



Counting rational points on projective varieties

Downloaded from: <https://research.chalmers.se>, 2026-05-09 21:39 UTC

Citation for the original published paper (version of record):

Salberger, P. (2023). Counting rational points on projective varieties. *Proceedings of the London Mathematical Society*, 126(4): 1092-1133. <http://dx.doi.org/10.1112/plms.12508>

N.B. When citing this work, cite the original published paper.

Counting rational points on projective varieties

Per Salberger

Mathematics, Chalmers University of
Technology, Göteborg, Sweden

Correspondence

Per Salberger, Mathematics, Chalmers
University of Technology, SE-412 96
Göteborg, Sweden.

Email: salberg@chalmers.se

Abstract

We develop a global version of Heath-Brown's p -adic determinant method to study the asymptotic behaviour of the number $N(W; B)$ of rational points of height at most B on certain subvarieties W of \mathbf{P}^n defined over \mathbf{Q} . The most important application is a proof of the dimension growth conjecture of Heath-Brown and Serre for all integral projective varieties of degree $d \geq 2$ over \mathbf{Q} . For projective varieties of degree $d \geq 4$, we prove a uniform version $N(W; B) = O_{d,n,\varepsilon}(B^{\dim W + \varepsilon})$ of this conjecture. We also use our global determinant method to improve upon previous estimates for quasi-projective surfaces. If, for example, X' is the complement of the lines on a non-singular surface $X \subset \mathbf{P}^3$ of degree d , then we show that $N(X'; B) = O_d(B^{3/\sqrt{d}}(\log B)^4 + B)$. For surfaces defined by forms $a_0x_0^d + a_1x_1^d + a_2x_2^d + a_3x_3^d$ with non-zero coefficients, then we use a new geometric result for Fermat surfaces to show that $N(X'; B) = O_d(B^{3/\sqrt{d}}(\log B)^4)$ for $B \geq e$.

MSC 2020

11D72, 11G35, 14G05, 14G40 (primary)

INTRODUCTION

In this paper, we shall study the number $N(W; B)$ of rational points of height at most $B \geq 1$ on quasi-projective subvarieties W of \mathbf{P}^n defined over \mathbf{Q} . The height $H(x)$ of a rational point x on W will always be given by $H(x) = \max(|x_0|, \dots, |x_n|)$ for a primitive integral $(n+1)$ -tuple (x_0, \dots, x_n)

representing x . We will use the O and \ll notation for functions defined for $B \geq 1$. If g is a function from $[1, \infty)$ to $[0, \infty)$, then we write $N(W; B) = O(g(B))$ or $N(W; B) \ll g(B)$ when there is a constant $C > 0$ such that $N(W; B) \leq Cg(B)$ for all $B \geq 1$. The constant C will depend on some parameters, which we include as indices.

It is trivial to show that $N(X; B) = O_{d,n}(B^{\dim X+1})$ and this is the best possible upper bound for projective varieties $X \subset \mathbf{P}^n$ of degree 1. For integral projective varieties of degree $d \geq 2$, one can obtain a better bound by means of the large sieve. Serre used in his book [46] arguments of S. Cohen to show that $N(X; B) = O_X(B^{\dim X+1/2}(\log X))$ and on p. 178 in [46] he asks if this can be improved to $N(X; B) = O_X(B^{\dim X}(\log X)^c)$ for some constant $c = c_X > 0$. The main result of this paper is the following bound conjectured by Serre in [47, p. 27].

Theorem 0.1. *Let $X \subset \mathbf{P}^n$ be an integral projective variety of degree $d \geq 2$ defined over \mathbf{Q} . Then, $N(X; B) = O_{X,\varepsilon}(B^{\dim X+\varepsilon})$.*

In the case of hypersurfaces, this was first conjectured by Heath-Brown in [25, p. 227] and later (see [12]) he went on to formulate the following uniform version of Serre’s conjecture.

Conjecture 0.2. *Let $X \subset \mathbf{P}^n$ be an integral projective variety defined over \mathbf{Q} of degree $d \geq 2$. Then, $N(X; B) = O_{d,n,\varepsilon}(B^{\dim X+\varepsilon})$.*

Conjecture 0.2 was shown for hypersurfaces of degree 2 in [27, theorem 2] and for geometrically integral varieties of degree $d = 2$ and $d \geq 6$ in a paper [12] of Browning, Heath-Brown and the author. In Section 7, we shall prove the following uniform bounds.

Theorem 0.3. *Let $X \subset \mathbf{P}^n$ be an integral projective variety over \mathbf{Q} of degree d . Then,*

$$\begin{aligned} N(X; B) &= O_{d,n,\varepsilon}(B^{\dim X+\varepsilon}) && \text{if } d \geq 4 \\ N(X; B) &= O_{n,\varepsilon}(B^{\dim X-1+2/\sqrt{3+\varepsilon}}) && \text{if } d = 3. \end{aligned}$$

This generalises the result in [42] for varieties of degree $d \geq 4$ with finitely many planes of codimension 1. To deduce Theorem 0.1 from Theorem 0.3, it remains to establish the non-uniform bound $O_{X,\varepsilon}(B^{\dim X+\varepsilon})$ for varieties of degree 3. This is done in Section 8 by ad hoc methods, very different from the methods in the first seven chapters.

If X is integral, but not geometrically integral, then the rational points on X will lie on a proper closed subset consisting of $O_{d,n}(1)$ components of degrees bounded in terms of d and n . It is thus enough to show Theorem 0.3 in the case where X is geometrically integral. By considering the affine cone of a suitable birational projection of X onto a hypersurface as in [12], we deduce Theorem 0.3 from the following result for affine hypersurfaces (see Theorem 7.4).

Theorem 0.4. *Let $f(y_1, \dots, y_n) \in \mathbf{Z}[y_1, \dots, y_n]$, $n \geq 3$ be a polynomial such that its homogeneous part $h(f)$ of maximal degree is irreducible over \mathbf{Q} . Let $d = \deg h(f)$ and $n(f; B)$ be the number of n -tuples $\mathbf{y} = (y_1, \dots, y_n)$ of integers such that $y_1, \dots, y_n \in [-B, B]$ and $f(\mathbf{y}) = 0$. Then,*

$$\begin{aligned} n(f; B) &= O_{d,n,\varepsilon}(B^{n-2+\varepsilon}) && \text{if } d \geq 4 \\ n(f; B) &= O_{n,\varepsilon}(B^{n-3+2/\sqrt{3+\varepsilon}}) && \text{if } d = 3. \end{aligned}$$

It suffices to show Theorem 0.4 when $n = 3$, thanks to a hyperplane section argument. In the case where $d \geq 6$, this was done in [12]. It is much more difficult to establish the theorem for affine surfaces of degree $d < 6$. To do this, we develop a global version of Heath-Brown's p -adic determinant method [27]. This global method is considerably more complicated to use than its local counterpart. But it gives sharper estimates for a number of Diophantine counting problems and it will also be used to establish the bounds in Theorems 0.5–0.9 for projective surfaces. As the proofs are very similar, we refer to the discussion of the proofs of those results for an introduction to the proof of Theorem 0.4 for affine surfaces.

If $X \subset \mathbf{P}^n$ contains a rational linear subspace of codimension 1, then $N(X; B) \gg B^{\dim X}$. We can thus not expect a lower growth order than $r = \dim X$ for X in Theorem 0.2. To obtain a better bound, we must count points on the complement $X' \subset X$ of all $(r - 1)$ -planes on X . In this paper, we shall use our determinant method and some new geometric results to improve upon previous bounds for $N(X'; B)$ for surfaces.

Let us first state our results for non-singular surfaces.

Theorem 0.5. *Let $X \subset \mathbf{P}^3$ be a non-singular projective surface of degree d defined over \mathbf{Q} and U be the complement of the union of all curves of degree at most $d - 2$ on X . Then,*

$$N(U; B) = O_d \left(B^{3/\sqrt{d}} (\log B)^4 + 1 \right).$$

Moreover, if X' is the complement of the union of all lines on X , then

$$N(X'; B) = O_d \left(B^{3/\sqrt{d}} (\log B)^4 + B \right).$$

Theorem 0.6. *Let $X \subset \mathbf{P}^4$ be a non-singular complete intersection of two hypersurfaces of degree d_1 and d_2 and let U be the complement of the union of all curves of degree at most $d_1 + d_2 - 3$ on X . Let $d = d_1 d_2$. Then,*

$$N(U; B) = O_{d,\varepsilon} \left(B^{3/\sqrt{d+\varepsilon}} \right).$$

Moreover, if X' is the complement of the union of all lines on X , then

$$N(X'; B) = O_{d,\varepsilon} \left(B^{3/\sqrt{d+\varepsilon}} + B \right).$$

The interest in U comes from the fact that there are $O_d(1)$ curves of degree $\leq d - 2$ on a non-singular surface X of degree d in \mathbf{P}^3 and $O_d(1)$ curves of degree $\leq d_1 + d_2 - 3$ on a non-singular complete intersection of two hypersurfaces of degree d_1 and d_2 in \mathbf{P}^4 (see [14] and [5]).

The estimates in Theorem 0.5 should be compared with the estimates $N(U; B) = O_{d,\varepsilon}(B^{3/\sqrt{d+2/(d-1)+\varepsilon}})$ and $N(X'; B) = O_{d,\varepsilon}(B^{3/\sqrt{d+2/(d-1)+\varepsilon}} + B^{1+\varepsilon})$ of Heath-Brown [27] and the estimates in Theorem 0.6 with the estimates $N(U; B) = O_{d,\varepsilon}(B^{3/\sqrt{d+2/(d_1+d_2-2)+\varepsilon}})$ and $N(X'; B) = O_{d,\varepsilon}(B^{3/\sqrt{d+2/(d_1+d_2-2)+\varepsilon}} + B^{1+\varepsilon})$ in [5].

An interesting consequence of Theorem 0.5 is the following uniform estimate for diagonal surfaces, which is sharper than the bound in [28, theorem 13].

Corollary 0.7. *Let $X \subset \mathbf{P}^3$ be the surface given by the equation $a_0 x_0^d + a_1 x_1^d + a_2 x_2^d + a_3 x_3^d = 0$ for a quadruple (a_0, a_1, a_2, a_3) of rational numbers different from zero. Let $X' \subset X$ be the open subset for which $a_i x_i^d + a_j x_j^d \neq 0$ for all $0 \leq i < j \leq 3$.*

Then,

$$N(X'; B) = O_d \left(B^{3/\sqrt{d}}(\log B)^4 + 1 \right).$$

We shall also study singular surfaces. The following result improves upon theorem 7 in [27].

Theorem 0.8. *Let $X \subset \mathbf{P}^n$ be a geometrically integral projective surface of degree d defined over \mathbf{Q} . Then,*

$$N(X'; B) = O_{d,n,\varepsilon} \left(B^{3/\sqrt{d+\varepsilon}} + B^{3/2\sqrt{d+2/3+\varepsilon}} + B^{1+\varepsilon} \right),$$

unless $d = 4$ and there is a two-dimensional family of conics on X . In that case

$$\begin{aligned} N(X'; B) &= O_{n,\varepsilon} \left(B^{43/28+\varepsilon} \right) \quad \text{and} \\ N(X'; B) &= O_X(B^{3/2}). \end{aligned}$$

It is not surprising that the exponent $3/\sqrt{d}$ occurs in the above estimates. If $X \subset \mathbf{P}^n$ is the \sqrt{d} -fold Veronese embedding of \mathbf{P}^2 for a perfect square d , then X is of degree d in \mathbf{P}^n and $N(U; B) \gg_U B^{3/\sqrt{d}}$ for any dense open subset U of X .

The most important ingredient in the proofs of Theorems 0.5–0.8 is the following result.

Theorem 0.9. *Let $X \subset \mathbf{P}^n$ be a geometrically integral projective surface of degree d defined over \mathbf{Q} and $B \geq 1$. Then there exists a set of $O_{d,n}(B^{3/2\sqrt{d}} \log B + 1)$ geometrically integral curves of degree $O_d(1)$ on X such that the following holds.*

- (a) *If $n = 3$ and X is non-singular, then all but $O_d(B^{3/\sqrt{d}}(\log B)^4 + 1)$ rational points of height $\leq B$ on X lie on one of these curves.*
- (b) *In general, there exists a constant $c > 0$ depending only on d and n such that all but $O_{d,n}(B^{3/\sqrt{d+c/\log(1+\log B)}})$ rational points of height $\leq B$ lie on one of these curves.*

The function $f(B) = B^{c/\log(1+\log B)}$ on $(1, \infty)$ extends to a continuous function on $[1, \infty)$ with $f(1) = e^c$ and $f(B) = O_{c,\varepsilon}(B^\varepsilon)$ for all $\varepsilon > 0$. We may thus replace $O_{d,n}(B^{3/\sqrt{d+c/\log(1+\log B)}})$ by $O_{d,n,\varepsilon}(B^{3/\sqrt{d+\varepsilon}})$ in (b).

We now present the main ideas behind the proofs of the uniform bounds for surfaces in Theorems 0.5–0.9. To describe the proof of the bounds in Theorem 0.5 for non-singular surfaces in \mathbf{P}^3 , let us first recall two basic results of Heath-Brown [27] (see theorems 14 and 5).

Theorem 0.10. *Let $X \subset \mathbf{P}^{r+1}$ be a geometrically integral projective hypersurface of degree d defined over \mathbf{Q} . Then there exists for all $\varepsilon > 0, B \geq 1$ a set of $O_{d,r,\varepsilon}(B^{(r+1)/d^{1/r+\varepsilon}})$ hypersurfaces $Y_i \subset \mathbf{P}^{r+1}$ of degree $O_{d,\varepsilon}(1)$ not containing X such that all rational points of height $\leq B$ on X lie on one of these hypersurfaces.*

Theorem 0.11. *Let $Z \subset \mathbf{P}^3$ be an integral curve of degree δ defined over \mathbf{Q} . Then $N(Z; B) = O_{\delta,\varepsilon}(B^{2/\delta+\varepsilon})$.*

From Theorems 0.10 and 0.11 it follows immediately (see [27, theorem 11]) that $N(U; B) = O_{d,\varepsilon}(B^{3/\sqrt{d+2/(d-1)+\varepsilon}})$ for non-singular surfaces $X \subset \mathbf{P}^3$ of degree d .

The following result was first presented in a talk at the Max-Planck institute 2002.

Theorem 0.12. *Let $X \subset \mathbf{P}^{r+1}$ be a geometrically integral projective hypersurface of degree d over \mathbf{Q} . Then there exists for $B \geq 1$ a hypersurface Y of degree $O_{d,r,\varepsilon}(B^{(r+1)/rd^{1/r}+\varepsilon})$, not containing X such that all rational points of height $\leq B$ on X lie on Y .*

To prove Theorem 0.12, we use our global version of Heath-Brown’s p -adic determinant method. The main difference is that we consider congruences between integral coordinates of the rational points of height $\leq B$ for (almost) all primes up to order $B^{(r+1)/rd^{1/r}+\varepsilon}$ instead of just one prime of order $B^{(r+1)/rd^{1/r}+\varepsilon}$ as in [27, theorem 14].

For surfaces X of degree d , there is thus an auxiliary surface $Y \subset \mathbf{P}^3$ of degree $O_{d,\varepsilon}(B^{3/2\sqrt{d}+\varepsilon})$, with $X \not\subset Y$ such that all rational points of height $\leq B$ on X lie on Y . If the implicit constant $k = k(\delta, \varepsilon)$ in 0.11 were of order $O_{\delta,\varepsilon}(\delta^{2+\varepsilon})$, then Theorem 0.5 would follow. But no one has been able to prove this in spite of the recent progress in [13].

Instead, we deduce Theorem 0.5 from Theorem 0.9(a) and a slight refinement of Theorem 0.11. For cubic surfaces, we need also a method based on Hilbert schemes from [43] to deal with the contribution from the conics in Theorem 0.9. To prove Theorem 0.9 in its turn, we use the following result, which is shown by combining the techniques in the proofs of Theorems 0.10 and 0.12.

Theorem 0.13. *Let $X \subset \mathbf{P}^{r+1}$ be a geometrically integral hypersurface over \mathbf{Q} of degree d and $B \geq 1$. Let $\Omega = \{p_1, \dots, p_t\}$ be (a possibly empty) set of primes and P_i be a non-singular \mathbf{F}_{p_i} -point on X_{p_i} for each $i \in \{1, \dots, t\}$. Let $q_t = p_1 \dots p_t$ if $t \geq 1$ with $q_t = 1$ if $t = 0$. Then there is a hypersurface $Y(P_1, \dots, P_t) \subset \mathbf{P}^{r+1}$ over \mathbf{Q} of degree $O_{d,r}(q^{-1}B^{(r+1)/rd^{1/r}} \log Bq + \log Bq + 1)$ not containing X such that all rational points on X of height $\leq B$ which specialise to P_i for $i \in \{1, \dots, t\}$ lie on $Y(P_1, \dots, P_t)$.*

Here $X_p \subset \mathbf{P}_{\mathbf{F}_p}^{r+1}$ denotes the reduction (mod p) of the scheme-theoretic closure of X in $\mathbf{P}_{\mathbf{Z}}^{r+1}$. If Ω is empty then we recover Theorem 0.12 while the case where Ω consists of a single prime of order $B^{(r+1)/rd^{1/r}+\varepsilon}$ is related to a statement in [27] used to establish Theorem 0.10.

We now sketch the proof of Theorem 0.9(a). Full details will be given in Section 3. We first use Siegel’s lemma to reduce to the case where X is defined by a primitive integral form F of discriminant $\Delta = O_d(B^k)$, where $k > 0$ only depends on d . Let $p_1 < p_2 < \dots$ be the sequence of primes not dividing Δ and u be the index for which $(p_1 \dots p_u)/p_u < eB^{3/2\sqrt{d}} \leq p_1 \dots p_u$. Then $u < p_u = O_d(1 + \log B)$ and $q_u = p_1 \dots p_u = O_d(B^{3/2\sqrt{d}} \log B + 1)$.

We now apply Theorem 0.13 for $r = 2$ and the sets $\Omega_t = \{p_1, \dots, p_t\}$ of the first t primes of this sequence for $t \in \{0, \dots, u\}$. Let x be a rational point on X of height $\leq B$ and (P_1, \dots, P_u) be the list of the specialisations of x to \mathbf{F}_{p_i} -points on X_{p_i} for $i \in \{1, \dots, u\}$. Let $Y(P_1, \dots, P_t)$, $0 \leq t \leq u$ be the surface in Theorem 0.13 and D_x be an irreducible component of $X \cap Y(\emptyset)$ containing x .

There are two cases.

Case I: $D_x \subseteq Y(P_1, \dots, P_u)$

Case II: There exists $t \in \{0, \dots, u-1\}$ such that $D_x \subseteq Y(P_1, \dots, P_t)$ but $D_x \not\subseteq Y(P_1, \dots, P_{t+1})$.

In case I, $D_x \subseteq X \cap Y(P_1, \dots, P_u)$ and $\deg D_x \leq d(\deg Y(P_1, \dots, P_u)) = O_d(1 + \log B)$. There are at most $d(\deg Y(\emptyset)) = O_d(B^{3/2\sqrt{d}} \log B + 1)$ such curves D_x . One may now show (see Lemma 3.13 and Theorem 3.16) that the total contribution from the curves D_x with $1 \ll_d \deg D_x \ll_d 1 + \log B$ is acceptable by applying the determinant method to each of these curves.

In case II, x belongs to $D_x \cap Y(P_1, \dots, P_{t+1})$, which is of codimension 2 in X . There are thus by the theorem of Bézout [19, 8.4] at most $\deg D_x \deg Y(P_1, \dots, P_{t+1})$ rational points on $D_x \cap Y(P_1, \dots, P_{t+1})$ and at most $d(\deg Y(P_1, \dots, P_t))(\deg Y(P_1, \dots, P_{t+1}))$ rational points on X , lying on different irreducible components of $X \cap Y(P_1, \dots, P_t)$ and $X \cap Y(P_1, \dots, P_{t+1})$.

By Theorem 0.13 we have that $\deg Y(P_1, \dots, P_{t+i}) = O_d(q_{t+1}^{-1} B^{3/2\sqrt{d}} (\log B)^{2-i} + \log B + 1)$ for $i = 0, 1$. We now sum over all $(t+1)$ -tuples (P_1, \dots, P_{t+1}) for $t \in \{0, \dots, u-1\}$. This will give $O_d(B^{3/\sqrt{d}} (\log B)^4 + 1)$ rational points as there are $O_d(q_{t+1}^2)$ sequences (P_1, \dots, P_{t+1}) and $q_{t+1}^2 = O_d(B^{3/\sqrt{d}} (\log B)^2 + 1)$. This completes our description of how to deduce Theorem 0.9(a) from Theorem 0.13. The proof of Theorem 0.9(b) is similar although somewhat more complicated.

We shall in fact in Theorem 3.16 prove a theorem for projective surfaces, which is more general than Theorem 0.9 for $n = 3$. This result is a consequence of a “main lemma” 3.2 for projective r -dimensional hypersurfaces, which concerns the set $X(\mathbf{Q}; B_0, \dots, B_{r+1})$ of rational points that can be represented by integral $(r+2)$ -tuples (x_0, \dots, x_{r+1}) in the region $|x_m| \leq B_m$, $m \in \{0, \dots, r+1\}$. This generalisation is used to prove Theorem 0.4 for surfaces. We then apply Lemma 3.2 to the case when $B_0 = 1$ and $B_1 = B_2 = B_3 = B$.

The proofs of Theorem 0.6 and Theorem 0.8 resemble the proof of Theorem 0.5. For some surfaces, we use again the method with Hilbert schemes in [43] to deal with the contribution from the conics which appear in Theorem 0.9. To obtain Theorem 0.7 from Theorem 0.6, we also use the new result (see Theorem 9.4) that there are no curves of degree $< (d+1)/3$ apart from the obvious lines on a non-singular Fermat surface of degree d .

The central technical result of the paper is thus Lemma 3.2. It is an improvement of the important theorem 14 in [27]. In this paper we shall only apply it for surfaces, although there are interesting applications to varieties of higher dimensions. For some applications of Theorem 0.9 to threefolds, see [44].

Here is a short description of the contents of the sections. In Section 1, we construct auxiliary hypersurfaces containing $X(\mathbf{Q}; B_0, \dots, B_{r+1})$. In case $B_0 = \dots = B_{r+1} = B$, we get Theorem 0.12. In Section 2, we generalise this to subsets of $X(\mathbf{Q}; B_0, \dots, B_{r+1})$ satisfying congruence conditions. In particular, when $B_0 = \dots = B_{r+1} = B$, we obtain Theorem 0.13. In Section 3, we prove the central technical results discussed above and deduce Theorem 0.9. In Section 4, we study the geometry of the Hilbert scheme of conics on a surface in \mathbf{P}^3 by means of the relative Riemann–Roch theorem of Knudsen–Mumford. These results are used in Section 5 to count the rational points on the conics which may appear when we apply Theorem 0.9. In Section 6, we prove Theorems 0.5–0.8 and in Section 7 we establish Theorems 0.3 and 0.4. In Section 8, we prove Theorem 0.1 for the remaining class of varieties of degree 3. Finally, in Section 9, we show a result on the degrees of curves on Fermat surfaces, which we need for the proof of Corollary 0.7 in Section 6. This section is purely geometric and independent of the previous sections.

The first version of this paper appeared around 2010 and this version is almost identical to a version from 2011 except that we have added references to some later papers, which were influenced by this paper. Walsh [50] and Castryck, Cluckers, Dittman and Nguyen [13] have managed to remove $\log B$ - and B^ϵ -factors in some of the estimates here. The latter paper contains precise results on how the implicit constants depend on the degree of the variety. There is also a recent paper by Paredes and Sasyk [36] devoted to the dimension growth conjecture for projective varieties over arbitrary global fields.

For other applications of the global determinant method the reader may consult the papers of Browning [8] and Xiao [50].

1 | A NEW VERSION OF THE DETERMINANT METHOD

In this section, we shall describe a new global version of Heath-Brown’s p -adic determinant method in which we make simultaneous use of congruence modulo all small primes. The main goal is to prove Theorem 1.2. To formulate the theorem, we shall need the following notation which will be used throughout the paper. In the sequel, a hypersurface X in \mathbf{P}^{r+1} will mean an equi-dimensional closed subscheme of codimension 1.

Notation 1.1. Let $X \subset \mathbf{P}^{r+1}$ be a hypersurface over \mathbf{Q} and $F(x_0, \dots, x_{r+1}) \in \mathbf{Q}[x_0, \dots, x_{r+1}]$ be a form which defines X . Let $B_0, \dots, B_{r+1} \in \mathbf{R}_{\geq 1}$. Then,

- (i) $X(\mathbf{Q}; B_0, \dots, B_{r+1})$ is the set of rational points on X which may be represented by an integral $(r+2)$ -tuple (x_0, \dots, x_{r+1}) with $|x_m| \leq B_m$ for $m \in \{0, \dots, r+1\}$. If $B_0 = \dots = B_{r+1} = B$, then we denote this set by $X(\mathbf{Q}; B)$.
- (ii) $N(X; B_0, \dots, B_{r+1}) = \#X(\mathbf{Q}; B_0, \dots, B_{r+1})$ and $N(X; B) = \#X(\mathbf{Q}; B)$.
- (iii) $V = B_0 \dots B_{r+1}$.
- (iv) $T = \max\{B_0^{f_0} \dots B_{r+1}^{f_{r+1}}\}$ with the maximum taken over all $(r+2)$ -tuples (f_0, \dots, f_{r+1}) for which the corresponding monomial $x_0^{f_0} \dots x_{r+1}^{f_{r+1}}$ occur in $F(x_0, \dots, x_{r+1})$ with non-zero coefficient.

Theorem 1.2. Let $X \subset \mathbf{P}^{r+1}$ be a geometrically integral hypersurface of degree d defined over \mathbf{Q} and let $(B_0, \dots, B_{r+1}) \in \mathbf{R}_{\geq 1}^{r+2}$. Then there exists a hypersurface $Y \subset \mathbf{P}^{r+1}$ over \mathbf{Q} of degree $O_{d,r}((V/T^{1/d})^{1/rd^{1/r}} \log V + 1)$ which contains $X(\mathbf{Q}; B_0, \dots, B_{r+1})$, but which does not contain X . In particular, if $B_0 = \dots = B_{r+1} = B$, then there exists a hypersurface $Y \subset \mathbf{P}^{r+1}$ over \mathbf{Q} of degree $O_{d,r}(B^{(r+1)/rd^{1/r}} \log B + 1)$ which contains $X(\mathbf{Q}; B)$, but which does not contain X .

Before we give the proof of the theorem, we first prove some lemmas. We shall use the following notation.

Notation 1.3.

- (i) Ξ is the scheme-theoretic closure of the hypersurface $X \subset \mathbf{P}_\mathbf{Q}^{r+1}$ in $\mathbf{P}_\mathbf{Z}^{r+1}$.
- (ii) $X_p = \Xi \times_{\mathbf{Z}} \mathbf{F}_p$ for a prime p .
- (iii) μ_p is the multiplicity of the \mathbf{F}_p -point P on X_p .
- (iv) $n_p = \sum_P \mu_p$ where P runs over all \mathbf{F}_p -points P on X_p .

There is always a primitive form $F(x_0, \dots, x_{r+1}) \in \mathbf{Z}[x_0, \dots, x_{r+1}]$ which defines $X \subset \mathbf{P}_\mathbf{Q}^{r+1}$ and we have then that $\Xi = \text{Proj}(\mathbf{Z}[x_0, \dots, x_{r+1}]/(F))$ and $X_p = \text{Proj}(\mathbf{F}_p[x_0, \dots, x_{r+1}]/(F_p))$ for the image F_p of F in $\mathbf{F}_p[x_0, \dots, x_{r+1}]$.

Lemma 1.4. Let $X \subset \mathbf{P}^{r+1}$ be as in 1.2 and p be a prime. Let ξ_1, \dots, ξ_s be primitive $(r+2)$ -tuples of integers representing integral points on Ξ . Let F_1, \dots, F_s be forms in (x_0, \dots, x_{r+1}) with integer coefficients and $\Delta = \det(F_j(\xi_l))$ be the determinant of the $s \times s$ -matrix $M = (F_j(\xi_l))_{1 \leq j, l \leq s}$. Then there exists a non-negative integer $N \geq r!^{1/r}(r/(r+1))s^{1+1/r}/n_p^{1/r} + O_{d,r}(s)$ such that $p^N \mid \Delta$.

Proof. Let P be an \mathbf{F}_p -point on X_p and $s_p = \#I_p$ for the subset $I_p \subseteq \{1, \dots, s\}$ of indices l such that $\xi_l + p\mathbf{Z}^{r+2}$ represents P . Then, by [42, 2.5], there exists a non-negative integer $N_p = (r!/\mu_p)^{1/r}(r/(r+1))s_p^{1+1/r} + O_{d,r}(s_p)$ such that $p^{N_p} \mid \det(A)$ for each $s_p \times s_p$ -submatrix A of

M with second indices l in I_p . If we apply this to all \mathbf{F}_p -points P on X_p and use Laplace expansion, then we get that $p^N \mid \Delta$ for $N = \sum_P N_P = r^{1/r}(r/(r+1))\sum_P s_P^{1+1/r}/\mu_P^{1/r} + O_{d,r}(s)$. By Hölder's inequality, $\sum_P s_P \leq (\sum_P \mu_P)^{1/(r+1)}(\sum_P s_P^{1+1/r}/\mu_P^{1/r})^{r/(r+1)}$. Hence, $\sum_P s_P^{1+1/r}/\mu_P^{1/r} \geq s^{1+1/r}/n_p^{1/r}$ and $N = \sum_P N_P \geq r^{1/r}(r/(r+1))s^{1+1/r}/n_p^{1/r} + O_{d,r}(s)$, which finishes the proof. \square

Lemma 1.5. *If X_p is geometrically integral, then $n_p^{1/r}/p - 1 = O_{d,r}(p^{-1/2})$.*

Proof. Let $X_{p,\text{sing}}$ be the singular locus of X_p . Then the sum of the degrees of the irreducible components of $X_{p,\text{sing}}$ is bounded in terms of d and r by the theorem of Bézout (see [19, 8.4]). Hence, by [32, lemma 1], we have $\#X_{p,\text{sing}}(\mathbf{F}_p) = O_{d,r}(p^{r-1})$ and $\sum_P (\mu_P - 1) \leq (d - 1)\#X_{p,\text{sing}}(\mathbf{F}_p) = O_{d,r}(p^{r-1})$. As $\#X_p(\mathbf{F}_p) = p^r + O_{d,r}(p^{r-1/2})$ by [32, theorem 1], we thus find that $n_p/p^r - 1 = O_{d,r}(p^{-1/2})$. To complete the proof, use the inequality $|\alpha^{1/r} - 1| \leq |\alpha - 1|$ for $\alpha = n_p/p^r \geq 0$. \square

Notation 1.6.

- (a) Let $\underline{m} = (m_0, \dots, m_{r+1})$ be an $(r + 2)$ -tuple of non-negative integers. Then, $\text{deg}(\underline{m}) = m_0 + \dots + m_{r+1}$. Also, if $x = (x_0, \dots, x_{r+1})$, then $x^{\underline{m}} = x_0^{m_0} \dots x_{r+1}^{m_{r+1}}$.
- (b) If $X \subset \mathbf{P}^{r+1}$ is a hypersurface over \mathbf{Q} defined by a primitive form $F(x_0, \dots, x_{r+1}) = \sum a_{\underline{m}}x^{\underline{m}}$ in $\mathbf{Z}[x_0, \dots, x_{r+1}]$, then $H(X) = \max |a_{\underline{m}}|$.

Lemma 1.7. *Let $X \subset \mathbf{P}^{r+1}$ be an integral hypersurface over \mathbf{Q} of degree d . Then one of the following two statements holds.*

- (a) *There exists a projective \mathbf{Q} -hypersurface $Y \subset \mathbf{P}^{r+1}$ of degree d different from X which contains $X(\mathbf{Q}; B_0, \dots, B_{r+1})$.*
- (b) *$H(X) = O_{d,r}(V^\theta)$ where $\theta = (d + r + 1)/(d - 1)!(r + 1)!$.*

Proof. This is proved in [27, theorem 4] for $r = 1$ and the same argument may be used to prove Lemma 1.7 for $r > 1$.

Lemma 1.8. *Let d be a positive integer and let $\underline{m} = (m_0, \dots, m_{r+1})$ run over all $(r + 2)$ -tuples of non-negative integers with $\text{deg}(\underline{m}) = d$. Then there exists a finite set of universal forms $\Phi_1(a_{\underline{m}}), \dots, \Phi_t(a_{\underline{m}})$ in $a_{\underline{m}}$ with integer coefficients with the following property. Whenever the variables $a_{\underline{m}}$ take values in a field K , then the form*

$$F(x_0, \dots, x_{r+1}) = \sum a_{\underline{m}}x^{\underline{m}}$$

is absolutely irreducible over K if and only if $\Phi_i(a_{\underline{m}}) \neq 0$ in K for some $i \in \{1, \dots, t\}$.

Proof. Let \mathbf{H}_k be the Hilbert scheme of hypersurfaces of degree k in \mathbf{P}^{r+1} and $\nu_k: \mathbf{H}_k \times \mathbf{H}_{d-k} \rightarrow \mathbf{H}_d$, $k \in \{1, \dots, d - 1\}$ be the morphism obtained by multiplying forms of degree k and $d - k$. Then $F(x_0, \dots, x_{r+1}) = \sum a_{\underline{m}}x^{\underline{m}}$ has a factor over K of degree k if and only if the corresponding K -point on \mathbf{H}_d belongs to $\nu_k(\mathbf{H}_k \times \mathbf{H}_{d-k})$. Also, as $\mathbf{H}_k \times \mathbf{H}_{d-k}$ is a projective scheme, $\nu_k(\mathbf{H}_k \times \mathbf{H}_{d-k})$ must be a closed subset of \mathbf{H}_d by the main theorem of elimination theory. The union of all images $\nu_k(\mathbf{H}_k \times \mathbf{H}_{d-k})$, $k \in \{1, \dots, d - 1\}$ is thus a closed subset of \mathbf{H}_d defined by a finite set of forms $\Phi_1(a_{\underline{m}}), \dots, \Phi_t(a_{\underline{m}})$ over \mathbf{Z} such that F is reducible over K if and only if all $\Phi_i(a_{\underline{m}}) = 0$ in K . This completes the proof. \square

Lemma 1.9. *Let $X \subset \mathbf{P}^{r+1}$ be a geometrically integral hypersurface over \mathbf{Q} of degree d and $(B_0, \dots, B_{r+1}) \in \mathbf{R}_{\geq 1}^{r+2}$. Then one of the following two statements holds.*

- (a) *There exists a projective \mathbf{Q} -hypersurface $Y \subset \mathbf{P}^{r+1}$ of degree d different from X which contains $X(\mathbf{Q}; B_0, \dots, B_{r+1})$.*
- (b) *The product π_X of all primes where X_p is not geometrically integral satisfies $\log \pi_X = O_{d,r}(1 + \log V)$.*

Proof. Let $F(x_0, \dots, x_{r+1}) = \sum a_m x^m$ be an integral primitive form defining X and $\Phi_1(a_m), \dots, \Phi_t(a_m)$ be the values of the universal forms in Lemma 1.8 of the coefficients a_m of F . Then $\Phi_i(a_m) \neq 0$ for some $i \in \{1, \dots, t\}$ as X is geometrically integral. Also, by applying Lemma 1.8 to F_p and $K = \mathbf{F}_p$ for the prime factors p of $\Phi_i(a_m)$, we obtain that π_X is a factor of $\Phi_i(a_m)$. But the degree D of Φ_i is bounded in terms of d and r . Hence, if $H(X) = O_{d,r}(V^\theta)$ for some $\theta = O_{d,r}(1)$, then there exists $\psi = O_{d,r}(1)$ such that $\Phi_i(a_m) = O_{d,r}(V^\psi)$. Therefore, by Lemma 1.7, we conclude that $\log \pi_X \leq \log |\Phi_i(a_m)| \ll_{d,r} 1 + \log V$ if (a) does not hold. This completes the proof. □

Lemma 1.10. *Let $\pi > 1$ be an integer and p run over all prime factors of π . Then,*

$$\sum_{p|\pi} \log p/p \leq \log \log \pi + 2.$$

Proof. We may and shall assume that π is square-free. Let m be a positive integer such that $m \leq \pi$ and $v_p(n)$ be the highest integer such that $p^{v_p(n)}|n$. We then have (cf. [49, pp. 13–14]):

$$m \sum_{p|\pi} \log p/p - \sum_{p|\pi} \log p \leq \sum_{p|\pi} v_p(m!) \log p \leq \sum_{p \leq \pi} v_p(m!) \log p = \log m! \leq m \log m,$$

$$\sum_{p|\pi} \log p/p \leq \log m + (1/m) \sum_{p|\pi} \log p \leq \log m + (1/m) \log \pi.$$

To obtain the assertion, let $m = \lceil \log \pi \rceil$ for $\pi > 2$. □

Lemma 1.11. *Let $X \subset \mathbf{P}^{r+1}$ be a hypersurface defined by a form $F(x_0, \dots, x_{r+1})$ of degree d and V, T be as in 1.1. Let $(B_0, \dots, B_{r+1}) \in \mathbf{R}_{\geq 1}^{r+2}$ and $\xi_l = (\xi_{l,0}, \dots, \xi_{l,r+1})$, $l \in \{1, \dots, s\}$ be primitive $(r + 2)$ -tuples of integers representing rational points in $X(\mathbf{Q}; B_0, \dots, B_{r+1})$. Then there exist monomials $F_1(x_0, \dots, x_{r+1}), \dots, F_s(x_0, \dots, x_{r+1})$ of the same degree $k = (r!/d)^{1/r} s^{1/r} + O_{d,r}(1)$ such that no non-trivial linear combination of these forms is divisible by F and such that*

$$\log |\det(F_j(\xi_l))| \leq (r!/d)^{1/r} s^{1+1/r} \log(V^{1/(r+1)}/T^{1/d(r+1)}) + s \log s + O_{d,r}(s \log V). \tag{1.12}$$

Proof. On applying [42, 3.4] in the case of hypersurfaces, $n = r + 1$ for the lexicographical ordering $<$ of the monomials in (x_0, \dots, x_{r+1}) we obtain:

$$\log |\det(F_j(\xi_l))| \leq (r!/d)^{1/r} (r/(r + 1))s[s^{1/r} \log W + O_{d,r}(\log V)] + s \log s, \tag{1.13}$$

where $W = (B_0^{a_0} \dots B_{r+1}^{a_{r+1}})^{(r+1)/r}$ and (a_0, \dots, a_{r+1}) are the invariants introduced by Broberg in [5, Section 2]. But as $B_0^{a_0} \dots B_{r+1}^{a_{r+1}} = (V/T^{1/d})^{1/(r+1)}$ (see [43, 1.4b]), we deduce that $(r/(r + 1)) \log W = \log (V^{1/(r+1)}/T^{1/d(r+1)})$, which completes the proof of Lemma 1.11. □

Proof of Theorem 1.2. We may assume that alternative (b) in 1.9 holds. Let π be the product of all primes p where X_p is not geometrically integral. Then,

$$\sum_{p|\pi} \log p/p \leq \log(1 + \log V) + O_{d,r}(1) \tag{1.14}$$

by 1.9(b) and 1.10.

Now suppose we are given $s \geq 1$ integral primitive $(r + 2)$ -tuples $\xi_l = (\xi_{l,0}, \dots, \xi_{l,r+1})$ with $l \in \{1, \dots, s\}$ such that $|\xi_{l,m}| \leq B_m$ for $m \in \{0, \dots, r + 1\}$ and $F(\xi_l) = 0$ for all l . We may then by Lemma 1.11 find monomials $F_j, 1 \leq j \leq s$ of degree $k = (r!/d)^{1/r} s^{1/r} + O_{d,r}(1)$ such that no non-trivial linear combination of these forms is divisible by F and such that (1.12) holds. We shall prove that the determinant $\det(F_j(\xi_l))$ of this $s \times s$ -matrix vanishes if s is large enough.

We first apply Lemma 1.4 to the primes $p \leq s^{1/r}$ where X_p is geometrically integral and write $\sum_{p \leq s^{1/r}}^*$ for a sum over these primes. We then obtain a positive factor D of $\det(F_j(\xi_l))$ which is relatively prime to π , such that

$$\log D \geq r^{1/r} (r/(r + 1)) s^{1+1/r} \sum_{p \leq s^{1/r}}^* (\log p)/n_p^{1/r} + O_{d,r}(s) \sum_{p \leq s^{1/r}}^* \log p.$$

The last term is $O_{d,r}(s^{1+1/r})$ since $\sum_{p \leq s^{1/r}} \log p = O(s^{1/r})$ (see [49, p. 31], for example). Also, $\log p/n_p^{1/r} \geq \log p/p$ if $n_p^{1/r} \leq p$ and $\log p/n_p^{1/r} \geq \log p/p - (n_p^{1/r} - p)(\log p)/p^2$ if $n_p^{1/r} \geq p$. Hence, by 1.5 we conclude that $\log p/n_p^{1/r} \geq \log p/p + O_{d,r}(\log p/p^{3/2})$ for all p where X_p is geometrically integral. Therefore, $\sum_{p \leq s^{1/r}}^* \log p/n_p^{1/r} \geq \sum_{p \leq s^{1/r}}^* \log p/p + O_{d,r}(1)$ and

$$\log D \geq r^{1/r} (r/(r + 1)) s^{1+1/r} \sum_{p \leq s^{1/r}}^* \log p/p + O_{d,r}(s^{1+1/r}).$$

But $\sum_{p \leq s^{1/r}} \log p/p - \sum_{p \leq s^{1/r}}^* \log p/p \leq \log(1 + \log V) + O_{d,r}(1)$ by (1.14) and $\sum_{p \leq s^{1/r}} \log p/p = (\log s)/r + O(1)$ (see [49, p.14]). Hence,

$$\log D \geq (r^{1/r}/(r + 1)) s^{1+1/r} [\log s - \log(1 + \log V)^r] + O_{d,r}(s^{1+1/r}). \tag{1.15}$$

If we combine this with (1.12) and use the fact that $\log s = O_r(s^{1/r})$, then we get that $\log(|\det(F_j(\xi_l))|/D) \leq (r^{1/r}/(r + 1)) s^{1+1/r} \log((V/T^{1/d})^{1/d^{1/r}} (1 + \log V)^r/s) + O_{d,r}(s(s^{1/r} + \log V))$, where in case $\det(F_j(\xi_l)) = 0$ we formally set $\log 0 = -\infty$.

For $s \geq (\log V)^r$, we conclude that there is a constant $C \geq 1$ depending only on d and r such that

$$\log(|\det(F_j(\xi_l))|/D) \leq (r^{1/r}/(r + 1)) s^{1+1/r} [\log((V/T^{1/d})^{1/d^{1/r}} (1 + \log V)^r/s) + \log C]. \tag{1.16}$$

In particular, $\det(F_j(\xi_l)) = 0$ for $s > C(V/T^{1/d})^{1/d^{1/r}} (1 + \log V)^r$. There is thus a positive integer

$$s \ll_{d,r} (V/T^{1/d})^{1/d^{1/r}} (\log V)^r + 1,$$

such that $\det(F_j(\xi_l)) = 0$ for any set of s integral $(r + 2)$ -tuples $\xi_l = (\xi_{l,0}, \dots, \xi_{l,r+1})$, $1 \leq l \leq s$ representing points in $X(\mathbf{Q}; B_0, \dots, B_{r+1})$. There are, therefore, integers $\lambda_1, \dots, \lambda_s$, not all zero, such that the form $G = \lambda_1 F_1 + \dots + \lambda_s F_s$ vanishes at $X(\mathbf{Q}; B_0, \dots, B_{r+1})$. By Lemma 1.11, G is a form of degree

$$k = (r!/d)^{1/r} s^{1/r} + O_{d,r}(1) = O_{d,r}((V/T^{1/d})^{1/r d^{1/r}} \log V + 1),$$

which is not divisible by F . It will therefore define a hypersurface $Y \subset \mathbf{P}^{r+1}$ with all the desired properties. If $B_0 = \dots = B_{r+1} = B$, then $V = B^{r+2}$ and $T = B^d$. Hence, $\deg Y = O_{d,r}(B^{(r+1)/rd^{1/r}} \log B + 1)$ in that case. This completes the proof of Theorem 1.2.

Theorem 1.17. *Let $C \subset \mathbf{P}^n$ be an integral curve of degree d over \mathbf{Q} . Then $N(C; B) = O_n(B^{2/d})$ if $d \leq 2$ and*

$$N(C; B) = O_{d,n}(B^{2/d} \log B + 1)$$

for general d . If C is not geometrically integral, then $\#C(\mathbf{Q}) = O_{d,n}(1)$.

Proof. Suppose first that C is geometrically integral and that $d \geq 3$. Then the assertion follows from Theorem 1.2 and the theorem of Bézout when $n = 2$. If $n > 2$, we apply the birational projection argument in [12, section 3] for a suitable linear projection $\lambda: C \rightarrow \mathbf{P}^2$ from a $(n - 3)$ -subspace not intersecting $C \subset \mathbf{P}^n$. It is shown there that we may choose λ such that the image $\lambda(C)$ is a geometrically integral plane curve of degree d and such that there exists a constant $c_0 \ll_{d,n} 1$ with $N(C; B) \leq dN(\lambda(C); c_0 B)$. Therefore, as $1 + \log c_0 B = O_{d,n}(1 + \log B)$, we conclude that:

$$N(C; B) \leq dN(\lambda(C); c_0 B) \ll_{d,n} (c_0 B)^{2/d} (1 + \log c_0 B) \ll_{d,n} B^{2/d} \log B + 1.$$

If $d = 2$, then we use the estimate $N(\lambda(C); c_0 B) \ll (c_0 B)^2$ in [9, theorem 6] instead of Theorem 1.2 and if $d = 1$, then we apply [27, lemma 1(iii)]. To prove the assertion for curves that are not geometrically integral, we apply the arguments in the proof of theorem 2.1 in [41]. This completes the proof. □

The uniform bound in 1.17 is a slight improvement of the bound $N(C; B) = O_{d,\varepsilon}(B^{2/d + \varepsilon})$ in [27, theorem 3] for space curves. It is easy to show that $B^{2/d} \ll_C N(C; B) \ll_C B^{2/d}$ for \mathbf{Q} -rational projective curves C . Hence the uniform bound in 1.17 is close to best possible for rational curves. For non-rational curves, there are sharper uniform bounds due to Ellenberg and Venkatesh [17].

2 | COUNTING FUNCTIONS WITH CONGRUENCE CONDITIONS

In this section, we shall prove a more precise version of 1.2 with congruence conditions.

Notation 2.1. Let $X, \Xi, (B_0, \dots, B_{r+1}) \in \mathbf{R}_{\geq 1}^{r+2}$ and $X(\mathbf{Q}; B_0, \dots, B_{r+1})$ be as in 1.1 and 1.3. Let p_1, \dots, p_t be primes and P_i be an \mathbf{F}_{p_i} -point on $X_{p_i} = \Xi \times_{\mathbf{Z}} \mathbf{F}_{p_i}$ for each $i \in \{1, \dots, t\}$.

- (i) $X(\mathbf{Q}; B_0, \dots, B_{r+1}; P_1, \dots, P_t)$ is the subset of points in $X(\mathbf{Q}; B_0, \dots, B_{r+1})$ which specialise to P_i on X_{p_i} for each $i \in \{1, \dots, t\}$. If $B_0 = \dots = B_{r+1} = B$, then we write $X(\mathbf{Q}; B; P_1, \dots, P_t)$ for this set.

(ii) $N(X; B_0, \dots, B_{r+1}; P_1, \dots, P_t) = \#X(\mathbf{Q}; B_0, \dots, B_{r+1}; P_1, \dots, P_t)$. If $B_0 = \dots = B_{r+1} = B$, then we denote this number by $N(X; B; P_1, \dots, P_t)$.

Theorem 2.2. *Let $X \subset \mathbf{P}^{r+1}$ be a geometrically integral hypersurface of degree d over \mathbf{Q} and $(B_0, \dots, B_{r+1}) \in \mathbf{R}_{\geq 1}^{r+2}$. Let (p_1, \dots, p_u) be a (possibly empty) strictly increasing sequence of primes and P_i be a non-singular \mathbf{F}_{p_i} -point on X_{p_i} for each $i \in \{1, \dots, u\}$. Let $q = p_1 \dots p_u$ if $u \geq 1$ and $q = 1$ if $u = 0$. Then there exists a hypersurface $Y \subset \mathbf{P}^{r+1}$ over \mathbf{Q} of degree*

$$O_{d,r} \left(q^{-1} \left(V/T^{1/d} \right)^{1/rd^{1/r}} \log Vq + \log Vq + 1 \right)$$

with $X(\mathbf{Q}; B_0, \dots, B_{r+1}; P_1, \dots, P_u) \subset Y(\mathbf{Q})$, which does not contain X .

Proof. It suffices by Lemma 1.9 to treat the case where the product π_X of all primes where X_p is not geometrically integral satisfies $\log \pi_X = O_{d,r}(1 + \log V)$. Set $\pi = q\pi_X$. Then, by Lemma 1.10 we get

$$\sum_{p|\pi} \log p/p = O_{d,r}(\log(1 + \log Vq)). \tag{2.3}$$

Now suppose that $\xi_l, 1 \leq l \leq s$ are primitive integral $(r + 2)$ -tuples representing rational points in $X(\mathbf{Q}; B_0, \dots, B_{r+1}; P_1, \dots, P_u) \neq \emptyset$. We may then by Lemma 1.11 find s monomials $F_j, 1 \leq j \leq s$ of degree $k = (r!/d)^{1/r} s^{1/r} + O_{d,r}(1)$ such that no non-trivial linear combination of these forms is divisible by F and such that

$$\log |\det(F_j(\xi_l))| \leq (r!/d)^{1/r} s^{1+1/r} \log \left(V^{1/(r+1)} / T^{1/d(r+1)} \right) + s \log s + O_{d,r}(s \log V). \tag{2.4}$$

By repeating the same argument that was used to obtain (1.15) we find a positive factor D of $\det(F_j(\xi_l))$, which is relatively prime to π , such that:

$$\log D \geq \left(r^{1/r} / (r + 1) \right) s^{1+1/r} [(\log s) - \log(1 + \log Vq)^r] + O_{d,r} \left(s^{1+1/r} \right). \tag{2.5}$$

There is also, by [42, 2.5], for each $i \in \{1, \dots, u\}$ a p_i -power d_i which divides $\det(F_j(\xi_l))$ such that

$$\log d_i / \log p_i = r^{1/r} (r / (r + 1)) s^{1+1/r} + O_{d,r}(s).$$

Hence if $D' = d_1 \dots d_u$, then D' is a positive factor of $\det(F_j(\xi_l))$ with $(D, D') = 1$ and

$$\log D' = r^{1/r} (r / (r + 1)) s^{1+1/r} \log q + O_{d,r}(s \log q). \tag{2.6}$$

If we compare (2.4) with (2.5) and (2.6), then we get that

$$\begin{aligned} \log(|\det(F_j(\xi_l))| / DD') &\leq \left(r^{1/r} / (r + 1) \right) s^{1+1/r} \log(q^r \left(V/T^{1/d} \right)^{1/d^{1/r}} (1 + \log Vq)^r / s) \\ &\quad + O_{d,r} \left(s \left(s^{1/r} + \log Vq \right) \right). \end{aligned}$$

There exists, therefore, a constant $C > 0$ depending only on d and r such that

$$\begin{aligned} & \log(|\det(F_j(\xi_l))|/DD') \\ & \leq (r!^{1/r}/(r+1))s^{1+1/r} \left[\log(q^r(V/T^{1/d})^{1/d^{1/r}}(1+\log Vq)^r/s) + \log C \right] \end{aligned} \tag{2.7}$$

for $s \geq (\log Vq)^r$. Hence $\det(F_j(\xi_l)) = 0$ for any $s > \max\{Cq^{-r}(V/T^{1/d})^{1/d^{1/r}}(1+\log Vq)^r, (\log Vq)^r\}$. There is thus a positive integer

$$s \ll_{d,r} \max \left\{ q^{-r}(V/T^{1/d})^{1/d^{1/r}}(1+\log Vq)^r, (\log Vq)^r, 1 \right\},$$

such that $\det(F_j(\xi_l)) = 0$ for any set of s integral $(r+2)$ -tuples $\xi_l, 1 \leq l \leq s$ representing points in $X(\mathbf{Q}; B_0, \dots, B_{r+1}; P_1, \dots, P_u)$. There are, therefore, integers $\lambda_1, \dots, \lambda_s$, not all zero, such that the form $G = \lambda_1 F_1 + \dots + \lambda_s F_s$ vanishes at $X(\mathbf{Q}; B_0, \dots, B_{r+1}; P_1, \dots, P_u)$. By Lemma 1.11, G is a form of degree

$$k = (r!/d)^{1/r} s^{1/r} + O_{d,r}(1) = O_{d,r} \left(q^{-1}(V/T^{1/d})^{1/rd^{1/r}}(1+\log Vq) + (\log Vq) + 1 \right),$$

which is not divisible by F . It will therefore define a hypersurface $Y \subset \mathbf{P}^{r+1}$ with all the required properties. This finishes the proof of Theorem 2.2.

Lemma 2.8. *Let $X \subset \mathbf{P}_Q^{r+1}$ be a geometrically integral hypersurface of degree d over \mathbf{Q} and $(B_0, \dots, B_{r+1}) \in \mathbf{R}_{\geq 1}^{r+2}$. Let (p_1, \dots, p_u) be a strictly increasing sequence of primes such that $q = p_1 \dots p_u \geq e(V/T^{1/d})^{1/rd^{1/r}}$ and P_i be a non-singular \mathbf{F}_{p_i} -point on X_{p_i} for each $i \in \{1, \dots, u\}$. Then there exists a hypersurface $Y \subset \mathbf{P}^{r+1}$ over \mathbf{Q} of degree $O_{d,r}(1 + \log V)$ and of height $H(Y) \leq q^f, f = O_r(1 + \log V)$ with $X(\mathbf{Q}; B_0, \dots, B_{r+1}; P_1, \dots, P_u) \subset Y(\mathbf{Q})$, which does not contain X .*

Proof. Let $\xi_l, 1 \leq l \leq s$ be primitive integral $(r+2)$ -tuples representing rational points in $X(\mathbf{Q}; B_0, \dots, B_{r+1}; P_1, \dots, P_u)$ and $F_j(x_0, \dots, x_{r+1}), 1 \leq j \leq s$ be monomials of the same degree $k = (r!/d)^{1/r} s^{1/r} + O_{d,r}(1)$ such that no non-trivial linear combination of these forms is divisible by F (see Lemma 1.11). By the Hadamard inequality:

$$\det(F_j(\xi_l))^2 \leq \prod_{j=1}^s \left(\sum_{l=1}^s F_j(\xi_l)^2 \right) \leq s^s \prod_{j=1}^s F_j(B_0, \dots, B_{r+1})^2. \tag{2.9}$$

From the proofs of [42, 3.4] and Lemma 1.11, we obtain the bound

$$\log \prod_{j=1}^s |F_j(B_0, \dots, B_{r+1})| \leq ((r!/d)^{1/r} r/(r+1))s^{1+1/r} \log W + O_{d,r}(s \log V)$$

where $(r/(r+1))\log W = \log(V^{1/(r+1)}/T^{1/d(r+1)})$. We have thus

$$\log \prod_{j=1}^s |F_j(B_0, \dots, B_{r+1})| \leq (r!/d)^{1/r} s^{1+1/r} \log(V^{1/(r+1)}/T^{1/d(r+1)}) + O_{d,r}(s \log V). \tag{2.10}$$

Let $A(t) = n_1 + \dots + n_t$ for the non-decreasing sequence $(n_i)_{i=1}^\infty$ of integers $N \geq 0$, where N occur $\binom{r+N-1}{r-1}$ times. Then, any $t \times t$ minor of $F_j(\xi_l)$ is divisible by $q^{A(t)}$ (see [42, 2.4]). Moreover, from $n_t^r/r \leq \binom{n_t+r-1}{r} < t \leq \binom{n_t+r}{r} \leq (n_t+r)^r/r!$, we deduce that $r!^{1/r}t^{1/r} - r \leq n_t < r!^{1/r}t^{1/r}$ and that $A(t) = r!^{1/r}(r/(r+1))t^{1+1/r} + O_r(t)$. We have thus for $q \geq e(V/T^{1/d})^{1/rd^{1/r}}$ that

$$\log q^{A(s)} \geq (r!/d)^{1/r} s^{1+1/r} \log \left(V^{1/(r+1)} / T^{1/d(r+1)} \right) + (r!^{1/r} (r/(r+1))) s^{1+1/r} + O_{d,r}(s(1 + \log V)). \tag{2.11}$$

There exists, therefore, for $q \geq e(V/T^{1/d})^{1/rd^{1/r}}$ a positive constant C depending solely on d and r such that $s^{s/2} \prod_{j=1}^s |F_j(B_0, \dots, B_{r+1})| < q^{A(s)}$ for $s > C(1 + \log V)^r$. For such q and s we have thus that $\det(F_j(\xi_l)) = 0$ since $\det(F_j(\xi_l)) < q^{A(s)}$ and $\det(F_j(\xi_l))$ is divisible by $q^{A(s)}$.

Now let s be the smallest integer with $s > C(1 + \log V)^r$. There are then s monomials F_j , $1 \leq j \leq s$ of degree $k = O_{d,r}(1 + \log V)$ such that $\det(F_j(\xi_l)) = 0$ for any set ξ_l , $1 \leq l \leq s$ of integral $(r + 2)$ -tuples representing points in $S = X(\mathbf{Q}; B_0, \dots, B_{r+1}; P_1, \dots, P_u)$. If $h: \mathbf{P}^{r+1} \rightarrow \mathbf{P}^{s-1}$ is the morphism defined by F_j , $1 \leq j \leq s$, there is thus no s -subset of $h(S)$ which spans \mathbf{P}^{s-1} . The linear span Π of $h(S)$ is therefore a $(t - 1)$ -plane in \mathbf{P}^{s-1} for some $t < s$.

Let x_l , $1 \leq l \leq t$ be rational points in S such that $\{h(x_1), \dots, h(x_t)\}$ spans Π and ξ_l , $1 \leq l \leq t$ be primitive integral $(r + 2)$ -tuples representing these points. Then the $s \times t$ matrix $A = (F_j(\xi_l))$ is of rank t . Let Δ be the greatest common divisor of all $t \times t$ minors of A . By [4, theorem 1] there exists a form $G(x_0, \dots, x_{r+1}) \in \mathbf{Z}F_1 + \dots + \mathbf{Z}F_s$ of degree k and height at most $(\Delta^{-1} \det(A^T A)^{1/2})^{1/(s-t)}$ which vanishes at ξ_1, \dots, ξ_t . This form will define a hyperplane Λ in \mathbf{P}^{s-1} containing Π such that $Y = h^{-1}(\Lambda)$ is a hypersurface in \mathbf{P}^{r+1} containing S . To estimate $H(Y)$, we note that

$$\det(A^T A) \leq \prod_{j=1}^s \left(\sum_{l=1}^t F_j(\xi_l)^2 \right) \leq t^s \prod_{j=1}^s F_j(B_0, \dots, B_{r+1})^2 < q^{2A(s)}.$$

Therefore, as Δ is divisible by $q^{A(t)}$, we conclude that

$$H(Y) \leq (\Delta^{-1} \det(A^T A)^{1/2})^{1/(s-t)} \leq \left(q^{-A(t)} t^{s/2} \prod_{j=1}^s |F_j(B_0, \dots, B_{r+1})| \right)^{1/(s-t)} < q^{(A(s)-A(t))/(s-t)},$$

where $(A(s) - A(t))/(s - t) \leq n_s < r!^{1/r} s^{1/r} = O_r(1 + \log V)$. Hence $H(Y) \leq q^f$ for some $f = O_r(1 + \log V)$. This completes the proof. □

In our applications of Theorem 2.2 and Lemma 2.8 we shall also need the following result.

Lemma 2.12. *Let $X \subset \mathbf{P}_Q^{r+1}$ be a geometrically integral hypersurface of degree d over \mathbf{Q} and $q = p_1 \dots p_u > 1$ be a square-free number such that X_{p_i} is geometrically integral for all primes factors p_i of q . Then the following holds.*

(a) $\prod_{i=1}^u \#X_{p_i}(\mathbf{F}_{p_i}) \leq q^r \exp(C(\log q)^{1/2} / \log \log q)$

for some positive constant C depending only on d and r .

(b) If the singular loci of all X_{p_i} , $i = 1, \dots, u$ are of codimension 3 or more and if

$$p_i \gg_{d,r} \log q \text{ for } i = 1, \dots, u, \text{ then } \prod_{i=1}^u \#X_{p_i}(\mathbf{F}_{p_i}) = O_{d,r}(q^r).$$

Proof.

(a) We have already seen in the proof of Lemma 1.5 that $\#X_p(\mathbf{F}_p)/p^r - 1 = O_{d,r}(p^{-1/2})$ for any prime p where X_p is geometrically integral. There is thus a positive constant A depending solely on d and r such that

$$\prod_{i=1}^u \#X_{p_i}(\mathbf{F}_{p_i}) \leq q^r \left(\prod_{i=1}^u (1 + p_i^{-1/2}) \right)^A.$$

The desired estimate therefore follows from the bound (see [49, I.5, theorem 5]):

$$\prod_{i=1}^u (1 + p_i^{-1/2}) \leq \exp((2 + o(1))(\log q)^{1/2} / \log \log q).$$

(b) By Hooley’s generalisation [29] of Deligne’s theorem, we have that $\#X_p(\mathbf{F}_p) - \#\mathbf{P}^r(\mathbf{F}_p) = O_{d,r}(p^{r-1})$ for any $p \in \{p_1, \dots, p_u\}$. There is thus a positive constant E depending solely on d and r such that

$$\prod_{i=1}^u \#X_{p_i}(\mathbf{F}_{p_i}) \leq q^r \left(\prod_{i=1}^u (1 + p_i^{-1}) \right)^E < q^r \left(\prod_{i=1}^u (1 - p_i^{-1}) \right)^{-E}.$$

By Lemma 1.10 and the last assumption:

$$\sum_{i=1}^u p_i^{-1} \leq (\log \log q + 2) / \log \min_i p_i = O_{d,r}(1).$$

As $a_p = -\log(1 - p^{-1}) - p^{-1} > 0$ and $\sum_p a_p$ converges, we have therefore that

$$-\log \left(\prod_{i=1}^u (1 - p_i^{-1}) \right) = \sum_{i=1}^u (-\log(1 - p_i^{-1})) = O_{d,r}(1),$$

Hence $(\prod_{i=1}^u (1 - p_i^{-1}))^{-E}$ is bounded in terms of d and r , which completes the proof. □

3 | CONSTRUCTION OF SOME HYPERSURFACES AND CODIMENSION 2 CYCLES

The main goal of this section is to prove Lemma 3.2, which forms the technical heart of this paper. It states that all rational points on a hypersurface $X \subset \mathbf{P}^{r+1}$ over \mathbf{Q} that may be represented by integral $(r + 2)$ -tuples in a box lie on a reasonably small number of subvarieties of codimension 1 or 2.

Lemma 3.1. *Let $X \subset \mathbf{P}^{r+1}$ be a geometrically integral hypersurface over \mathbf{Q} of degree d . Let $(B_0, \dots, B_{r+1}) \in \mathbf{R}_{\geq 1}^{r+2}$. Suppose that $X \subset \mathbf{P}^{r+1}$ is the only hypersurface of degree d containing $X(\mathbf{Q}; B_0, \dots, B_{r+1})$. Then the following holds.*

(a) *If x is a non-singular point in $X(\mathbf{Q}; B_0, \dots, B_{r+1})$, then the product π_x of all primes p where x specialises to a singular \mathbf{F}_p -point on X_p satisfies $\log \pi_x = O_{d,r}(1 + \log V)$.*

(b) If X is non-singular, then the product π_{sing} of all primes p where X_p is singular satisfies $\log \pi_{\text{sing}} = O_{d,r}(1 + \log V)$.

Proof. Let $F(x_0, \dots, x_{r+1}) = \sum a_m x^m \in \mathbf{Z}[x_0, \dots, x_{r+1}]$ be a primitive form defining $X \subset \mathbf{P}^{r+1}$. We have then by Lemma 1.7 that $\max |a_m| = O_{d,r}(V^{(d+r+1)/(d-1)(r+1)})$. To prove (a), let $\xi = (\xi_0, \dots, \xi_{r+1})$ be an integral primitive $(r+2)$ -tuple representing a non-singular point $x \in X(\mathbf{Q}; B_0, \dots, B_{r+1})$ and $m \in \{0, \dots, r+1\}$ be chosen such that $(\delta F / \delta x_m)(\xi) \neq 0$. Then, π_x is a factor of $(\delta F / \delta x_m)(\xi)$ and $\log \pi_x \leq \log |(\delta F / \delta x_m)(\xi)| = O_{d,r}(1 + \log V)$. To prove (b), we use that π_{sing} is a factor of the discriminant $\Delta \neq 0$ of F . Therefore, $\log \pi_{\text{sing}} \leq \log |\Delta| = O_{d,r}(1 + \log V)$. \square

The following lemma plays a central role in the proofs of the main theorems of this paper. By a prime divisor on X we shall mean a closed integral subscheme of codimension 1.

Main Lemma 3.2. *Let $r \geq 2$ and $X \subset \mathbf{P}^{r+1}$ be a geometrically integral hypersurface over \mathbf{Q} of degree d . Let $\mathbf{B} = (B_0, \dots, B_{r+1}) \in \mathbf{R}_{\geq 1}^{r+2}$. Then there exists a set of prime divisors D_γ , $\gamma \in \Gamma$ on X and a (possibly empty) set of effective codimension 2 cycles $Z(q)$ on X indexed by a set Q of square-free factors $q > 1$ of an integer q^* with the following properties:*

- (a) $\log q^* = O_{d,r}(\log V)$ and $\log V < p \ll_{d,r} \log V$ for all prime factors p of q^* .
- (b) $q = O_{d,r}((V/T^{1/d})^{1/rd^{1/r}} \log V)$ for all $q \in Q$.
- (c) $\log \#Q = O_{d,r}(\log V / \log \log V)$ if $V > e$ and $\#Q = 0$ if $V \leq e$.
- (d) There exists for each $q = p_1 \dots p_{t+1} \in Q$, $p_1 < p_2 < \dots < p_{t+1}$, a decomposition

$$Z(q) = \sum Z(P_1, \dots, P_{t+1})$$

of $Z(q)$ into effective cycles $Z(P_1, \dots, P_{t+1})$ with

$$\begin{aligned} \deg Z(P_1, \dots, P_{t+1}) &= O_{d,r}(q^{-2}(V/T^{1/d})^{2/rd^{1/r}} (\log V)^3 + q^{-1}(V/T^{1/d})^{1/rd^{1/r}} (\log V)^3 \\ &\quad + (\log V)^2), \end{aligned}$$

where (P_1, \dots, P_{t+1}) runs over all sequences of non-singular points in $\prod_{i=1}^{t+1} X_{P_i}(\mathbf{F}_{P_i})$.

- (e) $\sum_{q \in Q} \deg Z(q) = O_{d,r}((V/T^{1/d})^{1/d^{1/r}} V^{c/\log(1+\log V)})$

for some constant c depending only on d and r .

- (f) $\#\Gamma = O_{d,r}((V/T^{1/d})^{1/rd^{1/r}} \log V + 1)$ and $\deg D_\Gamma = O_{d,r}(1 + \log V)$ for $\gamma \in \Gamma$.
- (g) There exists for each $\gamma \in \Gamma$ a hypersurface of degree $O_{d,r}(1 + \log V)$ and of height $O_{d,r}((V/T^{1/d})^{f/rd^{1/r}} (\log V)^f)$, $f = O_r(1 + \log V)$ which contains D_γ but not X .
- (h) There exists for each non-singular point $x \in X(\mathbf{Q}; B_0, \dots, B_{r+1})$ outside $\bigcup_{\gamma \in \Gamma} D_\gamma$ an integer $q \in Q$ such that x belongs to the support of $Z(q)$ and such that x specialises to a non-singular \mathbf{F}_p -point on X_p for each prime factor p of q .
- (i) If X is non-singular, then we may obtain all the above conclusions for a set Q which is totally ordered with respect to $|$ and with

$$\sum_{q \in Q} \deg Z(q) = O_{d,r} \left((V/T^{1/d})^{1/d^{1/r}} (\log V)^{r+2} \right).$$

Proof. If $V \leq e$, then we choose a set of $O_r(1)$ hyperplane sections containing $X(\mathbf{Q}; \mathbf{B})$ and let $D_\gamma, \gamma \in \Gamma$ be the irreducible components of these hyperplane sections. If $X(\mathbf{Q}; \mathbf{B})$ is contained in another hypersurface Y of degree d , then we let $D_\gamma, \gamma \in \Gamma$ be the components of $X \cap Y$. We have then by the theorem of Bézout in [19, 8.4] that $\#\Gamma = O_d(1)$ and, moreover, that $\deg D_\gamma = O_d(1)$ for each $\gamma \in \Gamma$. In both cases, all the assertions of 3.2 will hold for the above divisors $D_\gamma, \gamma \in \Gamma$ and $Q = \emptyset$. We may and shall thus in the rest of the proof assume that $V > e$ and that $X \subset \mathbf{P}^{r+1}$ is the only hypersurface of degree d containing $X(\mathbf{Q}; \mathbf{B})$. Then, $\log \pi_X = O_{d,r}(\log V)$ by Lemma 1.9 and $\log \pi_{\text{sing}} = O_{d,r}(\log V)$ by Lemma 3.1. By the previous lemma we get also that $\log \pi_x = O_{d,r}(\log V)$ for any non-singular point x in $X(\mathbf{Q}; B_0, \dots, B_{r+1})$. There is thus when $V > e$ a positive constant k_1 depending only on d and r such that $\pi_x \leq V^{k_1}$ for any such point x .

For a prime p which does not divide π_X , let π_p be a product of all primes $p_i < p$ not dividing π_X with $p_i > \log V$ and $\pi_p = 1$ if there are no such primes. Let p^* be the largest prime not dividing π_X such that

$$\pi_{p^*} < eV^{k_1} (V/T^{1/d})^{1/rd^{1/r}}. \tag{3.3}$$

Such a prime must exist since $\pi_p = 1$ for the smallest prime p not dividing π_X .

For $q^* = p^* \pi_{p^*}$, we then have

$$q^* \geq eV^{k_1} (V/T^{1/d})^{1/rd^{1/r}}. \tag{3.4}$$

As p^* is the next prime after some prime factor of $\pi_X \pi_{p^*}$, we obtain from Bertrand's postulate that $p^* \leq 2\pi_X \pi_{p^*}$ and then from (3.3) that $q^* < 2e\pi_X V^{k_1} (V/T^{1/d})^{1/rd^{1/r}}$. Hence $\log q^* = O_{d,r}(\log V)$ by 1.9.

To estimate p^* , we use standard results for $\theta(x) = \sum_{p \leq x} \log p$ (see [49, p. 31]). This gives

$$p^* \ll \theta(p^*) \leq \log \pi_X + \log q^* + \theta(\log V) \ll_{d,r} \log V,$$

thereby proving (a).

If $x \in X(\mathbf{Q}; \mathbf{B})$ is a non-singular point on X , then it follows from the bound $\pi_x \leq V^{k_1}$ and (3.4) that there exists a factor q of q^* , which is relatively prime to $\pi_x \pi_X$ with

$$q \geq e(V/T^{1/d})^{1/rd^{1/r}}. \tag{3.5}$$

Let Q be the set of all factors $q > 1$ of q^* such that $q/p < e(V/T^{1/d})^{1/rd^{1/r}}$ for any prime p dividing q . Then, as $p \leq p^* = O_{d,r}(\log V)$ for each prime factor p of q^* , we get that

$$q = O_{d,r}((V/T^{1/d})^{1/rd^{1/r}} \log V) \tag{3.6}$$

for all $q \in Q$. Hence (b) holds.

Now note that $\log d(q^*) = O(\log q^*/\log \log q^*)$ (see [34, p. 56]) for $q^* > e$ and that the function $f(x) = x/\log x$ is strictly increasing for $x > e$. Therefore, as $\#\mathcal{Q} \leq d(q^*)$, we conclude from (a) that $\log \#\mathcal{Q} = O_{d,r}(\log V/\log \log V)$ for $V > e$. This proves (c).

We now choose a hypersurface $Y(P_1, \dots, P_u)$ as in 2.2 for each sequence (P_1, \dots, P_u) where P_i is a non-singular \mathbf{F}_{p_i} -point on X_{p_i} for $i \in \{1, \dots, u\}$ and where $p_1 < p_2 < \dots < p_u$. We allow the sequence (P_1, \dots, P_u) to be empty in which case we will write $Y(\emptyset)$ instead of $Y(P_1, \dots, P_u)$. For (P_1, \dots, P_u) with $q = p_1 \dots p_u \geq e(V/T^{1/d})^{1/rd^{1/r}}$, we choose $Y(P_1, \dots, P_u)$ such that the stronger conditions in 2.8 hold. Then, as $V > e$ we get that:

$$\deg Y(P_1, \dots, P_u) = O_{d,r}(\log V), \tag{3.7}$$

$$H(Y(P_1, \dots, P_u)) = O_{d,r} \left(\left(V/T^{1/d} \right)^{f/rd^{1/r}} (\log V)^f \right), \quad f = O_r(\log V), \tag{3.8}$$

for sequences (P_1, \dots, P_u) with $q = p_1 \dots p_u \geq e(V/T^{1/d})^{1/rd^{1/r}}$ and $q \in Q$.

We may now define $Z(q)$ for $q \in Q$. Let $q = p_1 p_2 \dots p_{t+1}$ be the prime decomposition of q with increasing prime factors and (P_1, \dots, P_{t+1}) run over all sequences of non-singular points in $\prod_{i=1}^{t+1} X_{p_i}(\mathbf{F}_{p_i})$. Then $Z(q) = \sum Z(P_1, \dots, P_{t+1})$, where $Z(P_1, \dots, P_{t+1})$ is the formal sum of the components of all intersections $D \cap D'$ of irreducible components D of $X \cap Y(P_1, \dots, P_t)$ and irreducible components $D' \neq D$ of $X \cap Y(P_1, \dots, P_{t+1})$.

To establish the bound for $\deg Z(P_1, \dots, P_{t+1})$ in (d), we apply the theorem of Bézout in [19, 8.4], which gives the following bound:

$$\deg Z(P_1, \dots, P_{t+1}) \leq \deg X \cdot \deg Y(P_1, \dots, P_t) \cdot \deg Y(P_1, \dots, P_{t+1}).$$

Also, by Theorem 2.2 and the assumption $V > e$, we have for $u = t$ and $u = t + 1$ that

$$\deg Y(P_1, \dots, P_u) \ll_{d,r} \left(q_u^{-1} \left(V/T^{1/d} \right)^{1/rd^{1/r}} + 1 \right) \log V q_u,$$

where $q_u = p_1 p_2 \dots p_u$ for $u > 0$ and $q_u = 1$ for $u = 0$.

We have already seen that $q_{t+1} \mid q^*$ and $\log V q^* = O_{d,r}(\log V)$. Hence,

$$\begin{aligned} \deg Y(P_1, \dots, P_t) &\ll_{d,r} \left(q_t^{-1} \left(V/T^{1/d} \right)^{1/rd^{1/r}} + 1 \right) \log V, \\ \deg Y(P_1, \dots, P_{t+1}) &\ll_{d,r} \left(q_{t+1}^{-1} \left(V/T^{1/d} \right)^{1/rd^{1/r}} + 1 \right) \log V, \\ \deg Z(P_1, \dots, P_{t+1}) &\ll_{d,r} \left(q_t^{-1} \left(V/T^{1/d} \right)^{1/rd^{1/r}} + 1 \right) \left(q_{t+1}^{-1} \left(V/T^{1/d} \right)^{1/rd^{1/r}} + 1 \right) (\log V)^2. \end{aligned} \tag{3.9}$$

Therefore, as $q_t^{-1} = p_{t+1} q_{t+1}^{-1} = p_{t+1} q^{-1}$ and $p_{t+1} \ll_{d,r} \log V$ (cf. (a)), we obtain the bound for $\deg Z(P_1, \dots, P_{t+1})$ in (d).

By (b) and (d),

$$\deg Z(P_1, \dots, P_{t+1}) \ll_{d,r} q^{-r} \left(V/T^{1/d} \right)^{1/d^{1/r}} (\log V)^{r+2}.$$

As $\log V \leq \exp(16(\log V)^{1/2}/e^2 \log \log V)$ for $V > e$, we have thus

$$\deg Z(P_1, \dots, P_{t+1}) \ll_{d,r} q^{-r} \left(V/T^{1/d}\right)^{1/rd^{1/r}} \exp\left(16(r+2)(\log V)^{1/2}/e^2 \log \log V\right).$$

By Lemma 2.12(a), $\prod_{i=1}^{t+1} \#X_{p_i}(\mathbf{F}_{p_i}) \leq q^r \exp(C(\log q)^{1/2}/\log \log q)$ for some constant $C > 0$ depending only on d and r . Moreover, as $\log q = O_{d,r}(\log V)$, we have

$$(\log q)^{1/2}/\log \log q \ll_{d,r} (\log V)^{1/2}/\log \log V.$$

for $V > e$.

On summing over all sequences (P_1, \dots, P_{t+1}) of non-singular points in $\prod_{i=1}^{t+1} \#X_{p_i}(\mathbf{F}_{p_i})$, we therefore get

$$\deg Z(q) = O_{d,r} \left(V/T^{1/d}\right)^{1/rd^{1/r}} \exp\left(C_1(\log V)^{1/2}/\log \log V\right), \tag{3.10}$$

for some positive constant C_1 depending only on d and r .

For $V > e$, we have further by (c) that $\#Q = \exp(C_2 \log V/\log \log V)$ for some constant $C_2 > 0$ depending only on d . Hence for $V > e$, we obtain that

$$\sum_{q \in Q} \deg Z(q) = O_d \left(\left(V/T^{1/d}\right)^{1/d^{1/r}} \exp(c \log V/\log \log V)\right), \tag{3.11}$$

with $c = C_1 + C_2 > 0$ depending solely on d and r . This proves (e).

We now define the prime divisors $D_\gamma \subset X$, $\gamma \in \Gamma$. These will be the irreducible components of $X \cap Y(\emptyset)$ which are contained in $Y(P_1, \dots, P_{t+1})$ for one of the sequences (P_1, \dots, P_{t+1}) of non-singular points in $\prod_{i=1}^{t+1} X_{p_i}(\mathbf{F}_{p_i})$ with $q = p_1 p_2 \dots p_{t+1} \in Q$ satisfying $q \geq e(V/T^{1/d})^{1/rd^{1/r}}$.

By the theorem of Bézout in [19, 8.4], (3.7) and Theorem 2.2, we obtain that

$$\begin{aligned} \deg D_\gamma &\leq \deg X \cdot \deg Y(P_1, \dots, P_{t+1}) = O_{d,r}(\log V), \\ \text{Card } \Gamma &\leq \deg(X \cap Y(\emptyset)) \leq \deg X \cdot \deg Y(\emptyset) = O_{d,r} \left(\left(V/T^{1/d}\right)^{1/rd^{1/r}} \log V\right), \end{aligned}$$

thereby proving (f). Moreover, as $D_\gamma \subset Y(P_1, \dots, P_{t+1})$, we see from (3.7) and (3.8) that (g) is satisfied for the divisors D_γ .

To prove (h), let $x \in X(\mathbf{Q}; B_0, \dots, B_{r+1})$ be a non-singular point outside $\bigcup_{\gamma \in \Gamma} D_\gamma$. There exists then by (3.5) a square-free number $q = q_{t+1} = p_1 \dots p_{t+1} \geq e(V/T^{1/d})^{1/rd^{1/r}}$ in Q , such that x specialises to a non-singular \mathbf{F}_{p_i} -point P_i on X_{p_i} for $i = 1, \dots, t+1$. Let D_x be an irreducible component of $X \cap Y(\emptyset)$ containing x . One cannot have $D_x \subseteq Y(P_1, \dots, P_{t+1})$ since this would contradict the assumption that $x \notin \bigcup_{\gamma \in \Gamma} D_\gamma$. Hence there is an index $u \in \{0, \dots, t\}$ with $D_x \subseteq Y(P_1, \dots, P_u)$ but $D_x \not\subseteq Y(P_1, \dots, P_{u+1})$. This means that x belongs to the support of $Z(q_{u+1})$ for the factor $q_{u+1} = p_1 \dots p_{u+1} \in Q$ of $q_{t+1} \in Q$, thereby proving (h).

If X is non-singular, then we change the definition of the constant $k_1 > 0$ and choose it such that the stronger condition $\pi_{\text{sing}} \leq V^{k_1}$ holds. By (3.4), there exists then a factor q of q^* , which is relatively prime to π_{sing} such that (3.5) holds. Let $q = p_1 p_2 \dots p_u$ be the prime decomposition of

such a factor with increasing prime factors. If we choose such a q with $q/p_u < e(V/T^{1/d})^{1/rd^{1/r}}$, then (3.6) holds. For $V > e$, let Q be the set of products $q_t = p_1 p_2 \dots p_t$, $t \in \{1, \dots, u\}$. We may then use the same arguments as above to establish (a)-(h) for the new set Q . This set is totally ordered with respect to the relation $|$ and $\#Q = \omega(q) = O(\log q / \log \log q)$ (see [34, p. 55]). Hence for $V > e$, we obtain from (a) that $\#Q \ll_{d,r} \log V / \log \log V$.

Now let $\Sigma_i = q_1^i + \dots + q_u^i$ and $V > e$. Then, by (d) and Lemma 2.12(b) we have

$$\sum_{q \in Q} \deg Z(q) \ll_{d,r} \Sigma_{r-2} \left(V/T^{1/d} \right)^{2/rd^{1/r}} (\log V)^3 + \Sigma_{r-1} \left(V/T^{1/d} \right)^{1/rd^{1/r}} (\log V)^3 + \Sigma_r (\log V)^2. \tag{3.12}$$

Further, $\Sigma_0 \leq \#Q \ll_{d,r} \log V$ and

$$\Sigma_i = q_u^i \sum_{t=0}^{u-1} \left(\frac{q_{t+1}}{q_u} \right)^i \leq q_u^i \sum_{t=0}^{u-1} \left(\frac{1}{2} \right)^{i(u-1-t)} < 2 q_u^i$$

for $i > 0$. Hence, as $q_u = O_{d,r}((V/T^{1/d})^{1/rd^{1/r}} \log V)$ (cf. (3.6)), we obtain from (3.12) that $\sum_{q \in Q} \deg Z(q) \ll_{d,r} (V/T^{1/d})^{1/d^{1/r}} (\log V)^{r+2}$. This proves (i), thereby completing the proof of Lemma 3.2. \square

We now apply Lemma 3.2 to surfaces X in \mathbf{P}^3 . Then the divisors D_γ , $\gamma \in \Gamma$ in 3.2 are integral curves on X of degree $O_d(1 + \log V)$. The following lemma will be sufficient to estimate the contribution to $N(X; \mathbf{B})$ from curves D_γ , $\gamma \in \Gamma$ of high degree, such that it only remains to consider curves of low degree.

Lemma 3.13. *Let $B \in \mathbf{R}_{\geq 1}^4$ and $X \subset \mathbf{P}^3$ be a geometrically integral surface over \mathbf{Q} of degree d and of height $H(X) = O_d(V^\theta)$ for some $\theta = O(1)$. Let $Y \subset \mathbf{P}^3$ be a hypersurface over \mathbf{Q} not containing X of degree $O_d(1 + \log V)$ and of height $H(Y) = O_d(V^f)$ for some $f = O_d(1 + \log V)$. Let D be an integral component of degree δ on $X \cap Y$. Then,*

$$N(D; \mathbf{B}) = O_{d,\delta} \left(V^{8/(\delta+3)} (1 + \log V)^3 \right).$$

Proof. As $\#D(\mathbf{Q}) = O_d(1)$ if D is not geometrically integral (see 1.17), we may assume that D is geometrically integral. Let F (resp. \hat{F}) be primitive integral forms defining X (resp. Y). Then at least one of the determinants

$$\Phi_{ij} = \begin{vmatrix} \partial F / \partial x_i & \partial F / \partial x_j \\ \partial \hat{F} / \partial x_i & \partial \hat{F} / \partial x_j \end{vmatrix}$$

for $0 \leq i < j \leq 3$ will not vanish identically on D . Let Φ be one of these forms and U be the open subset of D where $\Phi \neq 0$. Also, let \tilde{D} be the scheme-theoretic closure of D in \mathbf{P}_Z^3 .

To estimate $N(U; \mathbf{B})$, we will use a set of $s = \binom{\delta+2}{2} - 1 = \delta(\delta+3)/2$ monomials F_1, \dots, F_s of degree $\delta = \deg D$ in (x_0, x_1, x_2, x_3) such that no non-trivial linear combination of these forms vanishes on D . It is easy to see that such a set of monomials exists by utilising a birational projection from D to a plane geometrically integral curve of degree δ .

Now let $p > 4V^{8/(\delta+3)}$ and P be a non-singular \mathbf{F}_p -point P on $D_p = \tilde{D} \times_{\mathbf{Z}} \mathbf{F}_p$. We shall then prove that there is a non-zero form $G = \lambda_1 F_1 + \dots + \lambda_s F_s$, which vanishes at $D(\mathbf{Q}; V; P)$. To see this, it suffices to show that $\det(F_j(\xi_l)) = 0$ for any set ξ_1, \dots, ξ_s of primitive integral quadruples representing $s = \delta(\delta + 3)/2$ rational points in $D(\mathbf{Q}; V; P)$. The integer $\det(F_j(\xi_l))$ is divisible by $p^{A(s)}$ for $A(s) = s(s - 1)/2$ and $|\det(F_j(\xi_l))| \leq s!V^{\delta s} \leq s^s V^{\delta s}$. Moreover, $s^{2/(s-1)} \leq 4$ for $s = \delta(\delta + 3)/2 \geq 2$ and $2\delta/(s - 1) \leq 4\delta/s = 8/(\delta + 3)$. Hence, if $p > 4V^{8/(\delta+3)}$, we get that

$$|\det(F_j(\xi_l))| \leq s^s V^{\delta s} = (s^{2/(s-1)} V^{2\delta/(s-1)})^{s(s-1)/2} \leq (4V^{8/(\delta+3)})^{s(s-1)/2} < p^{A(s)}$$

and $\det(F_j(\xi_l)) = 0$.

By the theorem of Bézout, we have therefore that

$$N(D; \mathbf{B}; P) \leq N(D; V; P) \leq \deg D \cdot \deg G = (\deg D)^2 \ll_d (1 + \log V)^2. \tag{3.14}$$

for such P .

We next show that there exists a constant $C \geq 1$ depending solely on d and δ such that for $\mathbf{B} \in \mathbf{R}_{\geq 1}^4$ with $V \geq C$, we may find a set Ω of $O_d(1 + \log V)$ primes in $(4V^{8/(\delta+3)}, 8V^{8/(\delta+3)})$ with the property that any point in $U(\mathbf{Q}; \mathbf{B})$ specialises to a non-singular \mathbf{F}_p -point P on $D_p = \tilde{D} \times_{\mathbf{Z}} \mathbf{F}_p$ for some $p \in \Omega$.

To see this, let ξ be an integral quadruple representing a point in $U(\mathbf{Q}; \mathbf{B}) \subset \mathbf{P}^3(\mathbf{Q}; \mathbf{B})$. The assumptions for $\deg X, H(X), \deg Y$ and $H(Y)$ imply that $|\Phi(\xi)| = O_{d,\delta}(V^g)$ for some $g = O_d(1 + \log V)$. We now use a quantitative form of Bertrand’s postulate. It is e.g. known [35, p. 38] that $\sum_{x < p \leq 2x} \log p \geq \frac{2}{3}(\log 2)x + O(x^{1/2} \log^2 x)$ and hence that $\prod_{x < p \leq 2x} p > e^{2x/5}$ if x is sufficiently large. There exists therefore a constant $C \geq 1$ depending only on d and δ such that $\prod_{x < p \leq 2x} p > |\Phi(\xi)|$ for $x = 4V^{8/(\delta+3)}$ with $V \geq C$. This implies in its turn that there is a set Ω of $O_{d,\delta}(1 + \log V)$ primes in $(4V^{8/(\delta+3)}, 8V^{8/(\delta+3)})$ with $\prod_{p \in \Omega} p > |\Phi(\xi)|$ for $\mathbf{B} \in \mathbf{R}_{\geq 1}^4$ with $V \geq C$. We may therefore for such \mathbf{B} find a prime $p \in \Omega$ such that $\Phi(\xi)$ is not divisible by p and such that $\xi(\text{mod } p)$ defines a non-singular \mathbf{F}_p -point P on D_p .

We have thus for $V \geq C$ that

$$N(U; \mathbf{B}) \leq \#\Omega \cdot \max_{p \in \Omega} \#D_p(F_p) \cdot \max_p N(D; \mathbf{B}; P).$$

This coupled with (3.14), $\#\Omega = O_{d,\delta}(1 + \log V)$ and $\#D_p(\mathbf{F}_p) = O_d(p)$ shows that

$$N(U; \mathbf{B}) = O_{d,\delta}(V^{8/(\delta+3)}(1 + \log V)^3) \tag{3.15}$$

for $\mathbf{B} \in \mathbf{R}_{\geq 1}^4$ with $V \geq C$ and hence for all $\mathbf{B} \in \mathbf{R}_{\geq 1}^4$ as $N(U; \mathbf{B}) \leq N(\mathbf{P}^3; \mathbf{B}) \ll V$.

To estimate $N(D - U; \mathbf{B})$, recall that Φ vanishes on $D - U$ but not on D and that D is an irreducible component of $X \cap Y$. Therefore,

$$N(D - U; \mathbf{B}) \leq \deg D \cdot \deg \Phi \leq \deg X \cdot \deg Y \cdot (\deg X + \deg Y - 2) \ll_d (1 + \log V)^2$$

by the theorem of Bézout. This finishes the proof. □

Theorem 3.16. *Let $X \subset \mathbf{P}^3$ be a geometrically integral surface over \mathbf{Q} of degree d and X_{ns} be the non-singular locus of X . Let $\mathbf{B} \in \mathbf{R}_{\geq 1}^4$. Then there exists for each $\varepsilon > 0$ a set of $O_d((V/T^{1/d})^{1/2} \sqrt{d} \log V + 1)$*

geometrically integral curves $D_\lambda \subset X$, $\lambda \in \Lambda = \Lambda_\varepsilon$ of degree $O(1/\varepsilon)$ such that

$$N\left(X_{ns} - \bigcup_{\lambda \in \Lambda} D_\lambda; \mathbf{B}\right) = O_d\left(\left(V/T^{1/d}\right)^{1/\sqrt{d}} V^{c/\log(1+\log V)}\right) + O_{d,\varepsilon}\left(\left(V/T^{1/d}\right)^{1/2\sqrt{d}} V^\varepsilon\right)$$

for some positive constant c depending only on d . Moreover, if X is non-singular, then we may find a set of geometrically integral curves $D_\lambda \subset X$, $\lambda \in \Lambda = \Lambda_\varepsilon$ of degree $O(1/\varepsilon)$ such that

$$N\left(X - \bigcup_{\lambda \in \Lambda} D_\lambda; \mathbf{B}\right) = O_d\left(\left(V/T^{1/d}\right)^{1/\sqrt{d}} (\log V)^4\right) + O_{d,\varepsilon}\left(\left(V/T^{1/d}\right)^{1/2\sqrt{d}} V^\varepsilon\right).$$

Proof. We apply the main lemma in the case $r = 2$. Let $D_\gamma \subset X$, $\gamma \in \Gamma$ and $Z(q) \subset X$, $q \in Q$ be as in 3.2 and as in 3.2(i) if X is non-singular. Then, $N(X_{ns} - \bigcup_{\gamma \in \Gamma} D_\gamma; \mathbf{B}) \leq \sum_{q \in Q} \deg Z(q)$ by 3.2(h). We have, therefore, by 3.2(i) that

$$N\left(X - \bigcup_{\gamma \in \Gamma} D_\gamma; \mathbf{B}\right) \leq O_d\left(\left(V/T^{1/d}\right)^{1/\sqrt{d}} (\log V)^4\right) \tag{3.17}$$

if X is non-singular and by 3.2(e) that

$$N\left(X_{ns} - \bigcup_{\gamma \in \Gamma} D_\gamma; \mathbf{B}\right) \leq O_d\left(\left(V/T^{1/d}\right)^{1/\sqrt{d}} V^{c/\log(1+\log V)}\right), \quad c = O_d(1) \tag{3.18}$$

in general.

Now let $\Lambda \subset \Gamma$ be the subset of all indices $\gamma \in \Gamma$ such that $\deg D_\gamma \leq 16/\varepsilon$. Then,

$$N\left(X_{ns} - \bigcup_{\lambda \in \Lambda} D_\lambda; \mathbf{B}\right) \leq N\left(X_{ns} - \bigcup_{\gamma \in \Gamma} D_\gamma; \mathbf{B}\right) + \#\Gamma \cdot \max_{\gamma \in \Gamma - \Lambda} N(D_\gamma; \mathbf{B}). \tag{3.19}$$

To estimate $N(D_\gamma; \mathbf{B})$ for $\gamma \in \Gamma - \Lambda$, let $Y_\gamma \subset \mathbf{P}^3$ be the surface containing D_γ in 3.2(g). Then, $X \not\subset Y_\gamma$, $\deg Y_\gamma = O_d(1 + \log V)$ and $H(Y_\gamma) = O_d(V^f)$ for some $f = O(1 + \log V)$. We may therefore apply Lemma 3.13 to $Y = Y_\gamma$ and $D = D_\gamma$ and conclude that

$$N(D_\gamma; \mathbf{B}) = O_{d,\varepsilon}\left(V^{\varepsilon/2}(1 + \log V)^3\right) \tag{3.20}$$

for any divisor D_γ with index $\gamma \in \Gamma - \Lambda$.

From 3.2(f) and (3.20), we obtain

$$\#\Gamma \cdot \max_{\gamma \in \Gamma - \Lambda} N(D_\gamma; \mathbf{B}) \ll_{d,\varepsilon} \left(V/T^{1/d}\right)^{1/2\sqrt{d}} V^{\varepsilon/2}(1 + \log V)^4 \ll_\varepsilon \left(V/T^{1/d}\right)^{1/2\sqrt{d}} V^\varepsilon. \tag{3.21}$$

The desired bounds for $N(X_{ns} - \bigcup_{\lambda \in \Lambda} D_\lambda; \mathbf{B})$ now follow from (3.17), (3.18), (3.19) and (3.21). □

We may now state two corollaries of Lemma 3.2, which will play a central role in the proofs of Theorems 0.5–0.9.

Corollary 3.22. *Let $X \subset \mathbf{P}^3$ be a geometrically integral surface over \mathbf{Q} of degree d and let $B \geq 1$. Then there exists a set of $O_d(B^{3/2\sqrt{d}} \log B + 1)$ geometrically integral curves $D_\lambda \subset X$, $\lambda \in \Lambda$ of degree $O_d(1)$ and a positive constant c depending only on d such that*

$$N\left(X - \bigcup_{\lambda \in \Lambda} D_\lambda; B\right) = O_d(B^{3/\sqrt{d+c/\log(1+\log B)}}) \quad \text{and such that}$$

$$N\left(X - \bigcup_{\lambda \in \Lambda} D_\lambda; B\right) = O_d(B^{3/\sqrt{d}}(\log B)^4 + 1) \quad \text{if } X \text{ is non-singular.}$$

Proof. Let $B_0 = B_1 = B_2 = B_3 = B$. Then $V = B^4$ and $V/T^{1/d} = B^3$. Now apply the previous theorem in the case $r = 2$, $B_0 = B_1 = B_2 = B_3 = B$ and for some value of ε not greater than $3/8\sqrt{d}$. Then we obtain a set of geometrically integral curves $D_\lambda \subset X$ as above except that we only get that

$$N\left(X_{ns} - \bigcup_{\lambda \in \Lambda} D_\lambda; B\right) = O_d(B^{3/\sqrt{d+c/\log(1+\log B)}})$$

for singular surfaces. But it follows from the Jacobian criterion and the theorem of Bézout that the singular locus of X is contained in a union of $O_d(1)$ integral curves $D \subset X$ of degree $O_d(1)$ and from Theorem 1.17 that $\#D(\mathbf{Q}) = O_d(1)$ if D is not geometrically integral. We may therefore obtain the same bound for $N(X - \bigcup_{\lambda \in \Lambda} D_\lambda; B)$ by including $O_d(1)$ geometrically integral curves of degree $O_d(1)$ in the singular locus of X to the set $D_\lambda \subset X$, $\lambda \in \Lambda$. This finishes the proof. □

Corollary 3.23. *Let $X \subset \mathbf{P}^n$ be a geometrically integral surface over \mathbf{Q} of degree d . Then there exists a positive constant C depending solely on d and n such that the following holds. There exists for each $B \geq 1$ a set of $O_{d,n}(B^{3/2\sqrt{d}} \log B + 1)$ geometrically integral curves of degree $O_{d,n}(1)$ on X such that there are $O_{d,n}(B^{3/\sqrt{d+C/\log(1+\log B)}})$ points in $X(\mathbf{Q}; B)$, which do not lie on any of these curves.*

Proof. We consider a linear birational projection $X \rightarrow \mathbf{P}^3$ from an $(n-4)$ -subspace not intersecting $X \subset \mathbf{P}^n$ as in [43, 8.1]. We may then reduce to the case of hypersurfaces in \mathbf{P}^3 by an argument similar to the reduction to plane curves in the proof of Theorem 1.17. □

4 | THE HILBERT SCHEME OF CONICS IN \mathbf{P}^3

The aim of this section is to prove some results on the geometry of the Hilbert scheme of conics in \mathbf{P}^3 , which we will need in Section 5 to count rational points on families of conics. We shall throughout this section work over an algebraically closed field K of characteristic 0.

Let \mathbf{P}^{3v} be the Grassmannian of planes in \mathbf{P}^3 . There is a universal rank 3 subbundle S and a quotient line bundle Q on \mathbf{P}^{3v} (cf. [15, p.198]). Let \mathbf{H} be the projective bundle $P(\text{Sym}^2 S^\vee)$ of $\text{Sym}^2 S^\vee$ and $\pi: \mathbf{H} \rightarrow \mathbf{P}^{3v}$ be the associated morphism. We use here and in the sequel the “classical” definition in [19, Appendix B5] (and not the dual one of Grothendieck) of a projective space bundle $P(E)$ associated to a vector bundle E on a scheme.

\mathbf{H} is a parameter space for one-dimensional closed subschemes $C \subset \mathbf{P}^3$ of degree 2 spanning a plane. Such subschemes have arithmetic genus 0 by the adjunction formula. We may therefore regard \mathbf{H} as the Hilbert scheme of closed subschemes of \mathbf{P}^3 with Hilbert polynomial $P(t) = 2t + 1$

(cf. [21, 1.b]). We shall in the rest of this section use the word conic for a closed subscheme of \mathbf{P}^3 with Hilbert polynomial $2t + 1$. A conic may thus be a union of two intersecting lines or a double line.

Let $p: \mathfrak{S} \rightarrow \mathbf{H}$ be the universal family of conics in \mathbf{P}^3 , $i: \mathfrak{S} \rightarrow \mathbf{H} \times \mathbf{P}^3$ its embedding in $\mathbf{H} \times \mathbf{P}^3$ and $q: \mathfrak{S} \rightarrow \mathbf{P}^3$ be the restriction to \mathfrak{S} of the projection $\text{pr}_2: \mathbf{H} \times \mathbf{P}^3 \rightarrow \mathbf{P}^3$. Then p is flat and projective with conics $C \subset \mathbf{P}^3$ as fibres. Hence as $H^1(C, \mathcal{O}_C(k)) = 0$ and $\dim H^0(C, \mathcal{O}_C(k)) = 2k + 1$ for $k \geq 0$ for conics, we conclude from the semicontinuity theorem [24, III.12.9] that $R^1 p_*(q^*(\mathcal{O}_{\mathbf{P}^3}(k))) = 0$ and that $p_*(q^*(\mathcal{O}_{\mathbf{P}^3}(k)))$ is locally free of rank of $2k + 1$ for $k \geq 0$.

To find embeddings of \mathbf{H} into projective spaces, we start with the functorial map from $\text{pr}_{1*}(\text{pr}_2^* \mathcal{O}_{\mathbf{P}^3}(k))$ to $\text{pr}_{1*}(i_* \text{pr}_2^* (\text{pr}_2^* \mathcal{O}_{\mathbf{P}^3}(k))) = p_*(q^*(\mathcal{O}_{\mathbf{P}^3}(k)))$ for $\text{pr}_1: \mathbf{H} \times \mathbf{P}^3 \rightarrow \mathbf{H}$. If $h \in \mathbf{H}$ represents the conic $C \subset \mathbf{P}^3_{k(h)}$ over the residue field $k(h)$ of \mathcal{O}_h , then the induced map from $\text{pr}_{1*}(\text{pr}_2^* \mathcal{O}_{\mathbf{P}^3}(k)) \otimes_{\mathbf{H}} k(h)$ to $p_*(q^*(\mathcal{O}_{\mathbf{P}^3}(k))) \otimes_{\mathbf{H}} k(h)$ can be identified with the restriction map from $H^0(\mathbf{P}^3_{k(h)}, \mathcal{O}_{\mathbf{P}^3}(k))$ to $H^0(C, \mathcal{O}_C(k))$. As this map is surjective for all $h \in \mathbf{H}$, the map from $\text{pr}_{1*}(\text{pr}_2^* \mathcal{O}_{\mathbf{P}^3}(k))$ to $p_*(q^*(\mathcal{O}_{\mathbf{P}^3}(k)))$ must also be surjective. By flat base change [24, III.9.3], there is further a functorial isomorphism $\mathcal{O}_{\mathbf{H}} \times_K f_* \mathcal{O}_{\mathbf{P}^3}(k) \rightarrow \text{pr}_{1*}(\text{pr}_2^* \mathcal{O}_{\mathbf{P}^3}(k))$ for $f: \mathbf{P}^3_K \rightarrow K$. The $\mathcal{O}_{\mathbf{H}}$ -module $\text{pr}_{1*}(\text{pr}_2^* \mathcal{O}_{\mathbf{P}^3}(k))$ is thus free of rank $\binom{k+4}{4} = \dim H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(k))$.

Let $\sigma_k: \Lambda^{2k+1} \text{pr}_{1*}(\text{pr}_2^* \mathcal{O}_{\mathbf{P}^3}(k)) \rightarrow \Lambda^{2k+1} p_*(q^*(\mathcal{O}_{\mathbf{P}^3}(k)))$ be the $(2k + 1)$ -th exterior product of the above map. Then σ_k is a surjective map from a free $\mathcal{O}_{\mathbf{H}}$ -module to an invertible $\mathcal{O}_{\mathbf{H}}$ -module, which defines to a morphism $\rho_k: \mathbf{H} \rightarrow P(\Lambda^{2k+1} H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(k)))^\vee$ for $k \geq 1$ (cf. [24, II.7.1] and [37, 2.3]). As $\text{Sym}^k V = H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(k))$ for $V = H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(1))$, we will regard ρ_k as a morphism from \mathbf{H} to $P(\Lambda^{2k+1} \text{Sym}^k V^\vee)$. The geometry of this morphism will be important when we apply the determinant method to families of conics. For $k = 1$, we recover the morphism $\pi: \mathbf{H} \rightarrow \mathbf{P}^{3\vee}$.

There is a canonical isomorphism between the projective bundles $\mathbf{H} = P(\text{Sym}^2 S^\vee)$ and $P(E)$ for $E = \text{Sym}^2(S^\vee \otimes Q^\vee)$. We shall in the sequel identify \mathbf{H} with $P(E)$ and let $\mathcal{O}_E(1)$ denote the tautological line bundle. The Chow ring of $\mathbf{H} = P(E)$ is generated by the first Chern classes $c_1(\pi^* Q)$ and $c_1(\mathcal{O}_E(1))$ (see [19, 8.3.4]). In particular, $\text{Ch}^1(\mathbf{H}) = \mathbf{Z}c_1(\pi^* Q) \oplus \mathbf{Z}c_1(\mathcal{O}_E(1))$.

Lemma 4.1. *Let $[D_k] \in \text{Ch}^1(\mathbf{H})$, $k \geq 1$ be the inverse image of the hyperplane class in $\text{Ch}^1(P(\Lambda^{2k+1} \text{Sym}^k V^\vee))$ under the contravariant map induced by $\rho_k: \mathbf{H} \rightarrow P(\Lambda^{2k+1} \text{Sym}^k V^\vee)$. Then,*

$$[D_k] = kc_1(\pi^* Q) + \binom{k}{2} c_1(\mathcal{O}_E(1)) \quad \text{for } k \geq 1.$$

Proof. Let $L = q^*(\mathcal{O}_{\mathbf{P}^3}(1))$. Then $[D_k] = c_1(\Lambda^{2k+1}(p_*(q^* \mathcal{O}_{\mathbf{P}^3}(k)))) = c_1(\det(p_* L^{\otimes k}))$ and $R^i p_* L^{\otimes k} = R^i p_*(q^* \mathcal{O}_{\mathbf{P}^3}(k)) = 0$ when $i > 0$ and $k \geq 0$. There exist therefore by the Riemann–Roch theorem of Grothendieck and Knudsen-Mumford (see [31, theorem 4] and [18, p. 184]) elements $a_0, a_1, a_2 = p_*(c_1(L)^2)$ in $\text{Ch}^1(\mathbf{H})$ such that $c_1(\det(p_* L^{\otimes k})) = a_0 + ka_1 + \binom{k}{2} a_2$ for all $k \geq 0$. In particular, $a_0 = c_1(\det(p_* \mathcal{O}_{\mathfrak{S}})) = c_1(\mathcal{O}_{\mathbf{H}}) = 0$ and $a_1 = c_1(\det(p_* L)) = [D_1]$.

To determine $a_2 = p_*(c_1(L)^2)$, let $\Lambda \subset \mathbf{P}^3$ be a line and $D \subset \mathbf{H}$ be the scheme parameterising all conics $C \subset \mathbf{P}^3$ which meet Λ . Then, $p_*(c_1(L)^2) = [D]$ (cf. [23, p. 167]) and $a_2 = c_1(\mathcal{O}_E(1))$ as D is given by the vanishing of a global section of $\mathcal{O}_E(1)$ (cf. [19, 3.2.22]). Finally, as $\rho_1 = \pi$ we have that $D_1 \subset \mathbf{H}$ is given by the vanishing of a global section of $\pi^* Q$. Hence, $a_1 = c_1(\pi^* Q)$, thereby completing the proof. □

We shall write $\text{Ch}^p(\mathbf{H})$ (resp. $\text{Ch}_p(\mathbf{H})$) for the Chow group of cycles on \mathbf{H} of codimension p (resp. dimension p) and $\cup : \text{Ch}^p(\mathbf{H}) \times \text{Ch}^q(\mathbf{H}) \rightarrow \text{Ch}^{p+q}(\mathbf{H})$ for products in the Chow ring. This gives a natural intersection pairing $\cup : \text{Ch}^p(\mathbf{H}) \times \text{Ch}_p(\mathbf{H}) \rightarrow \text{Ch}_0(\mathbf{H}) = \mathbf{Z}$.

Lemma 4.2. *Let $X \subset \mathbf{P}^3$ be an integral projective surface of degree $d \geq 2$ and $\mathbf{H}_X \subset \mathbf{H}$ be the Hilbert scheme of conics on X . Let Y be an integral curve on \mathbf{H}_X and $F \subset X \times Y$ be the conic bundle surface over Y induced by the universal family of conics over \mathbf{H}_X and $k \geq 1$. Then the following holds.*

- (a) $\dim \pi(Y) = 1$.
- (b) *The projection $\text{pr}_1 : F \rightarrow X$ is surjective.*
- (c) $c_1(\pi^*Q) \cup [Y] = [K(Y) : K(\pi(Y))] \deg \pi(Y)$.
- (d) $c_1(O_E(1)) \cup [Y] = [K(F) : K(X)]d$.
- (e) $[D_k] \cup [Y] \geq k + \binom{k}{2}d$.
- (f) $[D_k] \cup [Y] \geq 2k + \binom{k}{2}d$ unless Y is mapped isomorphically onto a line in \mathbf{P}^{3v} under π .
- (g) *Assume that $\text{pr}_1 : F \rightarrow X$ is birational, $\text{pr}_2 : F \rightarrow Y$ smooth and that all fibres of pr_2 pass through a non-singular point P on X . Then there is a finite morphism $\phi : \mathbf{P}^2 \rightarrow \mathbf{P}^3$ of projective degree two which maps \mathbf{P}^2 birationally onto X .*

Proof. For (a), we use that the restriction of π to \mathbf{H}_X is finite and for (b) that X is integral. To prove (c), we interpret $c_1(\pi^*Q) \cup [Y]$ as the number of fibres C of $\text{pr}_2 : F \rightarrow Y$ whose linear span $\Pi = \langle C \rangle \subset \mathbf{P}^3$ contains a given point $P \in \mathbf{P}^3$. If P is sufficiently general, these conics will span $\deg \pi(Y)$ different planes Π and there will be $[K(Y) : K(\pi(Y))]$ different conics in each plane Π parameterised by points on Y . This proves (c). To obtain (d), we interpret $c_1(O_E(1)) \cup [Y]$ as the number of fibres of $\text{pr}_2 : F \rightarrow Y$, which meet a given line L . If we choose this line general enough, then it will intersect X in d different points and each fibre of $\text{pr}_2 : F \rightarrow Y$ in at most one point. This proves (d) as there are $[K(F) : K(X)]$ fibres of $\text{pr}_2 : F \rightarrow Y$ passing through a general point on X .

To obtain (e) and (f), we apply Lemmas 4.1 and 4.2(a)–(d). Then we get that $\deg \pi(Y) \geq 1$ and

$$[D_k] \cup [Y] = k[K(Y) : K(\pi(Y))](\deg \pi(Y)) + \binom{k}{2} d[K(F) : K(X)].$$

Hence (e) and (f) hold as $[K(Y) : K(\pi(Y))]\deg \pi(Y) \geq 2$ under the hypothesis in (f).

To show (g), let $p_2 : \tilde{F} \rightarrow \tilde{Y}$ be the base extension of pr_2 along the normalisation $\tilde{Y} \rightarrow Y$. Then \tilde{F} is non-singular and the composition $q_1 : \tilde{F} \rightarrow X$ of $\tilde{F} = F \times_Y \tilde{Y} \rightarrow F$ with pr_1 birational. There is thus a unique factorisation $q_1 = \eta \circ p_1$ (see [24, p. 91]) through the normalisation $\eta : \tilde{X} \rightarrow X$. Let $\tilde{P} = \eta^{-1}(P)$ and $\tilde{E} = p_1^{-1}(\tilde{P}) = q_1^{-1}(P)$. Then \tilde{P} is non-singular on \tilde{X} and the restriction of p_1 to $\tilde{F} - \tilde{E}$ quasi-finite as there are only finitely many fibres of pr_2 passing through a given point on $X - P$. As $p_1 : \tilde{F} \rightarrow \tilde{X}$ is birational, we have thus by Zariski’s main theorem [24, p. 280], that p_1 restricts to an isomorphism from $\tilde{F} - \tilde{E}$ to $\tilde{X} - \tilde{P}$ and that \tilde{X} is non-singular.

As \tilde{P} is a fundamental point of p_1^{-1} , there is a unique morphism $\pi_1 : \tilde{F} \rightarrow Z$ with $p_1 = \mu \circ \pi_1$ for the blow-up $\mu : Z \rightarrow \tilde{X}$ at \tilde{P} (see [24, p. 411]). π_1 is an isomorphism over $Z - \mu^{-1}(\tilde{P})$ and quasi-finite over $\mu^{-1}(\tilde{P})$ as only finitely many fibres of pr_2 pass through an infinitely near point of P . Hence p_1 is isomorphic to the blow-up at \tilde{P} by Zariski’s main theorem. As \tilde{E} is a section of the ruled surface $p_2 : \tilde{F} \rightarrow \tilde{Y}$, there is thus (cf. [24, p. 375]) an isomorphism $\phi_0 : \mathbf{P}^2 \rightarrow \tilde{X}$, where the lines through $\phi_0^{-1}(\tilde{P})$ are sent to the fibres of p_2 . Hence $\eta \circ \phi_0 : \mathbf{P}^2 \rightarrow X$ is finite and birational and its composition $\phi : \mathbf{P}^2 \rightarrow \mathbf{P}^3$ with $X \subset \mathbf{P}^3$ of projective degree 2 as it sends lines through $\phi_0^{-1}(\tilde{P})$ to conics on X . \square

Lemma 4.3. *Let $X \subset \mathbf{P}^3$ be an integral projective surface of degree $d \geq 3$ and $\mathbf{H}_X \subset \mathbf{H}$ be the Hilbert scheme of conics on X . Then the following holds.*

- (a) *If Y is an integral curve on \mathbf{H}_X which is mapped isomorphically onto a line on \mathbf{P}^{3v} under $\pi : \mathbf{H} \rightarrow \mathbf{P}^{3v}$, then $\pi(Y)$ is dual to a line in the singular locus of X .*
- (b) *If $\dim \mathbf{H}_X \geq 2$, then X is a scroll or a Steiner surface of degree 4 with singular locus of X consisting of three possibly coinciding lines.*
- (c) *Suppose that $\dim \mathbf{H}_X \geq 2$ and that X is not a scroll. Then there is exactly one component S of \mathbf{H}_X of dimension more than one. This component S is an integral rational surface with*

$$\deg \rho_k(S) = 4k^4 + 4k^3 - 2k^2 < k^2(2k + 1)^2 \quad \text{for all } k > 1.$$

Proof.

- (a) Let $L \subset \mathbf{P}^3$ be the line dual to $\pi(Y) \subset \mathbf{P}^{3v}$ and $F \subset X \times Y$ be the conic bundle surface over Y in Lemma 4.2. The hypothesis implies that any plane $\Pi \subset \mathbf{P}^3$ containing L is spanned by a unique fibre C of $\text{pr}_2 : F \rightarrow Y$ and hence that the surjective morphism $\text{pr}_1 : F \rightarrow X$ restricts to an isomorphism from $C \setminus L$ to $(\Pi \cap X) \setminus L$. But this can only occur if L is of multiplicity ≥ 2 on $\Pi \cap X$ as $\deg C < \deg(\Pi \cap X)$. Hence L must be in the singular locus of X as otherwise it would be simple on $\Pi \cap X$ for some plane Π containing L .
- (b) Suppose that X is not a scroll. There is then a non-singular point P on X not lying on any line on X and thus an integral curve Y on \mathbf{H}_X such that all conics parameterised by Y are non-singular and pass through P . As $\text{pr}_1 : F \rightarrow X$ is birational if $d \geq 3$ by [48, p. 158] we thus conclude from 4.2(g) that $X \subset \mathbf{P}^3$ is isomorphic to a birational projection of the Veronese surface $V_4 \subset \mathbf{P}^5$ from a line disjoint from V_4 . Hence X is a Steiner surface of degree 4 with a singular locus of X consisting of three possibly coinciding lines (cf. [48, p. 135]).
- (c) Let $\phi: \mathbf{P}^2 \rightarrow \mathbf{P}^3$ be a morphism as in 4.2(g) and X_{ns} be the non-singular locus of X . There is then a bijection between lines $\Lambda \subset \mathbf{P}^2$ with $\Lambda \cap \phi^{-1}(X_{ns}) \neq \emptyset$ and conics $C \subset X$ with $C \cap X_{ns} \neq \emptyset$ where $C = \phi(\Lambda)$. These conics will therefore be parameterised by an open subscheme Ω of \mathbf{H}_X isomorphic to an open subsurface of the dual projective plane \mathbf{P}^{2v} and the closure S of Ω is the only component of \mathbf{H}_X of dimension at least two. By the above bijection, there is further one conic on X passing through two given non-singular points on X and this conic is actually parameterized by a point on $\Omega \subset S$.

There are thus 16 conics on X meeting two disjoint lines in \mathbf{P}^3 which intersect X transversally. By the geometric interpretation of $c_1(O_E(1))$ in the proof of Lemma 4.1, we have therefore

$$c_1(O_E(1)) \cup c_1(O_E(1)) \cup [S] = 16. \tag{4.4}$$

Next, let $L \subset \mathbf{P}^3$ be a line that intersects X transversally in P_1, P_2, P_3 and P_4 . Then a conic C on X will span a plane $\Pi = \langle C \rangle \supset L$ if and only if C passes through two of these points. There are thus altogether six such conics. As $c_1(\pi^*Q)$ is the class of all conics on X , which span planes through a given point on \mathbf{P}^3 , we have thus that

$$c_1(\pi^*Q) \cup c_1(\pi^*Q) \cup [S] = 6. \tag{4.5}$$

Finally, let $L_0 \subset \mathbf{P}^3$ be a line that intersects X transversally in P_1, P_2, P_3 and P_4 and P be a point in $\mathbf{P}^3 - L_0$, such that the four lines $L_i = \langle P, P_i \rangle, 1 \leq i \leq 4$ spanned by P and any of P_i intersect X

transversally. There are then for each i three conics C on X , which contain P_i and for which $\langle C \rangle \supset L_i$. Hence

$$c_1(O_E(1)) \cup c_1(\pi^*Q) \cup [S] = 12. \tag{4.6}$$

For $[D_k] = kc_1(\pi^*Q) + \binom{k}{2}c_1(O_E(1))$, we now obtain from (4.4) to (4.6) that

$$\deg \rho_k(S) = [D_k] \cup [D_k] \cup [S] = 6k^2 + 24k \binom{k}{2} + 16 \binom{k}{2}^2 = 4k^4 + 4k^3 - 2k^2,$$

thereby completing the proof. □

Lemma 4.7. *Let $X \subset \mathbf{P}^3$ be a projective surface of degree $d \geq 2$. Then the Hilbert scheme of conics on X has $O_d(1)$ integral components and $\deg \rho_k(Y) = O_{d,k}(1)$ for each of these components Y .*

Proof. See [43, 3.5]. □

5 | RATIONAL POINTS ON THE UNION OF CONICS

In this section, we shall obtain an estimate for the contribution from the conics to $N(X; B)$ for a family $D_\lambda \subset X$, $\lambda \in \Lambda$ as in Corollary 3.22. We will get crucial savings by using that the rational points are less dense on conics of large height. These savings are especially important if the conics are parameterised by points on a curve or surface of high degree, which is the motivation for the geometric results in Section 4.

We keep the notation in Section 4 such that \mathbf{H} (resp. \mathbf{H}_X) is the Hilbert scheme of conics in \mathbf{P}^3 (resp. on $X \subset \mathbf{P}^3$), V is the vector space of linear forms in the homogeneous coordinates (x_0, x_1, x_2, x_3) of \mathbf{P}^3 . We shall also let $\rho_k : \mathbf{H} \rightarrow P(\Lambda^{2k+1}\text{Sym}^k V^\vee)$, $k \geq 1$ be the morphism described in Section 4.

The monomials of degree k in (x_0, x_1, x_2, x_3) form a basis of $\text{Sym}^k V$ and there is therefore a basis of $\Lambda^{2k+1}\text{Sym}^k V$ consisting of exterior products of $2k + 1$ monomials of degree k . Let $H : P(\Lambda^{2k+1}(\text{Sym}^k V)^\vee)(\mathbf{Q}) \rightarrow \mathbf{R}$ be the standard height function defined by the dual basis. Then H depends only on the homogeneous coordinates (x_0, x_1, x_2, x_3) . We let $H_k(C) = H(\rho_k(h))$, $k \geq 1$ for a conic $C \subset \mathbf{P}^3$ over \mathbf{Q} parameterised by $h \in \mathbf{H}(\mathbf{Q})$. It can thus be viewed as the height of the projective linear $2k$ -subspace spanned by the image of C under the k -fold Veronese embedding of \mathbf{P}^3 .

To compute $H_k(C)$, let F_j , $1 \leq j \leq (k + 1)(k + 2)(k + 3)/6$ be an ordering of the set of monomials in (x_0, x_1, x_2, x_3) of degree k and ξ_l , $1 \leq l \leq 2k + 1$ be quadruples of integers representing rational points on C whose images under the k -fold Veronese map span L_k . Then $H_k(C) = M/\Delta$ for the greatest absolute value M of all the $(2k + 1) \times (2k + 1)$ -minors of $(F_j(\xi_l))$ and the greatest common divisor Δ of these minors.

Lemma 5.1. *Let $C \subset \mathbf{P}^3$ be a non-singular conic defined over \mathbf{Q} . Then,*

$$N(C; B) = O_{k,\varepsilon}(B/H_k(C))^{1/(2k+1)k} + B^\varepsilon \quad \text{for any } k \geq 1.$$

Proof. See [43, 4.1(a)]. □

Lemma 5.2. *Let $X \subset \mathbf{P}^3$ be a geometrically integral projective surface of degree $d \geq 3$ and $S \subset \mathbf{H}_X$ be an integral component. Let $k > 1$. Then,*

- (a) $N(\rho_k(S); R^{\deg \rho_k(S)}) = O_{d,k}(R^2 \log R + 1)$ if $\dim S = 1$.
- (b) $N(\rho_k(S); R^{\deg \rho_k(S)^{1/2}}) = O_k(R^{7/2+1/k}(\log R)^2 + 1)$ if $\dim S = 2$,

and X is not a scroll.

Proof.

- (a) For $k > 1$ $\rho_k : \mathbf{H} \rightarrow P(\Lambda^{2k+1} \text{Sym}^k V^\vee)$ is a closed immersion into some projective space. Hence $N(\rho_k(S); R^{\deg \rho_k(S)}) = O_{d,k}(R^2 \log R + 1)$ by Theorem 1.17 and Lemma 4.7.
- (b) By Lemmas 4.3 and 4.7(b), we have $d = 4$ and $\deg \rho_k(S) = O_k(1)$. We now apply Corollary 3.23 to the closed immersion $\rho_k : S \subset P(\Lambda^{2k+1} \text{Sym}^k V^\vee)$ and $B = R^{\deg \rho_k(S)^{1/2}}$. We then obtain a set of $O_k(R^{3/2} \log R + 1)$ geometrically integral curves Y_γ , $\gamma \in \Gamma$ of degree $O_k(1)$ on $\rho_k(S) \subset P(\Lambda^{g(k)} \text{Sym}^k V)$ such that all but $O_k(R^{3+O_k(1/\log(1+\log R))})$ points of height $\leq B$ lie on one of these curves. It thus only remains estimate the contribution from the curves Y_γ , $\gamma \in \Gamma$ to $N(\rho_k(S); R^{\deg \rho_k(S)^{1/2}})$.

By Theorem 1.17 we have that $N(Y_\gamma; R^{\deg \rho_k(S)^{1/2}}) \ll_k R^{2 \deg \rho_k(S)^{1/2} / \deg \rho_k(Y_\gamma)} \log R + 1$ for all $\gamma \in \Gamma$. By Lemmas 4.2(e), 4.2(f) and 4.3(c), we have further that $\deg \rho_k(S)^{1/2} / \deg \rho_k(Y_\gamma) < (2k + 1)/(2k - 1)$ for all $\gamma \in \Gamma$ and that $\deg \rho_k(S)^{1/2} / \deg \rho_k(Y) < (2k + 1)/2k$ if Y_γ is not mapped isomorphically onto a line in \mathbf{P}^{3V} under π . By Lemmas 4.3(a) and (b) there are at most three curves $Y \subset S$ that are mapped isomorphically onto lines in \mathbf{P}^{3V} under π . Therefore,

$$\sum_{\gamma \in \Gamma} N(Y_\gamma; R^{\deg \rho_k(S)^{1/2}}) \ll_k (R^{3/2} \log R + 1)(R^{2(2k+1)/2k} \log R + 1) + R^{2(2k+1)/(2k-1)} \log R + 1,$$

which is acceptable for $k > 1$. This completes the proof.

The following result will be used to estimate the contribution from the conics $D_\lambda \subset X$, $\lambda \in \Lambda$ in Corollary 3.22.

Lemma 5.3. *Let $X \subset \mathbf{P}^3$ be a geometrically integral surface over \mathbf{Q} of degree $d \geq 3$ and E be a set of $O_{d,\varepsilon}(B^{f+\varepsilon})$ integral conics over \mathbf{Q} on X for some positive real number f . Then the following holds.*

- (a) $\sum_{C \in E} N(C; B) = O_{d,f,\varepsilon}(B^{1+f-df/8+3\varepsilon} + B^{1+3\varepsilon} + B^{f+3\varepsilon})$
if $\dim \mathbf{H}_X \leq 1$ or if all conics $C \in E$ are parameterised by points on irreducible components of dimension at most one on \mathbf{H}_X .
- (b) If $\dim \mathbf{H}_X \geq 2$ and X is not a scroll, then $d = 4$. Further, if $\#E = O_\varepsilon(B^{3/4+\varepsilon})$, then

$$\sum_{C \in E} N(C; B) = O_\varepsilon(B^{43/28+2\varepsilon})$$

Proof.

- (a) By Lemma 4.7 there are $O_d(1)$ integral components Y of \mathbf{H}_X . It is therefore enough to show the lemma in the case where all conics C in E are parameterised by points on an integral

one-dimensional component Y of \mathbf{H}_X . By Lemma 4.2(e),

$$\deg \rho_k(Y) \geq k + d \binom{k}{2} = (2k + 1)k \left(\frac{d}{4} + \frac{1 - 3d/4}{2k + 1} \right).$$

There exists thus for each pair $f > 0, \varepsilon > 0$ an integer $k = k(f, \varepsilon)$ such that

$$\deg \rho_k(Y)/(2k + 1)k \geq d/4 - 2\varepsilon/f. \tag{5.4}$$

Now let R be a real number with $1 \leq R \leq B^{f/2}$. There are, then, by Lemma 5.2(a) $O_{d,f,\varepsilon}(R^2 B^\varepsilon)$ conics C with $H_k(C)^{1/\deg \rho_k(Y)} \in [R, 2R]$ and we have by Lemma 5.1 and (5.4) that $N(C; B) = O_{f,\varepsilon}(BR^{2\varepsilon/f-d/4} + B^\varepsilon)$ for each of these conics. The total contribution from these conics is thus $O_{d,f,\varepsilon}(B^{1+2\varepsilon}R^{2-d/4} + R^2 B^{2\varepsilon})$. On summing over dyadic intervals $[R, 2R]$ which cover $[1, B^{f/2}]$, we obtain that the conics with $H_k(C)^{1/\deg \rho_k(Y)} \leq B^{f/2}$ contribute with $O_{d,f,\varepsilon}(B^{1+f-df/8+3\varepsilon} + B^{1+3\varepsilon} + B^{f+3\varepsilon})$ in total.

For conics with $H_k(C)^{1/\deg \rho_k(Y)} > B^{f/2}$, then $N(C; B) = O_{d,\varepsilon}(B^{1-df/8+\varepsilon} + B^\varepsilon)$ by Lemma 5.1. Therefore, as $\#E = O_{d,\varepsilon}(B^{f+\varepsilon})$ we get a total contribution of $O_{d,f,\varepsilon}(B^{1+f-df/8+2\varepsilon} + B^{f+2\varepsilon})$ from these conics C , thereby proving (a).

- (b) By Lemma 4.3(b), $d = 4$. The contribution to $\sum_{C \in E} N(C; B)$ from conics parameterised by points on components of dimension ≤ 1 is thus $O_\varepsilon(B^{1+3/8+3\varepsilon})$ by (a). We may thus assume that all $C \in E$ are parameterised by points on the unique two-dimensional irreducible component $S \subset \mathbf{H}_X$ described in Lemma 4.3(c) with $\deg \rho_k(S) < (2k + 1)^2$ for $k > 1$.

Now fix $k > 1$ and consider the conics $C \in E$ with $H_k(C) \in [R, 2R]$ for some R with $1 \leq R \leq B^{3/14}$. There are $O_k(R^{7/2+1/k}(\log R)^2 + 1)$ such conics by Lemma 5.2(b). We have also by Lemma 5.1 and (5.4) the uniform bound $N(C; B) = O_{k,\varepsilon}(BR^{8\varepsilon/3-1} + B^\varepsilon)$, $k > 1$ for each of these conics. The total contribution from these conics is thus $O_{k,\varepsilon}(BR^{5/2+1/k+7\varepsilon/3} + R^{7/2+1/k+\varepsilon} B^\varepsilon)$. On summing over $O(\log B)$ dyadic intervals $[R, 2R]$ which cover $[1, B^{3/14}]$, we get for each $k > 1$ a total contribution of $O_{k,\varepsilon}(B^{43/28+3/14k+\varepsilon/2})$ with $H_k(C)^{1/\rho_k(S)^{1/2}} \leq B^{3/14}$.

For conics C with $H_k(C)^{1/\rho_k(S)^{1/2}} > B^{3/14}$, then $N(C; B) = O_{k,\varepsilon}(B^{11/14+4\varepsilon/7})$ by Lemma 5.1 and (5.4). The total contribution from a set of $O_\varepsilon(B^{3/4+\varepsilon})$ such conics on X is thus $O_{k,\varepsilon}(B^{43/28+11\varepsilon/7})$. Hence, $\sum_{C \in E} N(C; B) = O_{k,\varepsilon}(B^{43/28+3/14k+11\varepsilon/7})$ for all $k > 1$, which suffices to deduce (b).

6 | RATIONAL POINTS ON PROJECTIVE SURFACES

The aim of this section is to prove Theorems 0.5–0.9. The following result improves upon theorem 0.1 in [43] and theorem 7 in [27].

Theorem 6.1. *Let $X \subset \mathbf{P}^n$ be a geometrically integral surface over \mathbf{Q} of degree d . Let X' be the complement of the union of all lines on X . Then,*

$$N(X'; B) = O_{d,n,\varepsilon}(B^{3/\sqrt{d+\varepsilon}} + B^{3/2\sqrt{d+2/3+\varepsilon}} + B^{1+\varepsilon}),$$

unless $d = 4$ and there is a two-dimensional family of conics on X . Then

$$N(X'; B) = O_{n,\varepsilon}(B^{43/28+\varepsilon}) \quad \text{and} \quad N(X; B) = O_X(B^{3/2}).$$

Proof. We first prove the theorem when $n = 3$. If $d \leq 2$, then $X' = \emptyset$. We may and shall therefore assume that $d \geq 3$. There exists by Corollary 3.22 a set of $\#\Gamma = O_{d,n}(B^{3/2\sqrt{d}} \log B + 1)$ geometrically integral curves $C_\gamma, \gamma \in \Gamma$ of degree $O_d(1)$ such that all but $O_{d,n}(B^{3/\sqrt{d}+O_{d,n}(1/\log(1+\log B))})$ rational points in $X(\mathbf{Q}; B)$ lie on one of these curves. As $N(C_\gamma; B) = O_{d,n}(B^{2/3} \log B + 1)$ for the curves $C_\gamma, \gamma \in \Gamma$ of degree ≥ 3 (see Theorem 1.17), there are thus $O_{d,n}(B^{3/2\sqrt{d}+2/3}(\log B)^2 + 1)$ points in $X(\mathbf{Q}; B)$, which lie on one of these curves.

To estimate the contribution from the conics, we apply Lemma 5.3 with $f = 3/2\sqrt{d}$. If $\dim \mathbf{H}_X \leq 1$, then we conclude from 5.3a) that there are $O_{d,\varepsilon}(B^{1+3/2\sqrt{d}-3\sqrt{d}/16+\varepsilon} + B^{1+\varepsilon})$ points in the set $X(\mathbf{Q}; B)$, which lie on one of the conics C_γ . The total contribution from all $C_\gamma, \gamma \in \Gamma$ of degree ≥ 2 to $N(X'; B)$ is thus acceptable, thereby proving the theorem when $n = 3$ and $\dim \mathbf{H}_X \leq 1$.

If $\dim \mathbf{H}_X \geq 2$, then by 5.3b) there are $O_\varepsilon(B^{43/28+\varepsilon})$ points in $X(\mathbf{Q}; B)$, which lie on one of the conics C_γ . As $d = 4$ (see Lemma 4.3), we have thus that $N(X'; B) = O_\varepsilon(B^{43/28+\varepsilon})$ for such surfaces. To establish the last assertion, we use the existence (see the proof of Lemma 4.3(c)) of a finite morphism $\phi: \mathbf{P}^2 \rightarrow \mathbf{P}^3$ of projective degree 2 which maps \mathbf{P}^2 birationally onto X . Then $H(y)^2 \ll_X H(\phi(y))$ for all $y \in \mathbf{P}^2(\mathbf{Q})$ (see [46, p. 15]). Hence $X(\mathbf{Q}; B) \cap \phi(\mathbf{P}^2(\mathbf{Q})) \ll_X B^{3/2}$. To get that $N(X'; B) = O_X(B^{3/2})$ it now only remains to that apply Theorem 1.17 to the components of a closed set $T \subset X$ such that ϕ maps $\mathbf{P}^2 - \phi^{-1}(T)$ isomorphically onto $X - T$.

To prove the theorem when $n > 3$, we use a projection $\lambda: \mathbf{P}^n \setminus \Lambda \rightarrow \mathbf{P}^3$ from a linear projective $(n-4)$ -plane $\Lambda \subset \mathbf{P}^n$ not intersecting X . By [12, Section 3] we may choose this morphism λ such that $Z = \lambda(X) \subset \mathbf{P}^3$ is of degree d with at most d points in X over each point of Z and such that there is a constant $c_0 = O_{d,n}(1)$ such that $H(\lambda(x)) \leq c_0 H(x)$ for all rational points x on $\mathbf{P}^n \setminus \Lambda$. By [43, 8.1d] we may further assume that there exists a proper closed subscheme T of Z such that $X \setminus \lambda^{-1}(T)$ is isomorphic to $Z \setminus T$ under λ and such that $T \subset \mathbf{P}^3$ is given by $O_{d,n}(1)$ equations of degree $O_{d,n}(1)$. Therefore, λ maps $(X' - \lambda^{-1}(T))(\mathbf{Q}; B)$ injectively into $Z'(\mathbf{Q}; c_0 B)$. By the Bézout theorem in [19, theorem 8.4.6] we have further that the sum of the degrees of the irreducible components of T is bounded in terms of d and n . It follows that this is also true for $\lambda^{-1}(T)$ such that $N(X' \cap \lambda^{-1}(T); B) = O_{d,n}(B)$ by Theorem 1.17. Hence as Theorem 6.1 holds for $Z \subset \mathbf{P}^3$, we conclude that it also holds for $X \subset \mathbf{P}^n$. □

Remark 6.2. If $d = 3, 4$ or 5 , then we get that $N(X'; B) = O_{n,\varepsilon}(B^{3/\sqrt{d}+\varepsilon})$ unless $d = 4$ and $\dim \mathbf{H}_X \geq 2$.

Theorem 6.3. *Let $X \subset \mathbf{P}^3$ be a non-singular surface over \mathbf{Q} of degree d and let U be the complement of the union of curves of degree $\leq d - 2$ on X . Then,*

$$N(U; B) = O_d \left(B^{3/\sqrt{d}} (\log B)^4 + 1 \right).$$

Proof. The result is known when $d \leq 2$ (see [27, theorem 2] for a sharper result when $d = 2$). We may thus assume that $d \geq 3$. Let $D_\lambda \subset X, \lambda \in \Lambda$ be a set of $O_d(B^{3/2\sqrt{d}} \log B + 1)$ geometrically integral curves $D_\lambda \subset X, \lambda \in \Lambda$ of degree $O_d(1)$ as in Corollary 3.22. Then,

$$N \left(X - \bigcup_{\lambda \in \Lambda} D_\lambda; B \right) = O_d \left(B^{3/\sqrt{d}} (\log B)^4 + 1 \right).$$

To estimate the contribution to $N(U; B)$ from the curves D_λ , we apply Theorem 1.17. Then we get $N(D_\lambda; B) = O_d(B^{2/(d-1)} \log B + 1)$ for curves D_λ of degree at least $d - 1$. The total contribution from

the curves D_λ , $\lambda \in \Lambda$ to $N(U; B)$ is thus $O_d(B^{3/2\sqrt{d} + 2/(d-1)}(\log B)^2 + 1)$, which is acceptable for $d \geq 4$. When $d = 3$, we get $O(B^{\sqrt{3/2} + 2/3}(\log B)^2 + 1)$ in total from the curves D_λ of degree ≥ 3 and for the conics D_λ we get $O_\varepsilon(B^{1 + \sqrt{3/2-3}\sqrt{3/16 + \varepsilon}} + B^{1 + \varepsilon})$ in total by Lemma 5.3(a) as $\dim \mathbf{H}_X \leq 1$. This gives the desired bound for $N(U; B)$ when $d = 3$, thereby completing the proof. \square

Corollary 6.4. *Let $X \subset \mathbf{P}^3$ be a non-singular surface over \mathbf{Q} of degree d . Then,*

$$N(X'; B) = O_d(B^{3/\sqrt{d}}(\log B)^4 + B).$$

Proof. It is proved in [14] that there are $O_d(1)$ curves of degree $\leq d - 2$ on X . The result thus follows from Theorems 6.3 and 1.17.

Corollary 6.5. *Let $\mathbf{a} = (a_0, a_1, a_2, a_3)$ be a quadruple of rational numbers different from zero and $n_{a,d}(B)$ be the number of primitive integer solutions in the region $\max(|x_0|, \dots, |x_3|) \leq B$ to the equation*

$$a_0x_0^d + a_1x_1^d + a_2x_2^d + a_3x_3^d = 0,$$

with $a_0x_0^d + a_jx_j^d \neq 0$ for $j = 1, 2, 3$.

Then,

$$n_{a,d}(B) = O_d\left(B^{3/\sqrt{d}}(\log B)^4 + 1\right).$$

Proof. The case $d = 1$ is trivial and the case $d = 2$ follows from [27, theorem 2]. So let $d \geq 3$ and $X \subset \mathbf{P}^3$ be the surface over \mathbf{Q} defined by the equation $a_0x_0^d + a_1x_1^d + a_2x_2^d + a_3x_3^d = 0$. Then it is known (see [16, Example 2.5.3(a)] or Theorem 9.4 below) that there are $3d^2$ lines on X and that their union is defined by the equation $(a_0x_0^d + a_1x_1^d)(a_0x_0^d + a_2x_2^d)(a_0x_0^d + a_3x_3^d) = 0$. It thus suffices to prove that $N(X'; B) = O_d(B^{3/\sqrt{d}}(\log B)^4 + 1)$.

There are $O_d(1)$ curves of degree $\leq d - 2$ on X (see [14]). We thus get the assertion from Theorem 6.3 provided we can prove that $N(C; B) = O_d(B^{3/\sqrt{d}}(\log B)^4 + 1)$ for any curve C on X of degree δ with $2 \leq \delta \leq d - 2$. But is shown in Theorem 9.4 below that $\delta \geq (d + 1)/3$ for any curve which is not a line. Hence $N(C; B) = O_d(B^{6/(d+1)}\log B)$ by Theorem 1.17, which is acceptable for $d \geq 3$. This finishes the proof. \square

Theorem 6.6. *Let $X \subset \mathbf{P}^4$ be a non-singular complete intersection of two hypersurfaces of degree d_1 and d_2 and let U be the complement of the union of all curves of degree at most $d_1 + d_2 - 3$ on X . Let $d = d_1d_2$. Then,*

$$N(U; B) = O_{d,\varepsilon}\left(B^{3/\sqrt{d+\varepsilon}}\right).$$

Proof. If $d_1 = 1$ or $d_2 = 1$, then $X \subset \mathbf{P}^4$ is given by equations $a_0x_0 + \dots + a_4x_4 = G(x_0, \dots, x_4) = 0$ where some $a_i \neq 0$. If $a_4 \neq 0$, then $G(x_0, \dots, x_3, -(a_0/a_4)x_0 - \dots - (a_3/a_4)x_3)$ is a form in (x_0, \dots, x_3) which defines a non-singular surface $\tilde{X} \subset \mathbf{P}^3$ of degree $d = d_1 + d_2 - 1 = d_1d_2$. The projection $\alpha: X \rightarrow \tilde{X}$ from $(0,0,0,0,1)$ is an isomorphism, which maps U onto the complement \tilde{U} of all curves of degree at most $d - 2$ on \tilde{X} and with $H(\alpha(x)) \leq H(x)$ for $x \in X(\mathbf{Q})$. We have thus by Theorem 6.3 that $N(U; B) \leq N(\tilde{U}; B) = O_d(B^{3/\sqrt{d}}(\log B)^4 + 1)$. If $d_1 = d_2 = 2$, then X is a del Pezzo surface of degree 4. It is well-known that there is no two-dimensional family of conics on such a surface. Hence $N(U; B) = O_\varepsilon(B^{3/\sqrt{d+\varepsilon}})$ by Theorem 6.1.

For the remaining pairs (d_1, d_2) we choose a set $\#\Lambda = O_d(B^{3/2\sqrt{d}} \log B + 1)$ geometrically integral curves D_λ , $\lambda \in \Lambda$ of degree $O_d(1)$ as in Corollary 3.23. Then, all but $O_{d,n}(B^{3/\sqrt{d}+O_{d,n}(1/\log(1+\log B))})$ rational points in $X(\mathbf{Q}; B)$ will lie on one of these curves. Moreover, by Theorem 1.17 we have that $N(D_\lambda; B) = O_d(B^{2/\delta} \log B + 1)$ for curves D_λ of degree $\geq \delta$. The total contribution from the curves D_λ , $\lambda \in \Lambda$ of degree $\geq \delta$ is thus $O_d(B^{3/2\sqrt{d} + 2/\delta} (\log B)^2 + 1)$, which is acceptable for $\delta \geq 4\sqrt{d}/3$. This completes the proof when $d = d_1 d_2 \geq 8$, since then $\deg D_\lambda \geq d_1 + d_2 - 2 \geq 4\sqrt{d}/3$ for all curves D_λ with $U \cap D_\lambda \neq \emptyset$.

We now treat the remaining case $\{d_1, d_2\} = \{2, 3\}$. The total contribution from the curves D_λ , $\lambda \in \Lambda$ of degree ≥ 4 is then $O_\varepsilon(B^{3/2\sqrt{6} + 2/4 + \varepsilon})$, which is acceptable. If $\deg D_\lambda = 3$, then D_λ is contained in a hyperplane $H \subset \mathbf{P}^4$ and either a non-singular twisted cubic in H or a plane cubic [22, proposition 18.9]. The self-intersection number $(D_\lambda \cdot D_\lambda) = 2p_a(D_\lambda) - 2$ (see Ex. II.8.4(c) and Ex.V.1.3 in [24]) such that $(D_\lambda \cdot D_\lambda) = -2$ for a twisted cubic. We may therefore apply the Hilbert scheme arguments in [14] and conclude that there are only $O(1)$ non-singular twisted cubics on X . The total contribution from these cubics is thus $O_\varepsilon(B^{2/3} \log B + 1)$ by Theorem 1.17.

Let $Q \subset \mathbf{P}^4$ be the unique quadric containing X . Any plane cubic C on X will span a plane $\Pi \subset Q$. Hence X can only contain a plane cubic when Q is singular. Also, as X is non-singular, the vertex v of Q must then be a point outside X . By projecting from v , we obtain a morphism $h: X \rightarrow Q'$ to a non-singular quadric in \mathbf{P}^3 , which maps the integral plane cubics on X onto lines on Q' . These lines are the fibres of two morphisms $g_i: Q' \rightarrow \mathbf{P}^1$, $i = 1, 2$ and any plane cubic on X is therefore a fibre of one of the two morphisms $f_i = g_i h$ to \mathbf{P}^1 . By [24, III.10.7] the set S of points $s \in \mathbf{P}^1(\mathbf{C})$ such that $X_s = X \times_{\mathbf{P}^1} k(s)$ is singular is finite and by [2, III.11.4] we have that $\chi(X_C) = \sum_{s \in S} \chi(X_s)$ with all $\chi(X_s) \geq 0$. Hence as $\chi(X_s) \geq 1$ for integral and singular fibres (cf. [20, p. 508]), we conclude that there are at most $\chi(X_C)$ such fibres of f_i . The Euler-characteristic $\chi(X_C) = 24$ as X is a K3-surface (cf. pp. 590–592 in [20]). There are thus at most 48 singular plane cubics on X and they contribute with $O_\varepsilon(B^{2/3} \log B + 1)$ to $N(X; B)$.

To count points on the non-singular plane cubics, we choose a birational projection $\beta: X \rightarrow \mathbf{P}^3$ as in [12, Section 3] and apply the arguments that we used for the conics in the proof of Theorem 6.1. Then $\deg \beta(X) = 6$ and all but $O(1)$ plane cubics on X are mapped isomorphically onto plane cubics on $\beta(X)$. There exists also a constant c_0 such that $N(D_\lambda; B) \leq N(\beta(D_\lambda); c_0 B)$ for all these cubics. It is thus enough to prove that the sum of $N(\beta(D_\lambda); c_0 B)$ over all non-singular plane cubics $\beta(D_\lambda)$ is of order $O_\varepsilon(B^{3/\sqrt{6} + \varepsilon})$. By [41, 1.8], we have $N(C; B) = O_\varepsilon((B/H(\Pi))^{1/3})^{2/3 + \varepsilon} + 1)$ for a non-singular cubic C on a plane $\Pi \subset \mathbf{P}^3$. The planes $\Pi_\lambda \subset \mathbf{P}^3$ spanned by the cubics $\lambda(D_\lambda)$ are images of planes on Q and hence parameterised by a one-dimensional subscheme of bounded degree of the dual space $\mathbf{P}^{3\vee}$. There are thus by Theorem 1.17 $O(R^2)$ such planes Π_λ of height $H(\Pi) \leq 2R$ for $R \geq 1$. The contribution from the non-singular cubics spanning planes of height $H(\Pi) \in [R, 2R]$ is therefore $O_\varepsilon(B^{2/3 + \varepsilon} R^{16/9} + R^2)$ and if we cover $[1, B^{3/4\sqrt{6}}]$ by $O(\log B)$ dyadic intervals $[R, 2R]$, then we get $O_\varepsilon(B^{2/3 + 4/3\sqrt{6} + 2\varepsilon})$ in total. If $H(\Pi) \geq B^{3/4\sqrt{6}}$, then $N(C; B) = O_\varepsilon(B^{2/3 - 1/6\sqrt{6} + \varepsilon})$ which gives $O_\varepsilon(B^{2/3 + 4/3\sqrt{6} + 2\varepsilon})$ in total as $\#\Lambda = O_\varepsilon(B^{3/2\sqrt{6} + \varepsilon})$. The contribution to $N(X; B)$ from the cubics is thus $O_\varepsilon(B^{2/3 + 4/3\sqrt{6} + 2\varepsilon})$, which is acceptable. This completes the proof. □

Corollary 6.7. *Let $X \subset \mathbf{P}^4$ be a non-singular complete intersection of two hypersurfaces of degree d_1 and d_2 and $d = d_1 d_2$. Then,*

$$N(X'; B) = O_{d,\varepsilon} \left(B^{3/\sqrt{d} + \varepsilon} + B \right).$$

Proof. It is proved in [5, lemma 12] that there are $O_d(1)$ curves of degree $\leq d_1 + d_2 - 3$ on X . The result therefore follows from Theorems 6.6 and 1.17. □

7 | INTEGRAL POINTS ON AFFINE SURFACES AND THE DIMENSION GROWTH CONJECTURE FOR PROJECTIVE VARIETIES

In this section, we shall prove Theorems 0.3 and 0.4. We will use the following notation.

Notation 7.1. Let $X \subset \mathbf{P}^{r+1}$ be a quasi-projective variety over \mathbf{Q} .

- (a) $S_1(X; B)$ is the set of rational points on X which may be represented by an integral $(r + 2)$ -tuple $(1, x_1, \dots, x_{r+1})$ with $|x_m| \leq B$ for $m \in \{1, \dots, r + 1\}$.
- (b) $N_1(X; B) = \# S_1(X; B)$.

Theorem 7.2. Let $X \subset \mathbf{P}^n$ be a geometrically integral surface over \mathbf{Q} of degree d and X_{ns} be the non-singular locus of X . Let $\mathbf{B} = (1, B, B, B)$. Suppose that the hyperplane Π_0 defined by $x_0 = 0$ intersects X properly. Then there exist a positive constant c bounded solely in terms of d such that the following holds.

There exists for each $B \geq 1$ a set of $O_d(B^{1/\sqrt{d}} \log B + 1)$ geometrically integral curves $D_\lambda \subset X$, $\lambda \in \Lambda$ of degree $O_d(1)$ such that $N(X_{ns} - \bigcup_{\lambda \in \Lambda} D_\lambda; \mathbf{B}) = O_d(B^{2/\sqrt{d+c}/\log(1+\log B)})$.

Proof. If $\mathbf{B} = (1, B, B, B)$, then $V = B^3$ and $T = B^d$ by the assumption on $\Pi_0 \cap X$. The theorem therefore follows from Theorem 3.16 applied to the case $\mathbf{B} = (1, B, B, B)$ and for some value of ϵ not exceeding $1/3\sqrt{d}$. □

Corollary 7.3. Let $X \subset \mathbf{P}^3$ be a geometrically integral surface over \mathbf{Q} of degree d . Suppose that the scheme-theoretic intersection $\Pi_0 \cap X$ defined is geometrically integral. Then

$$N_1(X; B) = O_{d,\epsilon} \left(B^{2/\sqrt{d+\epsilon}} + B^{1+\epsilon} \right).$$

Proof. Let $D_\lambda \subset X$, $\lambda \in \Lambda$, c and $\mathbf{B} = (1, B, B, B)$ be as in Theorem 7.2. Then,

$$N_1(X_{ns}; \mathbf{B}) \leq \sum_{\lambda \in \Lambda} N_1(D_\lambda; \mathbf{B}) + O_d \left(B^{2/\sqrt{d+c}/\log(1+\log B)} \right).$$

Further, if C is a geometrically integral space curve of degree δ , then $N_1(C; B) = O_{\delta,\epsilon}(B^{1/\delta + \epsilon})$ by the results on affine curves in [3, 38, 39]. We have thus that $N_1(D_\lambda; B) = O_{d,\epsilon}(B^{1/2 + \epsilon})$ for any curve D_λ , $\lambda \in \Lambda$ of degree > 1 . The total contribution to $N_1(X; B)$ from these curves is thus $O_{d,\epsilon}(B^{1/\sqrt{d} + 1/2 + \epsilon})$. Moreover, by proposition 1 in [12] there are $O_{d,\epsilon}(B^{1/\sqrt{d} + \epsilon} + B^{1 + \epsilon})$ points in $S_1(X; B)$ on the union of the curves D_λ , $\lambda \in \Lambda$ of degree 1. We have thus shown that $N_1(X_{ns}; \mathbf{B}) = O_{d,\epsilon}(B^{2/\sqrt{d} + \epsilon} + B^{1 + \epsilon})$.

The singular locus X_{sing} of X is contained in a union of $O_d(1)$ integral curves $D \subset X$ of degree $O_d(1)$ with $\#D(\mathbf{Q}) = O_d(1)$ if D is not geometrically integral (see the proof of Corollary 3.23). We have thus by the above estimates for affine curves that $N_1(X_{\text{sing}}; \mathbf{B}) = O_{d,\epsilon}(B^{1 + \epsilon})$, which completes the proof. □

Theorem 7.4. Let $f(y_1, y_2, y_3) \in \mathbf{Z}[y_1, y_2, y_3]$, be a polynomial such that its homogeneous part $h(f)$ of maximal degree is irreducible over $\overline{\mathbf{Q}}$. Let $d = \deg h(f)$ and $n(f; B)$ be the number of triples

$\mathbf{y} = (y_1, y_2, y_3)$ of integers such that $y_1, y_2, y_3 \in [-B, B]$ and $f(\mathbf{y}) = 0$. Then,

$$n(f; B) = O_{d,\varepsilon}(B^{1+\varepsilon}) \quad \text{if } d \geq 4$$

$$n(f; B) = O_\varepsilon(B^{2/\sqrt{3+\varepsilon}}) \quad \text{if } d = 3.$$

Proof. Let $F(x_0, x_1, x_2, x_3) = x_0^d f(x_1/x_0, x_2/x_0, x_3/x_0)$ and $X \subset \mathbf{P}^3$ be the surface over \mathbf{Q} defined by F . Then $\Pi_0 \cap X$ and X are geometrically integral since $h(f)$ and hence also f is irreducible over $\overline{\mathbf{Q}}$. It is clear from the definitions that $N_1(X; B) = n(f; B)$. The result is therefore just a reformulation of Corollary 7.3. □

Remark. Theorem 0.3 follows from Theorem 7.4 and [12, proposition 8].

The following result gives an almost complete answer to the dimension growth conjecture (Conjecture 0.2) of Heath-Brown and improves upon the previous results in [12, 42].

Theorem 7.5. *Let $X \subset \mathbf{P}^n$ be a geometrically integral projective variety of degree d and dimension r defined over \mathbf{Q} . Then,*

$$N(X; B) = O_{d,n,\varepsilon}(B^{r+\varepsilon}) \quad \text{if } d \geq 4$$

$$N(X; B) = O_{n,\varepsilon}(B^{r-1+2/\sqrt{3+\varepsilon}}) \quad \text{if } d = 3.$$

Proof. Use Theorem 7.4 and [12, theorem 2]. □

Remark. The birational projection argument in [12, section 3] implies that Conjecture 0.2 holds for projective varieties of degree d as soon as it holds for projective hypersurfaces of degree d . Conjecture 0.2 is thus known to be true for varieties of degree 2 [27, theorem 2] and for varieties of dimension at most 3 by [27, theorem 5], [27, theorem 9], [6] and [9, theorem 3]. It was first shown for non-singular hypersurfaces by Browning and Heath-Brown [11, theorem 1].

Castrick et al. [13] have recently been able to remove the B^ε -factor in Theorem 7.5 when $d \geq 5$ and there is also now a sharper bound $N(X; B) = O_{n,\varepsilon}(B^{r+1/7+\varepsilon})$ for $d = 3$ (see [45]).

8 | INTEGRAL AND RATIONAL POINTS ON CUBIC HYPERSURFACES

The aim of this section is to prove the dimension growth conjecture for projective cubic hypersurfaces and thereby obtain a proof of Theorem 0.1 for all projective geometrically integral varieties over \mathbf{Q} of degree $d \geq 2$.

The main new ingredient (see Theorem 8.11) will be an estimate for cubic polynomials $f \in \mathbf{Q}[y_1, \dots, y_n]$ where the homogeneous cubic part $h(f)$ vanishes on a two-dimensional affine linear subspace over \mathbf{Q} . As we will not require uniformity of the implicit constants, we may and shall assume that $h(f)$ vanishes on the linear subspace defined by $y_3 = \dots = y_n = 0$. The homogenisation $F(x_0, \dots, x_n) = x_0^3 f(x_1/x_0, \dots, x_n/x_0)$ of f is then a cubic form over \mathbf{Q} which will vanish on the line

$\Lambda \subset \mathbf{P}^n_{\mathbf{Q}}$ defined by $x_0 = x_3 = \dots = x_n = 0$. It will thus have an expansion

$$F = L_{11}x_1^2 + 2L_{12}x_1x_2 + L_{22}x_2^2 + 2Q_1x_1 + 2Q_2x_2 + C, \tag{8.1}$$

where $L_{11}, L_{12}, L_{22}, Q_1, Q_2$ and C are homogeneous polynomials in $\mathbf{Q}[x_0, x_3, \dots, x_n]$. There is also a similar expansion of $f(y_1, \dots, y_n) = F(1, y_1, \dots, y_n)$

$$f = l_{11}y_1^2 + 2l_{12}y_1y_2 + l_{22}y_2^2 + 2q_1y_1 + 2q_2y_2 + c, \tag{8.2}$$

where $l_{11}, l_{12}, l_{22}, q_1, q_2, c \in \mathbf{Q}[y_3, \dots, y_n]$ are the polynomials obtained from $L_{11}, L_{12}, L_{22}, Q_1, Q_2$ and C by letting $x_0 = 1$ and $x_i = y_i$ for $i = 3, \dots, n$.

Lemma 8.3. *Let $F \in \mathbf{Q}[x_0, x_1, \dots, x_n]$ be an irreducible cubic form as in (8.1) and suppose that at least one of the forms $L_{11}, L_{12}, L_{22}, Q_1, Q_2$ does not vanish. Then the closed subset of $\mathbf{A}^{n-2}_{\mathbf{Q}}$ defined by $l_{11} = l_{12} = l_{22} = q_1 = q_2 = c = 0$ is of codimension at least 2.*

Proof. It follows from the irreducibility of f and (8.2) that the highest common factor $(l_{11}, l_{12}, l_{22}, q_1, q_2, c) = 1$. There is thus no prime ideal in $\mathbf{Q}[y_3, \dots, y_n]$ of height 1 containing the ideal generated by $l_{11}, l_{12}, l_{22}, q_1, q_2$ and c and hence no irreducible component of codimension 1 in the given closed subset. □

Lemma 8.4. *Let $F \in \mathbf{Q}[x_0, x_1, \dots, x_n]$ be a cubic form as in (8.1) and $X \subset \mathbf{P}^n_{\mathbf{Q}}$ be the hypersurface defined by F . Let $Y \subset \mathbf{A}^n_{\mathbf{Q}}$ be the affine hypersurface with coordinates $y_i = x_i x_0$ for $i = 1, \dots, n$ defined by $f(y_1, \dots, y_n) = F(1, y_1, \dots, y_n)$.*

Suppose that the singular locus of Y is of codimension at least 2 in Y and that the line $\Lambda \subset X$ defined by $x_0 = x_3 = \dots = x_n = 0$ is not contained in the singular locus of X . Suppose also that X is not a cone with vertex on Λ . Then (see (8.2)):

$$D = \begin{vmatrix} l_{11} & l_{12} & q_1 \\ l_{12} & l_{22} & q_2 \\ q_1 & q_2 & c \end{vmatrix}$$

does not vanish identically.

Proof. The assertion is equivalent to the assertion that the ternary quadratic form

$$Q_{\text{gen}}(X_0, X_1, X_2) = cX_0^2 + 2q_1X_0X_1 + 2q_2X_0X_2 + l_{11}X_1^2 + 2l_{12}X_1X_2 + l_{22}X_2^2$$

defines a non-singular conic in \mathbf{P}^2_K over $K = \mathbf{Q}(y_3, \dots, y_n)$. To show this, let $p: Y \rightarrow \mathbf{A}^{n-2}_{\mathbf{Q}}$ be the morphism which sends (y_1, \dots, y_n) to (y_3, \dots, y_n) . Then, if we apply the theorem of generic smoothness [24, III.10.8] to the restriction of p to the non-singular locus of Y , we conclude that any singular point on the generic fibre of p belongs to the singular locus of Y . Therefore, as Y is non-singular in codimension 1, the generic fibre of p must be smooth and X_0 vanish at any singular point P on the conic in \mathbf{P}^2_K defined by $Q_{\text{gen}}(X_0, X_1, X_2)$. The ternary quadratic form $Q_{\text{gen}}(X_0, X_1, X_2)$ is thus non-singular if and only if the binary quadratic form $l_{11}X_1^2 + 2l_{12}X_1X_2 + l_{22}X_2^2$ is non-singular and $D \neq 0$ if and only if $L_{11}L_{22} - L_{12}^2 \neq 0$.

If $L_{11}L_{22} - L_{12}^2 = 0$, then there exists a linear form $L \in \mathbf{Q}[x_3, \dots, x_n]$ and rational numbers λ_{ij} such that $L_{ij} = \lambda_{ij}L$ for $1 \leq i \leq j \leq 2$. We may thus, when $D = 0$, assume that $L_{11} = L_{12} = 0$ after a

\mathbf{Q} -linear change of coordinates in x_1 and x_2 . Then (8.1) reduces to:

$$F = L_{22}x_2^2 + 2Q_1x_1 + 2Q_2x_2 + C.$$

But for such a form F we cannot have that $L_{22} = 0$ since this would imply that $\Lambda \subset \text{Sing } X$. Also, we cannot have that $Q_1 = 0$ since X would then be a cone with vertex $(0, 1, 0, \dots, 0)$ on Λ . Hence if $L_{11} = L_{12} = 0$, then $D = -l_{22}q_1^2 \neq 0$. This shows that D cannot vanish identically.

In what follows, we shall say that a line on X is *simple* if it is not contained in the singular locus of X .

Lemma 8.5. *Let $F \in \mathbf{Q}[x_0, x_1, \dots, x_n]$ be a cubic form form as in (8.1), $X \subset \mathbf{P}_{\mathbf{Q}}^n$ be the hypersurface defined by F and Λ be the line defined by $x_0 = x_3 = \dots = x_n = 0$. Suppose that Λ is disjoint to the singular locus $\text{Sing } X$ of $X = 0$ or that Λ is simple on X and there is more than one geometric point on $\Lambda \cap \text{Sing } X$. Then $L_{11}L_{22} - L_{12}^2$ does not vanish identically.*

Proof. If $L_{11}L_{22} - L_{12}^2 = 0$, then we may as in the previous proof assume that $L_{11} = L_{12} = 0$ and $F = L_{22}x_2^2 + 2Q_1x_1 + 2Q_2x_2 + C$. Then $\Lambda \subseteq \text{Sing } X$ when $L_{22} = 0$ while $\Lambda \cap \text{Sing } X = \{(0, 1, 0, \dots, 0)\}$ when $L_{22} \neq 0$. Hence $L_{11}L_{22} - L_{12}^2$ cannot vanish under the given assumptions. □

Notation 8.6. $Z_B = \mathbf{Z} \cap [-B, B]$ for $B > 1$.

$$n(f; B) := \#\{y = (y_1, \dots, y_n) \in \mathbf{Z}_B^n : f(y) = 0\} \text{ for polynomials } f \in \mathbf{Q}[y_1, \dots, y_n].$$

Lemma 8.7. *Let $e \geq 0$ and $f(y), y = (y_1, \dots, y_n)$ be a polynomial in $\mathbf{Q}[y_1, \dots, y_n]$ and $g(y) = f(\varphi(y))$ for a \mathbf{Q} -linear automorphism $\varphi \in GL_n(\mathbf{Q})$. Then $n(f; B) = O_f(B^e)$ if and only if $n(g; B) = O_g(B^e)$.*

Proof. This is trivial and left to the reader. □

Theorem 8.8. *Let $F(x_0, x_1, \dots, x_n)$ be an irreducible cubic form over \mathbf{Q} and $X \subset \mathbf{P}_{\mathbf{Q}}^n$ be the hypersurface defined by F . Let $Y \subset \mathbf{A}_{\mathbf{Q}}^n$ be the affine hypersurface with coordinates $y_i = x_i/x_0$ for $i = 1, \dots, n$ defined by $f(y_1, \dots, y_n) = F(1, y_1, \dots, y_n)$ and $Z \subset \mathbf{P}_{\mathbf{Q}}^{n-1}$ be the hypersurface defined by $h(f)(x_1, \dots, x_n) = F(0, x_1, \dots, x_n)$.*

Suppose that the following conditions hold.

- (i) Y has singular locus of codimension at least 2.
- (ii) There is a rational line Λ on Z , which is disjoint from the singular locus of X or which is simple on X with more than one geometric point of multiplicity 2 on X .

Then $n(f; B) = O_{f,\epsilon}(B^{n-2+\epsilon})$.

Proof. By Lemma 8.7 we may assume that Λ is given by $x_0 = x_3 = \dots = x_n = 0$ and that F has an expansion as in (8.1). We may also assume that $f \in \mathbf{Z}[y_1, \dots, y_n]$ after replacing F by mF for a suitable integer m .

Let $f_b(y_1, y_2) = f(y_1, y_2, b_3, \dots, b_n)$ for $b = (b_3, \dots, b_n) \in \mathbf{Z}_B^{n-2}$. Then,

$$n(f; B) = \sum_{b \in \mathbf{Z}_B^{n-2}} n(f_b; B) \tag{8.9}$$

where $n(f_b; B) = \#\{(y_1, y_2) \in \mathbf{Z}_B^2 : f_b(y_1, y_2) = 0\}$.

If $D \neq 0$ (cf. Lemma 8.4) and $L_{11}L_{22} - L_{12}^2 \neq 0$ at $b \in \mathbf{Z}_B^{n-2}$, then $n(f_b; B) = O_\varepsilon(B^\varepsilon)$ by lemma 13 in [12]. The total contribution to the sum in (8.9) from such $(n-2)$ -tuples is thus $O_\varepsilon(B^{n-2+\varepsilon})$.

For the remaining $b \in \mathbf{Z}_B^{n-2}$ with $f_b \neq 0$, we use the trivial estimate $n(f_b; B) = O(B)$. To estimate the number of such b , we first note that by (ii), X cannot be a cone with vertex on Λ as there cannot be more than two geometric points on $\Lambda \cap \text{Sing } X$. We may hence apply Lemmas 8.4 and 8.5 and conclude that D and $L_{11}L_{22} - L_{12}^2$ do not vanish. There are, therefore, $O_f(B^{n-3})$ $(n-2)$ -tuples $b \in \mathbf{Z}_B^{n-2}$ such that D or $L_{11}L_{22} - L_{12}^2$ vanishes at b . The total contribution to the sum in (8.9) from all $b \in \mathbf{Z}_B^{n-2}$ with $f_b \neq 0$ and $D(L_{11}L_{22} - L_{12}^2)(b) = 0$ is thus $O_f(B^{n-2})$.

It remains to estimate the contribution from the $(n-2)$ -tuples $b \in \mathbf{Z}_B^{n-2}$ with $f_b = 0$. These $(n-2)$ -tuples lie on an affine variety of dimension $\leq n-4$ (see Lemma 8.3). There are thus $O_f(B^{n-4})$ such $(n-2)$ -tuples and we have $n(f_b; B) \leq (2B+1)^2$ for such b . This gives $O_f(B^{n-2})$ in total. We have therefore shown that the right-hand side of (8.9) is of order $O_{f,\varepsilon}(B^{n-2+\varepsilon})$, thereby completing the proof of Theorem 8.8.

To treat the case when there is a rational line on Z not satisfying 8.8(ii), we apply the following result.

Proposition 8.10. *Let F be an irreducible cubic form in $\mathbf{Q}[x_0, x_1, \dots, x_n]$ and $X \subset \mathbf{P}_\mathbf{Q}^n$ be the hypersurface defined by F . Suppose that there is a rational point P of multiplicity two on X such that x_0 vanishes at P and such that the projective tangent cone of X at P is not a double hyperplane in $T_P(\mathbf{P}^n)$ defined by x_0^2 . Let $f(y_1, \dots, y_n) = F(1, y_1, \dots, y_n)$. Then $n(f; B) = O_{f,\varepsilon}(B^{n-2+\varepsilon})$.*

Proof. We may after a linear coordinate change in (x_1, \dots, x_n) (see Lemma 8.7) assume that $P = (0, 1, 0, \dots, 0)$. There are then forms Q, C in (x_0, x_2, \dots, x_n) with

$$F(x_0, x_1, \dots, x_n) = x_1Q(x_0, x_2, \dots, x_n) + C(x_0, x_2, \dots, x_n),$$

where after a further linear coordinate change in (x_2, \dots, x_n) we can suppose that

$$Q(x_0, x_2, \dots, x_n) = x_2L(x_0, x_2, \dots, x_n) + Q_1(x_0, x_3, \dots, x_n)$$

for some linear form $L \neq 0$. We may also after replacing F by mF for a suitable $m \in \mathbf{N}$ assume that Q and C have integer coefficients.

Let $A = \mathbf{Z}[x_0, x_3, \dots, x_n]$ and $R \in A$ be the resultant of $Q, C \in A[x_2]$. Then R is the determinant of the Sylvester matrix of Q and C and hence a sextic form in (x_0, x_3, \dots, x_n) . R cannot be the zero polynomial as Q and C have no common factor of positive degree for irreducible F . By the theory of resultants R belongs to the ideal in $A[x_2]$ generated by Q and C .

For $a = (a_3, \dots, a_n) \in \mathbf{Z}^{n-2}$, let $q_a(y_2) = Q(1, y_2, a_3, \dots, a_n)$ and $c_a(y_2) = C(1, y_2, a_3, \dots, a_n)$. Then $a_1q_a(a_2) + c_a(a_2) = 0$ for any pair $(a_1, a_2) \in \mathbf{Z}^2$ with $f(a_1, a_2, a_3, \dots, a_n) = 0$. As $r(a) := R(1, a_3, \dots, a_n)$ belongs to the ideal of $\mathbf{Z}[y_2]$ generated by $q_a(y_2)$ and $c_a(y_2)$, we have thus for such a that $q_a(a_2) | r(a)$. Also, if $a \in \mathbf{Z}_B^{n-2}$, then $r(a) = O_f(B^6)$. There are, thus, for $a \in \mathbf{Z}_B^{n-2}$ with $r(a) \neq 0$ only $O_{f,\varepsilon}(B^\varepsilon)$ possible values for $q_a(a_2)$ for $(a_1, a_2) \in \mathbf{Z}^2$ with $f(a_1, a_2, a_3, \dots, a_n) = 0$.

We may now count the contribution to $n(f; B)$ from all (y_1, \dots, y_n) with $R(1, y_3, \dots, y_n) \neq 0$ and $y_2L(1, y_2, y_3, \dots, y_n) \neq 0$. For $a = (a_3, \dots, a_n) \in \mathbf{Z}^{n-2}$, let $f_a(y_1, y_2) = f_a(y_1, y_2, a_3, \dots, a_n)$ and $l_a(y_2) = L(1, y_2, a_3, \dots, a_n)$. Then, $y_2l_a(y_2) = q_a(y_2) - Q_1(1, a_3, \dots, a_n)$ and $y_2l_a(y_2) = O_f(B^2)$ for $y_2 \in \mathbf{Z}_B$. There are, thus, $O_{f,\varepsilon}(B^\varepsilon)$ possible values of y_2 for each non-zero value of $y_2l_a(y_2)$. To conclude, we have therefore shown that there are $O_{F,\varepsilon}(B^{2\varepsilon})$ solutions $(y_1, y_2) = (a_1, a_2) \in \mathbf{Z}_B^2$ with

$y_2 l_a(y_2) \neq 0$ to the equation $f_a(y_1, y_2) = 0$ for each $a \in \mathbf{Z}_B^{n-2}$ with $r(a) \neq 0$. This gives $O_{f,\varepsilon}(B^{n-2+2\varepsilon})$ n -tuples in \mathbf{Z}_B^n in total with $f = 0$ and $y_2 L(1, y_2, y_3, \dots, y_n) R(y_3, \dots, y_n) \neq 0$.

The remaining contribution comes from n -tuples in \mathbf{Z}_B^n which lie on the closed subset $W \subset \mathbf{A}^n$ defined by the equations $f = y_2 L(1, y_2, y_3, \dots, y_n) R(y_3, \dots, y_n) = 0$. Here $f \notin \mathbf{Q}[y_3, \dots, y_n]$ as $L \neq 0$. It is therefore clear from the irreducibility of f that W is of dimension $\leq n-2$. There are thus $O_f(B^{n-2})$ n -tuples in \mathbf{Z}_B^n which lie on W . This completes the proof. \square

We now come to the most important new result of this section.

Theorem 8.11. *Let $F(x_0, \dots, x_n)$ be an irreducible cubic form over \mathbf{Q} and $f(y_1, \dots, y_n) = F(1, y_1, \dots, y_n)$. Let $X \subset \mathbf{P}_\mathbf{Q}^n$ be the hypersurface defined by F and $Z = H \cap X$ for the hyperplane $H \subset \mathbf{P}_\mathbf{Q}^n$ defined by $x_0 = 0$. Suppose that the singular locus of $Y = X - Z$ is of codimension at least 2 in Y and that there is a rational line Λ on Z with the following properties:*

- (i) X is not a cone with vertex at a point on Λ .
- (ii) If Λ is simple on X , then there is no rational point $P \in \Lambda$ of multiplicity two on X for which the projective tangent cone of X at P is a double hyperplane in $T_P(\mathbf{P}^n)$ defined by x_0^2 .

Then $n(f; B) = O_{f,\varepsilon}(B^{n-2+\varepsilon})$.

Proof. Suppose first that Λ is simple on X . We may then apply Theorem 8.8 if there is no or more than one singular geometric point on Λ and Proposition 8.10 when there is only one singular geometric point on Λ . It thus only remains to treat the case when Λ is a double line on X . We may then assume that $\Lambda \subset H$ is given by the equations $x_3 = \dots = x_n = 0$ such that

$$F(x_0, x_1, \dots, x_n) = x_1 Q_1(x_0, x_3, \dots, x_n) + x_2 Q_2(x_0, x_3, \dots, x_n) + C(x_0, x_3, \dots, x_n),$$

for some forms Q_1, Q_2 and C . If now $\lambda_1 Q_1 + \lambda_2 Q_2 = 0$ for $\lambda_1, \lambda_2 \in \mathbf{Q}$, then $\lambda_1 = \lambda_2 = 0$ since otherwise X would be a cone with vertex on $(0, \lambda_2, -\lambda_1, 0, \dots, 0) \in \Lambda$. Therefore, at least one of the quadratic forms Q_1 or Q_2 is not divisible by x_0^2 , such that we may apply Proposition 8.10 to $P = (0, 1, 0, \dots, 0)$ or $P = (0, 0, 1, \dots, 0)$. This finishes the proof. \square

If we combine this theorem with the results in the paper [9] of Browning and Heath-Brown, then we obtain the following general theorem.

Theorem 8.12. *Let $G(x_1, \dots, x_n)$ be an absolutely irreducible cubic form over \mathbf{Q} . Then, $n(G; B) = O_{G,\varepsilon}(B^{n-2+\varepsilon})$.*

Proof. If the hypersurface $Z \subset \mathbf{P}_\mathbf{Q}^{n-1}$ defined by G is a cone, then we may assume that its vertex is given by the equation $x_{m+1} = \dots = x_n = 0$, $m < n$ (see Lemma 8.7). Let $G_0(x_1, \dots, x_m) = G(x_1, \dots, x_m, 0, \dots, 0)$. Then G_0 is absolutely irreducible and $n(G; B) \leq (2B + 1)^{n-m} n(G_0; B)$. We may and shall therefore assume that the hypersurface Z defined by G is not a cone.

We now use the fact that any non-conical cubic hypersurface of dimension > 3 is normal (see [33, theorem 3.1]). We may thus apply Theorem 8.11 for non-conical cubic hypersurfaces of dimension > 3 with a rational line. If on the other hand there is no rational line on Z , then the assertion is already known by theorem 2 in [9].

It thus only remains to treat the case when $\dim Z \leq 3$. But then the result is already known thanks to theorems 3 and 9 in [27] and theorem 3 in [9]. \square

Remark. In the case of cubic hypersurfaces with singular locus of codimension ≥ 4 , this result was first obtained by Browning [7] by means of a version of the circle method.

Theorem 8.13. *Let $X \subset \mathbf{P}^m$ be a projective, geometrically integral, r -dimensional variety over \mathbf{Q} of degree $d \geq 2$. Then $N(X; B) = O_{X,\varepsilon}(B^{r+\varepsilon})$.*

Proof. It suffices by Theorem 7.5 and the following remark to prove this for varieties of degree 3. It is then proved in [12, section 3] that there exists a finite birational morphism $f: X \rightarrow \mathbf{P}^{r+1}$ over \mathbf{Q} which maps X onto a cubic hypersurface Z and a constant c such that $H(f(x)) \leq cH(x)$ for all $x \in X(\mathbf{Q})$. Therefore, as $N(Z; B) = O_{Z,\varepsilon}(B^{r+\varepsilon})$ by Theorem 8.12, we will also have that $N(X, B) = O_{X,\varepsilon}(B^{r+\varepsilon})$. □

9 | CURVES ON FERMAT SURFACES

In this section, we shall study the degrees of curves on Fermat surfaces.

Theorem 9.1. *Let K be an algebraically closed field of characteristic 0 and $X \subset \mathbf{P}^n$ be the hypersurface given by the equation $a_0x_0^d + \dots + a_nx_n^d = 0$ for an $(n + 1)$ -tuple (a_0, \dots, a_n) of non-zero elements in K . Let C be a closed integral curve on X of degree δ and geometric genus g which does not lie on any other hypersurface of degree d defined by a diagonal form $b_0x_0^d + \dots + b_nx_n^d$. Then the following holds.*

- (a) $(n + 1)\delta(d - (n - 1)) \leq n\delta d + n(n - 1)(g - 1)$.
- (b) $(n + 1)(d - (n - 1)) \leq nd + n(n - 1)(\delta - 3)/2$.

Proof.

- (a) Let $\Pi \subset \mathbf{P}^n$ be the hyperplane defined by the equation $y_0 + \dots + y_n = 0$ and $h: C \rightarrow \Pi$ the morphism which sends (x_0, \dots, x_n) to $(y_0, \dots, y_n) = (a_0x_0^d, \dots, a_nx_n^d)$. Let $\pi: C_1 \rightarrow C$ be the normalization of C and $f = h\pi$. Let V be the n -dimensional K -subspace of $H^0(C_1, f^*O_\Pi(1))$ spanned by $f^*y_i, i = 0, \dots, n$. For $s \in V \setminus \{0\}$, let $(s)_0 = \sum_P \text{ord}_P(s)P$ be its divisor of zeroes. The set $M_{P,V} = \{\text{ord}_P(s)\}_s \in V \setminus \{0\}$ consists of n non-negative integers. The ramification sequence $\alpha_0(P, V) \leq \alpha_1(P, V) \leq \dots \leq \alpha_{n-1}(P, V)$ of $f: C_1 \rightarrow \Pi$ at $P \in C_1$ is defined by

$$M_{P,V} = \{\alpha_0(P, V), 1 + \alpha_1(P, V), \dots, (n - 1) + \alpha_{n-1}(P, V)\}.$$

By the Plücker formula for $f: C_1 \rightarrow \Pi \approx \mathbf{P}^{n-1}$ (see (*) in exercise C13 in [1, Ch. I]), we get

$$\sum_{P \in C_1} \sum_{i=0}^{n-1} \alpha_i(P, V) = nD + n(n - 1)(g - 1), \tag{9.2}$$

where $D = \delta d$ is the degree $\sum_P \text{ord}_P(s)$ of the divisors $(s)_0$ of zeroes of $s \in V \setminus \{0\}$.

Let $(t)_0 = \sum_P m_P P$ be the divisor of zeroes of $t = \pi^*(x_0 \dots x_n) \in H^0(C_1, \pi^*O_C(n + 1))$. It is an effective Weil divisor of degree $(n + 1)\delta$ on C_1 . To obtain the desired result, it is enough to prove that

$$\sum_{i=0}^{n-1} \alpha_i(P, V) \geq m_P(d - (n - 1)) \tag{9.3}$$

for closed points P on C_1 . This is clear if $m_P = 0$, since $\alpha_i(P, V) \geq 0$ by definition. If $m_P \geq 1$, we reorder (x_0, \dots, x_n) such that $\text{ord}_P \pi^* x_n = 0$ and $\text{ord}_P \pi^* x_0 \leq \text{ord}_P \pi^* x_1 \leq \dots \leq \text{ord}_P \pi^* x_{n-1}$. Then, as $(f^* y_0, \dots, f^* y_{n-1})$ is a basis of V , we obtain that $\text{ord}_P f^* y_i \leq i + \alpha_i(P, V)$ for $i = 0, \dots, n-1$. Also, $m_P = \text{ord}_P \pi^* x_j + \dots + \text{ord}_P \pi^* x_{n-1}$ where $j = \max(n - m_P, 0)$. Hence,

$$\sum_{i=0}^{n-1} \alpha_i(P, V) \geq \sum_{i=j}^{n-1} \alpha_i(P, V) \geq \sum_{i=j}^{n-1} (\text{ord}_P \pi^* y_i - i) = dm_P - \sum_{i=j}^{n-1} i \geq m_P(d - (n - 1)).$$

- (b) It suffices by (a) to prove that $2g - 2 \leq \delta(\delta - 3)$. If C is a plane curve, then this follows from Exercise I.7.2(a) and Exercise IV.1.8(a) in [24]. If C is not contained in a plane, then we project it birationally to a plane curve C' and use that $g(C) = g(C')$ and $\text{deg } C \leq \text{deg } C'$. This completes the proof. □

The following theorem improves upon previous results (cf. [28]) on the degrees of curves on Fermat surfaces.

Theorem 9.4. *Let K be an algebraically closed field of characteristic 0 and $X \subset \mathbf{P}^3$ be the surface given by the equation $a_0 x_0^d + a_1 x_1^d + a_2 x_2^d + a_3 x_3^d = 0$ for a quadruple (a_0, a_1, a_2, a_3) of non-zero elements in K . Then the following holds.*

- (a) *Let $j = 1, 2$ or 3 . Then the subscheme of X defined by $a_0 x_0^d + a_j x_j^d = 0$ is a union of d^2 lines.*
- (b) *Let $d \geq 3$ and $C \subset X$ be a closed integral curve on X , which is not one of the $3d^2$ lines described in (a). Then the degree of C is at least $(d + 1)/3$.*

Proof. (b) Suppose first that C lies on another surface defined by a diagonal equation $b_0 x_0^d + b_1 x_1^d + b_2 x_2^d + b_3 x_3^d = 0$. Then the two diagonal forms define a one-dimensional subscheme Y of \mathbf{P}^3 , which is connected (see [24, Exercise II.8.4]). If $b_i/a_i \neq b_j/a_j$ for all $0 \leq i < j \leq 3$, then Y is also non-singular. Hence Y is integral and $C = Y$ of degree d^2 in this case. If $b_i/a_i = b_j/a_j = \lambda$ for some $i < j$, then $(b_0 - \lambda a_0)x_0^d + (b_1 - \lambda a_1)x_1^d + (b_2 - \lambda a_2)x_2^d + (b_3 - \lambda a_3)x_3^d$ is a binary form G in x_k and x_l for $k, l \notin \{i, j\}$. We may thus, then, find a linear factor L of G , which vanishes on C . Moreover, this linear form cannot divide $a_k x_k^d + a_l x_l^d$ as $a_k x_k^d + a_l x_l^d \neq 0$ on C . Hence the plane section of X defined by L is integral and C of degree d in this case.

It remains to consider the case where C lies on no other diagonal surface of degree d . Then $4(d - 2) \leq 3d + 3(\text{deg } C - 3)$ by 9.1(b), which is equivalent to the desired assertion. □

ACKNOWLEDGEMENT

The author would like to thank Tim Browning and Roger Heath-Brown for their interest in this paper, which was very much inspired by [12] and [27].

JOURNAL INFORMATION

The *Proceedings of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris, *Geometry of algebraic curves*, vol. I, Springer, Berlin, 1985.
2. W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces*, Springer, Berlin-Heidelberg, 1984.
3. E. Bombieri and J. Pila, *The number of integral points on arcs and ovals*, *Duke Math. J.* **59** (1989), 337–357.
4. E. Bombieri and J. Vaaler, *On Siegel’s lemma*, *Invent. Math.* **73** (1983), 11–32.
5. N. Broberg, *A note on a paper by R. Heath-Brown: “The density of rational points on curves and surfaces”*, *J. reine. angew. Math.* **571** (2004), 159–178.
6. N. Broberg and P. Salberger, *Counting rational points on threefolds*, in *Arithmetic of higher-dimensional varieties* (eds. B. Poonen, Y. Tschinkel) *Progress in Mathematics*, vol. 206, Birkhäuser, Boston, 2003.
7. T. D. Browning, *Counting rational points on cubic hypersurfaces*, *Mathematika* **54** (2007), 93–112.
8. T. D. Browning, *Power-free values of polynomials*, *Arch. Math.* **96** (2011), no. 2, 139–150
9. T. D. Browning and R. Heath-Brown, *Counting rational points on hypersurfaces*, *J. reine. angew. Math.* **584** (2005), 83–116.
10. T. D. Browning and R. Heath-Brown, *The density of rational points on non-singular hypersurfaces I*, *Bull. London Math. Soc.* **38** (2006), 401–410.
11. T. D. Browning and R. Heath-Brown, *The density of rational points on non-singular hypersurfaces II*, *Proc. London Math. Soc.* **93** (2006), 273–303.
12. T. D. Browning, R. Heath-Brown, and P. Salberger, *Counting rational points on algebraic varieties*, *Duke Math. J.* **132** (2006), 545–578.
13. W. Castryck, R., Cluckers, P. Dittman, and K. H. Nguyen, *The dimension growth conjecture, polynomial in degree and without logarithmic factors*, *Algebra Number Theory* **14**(8) (2020), 2261–2294.
14. J.-L. Colliot-Thélène, *Appendix to Heath-Brown’s paper “The density of rational points on curves and surfaces”*, *Ann. of Math.* **155** (2002), 596–598.
15. V. I. Danilov, *Algebraic varieties and schemes*, in *Algebraic geometry I* (ed. I. R. Sha-farevich), *Encyclopedia Math. Sci.* vol. 23, Springer, Berlin, 1994.
16. O. Debarre, *Higher-dimensional algebraic geometry*, Springer, New York, 2001.
17. J. Ellenberg and A. Venkatesh, *On uniform bounds for rational points on non-rational curves*, *Int. Math. Res. Not.* **35** (2005), 2163–2181.
18. J. Fogarty and D. Mumford, *Geometric invariant theory*, 2nd ed., Springer, New York, 1982.
19. W. Fulton, *Intersection theory*, Springer, Berlin, 1984.
20. P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.
21. J. Harris, *Curves in projective space*, Montreal Univ. Press, Montreal, 1982.
22. J. Harris, *Algebraic geometry*, *Graduate Texts in Math.*, vol. 133, Springer, New York, 1995.
23. J. Harris and I. Morrison, *Moduli of curves*, *Graduate Texts in Math.*, vol. 187, Springer, New York, 1998.
24. R. Hartshorne, *Algebraic geometry*, *Graduate Texts in Math.*, vol. 52, Springer, New York, 1977.
25. D. R. Heath-Brown, *Cubic forms in ten variables*, *Proc. Lond. Math. Soc.* **47** (1983), no. 3, 225–257.
26. D. R. Heath-Brown, *The density of rational points on non-singular hypersurfaces*, *Proc. Ind. Acad. Sci.* **104** (1994), 13–29.
27. D. R. Heath-Brown, *The density of rational points on curves and surfaces*, *Ann. of Math.* **155** (2002), 553–595.
28. D. R. Heath-Brown, *Counting rational points on algebraic varieties*, *Analytic number theory*, *Springer Lecture Notes in Mathematics*, vol. 1891, 2006, Springer, Berlin, pp. 51–96.
29. C. Hooley, *On the number of points on a complete intersection over a finite field, With an appendix by Nicholas M. Katz*, *J. Number Theory* **38** (1991), 338–358.
30. V. A. Iskovskikh and Y. G. Prokhorov, *Algebraic geometry V*, *Encyclopedia of Mathematical Sciences*, vol. 47, Springer, Berlin, 1999.
31. F. Knudsen and D. Mumford, *The projectivity of the moduli space of stable curves I*, *Math. Scand.* **39** (1976), 19–55.
32. S. Lang and A. Weil, *Number of points of varieties over finite fields*, *Amer. J. Math.* **76**, (1954), 819–827.
33. W. Lee, E. Park, and P. Schenzel, *On the classification of non-normal cubic hypersurfaces*, *J. Pure Appl. Algebra* **215** (2011), 2034–2042.
34. H. L. Montgomery and R. C. Vaughan, *Multiplicative number theory I. Classical theory*, *Cambridge Tracts in Advanced Mathematics*, vol. 97, Cambridge Univ. Press, Cambridge, 2007.

35. M. R. Murty, *Problems in analytic number theory*, Graduate Texts in Math, vol. 206, Springer, New York, 2001.
36. M. Paredes and R. Sasyk, *Uniform bounds for the number of rational points on varieties over global fields*, *Algebra Number Theory*. **16**(8) (2022), 1941–2000.
37. S. Paul and G. Tian, *CM stability and the generalized Futaki invariant I*, preprint 2006, arXiv:math. AG/0605278.
38. J. Pila, *Density of integral and rational points on varieties*, *Astérisque* **228** (1995), 183–187.
39. J. Pila, *Density of integral points on plane algebraic curves*, *Int. Math. Res. Not.* **96** (1996), 903–912.
40. E. Rogora, *Metodi proiettivi per lo studio di alcune questioni relative alla varietà immerse*, Ph.D. thesis, Università di Roma I, Rome, 1993.
41. P. Salberger, *Counting rational points on hypersurfaces of low dimension*, *Ann. Sci. Éc. Norm. Sup.* **38** (2005), 93–115.
42. P. Salberger, *On the density of rational and integral points on algebraic varieties*, *J. reine angew. Math.* **606** (2007), 123–147.
43. P. Salberger, *Rational points of bounded height on projective surfaces*, *Math. Zeitschrift*. **258** (2008), 805–826.
44. P. Salberger, *Rational points of bounded height on projective threefolds*, in *Analytic number theory; A tribute to Gauss and Dirichlet*, Clay Mathematics Proceedings, vol. 7, 2007, American Mathematical Society, Providence, R.I., pp. 207–216.
45. P. Salberger, *Uniform bounds for rational points on cubic hypersurfaces*, in *Arithmetic and geometry*, London Math Soc. Lecture Notes Ser, vol. 420, Cambridge University press, Cambridge, 2015, pp. 401–421.
46. J.-P. Serre, *Lectures on the Mordell-Weil theorem*, Vieweg, Braunschweig, 1990.
47. J.-P. Serre, *Topics in Galois theory*, AK Peters, Wellesley, MA, 2008.
48. J. Semple and L. Roth, *Introduction to algebraic geometry*, Oxford Univ. Press, Oxford, 1949.
49. G. Tennenbaum, *Introduction to analytic and probabilistic number theory*, Cambridge Univ. Press, Cambridge, 1995.
50. M. N. Walsh, *Bounded rational points on curves*, *Int. Math. Res. Not.* **2015** (2015), no. 14, 5644–5658.
51. S. Xiao, *Power-free values of binary forms and global determinant method*, *Int. Math. Res. Not.* (2017), no. 16, 5078–5135.