Critical branching as a pure death process coming down from infinity

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CRITICAL BRANCHING AS A PURE DEATH PROCESS COMING DOWN FROM INFINITY

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Abstract

We consider the critical Galton–Watson process with overlapping generations stemming from a single founder. Assuming that both the variance of the offspring number and the average generation length are finite, we establish the convergence of the finite-dimensional distributions, conditioned on non-extinction at a remote time of observation. The limiting process is identified as a pure death process coming down from infinity.

This result brings a new perspective on Vatutin's dichotomy, claiming that in the critical regime of age-dependent reproduction, an extant population either contains a large number of short-living individuals or consists of few long-living individuals.

Keywords: Galton–Watson process with overlapping generations; Bellman–Harris process; Sevastyanov process; Crump–Mode–Jagers process; convergence of finite-dimensional distributions; Vatutin's dichotomy

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1. Introduction

Consider a self-replicating system evolving in the discrete-time setting according to the following rules:

Rule 1: The system is founded by a single individual, the founder, born at time 0.

Rule 2: The founder dies at a random age L and gives a random number N of births at random ages τ_j satisfying $1 \leq \tau_1 \leq \ldots \leq \tau_N \leq L$.

Rule 3: Each new individual lives independently from others according to the same life law as the founder.

An individual that was born at time t_1 and dies at time t_2 is considered to be alive during the time interval $[t_1, t_2 - 1]$. Letting $Z(t)$ stand for the number of individuals alive at time t, we study the random dynamics of the sequence

$$Z(0) = 1, Z(1), Z(2), \ldots,$$

which is a natural extension of the well-known Galton–Watson process, or *GW process* for short; see [13]. The process $Z(\cdot)$ is the discrete-time version of what is usually called the

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Crump–Mode–Jagers process or the general branching process; see [5]. To emphasise the discrete-time setting, we call it a GW process with overlapping generations, or *GWO process* for short.

Put $b := \frac{1}{2}\mathrm{var}(N)$. This paper deals with the GWO processes satisfying

$$\mathrm{E}(N) = 1, \quad 0 < b < \infty. \tag{1}$$

The condition $\mathrm{E}(N) = 1$ says that the reproduction regime is critical, implying $\mathrm{E}(Z(t)) \equiv 1$ and making extinction inevitable, provided $b > 0$. According to [1, Chapter I.9], given (1), the survival probability

$$Q(t) := \mathrm{P}(Z(t) > 0)$$

of a GW process satisfies the asymptotic formula $tQ(t) \to b^{-1}$ as $t \to \infty$ (this was first proven in [6] under a third moment assumption). A direct extension of this classical result for the GWO processes,

$$tQ(ta) \to b^{-1}, \quad t \to \infty, \quad a := \mathrm{E}(\tau_1 + \ldots + \tau_N),$$

was obtained in [3, 4] under the conditions (1), $a < \infty$,

$$t^2\mathrm{P}(L > t) \to 0, \quad t \to \infty, \tag{2}$$

plus an additional condition. (Notice that by our definition, $a \geq 1$, and $a = 1$ if and only if $L \equiv 1$, that is, when the GWO process in question is a GW process.) Treating a as the *mean generation length* (see [5, 8]), we may conclude that the asymptotic behaviour of the critical GWO process with *short-living individuals* (see the condition (2)) is similar to that of the critical GW process, provided time is counted generation-wise.

New asymptotic patterns for the critical GWO processes are found under the assumption

$$t^2\mathrm{P}(L > t) \to d, \quad 0 \leq d < \infty, \quad t \to \infty, \tag{3}$$

which, compared to (2), allows the existence of *long-living individuals* given $d > 0$. The condition (3) was first introduced in the pioneering paper [12] dealing with the *Bellman–Harris processes*. In the current discrete-time setting, the Bellman–Harris process is a GWO process subject to two restrictions: (a) $\mathrm{P}(\tau_1 = \ldots = \tau_N = L) = 1$, so that all births occur at the moment of an individual's death, and (b) the random variables L and N are independent. For the Bellman–Harris process, the conditions (1) and (3) imply $a = \mathrm{E}(L)$, $a < \infty$, and according to [12, Theorem 3], we get

$$tQ(t) \to h, \quad t \to \infty, \qquad h := \frac{a + \sqrt{a^2 + 4bd}}{2b}. \tag{4}$$

As was shown in [11, Corollary B] (see also [7, Lemma 3.2] for an adaptation to the discrete-time setting), the relation (4) holds even for the GWO processes satisfying the conditions (1), (3), and $a < \infty$.

The main result of this paper, Theorem 1 of Section 2, considers a critical GWO process under the above-mentioned set of assumptions (1), (3), $a < \infty$, and establishes the convergence of the finite-dimensional distributions conditioned on survival at a remote time of observation. A remarkable feature of this result is that its limit process is fully described by a single parameter $c := 4bda^{-2}$, regardless of complicated mutual dependencies between the random variables τ_j, N, L.

Our proof of Theorem 1, requiring an intricate asymptotic analysis of multi-dimensional probability generating functions, is split into two sections for the sake of readability. Section 3 presents a new proof of (4) inspired by the proof of [12]. The crucial aspect of this approach, compared to the proof of [7, Lemma 3.2], is that certain essential steps do not rely on the monotonicity of the function $Q(t)$. In Section 4, the technique of Section 3 is further developed to finish the proof of Theorem 1.

We conclude this section by mentioning the illuminating family of GWO processes called the *Sevastyanov processes* [9]. The Sevastyanov process is a generalised version of the Bellman–Harris process, with possibly dependent L and N. In the critical case, the mean generation length of the Sevastyanov process, $a = E(LN)$, can be represented as

$$a = \text{cov}(L, N) + E(L).$$

Thus, if L and N are positively correlated, the average generation length a exceeds the average life length $E(L)$.

Turning to a specific example of the Sevastyanov process, take

$$P(L = t) = p_1 t^{-3} (\ln \ln t)^{-1}, \quad P(N = 0 | L = t) = 1 - p_2, \quad P(N = n_t | L = t) = p_2, \ t \geq 2,$$

where $n_t := \lfloor t(\ln t)^{-1} \rfloor$ and (p_1, p_2) are such that

$$\sum_{t=2}^{\infty} P(L = t) = p_1 \sum_{t=2}^{\infty} t^{-3} (\ln \ln t)^{-1} = 1, \quad E(N) = p_1 p_2 \sum_{t=2}^{\infty} n_t t^{-3} (\ln \ln t)^{-1} = 1.$$

In this case, for some positive constant c_1,

$$E(N^2) = p_1 p_2 \sum_{t=1}^{\infty} n_t^2 t^{-3} (\ln \ln t)^{-1} < c_1 \int_2^{\infty} \frac{d(\ln t)}{(\ln t)^2 \ln \ln t} < \infty,$$

implying that the condition (1) is satisfied. Clearly, the condition (3) holds with $d = 0$. At the same time,

$$a = E(NL) = p_1 p_2 \sum_{t=1}^{\infty} n_t t^{-2} (\ln \ln t)^{-1} > c_2 \int_2^{\infty} \frac{d(\ln t)}{(\ln t)(\ln \ln t)} = \infty,$$

where c_2 is a positive constant. This example demonstrates that for the GWO process, unlike for the Bellman–Harris process, the conditions (1) and (3) do not automatically imply the condition $a < \infty$.

2. The main result

Theorem 1. *For a GWO process satisfying* (1)*,* (3) *and* $a < \infty$*, there holds a weak convergence of the finite-dimensional distributions*

$$(Z(ty), 0 < y < \infty | Z(t) > 0) \xrightarrow{\text{fdd}} (\eta(y), 0 < y < \infty), \quad t \to \infty.$$

The limiting process is a continuous-time pure death process $(\eta(y), 0 \leq y < \infty)$*, whose evolution law is determined by a single compound parameter* $c = 4bda^{-2}$*, as specified next.*

The finite-dimensional distributions of the limiting process $\eta(\cdot)$ are given below in terms of the k-dimensional probability generating functions $\mathrm{E}\left(z_1^{\eta(y_1)} \cdots z_k^{\eta(y_k)}\right)$, $k \geq 1$, assuming

$$0 = y_0 < y_1 < \ldots < y_j < 1 \leq y_{j+1} < \ldots < y_k < y_{k+1} = \infty,$$
$$0 \leq j \leq k, \quad 0 \leq z_1, \ldots, z_k < 1. \tag{5}$$

Here the index j highlights the pivotal value 1 corresponding to the time of observation t of the underlying GWO process.

As will be shown in Section 4.2, if $j = 0$, then

$$\mathrm{E}\left(z_1^{\eta(y_1)} \cdots z_k^{\eta(y_k)}\right) = 1 - \frac{1 + \sqrt{1 + \sum_{i=1}^k z_1 \cdots z_{i-1}(1-z_i)\Gamma_i}}{(1 + \sqrt{1+c})y_1}, \quad \Gamma_i := c(y_1/y_i)^2,$$

and if $j \geq 1$,

$$\mathrm{E}\left(z_1^{\eta(y_1)} \cdots z_k^{\eta(y_k)}\right)$$
$$= \frac{\sqrt{1 + \sum_{i=1}^j z_1 \cdots z_{i-1}(1-z_i)\Gamma_i + cz_1 \cdots z_j y_1^2} - \sqrt{1 + \sum_{i=1}^k z_1 \cdots z_{i-1}(1-z_i)\Gamma_i}}{(1 + \sqrt{1+c})y_1}.$$

In particular, for $k = 1$, we have

$$\mathrm{E}\left(z^{\eta(y)}\right) = \frac{\sqrt{1 + c(1-z) + czy^2} - \sqrt{1 + c(1-z)}}{(1 + \sqrt{1+c})y}, \quad 0 < y < 1,$$
$$\mathrm{E}\left(z^{\eta(y)}\right) = 1 - \frac{1 + \sqrt{1 + c(1-z)}}{(1 + \sqrt{1+c})y}, \quad y \geq 1.$$

It follows that $\mathrm{P}(\eta(y) \geq 0) = 1$ for $y > 0$, and moreover, putting here first $z = 1$ and then $z = 0$ yields

$$\mathrm{P}(\eta(y) < \infty) = \frac{\sqrt{1 + cy^2} - 1}{(1 + \sqrt{1+c})y} \cdot 1_{\{0 < y < 1\}} + \left(1 - \frac{2}{(1 + \sqrt{1+c})y}\right) \cdot 1_{\{y \geq 1\}},$$
$$\mathrm{P}(\eta(y) = 0) = \frac{y-1}{y} \cdot 1_{\{y \geq 1\}},$$

implying that $\mathrm{P}(\eta(y) = \infty) > 0$ for all $y > 0$. In fact, letting $y \to 0$, we may set $\mathrm{P}(\eta(0) = \infty) = 1$.

To demonstrate that the process $\eta(\cdot)$ is indeed a pure death process, consider the function

$$\mathrm{E}\left(z_1^{\eta(y_1) - \eta(y_2)} \cdots z_{k-1}^{\eta(y_{k-1}) - \eta(y_k)} z_k^{\eta(y_k)}\right)$$

determined by

$$\mathrm{E}\left(z_1^{\eta(y_1) - \eta(y_2)} \cdots z_{k-1}^{\eta(y_{k-1}) - \eta(y_k)} z_k^{\eta(y_k)}\right) = \mathrm{E}\left(z_1^{\eta(y_1)} (z_2/z_1)^{\eta(y_2)} \cdots (z_k/z_{k-1})^{\eta(y_k)}\right).$$

This function is given by two expressions:

$$\frac{\left(1 + \sqrt{1+c}\right)y_1 - 1 - \sqrt{1 + \sum_{i=1}^{k}(1 - z_i)\gamma_i}}{\left(1 + \sqrt{1+c}\right)y_1}, \quad \text{for } j = 0,$$

$$\frac{\sqrt{1 + \sum_{i=1}^{j-1}(1 - z_i)\gamma_i + (1 - z_j)\Gamma_j + cz_j y_1^2} - \sqrt{1 + \sum_{i=1}^{k}(1 - z_i)\gamma_i}}{\left(1 + \sqrt{1+c}\right)y_1}, \quad \text{for } j \geq 1,$$

where $\gamma_i := \Gamma_i - \Gamma_{i+1}$ and $\Gamma_{k+1} = 0$. Setting $k = 2$, $z_1 = z$, and $z_2 = 1$, we deduce that the function

$$\mathrm{E}\left(z^{\eta(y_1) - \eta(y_2)}; \eta(y_1) < \infty\right), \quad 0 < y_1 < y_2, \quad 0 \leq z \leq 1, \tag{6}$$

is given by one of the following three expressions, depending on whether $j = 2$, $j = 1$, or $j = 0$:

$$\frac{\sqrt{1 + cy_1^2 + c(1 - z)\left(1 - (y_1/y_2)^2\right)} - \sqrt{1 + c(1 - z)\left(1 - (y_1/y_2)^2\right)}}{\left(1 + \sqrt{1+c}\right)y_1}, \quad y_2 < 1,$$

$$\frac{\sqrt{1 + cy_1^2 + c(1 - z)\left(1 - y_1^2\right)} - \sqrt{1 + c(1 - z)\left(1 - (y_1/y_2)^2\right)}}{\left(1 + \sqrt{1+c}\right)y_1}, \quad y_1 < 1 \leq y_2,$$

$$1 - \frac{1 + \sqrt{1 + c(1 - z)\left(1 - (y_1/y_2)^2\right)}}{\left(1 + \sqrt{1+c}\right)y_1}, \quad 1 \leq y_1.$$

Since the generating function (6) is finite at $z = 0$, we conclude that

$$\mathrm{P}(\eta(y_1) < \eta(y_2); \eta(y_1) < \infty) = 0, \quad 0 < y_1 < y_2.$$

This implies

$$\mathrm{P}(\eta(y_2) \leq \eta(y_1)) = 1, \quad 0 < y_1 < y_2,$$

meaning that unless the process $\eta(\cdot)$ is sitting at the infinity state, it evolves by negative integer-valued jumps until it gets absorbed at zero.

Consider now the conditional probability generating function

$$\mathrm{E}\left(z^{\eta(y_1) - \eta(y_2)} | \eta(y_1) < \infty\right), \quad 0 < y_1 < y_2, \quad 0 \leq z \leq 1. \tag{7}$$

In accordance with the three expressions given above for (6), the generating function (7) is specified by the following three expressions:

$$\frac{\sqrt{1 + cy_1^2 + c(1 - z)\left(1 - (y_1/y_2)^2\right)} - \sqrt{1 + c(1 - z)\left(1 - (y_1/y_2)^2\right)}}{\sqrt{1 + cy_1^2} - 1}, \quad y_2 < 1,$$

$$\frac{\sqrt{1 + cy_1^2 + c(1 - z)\left(1 - y_1^2\right)} - \sqrt{1 + c(1 - z)\left(1 - (y_1/y_2)^2\right)}}{\sqrt{1 + cy_1^2} - 1}, \quad y_1 < 1 \leq y_2,$$

$$1 - \frac{\sqrt{1 + c(1 - z)\left(1 - (y_1/y_2)^2\right)} - 1}{\left(1 + \sqrt{1+c}\right)y_1 - 2}, \quad 1 \leq y_1.$$

In particular, setting $z = 0$ here, we obtain

$$P(\eta(y_1) - \eta(y_2) = 0 | \eta(y_1) < \infty) = \begin{cases} \dfrac{\sqrt{1+c\left(1+y_1^2-(y_1/y_2)^2\right)}-\sqrt{1+c\left(1-(y_1/y_2)^2\right)}}{\sqrt{1+cy_1^2}-1} & \text{for } 0 < y_1 < y_2 < 1, \\[4mm] \dfrac{\sqrt{1+c}-\sqrt{1+c\left(1-(y_1/y_2)^2\right)}}{\sqrt{1+cy_1^2}-1} & \text{for } 0 < y_1 < 1 \le y_2, \\[4mm] 1 - \dfrac{\sqrt{1+c\left(1-(y_1/y_2)^2\right)}-1}{\left(1+\sqrt{1+c}\right)y_1-2} & \text{for } \quad 1 \le y_1 < y_2. \end{cases}$$

Notice that given $0 < y_1 \le 1$,

$$P(\eta(y_1) - \eta(y_2) = 0 | \eta(y_1) < \infty) \to 0, \quad y_2 \to \infty,$$

which is expected because of $\eta(y_1) \ge \eta(1) \ge 1$ and $\eta(y_2) \to 0$ as $y_2 \to \infty$.

The random times

$$T = \sup\{u : \eta(u) = \infty\}, \quad T_0 = \inf\{u : \eta(u) = 0\}$$

are major characteristics of a trajectory of the limit pure death process. Since

$$P(T \le y) = \mathrm{E}\big(z^{\eta(y)}\big)\Big|_{z=1}, \qquad P(T_0 \le y) = \mathrm{E}\big(z^{\eta(y)}\big)\Big|_{z=0},$$

in accordance with the above-mentioned formulas for $\mathrm{E}\big(z^{\eta(y)}\big)$, we get the following marginal distributions:

$$P(T \le y) = \frac{\sqrt{1+cy^2}-1}{\left(1+\sqrt{1+c}\right)y} \cdot 1_{\{0 \le y < 1\}} + \left(1 - \frac{2}{\left(1+\sqrt{1+c}\right)y}\right) \cdot 1_{\{y \ge 1\}},$$

$$P(T_0 \le y) = \frac{y-1}{y} \cdot 1_{\{y \ge 1\}}.$$

The distribution of T_0 is free from the parameter c and has the Pareto probability density function

$$f_0(y) = y^{-2} 1_{\{y > 1\}}.$$

In the special case (2), that is, when (3) holds with $d = 0$, we have $c = 0$ and $P(T = T_0) = 1$. If $d > 0$, then $T \le T_0$, and the distribution of T has the following probability density function:

$$f(y) = \begin{cases} \dfrac{1}{\left(1+\sqrt{1+c}\right)y^2}\left(1 - \dfrac{1}{\sqrt{1+cy^2}}\right) & \text{for} \quad 0 \le y < 1, \\[4mm] \dfrac{2}{\left(1+\sqrt{1+c}\right)y^2} & \text{for} \qquad y \ge 1, \end{cases}$$

which has a positive jump at $y = 1$ of size $f(1) - f(1-) = (1+c)^{-1/2}$; see Figure 1. Observe that $\frac{f(1-)}{f(1)} \to \frac{1}{2}$ as $c \to \infty$.

Intuitively, the limiting pure death process counts the long-living individuals in the GWO process, that is, those individuals whose life length is of order t. These long-living individuals may have descendants, however none of them would live long enough to be detected by the
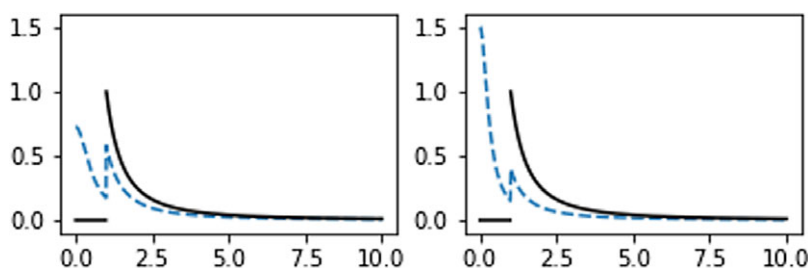
FIGURE 1. The dashed line is the probability density function of T; the solid line is the probability density function of T_0. The left panel illustrates the case $c = 5$, and the right panel illustrates the case $c = 15$.

finite-dimensional distributions at the relevant time scale, see Lemma 2 below. Theorem 1 suggests a new perspective on Vatutin's dichotomy (see [12]), claiming that the long-term survival of a critical age-dependent branching process is due to either a large number of short-living individuals or a small number of long-living individuals. In terms of the random times $T \leq T_0$, Vatutin's dichotomy discriminates between two possibilities: if $T > 1$, then $\eta(1) = \infty$, meaning that the GWO process has survived thanks to a large number of individuals, while if $T \leq 1 < T_0$, then $1 \leq \eta(1) < \infty$, meaning that the GWO process has survived thanks to a small number of individuals.

3. Proof that $tQ(t) \to h$

This section deals with the survival probability of the critical GWO process

$$Q(t) = 1 - P(t), \quad P(t) := \mathrm{P}(Z(t) = 0).$$

By its definition, the GWO process can be represented as the sum

$$Z(t) = 1_{\{L > t\}} + \sum_{j=1}^{N} Z_j(t - \tau_j), \quad t = 0, 1, \dots, \tag{8}$$

involving N independent daughter processes $Z_j(\cdot)$ generated by the founder individual at the birth times $\tau_j, j = 1, \dots, N$ (here it is assumed that $Z_j(t) = 0$ for all negative t). The branching property (8) implies the relation

$$1_{\{Z(t) = 0\}} = 1_{\{L \leq t\}} \prod_{j=1}^{N} 1_{\{Z_j(t - \tau_j) = 0\}},$$

which says that the GWO process goes extinct by the time t if, on one hand, the founder is dead at time t and, on the other hand, all daughter processes are extinct by the time t. After taking expectations of both sides, we can write

$$P(t) = \mathrm{E}\left(\prod_{j=1}^{N} P(t - \tau_j) ; L \leq t \right). \tag{9}$$

As shown next, this nonlinear equation for $P(\cdot)$ implies the asymptotic formula (4) under the conditions (1), (3), and $a < \infty$.

3.1. Outline of the proof of (4)

We start by stating four lemmas and two propositions. Let

$$\Phi(z) := \mathrm{E}\big((1-z)^N - 1 + Nz\big), \tag{10}$$

$$W(t) := \big(1 - ht^{-1}\big)^N + Nht^{-1} - \sum_{j=1}^{N} Q\big(t - \tau_j\big) - \prod_{j=1}^{N} P\big(t - \tau_j\big), \tag{11}$$

$$D(u, t) := \mathrm{E}\Big(1 - \prod_{j=1}^{N} P\big(t - \tau_j\big); u < L \le t\Big) + \mathrm{E}\big((1 - ht^{-1})^N - 1 + Nht^{-1}; L > u\big), \tag{12}$$

$$\mathrm{E}_u(X) := \mathrm{E}(X; L \le u), \tag{13}$$

where $0 \le z \le 1$, $u > 0$, $t \ge h$, and X is an arbitrary random variable.

Lemma 1. *Given* (10), (11), (12), *and* (13), *assume that* $0 < u \le t$ *and* $t \ge h$. *Then*

$$\Phi\big(ht^{-1}\big) = \mathrm{P}(L > t) + \mathrm{E}_u\Big(\sum_{j=1}^{N} Q\big(t - \tau_j\big)\Big) - Q(t) + \mathrm{E}_u(W(t)) + D(u, t).$$

Lemma 2. *If* (1) *and* (3) *hold, then* $\mathrm{E}(N; L > ty) = o\big(t^{-1}\big)$ *as* $t \to \infty$ *for any fixed* $y > 0$.

Lemma 3. *If* (1), (3), *and* $a < \infty$ *hold, then for any fixed* $0 < y < 1$,

$$\mathrm{E}_{ty}\Big(\sum_{j=1}^{N} \Big(\frac{1}{t - \tau_j} - \frac{1}{t}\Big)\Big) \sim at^{-2}, \quad t \to \infty.$$

Lemma 4. *Let* $k \ge 1$. *If* $0 \le f_j, g_j \le 1$ *for* $j = 1, \ldots, k$, *then*

$$\prod_{j=1}^{k}\big(1 - g_j\big) - \prod_{j=1}^{k}\big(1 - f_j\big) = \sum_{j=1}^{k}(f_j - g_j)r_j,$$

where $0 \le r_j \le 1$ *and*

$$1 - r_j = \sum_{i=1}^{j-1} g_i + \sum_{i=j+1}^{k} f_i - R_j,$$

for some $R_j \ge 0$. *If moreover* $f_j \le q$ *and* $g_j \le q$ *for some* $q > 0$, *then*

$$1 - r_j \le (k-1)q, \qquad R_j \le kq, \qquad R_j \le k^2 q^2.$$

Proposition 1. *If* (1), (3), *and* $a < \infty$ *hold, then* $\limsup_{t \to \infty} tQ(t) < \infty$.

Proposition 2. *If* (1), (3), *and* $a < \infty$ *hold, then* $\liminf_{t \to \infty} tQ(t) > 0$.

According to these two propositions, there exists a triplet of positive numbers (q_1, q_2, t_0) such that

$$q_1 \le tQ(t) \le q_2, \quad t \ge t_0, \quad 0 < q_1 < h < q_2 < \infty. \tag{14}$$

The claim $tQ(t) \to h$ is derived using (14) by accurately removing asymptotically negligible terms from the relation for $Q(\cdot)$ stated in Lemma 1, after setting $u = ty$ with a fixed $0 < y < 1$, and then choosing a sufficiently small y. In particular, as an intermediate step, we will show that

$$Q(t) = \mathrm{E}_{ty}\Big(\sum_{j=1}^{N} Q\big(t - \tau_j\big)\Big) + \mathrm{E}_{ty}(W(t)) - aht^{-2} + o\big(t^{-2}\big), \quad t \to \infty. \tag{15}$$

Then, restating our goal as $\phi(t) \to 0$ in terms of the function $\phi(t)$, defined by

$$Q(t) = \frac{h + \phi(t)}{t}, \quad t \geq 1, \tag{16}$$

we rewrite (15) as

$$\frac{h + \phi(t)}{t} = \mathrm{E}_{ty}\left(\sum_{j=1}^{N} \frac{h + \phi\left(t - \tau_j\right)}{t - \tau_j}\right) + \mathrm{E}_{ty}(W(t)) - aht^{-2} + o\left(t^{-2}\right), \quad t \to \infty. \tag{17}$$

It turns out that the three terms involving h, outside $W(t)$, effectively cancel each other, yielding

$$\frac{\phi(t)}{t} = \mathrm{E}_{ty}\left(\sum_{j=1}^{N} \frac{\phi\left(t - \tau_j\right)}{t - \tau_j} + W(t)\right) + o\left(t^{-2}\right), \quad t \to \infty. \tag{18}$$

Treating $W(t)$ in terms of Lemma 4 yields

$$\phi(t) = \mathrm{E}_{ty}\left(\sum_{j=1}^{N} \phi\left(t - \tau_j\right) r_j(t) \frac{t}{t - \tau_j}\right) + o\left(t^{-1}\right), \tag{19}$$

where $r_j(t)$ is a counterpart of r_j in Lemma 4. To derive from here the desired convergence $\phi(t) \to 0$, we will adapt a clever trick from Chapter 9.1 of [10], which was further developed in [12] for the Bellman–Harris process, with possibly infinite var(N). Define a non-negative function $m(t)$ by

$$m(t) := |\phi(t)| \ln t, \quad t \geq 2. \tag{20}$$

Multiplying (19) by $\ln t$ and using the triangle inequality, we obtain

$$m(t) \leq \mathrm{E}_{ty}\left(\sum_{j=1}^{N} m\left(t - \tau_j\right) r_j(t) \frac{t \ln t}{\left(t - \tau_j\right) \ln\left(t - \tau_j\right)}\right) + v(t), \tag{21}$$

where $v(t) \geq 0$ and $v(t) = o(t^{-1} \ln t)$ as $t \to \infty$. It will be shown that this leads to $m(t) = o(\ln t)$, thereby concluding the proof of (4).

3.2. Proof of lemmas and propositions

Proof of Lemma 1. For $0 < u \leq t$, the relations (9) and (13) give

$$P(t) = \mathrm{E}_u\left(\prod_{j=1}^{N} P\left(t - \tau_j\right)\right) + \mathrm{E}\left(\prod_{j=1}^{N} P\left(t - \tau_j\right); u < L \leq t\right). \tag{22}$$

On the other hand, for $t \geq h$,

$$\Phi\left(ht^{-1}\right) \stackrel{(10)}{=} \mathrm{E}_u\left(\left(1 - ht^{-1}\right)^N - 1 + Nht^{-1}\right) + \mathrm{E}\left(\left(1 - ht^{-1}\right)^N - 1 + Nht^{-1}; L > u\right).$$

Adding the latter relation to

$$1 = \mathrm{P}(L \leq u) + \mathrm{P}(L > t) + \mathrm{P}(u < L \leq t)$$

and subtracting (22) from the sum, we get

$$\Phi\left(ht^{-1}\right) + Q(t) = \mathrm{E}_u\left(\left(1 - ht^{-1}\right)^N + Nht^{-1} - \prod_{j=1}^{N} P\left(t - \tau_j\right)\right) + \mathrm{P}(L > t) + D(u, t),$$

with $D(u, t)$ defined by (12). After a rearrangement, we obtain the statement of the lemma. $\qquad\square$

Proof of Lemma 2. For any fixed $\epsilon > 0$,

$$\mathrm{E}(N; L > t) = \mathrm{E}(N; N \le t\epsilon, L > t) + \mathrm{E}\left(N; 1 < N(t\epsilon)^{-1}, L > t\right)$$
$$\le t\epsilon \mathrm{P}(L > t) + (t\epsilon)^{-1}\mathrm{E}\left(N^2; L > t\right).$$

Thus, by (1) and (3),

$$\limsup_{t \to \infty} \left(t\mathrm{E}(N; L > t)\right) \le d\epsilon,$$

and the assertion follows as $\epsilon \to 0$. $\qquad\square$

Proof of Lemma 3. For $t = 1, 2, \ldots$ and $y > 0$, put

$$B_t(y) := t^2 \, \mathrm{E}_{ty}\left(\sum_{j=1}^{N} \left(\frac{1}{t - \tau_j} - \frac{1}{t}\right)\right) - a.$$

For any $0 < u < ty$, using

$$a = \mathrm{E}_u(\tau_1 + \ldots + \tau_N) + A_u, \quad A_u := \mathrm{E}(\tau_1 + \ldots + \tau_N; L > u),$$

we get

$$B_t(y) = \mathrm{E}_u\left(\sum_{j=1}^{N} \frac{t}{t - \tau_j}\tau_j\right) + \mathrm{E}\left(\sum_{j=1}^{N} \frac{t}{t - \tau_j}\tau_j \, ; u < L \le ty\right)$$
$$- \mathrm{E}_u(\tau_1 + \ldots + \tau_N) - A_u$$
$$= \mathrm{E}\left(\sum_{j=1}^{N} \frac{\tau_j}{1 - \tau_j/t}; u < L \le ty\right) + \mathrm{E}_u\left(\sum_{j=1}^{N} \frac{\tau_j^2}{t - \tau_j}\right) - A_u.$$

For the first term on the right-hand side, we have $\tau_j \le L \le ty$, so that

$$\mathrm{E}\left(\sum_{j=1}^{N} \frac{\tau_j}{1 - \tau_j/t}; u < L \le ty\right) \le (1 - y)^{-1}A_u.$$

For the second term, $\tau_j \le L \le u$ and therefore

$$\mathrm{E}_u\left(\sum_{j=1}^{N} \frac{\tau_j^2}{t - \tau_j}\right) \le \frac{u^2}{t - u}\mathrm{E}_u(N) \le \frac{u^2}{t - u}.$$

This yields

$$-A_u \le B_t(y) \le (1 - y)^{-1}A_u + \frac{u^2}{t - u}, \quad 0 < u < ty < t,$$

implying

$$-A_u \le \liminf_{t \to \infty} B_t(y) \le \limsup_{t \to \infty} B_t(y) \le (1-y)^{-1} A_u.$$

Since $A_u \to 0$ as $u \to \infty$, we conclude that $B_t(y) \to 0$ as $t \to \infty$. □

Proof of Lemma 4. Let

$$r_j := \left(1 - g_1\right) \dots \left(1 - g_{j-1}\right) \left(1 - f_{j+1}\right) \dots \left(1 - f_k\right), \quad 1 \le j \le k.$$

Then $0 \le r_j \le 1$, and the first stated equality is obtained by telescopic summation of

$$\left(1 - g_1\right) \prod_{j=2}^{k} \left(1 - f_j\right) - \prod_{j=1}^{k} \left(1 - f_j\right) = (f_1 - g_1) r_1,$$

$$\left(1 - g_1\right)\left(1 - g_2\right) \prod_{j=3}^{k} \left(1 - f_j\right) - \left(1 - g_1\right) \prod_{j=2}^{k} \left(1 - f_j\right) = (f_2 - g_2) r_2, \dots,$$

$$\prod_{j=1}^{k} \left(1 - g_j\right) - \prod_{j=1}^{k-1} \left(1 - g_j\right)\left(1 - f_k\right) = (f_k - g_k) r_k.$$

The second stated equality is obtained with

$$R_j := \sum_{i=j+1}^{k} f_i \big(1 - \left(1 - f_{j+1}\right) \dots \left(1 - f_{i-1}\right)\big)$$

$$+ \sum_{i=1}^{j-1} g_i \big(1 - \left(1 - g_1\right) \dots \left(1 - g_{i-1}\right) \left(1 - f_{j+1}\right) \dots \left(1 - f_k\right)\big),$$

by performing telescopic summation of

$$1 - \left(1 - f_{j+1}\right) = f_{j+1},$$

$$\left(1 - f_{j+1}\right) - \left(1 - f_{j+1}\right)\left(1 - f_{j+2}\right) = f_{j+2} \left(1 - f_{j+1}\right), \dots,$$

$$\prod_{i=j+1}^{k-1} \left(1 - f_i\right) - \prod_{i=j+1}^{k} \left(1 - f_i\right) = f_k \prod_{i=j+1}^{k-1} \left(1 - f_i\right),$$

$$\prod_{i=j+1}^{k} \left(1 - f_i\right) - \left(1 - g_1\right) \prod_{i=j+1}^{k} \left(1 - f_i\right) = g_1 \prod_{i=j+1}^{k} \left(1 - f_i\right), \dots,$$

$$\prod_{i=1}^{j-2} \left(1 - g_i\right) \prod_{i=j+1}^{k} \left(1 - f_i\right) - \prod_{i=1}^{j-1} \left(1 - g_i\right) \prod_{i=j+1}^{k} \left(1 - f_i\right) = g_{j-1} \prod_{i=1}^{j-2} \left(1 - g_i\right) \prod_{i=j+1}^{k} \left(1 - f_i\right).$$

By the above definition of R_j, we have $R_j \ge 0$. Furthermore, given $f_j \le q$ and $g_j \le q$, we get

$$R_j \le \sum_{i=1}^{j-1} g_i + \sum_{i=j+1}^{k} f_i \le (k-1)q.$$

It remains to observe that

$$1 - r_j \le 1 - (1-q)^{k-1} \le (k-1)q,$$

and from the definition of R_j,

$$R_j \le q \sum_{i=1}^{k-j-1} \left(1 - (1-q)^i\right) + q \sum_{i=1}^{j-1} \left(1 - (1-q)^{k-j+i-1}\right) \le q^2 \sum_{i=1}^{k-2} i \le k^2 q^2.$$

□

Proof of Proposition 1. By the definition of $\Phi(\cdot)$, we have

$$\Phi(Q(t)) + P(t) = E_u\big(P(t)^N\big) + P(L > u) - E\big(1 - P(t)^N; L > u\big),$$

for any $0 < u < t$. This and (22) yield

$$\Phi(Q(t)) = E_u\bigg(P(t)^N - \prod_{j=1}^{N} P\left(t - \tau_j\right)\bigg) + P(L > u)$$
$$- E\big(1 - P(t)^N; L > u\big) - E\bigg(\prod_{j=1}^{N} P\left(t - \tau_j\right); u < L \le t\bigg). \qquad (23)$$

We therefore obtain the upper bound

$$\Phi(Q(t)) \le E_u\bigg(P(t)^N - \prod_{j=1}^{N} P\left(t - \tau_j\right)\bigg) + P(L > u),$$

which together with Lemma 4 and the monotonicity of $Q(\cdot)$ implies

$$\Phi(Q(t)) \le E_u\bigg(\sum_{j=1}^{N} \left(Q\left(t - \tau_j\right) - Q(t)\right)\bigg) + P(L > u). \qquad (24)$$

Borrowing an idea from [11], suppose to the contrary that

$$t_n := \min\{t : tQ(t) \ge n\}$$

is finite for any natural n. It follows that

$$Q(t_n) \ge \frac{n}{t_n}, \qquad Q(t_n - u) < \frac{n}{t_n - u}, \qquad 1 \le u \le t_n - 1.$$

Putting $t = t_n$ into (24) and using the monotonicity of $\Phi(\cdot)$, we find

$$\Phi\big(nt_n^{-1}\big) \le \Phi(Q(t_n)) \le E_u\bigg(\sum_{j=1}^{N} \left(\frac{n}{t_n - \tau_j} - \frac{n}{t_n}\right)\bigg) + P(L > u).$$

Setting $u = t_n/2$ here and applying Lemma 3 together with (3), we arrive at the relation

$$\Phi\big(nt_n^{-1}\big) = O\big(nt_n^{-2}\big), \quad n \to \infty.$$

Observe that under the condition (1), the L'Hospital rule gives

$$\Phi(z) \sim bz^2, \quad z \to 0. \qquad (25)$$

The resulting contradiction, $n^2 t_n^{-2} = O\big(nt_n^{-2}\big)$ as $n \to \infty$, finishes the proof of the proposition. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Proposition 2. The relation (23) implies

$$\Phi(Q(t)) \ge E_u\bigg(P(t)^N - \prod_{j=1}^{N} P\left(t - \tau_j\right)\bigg) - E\big(1 - P(t)^N; L > u\big).$$

By Lemma 4,

$$P(t)^N - \prod_{j=1}^{N} P\left(t - \tau_j\right) = \sum_{j=1}^{N} \left(Q\left(t - \tau_j\right) - Q(t)\right) r_j^*(t),$$

where $0 \leq r_j^*(t) \leq 1$ is a counterpart of the term r_j in Lemma 4. By the monotonicity of $P(\cdot)$, we have, again referring to Lemma 4,

$$1 - r_j^*(t) \leq (N-1)Q(t-L).$$

Thus, for $0 < y < 1$,

$$\Phi(Q(t)) \geq \mathrm{E}_{ty}\left(\sum_{j=1}^{N}(Q(t-\tau_j)-Q(t))r_j^*(t)\right) - \mathrm{E}\left(1-P(t)^N; L > ty\right). \quad (26)$$

The assertion $\liminf_{t\to\infty} tQ(t) > 0$ is proven by contradiction. Assume that $\liminf_{t\to\infty} tQ(t) = 0$, so that

$$t_n := \min\left\{t : tQ(t) \leq n^{-1}\right\}$$

is finite for any natural n. Plugging $t = t_n$ into (26) and using

$$Q(t_n) \leq \frac{1}{nt_n}, \quad Q(t_n - u) - Q(t_n) \geq \frac{1}{n(t_n - u)} - \frac{1}{nt_n}, \quad 1 \leq u \leq t_n - 1,$$

we get

$$\Phi\left(\frac{1}{nt_n}\right) \geq n^{-1}\mathrm{E}_{t_n y}\left(\sum_{j=1}^{N}\left(\frac{1}{t_n - \tau_j} - \frac{1}{t_n}\right)r_j^*(t_n)\right) - \frac{1}{nt_n}\mathrm{E}(N; L > t_n y).$$

Given $L \leq ty$, we have

$$1 - r_j^*(t) \leq NQ(t(1-y)) \leq N\frac{q_2}{t(1-y)},$$

where the second inequality is based on the already proven part of (14). Therefore,

$$\mathrm{E}_{t_n y}\left(\sum_{j=1}^{N}\left(\frac{1}{t_n - \tau_j} - \frac{1}{t_n}\right)(1 - r_j^*(t_n))\right) \leq \frac{q_2 y}{t_n^2(1-y)^2}\mathrm{E}(N^2),$$

and we derive

$$nt_n^2\Phi\left(\frac{1}{nt_n}\right) \geq t_n^2\mathrm{E}_{t_n y}\left(\sum_{j=1}^{N}\left(\frac{1}{t_n - \tau_j} - \frac{1}{t_n}\right)\right) - \frac{\mathrm{E}(N^2)q_2 y}{(1-y)^2} - t_n\mathrm{E}(N; L > t_n y).$$

Sending $n \to \infty$ and applying (25), Lemma 2, and Lemma 3, we arrive at the inequality

$$0 \geq a - yq_2\mathrm{E}(N^2)(1-y)^{-2}, \quad 0 < y < 1,$$

which is false for sufficiently small y. $\qquad\square$

3.3. Proof of (18) and (19)

Fix an arbitrary $0 < y < 1$. Lemma 1 with $u = ty$ gives

$$\Phi\left(ht^{-1}\right) = \mathrm{P}(L > t) + \mathrm{E}_{ty}\left(\sum_{j=1}^{N}Q(t-\tau_j)\right) - Q(t) + \mathrm{E}_{ty}(W(t)) + D(ty, t). \quad (27)$$

Let us show that

$$D(ty, t) = o(t^{-2}), \quad t \to \infty. \tag{28}$$

Using Lemma 2 and (14), we find that for an arbitrarily small $\epsilon > 0$,

$$\mathrm{E}\Big(1 - \prod_{j=1}^{N} P\left(t - \tau_j\right); ty < L \leq t(1 - \epsilon)\Big) = o(t^{-2}), \quad t \to \infty.$$

On the other hand,

$$\mathrm{E}\Big(1 - \prod_{j=1}^{N} P\left(t - \tau_j\right); t(1 - \epsilon) < L \leq t\Big) \leq \mathrm{P}(t(1 - \epsilon) < L \leq t),$$

so that in view of (3),

$$\mathrm{E}\Big(1 - \prod_{j=1}^{N} P\left(t - \tau_j\right); ty < L \leq t\Big) = o(t^{-2}), \quad t \to \infty.$$

This, (12), and Lemma 2 imply (28).

Observe that

$$bh^2 = ah + d. \tag{29}$$

Combining (27), (28), and

$$\mathrm{P}(L > t) - \Phi\left(ht^{-1}\right) \overset{(3)(25)}{=} dt^{-2} - bh^2 t^{-2} + o\left(t^{-2}\right) \overset{(29)}{=} -aht^{-2} + o\left(t^{-2}\right), \quad t \to \infty,$$

we derive (15), which in turn gives (17). The latter implies (18) since by Lemmas 2 and 4,

$$\mathrm{E}_{ty}\Big(\sum_{j=1}^{N} \frac{h}{t - \tau_j}\Big) - \frac{h}{t} = \mathrm{E}_{ty}\Big(\sum_{j=1}^{N} \Big(\frac{h}{t - \tau_j} - \frac{h}{t}\Big)\Big) - ht^{-1}\mathrm{E}(N; L > ty) = aht^{-2} + o(t^{-2}).$$

Turning to the proof of (19), observe that the random variable

$$W(t) = \left(1 - ht^{-1}\right)^N - \prod_{j=1}^{N} \Big(1 - \frac{h + \phi\left(t - \tau_j\right)}{t - \tau_j}\Big) + \sum_{j=1}^{N} \Big(\frac{h}{t} - \frac{h + \phi\left(t - \tau_j\right)}{t - \tau_j}\Big)$$

can be represented in terms of Lemma 4 as

$$W(t) = \prod_{j=1}^{N} \left(1 - f_j(t)\right) - \prod_{j=1}^{N} \left(1 - g_j(t)\right) + \sum_{j=1}^{N} \left(f_j(t) - g_j(t)\right)$$
$$= \sum_{j=1}^{N} \left(1 - r_j(t)\right)(f_j(t) - g_j(t)),$$

by assigning

$$f_j(t) := ht^{-1}, \quad g_j(t) := \frac{h + \phi\left(t - \tau_j\right)}{t - \tau_j}. \tag{30}$$

Here $0 \leq r_j(t) \leq 1$, and for sufficiently large t,

$$1 - r_j(t) \overset{(14)}{\leq} Nq_2 t^{-1}. \tag{31}$$

After plugging into (18) the expression

$$W(t) = \sum_{j=1}^{N} \left(\frac{h}{t} - \frac{h}{t - \tau_j} \right)(1 - r_j(t)) - \sum_{j=1}^{N} \frac{\phi\left(t - \tau_j\right)}{t - \tau_j}(1 - r_j(t)),$$

we get

$$\frac{\phi(t)}{t} = \mathrm{E}_{ty}\left(\sum_{j=1}^{N} \frac{\phi\left(t - \tau_j\right)}{t - \tau_j} r_j(t) \right) + \mathrm{E}_{ty}\left(\sum_{j=1}^{N} \left(\frac{h}{t - \tau_j} - \frac{h}{t} \right)(1 - r_j(t)) \right) + o(t^{-2}), \quad t \to \infty.$$

The latter expectation is non-negative, and for an arbitrary $\epsilon > 0$, it has the following upper bound:

$$\mathrm{E}_{ty}\left(\sum_{j=1}^{N} \left(\frac{h}{t - \tau_j} - \frac{h}{t} \right)(1 - r_j(t)) \right) \overset{(31)}{\leq} q_2\epsilon\mathrm{E}_{ty}\left(\sum_{j=1}^{N} \left(\frac{h}{t - \tau_j} - \frac{h}{t} \right) \right) + \frac{q_2 h}{(1 - y)t^2}\mathrm{E}(N^2; N > t\epsilon).$$

Thus, in view of Lemma 3,

$$\frac{\phi(t)}{t} = \mathrm{E}_{ty}\left(\sum_{j=1}^{N} \frac{\phi\left(t - \tau_j\right)}{t - \tau_j} r_j(t) \right) + o(t^{-2}), \quad t \to \infty.$$

Multiplying this relation by t, we arrive at (19).

3.4. Proof of $\phi(t) \to 0$

Recall (20). If the non-decreasing function

$$M(t) := \max_{1 \leq j \leq t} m(j)$$

is bounded from above, then $\phi(t) = O\left(\frac{1}{\ln t}\right)$, proving that $\phi(t) \to 0$ as $t \to \infty$. If $M(t) \to \infty$ as $t \to \infty$, then there is an integer-valued sequence $0 < t_1 < t_2 < \ldots$, such that the sequence $M_n := M(t_n)$ is strictly increasing and converges to infinity. In this case,

$$m(t) \leq M_{n-1} < M_n, \quad 1 \leq t < t_n, \quad m(t_n) = M_n, \quad n \geq 1. \tag{32}$$

Since $|\phi(t)| \leq \frac{M_n}{\ln t_n}$ for $t_n \leq t < t_{n+1}$, to finish the proof of $\phi(t) \to 0$, it remains to verify that

$$M_n = o(\ln t_n), \quad n \to \infty. \tag{33}$$

Fix an arbitrary $y \in (0, 1)$. Putting $t = t_n$ in (21) and using (32), we find

$$M_n \leq M_n\mathrm{E}_{t_n y}\left(\sum_{j=1}^{N} r_j(t_n) \frac{t_n \ln t_n}{(t_n - \tau_j) \ln\left(t_n - \tau_j\right)} \right) + \left(t_n^{-1} \ln t_n\right)o_n.$$

Here and elsewhere, o_n stands for a non-negative sequence such that $o_n \to 0$ as $n \to \infty$. In different formulas, the sign o_n represents different such sequences. Since

$$0 \leq \frac{t \ln t}{(t - u) \ln\left(t - u\right)} - 1 \leq \frac{u(1 + \ln t)}{(t - u) \ln\left(t - u\right)}, \quad 0 \leq u < t - 1,$$

and $r_j(t_n) \in [0, 1]$, it follows that

$$M_n - M_n\mathrm{E}_{t_n y}\left(\sum_{j=1}^{N} r_j(t_n) \right) \leq M_n\mathrm{E}_{t_n y}\left(\sum_{j=1}^{N} \frac{\tau_j(1 + \ln t_n)}{t_n(1 - y) \ln\left(t_n(1 - y)\right)} \right) + \left(t_n^{-1} \ln t_n\right)o_n.$$

Recalling that $a = \mathrm{E}(\sum_{j=1}^{N} \tau_j)$, observe that

$$\mathrm{E}_{t_n y}\left(\sum_{j=1}^{N} \frac{\tau_j(1 + \ln t_n)}{t_n(1 - y)\ln(t_n(1 - y))}\right) \leq \frac{a(1 + \ln t_n)}{t_n(1 - y)\ln(t_n(1 - y))} = \left(a(1 - y)^{-1} + o_n\right)t_n^{-1}.$$

Combining the last two relations, we conclude

$$M_n \mathrm{E}_{t_n y}\left(\sum_{j=1}^{N} (1 - r_j(t_n))\right) \leq a(1 - y)^{-1} t_n^{-1} M_n + t_n^{-1}(M_n + \ln t_n)o_n. \qquad (34)$$

Now it is time to unpack the term $r_j(t)$. By Lemma 4 with (30),

$$1 - r_j(t) = \sum_{i=1}^{j-1} \frac{h + \phi(t - \tau_i)}{t - \tau_i} + (N - j)\frac{h}{t} - R_j(t),$$

where, provided $\tau_j \leq ty$,

$$0 \leq R_j(t) \leq N q_2 t^{-1}(1 - y)^{-1}, \quad R_j(t) \leq N^2 q_2^2 t^{-2}(1 - y)^{-2}, \quad t > t^*,$$

for a sufficiently large t^*. This allows us to rewrite (34) in the form

$$M_n \mathrm{E}_{t_n y}\left(\sum_{j=1}^{N}\left(\sum_{i=1}^{j-1} \frac{h + \phi(t_n - \tau_i)}{t_n - \tau_i} + (N - j)\frac{h}{t_n}\right)\right)$$
$$\leq M_n \mathrm{E}_{t_n y}\left(\sum_{j=1}^{N} R_j(t_n)\right) + a(1 - y)^{-1} t_n^{-1} M_n + t_n^{-1}(M_n + \ln t_n)o_n.$$

To estimate the last expectation, observe that if $\tau_j \leq ty$, then for any $\epsilon > 0$,

$$R_j(t) \leq N q_2 t^{-1}(1 - y)^{-1} 1_{\{N > t\epsilon\}} + N^2 q_2^2 t^{-2}(1 - y)^{-2} 1_{\{N \leq t\epsilon\}}, \quad t > t^*,$$

implying that for sufficiently large n,

$$\mathrm{E}_{t_n y}\left(\sum_{j=1}^{N} R_j(t_n)\right) \leq q_2 t_n^{-1}(1 - y)^{-1} \mathrm{E}(N^2; N > t_n\epsilon) + q_2^2 \epsilon t_n^{-1}(1 - y)^{-2} \mathrm{E}(N^2),$$

so that

$$M_n \mathrm{E}_{t_n y}\left(\sum_{j=1}^{N}\left(\sum_{i=1}^{j-1} \frac{h + \phi(t_n - \tau_i)}{t_n - \tau_i} + (N - j)\frac{h}{t_n}\right)\right)$$
$$\leq a(1 - y)^{-1} t_n^{-1} M_n + t_n^{-1}(M_n + \ln t_n)o_n.$$

Since

$$\sum_{j=1}^{N} \sum_{i=1}^{j-1} \left(\frac{h}{t_n - \tau_i} - \frac{h}{t_n}\right) \geq 0,$$

we obtain

$$M_n \mathrm{E}_{t_n y}\left(\sum_{j=1}^{N}\left(\sum_{i=1}^{j-1} \frac{\phi(t_n - \tau_i)}{t_n - \tau_i} + (N - 1)\frac{h}{t_n}\right)\right)$$
$$\leq a(1 - y)^{-1} t_n^{-1} M_n + t_n^{-1}(M_n + \ln t_n)o_n.$$

By (16) and (14), we have $\phi(t) \geq q_1 - h$ for $t \geq t_0$. Thus, for $\tau_j \leq L \leq t_n y$ and sufficiently large n,

$$\frac{\phi(t_n - \tau_i)}{t_n - \tau_i} \geq \frac{q_1 - h}{t_n(1 - y)}.$$

This gives

$$\sum_{j=1}^{N} \left(\sum_{i=1}^{j-1} \frac{\phi(t_n - \tau_i)}{t_n - \tau_i} + (N-1)\frac{h}{t_n} \right) \geq \left(h + \frac{q_1 - h}{2(1 - y)} \right) t_n^{-1} N(N-1),$$

which, after multiplying by $t_n M_n$ and taking expectations, yields

$$\left(h + \frac{q_1 - h}{2(1 - y)} \right) M_n \mathrm{E}_{t_n y}(N(N-1)) \leq a(1 - y)^{-1} M_n + (M_n + \ln t_n) o_n.$$

Finally, since

$$\mathrm{E}_{t_n y}(N(N-1)) \to 2b, \quad n \to \infty,$$

we derive that for any $0 < \epsilon < y < 1$, there is a finite n_ϵ such that for all $n > n_\epsilon$,

$$M_n\big(2bh(1 - y) + bq_1 - bh - a - \epsilon\big) \leq \epsilon \ln t_n.$$

By (29), we have $bh \geq a$, and therefore

$$2bh(1 - y) + bq_1 - bh - a - \epsilon \geq bq_1 - 2bhy - y.$$

Thus, choosing $y = y_0$ such that $bq_1 - 2bhy_0 - y_0 = \frac{bq_1}{2}$, we see that

$$\limsup_{n \to \infty} \frac{M_n}{\ln t_n} \leq \frac{2\epsilon}{bq_1},$$

which implies (33) as $\epsilon \to 0$, concluding the proof of $\phi(t) \to 0$.

4. Proof of Theorem 1

We will use the following notational conventions for the k-dimensional probability generating function

$$\mathrm{E}\left(z_1^{Z(t_1)} \cdots z_k^{Z(t_k)} \right) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_k=0}^{\infty} \mathrm{P}(Z(t_1) = i_1, \ldots, Z(t_k) = i_k) z_1^{i_1} \cdots z_k^{i_k},$$

with $0 < t_1 \leq \ldots \leq t_k$ and $z_1, \ldots, z_k \in [0, 1]$. We define

$$P_k(\bar{t}, \bar{z}) := P_k(t_1, \ldots, t_n; z_1, \ldots, z_k) := \mathrm{E}\left(z_1^{Z(t_1)} \cdots z_k^{Z(t_k)} \right)$$

and write, for $t \geq 0$,

$$P_k(t + \bar{t}, \bar{z}) := P_k(t + t_1, \ldots, t + t_k; z_1, \ldots, z_k).$$

Moreover, for $0 < y_1 < \ldots < y_k$, we write

$$P_k(t\bar{y}, \bar{z}) := P_k(ty_1, \ldots, ty_k; z_1, \ldots, z_k),$$

and assuming $0 < y_1 < \ldots < y_k < 1$,

$$P_k^*\big(t, \bar{y}, \bar{z}\big) := \mathrm{E}\Big(z_1^{Z(ty_1)} \cdots z_k^{Z(ty_k)}; Z(t) = 0\Big) = P_{k+1}(ty_1, \ldots, ty_k, t; z_1, \ldots, z_k, 0).$$

These conventions will be similarly applied to the functions

$$Q_k\big(\bar{t}, \bar{z}\big) := 1 - P_k\big(\bar{t}, \bar{z}\big), \quad Q_k^*\big(t, \bar{y}, \bar{z}\big) := 1 - P_k^*\big(t, \bar{y}, \bar{z}\big). \tag{35}$$

Our special interest is in the function

$$Q_k(t) := Q_k\big(t + \bar{t}, \bar{z}\big), \quad 0 = t_1 < \ldots < t_k, \quad z_1, \ldots, z_k \in [0, 1), \tag{36}$$

to be viewed as a counterpart of the function $Q(t)$ treated by Theorem 2. Recalling the compound parameters

$$h = \frac{a + \sqrt{a^2 + 4bd}}{2b}$$

and $c = 4bda^{-2}$, put

$$h_k := h\frac{1 + \sqrt{1 + cg_k}}{1 + \sqrt{1 + c}}, \quad g_k := g_k(\bar{y}, \bar{z}) := \sum_{i=1}^{k} z_1 \cdots z_{i-1}(1 - z_i)y_i^{-2}. \tag{37}$$

The key step of the proof of Theorem 1 is to show that for any given $1 = y_1 < y_2 < \ldots < y_k$,

$$tQ_k(t) \to h_k, \quad t_i := t(y_i - 1), \quad i = 1, \ldots, k, \quad t \to \infty. \tag{38}$$

This is done following the steps of our proof of $tQ(t) \to h$ given in Section 3.

Unlike $Q(t)$, the function $Q_k(t)$ is not monotone over t. However, monotonicity of $Q(t)$ was used in the proof of Theorem 2 only for the proof of (14). The corresponding statement

$$0 < q_1 \le tQ_k(t) \le q_2 < \infty, \quad t \ge t_0,$$

follows from the bounds $(1 - z_1)Q(t) \le Q_k(t) \le Q(t)$, which hold by the monotonicity of the underlying generating functions over $z_1, \ldots, z_n$. Indeed,

$$Q_k(t) \le Q_k(t, t + t_2, \ldots, t + t_k; 0, \ldots, 0) = Q(t),$$

and on the other hand,

$$Q_k(t) = Q_k(t, t + t_2, \ldots, t + t_k; z_1, \ldots, z_k) = \mathrm{E}\Big(1 - z_1^{Z(t)}z_2^{Z(t+t_2)} \cdots z_k^{Z(t+t_k)}\Big) \ge \mathrm{E}\Big(1 - z_1^{Z(t)}\Big),$$

where

$$\mathrm{E}\Big(1 - z_1^{Z(t)}\Big) \ge \mathrm{E}\Big(1 - z_1^{Z(t)}; Z(t) \ge 1\Big) \ge (1 - z_1)Q(t).$$

4.1. Proof of $tQ_k(t) \to h_k$

The branching property (8) of the GWO process gives

$$\prod_{i=1}^{k} z_i^{Z(t_i)} = \prod_{i=1}^{k} z_i^{\mathbf{1}_{\{L > t_i\}}} \prod_{j=1}^{N} z_i^{Z_j(t_i - \tau_j)}.$$

Given $0 < t_1 < \ldots < t_k < t_{k+1} = \infty$, we use

$$\prod_{i=1}^{k} z_i^{1_{\{L > t_i\}}} = 1_{\{L \le t_1\}} + \sum_{i=1}^{k} z_1 \cdots z_i 1_{\{t_i < L \le t_{i+1}\}}$$

to deduce the following counterpart of (9):

$$P_k(\bar{t}, \bar{z}) = \mathrm{E}_{t_1}\left(\prod_{j=1}^{N} P_k(\bar{t} - \tau_j, \bar{z})\right) + \sum_{i=1}^{k} z_1 \cdots z_i \mathrm{E}\left(\prod_{j=1}^{N} P_k(\bar{t} - \tau_j, \bar{z}); t_i < L \le t_{i+1}\right).$$

This implies

$$\begin{aligned}
P_k(\bar{t}, \bar{z}) = {} & \mathrm{E}_{t_1}\left(\prod_{j=1}^{N} P_k(\bar{t} - \tau_j, \bar{z})\right) + \sum_{i=1}^{k} z_1 \cdots z_i \mathrm{P}(t_i < L \le t_{i+1}) \\
& - \sum_{i=1}^{k} z_1 \cdots z_i \mathrm{E}\left(1 - \prod_{j=1}^{N} P_k(\bar{t} - \tau_j, \bar{z}); t_i < L \le t_{i+1}\right).
\end{aligned} \tag{39}$$

Using this relation we establish the following counterpart of Lemma 1.

Lemma 5. *Consider the function* (36) *and put* $P_k(t) := 1 - Q_k(t) = P_k(t + \bar{t}, \bar{z})$. *For* $0 < u < t$, *the relation*

$$\begin{aligned}
\Phi(h_k t^{-1}) = {} & \mathrm{P}(L > t) - \sum_{i=1}^{k} z_1 \cdots z_i \mathrm{P}(t + t_i < L \le t + t_{i+1}) \\
& + \mathrm{E}_u\left(\sum_{j=1}^{N} Q_k(t - \tau_j)\right) - Q_k(t) + \mathrm{E}_u(W_k(t)) + D_k(u, t)
\end{aligned} \tag{40}$$

holds with $t_{k+1} = \infty$,

$$W_k(t) := (1 - h_k t^{-1})^N + N h_k t^{-1} - \sum_{j=1}^{N} Q_k(t - \tau_j) - \prod_{j=1}^{N} P_k(t - \tau_j), \tag{41}$$

and

$$\begin{aligned}
D_k(u, t) := {} & \mathrm{E}\left(1 - \prod_{j=1}^{N} P_k(t - \tau_j); u < L \le t\right) + \mathrm{E}\left((1 - h_k t^{-1})^N - 1 + N h_k t^{-1}; L > u\right) \\
& + \sum_{i=1}^{k} z_1 \cdots z_i \mathrm{E}\left(1 - \prod_{j=1}^{N} P_k(t - \tau_j); t + t_i < L \le t + t_{i+1}\right).
\end{aligned} \tag{42}$$

Proof. According to (39),

$$\begin{aligned}
P_k(t) = {} & \mathrm{E}_u\left(\prod_{j=1}^{N} P_k(t - \tau_j)\right) + \mathrm{E}\left(\prod_{j=1}^{N} P_k(t - \tau_j); u < L \le t\right) \\
& + \sum_{i=1}^{k} z_1 \cdots z_i \mathrm{P}(t + t_i < L \le t + t_{i+1}) \\
& - \sum_{i=1}^{k} z_1 \cdots z_i \mathrm{E}\left(1 - \prod_{j=1}^{N} P_k(t - \tau_j); t + t_i < L \le t + t_{i+1}\right).
\end{aligned}$$

By the definition of $\Phi(\cdot)$,

$$\Phi\left(h_k t^{-1}\right) + 1 = \mathrm{E}_u\left(\left(1 - h_k t^{-1}\right)^N + N h_k t^{-1}\right) + \mathrm{P}(L > t)$$
$$+ \mathrm{E}\left(\left(1 - h_k t^{-1}\right)^N - 1 + N h_k t^{-1}; L > u\right) + \mathrm{P}(u < L \le t),$$

and after subtracting the two last equations, we get

$$\Phi\left(h_k t^{-1}\right) + Q_k(t) = \mathrm{E}_u\left(\left(1 - h_k t^{-1}\right)^N + N h_k t^{-1} - \prod_{j=1}^{N} P_k\left(t - \tau_j\right)\right) + \mathrm{P}(L > t)$$
$$- \sum_{i=1}^{k} z_1 \cdots z_i \mathrm{P}(t + t_i < L \le t + t_{i+1}) + D_k(u, t),$$

with $D_k(u, t)$ satisfying (42). After a rearrangement, the relation (40) follows together with (41). □

With Lemma 5 in hand, the convergence (38) is proven by applying almost exactly the same argument as used in the proof of $tQ(t) \to h$. An important new feature emerges because of the additional term in the asymptotic relation defining the limit h_k. Let $1 = y_1 < y_2 < \ldots < y_k < y_{k+1} = \infty$. Since

$$\sum_{i=1}^{k} z_1 \cdots z_i \mathrm{P}\left(t y_i < L \le t y_{i+1}\right) \sim d t^{-2} \sum_{i=1}^{k} z_1 \cdots z_i\left(y_i^{-2} - y_{i+1}^{-2}\right),$$

we see that

$$\mathrm{P}(L > t) - \sum_{i=1}^{k} z_1 \cdots z_i \mathrm{P}\left(t y_i < L \le t y_{i+1}\right) \sim d g_k t^{-2},$$

where g_k is defined by (37). Assuming $0 \le z_1, \ldots, z_k < 1$, we ensure that $g_k > 0$, and as a result, we arrive at a counterpart of the quadratic equation (29),

$$b h_k^2 = a h_k + d g_k,$$

which gives

$$h_k = \frac{a + \sqrt{a^2 + 4 b d g_k}}{2b} = h \frac{1 + \sqrt{1 + c g_k}}{1 + \sqrt{1 + c}},$$

justifying our definition (37). We conclude that for $k \ge 1$,

$$\frac{Q_k(t\bar{y}, \bar{z})}{Q(t)} \to \frac{1 + \sqrt{1 + c \sum_{i=1}^{k} z_1 \cdots z_{i-1}(1 - z_i) y_i^{-2}}}{1 + \sqrt{1 + c}},$$
$$1 = y_1 < \ldots < y_k, \quad 0 \le z_1, \ldots, z_k < 1. \qquad (43)$$

4.2. Conditioned generating functions

To finish the proof of Theorem 1, consider the generating functions conditioned on the survival of the GWO process. Given (5) with $j \ge 1$, we have

$$Q(t)\mathrm{E}\left(z_1^{Z(ty_1)} \cdots z_k^{Z(ty_k)} | Z(t) > 0\right) = \mathrm{E}(z_1^{Z(ty_1)} \cdots z_k^{Z(ty_k)}; Z(t) > 0)$$
$$= P_k(t\bar{y}, \bar{z}) - \mathrm{E}\left(z_1^{Z(ty_1)} \cdots z_k^{Z(ty_k)}; Z(t) = 0\right) \stackrel{(35)}{=} Q_j^*\left(t, \bar{y}, \bar{z}\right) - Q_k(t\bar{y}, \bar{z}),$$

and therefore,

$$\mathrm{E}\Big(z_1^{Z(ty_1)}\cdots z_k^{Z(ty_k)}|Z(t)>0\Big) = \frac{Q_j^*\big(t,\bar{y},\bar{z}\big)}{Q(t)} - \frac{Q_k(t\bar{y},\bar{z})}{Q(t)}.$$

Similarly, if (5) holds with $j=0$, then

$$\mathrm{E}\Big(z_1^{Z(ty_1)}\cdots z_k^{Z(ty_k)}|Z(t)>0\Big) = 1 - \frac{Q_k(t\bar{y},\bar{z})}{Q(t)}.$$

Letting $t'=ty_1$, we get

$$\frac{Q_k(t\bar{y},\bar{z})}{Q(t)} = \frac{Q_k(t', t'y_2/y_1, \ldots, t'y_k/y_1)}{Q(t')}\frac{Q(ty_1)}{Q(t)},$$

and applying the relation (43), we have

$$\frac{Q_k(t\bar{y},\bar{z})}{Q(t)} \to \frac{1 + \sqrt{1 + \sum_{i=1}^k z_1\cdots z_{i-1}(1-z_i)\Gamma_i}}{\big(1+\sqrt{1+c}\big)y_1},$$

where $\Gamma_i = c(y_1/y_i)^2$. On the other hand, since

$$Q_j^*\big(t,\bar{y},\bar{z}\big) = Q_{j+1}(ty_1, \ldots, ty_j, t; z_1, \ldots, z_j, 0), \quad j\geq 1,$$

we also get

$$\frac{Q_j^*\big(t,\bar{y},\bar{z}\big)}{Q(t)} \to \frac{1 + \sqrt{1 + \sum_{i=1}^j z_1\cdots z_{i-1}(1-z_i)\Gamma_i + cz_1\cdots z_j y_1^2}}{\big(1+\sqrt{1+c}\big)y_1}.$$

We conclude that as stated in Section 2,

$$\mathrm{E}\Big(z_1^{Z(ty_1)}\cdots z_k^{Z(ty_k)}|Z(t)>0\Big) \to \mathrm{E}\Big(z_1^{\eta(y_1)}\cdots z_k^{\eta(y_k)}\Big).$$

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