A Minimal b Ghost

A Minimal b Ghost

Martin Cederwall

The b ghost, or b operator, used for fixing Siegel gauge in the pure spinor superfield formalism, is a composite operator of negative ghost number, satisfying $\{q, b\} = \square$, where q is the pure spinor differential (BRST operator). It is traditionally constructed using non-minimal variables. However, since all cohomology has minimal representatives, it seems likely that there should be versions of physically meaningful operators, also with negative ghost number, using only minimal variables. The purpose of this letter is to demonstrate that this statement holds by providing a concrete construction in $D = 10$ super-Yang–Mills theory, and to argue that it is a general feature in the pure spinor superfield formalism.

1. Introduction

The pure spinor superfield formalism (see e.g. refs. [1–3]) is a general method to formulate supersymmetric field theory, which for certain theories allows for an off-shell superfield formulation, even with maximal supersymmetry. Due to the absence of world-line reparametrisation invariance in the corresponding particle models, the b ghost is absent as a primitive operator, and the analogous operator (with the same name) needs to be constructed as a composite operator. Even the classical formulation of interactions sometimes demands negative ghost number operators. This applies to $D = 11$ supergravity,[4,5] and to higher derivative terms like Born–Infeld theory.[6] Such operators typically has a physical meaning in terms of mapping between different cohomology classes. They are traditionally constructed using non-minimal variables, and have quite complicated expressions. The purpose of this letter is to demonstrate how this may be done in the minimal picture. The main focus will be on $D = 10$ super-Yang–Mills theory.

M. Cederwall
Department of Physics
Chalmers Univ. of Technology, SE-412 96
Gothenburg Sweden
E-mail: martin.cederwall@chalmers.se

M. Cederwall
NORDITA
Hannes Alfvéns väg 12, SE-106 91
Stockholm, Sweden

The ORCID identification number(s) for the author(s) of this article can be found under https://doi.org/10.1002/prop.202300056

DOI: 10.1002/prop.202300056

2. The Non-Minimal b Operator

The b ghost, or b operator, used for fixing Siegel gauge in the pure spinor superfield formalism, is a composite operator of negative ghost number, satisfying $\{Q, b\} = \square$, where Q is some version of the pure spinor differential (BRST operator). In minimal variables, consisting of x^a, θ^α and λ^α with $(\lambda\gamma^a\lambda) = 0$, the differential is

$$q = \lambda^\alpha D_\alpha \,. \tag{2.1}$$

The b operator is traditionally constructed using non-minimal variables.[7] Then, also $\bar\lambda_\alpha$ and $r_\alpha = d\bar\lambda_\alpha$ are included. This is natural from many points of view, including integration.[7,8] The differential is modified to $Q = \lambda^\alpha D_\alpha + \bar\partial$, where $\bar\partial = d\bar\lambda_\alpha \frac{\partial}{\partial\bar\lambda_\alpha}$ is the Dolbeault operator on pure spinor space.

However, since all cohomology has minimal representatives, it seems likely that there should be versions of physically meaningful operators, also with negative ghost number, using only minimal variables, i.e., there should exist an operator b' acting on holomorphic functions of λ, such that $\{q, b'\} = \square$. The purpose of this letter is to demonstrate that this statement holds by providing a concrete construction in $D = 10$ super-Yang–Mills theory, and arguing that it is a general feature in the pure spinor superfield formalism.

The construction and examination of the b operator (mainly for $D = 10$ super-Yang–Mills theory and $D = 11$ supergravity) has been the subject of many papers,[9–20] mainly because this and similar issues is where the pure spinor superfield formalism becomes more complicated. In the present letter, we will work in $D = 10$ super-Yang–Mills theory, although analogous minimal operators should always exist.

The b operator is given in non-minimal pure spinor variables as the cochain[7]

$$\begin{aligned} b &= b_0 + b_1 + b_2 + b_3 \\ &= -\frac{1}{2}(\lambda\bar\lambda)^{-1}(\bar\lambda\gamma^a D)\partial_a + \frac{1}{16}(\lambda\bar\lambda)^{-2}(\bar\lambda\gamma^{abc}d\bar\lambda)\Big[N_{ab}\partial_c - \frac{1}{24}(D\gamma_{abc}D)\Big] \\ &\quad + \frac{1}{64}(\lambda\bar\lambda)^{-3}(d\bar\lambda\gamma^{abc}d\bar\lambda)(\bar\lambda\gamma_a D)N_{bc} \\ &\quad - \frac{1}{1024}(\lambda\bar\lambda)^{-4}(\bar\lambda\gamma^{ab}{}_i d\bar\lambda)(d\bar\lambda\gamma^{cdi}d\bar\lambda)N_{ab}N_{cd}. \end{aligned} \tag{2.2}$$

For useful rewritings, see in particular ref. [21]. We use conventions where the torsion is $T_{\alpha\beta}{}^a = 2\gamma^a_{\alpha\beta}$, i.e., $\{D_\alpha, D_\beta\} = -2\gamma^a_{\alpha\beta}$. We also write $N = (\lambda w)$ and $N^{ab} = (\lambda\gamma^{ab}w)$. The derivative $w_\alpha = \frac{\partial}{\partial\lambda^\alpha}$

does not respect the pure spinor constraint and is usually demanded to occur in such combinations.

3. The Minimal *b* Operator

Let V be the space of linear functions of an unconstrained spinor λ^α. Define $R = \mathrm{Sym}^\bullet(V)$ as the functions of λ. Functions of a pure spinor belong to the space $S = R/I$, where I is the ideal generated by $(\lambda\gamma^a\lambda)$. The derivative w_α is an operator on R, but not on S.

There is, however, a generalisation of the derivative,

$$\tilde{w}_\alpha = w_\alpha - \frac{1}{4(N+3)}(\gamma^a\lambda)_\alpha(w\gamma_a w), \tag{3.1}$$

that is well defined.[22,23] Its action between monomials $\lambda^{\alpha_1\cdots\alpha_n}$ in the modules $(0000n)$ is identical to the action of w, but maps the ideal generated by $(\lambda\gamma^a\lambda)$ to itself. This is indeed the most proper definition of $\tilde{w}$, since neither of the terms in Equation (3.1) is well defined on S. The concrete form is useful for explicit calculations, which then formally are performed in the space R, after which I is modded out. This is consistent thanks to $(\tilde{w}I)/I = 0$. Note that double contractions are needed for the gauge invariance (well-definedness) of $\tilde{w}$, so it is a genuinely quantum mechanical operator, and that it is not a "covariant derivative"—it does not satisfy a Leibniz rule. It follows from $\tilde{w}$ respecting the ideal that $(\tilde{w}\gamma^a\tilde{w}) = 0$. This is also straightforwardly obtained from a direct calculation. A useful relation in calculations is

$$[\tilde{w}_\alpha, \lambda^\beta] = \delta_\alpha{}^\beta - \frac{1}{2(N+3)}(\gamma^a\lambda)_\alpha(\gamma_a\tilde{w})^\beta. \tag{3.2}$$

It immediately leads to

$$[\tilde{w}_\alpha, \lambda^\alpha] = 16 - \frac{5N}{N+3}\ ,$$

interpolating between 16 (the dimension of the spinor module) at $N = 0$ and 11 (the dimension of pure spinor space) as $N \to \infty$.

Using $\tilde{w}$, it is possible to form operators of negative ghost number, built entirely out of minimal pure spinor variables. One such operator is the b operator. In $D = 11$ supergravity,[4,5] or in any supersymmetric model where the constraint on λ does not put it on a minimal orbit, demanding $\tilde{w}$ to respect the ideal does not uniquely define it,[22] and one will have access to more than one operator on S with ghost number -1. The condition that it acts as w between the modules in S will specify it uniquely.

We will now show concretely how a minimal b operator can be constructed, first by adding trivial terms to the non-minimal b, then by making a general Ansatz and solving it.

Normally, it is stated that b_3 represents $\bar{\partial}$ (operator) cohomology. This may be slightly confusing—since there is no 3-form cohomology, and any operator cohomology should map between cohomologies, one might think that b_3 is trivial. To understand this better, it is instructive to examine exactly why b_3 is closed, and in what sense it is not exact.

We use Dynkin labels for denoting highest weight modules, where $\lambda \in (00001)$, $\bar{\lambda} \in (00010)$. The factor $\bar{\lambda}(d\bar{\lambda})^3$ in b_3 comes in the module (02000). The prefactor $(\lambda\bar{\lambda})^{-4}$ implies that $\{\bar{\partial}, b_3\}$ will contain $\bar{\lambda}(d\bar{\lambda})^4$ antisymmetrised in the five spinor indices. There are two compeletly antisymmetric modules, (11001) and (00003). The second of these is not reached from $(00010) \otimes (02000)$, so one is left with the first one. It is a γ-traceless 3-index (hook) spinor, contracted with an object in the conjugate module (11010), formed from one λ and two N_{ab}'s, which thus takes the form containing a leading term $(\gamma^d\lambda)_\alpha N_{ab}N_{cd}$. However, since N_{ab} is constructed from λ, one has $(\gamma^b\lambda)_\alpha N_{ab} \propto (\gamma_a\lambda)_\alpha N$. There is no room for the module (11010), and $\{\bar{\partial}, b_3\} = 0$. The closedness of b_3 relies on N_{ab} containing λ.

Why, on the other hand, is b_3 non-trivial? If "naked" w's where allowed, b_3 would be trivial and formed as $[\bar{\partial}, a_2]$, where $a_2 \propto (\lambda\bar{\lambda})^{-3}(d\bar{\lambda}\gamma^{abc}d\bar{\lambda})(\bar{\lambda}\gamma_a w)N_{bc}$. This is of course not allowed since this operator is ill defined. But if we replace w by the well defined operator $\tilde{w}$ of Equation (3.1), we have a well defined operator a_2 such that $b_3 + [\bar{\partial}, a_2] = 0$. It is well defined on all holomorphic functions, i.e., on all elements of the cohomology of $\bar{\partial}$, but not on arbitrary cochains, since it it is singular on functions of a certain integer negative degree of homogeneity (-2) in λ. This makes it likely that it is possible to continue the procedure, and find $a = a_0 + a_1 + a_2$ such that $b' = b + [Q, a]$ only contains a 0-form term and is holomorphic, $\{q, b'\} = \Box$ and $\{\bar{\partial}, b'\} = 0$. This can indeed be done, but at the price of obtaining an operator which is singular on functions of some negative degree of homogeneity in λ.

At each step in the calculation (starting from a_2 and going down in form degree) one needs $b_i + [q, a_i] + [\bar{\partial}, a_{i-1}] = 0$. The $\bar{\partial}$-exactness of the first two terms follows, roughly speaking, from the extra factors λ introduced by the commmutator (3.2). The operator a is determined by a lengthy calculation to be

$$\begin{aligned}
a &= a_0 + a_1 + a_2,\\
a_0 &= (\lambda\bar{\lambda})^{-1}\Bigg[-\frac{1}{2}\left(1 - \frac{N+6}{(N+4)(N+5)}\right)(\bar{\lambda}\gamma^a\tilde{w})\partial_a\\
&\quad - \frac{1}{384(N+4)}(\bar{\lambda}\gamma^{abc}\tilde{w})(D\gamma_{abc}D)\\
&\quad + \frac{1}{128(N+4)(N+5)}(\bar{\lambda}\gamma^a\tilde{w})N^{bc}(D\gamma_{abc}D)\Bigg],\\
a_1 &= \frac{1}{384}(\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma^{abc}d\bar{\lambda})\left[(\tilde{w}\gamma_{abc}D) - \frac{3}{N+4}N_{ab}(\tilde{w}\gamma_c D)\right],\\
a_2 &= \frac{1}{128}(\lambda\bar{\lambda})^{-3}(d\bar{\lambda}\gamma^{abc}d\bar{\lambda})(\bar{\lambda}\gamma_a\tilde{w})N_{bc}.
\end{aligned} \tag{3.3}$$

The holomorphic b' operator now takes the form

$$\begin{aligned}
b' &= \frac{1}{(N+4)(N+5)(N+6)}\\
&\quad \left[-\frac{1}{2}(N^2+9N+15)(\tilde{w}\gamma^a D)\partial_a + \frac{1}{128}N^{ab}(\tilde{w}D^3_{ab})\right],
\end{aligned} \tag{3.4}$$

where D^3_{ab} is the antisymmetric product of three D's in (01001),

$$\begin{aligned}
(D^3)^\alpha_{ab} &= (\gamma^i)^{\alpha[\beta}(\gamma_{abi})^{\gamma\delta]}D_\beta D_\gamma D_\delta\\
&= (\gamma^i D)^\alpha(D\gamma_{abi}D) + \frac{1}{4}(\gamma_{[a}\gamma^{ij}D)^\alpha(D\gamma_{b]ij}D)\\
&\quad - \frac{1}{72}(\gamma_{ab}\gamma^{ijk}D)^\alpha(D\gamma_{ijk}D)\\
&= (\gamma^i D)^\alpha(D\gamma_{abi}D) + 32(\gamma_{[a}D)^\alpha\partial_{b]} - 12(\gamma_{ab}{}^i D)^\alpha\partial_i.
\end{aligned} \tag{3.5}$$

Seen just as an operator on functions of λ, x and θ, it is not at all a priori obvious that a b' with $\{q, b'\} = \square$ should exist. A general Ansatz would contain the two structures in Equation (3.4), each multiplied by an arbitrary function of N:

$$b' = f(N)(\tilde{w}\gamma^a D)\partial_a + g(N)N^{ab}(\tilde{w}D^3_{ab}). \tag{3.6}$$

The result of the anticommutator $\{q, b'\}$ potentially contains three other structures than $\square$, two with $D^2\partial$ and one with D^4. The system naïvely looks overdetermined. The vanishing of the D^4 term fixes the function $g(N)$ to be proportional to $\frac{1}{(N+4)(N+5)(N+6)}$, while the $\square$ term gives an recursion equation for the function $f(N)$,

$$f(N) = -\frac{N+3}{(N+6)(N+8)}\left(1 + Nf(N-1)\right) . \tag{3.7}$$

The general solution to Equation (3.7) is

$$f(N) = -\frac{N^2+9N+15}{2(N+4)(N+5)(N+6)} + \frac{c(-1)^N}{\begin{bmatrix} (N+1)(N+2)(N+3)(N+4)^2 \\ \times(N+5)^2(N+6)^2(N+7)(N+8) \end{bmatrix}} \tag{3.8}$$

(for some reason, there is a close resemblance of the homogeneous term with the coefficients in the pure spinor Hilbert series,

$$Z(t) = \sum_{N=0}^{\infty} \frac{\begin{bmatrix} (N+1)(N+2)(N+3)^2(N+4)^2 \\ \times(N+5)^2(N+6)(N+7) \end{bmatrix}}{2\cdot 3^2\cdot 4^2\cdot 5^2\cdot 6\cdot 7} t^N.) \tag{3.9}$$

Remarkably, when $c = 0$. the respective contributions of the two terms to each of the $D^2\partial$ terms in $\{q, b'\}$ take the same functional form in N, and the result is b' according to Equation (3.4).

Acting on a field $\psi = \lambda^\alpha A_\alpha$, the second term in Equation (3.4) gives 0, and one immediately gets

$$b'\psi = -\frac{1}{16}\partial^a(D\gamma_a A), \tag{3.10}$$

which shows that b' implies exactly Lorenz gauge.

4. Physical Operators

In ref. [6], so called "physical operators" were defined, then with the purpose of writing possible higher-derivative corrections. For $D = 10$ super-Yang–Mills theory, these are ghost number -1 operators obeying a sequence of relations obtained from inspection of the pure spinor superfield equations of motion,

$$\begin{aligned} &[Q, \hat{A}_\alpha] = -D_\alpha - 2(\gamma^i\lambda)_\alpha \hat{A}_i, \\ &\{Q, \hat{A}_a\} = \partial_a - (\lambda\gamma_a\hat{\chi}), \\ &[Q, \hat{\chi}^\alpha] = -\frac{1}{2}(\gamma^{ij}\lambda)^\alpha \hat{F}_{ij}, \\ &\{Q, \hat{F}_{ab}\} = \cdots \end{aligned} \tag{4.1}$$

with non-minimal solutions

$$\begin{aligned} \hat{A}_\alpha &= -(\lambda\bar{\lambda})^{-1}\left[\frac{1}{8}(\gamma^{ij}\bar{\lambda})_\alpha N_{ij} + \frac{1}{4}\bar{\lambda}_\alpha N\right], \\ \hat{A}_a &= -\frac{1}{4}(\lambda\bar{\lambda})^{-1}(\bar{\lambda}\gamma_a D) + \frac{1}{32}(\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma_a{}^{ij}d\bar{\lambda})N_{ij}, \\ \hat{\chi}^\alpha &= \frac{1}{2}(\lambda\bar{\lambda})^{-1}(\gamma^i\bar{\lambda})^\alpha\partial_i - \frac{1}{192}(\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma^{ijk}d\bar{\lambda})(\gamma_{ijk}D)^\alpha \\ &\quad - \frac{1}{64}(\lambda\bar{\lambda})^{-3}(\gamma_i\bar{\lambda})^\alpha(d\bar{\lambda}\gamma^{ijk}d\bar{\lambda})N_{jk}, \\ \hat{F}_{ab} &= \frac{1}{8}(\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma_{ab}{}^i d\bar{\lambda})\partial_i + \frac{1}{32}(\lambda\bar{\lambda})^{-3}(d\bar{\lambda}\gamma_{ab}{}^i d\bar{\lambda})(\bar{\lambda}\gamma_i D) \\ &\quad - \frac{1}{256}(\lambda\bar{\lambda})^{-4}(\bar{\lambda}\gamma_{abi}d\bar{\lambda})(d\bar{\lambda}\gamma^{ijk}d\bar{\lambda})N_{jk}, \\ &\ldots \end{aligned} \tag{4.2}$$

Suppose we would like to find physical operators constructed from minimal variables. We then observe that Equation (3.2) leads to

$$[q, \tilde{w}_\alpha] = -D_\alpha + \frac{1}{2(N+3)}(\gamma^i\lambda)_\alpha(\tilde{w}\gamma_i D) . \tag{4.3}$$

So, in the minimal picture we can identity

$$\begin{aligned} \hat{A}_\alpha &= \tilde{w}_\alpha , \\ \hat{A}_a &= -\frac{1}{4(N+4)}(\tilde{w}\gamma_a D) . \end{aligned} \tag{4.4}$$

The rewriting of the b operator in terms of physical operators in refs. [20, 21] is probably one of the deepest and physically most meaningful ones. We expect it to hold equally in minimal variables.

5. Discussion

We have seen how, contrary to what is sometimes stated in the literature, negative ghost number operators, such as the b operator or physical operators, are quite naturally constructed in minimal pure spinor superspace. This will certainly apply also to the operators used in the action of $D = 11$ supergravity[4,5] and to any other models.[24–29] There may be a certain advantage in using the minimal operators. The interpretation of the operators typically is that they map between different cohomology classes, and those are most simply represented in the minimal picture. Therefore, using minimal operators yields again minimal representatives, without the need of adding trivial terms.

However, most likely, $b'^2 = \{q, \Omega\}$, $\Omega \neq 0$. The first term in Equation (3.4) squares to zero, but we have not been able to show this for the rest. This may present a drawback in comparison with the non-minimal b.

The algebraic behaviour of b are essential for proving properties of amplitudes like color-kinematics duality and double copy.[21,30–32] Its failure to be a derivation defines the brackets for the kinetic Lie algebra. We have not examined this in detail for b'.

Finally, other versions of the pure spinor differential may present other possibilities to construct negative ghost number operators. In ref. [33], instead of using constrained spinors, the Tate resolution of the constraint gives an alternative differential.

Acknowledgements

The author is grateful to Max Guillen and Leron Borsten for helpful discussions, and to Nordita, where part of this work was done, for hospitality.

Conflict of Interest

The authors have declared no conflict of interest.

Keywords

pure spinors, supersymmetry

Received: March 2, 2023

[1] M. Cederwall, *Springer Proc. Phys.* **2013**, *153*, 61 [arXiv:1307.1762].
[2] M. Cederwall, Pure spinors in classical and quantum supergravity. arXiv:2210.06141, to appear in Hand-book of quantum gravity. eds. C. Bambi, L. Modesto, I. Shapiro.
[3] R. Eager, F. Hahner, I. Saberi, B. Williams, *J. Geom. Phys.* **2022**, *180*, 104626[arXiv:2111.01162].
[4] M. Cederwall, *J. High Energy Phys.* **2010**, *1001*, 117[arXiv:0912.1814].
[5] M. Cederwall, *Mod. Phys. Lett.* **2010**, *A25*, 3201[arXiv:1001.0112].
[6] M. Cederwall, A. Karlsson, *J. High Energy Phys.* **2011**, *1111*, 134[arXiv:1109.0809].
[7] N. Berkovits, *J. High Energy Phys.* **2005**, *0510*, 089[arXiv:hep-th/0509120].
[8] M. Cederwall, *J. High Energy Phys.* **2012**, *1201*, 150[arXiv:1111.1932].
[9] I. Oda, M. Tonin, *Nucl. Phys.* **2007**, *B779*, 63[arXiv:0704.1219].
[10] N. Berkovits, L. Mazzucato, *J. High Energy Phys.* **2010**, *1011*, 019[arXiv:1004.5140].
[11] O. Chandia, *Phys. Lett.* **2011**, *B695*, 312[arXiv:1008.1778].
[12] R. Lipinski Jusinskas, *J. High Energy Phys.* **2013**, *1305*, 048[arXiv:1303.3966].
[13] N. Berkovits, *J. High Energy Phys.* **2013**, *1306*, 091[arXiv:1305.0693].
[14] R. Lipinski Jusinskas, *J. High Energy Phys.* **2013**, *1307*, 142[arXiv:1306.1963].
[15] I. Bakhmatov, N. Berkovits, *J. High Energy Phys.* **2013**, *1311*, 214[arXiv:1310.3379].
[16] R. L. Jusinskas, Exploring the properties of the pure spinor b ghost. PhD thesis, ITP São Paulo (**2014**).
[17] T. Fleury, *J. High Energy Phys.* **2016**, *1603*, 200[arXiv:1512.00807].
[18] N. Berkovits, M Guillen, *J. High Energy Phys.* **2017**, *1707*, 115[arXiv:1703.05116].
[19] O. Chandia, B. C. Vallilo, *J. High Energy Phys.* **2021**, *2103*, 165[arXiv:2101.01129].
[20] M. Guillen, Taming the 11D pure spinor b-ghost. arXiv:2212.13653.
[21] M. Ben-Shahar, M. Guillen, *J. High Energy Phys.* **2021**, *2112*, 014 [arXiv:2108.11708].
[22] M. Cederwall, *AIP Conf. Proc.* **2010**, *1243*, 51.
[23] Y. Aisaka, E. A. Arroyo, N. Berkovits, N. Nekrasov, *J. High Energy Phys.* **2008**, *0808*, 050[arXiv:0806.0584].
[24] M. Cederwall, *J. High Energy Phys.* **2008**, *0809*, 116[arXiv:0808.3242].
[25] M. Cederwall, *J. High Energy Phys.* **2008**, *0810*, 70[arXiv:0809.0318].
[26] M. Cederwall, *Fortschr. Phys.* **2018**, *66*, 1700082[arXiv:1707.00554].
[27] M. Cederwall, *J. High Energy Phys.* **2018**, *1805*, 115[arXiv:1712.02284].
[28] M. Cederwall, *J. High Energy Phys.* **2021**, *2103*, 56 [arXiv:2012.02719].
[29] M. Cederwall, *Fortschr. Phys.* **2021**, *69*, 2100116[arXiv:2107.09037].
[30] L. Borsten, B. Jurčo, H. Kim, T. Macrelli, C. Sämann, Kinematic Lie algebras from twistor spaces. arXiv:2211.13261.
[31] M. Reiterer, *A homotopy BV algebra for Yang-Mills and color-kinematics*. arXiv:1912.03110.
[32] R. Bonezzi, C. Chiaffrino, F. Diaz-Jaramillo, O. Hohm, *Gauge invariant double copy of Yang-Mills theory: the quartic theory*. arXiv:2212.04513.
[33] M. Cederwall, S. Jonsson, J. Palmkvist, I. Saberi, *Supersymmetry and Koszul duality*. to appear.