Thesis for the degree of Doctor of Philosophy

Infinite-dimensional Lie bialgebras and Manin pairs

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Abstract

This PhD thesis is devoted to the theory of infinite-dimensional Lie bialgebra structures as well as their close relatives such as r-matrices and Manin pairs. The thesis is based on three papers.

Paper I. The standard structure on an affine Kac-Moody algebra induces a Lie bialgebra structure on the underlying loop algebra and its parabolic subalgebras. We obtain a full classification of the induced twisted Lie bialgebra structures in terms of Belavin-Drinfeld quadruples.

First, we prove that the induced structures are pseudo quasi-triangular. Then, using the algebro-geometric theory of the classical Yang-Baxter equation (CYBE), we reduce the problem of classification to the well-known Belavin-Drinfeld list of trigonometric solutions.

Paper II. We classify topological Lie bialgebra structures on the Lie algebra of Taylor series $\mathfrak{g}[\![x]\!]$, where $\mathfrak{g}$ is a simple Lie algebra over an algebraically closed field F of characteristic 0.

We formalize the notion of a topological Lie bialgebra and introduce topological analogues of Manin pairs, Manin triples, Drinfeld doubles and twists.

By relating topological Manin pairs with trace extension of $F[\![x]\!]$ we obtain their complete classification. The classification of topological doubles, which was known before, becomes a special case of the classification of Manin pairs.

The classification of doubles tells us that there are only three non-trivial doubles over $\mathfrak{g}[\![x]\!]$, namely $\mathfrak{g}(\!(x)\!) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]$, $n \in \{0, 1, 2\}$. We prove that topological Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$ are in one-to-one correspondence with Lagrangian Lie subalgebras of these doubles complementary to the diagonal embedding Δ of $\mathfrak{g}[\![x]\!]$. The classification of topological Lie bialgebra structures is then obtained by associating the corresponding Lagrangian subalgebras with algebro-geometric datum. When the underlying field F is the field of complex numbers, the classification becomes explicit.

Paper III. In this paper we associate arbitrary subspaces of $\mathfrak{g}(\!(x)\!) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]$ complementary to Δ with so-called series of type (n, s).

We prove that skew-symmetric (n, s)-type series are in bijection with Lagrangian subspaces and topological quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$. We classify all quasi-Lie bialgebra structures using the classification of Manin pairs from Paper II.

We show that series of type (n, s), solving the generalized CYBE, correspond to Lie subalgebras.

Keywords: Lie bialgebra, loop algebra, classical twist, Yang-Baxter equation, Manin triple, Manin pair, r-matrix.

List of publications

Paper I. R. Abedin and S. Maximov. "Classification of classical twists of the standard Lie bialgebra structure on a loop algebra". In: *Journal of Geometry and Physics* 164 (2021), p. 104149

Paper II. R. Abedin, S. Maximov, A. Stolin, and E. Zelmanov. *Topological Lie bialgebra structures and their classification over* $\mathfrak{g}[\![x]\!]$. preprint. 2022. arXiv: 2203.01105

Paper III. R. Abedin, S. Maximov, and A. Stolin. *Topological quasi-Lie bialgebras and* (n, s)*-type series*. preprint. 2022. arXiv: 2211.08807

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Contents

1 Introduction

Lie bialgebras originate together with Poisson-Lie groups in mathematical physics as a part of a vast research program launched by L.Fadeev in 1970's. More precisely, Poisson-Lie groups were introduced by V. Drinfeld in [10]. The motivation to introduce and study such objects was two-fold: 1) They represent good candidates of Poisson manifolds, that can be quantized and 2) Groups of dressing transformations of some integrable systems possess Poisson-Lie structure and hence their study lead to a better understanding of Hamiltonian systems. A Lie bialgebra is the structure we get if we consider the Lie algebra of a Poisson-Lie group. Lie bialgebras were introduced in the same paper [10]. The study of Lie bialgebras, as well as Lie algebras, is interesting on its own. Moreover, we have an extension of Lie's third theorem: Given a finite-dimensional Lie bialgebra we can view it as a tangent Lie bialgebra of some Poisson-Lie group.

Lie bialgebra structures on finite-dimensional Lie algebras, as well as structures associated with them, like classical r-matrices, have been thoroughly studied and there exist well-known classification results. The same cannot be said about infinite-dimensional Lie bialgebra structures. There is an abundance of examples of such structures, for example, the standard Lie bialgebra structure on a symmetrizable Kac-Moody algebra introduced in [11]. There are also well-known classifications of classical r-matrices with spectral parameters [7, 18, 19]. However, the first systematic classification of infinite-dimensional Lie bialgebras appeared only in 2010 in paper [16] by F.Montaner, A. Stolin and E. Zelmanov, where they described all Lie bialgebra structures on the Lie algebra $\mathfrak{g}[x]$ of polynomials with coefficients in a simple $\mathbb{C}$-Lie algebra $\mathfrak{g}$.

The thesis is based on three papers, referred to as Paper I, II and III, that can be seen as an attempt to close the gap between finite and infinite-dimensional pictures.

This introduction is devoted to a description of the classical finite-dimensional theory of Lie bialgebras in such a way that the infinite-dimensional results of Papers I, II and III become a natural extension of the analogous finite-dimensional results.

1.1 Classical Lie bialgebras

Let F be a field of characteristic 0 and $\mathfrak{g}$ be a Lie algebra over F. We impose no restrictions on the dimension of $\mathfrak{g}$. The linear map $\delta\colon \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is called a *(classical) Lie bialgebra structure on* $\mathfrak{g}$ if the following conditions are satisfied:

1. the dual map $\delta^\vee\colon (\mathfrak{g} \otimes \mathfrak{g})^\vee \to \mathfrak{g}^\vee$, restricted to $\mathfrak{g}^\vee \otimes \mathfrak{g}^\vee$, is a Lie bracket on the algebraic dual $\mathfrak{g}^\vee$;

2. for all $x, y \in \mathfrak{g}$ we have

$$\delta([x,y]) = [x \otimes 1 + 1 \otimes x, \delta(y)] - [y \otimes 1 + 1 \otimes y, \delta(x)], \tag{1.1}$$

where $[x \otimes 1 + 1 \otimes x, a \otimes b] := x \cdot (a \otimes b) := [x,a] \otimes b + a \otimes [x,b]$. In other words, δ is a 1-cocycle with values in $\mathfrak{g} \otimes \mathfrak{g}$.

If δ satisfies all the properties above, we call the pair $(\mathfrak{g}, \delta)$ a *Lie bialgebra.* Since the category of finite-dimensional Lie algebras is anti-equivalent to the category of finite-dimensional Lie coalgebras we could equivalently say that Lie bialgebra is both a Lie algebra and a Lie coalgebra such that the compatibility condition Eq. (1.1) holds.

Equation (1.1) can be written more symmetrically using the adjoint and coadjoint representations for $\mathfrak{g}$ and $\mathfrak{g}^\vee$, namely

$$\langle \mathrm{ad}_f g, \mathrm{ad}_x y\rangle + \langle \mathrm{ad}^*_x f, \mathrm{ad}^*_g y\rangle - \langle \mathrm{ad}^*_x g, \mathrm{ad}^*_f y\rangle + \langle \mathrm{ad}^*_y g, \mathrm{ad}^*_f x\rangle - \langle \mathrm{ad}^*_y f, \mathrm{ad}^*_g x\rangle = 0. \tag{1.2}$$

Such a representation of the compatibility condition reveals the symmetry between the commutator map on $\mathfrak{g}$ and the cocommutator δ. It follows that a pair $(\mathfrak{g}, \delta)$ is a Lie bialgebra structure if and only if $(\mathfrak{g}^\vee, [\cdot\,,\cdot]^\vee)$ is.

Example 1.1.1. The easiest possible Lie bialgebra structure on $\mathfrak{g}$ is the trivial structure $\delta = 0$. ◊

Example 1.1.2. Consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ with the basis

$$e := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad h := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad f := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then for any choice of constants $c_1, c_2, c_3 \in \mathbb{C}$ we have the following Lie bialgebra structure on $\mathfrak{g}$:

$$\begin{aligned} \delta(e) &:= c_1 e \wedge f + c_2 e \wedge h, \\ \delta(h) &:= c_3 e \wedge h - c_1 f \wedge h, \\ \delta(f) &:= c_3 e \wedge f + c_2 f \wedge h. \end{aligned}$$

If we denote the dual basis for $\mathfrak{g}^\vee$ by $\{e^*, h^*, f^*\}$, then the bracket defined by $\delta^\vee$ is described explicitly by

$$\begin{aligned} [e^*, f^*] &= c_1 e^* + c_3 f^*, \\ [e^*, h^*] &= c_2 e^* + c_3 h^*, \\ [f^*, h^*] &= c_2 f^* - c_1 h^*. \end{aligned}$$

The Lie bialgebra structure corresponding to the choice $c_1 = c_3 = 0$ and $c_2 = \frac{1}{2}$ is usually called *the standard structure.* ◊

A *morphism* between two Lie bialgebras $(\mathfrak{g}_1, \delta_1)$ and $(\mathfrak{g}_2, \delta_2)$ is a Lie algebra homomorphism $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ which satisfies the relation

$$\delta_2 \varphi = (\varphi \otimes \varphi)\delta_1. \tag{1.3}$$

In other words, it is a homomorphism of both Lie algebra and Lie coalgebra structures.

Observe that if $(\mathfrak{g}, \delta)$ is a Lie bialgebra then so is the pair $(\mathfrak{g}, \xi\delta)$ for any $\xi \in F$. It is natural to identify Lie bialgebra structures if they differ only by a non-zero scalar multiple. We say that two Lie bialgebras $(\mathfrak{g}_1, \delta_1)$ and $(\mathfrak{g}_2, \delta_2)$ are *equivalent* if there is a constant $\xi \in F^\times$ such that $(\mathfrak{g}_1, \delta_1)$ is isomorphic to $(\mathfrak{g}_2, \xi\delta_2)$.

Example 1.1.3. It was shown in [20] that, up to equivalence, there are three Lie bialgebra structures on $\mathfrak{sl}(2,\mathbb{C})$:

1. $\delta = 0$;
2. $\delta(e) = \frac{1}{2} e \wedge h,\ \delta(h) = 0,\ \delta(f) = \frac{1}{2} f \wedge h$;
3. $\delta(e) = 0,\ \delta(h) = e \wedge h,\ \delta(f) = e \wedge f$.

Later we describe the methods of [20] in more detail because they can be generalized to the case of infinite-dimensional Lie bialgebras. ◊

1.1.1 Coboundary Lie bialgebras and constant r-matrices

The easiest way to find a 1-cocycle $\delta\colon \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is to take a 1-coboundary dr for some $r \in \mathfrak{g} \otimes \mathfrak{g}$. Lie bialgebra structures of $\mathfrak{g}$ of the form $\delta = dr$ are said to be *coboundary*. The requirement on $dr^\vee$ to define a Lie algebra structure on $\mathfrak{g}^\vee$ means that

1. for any $x \in \mathfrak{g}$ the tensor $dr(x) \in \mathfrak{g} \otimes \mathfrak{g}$ is skew-symmetric and
2. $dr^\vee$ restricted to $\mathfrak{g}^\vee \otimes \mathfrak{g}^\vee$ satisfies the Jacobi identity.

Note that any element $r = \sum_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$ defines a linear map $\underline{r}\colon \mathfrak{g}^\vee \to \mathfrak{g}$ by $f \mapsto \sum_i f(a_i) b_i$. The following lemma gives equivalent formulations of the first requirement above.

Lemma 1.1.4. *Write $r = a + s$, where a is the alternating part of r and s is its symmetric part. The following statements are equivalent:*

1. *for all $x \in \mathfrak{g}$ the element $dr(x)$ is skew-symmetric;*
2. *$ds = 0$, i.e. s is* ad*-invariant;*
3. *for all $x \in \mathfrak{g}$ we have $\underline{s} \circ \mathrm{ad}_x^* = \mathrm{ad}_x \circ \underline{s}$.*

Now, following [14], we represent the Jacobi identity for $dr^\vee$ in terms of r itself. We start with the skew-symmetric case $r = a$. Using the Einstein notation we write $a = x_i \otimes y^i$. Consider *the algebraic Schouten bracket* of a, i.e. the element

$$[\![a, a]\!] := -2([y^i, y^j] \otimes x_i \otimes x_j + x_j \otimes [y^i, y^j] \otimes x_i + x_i \otimes x_j \otimes [y^i, y^j]) \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}. \tag{1.4}$$

One can check that it is the unique element satisfying the equation

$$\begin{aligned} \langle f \otimes g \otimes h, [\![a, a]\!] \rangle &= -2\left(\langle f, [\underline{a}(g), \underline{a}(h)] \rangle + \langle g, [\underline{a}(h), \underline{a}(f)] \rangle + \langle h, [\underline{a}(f), \underline{a}(g)] \rangle \right) \\ &= -2 \circlearrowleft \langle f, [\underline{a}(g), \underline{a}(h)] \rangle \end{aligned} \tag{1.5}$$

for all $f, g, h \in \mathfrak{g}^\vee$. We use the symbol $\circlearrowleft$ to denote the summation over circular permutations of symbols f, g and h. A direct computation yields the relation

$$\langle f \otimes g \otimes h, d([\![a, a]\!])(x) \rangle = 2 \circlearrowleft \langle da^\vee(da^\vee(f \otimes g) \otimes h), x \rangle, \quad \forall f, g, h \in \mathfrak{g}^\vee. \tag{1.6}$$

Therefore, we have the following statement.

Proposition 1.1.5. *For any skew-symmetric tensor $a \in \mathfrak{g} \otimes \mathfrak{g}$, the restriction of the linear map $da^\vee$ to $\mathfrak{g}^\vee \otimes \mathfrak{g}^\vee$ satisfies the Jacobi identity if and only if $[\![a,a]\!]$ is* ad*-invariant.*

Combining Proposition 1.1.5 and Lemma 1.1.4 with the F-linearity of the differential map we can state the following.

Corollary 1.1.6. *Let $r = a + s \in \mathfrak{g} \otimes \mathfrak{g}$, where a is the alternating part of r and s is its symmetric part. The linear map dr is a Lie bialgebra structure on $\mathfrak{g}$ if and only if both s and $[\![a,a]\!]$ are* ad*-invariant.*

The invariance of $[\![a,a]\!]$ can be analyzed even further. Any element $r \in \mathfrak{g} \otimes \mathfrak{g}$ produces the skew-symmetric bilinear map $\underline{\langle r,r\rangle}\colon \mathfrak{g}^\vee \otimes \mathfrak{g}^\vee \to \mathfrak{g}$, given by

$$\underline{\langle r,r\rangle}(f \otimes g) = [\underline{r}(f), \underline{r}(g)] - (\underline{r} \circ dr^\vee)(f \otimes g). \tag{1.7}$$

Define $\langle r,r\rangle$ as the unique element of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ satisfying

$$\langle f \otimes g \otimes h, \langle r,r\rangle\rangle = \langle h, \underline{\langle r,r\rangle}(f \otimes g)\rangle, \quad \forall f,g,h \in \mathfrak{g}^\vee. \tag{1.8}$$

Explicitly, if $r = x_i \otimes y^i$ then

$$\langle r,r\rangle = [x_i, x_j] \otimes y^i \otimes y^j + x_i \otimes [y^i, x_j] \otimes y^j + x_i \otimes x_j \otimes [y^i, y^j]. \tag{1.9}$$

Observe that for skew-symmetric r we have

$$\langle r,r\rangle = -\frac{1}{2}[\![r,r]\!]. \tag{1.10}$$

Theorem 1.1.7.

1. *If $s \in \mathfrak{g}\otimes\mathfrak{g}$ is symmetric and* ad*-invariant, then $\langle s,s\rangle$ is skew-symmetric and ad-invariant;*
2. *Let $r = a + s \in \mathfrak{g} \otimes \mathfrak{g}$, where a is skew-symmetric and s is symmetric and* ad*-invariant, then $\langle r,r\rangle$ is skew-symmetric and $\langle r,r\rangle = \langle a,a\rangle + \langle s,s\rangle$;*
3. *For r as in 2. the equality $\langle r,r\rangle = 0$ is a sufficient condition for $[\![a,a]\!]$ to be* ad*-invariant.*

The equality $\langle r,r\rangle = 0$ is called *the (constant) Classical Yang-Baxter equation* (CYBE). Usually, it is written in another form. More precisely, define the following embeddings of $\mathfrak{g} \otimes \mathfrak{g}$ into $U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes U(\mathfrak{g})$:

$$\begin{aligned} (\,\cdot\,)^{12}&\colon x \otimes y \mapsto x \otimes y \otimes 1, \\ (\,\cdot\,)^{13}&\colon x \otimes y \mapsto x \otimes 1 \otimes y, \\ (\,\cdot\,)^{23}&\colon x \otimes y \mapsto 1 \otimes x \otimes y, \end{aligned} \tag{1.11}$$

where $U(\mathfrak{g})$ stands for the universal enveloping algebra of $\mathfrak{g}$. Let $\mathrm{CYB}\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ be the function given by

$$\mathrm{CYB}\colon r \mapsto [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}], \tag{1.12}$$

where e.g. $[x \otimes y \otimes 1, 1 \otimes a \otimes b] = x \otimes [y,a] \otimes b$. Then the CYBE is exactly the equation $\mathrm{CYB}(r) = 0$. Elements $r \in \mathfrak{g} \otimes \mathfrak{g}$ solving the CYBE are called *quasi-triangular r-matrices*. Skew-symmetric quasi-triangular r-matrices are called *triangular r-matrices* or simply *r-matrices*. Therefore, according to the theorem above any (constant) r-matrix r gives rise to the Lie bialgebra structure dr on $\mathfrak{g}$.

Remark 1.1.8. If $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a triangular r-matrix, then dr defines a Lie bialgebra structure on any Lie algebra $\mathfrak{e}$ containing $\mathfrak{g}$ as a subalgebra. The same is not true for quasi-triangular r-matrices, because in general the invariance of the symmetric part of r with respect to the action of $\mathfrak{g}$ does not imply the invariance with respect to $\mathfrak{e}$. For example, any invariant element of $\mathfrak{sl}(n, \mathbb{C})$ is a multiple of the quadratic Casimir element. But Casimir elements are different for different n. ◊

We now have defined a subclass of Lie bialgebra structures on $\mathfrak{g}$, namely the coboundary structures. The first cohomology group $H^1(\mathfrak{g}, \mathfrak{g} \wedge \mathfrak{g})$ measures the size of this subclass. Since for any finite-dimensional semi-simple Lie algebra over a field of characteristic 0 (not necessarily algebraically closed) we have $H^1(\mathfrak{g}, \mathfrak{g} \wedge \mathfrak{g}) = 0$, any Lie bialgebra structure on such a Lie algebra is automatically coboundary. Furthermore, if the field is assumed to be algebraically closed and $\mathfrak{g}$ is simple, then by [15, Theorem 11.2] the space of ad-invariant elements in $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ is 1-dimensional. These observations lead to the following theorem.

Theorem 1.1.9. *Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra over a field of characteristic* 0*. Then for any Lie bialgebra structure δ on $\mathfrak{g}$ there is $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that $\delta = dr$. Moreover, if $\mathfrak{g}$ is simple and the field is assumed to be algebraically closed, then r can be chosen to be quasi-triangular, i.e.* $\mathrm{CYB}(r) = 0$.

Example 1.1.10. The Lie bialgebra structures on $\mathfrak{sl}(2, \mathbb{C})$ from Example 1.1.3 are defined by quasi-triangular r-matrices:

1. $r = 0$;
2. $r = \frac{1}{4} h \otimes h + e \otimes f = (\frac{1}{4} h \otimes h + \frac{1}{2}(e \otimes f + f \otimes e)) + \frac{1}{2}(e \otimes f - f \otimes e)$;
3. $r = \frac{1}{2}(e \otimes h - h \otimes e)$.

◊

Remark 1.1.11. In view of Theorem 1.1.9, to classify all Lie bialgebra structures on a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, it is enough to describe all solutions to the classical Yang-Baxter equation

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0.$$

If we assume that $\mathfrak{g}$ has dimension n, then solving the CYBE amounts to solving n^3 quadratic equations in n^2 variables. This is what makes the search for solutions a hard task that cannot be solved by a straight-forward calculation. ◊

We say that two elements $r_1, r_2 \in \mathfrak{g} \otimes \mathfrak{g}$ are *gauge equivalent*, if there is $\varphi \in \mathrm{Aut}_{F-\mathrm{LieAlg}}(\mathfrak{g})$ such that

$$r_2 = (\varphi \otimes \varphi) r_1. \tag{1.13}$$

It is clear that if r defines a Lie bialgebra, then so does ξr for any $\xi \in F$. We say that r_1 and r_2 are *equivalent* if there is a constant $\xi \in F^\times$ such that r_1 is gauge equivalent to ξr_2. One can easily check that the relations

$$dr_2 \varphi = (\varphi \otimes \varphi) dr_1 \quad \text{and} \quad r_2 = (\varphi \otimes \varphi) r_1$$

are equivalent. Consequently, equivalent coboundary Lie bialgebra structures correspond to equivalent elements in $\mathfrak{g} \otimes \mathfrak{g}$ and, conversely, if $r_1, r_2 \in \mathfrak{g} \otimes \mathfrak{g}$ are equivalent and r_1 defines a Lie bialgebra structure on $\mathfrak{g}$ then r_2 also defines a Lie bialgebra equivalent to the first one.

Remark 1.1.12. By Theorem 1.1.9 Lie bialgebra structures on a finite-dimensional simple Lie algebra $\mathfrak{g}$ over an algebraically closed field F are completely determined by quasi-triangular r-matrices. For the case $F = \mathbb{C}$, there are well-known classification results.

For $r = a + s$ with $s \neq 0$ the classification was obtained in [7]. The case $s = 0$ is less friendly. It was shown in [18] that the classification of skew-symmetric solutions is equivalent to the classification of quasi-Frobenius subalgebras of $\mathfrak{g}$. The latter problem is known to be "representation-wild". Therefore, at the present there is no hope to obtain a full classification of triangular r-matrices for simple Lie algebras, except for some low-dimensional ones; see [18, 19]. $\Diamond$

1.1.2 Manin triples

Now we place Lie bialgebras and quasi-triangular r-matrices into a larger picture of Lie algebra splittings and generating functions.

Let W be a subspace of a vector space V equipped with a reflexive bilinear form B and define

$$W^\perp := \{v \in V \mid B(v, W) = 0\}.$$

Recall that the subspace W is called

- *isotropic* if $W \subseteq W^\perp$;
- *coisotropic* if $W^\perp \subseteq W$;
- *Lagrangian* if $W = W^\perp$.

A *(classical) Manin triple* is a triple $((L, B), L_+, L_-)$, where L is a Lie algebra with an invariant non-degenerate symmetric bilinear form B and $L_\pm$ are Lagrangian subalgebras of L such that $L = L_+ \dotplus L_-$. We use the sign "$\dotplus$" to denote the direct sum of vector space structures, but not necessarily of Lie algebra structures.

Each Lie bialgebra $(\mathfrak{g}, \delta)$ can be associated with a Manin triple. More precisely, equip the vector space $\mathfrak{g} \dotplus \mathfrak{g}^\vee$ with the form

$$B(x + f, y + g) := f(y) + g(x). \tag{1.14}$$

Then one can prove that there exists a unique Lie bracket on $\mathfrak{g} \dotplus \mathfrak{g}^\vee$ making the triple $((\mathfrak{g} \dotplus \mathfrak{g}^\vee, B), \mathfrak{g}, \mathfrak{g}^\vee)$ into a Manin triple, namely

$$[x, f] := -f \circ \mathrm{ad}_x + (f \otimes 1)\delta(x). \tag{1.15}$$

Such a Manin triple associated with a Lie bialgebra $(\mathfrak{g}, \delta)$ is called *the (classical) double* of $(\mathfrak{g}, \delta)$. The Lie algebra $\mathfrak{g} \dotplus \mathfrak{g}^\vee$ with the form above is usually denoted by $\mathfrak{D}(\mathfrak{g}, \delta)$.

The converse is not true: not every Manin triple $((L, B), L_+, L_-)$ defines a Lie bialgebra structure on L. However, if the dual map

$$[\cdot\,, \cdot]^\vee \colon L_-^\vee \to (L_- \otimes L_-)^\vee \tag{1.16}$$

of the Lie bracket on L_- restricts to the map $\delta \colon L_+ \to L_+ \otimes L_+$, where we use the inclusion $L_+ \subseteq L_-^\vee$ given by B, then (L_+, δ) is a Lie bialgebra. Equivalently, if there is a linear map $\delta \colon L_+ \to L_+ \otimes L_+$ satisfying the condition

$$B(\delta(a), b_1 \otimes b_2) = B(a, [b_1, b_2]), \quad \forall a \in L_+,\ \forall b_1, b_2 \in L_-, \tag{1.17}$$

then (L_+, δ) is a Lie bialgebra. In this case δ coincides with dual map Eq. (1.16) on L_+. When the condition above is satisfied we say that the Manin triple $((L, B), L_+, L_-)$ *defines* the Lie bialgebra (L_+, δ).

Remark 1.1.13. If $((L, B), L_+, L_-)$ is a Manin triple with finite-dimensional L_+, then the form B identifies L_- with $L_+^\vee$ and the triple does define a Lie bialgebra structure. It follows that the notions of a finite-dimensional Manin triple and a finite-dimensional Lie bialgebra coincide. $\diamondsuit$

There is no nice way of defining a homomorphism between two Manin triples, but there is a natural notion of isomorphism. We call Manin triples $((L, B), L_+, L_-)$ and $((M, Q), M_+, M_-)$ isomorphic if there is a Lie algebra isomorphism $\varphi\colon L \to M$ satisfying the conditions

$$\varphi(L_\pm) = M_\pm \quad \text{and} \quad B(x, y) = Q(\varphi(x), \varphi(y)).$$

Similar to the Lie bialgebra case it is natural to identify Manin triples $((L, B), L_+, L_-)$ and $((L, \xi B), L_+, L_-)$, where $\xi \in F^\times$. We say that two Manin triples $((L, B), L_+, L_-)$ and $((M, Q), M_+, M_-)$ are *equivalent* and write

$$((L, B), L_+, L_-) \sim ((M, Q), M_+, M_-)$$

if there is $\xi \in F^\times$ such that $((L, B), L_+, L_-)$ is isomorphic to $((M, \xi Q), M_+, M_-)$. It is not hard to see that equivalent Manin triples define equivalent Lie bialgebra structures.

1.1.3 Generating series

Assume $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a quasi-triangular r-matrix and write it as $r = x_i \otimes y^i$, where (x_i) and (y^i) are sequences of linearly independent vectors. Define the following subspaces of $\mathfrak{g}$:

$$\mathfrak{l} := \operatorname{span}_F\{x_i\}, \ \mathfrak{k} := \operatorname{span}_F\{y^i\}, \quad \text{and} \quad L := \mathfrak{l} + \mathfrak{k} = \operatorname{span}_F\{x_i, y^i\}.$$

Since $\mathrm{CYB}(r) = 0$ these subspaces are subalgebras of $\mathfrak{g}$. Moreover, dr restricts to the map $L \to L \otimes L$ making (L, dr) into a Lie bialgebra. This Lie bialgebra is defined by the Manin triple $(\mathfrak{D}(L, dr), L, L^\vee)$. Let us write $[f, g]^r := dr^\vee(f \otimes g)$ for the Lie bracket on $L^\vee$. Now we observe that the linear map $\underline{r}\colon L^\vee \to \mathfrak{k}$ is surjective and hence it gives rise to the isomorphism (of vector spaces)

$$\underline{r}\colon \mathfrak{l}^\vee \cong L^\vee/\ker(\underline{r}) \to \mathfrak{k}. \tag{1.18}$$

Again, since $\mathrm{CYB}(r) = 0$ we can directly compute that

$$[\underline{r}f, \underline{r}g] = \underline{r}[f, g]^r. \tag{1.19}$$

In other words, the map Eq. (1.18) is a Lie algebra isomorphism. Putting all these observations together we see that $(\mathfrak{l} \dotplus \mathfrak{k}, \mathfrak{l}, \mathfrak{k})$ is Manin triple, where the Lie bracket and the form on $\mathfrak{l} \dotplus \mathfrak{k}$ are defined in the same way as on $\mathfrak{D}(L, dr) = L \dotplus L^\vee$. If we now take a basis $\{v_i\}$ for $\mathfrak{l}$ and a dual basis $\{w^i\}$ in $\mathfrak{k}$ the matrix $v_i \otimes w^i \in \mathfrak{l} \otimes \mathfrak{k}$ will coincide with the original matrix r. Therefore, we have proven the following result.

Lemma 1.1.14. *Let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be a quasi-triangular r-matrix. Write it as a sum $r = x_i \otimes y^i$ with a minimal number of simple tensors. Then there exists a Lie algebra $\mathfrak{a}$ with an invariant non-degenerate symmetric bilinear form that splits into a direct sum*

$$\mathfrak{a} = \mathfrak{l} \dotplus \mathfrak{k} = \operatorname{span}_F\{x_i\} \dotplus \operatorname{span}_F\{y^i\}$$

of Lagrangian Lie subalgebras, generated by the left and right components respectively of r.

Example 1.1.15. Consider, for example, the quasi-triangular r-matrix $r = \frac{1}{4} h \otimes h + e \otimes f$ from Example 1.1.10. The construction above gives the Manin triple $(\mathfrak{b}_+ \dotplus \mathfrak{b}_-, \mathfrak{b}_+, \mathfrak{b}_-)$. The Lie bracket on $\mathfrak{b}_+ \dotplus \mathfrak{b}_-$ is given by

$$\begin{aligned}
[(e,0),(0,f)] &= \frac{1}{2}(h,h), & [(e,0),(0,h)] &= (-2e,0),\\
[(h,0),(0,f)] &= (0,-2f), & [(h,0),(0,h)] &= (0,0)
\end{aligned}$$

and r splits this Lie algebra into the direct sum of Lagrangian Lie subalgebras. $\Diamond$

1.1.4 Classical twists

Let $((L,B), L_+, L_-)$ be a finite-dimensional Manin triple. As we now know it defines a Lie bialgebra structure δ on L_+. Assume we can find another Lagrangian Lie subalgebra $L'_- \subset L$ such that $L_+ \dotplus L'_- = L$. Then, using the same arguments, $((L,B), L_+, L'_-)$ defines another Lie bialgebra structure δ' on L_+. It is natural to ask what is the relation between δ and δ'.

Let us now investigate this question. Each $f \in L_-$ can be decomposed as $f = f_+ + f'$, where $f_+ \in L_+$ and $f' \in L'_-$. Define the linear map

$$\begin{aligned}
\underline{t} \colon L_- &\to L_+\\
f &\mapsto -f_+.
\end{aligned}$$

Then $L'_- = \{(\underline{t}+1)f \mid f \in L_-\}$. The isomorphism $L_-^\vee \cong L_+$, given by the form B, associates the map $\underline{t}$ with a unique element $t \in L_+ \otimes L_+$. If $t = x_i \otimes y^i$, then for all $f, g \in L_-$ the Lagrangian property of L'_- gives

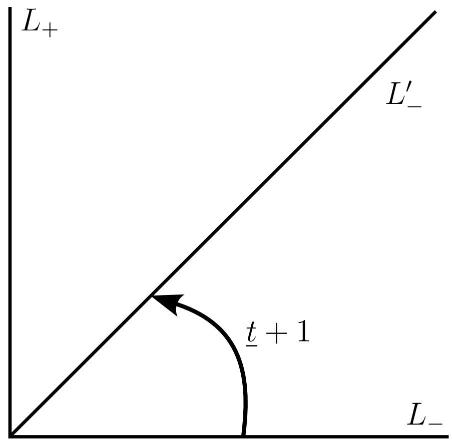
$$\begin{aligned}
0 &= B((\underline{t}+1)f, (\underline{t}+1)g)\\
&= B(\underline{t}f, g) + B(f, \underline{t}g)\\
&= B(x_i \otimes y^i + y^i \otimes x_i, f \otimes g).
\end{aligned}$$

In other words, the element t, corresponding to the map $\underline{t}$, is skew-symmetric. A direct calculation then yields the equality

$$\begin{aligned}
B(\delta(a) + dt(a), (\underline{t}+1)f \otimes (\underline{t}+1)g) &= B(a, [(\underline{t}+1)f, (\underline{t}+1)g])\\
&= B(\delta'(a), (\underline{t}+1)f \otimes (\underline{t}+1)g),
\end{aligned} \tag{1.20}$$

showing that $\delta' = \delta + dt$.

Note that if we simply add some 1-coboundary dt to δ the resulting linear map $\delta + dt$ does not need to be a Lie bialgebra structure. For that, the element t needs to satisfy certain



conditions which we will state very soon. The procedure of obtaining new Lie bialgebra structures from a given one by adding certain 1-coboundaries is called *twisting*. And we have just seen that at the level of Manin triples twisting is nothing else but a change of the third component in the triple.

Let us now present the exact statements. Consider a Lie bialgebra (L_+, δ) (not necessarily finite-dimensional). A skew-symmetric element $t \in L_+ \otimes L_+$ is called a *classical twist* of δ if it satisfies the equation

$$\mathrm{CYB}(t) = \circlearrowleft (\delta \otimes 1)t, \tag{1.21}$$

where $\circlearrowleft x \otimes y \otimes z \coloneqq x \otimes y \otimes z + y \otimes z \otimes x + z \otimes x \otimes y$. For a classical twist t the linear map $\delta_t \coloneqq \delta + dt$ is again a Lie bialgebra. Note that for $\delta = 0$ condition Eq. (1.21) turns into condition $\langle r, r \rangle = 0$ from Theorem 1.1.7. Therefore, classical twists of $\delta = 0$ are precisely the triangular r-matrices in $L_+ \otimes L_+$. In Paper I, based on the results from [13, 17, 18, 19], we prove the following statement.

Theorem 1.1.16. *Let (L_+, δ) be a Lie bialgebra defined by the Manin triple (L, L_+, L_-). Then we have the following one-to-one correspondences: Let (L_+, δ) be a Lie bialgebra defined by the Manin triple (L, L_+, L_-). Then there are the following one-to-one correspondences:*

Classical twists of δ, i.e. skew-symmetric tensors $t \in L_+ \otimes L_+$ satisfying $\mathrm{CYB}(t) = \circlearrowleft (\delta \otimes 1)t$

$\updownarrow$

Lagrangian Lie subalgebras $L_t \subseteq L$ complementary to L_+ and commensurable with L_-, i.e. $\dim(L_t + L_-)/(L_t \cap L_-) < \infty$

$\longleftrightarrow$

Linear maps $T \colon L_- \longrightarrow L_+$ such that $\dim(\mathrm{im}(T)) < \infty$ *and for all $w_1, w_2, w_3 \in L_-$ holds $B(Tw_1, w_2) + B(w_1, Tw_2) = 0$ and* $B([Tw_1 - w_1, Tw_2 - w_2], Tw_3 - w_3) = 0$

Remark 1.1.17. It is clear that the conditions on $t \in L_+ \otimes L_+$ to be a classical twist are sufficient but not necessary. Indeed, if $\delta = 0$ and $t = a + s \in L_+ \otimes L_+$ then, as we know from

Theorem 1.1.7, the map dt is a Lie bialgebra structure if and only if both s and $\mathrm{CYB}(t)$ are ad-invariant. In a similar vein, one can prove that $\delta_t = \delta + dt$, where $t = a + s \in L_+ \otimes L_+$, is again a Lie bialgebra structure if and only if both s and $\mathrm{CYB}(t) - \circlearrowleft (\delta \otimes 1)t$ are ad-invariant. However, twisting δ with such an element t does not correspond to a change of the third component in the corresponding Manin triple $(\mathfrak{D}(L_+, \delta), L_+, L_-)$. In other words, if we want twisting to preserve the first two components of the Manin triple we need the notion of a classical twist. ◊

We have explained how (classical) twisting looks at the level of cocycles and Manin triples. Let us finally explain how it is seen from the viewpoint of quasi-triangular r-matrices. Consider a Lie bialgebra $(\mathfrak{g}, \delta = dr)$ for some quasi-triangular r-matrix $r = a + s \in \mathfrak{g} \otimes \mathfrak{g}$. Adding a skew-symmetric $t \in \mathfrak{g} \otimes \mathfrak{g}$ to r does not change the invariance property of s. Moreover, a straight-forward computation shows that

$$\mathrm{CYB}(r + t) = \mathrm{CYB}(r) + \mathrm{CYB}(t) - \circlearrowleft (\delta \otimes 1)t = \mathrm{CYB}(t) - \circlearrowleft (\delta \otimes 1)t. \tag{1.22}$$

In other words, a skew-symmetric $t \in \mathfrak{g} \otimes \mathfrak{g}$ is a classical twist of a quasi-triangular structure $\delta = dr$ if and only if $r_t := r + t$ is again a quasi-triangular r-matrix. It is clear that $\delta_t = dr_t$.

1.2 Classification scheme for classical Lie bialgebras

Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra and $(\mathfrak{D}, \mathfrak{g}, \mathfrak{g}^\vee)$ be the associated Manin triple. Twisting δ, as we have seen, changes the third component of the triple and leaves the rest untouched. Therefore, if $t \in \mathfrak{g} \otimes \mathfrak{g}$ is a classical twist of δ, then $\mathfrak{D}(\mathfrak{g}, \delta + dt) = \mathfrak{D}(\mathfrak{g}, \delta)$ as a Lie algebra with form. In other words, classical doubles are invariant under twisting. For this reason, they are sometimes called *twisting classes*. This observation leads to the following classification scheme of Lie bialgebra structures on a given Lie algebra $\mathfrak{g}$.

1. Classify all classical doubles $\mathfrak{D}(\mathfrak{g}, \delta)$;
2. Choose a representative δ within each twisting class and describe its classical twists.

Remark 1.2.1. If (L_1, δ_1) and (L_2, δ_2) are isomorphic Lie bialgebras, then so are the associated Manin triples. Indeed, if $\varphi \colon L_1 \to L_2$ is a Lie bialgebra isomorphism, then the isomorphism between $(\mathfrak{D}(L_1, \delta_1), L_1, L_1^\vee)$ and $(\mathfrak{D}(L_2, \delta_2), L_2, L_2^\vee)$ is given by $\varphi \dotplus (\varphi^{-1})^\vee$. In particular, isomorphic Lie bialgebra structures will fall into the same twisting class. However, if δ_2 is a classical twist of δ_1 then, in general, we do not know if these structures are isomorphic or not. Therefore, if we want to classify Lie bialgebra structures not just up to twists, but up to equivalence we need to add an extra step to the classification scheme above:

3. Describe isomorphism classes of twists.

◊

By Theorem 1.1.16 we know that the second step of the scheme is equivalent to the classification of subalgebras $W \subseteq \mathfrak{D}(\mathfrak{g}, \delta)$ complementary to $\mathfrak{g}$ and commensurable with $\mathfrak{g}^\vee$. In case $\delta = dr$, for some quasi-triangular $r \in \mathfrak{g} \otimes \mathfrak{g}$, this problem is equivalent to the classification of quasi-triangular r-matrices of the form $r + t$, where $t \in \mathfrak{g} \wedge \mathfrak{g}$. Furthermore, if r

is triangular, then the description of the twists coincides with the description of all triangular r-matrices of the form $r + t$ with $t \in \mathfrak{g} \otimes \mathfrak{g}$. By our previous observations, isomorphic twists will correspond to gauge equivalent quasi-triangular r-matrices.

Remark 1.2.2. The division of quasi-triangular Lie bialgebra structures into different doubles can be used as a tool to determine if given r-matrices are not gauge equivalent. More precisely, if $\mathfrak{D}(\mathfrak{g}, dr_1)$ is not isomorphic to $\mathfrak{D}(\mathfrak{g}, dr_2)$, then r_1 cannot be gauge equivalent to r_2. $\lozenge$

There is no general solution for the problem in step 1. However, one of the key results in Paper II is the following theorem that allows us to approach this question for a large variety of Lie algebras.

Theorem 1.2.3. *Let $\mathfrak{g}$ be a central simple Lie algebra over a field F of characteristic 0 and A be a reduced unital associative commutative F-algebra. Equip the tensor product $\mathfrak{g} \otimes A$ with the Lie algebra bracket*

$$[a \otimes f, b \otimes g] = [a, b] \otimes fg.$$

Let L be a Lie algebra equipped with a non-degenerate symmetric invariant bilinear form B such that $\mathfrak{g} \otimes A \subseteq L$ is a coisotropic subalgebra. Then, as Lie algebras, $L \cong \mathfrak{g} \otimes \widetilde{A}$ for a unital associative commutative algebra extension $\widetilde{A} \supseteq A$.

Example 1.2.4. Consider a finite-dimensional simple Lie algebra $\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}$ over $\mathbb{C}$. Let δ be a Lie bialgebra structure on it. Then, by definition of the double, $\mathfrak{g} \otimes \mathbb{C}$ is a Lagrangian subalgebra of $\mathfrak{D}(\mathfrak{g}, \delta)$. Theorem 1.2.3 gives an isomorphism of $\mathfrak{g}$-modules $\mathfrak{D}(\mathfrak{g}, \delta) \cong \mathfrak{g} \otimes \widetilde{A}$ for some algebra extension $\mathbb{C} \subseteq \widetilde{A}$. Furthermore, if $\mathfrak{g}$ has dimension n over $\mathbb{C}$ then $\mathfrak{D}(\mathfrak{g}, \delta) = \mathfrak{g} \dotplus \mathfrak{g}^\vee$ has dimension $2n$ over $\mathbb{C}$. It now follows that $\widetilde{A}$ must have dimension 2 over $\mathbb{C}$. Up to isomorphism, these extensions are

$$\mathbb{C} \oplus \mathbb{C} \quad \text{and} \quad \mathbb{C}[x]/(x^2).$$

This means that classical doubles of $(\mathfrak{g}, \delta)$, as Lie algebras, are isomorphic to either

$$\mathfrak{g} \times \mathfrak{g} \quad \text{or} \quad \mathfrak{g}[x]/x^2\mathfrak{g}[x]. \tag{1.23}$$

In the first case $\mathfrak{g}$ sits inside $\mathfrak{g} \times \mathfrak{g}$ diagonally, i.e. $\mathfrak{g} \cong \Delta := \{(a, a) \mid a \in \mathfrak{g}\}$. In the second case we embed $\mathfrak{g}$ into $\mathfrak{g}[x]/x^2\mathfrak{g}[x]$ as $\mathfrak{g} \otimes 1$. Unfortunately, this is not a classification of classical doubles yet, because we need to classify all possible non-degenerate invariant symmetric bilinear forms on these Lie algebras as well. For that, fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and decompose $\mathfrak{g}$ into root spaces

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha.$$

For each $\alpha > 0$ choose a pair of vectors $(e_\alpha, e_{-\alpha}) \in \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}$ such that $\kappa(e_\alpha, e_{-\alpha}) = 1$. Let us denote by B a non-degenerate invariant symmetric bilinear form on $\mathfrak{D}(\mathfrak{g}, \delta)$. Choose some $\alpha \neq 0$ and consider the linear functional $\mathrm{t} \colon \widetilde{A} \to \mathbb{C}$, given by

$$\mathrm{t}(f) := B(e_\alpha \otimes 1, e_{-\alpha} \otimes f). \tag{1.24}$$

If $\widetilde{A} = \operatorname{span}_{\mathbb{C}}\{1, v\}$, then t is completely determined by its value $\lambda := \mathrm{t}(v)$. Indeed, since $\mathfrak{g} \otimes 1$ is a Lagrangian subalgebra of $\mathfrak{g} \otimes \widetilde{A}$ we have $\mathrm{t}(1) = 0$ and hence $\mathrm{t}(c_1 + c_2 v) = c_2 \lambda$ for all $c_1, c_2 \in \mathbb{C}$. It was shown in [20] that the functional t does not depend on our choice of the root α and, moreover, it completely determines the form B itself. More precisely, we have the equality

$$B(a \otimes f, b \otimes g) = \kappa(a, b)\, \mathrm{t}(fg). \tag{1.25}$$

Therefore, up to scaling, the only possible non-degenerate invariant symmetric bilinear forms on Lie algebras from Eq. (1.23) are

$$\begin{aligned} B(a \otimes (\lambda_1, \lambda_2), b \otimes (\mu_1, \mu_2)) &= \kappa(a, b)(\lambda_1\mu_1 - \lambda_2\mu_2), \\ B(a \otimes (\lambda_1 + \lambda_2 x), b \otimes (\mu_1 + \mu_2 x)) &= \kappa(a, b)(\lambda_1\mu_2 + \lambda_2\mu_1). \end{aligned} \tag{1.26}$$

This result completes the classification of classical doubles.

In the second step we need to choose some representatives within the two twisting classes above and describe their twists. One obvious representative from the second class is $\delta = 0$, which corresponds to the splitting

$$\mathfrak{g}[x]/x^2\mathfrak{g}[x] = \mathfrak{g} \otimes 1 \dotplus \mathfrak{g} \otimes [x],$$

where the bracket on the second factor is trivial: $[f \otimes [x], g \otimes [x]] = [f, g] \otimes [x^2] = 0$. Therefore, the second class corresponds to triangular r-matrices $r \in \mathfrak{g} \otimes \mathfrak{g}$. This also means that if we take a quasi-triangular r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$ which is not skew-symmetric, dr will fall into the first twisting class. An example of such an r-matrix is

$$r = \sum_{\alpha > 0} e_{-\alpha} \otimes e_\alpha + \sum_{\alpha > \beta > 0} (e_{-\alpha} \otimes e_\beta - e_\beta \otimes e_{-\alpha}) \in \mathfrak{g} \otimes \mathfrak{g}. \tag{1.27}$$

This example can be considered, in some sense, as a generalization of the quasi-triangular r-matrix $\frac{1}{4} h \otimes h + e \otimes f$ from Example 1.1.10.

To complete steps two and three we need to classify triangular and non-alternating quasi-triangular r-matrices up to gauge equivalence. We have already mentioned in Remark 1.1.11 that such classifications can be found in [7, 18] and [19]. In particular, for $\mathfrak{sl}(2, \mathbb{C})$ we get precisely 3 equivalence classes mentioned in Example 1.1.3. $\diamondsuit$

2 Pseudo quasi-triangular structure

A Lie bialgebra structure δ on $\mathfrak{g}$ is a coboundary structure if there is $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that $\delta = dr$, where

$$dr(a) = [a \otimes 1 + 1 \otimes a, r]. \tag{2.1}$$

Sometimes we can use Eq. (2.1) to define a Lie bialgebra structure on $\mathfrak{g}$ even when r is not an element of $\mathfrak{g} \otimes \mathfrak{g}$. In this case we refer to such a structure as *pseudo quasi-triangular*.

Example 2.0.1. Let $\mathfrak{a}$ be a finite-dimensional semi-simple Lie algebra over a field F of characteristic 0. Define $\mathfrak{g} := \mathfrak{a}[x] := \mathfrak{a} \otimes F[x]$ to be the Lie algebra of polynomials. Consider the formal power series

$$r(x, y) = \frac{\Omega}{x - y} = \Omega \sum_{k \geq 0} x^{-k-1} y^k \in (\mathfrak{a} \otimes \mathfrak{a})(\!(x)\!)[\![y]\!], \tag{2.2}$$

where $\Omega \in \mathfrak{a} \otimes \mathfrak{a}$ is the quadratic Casimir element of $\mathfrak{a}$. To see that Eq. (2.1) still makes sense, we need to use the well-known invariance of Ω, i.e.

$$[a \otimes 1 + 1 \otimes a, \Omega] = 0, \quad \forall a \in \mathfrak{a}. \tag{2.3}$$

Taking it into account we have

$$\begin{aligned}
\left[\sum_{i \geq 0} a_i x^i \otimes 1 + 1 \otimes \sum_{i \geq 0} a_i y^i, \frac{\Omega}{x - y}\right] &= \left[\sum_{i \geq 0} (a_i \otimes 1)x^i - (a_i \otimes 1)y^i + (a_i \otimes 1 + 1 \otimes a_i)y^i, \frac{\Omega}{x - y}\right] \\
&= \left[\sum_{i \geq 1} (a_i \otimes 1)(x^i - y^i), \frac{\Omega}{x - y}\right] \\
&= \sum_{i \geq 1} [a_i \otimes 1, \Omega](x^{i-1} + x^{i-2}y + \cdots + xy^{i-2} + y^{i-1}) \in \mathfrak{g} \otimes \mathfrak{g}.
\end{aligned} \tag{2.4}$$

Therefore, series Eq. (2.2) that does not lie in $\mathfrak{g} \otimes \mathfrak{g}$ produces a well-defined linear map $dr \colon \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ through the same formula Eq. (2.1). This map, by definition, is a 1-cocycle. Furthermore, since Ω is ad-invariant, the element $dr(f)$ is skew-symmetric for any $f \in \mathfrak{g}$. Finally, using once again the invariance of the Casimir element, we compute

$$\circlearrowleft (dr \otimes 1)dr(f) = -f \cdot \mathrm{CYB}(r) = -f \cdot \mathrm{CYB}(\Omega) \frac{x - y - x + z + y - z}{(x - y)(y - z)(x - z)} = 0, \quad \forall f \in \mathfrak{a}[x]. \tag{2.5}$$

In other words, the Jacobi identity for $dr^\vee$ restricted to $\mathfrak{g}^\vee \otimes \mathfrak{g}^\vee$ is equivalent to the ad-invariance of $\mathrm{CYB}(r)$, exactly as in the standard coboundary case. In our particular example $\mathrm{CYB}(r) = 0$ implying the invariance. Consequently, dr is indeed a Lie bialgebra structure on $\mathfrak{a}[x]$. $\lozenge$

In general, if we take $\mathfrak{a}$ as in the example above and consider a series of the form

$$r(x, y) = \frac{s(y)\Omega}{x - y} + g(x, y) \in (\mathfrak{a} \otimes \mathfrak{a})((x))[\![y]\!], \tag{2.6}$$

where $s \in F[y]$ and $g \in (\mathfrak{a} \otimes \mathfrak{a})[x, y]$, then $dr\colon f \mapsto [f(x) \otimes 1 + 1 \otimes f(y), r(x, y)]$ is again a well-defined linear map $\mathfrak{a}[x] \to (\mathfrak{a} \otimes \mathfrak{a})[x, y]$. Moreover, if $dr(f)$ is skew-symmetric for all $f \in \mathfrak{a}[x]$, then we have an identity similar to the one in the classical picture:

$$\circlearrowleft (dr \otimes 1)dr(f) = -f \cdot \mathrm{CYB}(r) \in (\mathfrak{a} \otimes \mathfrak{a} \otimes \mathfrak{a})[x, y, z]. \tag{2.7}$$

We can also view r in Eq. (2.6) as a meromorphic function $r\colon \mathbb{C} \times \mathbb{C} \to \mathfrak{a} \otimes \mathfrak{a}$. Considering elements $f \in \mathfrak{a}[x]$ as polynomial functions with values in $\mathfrak{a}$ we can again give meaning to Eq. (2.1). Namely, we define a polynomial function

$$dr(f)(x, y) = [f(x) \otimes 1 + 1 \otimes f(y), r(x, y)], \tag{2.8}$$

which we then interpret as an element of $(\mathfrak{a} \otimes \mathfrak{a})[x, y]$.

Therefore, we can define more Lie bialgebra structures on Lie algebra $\mathfrak{a}[x]$ (as well as $\mathfrak{a}[x, x^{-1}]$, $\mathfrak{a}[\![x]\!]$ etc.) using Eq. (2.1) and either meromorphic functions or series of the form Eq. (2.6). The first case leads to consideration of analytical solutions to the CYBE whereas the second - to formal. We now look at these two types of solutions in more detail. starting with the analytical case because it does not require any additional preparation.

2.1 Analytical solutions

Let $\mathfrak{g}$ be a Lie algebra over a field F. The two-parameter classical Yang-Baxter equation is the equation of the form

$$[r^{12}(x_1, x_2), r^{13}(x_1, x_3)] + [r^{12}(x_1, x_2), r^{23}(x_2, x_3)] + [r^{13}(x_1, x_3), r^{23}(x_2, x_3)] = 0, \tag{2.9}$$

where r is any function with the codomain $\mathfrak{g} \otimes \mathfrak{g}$. The notation $r^{ij}(x_i, x_j)$ means $(r(x_i, x_j))^{ij}$, where the maps $(\,\cdot\,)^{ij}$ are defined exactly as in Eq. (1.11). The same notation $\mathrm{CYB}(r)$, or $\mathrm{CYB}(r)(x, y, z)$ if one wants to specify the parameters, is used to denote the left-hand side of Eq. (2.9). Historically, one usually considers finite-dimensional Lie algebras over $\mathbb{C}$ and search for solutions in the set of meromorphic functions $r\colon U \to \mathfrak{g} \otimes \mathfrak{g}$ defined on a connected open neighbourhood $U \subseteq \mathbb{C} \times \mathbb{C}$.

Example 2.1.1. Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbb{C}$ and $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ be the corresponding quadratic Casimir element. Consider the function $r\colon \mathbb{C} \times \mathbb{C} \to \mathfrak{g} \otimes \mathfrak{g}$ given by

$$r(x, y) = \frac{\Omega}{x - y}. \tag{2.10}$$

As we have seen in Example 2.0.1 $\mathrm{CYB}(r) = (x - y)^{-1}(y - z)^{-1}(x - z)^{-1}\mathrm{CYB}(\Omega)(x - y - x + z + y - z) = 0.$ $\lozenge$

Example 2.1.2. Let $\mathfrak{g}$ be again a semi-simple $\mathbb{C}$-Lie algebra. Fix some triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \dotplus \mathfrak{h} \dotplus \mathfrak{n}_+$. Denote by $\Omega_\pm$ and $\Omega_\mathfrak{h}$ the projections of the Casimir element $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ on the subspaces $\mathfrak{n}_\pm \otimes \mathfrak{n}_\mp$ and $\mathfrak{h} \otimes \mathfrak{h}$ respectively. Another interesting example of a solution $r \colon \mathbb{C} \times \mathbb{C} \to \mathfrak{g} \otimes \mathfrak{g}$ to Eq. (2.9) is

$$r(x, y) = \frac{y\Omega}{x - y} + \frac{\Omega_\mathfrak{h}}{2} + \Omega_+. \tag{2.11}$$

Subalgebras $\mathfrak{n}_+$ and $\mathfrak{n}_-$ play symmetric roles, so we could equivalently have chosen Ω_- instead of Ω_+. The constant part $r_{DJ} := \Omega_\mathfrak{h}/2 + \Omega_+$ is a solution to the (constant) classical Yang-Baxter equation known as *Drinfeld-Jimbo r-matrix*. ♢

Constant r-matrices form a subset of solutions to Eq. (2.9), so there can be no question about their full classification. However, there are well-known classification results for non-degenerate solutions for simple Lie algebras. Let us present them in more detail.

The Killing form κ on a finite-dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ induces an isomorphism of vector spaces

$$\begin{aligned} \psi \colon \mathfrak{g} \otimes \mathfrak{g} &\to \operatorname{End}(\mathfrak{g}), \\ a \otimes b &\mapsto \kappa(\cdot, b)a. \end{aligned} \tag{2.12}$$

Let U be again a connected open subset of $\mathbb{C} \times \mathbb{C}$. A meromorphic function $r \colon U \to \mathfrak{g} \otimes \mathfrak{g}$ is called

1. *Non-degenerate* if the endomorphism $\psi(r(x, y))$ is invertible for some $(x, y) \in U$;
2. *Skew-symmetric* if for all $(x, y) \in U$ we have $r(x, y) = -\tau(r(y, x))$, where $\tau(a \otimes b) := b \otimes a$.

Similar to Eq. (1.13), we call two meromorphic functions $r_1, r_2 \colon U \to \mathfrak{g} \otimes \mathfrak{g}$ *(holomorphically) gauge equivalent* if there is a holomorphic map $\varphi \colon U \to \operatorname{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ such that

$$r_2(x, y) = (\varphi(x) \otimes \varphi(y)) r_1(x, y), \quad \forall (x, y) \in U. \tag{2.13}$$

It is easy to see, that if r_1 solves the CYBE, then so does r_2.

Example 2.1.3. Let $\mathfrak{g} = \mathfrak{n}_- \dotplus \mathfrak{h} \dotplus \mathfrak{n}_+$ be as in Example 2.1.2. The linear isomorphism of $\mathfrak{h}^\vee$ given by $\alpha \mapsto -\alpha$ induces an automorphism σ of $\mathfrak{g}$, which we can view as a constant holomorphic function. Applying $\sigma(x) \otimes \sigma(y)$ to Eq. (2.11) we obtain

$$r(x, y) = \frac{y\Omega}{x - y} + \frac{\Omega_\mathfrak{h}}{2} + \Omega_-. \tag{2.14}$$

♢

The following result, proven in [8], describes local behaviour of solutions to the CYBE.

Theorem 2.1.4. *Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra, U be a connected open subset of $\mathbb{C} \times \mathbb{C}$ and $r \colon U \to \mathfrak{g} \otimes \mathfrak{g}$ be a non-degenerate meromorphic solution to Eq.* (2.9)*. Then there exists an open neighbourhood $V \subseteq \mathbb{C}$ of 0, a holomorphic function $\varphi \colon V \to \operatorname{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ and a non-constant holomorphic function $f \colon V \to U$ such that*

$$(\varphi(v_1) \otimes \varphi(v_2)) r(f(v_1), f(v_2)) \tag{2.15}$$

depends on the difference $v_1 - v_2$.

The proof consists of three main steps. First, by translating and shrinking the domain U to $\widetilde{U} \times \widetilde{U} \subseteq \mathbb{C} \times \mathbb{C}$ one shows that

$$r(x, y) = \frac{s(y)\Omega}{x - y} + g(x, y), \tag{2.16}$$

for some holomorphic functions $s\colon \widetilde{U} \to \mathbb{C}^\times$ and $g\colon \widetilde{U} \times \widetilde{U} \to \mathfrak{g} \otimes \mathfrak{g}$. Second, setting $x = f(v_1)$ and $y = f(v_2)$, where f is the solution to the differential equation $f'(y) = s(f(y))$, we remove the coefficient in front of the Casimir element:

$$r(f(v_1), f(v_2)) = \frac{\Omega}{v_1 - v_2} + h(v_1, v_2).$$

Finally, solving another differential equation we obtain a gauge transformation $\varphi\colon V \to \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ such that

$$X(v_1 - v_2) := (\varphi(v_1) \otimes \varphi(v_2)) r(f(v_1), f(v_2)), \tag{2.17}$$

where $X\colon V \to \mathfrak{g} \otimes \mathfrak{g}$ is a meromorphic solution to

$$[X^{12}(v_1 - v_2), X^{13}(v_1 - v_3)] + [X^{12}(v_1 - v_2), X^{23}(v_2 - v_3)] + [X^{13}(v_1 - v_3), X^{23}(v_2 - v_3)] = 0. \tag{2.18}$$

Meromorphic functions in one variable solving Eq. (2.18) we thoroughly studied in [7]. The following statement is the famous trichotomy result for such solutions.

Theorem 2.1.5. *Let $V \subseteq \mathbb{C}$ be a connected neighbourhood of 0 and $r\colon V \to \mathfrak{g} \otimes \mathfrak{g}$ be a non-degenerate solution to Eq. (2.18). Then r is skew-symmetric, its poles form a lattice $\Gamma \subset \mathbb{C}$ and exactly one of the following three cases occurs:*

1. $\mathrm{rank}(\Gamma) = 2$. *Such solutions are called* elliptic*;*
2. $\mathrm{rank}(\Gamma) = 1$. *In this case there exists a rational function $f\colon V \to \mathfrak{g} \otimes \mathfrak{g}$ and a constant $\lambda \in \mathbb{C}$ such that $r(z)$ is gauge equivalent to $f(e^{\lambda z})$. Such r is called* trigonometric*;*
3. $\mathrm{rank}(\Gamma) = 0$ *and r is equivalent to a rational function $f\colon V \to \mathfrak{g} \otimes \mathfrak{g}$.*

Example 2.1.6. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$.

1. Elliptic solution of R. Baxter

$$r(z) = \frac{\mathrm{cn}(z)}{\mathrm{sn}(z)} h \otimes h + \frac{1 + \mathrm{dn}(z)}{\mathrm{sn}(z)}(e \otimes f + f \otimes e) + \frac{1 - \mathrm{dn}(z)}{\mathrm{sn}(z)}(e \otimes e + f \otimes f);$$

2. Trigonometric solution of R.Baxter

$$r(z) = \frac{\cot(z)}{2} h \otimes h + \frac{1}{\sin(z)}(e \otimes f + f \otimes e);$$

3. Rational solution of C. Yang

$$r(z) = \frac{1}{z}\left(\frac{1}{2} h \otimes h + e \otimes f + f \otimes e\right). \qquad \Diamond$$

Example 2.1.7. The two-parameter solution $\Omega/(x-y)$ from Example 2.1.1 clearly corresponds to a rational solution. It was shown in [13] that applying Theorems 2.1.4 and 2.1.5 to $y\Omega/(x-y) + r_{DJ}$ we get a trigonometric solution. $\Diamond$

A complete classification of elliptic and trigonometric solutions were achieved in the same paper [7]. These results are also stated in Section 7 of Paper II, so we do not repeat them here. However, to simplify further narration we should mention, that the authors of [7] present a list (BD list) of trigonometric solutions. These solutions are completely determined by: 1) an automorphism ν of the Dynkin diagram of $\mathfrak{g}$; 2) a BD quadruple $Q = (\Gamma_1, \Gamma_2, \gamma, t_0)$. It is then shown, that any non-trivial trigonometric solution of Eq. (2.17) is (holomorphically) gauge equivalent to a non-zero multiple of some solution from the list. We will denote the trigonometric solution from the list corresponding to a Dynkin diagram automorphism ν and a BD quadruple Q by X^ν_Q. When Q is the trivial quadruple $(\emptyset, \emptyset, \emptyset, 0)$ we will simply write X^ν_0.

Remark 2.1.8. One can check by direct calculation that the meromorphic function

$$\frac{y\Omega}{x-y} + t, \tag{2.19}$$

for $t \in \mathfrak{g}\otimes\mathfrak{g}$ is a (trigonometric) solution to CYBE if and only if $t = a+s$ is a quasi-triangular r-matrix with the symmetric part $s = \Omega$. Therefore, the classification of trigonometric solutions includes the classification of quasi-triangular r-matrices with a non-zero symmetric part, mentioned in Remark 1.1.12.

Similarly, the function

$$\frac{\Omega}{x-y} + t, \tag{2.20}$$

for $t \in \mathfrak{g}\otimes\mathfrak{g}$ is a (rational) solution if and only if t is a triangular r-matrix. As we now know, the classification of triangular r-matrices is a wild problem, so the class of rational solutions cannot be fully classified. However, there are strong structural results for such solutions developed in [18, 19]. $\Diamond$

2.1.1 Subtleties of meromorphic solutions

Dependence on changes in the domain. Given a solution to the CYBE of the form Eq. (2.16), solutions to the differential equations leading to the difference-dependent form Eq. (2.17) are unique. Meaning that assignment of one of the classes is unique. However, it is unclear why two different forms Eq. (2.16) for r, defined on disjoint subsets U_1 and U_2 of U, should lead to the same type of solution. In other words, the solution type may depend on how we translate and shrink the domain. To avoid this problem we can simply start with solutions, which are already in the form

$$\frac{s(y)\Omega}{x-y} + g(x,y), \tag{2.21}$$

for holomorphic functions $s\colon U \to \mathbb{C}^\times$ and $g\colon U \times U \to \mathbb{C}\times\mathbb{C}$, where $U \subseteq \mathbb{C}$ is an open connected neighbourhood of 0.

Dependence on the field. There is no strong reason to consider solutions of the CYBE only for the field of complex numbers. Historically, this was done in order to use the tools of complex analysis. In particular, the trichotomy result Theorem 2.1.5 depends heavily on the field of complex numbers. At the same time, two solutions we have seen

$$\frac{\Omega}{x-y} \quad \text{and} \quad \frac{y\Omega}{x-y} + r_{DJ}$$

make sense for any finite-dimensional semi-simple Lie algebra $\mathfrak{g}$ over a field F of characteristic 0. And they both produce pseudo quasi-triangular Lie bialgebra structures on $\mathfrak{g}[x]$. For that reason we want to consider functions defined over any field of characteristic 0.

Convergence. We obtained a Lie bialgebra structure dr on $\mathfrak{g}[x]$ with the help of $r(x,y) = \Omega/(x-y)$ by defining pointwise

$$dr(f)(x,y) = [f(x) \otimes 1 + 1 \otimes f(y), r(x,y)].$$

This approach fails if we try to define in the same way a Lie bialgebra structure on the Lie algebra of Taylor series $\mathfrak{g}[\![x]\!]$, because we cannot in general evaluate elements $f \in \mathfrak{g}[\![x]\!]$ even over $\mathbb{C}$. Viewing r as a series and computing formally $[f \otimes 1 + 1 \otimes f, r]$ in the space $(\mathfrak{g} \otimes \mathfrak{g})[\![x,y]\!][(x-y)^{-1}] \subseteq (\mathfrak{g} \otimes \mathfrak{g})(\!(x)\!)[\![y]\!]$ does not lead to such a problem and yields a Lie bialgebra structure on $\mathfrak{g}[\![x]\!]$.

Equivalences. Lie bialgebras are algebraic structures and hence we want an algebraic notion of equivalence. Assume we have two Lie bialgebra structures dr_1 and dr_2 on $\mathfrak{g}[x]$ given by two meromorphic functions r_1 and r_2 of the form Eq. (2.21). If these functions are gauge equivalent, i.e. $(\varphi(x) \otimes \varphi(y))r_2(x,y) = r_1(x,y)$ for some holomorphic φ, then, in general, this gives us no information about the relation between dr_1 and dr_2. To apply φ to $\mathfrak{g}[x]$ we need to express it as a formal power series. And even if we do that, there is no guarantee that it will preserve $\mathfrak{g}[x]$.

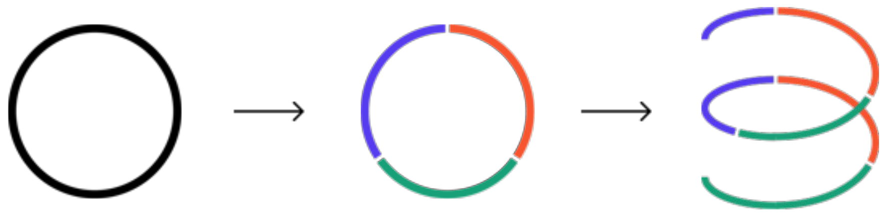
2.2 Paper I

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ and σ be its automorphism of finite order $|\sigma| = m$. The eigenvalues of σ are $\varepsilon_\sigma^k = \exp(2\pi i k/m)$, $1 \leq k \leq m$, and we have the following splitting of $\mathfrak{g}$ into a direct sum of eigenspaces under the action of σ

$$\mathfrak{g} = \bigoplus_{k=0}^{m-1} \mathfrak{g}_k^\sigma. \tag{2.22}$$

Denote by $\mathfrak{L}$ the Lie algebra of Laurent polynomials $\mathfrak{g}[x,x^{-1}] := \mathfrak{g} \otimes \mathbb{C}[x,x^{-1}]$ with coefficients in $\mathfrak{g}$. *The loop algebra* $\mathfrak{L}^\sigma$ is the subalgebra of $\mathfrak{L}$ defined by

$$\mathfrak{L}^\sigma := \bigoplus_{k\in\mathbb{Z}} z^k \mathfrak{g}_k^\sigma,$$



$$\mathfrak{g} \qquad \bigoplus_{k=0}^{|\sigma|-1} \mathfrak{g}_k^\sigma \qquad \mathfrak{L}^\sigma := \bigoplus_{k\in\mathbb{Z}} z^k \mathfrak{g}_k^\sigma$$

where we put $\mathfrak{g}_{k+\ell m}^\sigma = \mathfrak{g}_k^\sigma$ for any $\ell \in \mathbb{Z}$. In particular, $\mathfrak{L}^{\mathrm{id}} = \mathfrak{L}$.

If we fix an automorphism ν of the Dynkin diagram of $\mathfrak{g}$, and take two finite-order automorphisms σ and σ' of $\mathfrak{g}$, whose cosets are conjugate to $\nu\mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$, then the theory of loop algebras tells us exactly what the relation between $\mathfrak{L}^\sigma$ and $\mathfrak{L}^{\sigma'}$ is: it is a composition of a conjugation and an operation known as *regrading*.

It was explained in [11] that any symmetrizable Kac-Moody algebra $\mathfrak{K}(A)$ admits a so-called *standard Lie bialgebra structure* δ_0. If we fix a non-degenerate symmetric invariant bilinear form B on $\mathfrak{K}(A)$ and a set $\{X_i^-, H_i, X_i^+\} \cup \{D_i\}$ of standard generators, then δ_0 can be described as

$$\delta(H_i) = 0, \quad \delta_0(D_i) = 0 \;\text{ and }\; \delta(X_i^\pm) = \frac{B(\alpha_i^\vee, \alpha_i^\vee)}{2} H_i \wedge X_i^\pm.$$

Assume A is an affine matrix of type $X_N^{(m)}$, $\mathfrak{g}$ is a Lie algebra of type X_N and ν is an automorphism of the corresponding Dynkin diagram of order m. Then the loop algebra $\mathfrak{L}^\nu$ is isomorphic to the quotient $[\mathfrak{K}(A), \mathfrak{K}(A)]/Z(\mathfrak{K}(A))$ of the derived algebra of $\mathfrak{K}(A)$ by its center. Consequently, the loop algebra $\mathfrak{L}^\nu$ inherits the standard Lie bialgebra structure δ_0^ν from $\mathfrak{K}(A)$. Using conjugation and regrading we can then push this structure to $\mathfrak{L}^\sigma$ for any finite-order automorphism σ of $\mathfrak{g}$ whose coset is conjugate to $\nu\mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$. In this way, any loop algebra $\mathfrak{L}^\sigma$ is equipped with a standard Lie bialgebra structure δ_0^σ.

As we already know, these structures can be twisted. One of the main results of Paper I is the classification of classical twists of the standard Lie bialgebra structure δ_0^σ on the loop algebra $\mathfrak{L}^\sigma$. The process looks as follows.

First, we show that for any finite-order automorphism σ the corresponding standard structure δ_0^σ is pseudo quasi-triangular. More precisely, for each σ we define a meromorphic solution of CYBE $r_0^\sigma \colon \mathbb{C} \times \mathbb{C} \to \mathfrak{g} \otimes \mathfrak{g}$ such that

$$\delta_0^\sigma(f)(x, y) = [f(x) \otimes 1 + 1 \otimes f(y), r_0^\sigma(x, y)] \;\; \forall f \in \mathfrak{L}^\sigma, \; \forall x, y \in \mathbb{C}^\times.$$

Since the equality Eq. (1.22) and the discussion after it holds verbatim for pseudo quasi-triangular structures, twisted structures $\delta_t^\sigma := \delta_0^\sigma + t$ are defined by r-matrices $r_t^\sigma := r_0^\sigma + t$. We name these r-matrices σ-*trigonometric*.

Example 2.2.1. For $\sigma = \mathrm{id}$ we get the familiar r-matrix from Example 2.1.3

$$r_0^{\mathrm{id}} = \frac{y\Omega}{x - y} + \frac{\Omega_{\mathfrak{h}}}{2} + \Omega_-.$$

And when σ is the so-called *Coxeter automorphism* σ_1 corresponding to a Dynkin diagram automorphism ν, the function $r_0^{\sigma_1}$ corresponds to the trigonometric solution X_0^ν in the BD list [7, Eq. 6.1]. More precisely,

$$r_0^{\sigma_1}\left(e^{u/|\sigma_1|}, e^{v/|\sigma_1|}\right) = X_0^\nu(u-v).$$

$\Diamond$

Second, we show that the operations of regrading and conjugation, that relate two loop algebras $\mathfrak{L}^\sigma$ and $\mathfrak{L}^{\sigma'}$ can be performed at the level of the corresponding σ-trigonometric r-matrices.

$$\begin{tikzcd}
\mathfrak{L}^\sigma \arrow[r, leftrightarrow, "\substack{\text{regrading}\\ \text{conjugation}}"] \arrow[d, dotted, leftrightarrow] \arrow[dr, phantom, "\Downarrow"] & \mathfrak{L}^{\sigma'} \arrow[d, dotted, leftrightarrow] \\
r^\sigma \arrow[r, leftrightarrow] & r^{\sigma'}
\end{tikzcd}$$

Third, using regrading and conjugation at the level of σ-trigonometric r-matrices we push the BD list from Coxeter case to any other finite-order automorphism σ. More precisely, start with a Lie bialgebra structure δ_t^σ on $\mathfrak{L}^\sigma$ with the corresponding σ-trigonometric solution r_t^σ. Let ν be the automorphism of the Dynkin diagram of $\mathfrak{g}$ lying in the same coset $\sigma\mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$. We start by showing that there exists a trigonometric r-matrix $X \colon \mathbb{C} \to \mathfrak{g}\otimes\mathfrak{g}$ and a holomorphic function $\varphi_1 \colon \mathbb{C} \to \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ such that

$$(\varphi_1(u) \otimes \varphi_1(v)) r_0^\sigma\left(e^{u/|\sigma|}, e^{v/|\sigma|}\right) = X(u-v).$$

Since any trigonometric solution to Eq. (2.18) is (holomorphically) gauge equivalent to a multiple (in this case 1) of some solution from the BD list, there exists a BD quadruple Q and a holomorphic function $\varphi_2 \colon \mathbb{C} \to \mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ such that

$$(\varphi_2(u) \otimes \varphi_2(v)) X(u-v) = X_Q^\nu(u-v) = r_Q^{\sigma_1}\left(e^{u/|\sigma_1|}, e^{v/|\sigma_1|}\right).$$

We make an observation that the holomorphic maps φ_1 and φ_2 actually take their values in $\mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$. This implies that σ_1 is the Coxeter automorphism from the same coset $\sigma\mathrm{Aut}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$. Therefore, we can now regrade σ_1 back to σ by applying another holomorphic $\varphi_3 \colon \mathbb{C} \to \mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$, namely

$$(\varphi_3(u) \otimes \varphi_3(v)) r_Q^{\sigma_1}\left(e^{u/|\sigma_1|}, e^{v/|\sigma_1|}\right) = r_Q^\sigma\left(e^{u/|\sigma|}, e^{v/|\sigma|}\right).$$

Composing these gauge equivalences we obtain a holomorphic map $\varphi \colon \mathbb{C} \to \mathrm{Inn}_{\mathbb{C}-\mathrm{LieAlg}}(\mathfrak{g})$ making the following (holomorphic) transformation

$$(\varphi(u) \otimes \varphi(v)) r_t^\sigma\left(e^{u/|\sigma_1|}, e^{v/|\sigma_1|}\right) = r_Q^\sigma\left(e^{u/|\sigma|}, e^{v/|\sigma|}\right).$$

Thus we now have a copy of BD list at each σ.

We have explained in Section 2.1.1 that holomorphic equivalences are not suitable for purposes of classification of Lie bialgebra structures, since in general they do not produce

$$
\begin{tikzcd}
 & (\mathfrak{L}^\sigma, \delta^\sigma) \arrow[rr, leftrightarrow, "\text{regular}"] \arrow[dl, dotted, leftrightarrow] & \arrow[d, phantom, "\Updownarrow"] & (\mathfrak{L}^\sigma, \delta^\sigma_Q) \arrow[dr, dotted, leftrightarrow] & \\
r^\sigma(x,y) \arrow[rrrr, leftrightarrow, "\text{regular}"] \arrow[dr, leftrightarrow, "\text{holomorphic}"] & & & & r^\sigma_Q(x,y) \arrow[dl, leftrightarrow, "\text{holomorphic}"'] \\
 & X(x-y) \arrow[rr, leftrightarrow, "\text{holomorphic}"'] & \arrow[u, phantom, "\Uparrow \text{geometry}"] & r^{\sigma_1}_Q(x,y) &
\end{tikzcd}
$$

isomorphisms of Lie algebras. The automorphisms we are interested in are called *regular*. These are elements in $\mathrm{Aut}_{\mathbb{C}[x^{|\sigma|},x^{-|\sigma|}]-\mathrm{LieAlg}}(\mathfrak{L}^\sigma)$. The final step is to show that the existence of a holomorphic gauge equivalence φ between r^σ_t and r^σ_Q guarantees the existence of a regular gauge equivalence ϕ. This is done using the gemetric theory of CYBE [2, 9].

The regular equivalence ϕ gives the desired equivalence of the corresponding Lie bialgebra structures

$$\delta^\sigma_Q \phi = (\phi \otimes \phi)\delta^\sigma_t.$$

In this way, we have shown that all twists of the standard Lie bialgebra structure δ^σ_0 on $\mathfrak{L}^\sigma$ up to regular equivalence can be described by BD quadruples Q. Solutions of the form r^{id}_t, $t \in (\mathfrak{g} \otimes \mathfrak{g})[x,y]$, are called quasi-trigonometric. Our result gives an alternative proof of their classification presented in [17].

3 | Topological Lie bialgebras and formal solutions

In Paper II we introduce the notion of a topological Lie bialgebra. Roughly speaking, we equip $\mathfrak{g}$ with a topology and allow a Lie bialgebra structure on $\mathfrak{g}$ to be a continuous map $\delta\colon \mathfrak{g} \to \widehat{\mathfrak{g} \otimes \mathfrak{g}}$, where the codomain is a completion of $\mathfrak{g} \otimes \mathfrak{g}$ with respect to some linear topology. If $\mathfrak{g}$ is equipped with the discrete topology, the notion coincides with the classical one presented in Section 1.1.

The notion of a topological Lie bialgebra can be motivated in different ways. Our original motivation comes from paper [16], where the authors study Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$, for a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Instead of taking the algebraic dual $\mathfrak{g}[\![x]\!]^\vee$, which is uncountable, they take

$$\mathfrak{g}[\![x]\!]' := \{\text{linear } f\colon \mathfrak{g}[\![x]\!] \to \mathbb{C} \mid f(x^n\mathfrak{g}[\![x]\!]) = 0 \text{ for some } n > 0\}. \tag{3.1}$$

The corresponding "double" is defined as the space $\mathfrak{g}[\![x]\!] \dotplus \mathfrak{g}[\![x]\!]'$ with the same Lie bracket and bilinear form as in the classical case. By Theorem 1.2.3 we know that it has the form $\mathfrak{g} \otimes \widetilde{A}$, where $\widetilde{A}$ is an algebra extension of $F[\![x]\!]$ with a countable basis over F. Classifying such extensions they obtain a beautiful result: up to some notion of equivalence there are only three non-trivial doubles

$$\mathfrak{g}(\!(x)\!), \quad \mathfrak{g}(\!(x)\!) \times \mathfrak{g}, \quad \text{and} \quad \mathfrak{g}(\!(x)\!) \times \mathfrak{g}[x]/x^2\mathfrak{g}[x]. \tag{3.2}$$

This result is then used for classification of Lie bialgeba structures on $\mathfrak{g}[x]$ and their classical twists. For more information we refer to the introduction of Paper II.

Remark 3.0.1. We always have the trivial Lie bialgebra structure $\delta = 0$ on any Lie algebra. The corresponding double is called trivial or degenerate. All triangular Lie bialgebra structures fall into that double. $\Diamond$

Another possible motivation is given through formal r-matrices which we now present.

3.1 Formal r-matrices

Let $\mathfrak{g}$ be a finite-dimensional semi-simple Lie algebra over an algebraically closed field F of characteristic 0. A series of the form

$$r(x,y) = \frac{s(y)\Omega}{x-y} + g(x,y) = s(y)\Omega \sum_{k\geq 0} x^{-k-1}y^k + g(x,y) \in (\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!], \tag{3.3}$$

where $s \in F[\![y]\!]$ and $g \in (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$ is called a *formal r-matrix* if it solves the (formal) classical Yang-Baxter equation, i.e. $\mathrm{CYB}(r) = 0$.

We explained in Section 2.1 that any meromorphic solutions to the CYBE locally is of the form $s(y)\Omega/(x - y) + g(x, y)$, where s and g are some holomorphic functions. Taking its Taylor series expansion at $y = 0$ we obtain a formal r-matrix. Moreover, to define a Lie bialgebra structure, using a meromorphic solution, we need to know only its local behaviour. Therefore, by looking at formal solutions instead of analytical ones we do not loose any pseudo quasi-triangular Lie bialgebra structure. Instead, there are formal solutions to the CYBE having 0 radius of convergence. Furthermore, there is no restriction on the field anymore. In this way, we avoid most of the problems analytical solutions have and extend the class of pseudo quasi-triangular Lie bialgebra structures.

To each formal r-matrix r we associate another formal power series $\overline{r}$ by

$$\overline{r}(x, y) = \frac{s(x)\Omega}{x - y} - \tau(g(y, x)), \tag{3.4}$$

where τ stands for $F[\![x, y]\!]$-linear extension of the map $\tau(a \otimes b) = b \otimes a$. We call a formal r-matrix *skew-symmetric* if $r = \overline{r}$.

Remark 3.1.1. Note that a series of the form

$$\frac{u(x, y)\Omega}{x - y} + g(x, y),$$

where $s \in F[\![x]\!]$ can be easily rewritten in the standard form Eq. (3.3). Indeed,

$$\begin{aligned} \frac{u(x, y)\Omega}{x - y} + g(x, y) &= \frac{u(y, y)\Omega}{x - y} + \frac{u(x, y) - u(y, y)}{x - y} + g(x, y) \\ &= \frac{s(y)\Omega}{x - y} + h(x, y). \end{aligned}$$

Here we used the fact, that if $T \in (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$ satisfies $T(x, x) = 0$, then there exists $H \in (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$ such that $T(x, y) = (x - y)H(x, y)$. $\Diamond$

Motivated by the proof of skew-symmetry for analytical solutions, we prove in Paper II that any formal r-matrix is automatically skew-symmetric. In the classical scenario, if we have a triangular r-matrix, then by Lemma 1.1.14 it defines some Lagrangian Lie algebra splitting as well as a Lie bialgebra structure on an appropriate Lie algebra. Let us check if the same is true for a formal r-matrix.

Fix a basis $\{b_i\}_{i=1}^d$ for $\mathfrak{g}$ orthonormal with respect to its Killing form. Given a formal r-matrix r with $s \in F[\![y]\!]^\times$ we can write it in the form

$$r(x, y) = \sum_{k \geq 0} \sum_{i=1}^{d} r_{k,i}(x) \otimes b_i y^k, \tag{3.5}$$

where $r_{k,i} \in \mathfrak{g}(\!(x)\!)$. Define the subspace

$$\mathfrak{g}(r) := \mathrm{span}_F\{r_{k,i} \mid k \geq 0, 1 \leq i \leq d\} \subseteq g(\!(x)\!). \tag{3.6}$$

We show in Paper II, that $\mathfrak{g}(r)$ is a subalgebra of $\mathfrak{g}(\!(x)\!)$ complementary to $\mathfrak{g}[\![x]\!]$, i.e.

$$\mathfrak{g}(\!(x)\!) = \mathfrak{g}[\![x]\!] \dotplus \mathfrak{g}(r).$$

Moreover, equipping $\mathfrak{g}(\!(x)\!)$ with the form

$$B(a \otimes f, b \otimes g) = \kappa(a,b)\mathrm{coeff}_{-1}\{fg\}, \tag{3.7}$$

both $\mathfrak{g}[\![x]\!]$ and $\mathfrak{g}(r)$ become Lagrangian subalgebras. Consequently, $(\mathfrak{g}(\!(x)\!), \mathfrak{g}[\![x]\!], \mathfrak{g}(r))$ is a Manin triple. This can be viewed as a topological analogue for Lemma 1.1.14. However, the dual of the Lie bracket on $\mathfrak{g}(r)$ does not produce a map $\mathfrak{g}[\![x]\!] \to \mathfrak{g}[\![x]\!] \otimes \mathfrak{g}[\![x]\!]$, but a map $\mathfrak{g}[\![x]\!] \to (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$. This can be seen, for example, by letting f in the formula Eq. (2.4) be a formal power series. In other words, dr is not a Lie bialgebra on $\mathfrak{g}[\![x]\!]$ in the classical sense. Moreover, $\mathfrak{g}(\!(x)\!)$ is not a classical double either, because $\mathfrak{g}(r)$ has countable dimension, not uncountable as the algebraic dual $\mathfrak{g}[\![x]\!]^\vee$.

To fix this issue, we equip $\mathfrak{g}[\![x]\!]$ with the (x)-adic topology and the tensor product $\mathfrak{g}[\![x]\!] \otimes \mathfrak{g}[\![x]\!]$ with the unique linear topology making $\otimes$-map continuous. Then the completion of the tensor product $\mathfrak{g}[\![x]\!] \otimes \mathfrak{g}[\![x]\!]$ is exactly $(\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$. The continuous dual $\mathfrak{g}[\![x]\!]'$ coincides with the space Eq. (3.1). We equip it with the discrete topology. A linear map

$$\delta \colon \mathfrak{g}[\![x]\!] \to (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$$

is called a *topological Lie bialgebra structure* on $\mathfrak{g}[\![x]\!]$ if δ is a 1-cocycle and the dual map $\delta' \colon \mathfrak{g}[\![x]\!]' \otimes \mathfrak{g}[\![y]\!]' \to \mathfrak{g}[\![x]\!]'$ is a Lie bracket. In this setting we get a very beautiful result.

Lemma 3.1.2. *Any formal r-matrix r defines a topological Lie bialgebra structure dr on the Lie algebra $\mathfrak{g}[\![x]\!]$.*

In Paper II we also introduce the notion of a topological double for a topological Lie bialgebra structure. We know that in the finite-dimensional case the construction of a classical double gives a bijection between Manin triples and Lie bialgebra structure. This bijection is not true as soon as we take into account infinite-dimensional Lie algebras. We explained in Section 1.1.2 that every Lie bialgebra produces a Manin triple, but not any Manin triple gives rise to a Lie bialgebra structure. Making one more step forward to the topological setting, these relations are completely lost in general. It is not true anymore, that a topological Lie bialgebra gives rise to a (topological) Manin triple. Luckily, when we consider topological Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$ this is still the case. For each such structure we can define a topological double $\mathfrak{D}(\mathfrak{g}[\![x]\!], \delta)$, with a commutator map and a form given by the same Eqs. (1.14) and (1.15) and a (topological) Manin triple $(\mathfrak{D}(\mathfrak{g}[\![x]\!], \delta), \mathfrak{g}[\![x]\!], \mathfrak{g}[\![x]\!]')$.

From this viewpoint, paper [16] becomes the very first step in our classification scheme, namely, the authors classify all topological doubles for Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$. Furthermore, they find pseudo quasi-triangular representatives within each of the non-trivial doubles:

$$\frac{\Omega}{x-y}, \quad \frac{y\Omega}{x-y} + r_{DJ} \text{ and }, \ \frac{y^2\Omega}{x-y} + y\Omega \tag{3.8}$$

Having such a picture in front of you, it is tempting to extend the notion of a twist and apply the same classification scheme as in Section 1.2. This can be considered as the start of the main part of Paper II.

3.2 Paper II

Let δ be a topological Lie bialgebra structure on $\mathfrak{g}[\![x]\!]$. A topological twist is simply a skew-symmetric element $t \in (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$ of the completed tensor product satisfying the same relation as in the classical case, namely

$$\mathrm{CYB}(t) = \circlearrowleft (\delta \otimes 1)t.$$

We prove the following analogue of Theorem 1.1.16.

Theorem 3.2.1. *Let $(\mathfrak{g}[\![x]\!], \delta)$ be a topological Lie bialgebra with the classical double $\mathfrak{D}$. Then there are the following one-to-one correspondences:*

$$\begin{tikzcd}
& \boxed{\textit{Topological twists of } \delta} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\boxed{\begin{array}{c}\textit{Lagrangian} \\ \textit{Lie subalgebras } W \subseteq \mathfrak{D} \\ \textit{complementary to } \mathfrak{g}[\![x]\!]\end{array}} \arrow[rr, leftrightarrow] & & \boxed{\begin{array}{c}\textit{Linear maps } T \colon \mathfrak{g}[\![x]\!]' \longrightarrow \mathfrak{g}[\![x]\!] \textit{ such that} \\ \textit{for all } p_1, p_2, p_3 \in \mathfrak{g}[\![x]\!]' \\ B(Tp_1, p_2) + B(p_1, Tp_2) = 0 \textit{ and} \\ B([Tp_1 - p_1, Tp_2 - p_2], Tp_3 - p_3) = 0\end{array}}
\end{tikzcd}$$

If W defines a topological structure δ, then the twist W_t of W by t defines the topological structure $\delta + dt$. And similarly, if $\delta = dr$ for some formal r-matrix r, then $r + t$ is a formal r-matrix if and only if t is a topological twist of δ. Therefore, topological twists behave in the exact same way as classical twists. Furthermore, since we have pseudo quasi-triangular representatives in each non-trivial twisting class, given by r-matrices Eq. (3.8), we obtain the converse of Lemma 3.1.2.

Lemma 3.2.2. *For each topological Lie bialgebra structure δ on $\mathfrak{g}[\![x]\!]$ there exists a formal r-matrix, such that $\delta = dr$.*

In other words, we have a picture similar to the one described in Theorem 1.1.9. The classification of topological Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$ reduces completely to the classification of formal r-matrices.

We have seen that formal r-matrices

$$r(x, y) = \frac{s(y)\Omega}{x - y} + g(x, y)$$

with $s \in F[\![y]\!]^\times$ lead to Lagrangian subalgebras in the first double $\mathfrak{g}(\!(x)\!)$. We prove that if $s \in y^m F[\![y]\!]^\times$, then it gives rise to a Lagrangian splitting of $\mathfrak{g}(\!(x)\!) \times \mathfrak{g}[x]/x^m$. If $m > 2$, such a splitting would contradict the fact, that there are only 3 non-trivial doubles. This imposes an interesting limitation on formal r-matrices.

Lemma 3.2.3. *Let*

$$r(x,y) = \frac{s(y)\Omega}{x-y} + g(x,y)$$

be a formal r*-matrix with* $s \in y^m F[\![y]\!]$*, then* $m \in \{0,1,2\}$*.*

As a consequence, there are three classes of formal r-matrices, corresponding to $m = 0, 1$ and 2 that we need to describe. We remind that the base field F is an arbitrary algebraically closed field of characteristic 0.

Formal r-matrices with $m = 0$ were thoroughly studied and classified in [1]. Such formal r-matrices, after an appropriate coordinate transformation $x \mapsto x + a_2x^2 + \ldots$, $a_i \in F$, are of the form

$$\frac{\Omega}{x-y} + g(x,y), \tag{3.9}$$

for some $g \in (\mathfrak{g} \otimes \mathfrak{g})[\![x,y]\!]$. Such solutions were geometrized in [1], i.e. it was shown how to attach a certain geometric datum to them. In particular, every such solution is associated with a sheaf of Lie algebras on a projective curve X of arithmetic genus 1, constructed using the subalgebra $\mathfrak{g}(r) \subset \mathfrak{g}(\!(x)\!)$. This results in a trichotomy of such solutions, namely X is either elliptic or has a singularity, which is either nodal or cuspidal. In the special case $F = \mathbb{C}$ one restores Theorem 2.1.5. More precisely, it is shown that any formal r-matrix is formally equivalent to a Taylor series expansion at $y = 0$ of a meromorphic r-matrix and that this correspondence respects the classes.

In Paper II we use similar geometrization techniques for cases $m = 1$ and 2 to show that Lagrangian subalgebras W of $\mathfrak{g}(\!(x)\!) \times \mathfrak{g}[x]/x^m\mathfrak{g}[x]$, complementary to the diagonal embedding $\Delta \coloneqq \{(f,f) \mid f \in \mathfrak{g}[\![x]\!]\}$ of $\mathfrak{g}[\![x]\!]$, carry a structure of an $F[x^{-1}]$-module. Using this fact one can prove that up to an appropriate equivalence any such W is bounded. This means that topological Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$ with doubles $\mathfrak{g}(\!(x)\!) \times \mathfrak{g}$ and $\mathfrak{g}(\!(x)\!) \times \mathfrak{g}[x]/x^2\mathfrak{g}[x]$ correspond respectively to formal r-matrices of the forms

$$\frac{y\Omega}{x-y} + p(x,y) \quad \text{and} \quad \frac{y^2\Omega}{x-y} + q(x,y),$$

for some polynomials $p, q \in (\mathfrak{g} \otimes \mathfrak{g})[x,y]$. Solutions of the first type are precisely the quasi-trigonometric solutions. For $F = \mathbb{C}$ they were classified in [17] and Paper I. Solutions of the second kind are called quasi-rational. They are described in [21]. This completes the classification of topological Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$ with non-trivial doubles.

3.3 Generalizations and Paper III

Let F be again an algebraically closed field of characteristic 0 and $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over F with an orthonormal basis $\{b_i\}_{i=1}^d$. We equip $\mathfrak{g}[\![x]\!]$ with the (x)-adic topology.

Results of Paper II tell us that a topological Lie algebra L

- equipped with a separately continuous non-degenerate invariant symmetric bilinear form B
- containing $\mathfrak{g}[\![x]\!]$ as a Lagrangian (topological) Lie subalgebra

- and satisfying $\mathfrak{g}[\![x]\!]' \subseteq B(L,\cdot)$

is necessarily of the form $L(n,\alpha)$ or $L(\infty)$. Here

$$L(n,\alpha) \coloneqq \mathfrak{g}(\!(x)\!) \times \mathfrak{g}[x]/x^n\mathfrak{g}[x]$$

and it is equipped with a bilinear form

$$B\big(a \otimes (f,[p]), b \otimes (g,[q])\big) = \kappa(a,b)\,\mathrm{t}_\alpha(fg - pq)$$

completely determined by a sequence $\alpha = (\alpha_i \in F \mid -\infty < i \leq n-2)$, $a_i \in F$, and $L(\infty)$ is the double of the trivial topological Lie bialgebra structure $\delta = 0$ on $\mathfrak{g}[\![x]\!]$. The latter Lie algebra admits no clear additional structure so we ignore it as we did while classifying topological Lie bialgebra structures.

Remark 3.3.1. Lie algebras L satisfying the conditions presented above are called *topological Manin pairs* over $\mathfrak{g}[\![x]\!]$. In the classical case each Manin triple is a Manin pair by forgetting the third component. However, by our conventions, topological Manin triples are not necessarily topological Manin pairs because we do not require from Manin triples to satisfy the last inclusion. $\lozenge$

By this point we have seen that Lagrangian Lie subalgebras of $L(n,\alpha)$ complementary to

$$\Delta = \{(f,[f]) \mid f \in \mathfrak{g}[\![x]\!]\} \cong \mathfrak{g}[\![x]\!].$$

exist only for $n \in \{0,1,2\}$ and they correspond on one hand to topological Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$, on the other hand to formal r-matrices, which in some sense are generating functions for such subalgebras.

Instead of looking at Lagrangian Lie subalgebras, we can look separately at Lagrangian subspaces and Lie subalgebras of $L(n,\alpha)$ complementary to Δ. This leads to a number of interesting questions: Do they exist for each n? What are the corresponding algebraic structures? What conditions do their generating functions satisfy? These are the main questions considered in Paper III.

We start by defining *series of type* (n,s). These are series of the form

$$\frac{s(x)y^n\Omega}{x-y} + g(x,y).$$

We have already met such series before, but this time we interpret them not as elements in $(\mathfrak{g}\otimes\mathfrak{g})(\!(x)\!)[\![y]\!]$ but in $(L(n,\alpha)\otimes\mathfrak{g})[\![(y,[y])]\!]$. More precisely, we have

$$\frac{y^n\Omega}{x-y} = \sum_{k=0}^{n-1}\sum_{i=1}^{d} b_i(0,-[x]^{(n-1)-k}) \otimes b_i(y^k,[y]^k) + \sum_{k=n}^{\infty}\sum_{i=1}^{d} b_i(x^{(n-1)-k},0) \otimes b_i(y^k,0). \tag{3.10}$$

Since $((L(n,\alpha)\otimes\mathfrak{g})[\![(y,[y])]\!]$ is an $F[x] \cong F[(x,[x])]$-module and $(\mathfrak{g}\otimes\mathfrak{g})[\![x,y]\!] \cong (\Delta \otimes \mathfrak{g})[\![(y,[y])]\!]$, the series

$$\frac{s(x)y^n\Omega}{x-y} + g(x,y),$$

with $g \in (\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$ and $s \in F[x]^\times$ is again in $((L(n, \alpha) \otimes \mathfrak{g})[\![(y, [y])]\!]$. If $n = 0$ we get a series of the form Eq. (3.3) discussed earlier.

For each (n, s)-type series r we define another (n, s)-type series $\overline{r}$:

$$\overline{r}(x, y) := \frac{s(y)x^n\Omega}{x - y} - \tau(g(y, x)) \in (L(n, \alpha) \otimes \mathfrak{g})\,[\![(y, [y])]\!], \tag{3.11}$$

where τ is the $F[\![x, y]\!]$-linear extension of the map $a \otimes b \mapsto b \otimes a$. A series of type (n, s) is called *skew-symmetric* if $r = \overline{r}$.

Each series of type (n, s) produces in a natural way a subspace of $L(n, \alpha)$. More precisely, we associate a series

$$\frac{s(x)y^n\Omega}{x - y} + g(x, y) = \sum_{k=0}^{\infty}\sum_{i=1}^{d} f_{k,i} \otimes b_i(y^k, [y]^k) \tag{3.12}$$

with the subspace

$$W(r) := \operatorname{span}_F\{f_{k,i} \mid k \geq 0,\ 1 \leq i \leq d\} \subseteq L(n, \alpha). \tag{3.13}$$

Again, for $n = 0$ this construction coincides with the construction Eq. (3.6) of $\mathfrak{g}(r)$.

The main theorem of Paper III describes the relation between (n, s)-type series and subspaces of $L(n, \alpha)$ complementary to Δ.

Theorem 3.3.2.

1. *W defines a bijection between series of type $\left(n, \frac{1}{x^n\alpha(x)}\right)$ and subspaces $V \subset L(n, \alpha)$ complementary to the diagonal Δ, i.e. $L(n, \alpha) = \Delta \dotplus V$;*
2. *For any series r of type $\left(n, \frac{1}{x^n\alpha(x)}\right)$ we have $W(r)^\perp = W(\overline{r})$ inside $L(n, \alpha)$;*
3. *W defines a bijection between skew-symmtric $\left(n, \frac{1}{x^n\alpha(x)}\right)$-type series and Lagrangian subspaces $V \subset L(n, \alpha)$ complementary to the diagonal Δ;*
4. *W defines a bijection between $\left(n, \frac{1}{x^n\alpha(x)}\right)$-type series solving* the generalized Yang-Baxter equation

   $$\mathrm{GCYB}(r) := [r^{12}(x_1, x_2), r^{13}(x_1, x_3)] + [r^{12}(x_1, x_2), r^{23}(x_2, x_3)] + [r^{13}(x_1, x_3), \overline{r}^{23}(x_2, x_3)] = 0$$

   and subalgebras $V \subset L(n, \alpha)$ complementary to the diagonal Δ.

In other words, if we split the condition of being a Lagrangian subalgebra into being a Lagrangian subspace and a Lie subalgebra, the classical Yang-Baxter equation splits into skew-symmetry condition and the generalized Yang-Baxter equation.

Remark 3.3.3. To restore the original $\left(n, \frac{1}{x^n\alpha(x)}\right)$-type series from a subspace W of $L(n, \alpha)$ complementary to Δ, we just need to construct a basis $\{f_{i,k}\}$ of W dual to $\{b_i(y^k, [y]^k)\} \subseteq \Delta$. Then

$$r = \sum_{k \geq 0}\sum_{i=1}^{d} f_{i,k} \otimes b_i(y^k, [y]^k).$$

is the original series of type $\left(n, \frac{1}{x^n\alpha(x)}\right)$. $\Diamond$

Following ideas from [6] and [12] we define a *topological quasi-Lie bialgebra structure* on $\mathfrak{g}[\![x]\!]$ by relaxing the axioms for a topological Lie bialgebra. It then turns out, that Lagrangian subspaces of $L(n, \alpha)$ are in bijection with topological quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$.

We prove the existence of Lagrangian subspaces and Lie subalgebras for all n and α by explicitly constructing examples for each n and $\alpha = (\dots, 0, \alpha_0, 0, \dots, 0)$. These constructions can be coordinate-transformed to any other sequence α.

Topological quasi-Lie bialgebras can be twisted in a much easier way than topological Lie bialgebras. If δ is a quasi-Lie bialgebra, then so is $\delta + dt$, where t is an arbitrary skew-symmetric element in $(\mathfrak{g} \otimes \mathfrak{g})[\![x, y]\!]$. In terms of subspaces: If W is a Lagrangian subspace of $L(n, \alpha)$ defining a quasi-Lie bialgebra structure δ, then any other Lagrangian subspace of $L(n, \alpha)$ corresponds to a twist of δ. Therefore, the explicit examples of Lagrangian subspaces for each pair (n, α) give us a classification of topological quasi-Lie bialgebra structures on $\mathfrak{g}[\![x]\!]$ up to twisting.

The last part of Paper III is devoted to construction of Lie subalgebras of $L(n, \alpha)$. In particular, we construct Lie subalgebras of $L(n, \alpha)$ complementary to Δ having unbounded orthogonal complements. We hope such subalgebras will be interesting from the viewpoint of Adler-Konstant-Symes scheme.

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