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The dimension of the moduli space of pointed algebraic curves of low genus

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ABSTRACT

We explicitly compute the moduli space pointed algebraic curves with a given numerical semigroup as Weierstrass semigroup for many cases of genus at most seven and when possible describe the structure as cone over an explicitly given variety. We determine the dimension of the moduli space for all semigroups of genus seven.

1. Introduction

On a smooth projective curve C the pole orders of rational functions with poles only at a given point P form a numerical semigroup, the Weierstrass semigroup. The space $\mathcal{M}_{g,1}^{\Gamma}$ parametrising pointed smooth curves with Weierstrass semigroup at the marked point equal to Γ is a locally closed subspace of the moduli space $\mathcal{M}_{g,1}$. In this paper we compute the dimension of $\mathcal{M}_{g,1}^{\Gamma}$ for all semigroups of genus at most seven.

By the famous result of Pinkham [22] the space $\mathcal{M}_{g,1}^{\Gamma}$ is closely related to the negative weight part of the versal deformation of the monomial curve singularity C_{Γ} with semigroup Γ. This connection has been used in a series of papers by Nakano–Mori [21] and Nakano [19, 20] to explicitly determine $\mathcal{M}_{g,1}^{\Gamma}$ for many semigroups of genus at most six, using the SINGULAR [6] package deform.lib [18]. In all these cases $\mathcal{M}_{g,1}^{\Gamma}$ is irreducible and rational. For the remaining cases (with two exceptions) irreduciblity and stably rationality was shown by Bullock [2], with different methods.

We extend the computations of Nakano [19]. One quickly runs into the limits of what can be computed in reasonable time. Therefore, we also use other approaches to compute deformations. One method is to use Hauser's algorithm [10]; the method of Contiero-Stöhr [3] to compute $\mathcal{M}_{g,1}^{\Gamma}$ is closely related. In this method one first perturbs the equations in all possible ways, and takes care of flatness only later. This means introducing may new variables, most of which can be eliminated. In a number of cases this approach is successful. In one case it is more convenient to use the projection method developed by De Jong and Van Straten [14], as applied to curves in [24].

We list the semigroups of genus at most 7 in Tables 1 and 2. They are easily computed by hand, but one can also check that the results are correct with the package NumericalSpgs [7] for the GAP system. For $g \leq 6$ we follow the notation of [19]. The corresponding gap sequences are for $g \leq 7$ already (correctly)

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listed by Haure [9], in the first published paper containing the term Weierstrass points. Some of his conclusions are clearly wrong, but his style of writing, common for the time, makes it difficult to assess what is correct. Haure determines a plane model and finds in that way the number of moduli on which curves with given Weierstrass semigroup depend. The end of [9, Chapter IV] gives (mostly without proof) a list of gap sequences and the corresponding plane model for $g \leq 7$. We did not study this plane model, but only compared the number of moduli with the dimension we found. Our computations show that Haure's numbers are correct except in two consecutive cases. The non-emptiness of $\mathcal{M}_{g,1}^{\Gamma}$ for all semigroups with $g \leq 7$ was established in a different way by Komeda [16].

We describe the structure of $\mathcal{M}_{g,1}^{\Gamma}$ in the cases we have been able to determine it. In many cases, e.g. if the monomial curve C_{Γ} is a complete intersection, the space $\mathcal{M}_{g,1}^{\Gamma}$ is smooth. The next common case is that $\mathcal{M}_{g,1}^{\Gamma}$ is a weighted cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$; the curve C_{Γ} has then codimension 3 and is given by 6 equations. For codimension 4 and 10 equations the moduli space is typically given by 20 equations and the exact structure depends on the curve. Except for the curve already studied in [24] these equations are too complicated, with too many monomials, to be useful. Other interesting occurring varieties are the Grassmannian $G(2,5)$ and a degeneration of it.

As to the dimension of $\mathcal{M}_{g,1}^{\Gamma}$, in general the following bounds are known [4, 23]:

$$2g - 2 + t - \dim \mathrm{T}^{1,+} \leq \dim \mathcal{M}_{g,1}^{\Gamma} \leq 2g - 2 + t .$$

where t is the rank of the highest syzygy module of the ideal of C_{Γ}, and $\dim \mathrm{T}^{1,+}$ the number of deformations of positive weight, both easily computable with SINGULAR [6] or *Macaulay2* [5]. The result of our computations is that for all semigroups with $g \leq 7$ the dimension is given by the lower bound.

In the first section we recall the relation between the moduli space $\mathcal{M}_{g,1}^{\Gamma}$ and deformations of the monomial curve with semigroup Γ. The next section describes the computation methods used in this paper. The main part of the paper discusses the computation of the moduli space or of its dimension for the different types of semigroup.

2. The moduli space $\mathcal{M}_{g,1}^{\Gamma}$

Let P be a smooth point on a possibly singular integral complete curve C of arithmetic genus $g > 1$, defined over an algebraically closed field $\mathbf{k}$ of characteristic zero. An integer $n \in \mathbb{N}$ is a *gap* if there does not exist rational function on C with pole divisor nP, or equivalently $H^0(C, \mathcal{O}_C(n-1)P)) = H^0(C, \mathcal{O}_C(nP))$. There are exactly g gaps by the Weierstrass gap theorem, an easy consequence of Riemann-Roch. The nongaps form a numerical semigroup Γ, the *Weierstrass semigroup* of C at P; this is the set of nonnegative integers $n \in \mathbb{N}$ such that there is a rational function on C with pole divisor nP. For any numerical semigroup the *genus* is defined as the number of gaps.

Given a numerical semigroup Γ of genus $g > 1$, let $\mathcal{M}_{g,1}^{\Gamma}$ be the space parameterizing pointed smooth curves with Γ as Weierstrass semigroup at the marked point. It is a locally closed subspace of the moduli space $\mathcal{M}_{g,1}$ of pointed smooth curves of genus g. Note that $\mathcal{M}_{g,1}^{\Gamma}$ can be empty.

The connection between the moduli space $\mathcal{M}_{g,1}^{\Gamma}$ and deformations of negative weight of monomial curves was first observed by Pinkham [22, Chapter 13]. Given a numerical semigroup $\Gamma = \langle n_1, \ldots, n_r \rangle$ we form the semigroup ring $\mathbf{k}[\Gamma] := \oplus_{n \in \Gamma} \mathbf{k} t^n$ and denote by $C_{\Gamma} := \mathrm{Spec}\,\mathbf{k}[\Gamma]$ its associated affine monomial curve. Consider the versal deformation of C_{Γ}

$$
\begin{array}{ccc}
\mathcal{X}_{t_0} \cong C_{\Gamma} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
\{t_0\} = \mathrm{Spec}\,\mathbf{k} & \longrightarrow & B
\end{array}
$$

where $B = \mathrm{Spec}\,A$ is the spectrum of local, complete noetherian $\mathbf{k}$-algebra. Pinkham [22] showed that the natural $\mathbb{G}_m$-action on C_{Γ} can be extended to the total and parameter spaces. This induces a grading on the tangent space $T_{C_{\Gamma}}^1$ to B. The convention here is that a deformation has negative weight $-e$ if it

decreases the weights of the equations of the curve by e; the corresponding deformation variable has then (positive) weight e. A numerical semigroup Γ is called *negatively graded* if $T^1_{C_\Gamma}$ has no positive graded part.

Let B^- be the subspace of B with negative weights. Then the restriction $\mathcal{X}^- \to B^-$ is versal for deformations with good $\mathbb{G}_m$-action. Both $\mathcal{X}^-$ and B^- are defined by polynomials and we use the same symbols for the corresponding affine varieties. The deformation $\mathcal{X}^- \to B^-$ can be fiberwise compactified to $\overline{\mathcal{X}}^- \to B^-$; each fiber is an integral curve in a weighted projective space with one point P at infinity and this is a point with semigroup Γ. All the fibers over a given $\mathbb{G}_m$ orbit in $\mathcal{T}^-$ are isomorphic, and two fibers are isomorphic if and only if they lie in the same orbit. This is proved in [22] for smooth fibers and in general in the Appendix of [17].

Each pointed curve from $\mathcal{M}^\Gamma_{g,1}$ occurs as fiber by the following construction. Consider the section ring $\mathcal{R} = \oplus_{n=0}^\infty H^0(C, \mathcal{O}(nP))$. It gives an embedding of $C = \operatorname{Proj}\mathcal{R}$ in a weighted projective space, with coordinates $X_0, \ldots, X_r$ where $\deg X_0 = 1$. The space $\operatorname{Spec}\mathcal{R}$ is the corresponding quasi-cone in affine space. Setting $X_0 = 0$ defines the monomial curve C_Γ, all other fibers are isomorphic to $C \setminus P$. In particular, if C is smooth, this construction defines a smoothing of C_Γ.

Theorem 2.1 ([22, Theorem 13.9]). *Let $\mathcal{X}^- \to B^-$ be the equivariant negative weight miniversal deformation of the monomial curve C_Γ for a given semigroup Γ and denote by B^-_s the open subset of B^- given by the points with smooth fibers. Then the moduli space $\mathcal{M}^\Gamma_{g,1}$ is isomorphic to the quotient $\mathcal{M}^\Gamma_{g,1} = B^-_s/\mathbb{G}_m$ of B^-_s by the $\mathbb{G}_m$-action.*

The closure of a component of B^-_s is a smoothing component and is itself contained in a smoothing component in B. For quasihomogeneous curve singularities there is a simple formula for the dimension of smoothing components: it is $\mu + t - 1$ [8], with $\mu = 2\delta - r + 1$ the Milnor number and $t = \dim_\mathbf{k} \operatorname{Ext}^1_\mathcal{O}(\mathbf{k}, \mathcal{O})$ the type. For monomial curves $\delta = g$ and $r = 1$, and the type can be computed from the semigroup [1, 4.1.2] : $t = \lambda(\Gamma)$, the number of gaps ℓ of Γ such that $\ell + n \in \Gamma$ whenever n is a nongap. Given the equations of C_Γ (anyway needed for deformation computations) the type is easily found as the rank of the highest syzygy module.

Let $\dim T^{1,+}$ be the dimension of the space of infinitesimal deformations of C_Γ of positive weight. Then we have the following bounds for the dimension of components of M [4]; the upper bound is due to Rim–Vitulli [23].

Theorem 2.2. *Let Γ be a numerical semigroup Γ of genus bigger than 1. If $\mathcal{M}^\Gamma_{g,1}$ is nonempty, then for any irreducible component E of $\mathcal{M}^\Gamma_{g,1}$*

$$2g - 2 + t - \dim T^{1,+} \leq \dim E \leq 2g - 2 + t.$$

3. Computing deformation spaces in negative weight

By Pinkham's theorem (Theorem 2.1), to explicitly describe the moduli space $\mathcal{M}^\Gamma_{g,1}$ one can compute the negative weight part of the versal deformation of the monomial curve C_Γ. For many semigroups of low genus this was done by Nakano-Mori [21] and Nakano [19, 20], using the computer algebra system SINGULAR [6]. The main obstacle in the remaining cases is that the computations take too long, and result in long formulas without apparent structure. In this section we describe several methods to determine versal deformations, with comments on computational matters.

3.1. The standard approach

We recall the main steps, see also [25, Chapter 3]. Let X be a variety with $\mathbb{G}_m$-action with isolated singularity at the origin in $\mathbb{A}^n$. Let $S = \mathbf{k}[X_1, \ldots, X_n]$ be the polynomial ring in n variables. Let

$f = (f_1, \ldots, f_r)$ generate the ideal $I(X)$ of X. The first few terms of the resolution of $\mathbf{k}[X] = S/I(X)$ are

$$0 \longleftarrow \mathbf{k}[X] \longleftarrow S \xleftarrow{f} S^k \xleftarrow{r} S^l$$

where the columns of the matrix r generate the module of relations. Let $X_B \to B$ be a deformation of X over $B = \operatorname{Spec} A$. The flatness of the map $X_B \to B$ translates into the existence of a lifting of the resolution to

$$0 \longleftarrow \mathbf{k}[X_B] \longleftarrow S \otimes A \xleftarrow{F} (S \otimes A)^k \xleftarrow{R} (S \otimes A)^l \, .$$

To find the versal deformation we must find a lift $FR = 0$ in the most general way. The first step is to compute infinitesimal deformations. We write $F = f + \varepsilon f'$ and $R = r + \varepsilon r'$. As $\varepsilon^2 = 0$, the condition $FR = 0$ gives

$$FR = (f + \varepsilon f')(r + \varepsilon r') = fr + \varepsilon(fr' + f'r) = 0 \, .$$

Because $fr = 0$, we obtain the equation $fr' + f'r = 0$ in S. We first solve the equation $f'r = 0$ or rather its transpose $r^t(f')^t = 0$ in $\mathbf{k}[X]$. This means finding syzygies between the columns of the matrix r^t; then we find r' by lifting $f'r$ with f. After this we lift order for order. Obstructions to do this may come up, leading to equations in the deformation parameters.

All these computations can be done with a computer algebra system. Indeed, they are implemented in the packages *VersalDeformations* [12] for *Macaulay2* [5] and `Deform.lib` [18] for SINGULAR [6]. The specific outcome of a computation, which depends on Groebner basis calculations, is governed by the chosen monomial ordering and also by the choice of the generators $(f_1, \ldots, f_k)$ of the ideal $I(X)$. The algorithm tries to find the row vector F, equations of the base space come from obstructions to do that. Typically a computer computation will not choose the easiest form of the base equations.

When restricting to deformations of negative weight all resulting equations are polynomial and the computation is finite; it might be undoable in practice, even with a powerful computer.

3.2. Hauser's algorithm

An alternative method was developed in the complex analytic setting by Hauser [10, 11]. One can see the method of Contiero-Stöhr [3] to compute a compactification of the moduli space $\mathcal{M}_{g,1}^{\Gamma}$ as a variant. It has been used in [4] to compute the base space for several families of Gorenstein monomials curves. We start again from the generators f of the ideal $I(X)$, but now we perturb f in the most general way, modulo trivial perturbations, that is we take a semi-universal unfolding of the associated map $f : \mathbb{A}^n \to \mathbb{A}^k$. Except when X is zero dimensional, the base space of this unfolding will be infinite dimensional; this problem is handled carefully by Hauser [11]. In our situation f is weighted homogeneous and we restrict ourselves to an unfolding with terms of lower degree. Therefore we are back in a finite dimensional situation, and we can work over any field $\mathbf{k}$. So we have an unfolding $F : \mathbb{A}^n \times \mathbb{A}^s \to \mathbb{A}^k$ of f. Now we determine the locus B containing $0 \in \mathbb{A}^s$ over which F is flat. The restriction of F to B is then the versal deformation of X of negative weight.

In our situation we have a monomial curve X of multiplicity m and embedding dimension $e + 1$. We take coordinates $x, y_1, \ldots, y_e$. An Apry basis of the semigroup leads to an additive realization of $\mathbf{k}[X]$ as $\sum_{i=0}^{m} y^{(i)} \mathbf{k}[x]$, where $y^{(0)} = 1$, $y^{(1)} \ldots y^{(m-1)}$ are expressions in the variables $y_1, \ldots, y_e$. The equations of X are then (in multi-index notation) of the form $y^\alpha = \varphi$ with $\varphi \in \sum_{i=0}^{m} y^{(i)} \mathbf{k}[x]$. The unfolding is also done only with terms from $\sum_{i=0}^{m} y^{(i)} \mathbf{k}[x]$. We start from generators f of the said form, compute the relation matrix r and write the unfolding F. We have to lift $fr = 0$ to $FR = 0$. To this end we compute Fr and reduce this column vector to normal form with respect to the list F. It is important that we do not compute a Groebner basis of the ideal generated by F, as this will take too long. But reducing with respect to F will result in a vector with entries of bounded degree lying in $\sum_{i=0}^{m} y^{(i)} R[x]$ with coefficients from $R = \mathbf{k}[t_1, \ldots, t_s]$, where the t_j are coordinates on the base $\mathbb{A}^s$. The vanishing of these coefficients define the locus where $FR = 0$, so where F is flat.

This procedure leads to a rather large number of relatively simple equations in a large number of variables, most of which occur linearly and can be eliminated. It is this process of elimination which can lead to few equations in a limited number of variables, but with many monomials, see the proof of Proposition 4.5 for an example.

Also here most computations are easily done with a computer algebra system. The first step, to find the unfolding, can be automatised, but for the not too complicated cases relevant for this paper it seems preferable to do it by hand, choosing names for the deformation variables reflecting their weights.

3.3. The projection method

Computing deformations using projections onto a hypersurface is a method developed in a series of papers by Theo de Jong and Duco van Straten, see [13, 14]. The application to curves is in [24], see also [25, Chapter 11]. Let again X be a monomial curve and $X \to Y$ a projection onto a plane curve, which is a finite generically injective map. Let Σ be the subspace of Y by the conductor ideal $I = \mathrm{Hom}_Y(\mathcal{O}_X, \mathcal{O}_Y)$ in $\mathcal{O}_Y$. This makes it possible to reconstruct X, as $\mathcal{O}_X = \mathrm{Hom}_Y(I, \mathcal{O}_Y)$. Because we use this method only once, we refer to [25, Chapter 11] for a description how to use deformations the plane curve Y together with Σ to get deformations of the original curve X.

The space Σ is a fat point, so in particular Cohen-Macaulay of codimension 2. Therefore the ideal I defining Σ in $\mathbb{A}^2$ is generated by the maximal minors $\Delta_1 \ldots \Delta_k$ of an $k \times (k-1)$ matrix M. We write these generators as row vector Δ. The curve Y is defined by a function of the form $f = \Delta\alpha$ with α a column vector, or equivalently by the determinant of the matrix (M, α). We write an element $n \colon \Delta_i \mapsto n_i$ of the normal module $N := \mathrm{Hom}_\Sigma(I/I^2, \mathcal{O}_\Sigma)$ as row vector n. A deformation $\Sigma_B \to Y_B$ comes from a deformation of the curve X if for every normal vector n_B there exists a vector γ_B on the ambient space, satisfying

$$n_B\alpha_B + \Delta_B\gamma_B = 0 \,. \tag{1}$$

This is the basic deformation equation, which can be solved step by step.

When restricting to deformations of negative weight the result of the computation is again given by quasihomogeneous matrices with polynomial entries. Once setup correctly the computation is easily done with a computer algebra system.

An important concept here is that of I^2-equivalence [13, Definition 1.14]: two functions f and g are I^2-equivalent, if and only if $f - g \in I^2$. Suppose $f = \Delta\alpha$ and $g = \Delta\beta$ are I^2-equivalent. Then $\alpha - \beta = A\Delta^t$ for some matrix A. Suppose $n_B\alpha_B + \Delta_B\gamma_B$ is a lift of $n\alpha + \Delta\gamma$ over a base space B. Choose any lift A_B of A. Then

$$n_B(\alpha_B - A_B\Delta_B^t) + \Delta_B(\gamma_B + A_B^t n_B^t) \tag{2}$$

is a lift of $n\beta + \Delta(\gamma + A^t n^t)$. In particular, for curves with projections defined by I^2-equivalent functions, the base spaces of the versal deformation are the same up to a smooth factor.

4. Semigroups of genus $g \le 7$

In Tables 1 and 2 we list the semigroups of genus at most 7. For $g \le 6$ we follow the notation of [19]. The tables also contain also the dimension d of $\mathcal{M}_{g,1}^\Gamma$ and the type t of the semigroup. Furthermore they give under the heading base the structure of $\mathcal{M}_{g,1}^\Gamma$ in the cases we have been able to determine it; the entries indicating the different possibilities are discussed below. Inspection of the tables shows that the main parameters governing the structure of $\mathcal{M}_{g,1}^\Gamma$ are the number of generators of Γ and the type t. The first step in our computations is always to find equations for the monomial curve C_Γ, followed by the free resolution. This gives the type t. The next step is to find the graded parts of the vector space T^1 of infinitisemal deformations.

Proposition 4.1. *For all semigroups with $g \le 7$ the dimension of $\mathcal{M}_{g,1}^\Gamma$ is given by the lower bound $2g - 2 + t - \dim T^{1,+}$ of Theorem 2.2.*

Table 1. semigroups of genus ≤ 6.

[19]	semigroup	d	t	base	[19]	semigroup	d	t	base
$N(1)_1$	2,3	1	1	sm	$N(6)_1$	2,13	11	1	sm
$N(2)_1$	2,5	3	1	sm	$N(6)_2$	3,10,11	12	2	sm
$N(2)_2$	3,4,5	4	2	sm	$N(6)_3$	3,8,13	11	2	sm
$N(3)_1$	2,7	5	1	sm	$N(6)_4$	3,7	10	1	sm
$N(3)_2$	3,5,7	6	2	sm	$N(6)_5$	4,9,10,11	13	3	B_2
$N(3)_3$	3,4	5	1	sm	$N(6)_6$	4,7,10,13	12	3	B_1
$N(3)_4$	4,5,6,7	7	3	B_1	$N(6)_7$	4,7,9	11	2	sm
$N(4)_1$	2,9	7	1	sm	$N(6)_8$	4,6,11,13	11	3	B_1^*
$N(4)_2$	3,7,8	8	2	sm	$N(6)_9$	4,6,9	10	1	sm
$N(4)_3$	3,5	7	1	sm	$N(6)_{10}$	4,5	10	1	sm
$N(4)_4$	4,6,7,9	9	3	B_1	$N(6)_{11}$	5,8,9,11,12	14	4	?
$N(4)_5$	4,5,7	8	2	sm	$N(6)_{12}$	5,7,9,11,13	13	4	!
$N(4)_6$	4,5,6	7	1	sm	$N(6)_{13}$	5,7,8,11	12	3	B_1
$N(4)_7$	5,6,7,8,9	10	4	!	$N(6)_{14}$	5,7,8,9	11	1	sm
$N(5)_1$	2,11	9	1	sm	$N(6)_{15}$	5,6,9,13	12	3	B_1
$N(5)_2$	3,8,10	10	2	sm	$N(6)_{16}$	5,6,8	11	2	sm
$N(5)_3$	3, 7, 11	9	1	sm	$N(6)_{17}$	5,6,7	10	2	sm
$N(5)_4$	4,7,9,10	11	3	B_1	$N(6)_{18}$	6,8,9,10,11,13	15	5	?
$N(5)_5$	4,6,9,11	10	3	B_1	$N(6)_{19}$	6,7,9,10,11	14	4	?
$N(5)_6$	4,6,7	9	1	sm	$N(6)_{20}$	6,7,8,10,11	13	3	G'
$N(5)_7$	4,5,11	9	2	sm	$N(6)_{21}$	6,7,8,9,11	12	2	G
$N(5)_8$	5,7,8,9,11	12	4	?	$N(6)_{22}$	6,7,8,9,10	11	1	G
$N(5)_9$	5,6,8,9	11	3	B_1	$N(6)_{23}$	7,...,13	16	6	?
$N(5)_{10}$	5,6,7,9	10	2	sm					
$N(5)_{11}$	5,6,7,8	9	1	sm					
$N(5)_{12}$	6,...,11	13	5	?					

Table 2. Semigroups of genus 7.

name	semigroup	d	t	base	name	semigroup	d	t	base
$N(7)_1$	2,15	13	1	sm	$N(7)_{21}$	5,6,9	12	2	sm
$N(7)_2$	3,11,13	14	2	sm	$N(7)_{22}$	6,9,10,11,13,14	17	5	?
$N(7)_3$	3,10,14	13	2	sm	$N(7)_{23}$	6,8,10,11,13,15	16	5	?
$N(7)_4$	3,8	12	1	sm	$N(7)_{24}$	6,8,9,11,13	15	4	?
$N(7)_5$	4,10,11,13	15	3	B_2	$N(7)_{25}$	6,8,9,10,13	14	2	G'
$N(7)_6$	4,9,11,14	14	3	B_1	$N(7)_{26}$	6,8,9,10,11	13	1	G
$N(7)_7$	4,9,10,15	13	3	B_1	$N(7)_{27}$	6,7,10,11,15	15	4	?
$N(7)_8$	4,7,13	13	2	sm	$N(7)_{28}$	6,7,9,11	14	3	B_1
$N(7)_9$	4,7,10	12	1	sm	$N(7)_{29}$	6,7,9,10	13	3	B_1
$N(7)_{10}$	4,6,13,15	12	3	B_1^*	$N(7)_{30}$	6,7,8,11	13	3	B_1
$N(7)_{11}$	4,6,11	10	1	sm	$N(7)_{31}$	6,7,8,10	12	2	sm
$N(7)_{12}$	5,9,11,12,13	16	4	?	$N(7)_{32}$	6,7,8,9	11	2	sm
$N(7)_{13}$	5,8,11,12,14	15	4	?	$N(7)_{33}$	7,9,...,13,15	18	6	?
$N(7)_{14}$	5,8,9,12	14	3	B_1	$N(7)_{34}$	7,8,10,11,12,13	17	5	?
$N(7)_{15}$	5,8,9,11	13	2	sm	$N(7)_{35}$	7,8,9,11,12,13	16	4	?
$N(7)_{16}$	5,7,11,13	14	3	B_1	$N(7)_{36}$	7,8,9,10,12,13	15	3	?
$N(7)_{17}$	5,7,9,13	13	2	sm	$N(7)_{37}$	7,8,9,10,11,13	14	2	?
$N(7)_{18}$	5,7,9,11	12	1	sm	$N(7)_{38}$	7,8,9,10,11,12	13	1	?
$N(7)_{19}$	5,7,8	12	2	sm	$N(7)_{39}$	8,...,15	19	7	?
$N(7)_{20}$	5,6,13,14	13	3	B_1					

This result follows from the computations discussed in the rest of this section.

4.1. Smooth base space

The base space of the versal deformation of C_Γ is smooth (indicated with 'sm' in the tables) if the obstruction space T^2 vanishes. This happens if the curve is a complete intersection, or Gorenstein of codimension three, or Cohen-Macaulay of codimension two. In the latter case the equations are the vanishing the minors of a 2×3 matrix, and $t = 2$. Also $T^2 = 0$ for codimension 3 curves with $t = 2$.

In all these cases the dimension of $\mathcal{M}_{g,1}^{\Gamma}$ is dim $T^{1,-} - 1$ and therefore the lower bound of Theorem 2.2 holds. For $N(7)_{11}$ the dimension is 12, whereas Haure gives 10 [9].

4.2. Cone over a Segre embedding

For codimension 3 curves C_{Γ} with $t = 3$ and $g \leq 7$ it is possible to explicitly determine the structure of the base space. Most cases with $g \leq 6$ were computed by Nakano [19].

The result for $N(6)_6 = \langle 4, 7, 10, 13 \rangle$ was not given in [19]. We computed the deformation using Bernd Martin's package `deform.lib` [18] for SINGULAR [6]. The equations for the curve are determinantal:

$$\begin{bmatrix} x & y_7 & y_{10} & y_{13} \\ y_7 & y_{10} & y_{13} & x^4 \end{bmatrix}$$

The speed of the computation in SINGULAR depends very much on the chosen ordering. A good choose is using the variables (y_{13}, y_{10}, y_7, x) in this order with graded reverse lexicographic order, but with weights of the variables all equal to 1, not using the weights $13, 10, 7, 4$. SINGULAR returns an ideal Js in 16 variables $A, \ldots, P$ of weight 2, 6, 10, 1, 5, 9, 13, 8, 12, 16, 7, 10, 4, 1, 4, 7. It is generated by the minors of

$$\begin{bmatrix} N & -K + P \\ O & C - L - BM + AM^2 \\ P & 2G - 2FM + 2EM^2 - 2DM^3 - M^3N \\ C - BM + AM^2 & -J + IM - HM^2 - M^4 + M^3O \end{bmatrix} \tag{3}$$

We are allowed to simplify the equations of the base space by a coordinate transformation. An obvious transformation gives the matrix of the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$.

Observe that the coordinate ring of the Segre cone has a resolution of the form

$$0 \longleftarrow S/I \longleftarrow S \xleftarrow{f} S^6 \xleftarrow{r} S^8 \xleftarrow{s} S^3 \longleftarrow 0$$

where f is the row vector of minors of the matrix

$$\begin{bmatrix} X_1 & X_2 & X_3 & X_4 \\ Y_1 & Y_2 & Y_3 & Y_4 \end{bmatrix}$$

and the 8×3 matrix s is the transpose of a matrix of the form

$$\begin{bmatrix} X_1 & X_2 & X_3 & X_4 & 0 & 0 & 0 & 0 \\ Y_1 & Y_2 & Y_3 & Y_4 & X_1 & X_2 & X_3 & X_4 \\ 0 & 0 & 0 & 0 & Y_1 & Y_2 & Y_3 & Y_4 \end{bmatrix}$$

Computing the resolution of the ideal Js with SINGULAR gives indeed a 8×3 matrix with some zeroes, but of course not exactly in the form given above. The matrix (3) was found by column and row operations on the matrix computed with SINGULAR.

The Segre cone occurs for many curves as base space. A necessary condition is that dim $T^2 = 6$. For some curves dim $T^2 = 12$. This happens for the semigroups $\langle n_1, n_2, n_3, n_4 \rangle$ in the tables for which the first blow-up also has four generators. Then the base space has a more complicated structure.

Proposition 4.2. *For monomial curve singularity of genus at most 7 of codimension 3, with type $t = 3$, such that the first blow up has lower embedding dimension, the base space of negative weight is up to a smooth factor the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^3$, except in the cases $N(6)_8$ and $N(7)_{10}$, where it has two components, being the intersection of the Segre cone with coordinate hyperplanes (B_1^* in Tables 1 and 2).*

Proof. By the assumption in the statement dim $T^2 = 6$. Most cases with $g \leq 6$ were computed by Nakano [19]. It can be checked that systems of generators given in [19] are minors of 2×4 matrices.

To identify the base space as Segre cone it in fact suffices to show that the quadratic part of the equations defines a Segre cone. If it does, the Segre cone has to be the tangent cone of the base space, for otherwise the dimension of the tangent cone should be less than the dimension of the Segre cone, but this dimension is in all cases equal to the lower bound of Theorem 2.2. Because the Segre cone is rigid and the base space itself is a deformation of its tangent cone, they are isomorphic.

For the semigroups with 4 generators and $t = 3$ of genus $g = 7$ (see Table 2) the versal deformation up to degree 2 is easily computed with SINGULAR, and the cases where the base space is B_1 identified.

For $N(6)_8 = \langle 4, 6, 11, 13 \rangle$ the versal deformation in all degrees can be computed. The generators of the ideal of the curve are the 2×2 minors of the symmetric matrix

$$\begin{bmatrix} x & y_6 & y_{11} \\ y_6 & x^2 & y_{13} \\ y_{11} & y_{13} & x^3 y_6 \end{bmatrix}$$

With these generators and the graded reverse lexicographic order with variables (y_{13}, y_{11}, y_6, x) of weights $(13, 11, 6, 4)$ SINGULAR succeeds in computing rather quickly the versal deformation in all degrees. Replacing the generator $y_{11} y_{13} - x^3 y_6^2$ by $y_{11} y_{13} - x^6$ results in a computation which does not finish in reasonable time. After a coordinate transformation the base space is given by the minors of

$$\begin{bmatrix} T_{-1} & T_1 & T_6 & T_8 \\ T_9 & T_{11} & T_{16} & T_{18} \end{bmatrix}$$

where the indices indicate the weight of the deformation variables. The base space is again a Segre cone, but the base space in negative weight lies in the hyperplane $T_{-1} = 0$ and consists of two components, one smooth given by $T_1 = T_6 = T_8 = 0$ and the other by $T_9 = 0$ and the vanishing of the three minors of the matrix not involving T_9. Note that the last generator of the ideal of the second component given in [19, p. 159] can be expressed in the previous ones.

For $N(7)_{10} = \langle 4, 6, 13, 15 \rangle$ the quadratic part of the equations for the bases space are the minors of

$$\begin{bmatrix} T_{-3} & T_{-1} & T_6 & T_8 \\ T_{11} & T_{13} & T_{20} & T_{22} \end{bmatrix}$$

and in negative weight there are two components of different dimension, given by $T_{11} = T_{13} = T_6 T_{22} - T_8 T_{20} = 0$ and $T_6 = T_8 = 0$. Over the largest component the equations of the total space can be written in rolling factors format (see, e.g., [25, p. 95]): three equations are the minors of the matrix

$$\begin{bmatrix} x & y_6 + T_2 x & y_{13} \\ y_6 & x^2 + T_4 x & y_{15} \end{bmatrix}$$

while the fifth and sixth equations are obtained by replacing in each monomial a factor occurring in the top row of the matrix by one of the bottom row. From the equations in the matrix one finds that $x(xy_{13} + T_4 y_{13}) = y_6(y_{15} + T_2 y_{13})$, so $y_{15} + T_2 y_{13}$ rolls to $xy_{13} + T_4 y_{13}$. This gives:

$$x^2 y_6^3 - y_{13}^2 \quad + P_{22} x + P_{20} y_6 + T_{13} y_{13} + T_{11} y_{15}$$
$$xy_6^4 - y_{13} y_{15} + P_{22} y_6 + \dots$$
$$y_6^5 - y_{15}^2 \quad + P_{22}(x^2 + T_4 x - T_2 y_6) + \dots$$

Here P_{22} and P_{20} are polynomials containing deformation variables of degree $4, 6, \dots, 22$. It follows in particular that the origin is a singular point of all fibers, in general an ordinary double point. Only the smallest component is a smoothing component. □

Remark 4.3. For $N(7)_{10}$ the gap sequence is 1, 2, 3, 5, 7, 9, 11 and Haure [9] gives a plane model of degree 13 with 11 moduli, whereas $\dim \mathcal{M}_{g,1}^\Gamma = 12$. This is the other case where Haure's result differs from our result.

4.3. Curves with first blow-up of multiplicity four

For the cases $N(6)_5 = \langle 4, 9, 10, 11 \rangle$ and $N(6)_5 = \langle 4, 10, 11, 13 \rangle$ the first blow-up is $N(3)_4 = \langle 4, 5, 6, 7 \rangle$ and $N(4)_4 = \langle 4, 6, 7, 9 \rangle$ respectively. For the first curve we compute the base space with Hauser's algorithm; we do it in fact for all semigroups $\langle 4, 1 + 4\tau, 2 + 4\tau, 3 + 4\tau \rangle$. A similar, but more complicated computation is in [4].

The equations of the curve are given by the minors of the matrix

$$\begin{bmatrix} x^\tau & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & x^{\tau+1} \end{bmatrix}$$

We write the unfolding with variables which are polynomials in x, where $f_i^{(j)}$ with $i \neq j$ has degree $4\tau + j$. We use coordinate transformations to remove as many terms as possible. The result is

$$y_1^2 - y_2 x^\tau + f_2^{(1)} y_1 + f_2^{(2)} + f_2^{(-1)} y_3 + f_2^{(0)} y_2$$
$$y_1 y_2 - y_3 x^\tau + f_3^{(1)} y_2 + f_3^{(2)} y_1 + f_3^{(3)} + f_3^{(0)} y_3$$
$$y_1 y_3 - x^{2\tau+1} + f_4^{(2)} y_2 + f_4^{(3)} y_1 + f_4^{(4)}$$
$$y_2^2 - x^{2\tau+1} + g_4^{(1)} y_3 + g_4^{(2)} y_2 + g_4^{(3)} y_1 + g_4^{(4)}$$
$$y_2 y_3 - y_1 x^{\tau+1} + f_5^{(5)} + f_5^{(4)} y_1$$
$$y_3^2 - y_2 x^{\tau+1} + f_6^{(5)} y_1 + f_6^{(6)} + f_6^{(3)} y_3 + f_6^{(4)} y_2$$

We have four transformations left, which we cannot show in the above notation. They act on the unfolding as $f_3^{(1)} - x^\tau a_{3,1}$, $f_2^{(1)} - x^\tau a_{2,1}$, $f_4^{(2)} + x^\tau a_{3,2}$ and $f_2^{(0)} + \tau a_0 x^{\tau-1}$; we use them to remove the lowest weight variables from $f_2^{(0)}, f_2^{(1)}, f_3^{(1)}$ and $f_4^{(2)}$.

We proceed as explained in Section 3.2: we compute the relation matrix for the unperturbed generators of the ideal, multiply with the perturbed generators and reduce the result with them. The result does not contain quadratic monomials in the y_i and for flatness it has to vanish identically, giving conditions on the coefficients. We write these as equations for the polynomials $f_i^{(j)}, g_i^{(j)}$. The polynomials $f_i^{(i)}$ and $g_4^{(4)}$ can be eliminated. We obtain fifteen equations.

The first one is $(x^\tau - f_3^{(0)})(f_2^{(1)} - f_3^{(1)}) + (x^\tau - f_2^{(0)})g_4^{(1)} + f_2^{(-1)} f_3^{(2)} = 0$. We will use this equation to eliminate $g_4^{(1)}$. To this end we rewrite it, and do the same with five other equations containing $x^\tau - f_2^{(0)}$. We obtain

$$(x^\tau - f_2^{(0)})(g_4^{(1)} + f_2^{(1)} - f_3^{(1)}) = -(f_2^{(0)} - f_3^{(0)})(f_2^{(1)} - f_3^{(1)}) - f_2^{(-1)} f_3^{(2)}$$
$$(x^\tau - f_2^{(0)})(f_3^{(2)} - g_4^{(2)}) = -(f_2^{(1)} - f_3^{(1)})f_3^{(1)} - f_2^{(-1)}(f_4^{(3)} - f_6^{(3)})$$
$$(x^\tau - f_2^{(0)})(g_4^{(3)} - f_4^{(3)} + f_6^{(3)}) = (f_2^{(0)} - f_3^{(0)})(f_4^{(3)} - f_6^{(3)}) - f_3^{(1)} f_3^{(2)}$$
$$(x^\tau - f_2^{(0)})(f_4^{(3)} - x f_2^{(-1)}) = (x f_2^{(0)} - f_6^{(4)})f_2^{(-1)} - (f_2^{(1)} - f_3^{(1)})f_4^{(2)}$$
$$(x^\tau - f_2^{(0)})(f_6^{(4)} - f_5^{(4)} - x f_2^{(0)} + x f_3^{(0)}) = f_3^{(2)} f_4^{(2)} + (f_2^{(0)} - f_3^{(0)})(x f_2^{(0)} - f_6^{(4)})$$
$$(x^\tau - f_2^{(0)})(f_6^{(5)} + x f_3^{(1)}) = -(x f_2^{(0)} - f_6^{(4)})f_3^{(1)} - f_4^{(2)}(f_4^{(3)} - f_6^{(3)})$$

It can be checked that the remaining equations are consequences of these ones. All the above equations are of the form

$$L \cdot (x^\tau - f_2^{(0)}) = R$$

with L and R polynomials in x satisfying $\deg_x(R) \leq \deg_x(L) + t$. Division with remainder gives $R = Q(x^\tau - f_2^{(0)}) + \overline{R}$, and therefore we can solve $L = Q$ and find the coefficients of $\overline{R}$ as equations for the base space. In other words, the condition leading to the equations of the base space is that the right hand side of the above equations is divisible by $x^\tau - f_2^{(0)}$. A similar structure first appeared for the base spaces of rational surface singularities of multiplicity four [15].

The right hand side of the equations are the minors of the matrix

$$\begin{bmatrix} -f_2^{(-1)} & (f_2^{(0)} - f_3^{(0)}) & f_3^{(1)} & f_4^{(2)} \\ (f_2^{(1)} - f_3^{(1)}) & f_3^{(2)} & (f_4^{(3)} - f_6^{(3)}) & -(xf_2^{(0)} - f_6^{(4)}) \end{bmatrix}$$

It seems that the eliminated variable $f_4^{(3)}$ occurs in the matrix, but we can take $f_4^{(3)} - f_6^{(3)}$ as independent variable.

We make the divisibility conditions explicit for $\tau = 1$ ($N(3)_4$) and $\tau = 2$ ($N(6)_5$). The $f_i^{(j)}$ are polynomials in x, of the form $f_i^{(j)} = f_{i,j+4\tau} + f_{i,j+4\tau-4}x + \cdots + f_{i,r}x^k$ if $j + 4\tau = 4k + r$ with $1 \le r \le 4$. Recall that we removed the variables of lowest weight in $f_2^{(0)}, f_2^{(1)}, f_3^{(1)}$ and $f_4^{(2)}$.

For $\tau = 1$ the matrix becomes

$$\begin{bmatrix} -f_{2,3} & -f_{3,4} & f_{3,5} & f_{4,6} \\ f_{2,5} - f_{3,5} & f_{3,6} + f_{3,2}x & f_{4,7} - f_{6,7} + (f_{4,3} - f_{6,3})x & f_{6,8} + f_{6,4}x \end{bmatrix}$$

and the condition that the minors are divisible by x is obviously that they vanish when $x = 0$ is substituted. Therefore the base space is given by the vanishing of the minors of

$$\begin{bmatrix} -f_{2,3} & -f_{3,4} & f_{3,5} & f_{4,6} \\ f_{2,5} - f_{3,5} & f_{3,6} & -f_{6,4}f_{2,3} - f_{6,7} & f_{6,8} \end{bmatrix}$$

where we substituted the value for $f_{4,7}$. Note that the variables $f_{6,4}$, $f_{6,3}$ and $f_{3,2}$ do not occur in the equations. We recover the result that the base space for $N(3)_4$ is the Segre cone.

For $\tau = 2$ the last entry of matrix becomes $f_{6,12} + f_{6,8}x + f_{6,4}x^2 - f_{2,8}x$. We apply division with remainder by $x^2 - f_{2,8}$, leading to $f_{6,12} + f_{6,4}f_{2,8} + (f_{6,8} - f_{2,8})x$. Doing the same for other entries we obtain the transpose of the matrix

$$\begin{bmatrix} -f_{2,7} - f_{2,3}x & f_{2,9} - f_{3,9} + (f_{2,5} - f_{3,5})x \\ f_{2,8} - f_{3,8} - f_{3,4}x & f_{3,10} + f_{3,2}f_{2,8} + f_{3,6}x \\ f_{3,9} + f_{3,5}x & f_{4,11} - f_{6,11} + (f_{4,3} - f_{6,3})f_{2,8} + (f_{4,7} - f_{6,7})x \\ f_{4,10} + f_{4,6}x & f_{6,12} + f_{6,4}f_{2,8} + (f_{6,8} - f_{2,8})x \end{bmatrix}$$

Making a coordinate transformation and renaming the variables gives a matrix of the form

$$\begin{bmatrix} a_7 + a_3x & a_8 + a_4x & a_9 + a_5x & a_{10} + a_6x \\ b_9 + b_5x & b_{10} + b_6x & b_{11} + b_7x & b_{12} + b_8x \end{bmatrix}$$

Division with remainder and taking the x-coefficient leads to two equations from each minor:

$$a_{i+4}b_{j+2} + a_ib_{j+6} - a_{j+4}b_{i+2} + a_jb_{i+6}$$
$$a_{i+4}b_{j+6} + a_ib_{j+2}f_{2,8} - a_{j+4}b_{i+6} - a_jb_{i+2}f_{2,8}$$

For $N(7)_5$ a computation of the versal deformation up to order 3 allows to recognize the base to be B_2 also in this case.

4.4. *The cone over a Grassmannian*

The semigroup $N(6)_{22} = \langle 6, 7, 8, 9, 10 \rangle$ is the first of the second family of curves studied in [4]. The computation can also easily be done with SINGULAR. The result is that the base space is the cone over the Grassmannian $G(2, 5)$. In Tables 1 and 2 this base space is denote by G. This shows that $\mathcal{M}_{6,1}^{N(6)_{22}}$ is rational. Equations for the base space can be recognized because they are the Pfaffians of a skew-symmetric 5×5 matrix, which is the relation matrix between the equations. Again, a computer computation will in general not lead to a skew matrix, but one can obtain that form by row and column operations.

The curve $N(7)_{26} = \langle 6, 8, 9, 10, 11 \rangle$ is also Gorenstein and has as base space a cone over $G(2, 5)$. While $N(6)_{21} = \langle 6, 7, 8, 9, 11 \rangle$, which deforms into $N(6)_{22}$, is not Gorenstein, but has type $t = 2$, the dimension of T^2 is also five, and a computation with SINGULAR shows that the base space has the same structure: it is a cone over the Grassmannian.

4.5. A codimension four base space

For $N(6)_{20} = \langle 6,7,8,10,11 \rangle$ $(t=3)$ and $N(7)_{25} = \langle 6,8,9,10,13 \rangle$ $(t=2)$ one has dim $T^2 = 9$. We compute the base spaces with Hauser's algorithm. It turns out that they have the same structure, called G' in the tables. We give here the details for the first curve. An additive basis over $\mathbf{k}[x]$ of the coordinate ring is $(1, y_7, y_8, y_{10}, y_{11}, y_7 y_8)$. We take the following unfolding of the generators of the ideal:

$$y_7^2 - y_8 x + (f_{14,1}x + f_{14,7})y_7 + f_{14,14} + f_{14,3}y_{11} + f_{14,4}y_{10} + f_{14,6}y_8$$

$$y_8^2 - y_{10}x + (f_{16,2}x + f_{16,8})y_8 + (f_{16,3}x + f_{16,9})y_7 + f_{16,16} + f_{16,5}y_{11} + f_{16,6}y_{10}$$

$$y_7 y_{10} - y_{11}x + (f_{17,1}x + f_{17,7})y_{10} + (f_{17,3}x + f_{17,9})y_8 + (f_{17,4}x + f_{17,10})y_7 + f_{17,17} + f_{17,6}y_{11}$$

$$y_{11}y_7 - x^3 + (f_{18,2}x + f_{18,8})y_{10} + (f_{18,4}x + f_{18,10})y_8 + f_{18,18}$$

$$y_8 y_{10} - x^3 + (g_{18,1}x + g_{18,7})y_{11} + (g_{18,5}x + g_{18,11})y_7 + g_{18,18}$$

$$y_{11}y_8 - y_7 x^2 + f_{19,19} + (f_{19,2}x + f_{19,8})y_{11} + (f_{19,3}x + f_{19,9})y_{10}$$
$$+ (f_{19,5}x + f_{19,11})y_8 + (f_{19,6}x + f_{19,12})y_7$$

$$y_{10}^2 - y_8 x^2 + (f_{20,1}x^2 + f_{20,7}x + f_{20,13})y_7 + f_{20,20} + (f_{20,3}x + f_{20,9})y_{11}$$
$$+ (f_{20,4}x + f_{20,10})y_{10} + f_{20,5}y_7 y_8 + (f_{20,6}x + f_{20,12})y_8$$

$$y_{11}y_{10} - y_8 y_7 x + (f_{21,1}x^2 + f_{21,7}x + f_{21,13})y_8 + (f_{21,5}x + f_{21,11})y_{10}$$
$$+ (f_{21,2}x^2 + f_{21,8}x + f_{21,14})y_7 + (f_{21,4}x + f_{21,10})y_{11} + f_{21,21}$$

$$y_{11}^2 - y_{10}x^2 + f_{22,7}y_7 y_8 + (f_{22,6}x + f_{22,12})y_{10} + (f_{22,5}x + f_{22,11})y_{11}$$
$$+ (f_{22,3}x^2 + f_{22,9}x + f_{22,15})y_7 + (f_{22,2}x^2 + f_{22,8}x + f_{22,14})y_8 + f_{22,22}$$

This shows the variables involved, except that the $f_{i,i}$ and $g_{18,18}$ are polynomials in x. In practice it is easier to first work with the coefficients of the y_i as polynomials. On this level the variables $f_{i,i}$ and $g_{18,18}$ can be eliminated. After that step the coefficients of x can be taken. Most variables can be eliminated. What is left are nine rather long polynomials with in total 134 monomials, but on closer inspection a coordinate transformation can be found, leading to the following generators of the ideal of the base space:

$$f_{18,8}f_{20,5} - f_{17,6}f_{22,7}$$
$$f_{14,4}f_{20,9} - f_{19,3}f_{21,10} + g_{18,7}f_{22,6}$$
$$-f_{16,6}f_{19,3}f_{20,5} + f_{14,4}f_{20,5}f_{21,5} - g_{18,7}f_{22,7}$$
$$f_{18,8}g_{18,7} + f_{16,6}f_{17,6}f_{19,3} - f_{14,4}f_{17,6}f_{21,5}$$
$$-f_{16,8}g_{18,7} - f_{16,6}f_{20,9} + f_{21,10}f_{21,5}$$
$$-f_{16,8}f_{19,3}f_{20,5} + f_{20,9}f_{22,7} + f_{20,5}f_{21,5}f_{22,6}$$
$$f_{14,4}f_{16,8}f_{20,5} - f_{21,10}f_{22,7} - f_{16,6}f_{20,5}f_{22,6}$$
$$-f_{16,8}f_{17,6}f_{19,3} + f_{18,8}f_{20,9} + f_{17,6}f_{21,5}f_{22,6}$$
$$f_{14,4}f_{16,8}f_{17,6} - f_{18,8}f_{21,10} - f_{16,6}f_{17,6}f_{22,6}$$

The singular locus consists of two components, the $(f_{20,5}, f_{17,6})$-plane and the Segre cone

$$\begin{bmatrix} f_{21,5} & f_{16,6} & f_{16,8} \\ f_{19,3} & f_{14,4} & f_{22,6} \end{bmatrix}$$

the other variables being zero. If $f_{20,5} = 1$ then $f_{18,8} = f_{17,6}f_{22,7}$ and the generators reduce to the Pfaffians of the matrix

$$\begin{bmatrix} 0 & f_{19,3} & f_{14,4} & f_{22,6} & f_{22,7} \\ -f_{19,3} & 0 & -g_{18,7} & f_{20,9} & -f_{21,5} \\ -f_{14,4} & g_{18,7} & 0 & f_{21,10} & -f_{16,6} \\ -f_{22,6} & -f_{20,9} & -f_{21,10} & 0 & -f_{16,8} \\ -f_{22,7} & f_{21,5} & f_{16,6} & f_{16,8} & 0 \end{bmatrix}$$

in accordance with the fact that the curve deforms into $N(6)_{21}$ and $N(6)_{22}$.

On the other component of the singular locus we take $f_{19,3} = 1$ and find $f_{16,6} = f_{14,4}f_{21,5}$, $f_{16,8} = f_{21,5}f_{22,6}$ while the generators reduce to the minors of

$$\begin{bmatrix} f_{22,7} & f_{18,8} & f_{16,6} - f_{14,4}f_{21,5} & f_{16,8} - f_{21,5}f_{22,6} \\ f_{20,5} & f_{17,6} & g_{18,7} & f_{20,9} \end{bmatrix}$$

4.6. Codimension 4 and type 4

For most of the semigroups with 5 generators and type 4 in the list the associated monomial curve has $\dim T^2 = 20$. Only for $N(6)_{19} = \langle 6, 7, 9, 10, 11 \rangle$ and $N(7)_{24} = \langle 6, 8, 9, 11, 13 \rangle$ one has $\dim T^2 = 21$, while $\dim T^2 = 26$ for $N(7)_{12} = \langle 5, 9, 11, 12, 13 \rangle$. The first two curves deform into $N(6)_{20}$ respectively $N(7)_{25}$, which are curves with base space G'. We have not been able to determine the exact structure of the base space; in the tables this is marked by a question mark (?). Only for two cases ($N(4)_7$ and $N(6)_{12}$, marked !) we give here explicit equations. For $N(4)_7$ Nakano computed the base computing in characteristic 7 [20]. The versal deformation of the monomial curve with semigroup $N(4)_7 = \langle 5, 6, 7, 8, 9 \rangle$ was computed with the projection method in [24]. This computation also takes care of $N(6)_{12}$ by the following result.

Lemma 4.4. *The curves $N(4)_7$ and $N(6)_{12}$ have I^2-equivalent plane projections.*

Proof. The projection onto the plane of the first two coordinates has equation $y_6^5 - x^6 = 0$ for $N(4)_7$ and $y_7^5 - x^7 = 0$ for $N(6)_{12}$. The conductor ideal I has in both cases length 6, being the difference of the δ-invariant of the plane curve and $\delta = g$ of the monomial curve. An easy computation gives that $I = \mathfrak{m}^3$, and therefore $x^7 - x^6 \in I^2$. □

We slightly modify the computation given in [24] by disregarding all terms in I^2. We start by describing the deformation of the matrix defining Σ:

$$\begin{bmatrix} y & e_{01} & e_{02} \\ -(x + e_{10}) & y + e_{11} & e_{12} \\ -e_{20} & -(x + e_{21}) & y + e_{22} \\ -e_{30} & -e_{31} & -x \end{bmatrix}$$

We consider only consider deformations of negative weight, so we deform the equation of the plane curve in the following way:

$$y^5 + c_0 x^5 + c_1 x^4 y + c_2 x^3 y^2 + c_3 x^2 y^3 + d_0 x^4 + d_1 x^3 y + d_2 x^2 y^2 + d_3 x y^3 + d_4 y^4$$

With the help of SINGULAR [6] the deformation equation (1) was solved for all generators of N. The equation holds modulo the ideal J of the base space, described below. We give here the vector α as direct result of the computation.

$$\alpha_1 = x^2 c_0 + \tfrac{1}{2} x c_2 e_{01} + x c_3 e_{02} + x d_0 - y c_3 e_{01} - y c_1 e_{10} + y c_2 e_{11} + y c_3 e_{12} - c_3 e_{02} e_{10} + c_2 e_{02} e_{20} + c_3 e_{02} e_{21}$$
$$- c_0 e_{01} e_{30} - c_1 e_{02} e_{30} + c_0 e_{12} e_{30} - e_{10} e_{12} e_{30} - c_1 e_{01} e_{31} - c_2 e_{02} e_{31} + c_0 e_{22} e_{31} + \tfrac{1}{2} d_2 e_{01} + d_3 e_{02}$$

$$\alpha_2 = x^2 c_1 + \tfrac{1}{2}xyc_2 + xc_3 e_{01} + xc_1 e_{10} - \tfrac{1}{2}xc_2 e_{11} + xd_1 + \tfrac{1}{2}yd_2 + d_3 e_{01} - e_{01}^2 + d_4 e_{02} + d_1 e_{10} - \tfrac{1}{2}d_2 e_{11}$$

$$\alpha_3 = \tfrac{1}{2}x^2 c_2 + xyc_3 + xc_1 e_{20} + \tfrac{1}{2}xc_2 e_{21} + \tfrac{1}{2}xd_2 - yc_3 e_{10} + yc_2 e_{20} + yc_3 e_{21} + yd_3 - c_3 e_{02} e_{30} + d_4 e_{01}$$
$$\qquad + e_{02} e_{10} - e_{01} e_{11} + d_1 e_{20} + \tfrac{1}{2}d_2 e_{21} + e_{01} e_{22}$$

$$\alpha_4 = y^2 + xc_1 e_{30} + \tfrac{1}{2}xc_2 e_{31} + yc_2 e_{30} + yc_3 e_{31} + yd_4 + e_{01} e_{10} + e_{02} e_{20} + d_1 e_{30} + \tfrac{1}{2}d_2 e_{31} - e_{02} e_{31}$$

To describe the base space it is useful to apply a coordinate transformation, given by:

$$d_0 \mapsto d_0 + c_0 e_{21}$$
$$d_1 \mapsto d_1 + c_1 e_{21} + c_0 e_{31}$$
$$d_2 \mapsto d_2 + c_1 e_{31}$$
$$d_3 \mapsto d_3 + e_{01} + c_3 e_{10}$$
$$d_4 \mapsto d_4 + e_{11}$$

In the new coordinates ideal of the base is given by the following 20 generators, which we write as sum of corresponding minors:

$$\begin{bmatrix} d_0 + c_3 e_{02} - c_0 e_{10} & d_1 - c_1 e_{10} + c_3 e_{12} - c_0 e_{20} & d_2 - c_2 e_{10} - c_0 e_{30} & d_3 & d_4 \\ e_{01} & e_{11} & e_{10} - e_{21} & e_{20} - e_{31} & e_{30} \end{bmatrix}$$
$$+ \begin{bmatrix} -c_0 e_{31} & e_{02} - c_1 e_{31} & -e_{01} + e_{12} - c_2 e_{31} & -e_{11} + e_{22} - c_3 e_{31} & e_{21} \\ e_{02} - c_0 e_{30} & e_{12} - c_1 e_{30} & e_{22} - c_2 e_{30} & e_{10} - c_3 e_{30} & e_{20} \end{bmatrix}$$

$$\begin{bmatrix} d_0 & d_1 & d_2 - e_{02} + c_1 e_{20} - c_3 e_{22} & d_3 - e_{12} + c_2 e_{20} + c_1 e_{30} & d_4 - e_{22} + c_2 e_{30} \\ e_{02} & -e_{01} + e_{12} & -e_{11} + e_{22} & e_{21} & e_{31} \end{bmatrix}$$
$$+ \frac{1}{c_0} \begin{bmatrix} c_0 e_{12} - c_1 e_{02} & c_0 e_{22} - c_2 e_{02} & c_0 e_{10} - c_3 e_{02} & c_0 e_{20} & c_0 e_{30} - e_{02} \\ c_1 e_{01} - c_0 e_{11} & c_2 e_{01} - c_0 e_{10} + c_0 e_{21} & c_3 e_{01} - c_0 e_{20} + c_0 e_{31} & -c_0 e_{30} & e_{01} \end{bmatrix}$$

The $\frac{1}{c_0}$ in front of the last matrix means that each minor has to be divided by c_0. The structure of this base space is discussed in [24].

According to the formula (2) we obtain the deformation for $N(4)_7$ by adding Δ_1 to α_1. For $N(6)_{12}$ the terms in I^2 are $x^7 + b_0 x^6 + b_1 x^5 y + b_2 x^4 y^2$, so we add the vector $\Delta_1(x + b_0, b_1, b_2, 0)^t$ to α. In particular we find that the codimension of the base space is the same for $N(4)_7$ and $N(6)_{12}$.

The curve $N(6)_{11} = \langle 5, 8, 9, 11, 12 \rangle$ deforms with the deformation $(t^5, t^8 + st^7, t^9, t^{11}, t^{12})$ of the parametrisation to a curve with semigroup $\langle 5, 7, 9, 11, 13 \rangle$ but not to the monomial curve $N(6)_{12}$ with this semigroup: the deformation $(t^5, t^7 + s't^8, t^9, t^{11}, t^{13})$ of $N(6)_{12}$ is non-trivial of positive weight. We did not compute the base space for $N(6)_{11}$, but we determined the quadratic part of the equations. It contains quadrics of rank two, so the base space is definitely more complicated. It is feasible to compute the deformation with Hauser's algorithm, but the problem is to simplify the resulting equations and write them in a systematic way. Even for $N(4)_7$ it is very hard to see that the equations from Hauser's algorithm give the same base space as the one above from the projection method.

Proposition 4.5. *For all semigroups of genus $g \leq 7$ with 5 generators and type 4 one has* $\dim M = 2g + 2 - \dim T^{1,+}$.

Proof. If the monomial curve is negatively graded the result follows directly from the Rim–Vitulli formula in Theorem 2.2. For $N(7)_{13}$ and $N(7)_{27}$ a computation of the deformation up to order two yields 20 quadratic equations which are among the equations for the tangent cone to the base space (and probably give the tangent cone exactly). These 20 equations define a projective scheme of dimension 15, which is therefore an upper bound for the dimension of $\mathcal{M}^{\Gamma}_{g,1}$. At the same time Theorem 2.2 gives 15 as lower bound.

For $N(7)_{24}$ the situation is more complicated. One of the 21 equations starts with cubic terms. We did compute the base space with Hauser's method. It leads to 256 equations in 63 variables, with 6923 monomials in total. These equations are not independent, in fact they can be reduced to 59 equations. Thirty eight variables occur linearly and can be eliminated. The result consists of 21 equations in 25 variables, with 24829 monomials in total; the equation starting with cubic terms has 3196 monomials. Taking the lowest degree part of each equation gives a manageable system with 163 monomials defining a scheme of dimension 15. □

4.7. Higher codimension

For the remaining curves we did not determine the base space. For the curves $N(5)_{12}$, $N(6)_{18}$, $N(7)_{22}$ and $N(7)_{23}$ with type 5 the dimension of T^2 is 45, for $N(7)_{34}$ it is 46. The curves $N(6)_{22}$ and $N(7)_{33}$ have type 6 and dim $T^2 = 84$, while for $N(7)_{39}$, the only type 8 curve, dim $T^2 = 140$. With decreasing type the dimension of T^2 also decreases: for $N(7)_{35}$, $N(7)_{36}$, $N(7)_{37}$ and $N(7)_{38}$ the dimensions are 28, 19, 14, 14 respectively. All the curves discussed here are negatively graded, except $N(7)_{23}$. For this case the base space was computed up to order two, and a standard basis of the resulting ideal was computed in finite characteristic, to speed up the computation. The resulting upper bound for the dimension of $\mathcal{M}_{7,1}^{N(7)_{23}}$ again coincides with the lower bound of Theorem 2.2. Herewith Proposition 4.1 is completely established.

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