mu_0}\phi - \phi) = 0$. $\qquad\square$

We shall also need the following

Lemma 3.9. *Suppose that K is compact and (K, ϕ) is regular. Then*

$$\inf_{\mathcal{P}(K)} E_{\omega_\phi} = \sup_{\psi \in \mathcal{L}_c(\mathbb{R}^d)} \mathcal{G}_K = \sup_{\psi \in \mathcal{L}_K(\mathbb{R}^d)} \mathcal{G}_K, \tag{3.10}$$

where

$$\mathcal{G}_K(\psi) := \mathcal{E}(\psi) - \sup_K(\psi - \phi). \tag{3.11}$$

Moreover, $P_K \phi$ is the unique maximizer of the functional $\mathcal{G}_K$ subject to the normalization $\sup_K(\psi - \phi) = 0$. Similarly, if μ_0 has compact support and does not charge polar subsets, then $P_{\mu_0}\phi$ is the unique maximizer of the functional

$$\mathcal{G}_\infty(\psi) := \mathcal{E}(\psi) - \sup_{\mu_0}(\psi - \phi), \tag{3.12}$$

subject to the normalization $\sup_{\mu_0}(\psi - \phi) = 0$.

Proof. By Theorem 2.12 the lhs in formula (3.10) is given by $\mathcal{E}(P_K \phi)$ which in turn is given by $\mathcal{G}_K(P_K \phi)$, by the regularity assumption. Moreover, if $\psi \in \mathcal{L}_c(\mathbb{R}^d)$ and $\sup_K(\psi - \phi) = 0$, then $\psi \leq P_K \phi$ (by the very definition of $P_K \phi$) and hence $\mathcal{G}_K(\psi) \leq \mathcal{G}_K(P_K \phi)$, since $\mathcal{E}$ is increasing (Prop 2.8). Moreover, the uniqueness in question follows from the fact that $\mathcal{E}$ is strictly increasing. The corresponding results for $\mathcal{G}_\infty(\psi)$ are shown in a similar way, now using that $\sup_{\mu_0}(P_{\mu_0}\phi - \phi) = 0$, by the previous lemma. $\qquad\square$

We are now ready for the proof of the core analytic result of the present paper:

Theorem 3.10. *Let μ_0 be a measure on $\mathbb{R}^d$ which does not charge polar subsets and assume that μ_0 has compact support S_0. Given a continuous function ϕ on $\mathbb{R}^d$ the following is equivalent:*

(1) *μ_0 is determining for (S_0, ϕ)*

(2) (S_0, ϕ) *is regular and*

$$\lim_{\beta \to \infty} \inf_{\mathcal{P}(S_0)} F_{\phi,\beta} = \inf_{\mathcal{P}(S_0)} E_\phi \tag{3.13}$$

(3) (S_0, ϕ) *is regular and the minimizers* $\mu_{\phi,\beta}$ *of* $F_{\phi,\beta}$ *converge weakly towards the minimizer* $\mu_{(S_0,\phi)}$ *of* E_ϕ *as* $\beta \to \infty$ *(and then convergence in energy automatically holds)*

(4) (S_0, ϕ) *is regular and the solution* $\psi_{\beta,\phi}$ *of the equation (3.4) converges towards* $P_{S_0}\phi$ *in energy, as* $\beta \to \infty$.

Proof. First note that since $F_{\phi,\beta}$ coincides with $F_{\omega_\phi,\beta}$ up to an additive constant, which is independent of β, it is equivalent to prove the theorem with $F_{\phi,\beta}$ and E_ϕ replaced by $F_{\omega_\phi,\beta}$ and E_{ω_ϕ}, respectively.

Step 1: 1 implies 2 and 3 and 4.

By Lemma 3.5

$$\inf_{\mathcal{M}(S_0)} F_{\omega_\phi,\beta} = \sup_{\mathcal{E}_{S_0}(\mathbb{R}^d)} \mathcal{G}_{\phi,\beta}$$

and hence, by the previous lemma, the convergence (3.13) may be reformulated as

$$\lim_{\beta \to \infty} \sup_{\mathcal{L}_{S_0}(\mathbb{R}^d)} \mathcal{G}_\beta = \sup_{\psi \in \mathcal{L}_{S_0}(\mathbb{R}^d)} \mathcal{G}_{S_0} \tag{3.14}$$

Now, if μ_0 is determining for (S, ϕ_0), then, by Prop 3.6, for any $\epsilon > 0$ there exists a constant C such that

$$\sup_{S_0}(\psi - \phi) - C/\beta - \epsilon \leq \mathcal{I}_\beta(\psi) \leq \sup_{S_0}(\psi - \phi) + C/\beta \tag{3.15}$$

Since the functional $\mathcal{E}$ is usc this immediately implies the convergence in item 2. Moreover, by compactness (Prop 2.2) we may, after perhaps passing to a subsequence, assume that the maximizer $\psi_{\beta,\phi}$ of $\mathcal{G}_{\phi,\beta}$ converges towards a maximizer of $\mathcal{G}_{S_0}$. Hence, by the previous lemma, $\psi_{\beta,\phi}$ converges towards $P_{S_0}\phi$, which combined with (3.15) gives $\mathcal{E}(\psi_{\beta,\phi}) \to \mathcal{E}(P_{S_0}\phi)$. This implies (Prop 2.8) that $E_{\omega_\phi}(\mu_{\beta,\phi}) \to E_{\omega_\phi}(\Delta P_{S_0}\phi) = E_\omega(\mu_{(S_0,\phi)})$ and hence $2, 3$ and 4 follow.

Step 2: 2 implies 1

First note that, since trivially, $\mathcal{I}_\beta \leq \mathcal{L}_\infty \leq \mathcal{L}_{S_0}$ we have that

$$\liminf_{\beta \to \infty} \sup_{\mathcal{L}_{S_0}(\mathbb{R}^d)} \mathcal{G}_\beta \geq \sup_{\psi \in \mathcal{L}_{S_0}(\mathbb{R}^d)} \mathcal{G}_\infty \geq \sup_{\psi \in \mathcal{L}_{S_0}(\mathbb{R}^d)} \mathcal{G}_{S_0} \tag{3.16}$$

Combined with (3.14) this means that if item 2 holds, then the inequalities above must be equalities and hence

$$\sup_{\mathcal{L}_{S_0}(\mathbb{R}^d)} \mathcal{G}_\infty = \sup_{\psi \in \mathcal{L}_{S_0}(\mathbb{R}^d)} \mathcal{G}_{S_0} \tag{3.17}$$

But this implies that $P_{\mu_0}\phi = P_{S_0}\phi$. Indeed, by definition, we have $P_{\mu_0}\phi \geq P_{S_0}\phi$ and since $\mathcal{L}_{S_0}(P_{S_0}\phi) = 0 = \mathcal{L}_\infty(P_{\mu_0}\phi)$ the equality (3.17) forces $\mathcal{E}(P_{\mu_0}\phi) \geq \mathcal{E}(P_{S_0}\phi)$. By the strict monotonicity of $\mathcal{E}$ this means that $P_{\mu_0}\phi = P_{S_0}\phi$. Since, by definition, $P_{\mu_0}\phi$ and $P_{S_0}\phi$ are defined as the upper semi-continuous regularizations of $\Pi_{\mu_0}\phi$ and $\Pi_{S_0}\phi$, respectively, it thus follows that

$$\Pi_{\mu_0}\phi \leq (\Pi_{\mu_0}\phi)^* = (\Pi_{S_0}\phi)^* \leq \phi$$

using that (S_0, ϕ) is assumed regular in the last equality. Hence, μ_0 is determining for (S_0, ϕ).

Step 3: The weak convergence in item 3 implies convergence in energy and items 4 and 2

Assume that $\mu_{\phi,\beta}$ converges towards $\mu_{(S_0,\phi)}$. By compactness (Prop 2.2) this means that there exist constants C_β such that

$$\psi_{\beta,\phi} + C_\beta \to P_{S_0}\phi$$

in L_{loc}^1. Since $\mathcal{L}_{\beta,\phi}(\psi_{\beta,\phi}) = 0$ it follows that

$$\lim_{\beta\to\infty} C_\beta = \lim_{\beta\to\infty} \mathcal{L}_{\beta,\phi}(P_{S_0}\phi) = \mathcal{L}_{\infty,\phi}(P_{S_0}\phi) = \mathcal{L}_{S_0,\phi}(P_{S_0}\phi) = 0$$

using in the next to last equality that $P_{S_0}\phi$ is continuous (by Lemma 2.13), since (S_0, ϕ) is assumed regular. Hence, $\psi_{\beta,\phi}$ converges towards $P_{S_0}\phi$ in L_{loc}^1 and the lower bound (3.16) gives

$$\liminf_{\beta\to\infty} \mathcal{E}(\psi_{\beta,\phi}) \geq \mathcal{E}(P_{S_0}\phi).$$

Since $\mathcal{E}$ is usc this shows that, in fact,

$$\mathcal{E}(\psi_{\beta,\phi}) \to \mathcal{E}(P_{S_0}\phi). \tag{3.18}$$

Hence, item 4 holds. Now, by Lemma 3.5,

$$\beta^{-1} D_{\mu_0}(\mu_{\beta,\phi}) = \int (\psi_{\beta,\phi} - \phi)\mu_{\beta,\phi} \to \int (P_{S_0}\phi - \phi)\Delta(P_{S_0}\phi) = 0$$

using (3.18) in the convergence step and the orthogonality relation (2.14) in the last equality. All in all this means the weak convergence in 3 implies the convergence in energy of $\mu_{\beta,\phi}$, as well as the convergence of free energies in item 2. $\qquad\square$

Finally, combining the previous theorem with Prop 3.3 (and the subsequent discussion) and Lemma 3.2 we arrive at the following result, which contains, in particular, Theorem 1.1 and Theorem 1.2 stated in the introduction, except the LDP statement proved in Section 4.1.

Theorem 3.11. *Let μ_0 be a measure on $\mathbb{R}^d$ which does not charge polar subsets and assume that the support S_0 of μ_0 is compact. Then the following is equivalent:*

(1) *The measure μ_0 is strongly determining*

(2) *S_0 is locally regular and $\inf_{\mathcal{P}(S_0)} F_{\phi,\beta} \to \inf_{\mathcal{P}(S_0)} E_\phi$, as $\beta \to \infty$, for any given $\phi \in C(S_0)$.*

(3) *S_0 is locally regular and the functional F_β converges towards E, as $\beta \to \infty$, in the sense of Gamma-convergence.*

(4) *S_0 is locally regular and the measure μ_0 has the Energy Approximation Property.*

(5) *S_0 is locally regular and for any given $\phi \in \mathcal{C}(S_0)$ the measures $\mu_{\phi,\beta}$ converge weakly towards $\mu_{(S_0,\phi)}$, as $\beta \to \infty$ (and then convergence in energy automatically holds)*

3.3 Quasi-explicit approximations using finite energy weights ϕ

In this section we provide a constructive procedure for obtaining the approximation in Theorem 1.1. To this end we assume, as before, that $\alpha \leq 2$ and consider *generalized weights*

$$\phi \in \mathcal{C}(S) + \mathcal{E}_S(\mathbb{R}^d) - \mathcal{E}_S(\mathbb{R}^d).$$

(for the construction in question it is enough to work with $\mathcal{C}(S) + \mathcal{E}_S(\mathbb{R}^d)$, but since it requires not extra effort we will consider the more general setting). First recall that by basic Hilbert space duality theory we have the following

Lemma 3.12. *Let S be a compact subspace of $\mathbb{R}^d$. Then a function ϕ is in $\mathcal{E}_S(\mathbb{R}^d)$ iff $|\langle \phi, \mu \rangle| < \infty$ for all measures $\mu \in \mathcal{P}(S)$ satisfying $E(\mu) < \infty$. Moreover, if $\phi \in \mathcal{E}_S(\mathbb{R}^d)$, then the functional $\langle \phi, \cdot \rangle$ is continuous wrt the weak topology on any sublevel set $\{E \leq C\}$ in $\mathcal{P}(S)$.*

Now, given a generalized weight $\phi \in \mathcal{C}(S) + \mathcal{E}_S(\mathbb{R}^d) - \mathcal{E}_S(\mathbb{R}^d)$ we define the corresponding weighted energy by

$$E_\phi(\mu) := E(\mu) + \langle \phi, \mu \rangle$$

if $E(\mu) < \infty$ and otherwise $E_\phi(\mu) := \infty$.

Lemma 3.13. *Let S be a non-polar subset. Then the restriction of E_ϕ to $\mathcal{P}(S)$ is lsc and strictly convex.*

Proof. This is shown using Hilbert space theory exactly as in the proof of [10, Thm 3.21] (which concerns the case when $d = \alpha = 2$). $\qquad\square$

In particular, if S is a non-polar compact set, then E_ϕ admits a unique minimizer on $\mathcal{P}(X)$ that we shall call denote, as before, by $\mu_{(S,\phi)}$. Combining the previous two lemmas yields the following generalization of the convergence in item 5 of Theorem 3.11 to generalized weights ϕ :

Proposition 3.14. *Let μ_0 be a measure in $\mathbb{R}^d$ with compact support S_0 such that μ_0 does not charge polar sets and μ_0 is determining and assume that*

$$\phi \in \mathcal{C}(S_0) + \mathcal{E}_{S_0}(\mathbb{R}^d) - \mathcal{E}_{S_0}(\mathbb{R}^d).$$

Then the corresponding free energies $F_{\beta,\phi}$ converge, as $\beta \to \infty$, to E_ϕ in the sense of Gamma-convergence. As a consequence, the minimizer $\mu_{\beta,\phi}$ of $F_{\beta,\phi}$ converges in energy towards the weighted equilibrium measure $\mu_{(S_0,\phi)}$ of S_0.

Proof. Since $D_{\mu_0} \geq 0$ the lower bound in the Gamma-convergence follows directly from the lower semi-continuity of E_ϕ in the previous lemma. The reconstruction property then follows from the reconstruction property in the case when $\phi = 0$ using the continuity statement in Lemma 3.12. That is to say that any reconstruction sequence for $E(\mu)$ is also a reconstruction sequence for $E_\phi(\mu)$.

We thus arrive at the following constructive version of the approximation in Theorem 1.1: $\qquad\square$

Theorem 3.15. *Let μ_0 be a measure in $\mathbb{R}^d$ with compact support S_0 such that μ_0 does not charge polar sets and μ_0 is determining. Given $\mu \in \mathcal{P}(S_0)$ such that $E(\mu) < \infty$, let μ_β be the minimizer of the free energy functional F_{β,ψ_μ} on $\mathcal{P}(S_0)$, i.e. $F_{\beta,\psi_\mu} := E_\psi + \beta^{-1} D_{\mu_0}$. Then μ_β converges in energy towards μ, as $\beta \to \infty$.*

Proof. By definition $\mu = \Delta \psi_\mu$ and μ minimizes E_{ψ_μ} on $\mathcal{P}(S_0)$. Hence, the convergence follows directly from the convergence in the previous proposition. $\qquad\square$

This is a constructive approximation in the sense that there are quasi-explicit ways of approximating the minimizer $\mu_{\beta,\psi}$ of $F_{\beta,\psi}$, for a given $\psi \in \mathcal{E}_{S_0}(\mathbb{R}^d)$. For example, when $\psi \in \mathcal{E}_{S_0}(\mathbb{R}^d)$ is assumed bounded on S_0 it follows from Cor 4.9 below that the measure $\mu_{N,\beta}$ defined as the expectations $\mathbb{E}(\delta_N)$ of the empirical measure of the Riesz gas associated to the finite measure $e^{-\beta\psi}\mu_0$, converges in energy towards μ_β :

$$\mu_{N,\beta} := \mathbb{E}(\delta_N) = \frac{\int_{(\mathbb{R}^d)^{N-1}} e^{-\beta H(x,x_2,\ldots x_N)} \left(e^{-\beta\psi}\mu_0\right)^{\otimes N-1}}{\int_{(\mathbb{R}^d)^N} e^{-\beta H(x_1,\ldots x_N)} \left(e^{-\beta\psi}\mu_0\right)^{\otimes N}} e^{-\beta\psi(x)}\mu_0,$$

where H denotes the Hamiltonian of the Riesz gas. In the general case we can simply replace ψ with $\max\{\psi, -R\}$ for a given parameter $R > 0$ and obtain $E(\mu_\beta)$ from the double limit where first $N \to \infty$ and then $R \to \infty$. By a diagonal argument this yields a sequence of measures μ_{N,R_N}, absolutely continuous with respect to μ_0 and converging in energy towards a given measure $\mu \in \mathcal{P}(S_0)$ of finite energy. This is a quasi-explicit approximation in the sense that $\mu_{N,\beta}$ is given by a quotient of two integrals, whose integrands are explicitly given.

3.4 The limit $\beta \to 0$

Before turning to large deviations we note that in the opposite (infinite temperature) limit $\beta \to 0$ the minimizers μ_β always converge towards the reference measure μ_0.

Proposition 3.16. *Let μ_0 be a measure on $\mathbb{R}^d$ which does not charge polar subsets and of compact support S_0 and assume that $\phi \in \mathcal{C}(S_0)$. Then $\beta F_{\beta,\phi} \to D_{\mu_0}$ as $\beta \to 0$ in the sense of Gamma-convergence on $\mathcal{P}(S_0)$. In particular, the minimizer μ_β of F_β converges weakly towards μ_0.*

Proof. First observe that, since $D_{\mu_0} \leq \beta F_\beta := D_{\mu_0} + \beta E_\phi$ and D_{μ_0} is lsc, the lower bound in the definition of Gamma-convergence is satisfied. All that remains is thus to show that for any $\mu \in \mathcal{P}(S_0)$ such that $D_{\mu_0}(\mu) < \infty$ there exists a recovery family μ_β. In the case when $E(\mu) < \infty$ we can trivially take $\mu_\beta = \mu$. Next, note that there exists a sequence $\nu_j \in \mathcal{P}(S_0)$ such that $E(\nu_j) < \infty$ and $D_{\mu_0}(\nu_j) \to D_{\mu_0}(\mu_0)$ in $\mathcal{P}(S_0)$, as $j \to \infty$. Indeed applying Lemma 3.5 to $\phi = 0$ and a fixed β, say $\beta = 1$, shows that there exists a measure ν of finite energy which is absolutely continuous wrt μ_0 :

$$\nu = \rho\mu_0, \ \ \rho \in L^1(\mu_0)$$

Hence, the truncated sequence $\nu_j := \max(\rho, j)/\int \nu_j$ has the required properties. Using a truncation argument again and the monotone convergence theorem then shows that any μ such that $D_{\mu_0}(\mu) < \infty$ has the property that there exists a sequence $\mu_j \in \mathcal{P}(S_0)$ such that $E(\mu_j) < \infty$ and $D_{\mu_0}(\mu_j) \to D_{\mu_0}(\mu)$. But then the recovery property for any μ follows by a simple diagonal argument. $\qquad\square$

4 Large deviations

We start with the following general setup. Let X be a compact topological space and W a symmetric proper lsc function on $X \times X$ called the *pair interaction potential.* Given a probability measure μ_0 with support X the corresponding *Gibbs measures* at inverse temperature $\beta_N \in]0, \infty[$ are defined as the following sequence of symmetric probability measures on X^N :

$$\mu_{\beta_N}^{(N)} := \frac{1}{Z_{N,\beta_N}} e^{-\beta_N H^{(N)}} \mu_0^{\otimes N},$$

where

$$H^{(N)}(x_1, ...x_N) := \frac{1}{(N-1)} \frac{1}{2} \sum_{i \neq j} W(x_i, x_j) \tag{4.1}$$

and the normalization constant Z_{N,β_N} is assumed to be non-zero (it is automatically finite, since W is lsc and X is compact). We also assume that the following limit exists:

$$\beta := \lim_{N \to \infty} \beta_N \in]0, \infty]$$

Setting

$$E(\mu) := \frac{1}{2} \int_{X^2} W \mu^{\otimes 2},$$

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the corresponding free energy functional F_β on $\mathcal{P}(X)$ is defined as in formula (3.1) (with $\phi = 0$). The empirical measure δ_N (formula (1.4)) defines a $\mathcal{P}(X)$-valued random variable on $(X^N, \mu_{\beta_N}^{(N)})$. By definition, its law is the probability measure

$$\Gamma_N := (\delta_N)_* \mu_{\beta_N}^{(N)} \tag{4.2}$$

on $\mathcal{P}(X)$.

We recall the general definition of a *Large Deviation Principle (LDP)* for a sequence of measures [25], which is modeled on the classical Laplace steepest descent principle for integrals:

Definition 4.1. Let Y be a compact Polish space, i.e. a compact complete separable metric space.

(i) A function $I : Y \to]-\infty, \infty]$ is a *rate function* if it is lower semi-continuous and $\inf_Y I = 0$

(ii) A sequence Γ_N of measures on Y satisfies a *large deviation principle* with *speed* r_N and *rate function I* if

$$\limsup_{N \to \infty} \frac{1}{r_N} \log \Gamma_N(\mathcal{F}) \leq - \inf_{\mu \in \mathcal{F}} I(\mu)$$

for any closed subset $\mathcal{F}$ of Y and

$$\liminf_{N \to \infty} \frac{1}{r_N} \log \Gamma_N(\mathcal{G}) \geq - \inf_{\mu \in \mathcal{G}} I(\mu)$$

for any open subset $\mathcal{G}$ of Y.

Fixing a metric on Y The LDP may also be equivalently expressed in terms of $\Gamma_N(B_\epsilon(\mu))$, where $B_\epsilon(\mu)$ denotes the closed ball of radius ϵ centered at $\mu \in Y$. For example, if Γ_N is the law of the empirical measure δ_N of a random point process, then the LDP is equivalent [25, Theorems 4.1.11, 4.1.18] to

$$\lim_{\epsilon \to 0} \liminf_{N \to \infty} \frac{1}{r_N} \log \mathbb{P}\left(\frac{1}{N} \sum_{i=1}^N \delta_{x_i} \in B_\epsilon(\mu) \right) = \lim_{\epsilon \to 0} \limsup_{N \to \infty} \frac{1}{r_N} \log \mathbb{P}\left(\frac{1}{N} \sum_{i=1}^N \delta_{x_i} \in B_\epsilon(\mu) \right)$$
$$= -I(\mu)$$

for some functional $I(\mu)$ (which, as a consequence, thus has to be lower semi-continuous).

Given W and μ_0 and a sequence β_N as above we will say that *the corresponding LDP holds at inverse temperature β* if the Gibbs measures $\mu_{\beta_N}^{(N)}$ are well-defined and the laws Γ_N of the corresponding empirical measures on X^N satisfy a LDP.

Theorem 4.2. *Assume given a proper lsc pair interaction potential W and a measure μ_0 with compact support X.*

- *When $\beta \in]0, \infty[$ the corresponding LDP holds with speed βN iff the functional F_β is proper lsc on $\mathcal{P}(S_0)$ iff there exists a measure of finite energy, which is absolutely continuous wrt μ_0. Then the rate functional is given by $F_\beta - \inf_{\mathcal{P}(X)} F_\beta$.*
- *When $\beta = \infty$ the corresponding LDP holds with speed $N\beta_N$ if F_β is proper lsc for all $\beta \in]0, \infty]$ and Gamma-continuous as $\beta \to \infty$. The rate functional is then given by $E - \inf_{\mathcal{P}(X)} E$.*

Proof. This result is essentially contained in [29, 7, 32]. But for completeness we provide some details. First observe that since X is compact F_β is proper lsc iff $\inf_{\mathcal{P}(S_0)} F_\beta < \infty$. Since E is bounded from below on $\mathcal{P}(S_0)$, the latter condition immediately implies

the existence of a measure μ of finite energy and which is absolutely continuous wrt μ_0. Conversely, if such a measure μ exists then writing $\mu = \rho\mu_0$ and setting $\nu := \max\{1, \rho\}\mu_0/C$, where C ensures that $\nu \in \mathcal{P}(K)$ gives $F_\beta(\nu) < \infty$. Indeed, $D_{\mu_0}(\nu) < \infty$ and $E(\nu) < \infty$, using that W is bounded from below on $S_0 \times S_0$.

Next, if $\inf_{\mathcal{P}(S_0)} F_\beta < \infty$ then the LDP for $\beta < \infty$ essentially follows from the results in [29, 7, 32] (the converse is trivial since the rate functional of an LDP is proper lsc). For completeness let us recall the argument given in [7], which builds on the variational approach introduced in [47] (see also [40, 23] for similar results). Fix a continuous functional Φ on $\mathcal{P}(S_0)$ and set $H_\Phi^{(N)} := H^{(N)} + N\delta_N^*\Phi$, $F_{\beta,\Phi} := F_\beta + \Phi$ and

$$\mathcal{F}_{\beta_N}^{(N)}[\Phi] := -\frac{1}{N\beta_N} \log \int e^{-\beta_N H_\Phi^{(N)}} \mu_0^{\otimes N}$$

Using Bryc's criterion for a LDP it is, as explained in [7], enough to prove that

$$\lim_{N\to\infty} \mathcal{F}_{\beta_N}^{(N)}[\Phi] = \inf_{\mathcal{P}(S_0)} F_{\beta,\Phi} \tag{4.3}$$

Note if F_β is proper lsc, then so is $F_{\beta,\Phi}$. The starting point of the proof of the asymptotics (4.3) is *Gibbs variational principle* (which follows from Jensen's inequality):

$$\mathcal{F}_{\beta_N}^{(N)}[\Phi] = N^{-1} \inf_{\mu_N} \left(\int_{X^N} H_\Phi^{(N)} \mu_N + D_{\mu_0^{\otimes N}}(\mu_N) \right) \tag{4.4}$$

It implies, using that W is lsc (to handle the energy term) and the sub-additivity of the entropy (see [7]), the lower bound

$$\inf_{\mathcal{P}(S_0)} F_{\beta,\Phi} \leq \liminf_{N\to\infty} \mathcal{F}_{\beta_N}^{(N)}[\Phi] \tag{4.5}$$

As for the corresponding upper bound

$$\limsup_{N\to\infty} \mathcal{F}_{\beta_N}^{(N)}[\Phi] \leq \inf_{\mathcal{P}(S_0)} F_{\beta,\Phi} \tag{4.6}$$

it is shown by taking $\mu_N = \mu^{\otimes N}$ in the rhs of formula (4.4), where μ realizes the infimum of the proper lsc functional $F_{\beta,\Phi}$ using that

$$N^{-1} \int_{X^N} H^{(N)} \mu^{\otimes N} + \beta^{-1} N^{-1} D_{\mu_0^{\otimes N}}(\mu^{\otimes N}) = E(\mu) + \beta^{-1} D_\mu(\mu),$$

if $E(\mu) < \infty$ (by the Fubini-Tonelli theorem) together with the basic fact $(\delta_N)(\mu^{\otimes N}) \to \delta_\mu$ weakly on $\mathcal{P}(X)$ to handle the term depending on Φ.

Next consider the case when $\beta = \infty$. As pointed out above, in order to establish the LDP in question, it is enough to show that the limit (4.3) also holds for $\beta = \infty$. To this end first observe that the corresponding lower bound is easier since the entropy term can be dropped. Moreover, to prove the corresponding upper bound fix $\beta > 0$ and note that, by Hölder's inequality,

$$\mathcal{F}_{\beta_N}^{(N)}[\Phi] \leq \mathcal{F}_\beta^{(N)}[\Phi]$$

for N sufficiently large. Hence, the upper bound (4.6) for $\beta = \infty$ is obtained by first letting $N \to \infty$, then using the corresponding upper bound for $\beta < \infty$ and finally letting $\beta \to \infty$ and using the assumed Gamma-convergence of F_β towards F_∞. $\square$

Example 4.3. Even if W is assumed bounded, the LDP may hold at $\beta = \infty$ with a rate functional which is different than $E - \inf E$. A simple such example is obtained by taking $X = [0,1]$, $\mu_0 = dx$ and $W(x,y) := V(x) + V(y)$ where V is the proper lsc function

defined by $V(x) = 0$ for $x \neq 0$ and $V(0) = -1$, say. Since $V = 0$ a.e. wrt dx we have that $\mu_\beta^{(N)} = dx^{\otimes N}$. But if the LDP would hold with a rate functional $E - \inf E$, then

$$\lim_{N \to \infty} -\frac{1}{N\beta_N} \log Z_N = \inf_{\mathcal{P}([0,1])} E = \inf_{\mathcal{P}([0,1])} \int_{[0,1]} V\mu = \inf_{[0,1]} V = -1,$$

which contradicts $Z_N = \int_{[0,1]} dx = 1$. This example also illustrates that the expectations of the empirical measure δ_N (which here equals dx) may, in general, not converge to a minimizer of E (which here equals δ_0). Also note that in this example, the measure δ_0 does not have the Energy Approximation property (since $E(\delta_0) = -1$, while $E(\mu) = 0$ if $\mu = \rho dx$). Similarly, F_β Gamma-converges to the constant functional 0. Moreover, in this setting the measure dx is not determining, since setting $\psi := \psi_{\delta_0} - 1$ gives $\psi = -V$ and hence $\psi = 0$ a.e. dx, while $\psi(0) > 0$. Moreover, the Bernstein-Markov inequality fails in this example (with $\phi = 0$), as is seen by taking $\mu = \delta_0$.

Modifying the previous example we also have the following example involving the Coulomb gas in the plane $\mathbb{C}$ subject to an exterior potential ϕ, showing that the corresponding LDP does not always hold at $\beta = \infty$ if ϕ is lsc, but not continuous.

Example 4.4. Set $W(z, w) := -\log |z - w|^2 + \phi(z) + \phi(w)$ for a given function ϕ in $\mathbb{C}$ and denote by $H_\phi^{(N)}$ and E_ϕ the corresponding N-particle Hamiltonian and energy functional, respectively. Consider the corresponding Gibbs measure at $\beta_N = N$, say, with μ_0 given by Lebesgue measure on a given compact subset X in $\mathbb{C}$. Fix a continuous function ϕ_0 on X and $t \geq 0$ and set $\phi := \phi_0(z) - t\chi_{S^1}(z)$ in $\mathbb{C}$, where $\chi = 1$ on the unit-circle S^1 and $\chi = 0$ on the complement of S^1. Assume to get a contradiction that

$$\liminf_{N \to \infty} N^{-2} \log Z_N[\phi] \geq -\inf_{\mathcal{P}(X)} E_\phi, \quad Z_N[\phi] := \frac{1}{Z_{N,\beta_N}} \int e^{-\beta_N H^{(N)}} \mu_0^{\otimes N} \qquad (4.7)$$

Now, since the measure ν defined by the uniform measure on S^1 is a candidate for the inf of E_ϕ, the rhs in formula (4.7) is bounded from below by $-E_0(\nu) - \phi_0(r) + t$ and $E_0(\nu)$ is finite. But $\phi_t = \phi_0$ a.e. wrt μ_0 and hence $Z_N[\phi] = Z_N[\phi_0]$, which implies that the lhs in formula (4.7) is uniformly bounded from above by a finite constant C_0. Taking t sufficiently large thus gives the desired contradiction. Finally, note that the same example applies in the non-compact case where $X = \mathbb{C}$ if $\phi_0(z) \geq (1 + \epsilon) \log(|1 + |z|^2) - C$ for some $\epsilon > 0$ and $C > 0$.

As we will show in Section 4.2 the Gamma-continuity assumption on F_β is not necessary for the existence of a LDP at $\beta = \infty$ with rate functional $F_\beta - \inf F_\beta$. On the other hand, by the previous example it it is not enough to assume that F_∞ is proper lsc. In the case of the 2d Coulomb gas, this will be illustrated using the well-known notion of Bernstein-Markov inequalities. This notion can be extended to a general pair interaction potential W as follows (the case of the Riesz gas was introduced in [18]).

Definition 4.5. Given $\phi \in \mathcal{C}(X)$ we will say that a measure μ_0 satisfies the *weighted Bernstein-Markov inequality (wrt the pair interaction W)* if for any $\epsilon > 0$ there exists a constant $C > 0$ such that for any $p > 0$

$$\sup_X e^{\psi_\mu - \phi} \leq C^{1/p} e^\epsilon \left\| e^{\psi_\mu - \phi} \right\|_{L^p(S_0, \mu_0)} \qquad (4.8)$$

for all discrete measures μ of the form $\mu = N^{-1} \sum_{i=1}^N \delta_{x_i}$, for some $x_i \in X$. We say that μ_0 satisfies the *strong Bernstein-Markov property* if it satisfies the weighted Bernstein-Markov inequality for all $\phi \in \mathcal{C}(X)$.

The following result shows, in particular, that the Bernstein-Markov inequality is a sufficient condition for the LDP to hold at zero-temperature if E is proper lsc and strictly convex (see [5, 16] for the logarithmic case and complex case and [18] for the case of the Riesz gas).

Theorem 4.6. *Assume that E is proper lsc on $\mathcal{P}(X)$.*

- *If μ_0 has support X and satisfies the Bernstein-Markov inequality, then*

$$- \lim_{N \to \infty} \frac{1}{N\beta_N} \log Z_N = \inf_{\mathcal{P}(X)} E \tag{4.9}$$

  and the following concentration property holds: any limit point Γ in $\mathcal{P}(\mathcal{P}(X))$ of the law Γ_N of the empirical measure δ_N is supported in $\arg\inf_{\mathcal{P}(X)} E$. In particular, if E admits a unique minimizer μ, then δ_N converges in law towards μ.

- *If μ_0 has the strong Bernstein-Markov property and E is strictly convex on $\mathcal{P}(X)$, then the LDP holds at a speed $\beta_N N$ and with rate functional $E - \inf_{\mathcal{P}(X)} E$.*

Proof. Set $\Phi(\mu) = \langle \mu, \phi \rangle$ for a given $\phi \in C(X)$ and assume that μ_0 satisfies the weighted Bernstein-Markov-inequality for the weight ϕ. Then,

$$\limsup_{N \to \infty} \mathcal{F}_{\beta_N}^{(N)}[\Phi] \leq \inf_{\mathcal{P}(X)} (E + \Phi) \tag{4.10}$$

To see this, first observe that the function ψ on X obtained by freezing all but one of arguments in $H^{(N)}(x_1, x_2, ..., x_N)$ is of the form ψ_μ for μ a discrete measure of the form appearing in the definition of the Bernstein-Markov-inequality. Hence, using the weighted Bernstein-Markov-inequality N times gives

$$\limsup_{N \to \infty} \mathcal{F}_{\beta_N}^{(N)}[\Phi] \leq \limsup_{N \to \infty} N^{-1} \inf_{X^N} H_\Phi^{(N)}$$

The bound (4.10) now follows from the following fact, which holds for any $\Phi \in C(\mathcal{P}(X))$:

$$\lim_{N \to \infty} N^{-1} \inf_{X^N} H_\Phi^{(N)} = \inf_{\mathcal{P}(X)} (E + \Phi) \tag{4.11}$$

This is essentially well-known and classical (a proof is provided below). Now, combining the upper bound (4.10) with the corresponding lower bound (4.5) (which always holds) gives

$$\lim_{N \to \infty} \mathcal{F}_{\beta_N}^{(N)}[\Phi] = \inf_{\mathcal{P}(X)} (E + \Phi) \tag{4.12}$$

for all linear and continuous Φ. In particular, specializing to $\Phi = 0$ proves (4.9). To prove the concentration property in the first point we note that the lower bound (4.5) can be refined to give

$$\int_{\mathcal{P}(X)} E\Gamma \leq \liminf_{N \to \infty} \mathcal{F}_{\beta_N}^{(N)}[0]$$

Combining this inequality with (4.12) (for $\Phi = 0$) and using that E is lsc gives the concentration property in question.

Finally, if the Bernstein-Markov-property holds for all ϕ, then the asymptotics (4.12) hold for all linear bounded functionals Φ. Hence, if E is strictly convex the LDP in question follows from the Gärtner-Ellis theorem (see [6, Lemma 4.7] for a convenient reformulation of the Gärtner-Ellis theorem).

Proof of the asymptotics (4.11):

We follow the argument in the proof of Theorem 4.2. By (4.5) it is enough to prove the corresponding upper bound. To this end fix $\beta > 0$ and note that, since, $\inf_{X^N} H_\Phi^{(N)}$ is trivially bounded from above by $\mathcal{F}_\beta^{(N)}[\Phi]$ the upper bound (4.6) gives, for a fixed μ_0 on X,

$$\limsup_{N \to \infty} N^{-1} \inf_{X^N} H_\Phi^{(N)} \leq \inf_{\mathcal{P}(X)} \left(E_\Phi + \beta^{-1} D_{\mu_0} \right)$$

Thus, the upper bound in question is obtained by taking μ_0 as the minimizer of E_Φ (using that $D_{\mu_0}(\mu_0) = 0$). We note that this proof of (4.11) is closely related to the proof of the result in [7, Cor 1.6], saying that $N^{-1} H^{(N)}$, identified with a functional on $\mathcal{P}(X)$, Gamma-converges towards E (which implies (4.11) and is, in fact, equivalent to (4.11) for all Φ). □

Remark 4.7. The proof of the first point is similar to the proof of the corresponding result in [41], which is claimed without any assumptions on μ_0 (see the discussion in Section 1.4). The main difference is that the Bernstein-Markov-property of μ_0 is used here to justify the upper bound in [41, Lemma 4], which does not hold for a general μ_0 (by Example 4.3) and which corresponds to (4.10) here. See also [18, Section 4] for another approach based on the Bernstein-Markov-property.

4.1 The case of the Riesz gas

Let us now specialize to the case of the Riesz gas, i.e. the case when the pair interaction potential $W(x, y)$ is taken as the Riesz kernel W_α (section 2). The following result contains, in particular, the LDP for the Coulomb gas ($\alpha = 2$) appearing in Theorem 1.2 in the introduction.

Theorem 4.8. *Assume that* $\alpha \in]0, d[$. *Given a measure* μ_0 *with compact support* S_0, *not charging polar subsets, the following holds for the corresponding Riesz gas:*

- *For any* $\beta \in]0, \infty[$, *the LDP holds with speed* βN *and rate functional* $F_\beta - \inf_{\mathcal{P}(X)} F_\beta$.
- *If* $\alpha \leq 2$, *then the LDP holds for* $\beta \in]0, \infty]$ *at a speed* $\beta_N N$ *with a rate functional which is continuous wrt Gamma-convergence iff* μ_0 *is strongly determining.*

Proof. Combining Theorems 3.11, 4.2 we just have to verify that if μ_0 does not charge polar subsets, then the assumption in the first point of Theorem 4.2 is satisfied. But this follows from Lemma 3.5, by taking $\mu = \mu_\beta$. □

Corollary 4.9. *Assume that* $\alpha \in]0, d[$ *and let* μ_0 *be as in the previous theorem and denote by* $\mu_{N,\beta}$ *the expectation of the empirical measure of the corresponding Riesz gas. In other words,* $\mu_{N,\beta}$ *is the push-forward to* $\mathbb{R}^d$ *of the Gibbs measure defining the Riesz gas. Then* $\mu_{N,\beta}$ *converges in energy towards the minimizer* μ_β *of the free energy functional* F_β.

Proof. The weak convergence of $\mu_{N,\beta}$ towards μ_β follows directly form the LDP in the previous theorem. To prove that $E(\mu_{N,\beta}) \to E(\mu_\beta)$ it is, by basic integration theory, enough to show that there exists a constant C_β such that

$$\mu_{N,\beta} \leq C_\beta \mu_0 \tag{4.13}$$

To prove this inequality first observe that

$$\frac{\mu_{N,\beta}}{\mu_0} \leq \left\{ \sup_{S_0} e^{\beta\psi} : \ \psi \in \mathcal{L}_{S_0}(\mathbb{R}) : \ \int e^{\beta\psi} \mu_0 = 1 \right\}$$

using that $H(x, x_2, \ldots x_N)$ is in $\mathcal{L}_{S_0}(\mathbb{R})$ for any fixed $(x_2, \ldots, x_N)$ and integrating over S_0^{N-1}. By Prop 3.6 thus shows that, for any given $\epsilon > 0$ there exists a constant C_ϵ such that $\mu_{N,\beta} \leq C_\epsilon e^{\epsilon\beta} \mu_0$ which, in particular, implies the inequality (4.13). □

Remark 4.10. It follows from the LDP above, when $\alpha \leq 2$ (and its proof), that the functional $\mathcal{F}_{(S,\phi)}$ defined by formula (2.17) can be expressed in terms of the moment generating function of the corresponding empirical measure:

$$\mathcal{F}_{(S_0,\phi)}(u) = \lim_{N\to\infty} \frac{1}{\beta_N N} \log \mathbb{E}\left(e^{N\beta_N \sum_{i=1}^{N} u(x_i)}\right) \tag{4.14}$$

In the complex-geometric setting in [5] (which covers in particular the case when $d = \alpha = 2$) the proof of the corresponding LDP goes the other way around: first the analog of (4.14) is established and then the LDP is deduced from the Gärtner-Ellis theorem.

According to Theorem 4.6 the Bernstein-Markov-property of a measure μ_0 is a sufficient criterion for the LDP to hold at $\beta = \infty$. However, in general, the corresponding rate functional is not Gamma-continuous up to $\beta = \infty$, even if μ_0 is assumed to be absolutely continuous wrt dx. This will be exemplified in the following section.

4.2 The 2d Coulomb gas and orthogonal polynomials on the real line

Now consider the "logarithmic case" $\alpha = 2 = d$, i.e. the Coulomb gas on a measure μ_0 in $\mathbb{R}^2$ that we shall identify with $\mathbb{C}$. In this section we assume that the support S_0 of μ_0 is contained in $\mathbb{R} \subset \mathbb{C}$.

Lemma 4.11. *Assume that μ_0 has compact support S_0 contained in $\mathbb{R}$. Then it has the Bernstein-Markov inequality iff it satisfies the weighted Bernstein-Markov inequality for all weights ϕ (i.e. iff it has the strong Bernstein-Markov property). Similarly, μ_0 is determining iff it is strongly determining.*

Proof. This is well-known, but for completeness we recall the argument. First assume that the Bernstein-Markov inequality holds in the non-weighted case, $\phi = 0$. Now take a general continuous function ϕ on $\mathbb{R}$. In a neighborhood of S_0 we can express $-\phi$ as the uniform limit of $\log(|q_k|^2)$ for some polynomials q_k on $\mathbb{C}$ of degree k (using the Stone-Weierstrass theorem). The Bernstein-Markov inequality wrt ϕ then follows from the non-weighted one by replacing p_k in formula (1.11) with $p_k q_k$. Similarly, if μ_0 is determining for $(S_0, 0)$, then it is also determining for (S_0, ϕ), as shown by replacing ψ in formula (1.2) with $\psi + k^{-1} \log(|q_k|^2)$. $\qquad\square$

Combining Theorem 4.6 with Proposition 4.13 below now gives the following characterization of measures μ_0 on $\mathbb{R}$ such that the corresponding LDP holds at $T = 0$:

Theorem 4.12. *Let μ_0 be a measure whose support is a compact regular subset S_0 of $\mathbb{R}$ and such that μ_0 does not charge polar subsets. Then the LDP for the corresponding Coulomb gas at $T = 0$ holds with rate functional $E - \inf E$ iff μ_0 satisfies the Bernstein-Markov-inequality. As a consequence, it is not enough to assume that E is proper lsc (i.e. that S_0 is non-polar) for the LDP to hold at $T = 0$. More precisely, there exists a measure μ_0 with support $[-1, 1]$, which is absolutely continuous wrt dx and such that the corresponding expectations $\mathbb{E}(\delta_{N_k})$ do not converge towards the equilibrium measure of $[-1, 1]$ when $\beta_N = N - 1$ and $N \to \infty$.*

Proof. To prove the "only if" direction we set $\beta_N = N - 1$ (and hence $\beta = \infty$) and note that $p_N = 2$ in formula (1.10). This means that the corresponding Coulomb gas in $\mathbb{C}$ defines a determinantal point process with correlation kernel $K_k(z, w)$, where K_k is the integral kernel of the orthogonal projection from $L^2(\mathbb{C}, \mu_0)$ onto the space $\mathcal{P}_k(\mathbb{C})$ of all polynomials $p_k(z)$ on $\mathbb{C}$ of degree at most $k := N - 1$:

$$K_k(z, w) = \sum_{j=0}^{k} p_j(z)\overline{p_j(w)}, \tag{4.15}$$

for an orthonormal base p_j in $\mathcal{P}_k(\mathbb{C})$ (known as the Christoffel-Darboux kernel in the literature on orthogonal polynomials and the Bergman kernel in the complex analysis literature). In fact, this is the case for any measure μ_0 on $\mathbb{C}$ not charging polar subsets (see, for example, [9]). Accordingly, it follows from general properties of determinantal point processes that

$$\mathbb{E}(\delta_{N_k}) = \frac{1}{k+1} K_k(x, x) \mu_0 \tag{4.16}$$

Now, if the LDP holds at $T = 0$ with rate functional E, then it follows, in particular, that $\mathbb{E}(\delta_{N_k})$ converges towards the equilibrium measure μ_{S_0}. But combining formula (4.16) with Prop 4.13 below then implies that μ_0 satisfies the Bernstein-Markov-inequality. For the last statement it is enough to construct a measure μ_0 on $\mathbb{R}$ not charging polar subsets and not satisfying the BM-inequality. The existence of such a measure is without doubt well-known to experts, but for completeness a concrete such measure is constructed in the appendix. □

The following proposition, used in the proof of the previous theorem, is an unpublished result of Totik (thanks to Norman Levenberg for pointing this out).

Proposition 4.13. *Let μ_0 be a measure whose support is a compact regular subset S_0 of $\mathbb{R}$ and such that μ_0 does not charge polar subsets. Denote by K_k the corresponding kernel defined by formula (4.15). If $\frac{1}{k+1} K_k(x, x) \mu_0$ converges weakly towards the equilibrium measure μ_{S_0}, then μ_0 satisfies the Bernstein-Markov inequality.*

Proof. Let us explain how to deduce this from the results in [17] concerning measures μ_0 with compact support $S_0 \subset \mathbb{R}$. We denote by p_k the sequence of orthonormal polynomials in $L^2(\mu_0)$ associated to μ_0 of degree k, by γ_k the positive non-vanishing leading coefficient of p_k, i.e. $p_k = \gamma_k x^k + O(x^{k-1})$ and by ν_k the empirical measure on the zeroes of p_k. The proposition then follows directly from combining the following three results proved in [55, Thm 3.2.3], [53, Thm 13.1] and [55, Cor 2.2.3], respectively:

(1) If S_0 is regular, then μ_0 satisfies the BM-inequality iff μ_0 is regular in the sense of Saff-Totik i.e.

$$\lim_{m \to \infty} m^{-1} \log \gamma_m = \inf_{S_0} E$$

(the lower bound holds for any μ_0)

(2) $\frac{1}{k+1} K_k(x, x) \mu_0$ converges weakly towards $\mu \in \mathcal{P}(S)$ iff ν_k converges weakly towards $\mu \in \mathcal{P}(S)$

(3) If S_0 has non-zero capacity (i.e $\inf_{S_0} E$ is finite) and ν_k converges weakly towards the equilibrium measure μ_{S_0}, then either μ_0 is regular or there exists a polar Borel subset $C \subset S_0$ such that $\mu(C) = \mu_0(S_0)$. □

The proof of the previous proposition relies on special properties of orthonormal polynomials on subsets of real line, not shared by general orthonormal polynomials on subsets of $\mathbb{C}$. Accordingly, the equivalence in Theorem 4.12 is widely open in the general logarithmic setting in $\mathbb{C}$ (as well as in higher dimensions). This being said, Theorem 1.2 can be viewed as a general variant of Theorem 4.12 where the property of being Bernstein-Markov property is replaced by the stronger property of being determining (and then the conclusion is also stronger). By Prop 3.6 this amounts to demanding that the Bernstein-Markov inequality (4.8) holds for *all* potentials ψ_μ.

4.3 Proof of Cor 1.4

By Theorem 1.2 we just have to provide a measure μ_0 with support $K \subset \mathbb{C}$, which is absolutely continuous wrt Lebesgue measure (and, hence does not charge polar subsets) with the BM-property, but which is not determining. When $K = [-1, 1]$ such an example has been constructed by Totik (reported in [17]) and as indicated in [17], the general case is similar (for completeness a proof is provided in the appendix).

5 Relations to the Ehrenfest classification of phase transitions

5.1 The general setting

Let us start by recalling the classical Ehrenfest classification of phase transitions in a general statistical mechanical setting, where the Hamiltonian $H^{(N)}$ in formula (4.1) is replaced by a general measurable (not necessarily symmetric) function on $(X^N, \mu_0^{\otimes N})$. The corresponding *free energy at temperature* T_N is defined by

$$F_{N,T_N} = -\frac{T_N}{N} \log \int_{X_N} e^{-\frac{1}{T_N} H^{(N)}} \mu_0^{\otimes N},$$

assuming that it is finite. By definition, there is a *phase transition of order* m at temperature $T \in]0, \infty]$ if, for any sequence $T_N \to T$ the limit

$$f(T) := \lim_{N \to \infty} F_{N,T_N} \tag{5.1}$$

exists and the derivatives of order $j = 1, ..., m$ exist at T, but not the derivative of order $m + 1$. It should be stressed that the notion of a phase transitions is often used in a broader sense (as discussed in the case of the Coulomb case in [54]), but here we shall be concerned only with the Ehrenfest classification.

We recall that phase transitions have been studied extensively in the setting of spin models, such as the *Ising* and *Potts* models on graphs, where the space X is finite. For example, on the complete graph with N nodes the (ferromagnetic) Potts model is defined by the Hamiltonian $H^{(N)}$ of the form (4.1) with pair-interaction $W(x, y) = -x \cdot y$ and space $X = \{1, 2, ..q\}$ for a given integer $q \geq 2$, endowed with the counting measure μ_0. The case $q = 2$ is the Ising model on the complete graph (known as the *Curie-Weiss model* for magnetism). As is well-known, there is a critical critical temperature $T_c \in]0, \infty[$ such that $f(T)$ is smooth for $T > T_c$ and a phase transition occurs at $T = T_c$, which is of order two when $q = 2$ and order one when $q \geq 3$ [58]. Moreover, according to the "mean-field philosophy" this implies phase transitions for the Ising and Potts model on $\mathbb{Z}^d$, when d is sufficiently large [14].

However, by the following basic lemma, there are no zeroth-order phase transitions when $T > 0$, i.e. no points where f is discontinuous:

Lemma 5.1. *If the limit* (5.1) *exists for any* $T \in]0, \infty[$, *then* f *is concave and increasing on* $]0, \infty[$ *and, in particular, continuous.*

Proof. If the limits exists then we can take $T_N = T$ for all T and observe that $T \mapsto F_{N,T}$ is concave and increasing (as follows, for example, from Gibbs variational principle (4.4)). Since these properties are preserved by point-wise convergence the lemma follows. □

Moreover, the following lemma explains why zeroth-order phase transitions do not appear, even at $T = 0$, in the spin models discussed above.

Lemma 5.2. *For a Hamiltonian of the form* (4.1)*, with lower semi-continuous pair interaction* W*, there is no zeroth-order phase transition under the following condition:*

$$\lim_{T \to 0} \inf_{\mathcal{P}(X)} (E + T D_{\mu_0}) = \inf_{\mathcal{P}(X)} E$$

In particular, this is the case if the pair interaction potential W is continuous.

Proof. By (4.5) (for $\Phi = 0$)

$$\inf_{\mathcal{P}(X)} E \le f(0).$$

Since $f(0) \le f(T)$ letting $T \to 0$ it follows from the assumption that f is continuous at $T = 0$, as desired. To prove the last statement note that, since D is lsc we have, in general, that

$$\lim_{T \to 0} \inf_{\mathcal{P}(X)} (E + T D_{\mu_0}) = \lim_{T \to 0} (E(\mu_T) + T D_{\mu_0}(\mu_T)) \le \liminf_{T \to 0} E(\mu_T)$$

But if E is continuous, then it follows from the compactness of $\mathcal{P}(X)$ that the rhs above is equal to the infimum of E. $\square$

Finally, we make the following observation (which applies in particular to Riesz interactions when μ_0 satisfies a Bernstein-Markov inequality):

Lemma 5.3. *Assume that $e^{-H^{(N)}}$ is continuous on X^N and that there exists a sequence ϵ_N in $\mathbb{R}$, tending to zero, such that*

$$F_{N,T_N} \le \inf_{X^N} \frac{1}{N} H^{(N)} + \epsilon_N$$

Then there is a zeroth-order phase transition at $T = 0$ iff

$$\lim_{T \to 0} \lim_{N \to \infty} F_{N,T} \ne \lim_{N \to \infty} \lim_{T \to 0} F_{N,T} \tag{5.2}$$

Proof. By the continuity assumption $\lim_{T \to 0} F_{N,T} = \inf_{X^N} \frac{1}{N} H^{(N)}$. Indeed, in general, the $L^p(\mu_0)$-norms of a bounded function f on a compact set K converge, as $p \to \infty$, to the essential sup $\|f\|_{L^\infty(K,\mu_0)}$ (compare Step 3 in the proof of Prop 3.6). In the present case $f = e^{-H^{(N)}}$ is continuous and hence the essential sup coincides with the ordinary sup. Moreover, if the inequality in the lemma holds then necessarily $f(0) = \lim_{N \to \infty} \inf_{X^N} \frac{1}{N} H^{(N)}$. Hence, the rhs in (5.2) is equal to $f(0)$, while the lhs is equal to $\lim_{T \to 0} f(T)$. $\square$

While there is an abundance of first and second order phase transitions in the physics and mathematics literature, zeroth-order phase transition appear to be of a rather pathological nature. Still, there has been some speculations on zeroth-order phase transitions in the physics literature in the context of superfluidity (see [45]) and black holes [36]. To the best of the authors knowledge there are, however, no previous examples of zeroth-order phase transitions in the rigorous sense described above.

5.2 Phase transitions for the 2d Coulomb gas

Now consider the setting of the Coulomb gas in $\mathbb{R}^2$ with a given exterior continuous potential ϕ and fix a measure μ_0 on $\mathbb{R}^2$ which has the Bernstein-Markov property. Then the corresponding free energy $f_\phi(T)$ exists for all $T \in [0, \infty[$ (by Theorem 4.6). Any measure μ_0 as in Corollary 1.4 provides an example of such a measure, for which the corresponding Coulomb gas has a zeroth-order phase transition.

We recall that phase transitions are also frequently studied as the strength of ϕ is varied (in the standard case of spin systems $\phi(x) := -x$). This means that ϕ is replaced by

$$\phi_h := \phi_0 + h\phi$$

for a given parameter $h \in \mathbb{R}$ and continuous functions ϕ_0 and ϕ. We then set

$$f(T, h) := f_{\phi_h}(T)$$

for $(T, h) \in [0, \infty[\times \mathbb{R}$. Set $T = 0$ and consider the function $h \mapsto f(T, h)$. By Prop 2.10 there is no zeroth or first order phase transitions. A third order phase transition was discovered by Gross-Witten in the context of lattice gauge theories and unitary random matrices [34] (and used in [39] to study the expected length of the longest increasing subsequence in a random permutation). This phase transition concerns the case when μ_0 is the invariant measure on the unit-circle S^1 in $\mathbb{C}$, $\phi_0 = 0$ and $\phi(z)$ is half the real part of z, i.e. equal to $\cos \theta$ on S^1 (the phase transition appears at $h = 2$). See [44] or a general discussion about third-order phase transitions for 2d Coulomb gases. Here we give simple examples of *second order* phase transitions on the unit-disc.

Proposition 5.4. *Consider the Coulomb gas in $\mathbb{R}^2$ and let μ_0 be normalized Lebesgue measure on the closed unit-disc K and ϕ a non-constant radial subharmonic function ϕ on a neighborhood of K. Set $\phi_h = h\phi$. Then the corresponding function $h \mapsto f(0, h)$ is differentiable at $h = 0$, but not two times differentiable.*

Proof. In order to use standard complex analytic normalizations it will be convenient to use a normalization where $\Delta := \frac{1}{4\pi}(\partial_x^2 + \partial_y^2)$. These normalizations ensure that $\Delta \log |z|^2$ is the uniform probability measure on S^1. First observe that without loss of generality we may, by replacing ϕ by $A\phi + B$ assume that $\int \Delta \phi \leq 1$ on K and $\phi = 0$ on ∂K. By the maximum principle it then follows that $\phi \leq 0$ in K. Set $\psi_h := \log |z|^2$ when $|z| \geq 1$. For $|z| \leq 1$ we set $\psi_h = h\phi$ when $h \geq 0$ and $\psi_h = 0$ if $h < 0$ and make the following

$$\text{Claim: } \psi_h = P_K(h\phi).$$

First observe that $\psi_h(z)$ is subharmonic. Indeed, writing $\psi_h(z) = \Phi(x)$ for $x := \log |z|^2$ the subharmonicity of ψ_h is equivalent to the convexity of Φ, which in turn follows from noting that $\Phi(x) = x$ when $x \geq 0$ and when $x \leq 0$ we have $\partial_x^2 \Phi \geq 0$ and

$$\partial_x \Phi(0) = \int_{-\infty}^0 \partial_x^2 \Phi = \int_K \Delta \phi \leq 1$$

Moreover, this implies that, when $h \geq 0$,

$$\Delta \psi_h = (1 - ch)\delta_{\partial K} + h\Delta\phi, \quad c = \int_K \Delta\phi (= 1 - \partial_x \Phi(0))$$

where $\delta_{\partial K}$ denotes the uniform measure on the unit-circle ∂K. Moreover, when $h < 0$,

$$\Delta \psi_h = \delta_{\partial K}$$

Hence, $\psi_h \leq h\phi$ on K and $\psi_h = h\phi$ almost everywhere with respect to $\Delta\psi_h$. The claim above thus follows from the domination principle. Now, by Prop 2.10 we have that

$$\frac{df(0, h)}{dh} = \int_K \phi \Delta \psi_h = \int_{|z|<1} \phi \Delta \psi_h,$$

using that $\phi = 0$ on the boundary of the unit-disc K. By the previous discussion this means that $\frac{df(0,h)}{dh}$ vanishes identically when $h < 0$ and is equal to h times the non-zero number $\int_{|z|<1} \phi \Delta \phi$ when $h \geq 0$. Hence, $\frac{df(0,h)}{dh}$ is not differentiable at $h = 0$, as desired. $\square$

Remark 5.5. The second order phase transition above can be contributed to the fact that the support of the weighted equilibrium measure μ_h changes drastically at $h = 0$: for $h > 0$ it contains a disc inside K, which disappears when $h \leq 0$.

In particular, if $\phi = |z|^2$, say, and if μ_0 is taken as the measure whose support is the unit-disc, provided by Corollary 1.4, then the corresponding Coulomb gas exhibits a rather peculiar phase diagram in the (T, h)-plane. Indeed, for any fixed h there is a zeroth-order phase transition as $T \to 0^+$ and moreover, when $T = 0$ there is a second order phase transition as $h \to 0$. Let us also remark that, comparing with standard physics terminology, the measure

$$\mu_{h,T} := \lim_{N \to \infty} \mathbb{E}(\delta_N)$$

(which minimizes the corresponding free energy functional) plays the role of an *order parameter* for the phase transitions (which usually appears as a physical observable). By Theorem 3.10, a zeroth-order phase transition at $T = 0$ is equivalent to the discontinuity of $T \mapsto \mu_{T,0}$, viewed as a curve in $\mathcal{P}(S_0)$. Equivalently, this means that there exists some (smooth) exterior potential ϕ such that the corresponding free energy $f(T, h)$ satisfies

$$\lim_{T \to 0^+} \frac{\partial f(T,0)}{\partial h} \neq \frac{\partial f(0,0)}{\partial h}$$

6 Appendix

6.1 Capacities and determining measures

We start by recalling the notion of (non-weighted) capacity, mainly following [42]. Given a parameter $\alpha \in]0, d[$ the corresponding *capacity* of a compact set $K \subset \mathbb{R}^d$ is defined by

$$\mathcal{C}_\alpha(K) := 1/\inf_{\mu \in \mathcal{P}(K)} E(\mu),$$

where E is the energy of μ. The *inner capacity* of a general bounded set $S \subset \mathbb{R}^d$ is defined by

$$\mathcal{C}_\alpha(S)_* = \sup_{K \subset S} \mathcal{C}_\alpha(K)$$

where the sup ranges over all compact subsets K of S. Similarly, the *outer capacity* is defined by

$$\mathcal{C}_\alpha(S)^* := \inf_{S \subset U} \mathcal{C}_\alpha(U),$$

where the sup ranges over all bounded open sets U containing S. A bounded subset S is said to be *polar* if $\mathcal{C}_\alpha(S)^* = 0$. This equivalently means that there exists a potential ψ such that $S \Subset \{\psi = -\infty\}$. A subset S is said to be *capacitable* if $\mathcal{C}_\alpha(S)_* = \mathcal{C}_\alpha(S)^*$. Any bounded Borel set S is capacitable. The set functional $\mathcal{C}_\alpha$ is *invariant under translations* and satisfies Choquet's capacity axioms on Borel sets:

- (monotonicity) If $E \subset F$ then $\mathcal{C}_\alpha(E) \leq \mathcal{C}_\alpha(F)$.
- (inner continuity) If S_i is a sequence of sets increasing to S and $S = \bigcup_i S_i$, then $\mathcal{C}_\alpha(S_i) \to \mathcal{C}_\alpha(S)$
- (outer continuity). If K_i is a sequence of compact sets decreasing to the compact set K, then $\mathcal{C}_\alpha(K_i) \to \mathcal{C}_\alpha(K)$

Moreover, $\mathcal{C}_\alpha$ is *sub-additive:* given a sequence of compact subset K_j

$$\mathcal{C}_\alpha(\bigcup_j K_j) \leq \sum_j \mathcal{C}_\alpha(K_j), \tag{6.1}$$

assuming in the case $d = \alpha = 2$ that the diameter of K is at most one [49, Thm 5.1.4 a] (then $\mathcal{C}_\alpha$ is usually called the Wiener capacity).

Example 6.1. The capacity of a ball B_r of radius r centered at $x \in \mathbb{R}^d$ is given by $\mathcal{C}_\alpha(B_r(x)) = A(d, \alpha) r^{d-\alpha}$ for an explicit constant $A(d, \alpha)$ [42], unless $\alpha = d = 2$, in which case $\mathcal{C}_\alpha(B_r(x)) = -1/\log r$. In particular, $\mathcal{C}_\alpha(B_r(x))$ decreases to $0 (= \mathcal{C}_\alpha(\{x\}))$ as $r \to 0$ (which is consistent, as it must with outer continuity).

Similarly, we define the *weighted capacity* $\mathcal{C}_\alpha(K, \phi)$ of a compact weighted set (K, ϕ) by replacing the energy $E(\mu)$ with its weighted analog $E_\phi(\mu)$. Inner and outer weighted capacities are then defined just as before. It follows from the previous case $\phi = 0$ that $S \mapsto \mathcal{C}_\alpha(S, \phi)$ satisfies Choquet's axioms on bounded Borel sets, for any given continuous function ϕ on $\mathbb{R}^d$.

Next recall that if μ is supported on a compact set K, then a Borel subset C of K is said to be a *μ-carrier* if $\mu(C) = \mu(K)$.

Proposition 6.2. *Assume that $\alpha \leq 2$. Then μ is determining for a regular compact weighted set (K, ϕ) iff for any μ-carrier C of K which is the union of increasing compact subsets of K,*

$$\mathcal{C}_\alpha(C, \phi) = \mathcal{C}_\alpha(K, \phi) \tag{6.2}$$

Proof. Assume first that μ is determining and let C be a μ-carrier C, which is the union of increasing compact subsets of K_i of K. Since μ is assumed determining we have $P_C \phi = P_K \phi$ and by the outer continuity of $\mathcal{C}$ we have $\mathcal{C}_\alpha(K_j, \phi) \to \mathcal{C}_\alpha(C, \phi)$. But, by Prop 2.6 and its proof) $P_{K_i} \phi$ decreases to $P_C \phi$. As a consequence, $\mathcal{C}_\alpha(K_j, \phi)^{-1} := E(\Delta(P_{K_i}\phi)) \to E(\Delta(P_C\phi)) = E(\Delta(P_K\phi)) =: \mathcal{C}_\alpha(K, \phi)^{-1}$. Hence, (6.2) holds. Conversely, assume that (6.2) holds for any carrier C as above. By Lemma 3.8 (and its proof) there exists such a carrier C with the property that $P_C \phi = P_\mu \phi$. Hence, by the previous argument $E(\Delta(P_\mu\phi))$ is equal to the infimum of E_ϕ on $\mathcal{P}(K)$. This means, by uniqueness of minimizers, that $\Delta(P_\mu\phi) = \Delta(P_K\phi)$ and hence there exists a constant c such that $P_C\phi + c = P_K\phi$. But, then it follows from Lemma 3.8 that $c = \sup_\mu(P_K\phi - \phi)$ and since $P_K\phi$ is continuous (by Lemma 2.13) this means that $c = \sup_K(P_K\phi - \phi) = 0$. Hence, $P_\mu\phi = P_K\phi$, which implies that μ is determining for (K, ϕ) (just as in the proof of Theorem 3.10). $\qquad\square$

Remark 6.3. The capacity criterion above goes back to Ullman in the case when $d = \alpha = 2$ and $\phi = 0$ (see [56, Thm 2]) and is usually called *Ullman's criterion* in the theory of orthogonal polynomials on the real line [55].

6.2 Explicit construction of the measures in Corollary 1.4 and Theorem 4.12

Following Totik's example for $K = [-1, 1]$ (reported in [17]) and the general discussion in [17], the idea of the construction is to start with a sufficiently dense set of points on K and then replacing them by balls of sufficiently small radius, ensuring that the corresponding measure μ_0 is carried by a measure which has small capacity. Since it requires no more effort we will consider the general setting in $\mathbb{R}^d$ and the Riesz gas with $\alpha \leq 2$, using the general notion of Bernstein-Markov inequalities (definition 4.5).

Lemma 6.4. *Let K be a compact domain in $\mathbb{R}^d$ and fix $\alpha \leq 2$. For any $\phi \in \mathcal{C}(K)$, there exists a measure μ_0 with support K such that μ_0 is absolutely continuous wrt dx and satisfies the strong Bernstein-Markov property, but μ_0 is not determining for (K, ϕ).*

Proof. We will use the following sufficient criterion for a measure μ whose support K is assumed locally regular to have the strong Bernstein-Markov-property: there exists

$r_0, a, C > 0$ such that for any $z \in K$ and $r \in [0, r_0]$

$$\mu(B_R(z)) \geq Cr^a \tag{6.3}$$

(see [17] for the case $d = \alpha = 2$ and [18] for the case of a general α). This will be contrasted with the capacity criterion in Prop 6.2. Fix a positive integer k and consider the "grid" $K \cap (\mathbb{Z}k^{-1})^d$. We let Λ_k be the finite set contained in the interior of K obtained by removing from $K \cap (\mathbb{Z}k^{-1})^d$ all points with distance less than k^{-1} to ∂K and denote by ν_k the empirical measure on Λ_k. Next take a sequence λ_k with polynomial decay such that $\sum_{k=1}^{\infty} \lambda_k < \infty$, say $\lambda_k = k^{-2}$, and set

$$\nu = \sum_{k=1}^{\infty} \lambda_k \nu_k \tag{6.4}$$

Then, for k sufficiently large the mass criterion (6.3) is satisfied. Indeed, if $k^{-1} \leq 10r$ say, then (6.3) holds for μ_k with with $a = d$ and a constant C independent of k. Hence,

$$\nu(B_R(z)) \geq \sum_{k^{-1} \leq 10r} \lambda_k \nu_k \geq Cr^d \sum_{k^{-1} \leq 10r} \lambda_k \geq C'r^{d+1}, \tag{6.5}$$

showing that ν satisfies the mass criterion (6.3). Next, we will modify the construction to get a measure μ not charging polar subsets. To this end fix a sequence ϵ_k of positive numbers such that $\epsilon_k < k^{-1}$ and define μ_k as the measure obtained by replacing each Dirac mass at a point x in the definition of ν_k by the normalized Lebesgue measure on a ball or radius ϵ_k, centered at x. Equivalently, this means that

$$\mu_k := \int_{|s| \leq \epsilon_k} (T_s)_* \nu_k \, ds,$$

where T_s denote the translation map $x \mapsto x + s$ for a given $s \in \mathbb{R}^d$. We then define μ as in the decomposition in (6.4). By translation invariance the same estimate (6.5) holds for μ and hence μ satisfies the strong Bernstein-Markov-inequality, according to the mass criterion (6.3). Moreover, since μ_k is absolutely continuous wrt Lebesgue measure so is μ. Hence, μ does not charge polar subsets and clearly has support K. Finally, we note that if ϵ_k is sufficiently small, then the capacity criterion (6.2) is not satisfied and hence μ is not determining for (K, ϕ). Indeed, by construction, the set

$$C := \bigcup_k B_{\epsilon_k}(\Lambda_k)$$

is a carrier for μ, where $B_{\epsilon_k}(\Lambda_k)$ denotes an ϵ_k-neighborhood of Λ_k, i.e a disjoint union of M_k balls in K of radius ϵ_k, where $M_k \sim k^d$. After a harmless scaling we may as well assume that the diameter of K is equal to one. Then, as recalled in the previous section, $\mathcal{C}_\alpha$ is sub-additive and invariant under translations. Hence,

$$\mathcal{C}_\alpha(C) \leq \sum_k M_k \mathcal{C}_\alpha(B_{\epsilon_k}) \leq \sum_k Ck^d \mathcal{C}_\alpha(B_{\epsilon_k})$$

where B_{ϵ_k} the closed ball of radius ϵ_k centered at 0. Since $\epsilon \mapsto \mathcal{C}_\alpha(B_\epsilon)$ strictly decreases to 0 as $\epsilon \to 0$ (see Example 6.1) this means that, given a continuous function ϕ on $\mathbb{R}^d$ and a positive number $\delta > 0$ we can take ϵ_k sufficiently small to ensure that $\mathcal{C}_\alpha(C, \phi) < \delta$. In particular, taking $\delta = \mathcal{C}_\alpha(K, \phi)$ the capacity criterion (6.2) is violated and hence ν is not determining for (K, ϕ). □

A similar construction yields the following

Lemma 6.5. *There exists a measure μ_0 with support $[0,1]$ such that μ_0 is absolutely continuous wrt dx, but which does not satisfy the Bernstein-Markov-inequality.*

Proof. We recall the following necessary condition for a measure μ_0 on $[0,1]$ to satisfy the Bernstein-Markov-inequality [55, Thm 4.2.8]: for any $\eta > 0$

$$\lim_{r \to \infty} \mathcal{C}_\alpha \left(\left\{ x \in [0,1] : \ \mu(B_r(x)) \ge e^{-\eta r^{-1}} \right\} \right) = \mathcal{C}_\alpha([0,1]) \tag{6.6}$$

Denote by k an integer of the form $k = 2^m$ for some positive integer m. We will show that the necessary condition above is not satisfied for a measure of the form

$$\mu = \sum_k \lambda_k \mu_k$$

as defined in the previous construction, if ϵ_k and λ_k are both taken sufficiently small. More precisely, we will show that this happens if $\lambda_k = e^{-\epsilon_k^{-2}}$ and ϵ_k satisfies

$$k\mathcal{C}_\alpha(B_{2\epsilon_k}) < \mathcal{C}_\alpha([0,1])/2,$$

say. To see this fix $r > 0$ and first note that for any $x \in [0,1]$

$$\left(\sum_{k: \ \epsilon_k \le r} \lambda_k \mu_k \right)(B_r(x)) \le Ce^{-r^{-2}/2} \tag{6.7}$$

Next, consider the set A_k defined as an ϵ_k-neighborhood of the support $B_{\epsilon_k}(\Lambda_k)$ of μ_k (which contains the support of μ_j for $j \le k$), i.e. $A_k = B_{2\epsilon_k}(\Lambda_k)$. The definition is made so that, if $r < \epsilon_k$ then the r-neighborhood of $([0,1] - A_k)$ does not intersect $B_{\epsilon_k}(\Lambda_k)$. Hence,

$$x \in ([0,1] - A_k) \implies \left(\sum_{k: \ \epsilon_k > r} \lambda_k \mu_k \right)(B_r(x)) = 0,$$

which, combined with (6.7), means that the inequality in condition (6.6) fails when $x \in [0,1] - A_k$. But if ϵ_k is sufficiently small, then we get, by the sub-additive of the capacity (just as in the previous construction) that

$$\mathcal{C}_\alpha(A_k) < \mathcal{C}_\alpha([0,1])/2$$

say, for all k. Hence the capacity condition (6.2) is violated, showing that μ does not satisfy the Bernstein-Markov-inequality. $\square$

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