A space-time calculus based on symmetric 2-spinors

General Relativity and Gravitation (2023) 55:51
https://doi.org/10.1007/s10714-023-03088-6

RESEARCH

A space-time calculus based on symmetric 2-spinors

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Received: 2 November 2022 / Accepted: 7 February 2023

Abstract
In this paper we present a space-time calculus for symmetric spinors, including a product with a number of index contractions followed by symmetrization. As all operations stay within the class of symmetric spinors, no involved index manipulations are needed. In fact spinor indices are not needed in the formalism. It is also general because any covariant tensor expression in a 4-dimensional Lorentzian spacetime can be translated to this formalism. The computer algebra implementation *SymSpin* as part of *xAct* for *Mathematica* is also presented.

Keywords Symmetric spinors · Spinor algebra · Symbolic computer algebra

Contents

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Published online: 17 March 2023

1 Introduction

When working with tensorial expressions, one usually encounters difficulties handling index manipulations due to complicated symmetries. Techniques including group theoretical calculations and Young tableaux have been introduced to try to tackle these problems. However, their complexity grows quickly with the size of the problem. The purpose of this paper is to present a formalism based on 2-spinors that aims to simplify the situation by utilizing the symmetry properties of irreducible spinors.

Let $(\mathcal{M}, g_{ab})$ be a 4-dimensional manifold with metric g_{ab} of Lorentzian signature and admitting a spin structure with spin metric ϵ_{AB}. It is well known that any tensor field on $\mathcal{M}$ can be expressed in terms of 2-spinors, which in turn can be decomposed into irreducible symmetric spinors [9, Prop 3.3.54]. For instance a valence $(3, 0)$ spinor can be decomposed as

$$T_{ABC} = T_{(ABC)} + \tfrac{1}{3}T^{D}{}_{D(B}\epsilon_{C)A} - \tfrac{1}{3}\epsilon_{A(B}T^{D}{}_{C)D} - \tfrac{1}{2}T_{A}{}^{D}{}_{D}\epsilon_{BC}. \tag{1}$$

Therefore, it is sufficient to work with with symmetric spinors. To fully establish this perspective, a symmetric product for symmetric spinors with a number of contractions is needed. It is the intention of this work to introduce the corresponding algebra and to derive its basic properties. In particular, with these operations we stay within the algebra of symmetric spinors. This offers great simplifications, and speeds up the calculations. Furtheremore, no relevant information is left in the indices, and we therefore get an index-free compact formalism.

We would like to point out that we use the 2-spinor formalism with all its benefits, but we only focus on the irreducible parts. This perspective naturally leads to the symmetric product introduced below. It also includes differential operators and their commutators. It turns out to be a convenient tool to perform lengthy spinor computations, either by hand or by computer. All index manipulations, like index raising and lowering and symmetrizations are no longer needed.

We have previously described the decomposition of the covariant derivative [4], leading to four fundamental spinor operators, which can be viewed as a special case. Also, the symmetric product is a generalization of some special operators, like the $\mathcal{K}^i$ operators defined in [1, Definition II.4]. Therefore, all properties of such operators can easily be derived from the corresponding properties of the symmetric product described in this paper.

Even partial implementations of the formalism proved to be very helpful. In [1] we studied linearized gravity on the Kerr spacetime and derived a covariant form of the Teukolsky-Starobinsky identities (TSI) using projection operators $\mathcal{K}^i$ and the fundamental spinor operators. For that work, the commutation relations between all the operators where very important. As the TSI equations are fourth order, the corresponding calculations would have been unfeasible with standard indexed spinor formalism. In [3] we used similar techniques to derive symmetry operators for linearized gravity on vacuum Petrov type D spacetimes. In [2], we used the formalism to analyze a differential complex leading to a completeness result regarding a set of local gauge invariant quantities for linearized gravity on the Kerr spacetime.

As a simpler example, consider a condition of the form

$$0 = K_{AB}{}^{FH} L_F{}^C \varphi_{HC} + M_{(A}{}^C \varphi_{B)C}, \tag{2}$$

for symmetric spinors K, L, M, φ. For arbitrary φ a systematic computation, using the techniques of this paper, shows that the conditions on K, L, M are of the form

$$K^G{}_{(ABC} L_{|G|F)} = 0, \qquad M_{AB} = \tfrac{1}{2} K^{CF}{}_{AB} L_{CF}, \tag{3}$$

see Sect. 3.2 for details. The same techniques have been used in [7] to derive conditions on the spacetime for the existence of second order symmetry operators for the massive Dirac equation.

The formalism is implemented in the *SymSpin* [5] package for *xAct* [8] for *Mathematica*.

In Sect. 2 we introduce the symmetric product and state basic properties in Theorem 3. The expansion of a product into symmetric products is discussed in Lemma 6. The irreducible parts of the Levi-Civita connection, its commutators, curvature and Leibniz rules are discussed in Sect. 2.4. A concise form the the dyad components of such symmetric spinors is given in Sect. 2.5. The computer algebra implementation is discussed in Sects. 3 and 4 contains some conclusions.

2 Symmetric spinor algebra

Let $\mathcal{S}_{k,l}$ be the space of symmetric valence (k, l) spinors. In abstract index notation, elements are of the form $\phi_{A_1\dots A_k A'_1\dots A'_l} \in \mathcal{S}_{k,l}$. Sometimes it is convenient to suppress the valence and/or indices and we write e.g. $\phi \in \mathcal{S}$ or $\phi \in \mathcal{S}_{k,l}$.

2.1 Symmetric product

Given two symmetric spinors, we introduce a product which involves a given number of contractions and symmetrization afterwards.

Definition 1 Let k, l, n, m, i, j be integers with $i \le min(k, n)$ and $j \le min(l, m)$. The symmetric product is a bilinear form

$$\overset{i,j}{\odot} : \mathcal{S}_{k,l} \times \mathcal{S}_{n,m} \to \mathcal{S}_{k+n-2i,l+m-2j}. \tag{4}$$

For $\phi \in \mathcal{S}_{k,l}$, $\psi \in \mathcal{S}_{n,m}$, it is given by

$$(\phi \overset{i,j}{\odot} \psi)_{A_1\dots A_{k+n-2i}}{}^{A'_1\dots A'_{l+m-2j}} = \phi_{(A_1\dots A_{k-i-1}}{}^{(A'_1\dots A'_{l-j-1}|B_1\dots B_i B'_1\dots B'_j|} \psi_{A_{k-i}\dots A_{k+n-2i})B_1\dots B_i B'_1\dots B'_j}{}^{A'_{l-j}\dots A'_{l+m-2j})} \tag{5}$$

For many commutator relations we will need the following coefficients.

Definition 2 Define the associativity coefficients

$$F^{t,m,M}_{i,r,k} = \sum_{p=0}^{M} \sum_{q=0}^{M-p} \frac{(-1)^{t-p+q}\binom{k-m}{p}\binom{m}{M-p-q}\binom{k-m-p}{q}\binom{r-m}{t-p}\binom{i-t}{M-p-q}\binom{t-p}{q}}{\binom{i+k-M-p+1}{M-p}\binom{M-p}{q}\binom{k-2m+r}{t}}. \quad (6)$$

Observe that the limits can be restricted to $\max(0, m-r+t) \le p \le \min(k-m, M, t)$ and $\max(0, M-m-p, M-i-p+t) \le q \le \min(k-m-p, M-p, t-p)$ because the terms are zero outside this range.

For multiple products we will use the convention $\omega \overset{m,n}{\odot} \varphi \overset{t,u}{\odot} \phi = \omega \overset{m,n}{\odot} (\varphi \overset{t,u}{\odot} \phi)$.

Theorem 3 *Let $\phi \in \mathcal{S}_{i,j}$, $\omega \in \mathcal{S}_{r,s}$, $\varphi \in \mathcal{S}_{k,l}$. The symmetric product $\odot$ of Definition 1 has the following properties:*

1. *It is graded anti-commutative:*

$$\phi \overset{m,n}{\odot} \omega = (-1)^{m+n} \omega \overset{m,n}{\odot} \phi \quad (7a)$$

2. *It is non-associative:*

$$(\omega \overset{m,n}{\odot} \varphi) \overset{t,u}{\odot} \phi = \sum_{M=0}^{\min(i,k)} \sum_{N=0}^{\min(j,l)} (-1)^{t+u+M+N} F^{t,m,M}_{i,r,k} F^{u,n,N}_{j,s,l} \, \omega \overset{t+m-M,u+n-N}{\odot} \varphi \overset{M,N}{\odot} \phi. \quad (7b)$$

3. *It is Hermitian:*

$$\overline{\phi \overset{m,n}{\odot} \omega} = \overline{\phi} \overset{n,m}{\odot} \overline{\omega} \quad (7c)$$

Combining the first two points, we get the following useful relation.

Corollary 4

$$\phi \overset{t,u}{\odot} \omega \overset{m,n}{\odot} \varphi = \sum_{M=0}^{\min(i,k)} \sum_{N=0}^{\min(j,l)} F^{t,m,M}_{i,r,k} F^{u,n,N}_{j,s,l} \, \omega \overset{t+m-M,u+n-N}{\odot} \phi \overset{M,N}{\odot} \varphi \quad (8)$$

2.2 Irreducible decomposition

A key property of the symmetric product is that the product of two symmetric spinors can always be decomposed in terms of symmetric products and spin metrics.

Definition 5 We will use the following notation for products of spin metrics.

$$\epsilon_{A_1 \dots A_p}{}^{B_1 \dots B_p} = \epsilon_{A_1}{}^{B_1} \dots \epsilon_{A_p}{}^{B_p}, \quad (9a)$$

$$\bar{\epsilon}_{A'_1 \dots A'_q}{}^{B'_1 \dots B'_q} = \bar{\epsilon}_{A'_1}{}^{B'_1} \dots \bar{\epsilon}_{A'_q}{}^{B'_q}. \quad (9b)$$

Lemma 6 *For $\phi \in \mathcal{S}_{i,j}, \varphi \in \mathcal{S}_{k,l}$ with p unprimed and q primed contractions, we have the irreducible decomposition*

$$\begin{aligned}
&\phi^{C_1\dots C_p C'_1\dots C'_q}{}_{A_1\dots A_{i-p}A'_1\dots A'_{j-q}}\varphi^{B_1\dots B_{k-p}B'_1\dots B'_{l-q}}{}_{C_1\dots C_p C'_1\dots C'_q}\\
&= (-1)^{p+q}\sum_{m=p}^{\min(i,k)}\sum_{n=q}^{\min(j,l)}\Bigg(\frac{(-1)^{m+n}\binom{i-p}{m-p}\binom{k-p}{m-p}\binom{j-q}{n-q}\binom{l-q}{n-q}}{\binom{i+k-m-p+1}{m-p}\binom{j+l-n-q+1}{n-q}}\\
&\quad\times \epsilon_{(A_1\dots A_{m-p}}{}^{(B_1\dots B_{m-p}}(\phi \overset{m,n}{\odot} \varphi)_{A_{m-p+1}\dots A_{i-p})(A'_{n-p+1}\dots A'_{j-q}}{}^{B_{m-p+1}\dots B_{k-p})(B'_{n-q+1}\dots B'_{l-q}}\bar{\epsilon}_{A'_1\dots A'_{n-q})}{}^{B'_1\dots B'_{n-q})}\Bigg).
\end{aligned} \tag{10}$$

Proof Let ϕ and φ be symmetric of valence $(i,0)$ and $(k,0)$ respectively. By [9, Prop 3.3.54] the irreducible decomposition of the product must have the following form

$$\phi_{A_1\dots A_i}\varphi^{B_1\dots B_k} = \sum_{m=0}^{\min(i,k)} c_m \epsilon_{(A_1\dots A_m}{}^{(B_1\dots B_m}(\phi \overset{m,0}{\odot} \varphi)_{A_{m+1}\dots A_i)}{}^{B_{m+1}\dots B_k)} \tag{11}$$

Taking a trace of the summand, we find by partial expansions of the symmetrizations that

$$\begin{aligned}
&\epsilon_{(A_1\dots A_m}{}^{(B_1\dots B_m}(\phi \overset{m,0}{\odot} \varphi)_{A_{m+1}\dots A_i)}{}^{B_{m+1}\dots B_{k-1}A_i)}\\
&= \tfrac{m}{i}\epsilon_{A_i(A_1\dots A_{m-1}}{}^{(B_1\dots B_m}(\phi \overset{m,0}{\odot} \varphi)_{A_m\dots A_{i-1})}{}^{B_{m+1}\dots B_{k-1}A_i)} + \tfrac{i-m}{i}\epsilon_{(A_1\dots A_m}{}^{(B_1\dots B_m}(\phi \overset{m,0}{\odot} \varphi)_{A_{m+1}\dots A_{i-1})A_i}{}^{B_{m+1}\dots B_{k-1}A_i)}\\
&= \tfrac{m}{ik}\epsilon_{A_i(A_1\dots A_{m-1}}{}^{A_i(B_1\dots B_{m-1}}(\phi \overset{m,0}{\odot} \varphi)_{A_m\dots A_{i-1})}{}^{B_m\dots B_{k-1})} + \tfrac{m(m-1)}{ik}\epsilon_{A_i(A_1\dots A_{m-1}}{}^{(B_1|A_i|\dots B_{m-1}}(\phi \overset{m,0}{\odot} \varphi)_{A_m\dots A_{i+1})}{}^{B_m\dots B_{k-1})}\\
&\quad + \tfrac{m(k-m)}{ik}\epsilon_{A_i(A_1\dots A_{m-1}}{}^{(B_1\dots B_m}(\phi \overset{m,0}{\odot} \varphi)_{A_m\dots A_{i-1})}{}^{B_{m+1}\dots B_{k-1})A_i}\\
&\quad + \tfrac{(i-m)m}{ik}\epsilon_{(A_1\dots A_m}{}^{A_i(B_1\dots B_{m-1}}(\phi \overset{m,0}{\odot} \varphi)_{A_{m+1}\dots A_{i-1})A_i}{}^{B_m\dots B_{k-1})}\\
&= \tfrac{m(i+k-m+1)}{ik}\epsilon_{(A_1\dots A_{m-1}}{}^{(B_1\dots B_{m-1}}(\phi \overset{m,0}{\odot} \varphi)_{A_m\dots A_{i-1})}{}^{B_m\dots B_{k-1})}.
\end{aligned} \tag{12}$$

Recursively for $p \leq \min(i,k)$ traces we get

$$\begin{aligned}
&\epsilon_{(A_1\dots A_m}{}^{(B_1\dots B_m}(\phi \overset{m,0}{\odot} \varphi)_{A_{m+1}\dots A_i)}{}^{B_{m+1}\dots B_{k-p}A_{i-p+1}\dots A_i)}\\
&= \tfrac{m(i+k-m+1)}{ik}\epsilon_{(A_1\dots A_{m-1}}{}^{(B_1\dots B_{m-1}}(\phi \overset{m,0}{\odot} \varphi)_{A_m\dots A_{i-1})}{}^{B_m\dots B_{k-p}A_{i-p+1}\dots A_{i-1})}\\
&= \tfrac{m(i+k-m+1)}{ik}\,\tfrac{(m-1)(i+k-m)}{(i-1)(k-1)}\epsilon_{(A_1\dots A_{m-2}}{}^{(B_1\dots B_{m-2}}(\phi \overset{m,0}{\odot} \varphi)_{A_{m-1}\dots A_{i-2})}{}^{B_{m-1}\dots B_{k-p}A_{i-p+1}\dots A_{i-2})}\\
&= \epsilon_{(A_1\dots A_{m-p}}{}^{(B_1\dots B_{m-p}}(\phi \overset{m,0}{\odot} \varphi)_{A_{m-p+1}\dots A_{i-p})}{}^{B_{m-p+1}\dots B_{k-p})}\prod_{q=0}^{p-1}\frac{(m-q)(i+k-m+1-q)}{(i-q)(k-q)}.\\
&= \epsilon_{(A_1\dots A_{m-p}}{}^{(B_1\dots B_{m-p}}(\phi \overset{m,0}{\odot} \varphi)_{A_{m-p+1}\dots A_{i-p})}{}^{B_{m-p+1}\dots B_{k-p})}\frac{\binom{1+i+k-m}{p}\binom{m}{p}}{\binom{i}{p}\binom{k}{p}}
\end{aligned} \tag{13}$$

Taking $p \leq \min(i,k)$ traces in (11) gives

$$\begin{aligned}
&\phi_{A_1\dots A_{i-p}}{}^{C_1\dots C_p}\varphi^{B_1\dots B_{k-p}}{}_{C_1\dots C_p}\\
&\quad=(-1)^p\phi_{A_1\dots A_i}\varphi^{B_1\dots B_{k-p}A_{i-p+1}\dots A_i}\\
&\quad=(-1)^p\sum_{m=0}^{\min(i,k)}c_m\epsilon_{(A_1\dots A_m}{}^{(B_1\dots B_m}(\phi\overset{m,0}{\odot}\varphi)_{A_{m+1}\dots A_i)}{}^{B_{m+1}B_{k-p}A_{i-p+1}\dots A_i)}\\
&\quad=(-1)^p\sum_{m=p}^{\min(i,k)}c_m\frac{\binom{i+k-m+1}{p}\binom{m}{p}}{\binom{i}{p}\binom{k}{p}}\epsilon_{(A_1\dots A_{m-p}}{}^{(B_1\dots B_{m-p}}(\phi\overset{m,0}{\odot}\varphi)_{A_{m-p+1}\dots A_{i-p})}{}^{B_{m-p+1}\dots B_{k-p})}.
\end{aligned}\tag{14}$$

With $m < p$, we get at least one contraction of the symmetric spinor $(\phi \overset{m,0}{\odot} \varphi)$ and the term drops out. If we symmetrize over all free indices, only the $m = p$ term survives, and we get

$$c_m=\frac{(-1)^m\binom{i}{m}\binom{k}{m}}{\binom{i+k-m+1}{m}}.\tag{15}$$

Hence

$$\begin{aligned}
&\phi_{A_1\dots A_{i-p}}{}^{C_1\dots C_p}\varphi^{B_1\dots B_{k-p}}{}_{C_1\dots C_p}\\
&\quad=(-1)^p\sum_{m=p}^{i}\frac{(-1)^m\binom{i}{m}\binom{k}{m}}{\binom{i+k-m+1}{m}}\frac{\binom{i+k-m+1}{p}\binom{m}{p}}{\binom{i}{p}\binom{k}{p}}\epsilon_{(A_1\dots A_{m-p}}{}^{(B_1\dots B_{m-p}}(\phi\overset{m,0}{\odot}\varphi)_{A_{m-p+1}\dots A_{i-p})}{}^{B_{m-p+1}\dots B_{k-p})}\\
&\quad=(-1)^p\sum_{m=p}^{\min(i,k)}\frac{(-1)^m\binom{i-p}{m-p}\binom{k-p}{m-p}}{\binom{i+k-m-p+1}{m-p}}\epsilon_{(A_1\dots A_{m-p}}{}^{(B_1\dots B_{m-p}}(\phi\overset{m,0}{\odot}\varphi)_{A_{m-p+1}\dots A_{i-p})}{}^{B_{m-p+1}\dots B_{k-p})}.
\end{aligned}\tag{16}$$

By complex conjugation we get the corresponding decomposition for the primed indices. □

2.3 Proof of Theorem 3

To proof the main theorem and in particular (7b), we need the following intermediate identities. We restrict to unprimed indices, as the effect of primed indices can be superimposed. We begin with a partial expansion of symmetrization of B indices.

Proposition 7 *Let $\omega \in \mathcal{S}_{r,0}$, $\varphi \in \mathcal{S}_{k,0}$. We have the partial expansion*

$$\begin{aligned}
&(\omega\overset{m,0}{\odot}\varphi)_{A_1\dots A_{k+r-2m-t}B_1\dots B_t}\\
&\quad=\sum_{p=0}^{t}\frac{\binom{k-m}{p}\binom{r-m}{t-p}}{\binom{k-2m+r}{t}}\omega_{B_{p+1}\dots B_t(A_1\dots A_{r-m-t+p}}{}^{C_1\dots C_m}\varphi_{A_{r-m-t+p+1}\dots A_{k+r-2m-t})B_1\dots B_pC_1\dots C_m}.
\end{aligned}\tag{17}$$

The sum can be limited to the range $\max(0, t+m-r) \leq p \leq \min(t, k-m)$.

Proof Partial expansion of the symmetry for the indices $B_t, B_{t-1}, \dots B_1$ gives

$$
\begin{aligned}
&(\omega \overset{m,0}{\odot} \varphi)_{A_1\ldots A_{k+r-2m-t}B_1\ldots B_t} \\
&= \tfrac{r-m}{k+r-2m}\omega^{C_1\ldots C_m}_{B_t(A_1\ldots A_{r-m-1}}\varphi_{A_{r-m}\ldots A_{k+r-2m-t}B_1\ldots B_{t-1})C_1\ldots C_m} \\
&\quad + \tfrac{k-m}{k+r-2m}\omega^{C_1\ldots C_m}_{(A_1\ldots A_{r-m}}\varphi_{A_{r-m+1}\ldots A_{k+r-2m-t}B_1\ldots B_{t-1})B_tC_1\ldots C_m} \\
&= \sum_{p=0}^{t}\Bigg(\binom{t}{p}\tfrac{(r-m)!(k-m)!(k+r-2m-t)!}{(r-m-t+p)!(k-m-p)!(k+r-2m)!} \\
&\quad \times \omega^{C_1\ldots C_m}_{B_{p+1}\cdots B_t(A_1\ldots A_{r-m-t+p}}\varphi_{A_{r-m-t+p+1}\ldots A_{k+r-2m-t})B_1\ldots B_pC_1\ldots C_m}\Bigg),
\end{aligned}
\tag{18}
$$

which can be simplified to (17). ◻

We aso need to make an irreducible decomposition of a product of two spinors with some contractions and symmetrizations.

Proposition 8 *Let* $\phi \in \mathcal{S}_{i,0}$, $\varphi \in \mathcal{S}_{k,0}$.

$$
\begin{aligned}
&\phi^{C_1\ldots C_p(B_1\ldots B_{t-p}}_{(A_1\ldots A_{i-t}}\varphi^{B_{t-p+1}\ldots B_{t-p+m})}_{A_{i-t+1}\ldots A_{i+k-t-m-p})C_1\ldots C_p} \\
&\quad = \sum_{M=0}^{\min(i,k)-p}\sum_{q=0}^{M}\frac{(-1)^{q+M}\binom{m}{M-q}\binom{k-m-p}{q}\binom{i-t}{M-q}\binom{t-p}{q}}{\binom{i+k-M-2p+1}{M}\binom{M}{q}} \\
&\qquad \epsilon^{(B_1\ldots B_M}_{(A_1\ldots A_M}(\phi \overset{M+p,0}{\odot} \varphi)^{B_{M+1}\ldots B_{t-p+m})}_{A_{M+1}\ldots A_{i+k-t-m-p})}
\end{aligned}
\tag{19}
$$

Proof Let $\doteq$ mean equal after lowering the A indices, raising the B indices and symmetrizing over the A and B index sets separately. Using Lemma 6, performing a partial expansion of the symmetries and noticing that $\epsilon_{A_i}^{\ A_j} \doteq 0$ and $\epsilon_{B_i}^{\ B_j} \doteq 0$ if $i \neq j$, we get

$$
\begin{aligned}
&\phi^{C_1\ldots C_p}_{A_1\ldots A_{i-t}B_1\ldots B_{t-p}}\varphi^{A_{i-t+1}\ldots A_{i+k-t-m-p}B_{t-p+1}\ldots B_{t-p+m}}_{C_1\ldots C_p} \\
&= (-1)^p\sum_{M=0}^{\min(i,k)-p}\Bigg(\frac{(-1)^{M+p}\binom{i-p}{M}\binom{k-p}{M}}{\binom{i+k-M-2p+1}{M}} \\
&\quad \times \epsilon^{(A_{i-t+1}\ldots A_{i-t+M}}_{(A_1\ldots A_M}(\phi \overset{M+p,0}{\odot} \varphi)^{A_{i-t+M+1}\ldots A_{i+k-t-m-p}B_{t-p+1}\ldots B_{t-p+m})}_{A_{M+1}\ldots A_{i-t}B_1\ldots B_{t-p})}\Bigg) \\
&\doteq \sum_{M=0}^{\min(i,k)-p}\frac{(-1)^M\binom{i-p}{M}\binom{k-p}{M}}{\binom{i+k-M-2p+1}{M}}\Bigg(\frac{(t-p)(k-m-p)}{(i-p)(k-p)}\epsilon^{A_{i-t+1}(A_{i-t+2}\ldots A_{i-t+M}}_{B_1(A_1\ldots A_{M-1}} \\
&\quad (\phi \overset{M+p,0}{\odot} \varphi)^{A_{i-t+M+1}\ldots A_{i+k-t-m-p}B_{t-p+1}\ldots B_{t-p+m})}_{A_M\ldots A_{i-t}B_2\ldots B_{t-p})} \\
&\quad + \frac{(i-t)m}{(i-p)(k-p)}\epsilon^{B_{t-p+1}(A_{i-t+1}\ldots A_{i-t+M-1}}_{A_1(A_2\ldots A_M}
\end{aligned}
$$

$$(\phi \overset{M+p,0}{\odot} \varphi)_{A_{M+1}\dots A_{i-t}B_1\dots B_{t-p})}{}^{A_{i-t+M}\dots A_{i+k-t-m-p}B_{t-p+2}\dots B_{t-p+m})}\Big). \tag{20}$$

Repeatedly expanding, we find

$$\begin{aligned}
&\phi_{A_1\dots A_{i-t}B_1\dots B_{t-p}}{}^{C_1\dots C_p}\varphi_{C_1\dots C_p}{}^{A_{i-t+1}\dots A_{i+k-t-m-p}B_{t-p+1}\dots B_{t-p+m}}\\
&\quad \hat{=} \sum_{M=0}^{\min(i,k)-p}\frac{(-1)^M\binom{i-p}{M}\binom{k-p}{M}}{\binom{i+k-M-2p+1}{M}}\Big(\sum_{q=0}^{M}\binom{M}{q}\\
&\quad \frac{(t-p)!(k-m-p)!(i-t)!m!(i-p-M)!(k-p-M)!}{(t-p-q)!(k-m-p-q)!(i-t-M+q)!(m-M+q)!(i-p)!(k-p)!}\\
&\quad \epsilon_{B_1\dots B_q A_1\dots A_{M-q}}{}^{A_{i-t+1}\dots A_{i-t+q}B_{t-p+1}\dots B_{t-p+M-q}}\\
&\quad (\phi \overset{M+p,0}{\odot} \varphi)_{A_{M-q+1}\dots A_{i-t}B_{q+1}\dots B_{t-p}}{}^{A_{i-t+q+1}\dots A_{i+k-t-m-p}B_{t-p+M-q+1}\dots B_{t-p+m}}\Big).
\end{aligned} \tag{21}$$

Moving the A indices down and the B indices up and writing out the symmetrizations, we get

$$\begin{aligned}
&\phi_{(A_1\dots A_{i-t}}{}^{C_1\dots C_p(B_1\dots B_{t-p}}\varphi_{A_{i-t+1}\dots A_{i+k-t-m-p})C_1\dots C_p}{}^{B_{t-p+1}\dots B_{t-p+m})}\\
&\quad = \sum_{M=0}^{\min(i,k)-p}\frac{(-1)^M\binom{i-p}{M}\binom{k-p}{M}}{\binom{i+k-M-2p+1}{M}}\Big(\sum_{q=0}^{M}\frac{(-1)^q\binom{m}{M-q}\binom{k-m-p}{q}\binom{i-t}{M-q}\binom{t-p}{q}}{\binom{M}{q}\binom{i-p}{M}\binom{k-p}{M}}\\
&\quad \epsilon_{(A_{i-t+1}\dots A_{i-t+q}A_1\dots A_{M-q}}{}^{(B_1\dots B_q B_{t-p+1}\dots B_{t-p+M-q}}(\phi \overset{M+p,0}{\odot}\\
&\quad \varphi)_{A_{M-q+1}\dots A_{i-t}A_{i-t+q+1}\dots A_{i+k-t-m-p})}{}^{B_{q+1}\dots B_{t-p}B_{t-p+M-q+1}\dots B_{t-p+m})}\Big).
\end{aligned} \tag{22}$$

After rearranging the indices, and simplifying, we get (19). ◻

Proof of Theorem 3 Part 1 follows from the zee-zaw rule on the $m+n$ contracted indices and part 3 follows from complex conjugation of (5). Part 2 follows from the following argument. Proposition 7, a renaming of the contracted indices and using the zee-zaw rule gives

$$\begin{aligned}
&(\phi \overset{t,0}{\odot} \omega \overset{m,0}{\odot} \varphi)_{A_1\dots A_{i+k+r-2m-2t}}\\
&\quad = \sum_{p=0}^{t}\frac{\binom{k-m}{p}\binom{r-m}{t-p}}{\binom{k-2m+r}{t}}\\
&\quad \times \omega_{B_{p+1}\cdots B_t(A_1\dots A_{r-m-t+p}}{}^{C_1\dots C_m}\phi_{A_{k+r-2m-t+1}\dots A_{i+k+r-2m-2t}}{}^{B_1\dots B_t}\varphi_{A_{r-m-t+p+1}\dots A_{k+r-2m-t})B_1\dots B_p C_1\dots C_m}\\
&\quad = \sum_{p=0}^{t}(-1)^m\frac{\binom{k-m}{p}\binom{r-m}{t-p}}{\binom{k-2m+r}{t}}\\
&\omega_{B_1\dots B_{m+t-p}(A_{i-t+k-m-p+1}\dots A_{i+k+r-2m-2t}}\phi_{A_1\dots A_{i-t}}{}^{C_1\dots C_p B_1 B_{t-p}}\varphi_{A_{i-t+1}\dots A_{i-t+k-m-p})C_1\dots C_p}{}^{B_{t-p+1}\dots B_{t-p+m}}.
\end{aligned} \tag{23}$$

Using Proposition 8, contracting the spin metrics, and using the zee-zaw rule, we get

$$
\begin{aligned}
&(\phi \overset{t,0}{\odot} \omega \overset{m,0}{\odot} \varphi)_{A_1 \dots A_{i+k+r-2m-2t}} \\
&= \sum_{p=0}^{t} \sum_{M=0}^{\min(i,k)-p} \sum_{q=0}^{M} (-1)^m \frac{\binom{k-m}{p}\binom{r-m}{t-p}}{\binom{k-2m+r}{t}} \frac{(-1)^{q+M}\binom{m}{M-q}\binom{k-m-p}{q}\binom{i-t}{M-q}\binom{t-p}{q}}{\binom{i+k-M-2p+1}{M}\binom{M}{q}} \\
&\quad \times \omega_{B_1 \dots B_{m+t-p}(A_{i-t+k-m-p+1} \dots A_{i+k+r-2m-2t}} \epsilon_{A_1 \dots A_M}^{B_1 \dots B_M} (\phi \overset{M+p,0}{\odot} \varphi)_{A_{M+1} \dots A_{i+k-t-m-p})}^{B_{M+1} \dots B_{t-p+m}} \\
&= \sum_{p=0}^{t} \sum_{M=0}^{\min(i,k)-p} \sum_{q=0}^{M} (-1)^{m+q+M} \frac{\binom{k-m}{p}\binom{r-m}{t-p}\binom{m}{M-q}\binom{k-m-p}{q}\binom{i-t}{M-q}\binom{t-p}{q}}{\binom{k-2m+r}{t}\binom{i+k-M-2p+1}{M}\binom{M}{q}} \\
&\quad \times \omega_{B_{M+1} \dots B_{m+t-p}(A_1 \dots A_{r+M+p-m-t}} (\phi \overset{M+p,0}{\odot} \varphi)_{A_{r+M+p-m-t+1} \dots A_{i+k+r-2m-2t})}^{B_{M+1} \dots B_{t-p+m}} \\
&= \sum_{p=0}^{t} \sum_{M=0}^{\min(i,k)-p} \sum_{q=0}^{M} (-1)^{q+t-p} \frac{\binom{k-m}{p}\binom{r-m}{t-p}\binom{m}{M-q}\binom{k-m-p}{q}\binom{i-t}{M-q}\binom{t-p}{q}}{\binom{k-2m+r}{t}\binom{i+k-M-2p+1}{M}\binom{M}{q}} \\
&\quad \times \omega_{(A_1 \dots A_{r+M+p-m-t}}^{B_{M+1} \dots B_{m+t-p}} (\phi \overset{M+p,0}{\odot} \varphi)_{A_{r+M+p-m-t+1} \dots A_{i+k+r-2m-2t})B_{M+1} \dots B_{t-p+m}}.
\end{aligned} \tag{24}
$$

Hence

$$
\begin{aligned}
&(\phi \overset{t,0}{\odot} \omega \overset{m,0}{\odot} \varphi) \\
&= \sum_{p=0}^{t} \sum_{M=0}^{\min(i,k)-p} \sum_{q=0}^{M} \frac{(-1)^{t-p+q}\binom{k-m}{p}\binom{r-m}{t-p}\binom{m}{M-q}\binom{k-m-p}{q}\binom{i-t}{M-q}\binom{t-p}{q}}{\binom{k-2m+r}{t}\binom{i+k-M-2p+1}{M}\binom{M}{q}} \\
&\quad (\omega \overset{t+m-p-M,0}{\odot} \phi \overset{M+p,0}{\odot} \varphi) \\
&= \sum_{p=0}^{t} \sum_{M=p}^{\min(i,k)} \sum_{q=0}^{M-p} \frac{(-1)^{t-p+q}\binom{k-m}{p}\binom{r-m}{t-p}\binom{m}{M-p-q}\binom{k-m-p}{q}\binom{i-t}{M-p-q}\binom{t-p}{q}}{\binom{k-2m+r}{t}\binom{i+k-M-p+1}{M-p}\binom{M-p}{q}} \\
&\quad (\omega \overset{t+m-M,0}{\odot} \phi \overset{M,0}{\odot} \varphi) \\
&= \sum_{M=0}^{\min(i,k)} F_{i,r,k}^{t,m,M} (\omega \overset{t+m-M,0}{\odot} \phi \overset{M,0}{\odot} \varphi),
\end{aligned} \tag{25}
$$

where we have made the change $M \to M - p$ and re-ordered the sums. The limits can be restricted to $\max(0, m - r + t) \leq p \leq \min(k - m, M, t)$ and $\max(0, M - m - p, M - i - p + t) \leq q \leq \min(k - m - p, M - p, t - p)$ because the terms are zero outside this range. The treatment of the primed indices is completely analogous.

2.4 Derivatives

In [4], the irreducible decomposition of the covariant derivative of a symmetric spinor was done in terms of fundamental spinor operators. By extending the symmetric product to the space of linear, symmetric differential operators of valence (k, l), $\mathcal{O}_{k,l}$, we can express the fundamental spinor operators in a compact way.

Remark 1 For $\nabla \in \mathcal{O}_{1,1}$ we have the fundamental spinor operators [4, Definition 13]

$$\mathscr{D}\varphi = \nabla \overset{1,1}{\odot} \varphi, \qquad \mathscr{C}\varphi = \nabla \overset{0,1}{\odot} \varphi, \qquad \mathscr{C}^\dagger\varphi = \nabla \overset{1,0}{\odot} \varphi, \qquad \mathscr{T}\varphi = \nabla \overset{0,0}{\odot} \varphi. \tag{26}$$

On $\varphi \in \mathcal{S}_{k,l}$ we have the irreducible decomposition of the covariant derivative into fundamental operators [4, Lemma 15],

$$\begin{aligned}
\nabla_{A_1}{}^{A'_1}\varphi_{A_2\dots A_{k+1}}{}^{A'_2\dots A'_{l+1}} = {} & (\mathscr{T}\varphi)_{A_1\dots A_{k+1}}{}^{A'_1\dots A'_{l+1}} \\
& - \tfrac{l}{l+1}\bar{\epsilon}^{A'_1(A'_2}(\mathscr{C}\varphi)_{A_1\dots A_{k+1}}{}^{A'_3\dots A'_{l+1})} \\
& - \tfrac{k}{k+1}\epsilon_{A_1(A_2}(\mathscr{C}^\dagger\varphi)_{A_3\dots A_{k+1})}{}^{A'_1\dots A'_{l+1}} \\
& + \tfrac{kl}{(k+1)(l+1)}\epsilon_{A_1(A_2}\bar{\epsilon}^{A'_1(A'_2}(\mathscr{D}\varphi)_{A_3\dots A_{k+1})}{}^{A'_3\dots A'_{l+1})}.
\end{aligned} \tag{27}$$

Next, we write the commutators in the new notation. Define the operator

$$\square = -(\nabla \overset{0,1}{\odot} \nabla) \in \mathcal{O}_{2,0}, \tag{28}$$

and its complex conjugate $\bar{\square} \in \mathcal{O}_{0,2}$.

In index notation, it reads $\square_{AB} = \nabla_{(A|A'|}\nabla_{B)}{}^{A'}$. Acting on $\varphi \in \mathcal{S}_{k,l}$ it can be expressed in terms of curvature via

$$\square \overset{0,0}{\odot} \varphi = -k\Psi \overset{1,0}{\odot} \varphi - l\Phi \overset{0,1}{\odot} \varphi, \tag{29a}$$

$$\square \overset{1,0}{\odot} \varphi = -(k-1)\Psi \overset{2,0}{\odot} \varphi - l\Phi \overset{1,1}{\odot} \varphi + (k+2)\Lambda \overset{0,0}{\odot} \varphi, \tag{29b}$$

$$\square \overset{2,0}{\odot} \varphi = -(k-2)\Psi \overset{3,0}{\odot} \varphi - l\Phi \overset{2,1}{\odot} \varphi. \tag{29c}$$

Lemma 9 [4, Lemma 18] *Let $\varphi \in \mathcal{S}_{k,l}$. The operators $\mathscr{D}$, $\mathscr{C}$, $\mathscr{C}^\dagger$ and $\mathscr{T}$ satisfy the commutator relations*

$$\mathscr{D}\mathscr{C}\varphi = \tfrac{k}{k+1}\mathscr{C}\mathscr{D}\varphi - \bar{\square} \overset{0,2}{\odot} \varphi, \qquad k \geq 0, l \geq 2, \tag{30a}$$

$$\mathscr{D}\mathscr{C}^\dagger\varphi = \tfrac{l}{l+1}\mathscr{C}^\dagger\mathscr{D}\varphi - \square \overset{2,0}{\odot} \varphi, \qquad k \geq 2, l \geq 0, \tag{30b}$$

$$\mathscr{C}\mathscr{T}\varphi = \tfrac{l}{l+1}\mathscr{T}\mathscr{C}\varphi - \square \overset{0,0}{\odot} \varphi, \qquad k \geq 0, l \geq 0, \tag{30c}$$

$$
\begin{aligned}
\mathscr{C}^{\dagger}\mathscr{T}\varphi &= \tfrac{k}{k+1}\mathscr{T}\mathscr{C}^{\dagger}\varphi - \overline{\Box}\overset{0,0}{\odot}\varphi, && k \geq 0, l \geq 0, \quad \text{(30d)}\\
\mathscr{D}\mathscr{T}\varphi &= -\left(\tfrac{1}{k+1}+\tfrac{1}{l+1}\right)\mathscr{C}\mathscr{C}^{\dagger}\varphi + \tfrac{l(l+2)}{(l+1)^2}\mathscr{T}\mathscr{D}\varphi \\
&\quad - \tfrac{l+2}{l+1}\Box\overset{1,0}{\odot}\varphi - \tfrac{l}{l+1}\overline{\Box}\overset{0,1}{\odot}\varphi, && k \geq 1, l \geq 0, \quad \text{(30e)}\\
\mathscr{D}\mathscr{T}\varphi &= -\left(\tfrac{1}{k+1}+\tfrac{1}{l+1}\right)\mathscr{C}^{\dagger}\mathscr{C}\varphi + \tfrac{k(k+2)}{(k+1)^2}\mathscr{T}\mathscr{D}\varphi \\
&\quad - \tfrac{k}{k+1}\Box\overset{1,0}{\odot}\varphi - \tfrac{k+2}{k+1}\overline{\Box}\overset{0,1}{\odot}\varphi, && k \geq 0, l \geq 1, \quad \text{(30f)}\\
\mathscr{C}\mathscr{C}^{\dagger}\varphi &= \mathscr{C}^{\dagger}\mathscr{C}\varphi + \left(\tfrac{1}{k+1}-\tfrac{1}{l+1}\right)\mathscr{T}\mathscr{D}\varphi - \Box\overset{1,0}{\odot}\varphi + \overline{\Box}\overset{0,1}{\odot}\varphi, && k \geq 1, l \geq 1. \quad \text{(30g)}
\end{aligned}
$$

Example 1 Let $\varphi \in \mathcal{S}_{2s,0}$ be a spin-s field, i.e. $\mathscr{C}^{\dagger}\varphi = 0$. Then (30b) is the algebraic consistency condition, also known as Buchdahl constraint, see [9, Section 5.8].

Let $\varphi \in \mathcal{S}_{k,0}$ be a Killing spinor, i.e. $\mathscr{T}\varphi = 0$. Then (30c) is the integrability condition restricting compatible geometries.

Lemma 10 *For symmetric spinors $\phi \in \mathcal{S}_{i,j}$, $\varphi \in \mathcal{S}_{k,l}$ we have the following Leibniz rules.*

$$
\begin{aligned}
\mathscr{T}(\phi\overset{m,n}{\odot}\varphi) &= (-1)^{m+n}\varphi\overset{m,n}{\odot}\mathscr{T}\phi + \tfrac{(-1)^{m+n}n}{j+1}\varphi\overset{m,n-1}{\odot}\mathscr{C}\phi + \tfrac{(-1)^{m+n}m}{i+1}\varphi\overset{m-1,n}{\odot}\mathscr{C}^{\dagger}\phi \\
&\quad + \tfrac{(-1)^{m+n}mn}{(i+1)(j+1)}\varphi\overset{m-1,n-1}{\odot}\mathscr{D}\phi + \phi\overset{m,n}{\odot}\mathscr{T}\varphi + \tfrac{n}{l+1}\phi\overset{m,n-1}{\odot}\mathscr{C}\varphi \\
&\quad + \tfrac{m}{k+1}\phi\overset{m-1,n}{\odot}\mathscr{C}^{\dagger}\varphi + \tfrac{mn}{kl+k+l+1}\phi\overset{m-1,n-1}{\odot}\mathscr{D}\varphi, \qquad \text{(31a)}\\
\mathscr{C}(\phi\overset{m,n}{\odot}\varphi) &= \tfrac{(-1)^{m+n+1}(l-n)}{j+l-2n}\varphi\overset{m,n+1}{\odot}\mathscr{T}\phi + \tfrac{(-1)^{m+n}(j-n)(j+l-n+1)}{(j+1)(j+l-2n)}\varphi\overset{m,n}{\odot}\mathscr{C}\phi \\
&\quad + \tfrac{(-1)^{m+n+1}m(l-n)}{(i+1)(j+l-2n)}\varphi\overset{m-1,n+1}{\odot}\mathscr{C}^{\dagger}\phi \\
&\quad + \tfrac{(-1)^{m+n}m(j-n)(j+l-n+1)}{(i+1)(j+1)(j+l-2n)}\varphi\overset{m-1,n}{\odot}\mathscr{D}\phi - \tfrac{j-n}{j+l-2n}\phi\overset{m,n+1}{\odot}\mathscr{T}\varphi \\
&\quad + \tfrac{(l-n)(j+l-n+1)}{(l+1)(j+l-2n)}\phi\overset{m,n}{\odot}\mathscr{C}\varphi + \tfrac{m(-j+n)}{(k+1)(j+l-2n)}\phi\overset{m-1,n+1}{\odot}\mathscr{C}^{\dagger}\varphi \\
&\quad + \tfrac{m(l-n)(j+l-n+1)}{(k+1)(l+1)(j+l-2n)}\phi\overset{m-1,n}{\odot}\mathscr{D}\varphi, \qquad \text{(31b)}\\
\mathscr{C}^{\dagger}(\phi\overset{m,n}{\odot}\varphi) &= \tfrac{(-1)^{m+n+1}(k-m)}{i+k-2m}\varphi\overset{m+1,n}{\odot}\mathscr{T}\phi + \tfrac{(-1)^{m+n+1}n(k-m)}{(j+1)(i+k-2m)}\varphi\overset{m+1,n-1}{\odot}\mathscr{C}\phi \\
&\quad + \tfrac{(-1)^{m+n}(i-m)(i+k-m+1)}{(i+1)(i+k-2m)}\varphi\overset{m,n}{\odot}\mathscr{C}^{\dagger}\phi \\
&\quad + \tfrac{(-1)^{m+n}n(i-m)(i+k-m+1)}{(i+1)(j+1)(i+k-2m)}\varphi\overset{m,n-1}{\odot}\mathscr{D}\phi - \tfrac{i-m}{i+k-2m}\phi\overset{m+1,n}{\odot}\mathscr{T}\varphi \\
&\quad + \tfrac{n(-i+m)}{(l+1)(i+k-2m)}\phi\overset{m+1,n-1}{\odot}\mathscr{C}\varphi + \tfrac{(k-m)(i+k-m+1)}{(k+1)(i+k-2m)}\phi\overset{m,n}{\odot}\mathscr{C}^{\dagger}\varphi \\
&\quad + \tfrac{n(k-m)(i+k-m+1)}{(k+1)(l+1)(i+k-2m)}\phi\overset{m,n-1}{\odot}\mathscr{D}\varphi, \qquad \text{(31c)}\\
\mathscr{D}(\phi\overset{m,n}{\odot}\varphi) &= \tfrac{(-1)^{m+n}(k-m)(l-n)}{(i+k-2m)(j+l-2n)}\varphi\overset{m+1,n+1}{\odot}\mathscr{T}\phi
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{(-1)^{m+n+1}(j-n)(k-m)(j+l-n+1)}{(j+1)(i+k-2m)(j+l-2n)}\varphi \overset{m+1,n}{\odot} \mathscr{C}\phi \\
&+\tfrac{(-1)^{m+n+1}(i-m)(l-n)(i+k-m+1)}{(i+1)(i+k-2m)(j+l-2n)}\varphi \overset{m,n+1}{\odot} \mathscr{C}^{\dagger}\phi \\
&+\tfrac{(-1)^{m+n}(i-m)(j-n)(i+k-m+1)(j+l-n+1)}{(i+1)(j+1)(i+k-2m)(j+l-2n)}\varphi \overset{m,n}{\odot} \mathscr{D}\phi \\
&+\tfrac{(i-m)(j-n)}{(i+k-2m)(j+l-2n)}\phi \overset{m+1,n+1}{\odot} \mathscr{T}\varphi \\
&+\tfrac{(-i+m)(l-n)(j+l-n+1)}{(l+1)(i+k-2m)(j+l-2n)}\phi \overset{m+1,n}{\odot} \mathscr{C}\varphi \\
&+\tfrac{(j-n)(-k+m)(i+k-m+1)}{(k+1)(i+k-2m)(j+l-2n)}\phi \overset{m,n+1}{\odot} \mathscr{C}^{\dagger}\varphi \\
&+\tfrac{(k-m)(l-n)(i+k-m+1)(j+l-n+1)}{(k+1)(l+1)(i+k-2m)(j+l-2n)}\phi \overset{m,n}{\odot} \mathscr{D}\varphi. \qquad (31d)
\end{aligned}
$$

Proof Collectively, the left hand sides can be written as $\nabla \overset{t,u}{\odot} (\phi \overset{m,n}{\odot} \varphi)$ where $t, u \in \{0, 1\}$. Let ∇_ϕ and ∇_φ be ∇ only differentiating ϕ respectively φ. From the relations (7a) and (8) we get

$$
\begin{aligned}
\nabla \overset{t,u}{\odot} (\phi \overset{m,n}{\odot} \varphi) &= \nabla_\phi \overset{t,u}{\odot} (\phi \overset{m,n}{\odot} \varphi) + \nabla_\varphi \overset{t,u}{\odot} (\phi \overset{m,n}{\odot} \varphi) \\
&= (-1)^{m+n}\nabla_\phi \overset{t,u}{\odot} (\varphi \overset{m,n}{\odot} \phi) + \nabla_\varphi \overset{t,u}{\odot} (\phi \overset{m,n}{\odot} \varphi) \\
&= (-1)^{m+n} \sum_{M=0}^{1}\sum_{N=0}^{1} F^{t,m,M}_{1,k,i} F^{u,n,N}_{1,l,j}\, \varphi \overset{t+m-M,u+n-N}{\odot} \nabla \overset{M,N}{\odot} \phi \\
&\quad + \sum_{M=0}^{1}\sum_{N=0}^{1} F^{t,m,M}_{1,i,k} F^{u,n,N}_{1,j,l}\, \phi \overset{t+m-M,u+n-N}{\odot} \nabla \overset{M,N}{\odot} \varphi. \qquad (32)
\end{aligned}
$$

Explicit calculations of the $F^{t,m,M}_{1,i,k}$ coefficients gives the relations (31). □

Example 2 Suppose $\phi \in \mathcal{S}_{s,0}$ is a spin-s field, $\mathscr{C}^{\dagger}\phi = 0$ and $\varphi \in \mathcal{S}_{k,0}$ is a Killing spinor, $\mathscr{T}\varphi = 0$. Then the right hand side of (31c) vanishes for $m = k$, i.e. the full contraction of the fields is a spin-($s - k/2$) field. This is known as Penrose's spin lowering, see [10, Equation (6.4.2)].

2.5 GHP expansion

In this section we collect equations to efficiently expand symmetric spinorial equations into GHP components. We point out that our older GHP package *SpinFrames* first expands spinors into a dyad and then takes components. The expansion was computationally expensive. In this section and the corresponding new package, there are closed forms for components of symmetric spinors, which is a huge improvement in performance.

Let us first briefly review the formalism, see [6] for details. Introducing a normalized spinor dyad (o_A, ι_A), $o_A\iota^A = 1$, a two dimensional subgroup of the Lorentz group is given by

$$o_A \to \lambda o_A, \qquad \iota_A \to \lambda^{-1}\iota_A, \tag{33}$$

with non-vanishing, complex scalar field λ. A field ϕ is said to be of GHP weight $\{p, q\}$ if it transforms via

$$\phi \to \lambda^p \bar{\lambda}^q \phi \tag{34}$$

under (33) and its complex conjugate. The Levi-Civita connection has a natural lift of the form

$$\Theta_{AA'} = \nabla_{AA'} - p\omega_{AA'} - q\bar{\omega}_{AA'}, \qquad \text{with } \omega_{AA'} = \iota^B \nabla_{AA'} o_B, \tag{35}$$

and is of weight zero in the sense that it maps $\{p, q\}$ weighted fields to $\{p, q\}$ fields. The GHP operators are given by the dyad expansion of (35),

$$\Theta_{AA'} = \iota_A \bar{\iota}_{A'} \,\text{þ} - \iota_A \bar{o}_{A'}\,\eth - o_A \bar{\iota}_{A'}\,\eth' + o_A \bar{o}_{A'}\,\text{þ}'. \tag{36}$$

The connection coefficients are defined as follows,

$$\Theta_{AA'} o_B = \Gamma_{AA'}\iota_B, \qquad \text{where } \Gamma_{AA'} = -\iota_A\bar{\iota}_{A'}\kappa + \iota_A\bar{o}_{A'}\sigma + o_A\bar{\iota}_{A'}\rho - o_A\bar{o}_{A'}\tau, \tag{37a}$$

$$\Theta_{AA'} \iota_B = \Gamma'_{AA'} o_B, \qquad \text{where } \Gamma'_{AA'} = -\iota_A\bar{\iota}_{A'}\tau' + \iota_A\bar{o}_{A'}\rho' + o_A\bar{\iota}_{A'}\sigma' - o_A\bar{o}_{A'}\kappa'. \tag{37b}$$

To express the dyad expansion of a general symmetric spinor, it is convenient to define a symmetric spinor basis $\mathcal{B}^{n,k}_{m,l}$ of weight $\{2n-k, 2m-l\}$ by

$$\mathcal{B}^{n,k}_{m,l}{}^{A'_1\dots A'_l}_{A_1\dots A_k} = o_{(A_1}\dots o_{A_n}\iota_{A_{n+1}}\dots\iota_{A_k)}\bar{o}^{(A'_1}\dots\bar{o}^{A'_m}\bar{\iota}^{A'_{m+1}}\dots\bar{\iota}^{A'_l)}. \tag{38}$$

In particular this allows us to mostly avoid spinor indices for the rest of this section. For a full contraction of two basis elements we find

$$\mathcal{B}^{n,k}_{m,l} \overset{k,l}{\odot} \mathcal{B}^{i,k}_{j,l} = (-1)^{(n+m)} \binom{k}{n}^{-1} \binom{l}{m}^{-1} \delta^n_{k-i}\delta^m_{l-j}, \tag{39}$$

where $\delta^a_b = 1$ if $a = b$ and zero otherwise. Now any $\phi \in \mathcal{S}_{k,l}$ can be expanded into

$$\phi = \sum_{i=0}^{k}\sum_{j=0}^{l}(-1)^{k-i+l-j}\binom{k}{i}\binom{l}{j}\phi_{ij'}\mathcal{B}^{i,k}_{j,l}, \tag{40}$$

where the scalar components of weight $\{k-2i, l-2j\}$ are defined by

$$\phi_{ij'} = \mathcal{B}^{k-i,k}_{l-j,l} \overset{k,l}{\odot} \phi. \tag{41}$$

The following two lemmas yield component expressions for general symmetric products and derivatives of symmetric spinors. This allows to expand general symmetric spinor differential equations into dyad components, without expanding the symmetrizations.

Lemma 11 *For $\phi \in \mathcal{S}_{i,j}$, $\varphi \in \mathcal{S}_{k,l}$ the symmetric product has components*

$$(\phi \overset{m,n}{\odot} \varphi)_{st'} = \sum_{p=0}^{k} \sum_{q=0}^{l} G^{m,p,s}_{i,k} G^{n,q,t}_{j,l} \phi_{(s+m-p)(t+n-q)'} \varphi_{pq'}, \tag{42}$$

with coefficients given by

$$G^{m,p,s}_{i,k} = \sum_{r=0}^{m} (-1)^r \frac{\binom{i}{s+m-p}\binom{i+p-s-m}{r}\binom{s+m-p}{m-r}\binom{k}{p}\binom{k-p}{m-r}\binom{p}{r}}{\binom{i}{m}\binom{k}{m}\binom{m}{r}\binom{i+k-2m}{s}}. \tag{43}$$

Proof For ease of notation we assume $\phi \in \mathcal{S}_{i,0}$, $\varphi \in \mathcal{S}_{k,0}$. Using the observation that $\mathcal{B}^{p,k}_{0,0} = \mathcal{B}^{0,k-p}_{0,0} \overset{0,0}{\odot} \mathcal{B}^{p,p}_{0,0}$, where $\mathcal{B}^{0,k-p}_{0,0}$ is a symmetric product of ι_A and $\mathcal{B}^{p,p}_{0,0}$ is a symmetric product of o_A, we can use (17) to obtain

$$\begin{aligned}
\mathcal{B}^{p,k}_{0,0}{}_{A_1\dots A_{k-m}}{}^{B_1\dots B_m} &= (\mathcal{B}^{0,k-p}_{0,0} \overset{0,0}{\odot} \mathcal{B}^{p,p}_{0,0})_{A_1\dots A_{k-m}}{}^{B_1\dots B_m} \\
&= \sum_{q=0}^{m} \frac{\binom{p}{q}\binom{k-p}{m-q}}{\binom{k}{m}} \mathcal{B}^{0,k-p}_{0,0}{}_{(A_1\dots A_{k-p-m+q}}{}^{(B_{q+1}\dots B_m} \mathcal{B}^{p,p}_{0,0}{}_{A_{k-p-m+q+1}\dots A_{k-m})}{}^{B_1\dots B_q)} \\
&= \sum_{q=0}^{m} \frac{\binom{p}{q}\binom{k-p}{m-q}}{\binom{k}{m}} \mathcal{B}^{p-q,k-m}_{0,0}{}_{A_1\dots A_{k-m}} \mathcal{B}^{q,m}_{0,0}{}^{B_1\dots B_m}
\end{aligned} \tag{44}$$

Using this in the expansion (40), we find

$$\varphi_{A_1\dots A_{k-m}B_1\dots B_m} = \sum_{p=0}^{k}\sum_{q=0}^{m} (-1)^{k-p} \frac{\binom{k}{p}\binom{k-p}{m-q}\binom{p}{q}}{\binom{k}{m}} \varphi_{p0'} \mathcal{B}^{p-q,k-m}_{0,0}{}_{A_1\dots A_{k-m}} \mathcal{B}^{q,m}_{0,0}{}_{B_1\dots B_m}, \tag{45a}$$

$$\phi_{A_1\dots A_{i-m}}{}^{B_1\dots B_m} = \sum_{r=0}^{i}\sum_{q=0}^{m} (-1)^{i-r} \frac{\binom{i}{r}\binom{i-r}{q}\binom{r}{m-q}}{\binom{i}{m}} \phi_{r0'} \mathcal{B}^{r-m+q,i-m}_{0,0}{}_{A_1\dots A_{i-m}} \mathcal{B}^{q,m}_{0,0}{}^{B_1\dots B_m}. \tag{45b}$$

Contracting the B indices, symmetrizing and using (39) yield

$$\begin{aligned}
&\phi_{(A_1\dots A_{i-m}}{}^{B_1\dots B_m} \varphi_{A_{i-m+1}\dots A_{i+k-2m})B_1\dots B_m} \\
&= \sum_{r=0}^{i}\sum_{p=0}^{k}\sum_{q=0}^{m} (-1)^{k+i-r-p} \frac{\binom{i}{r}\binom{i-r}{q}\binom{r}{m-q}\binom{k}{p}\binom{k-p}{m-q}\binom{p}{q}}{\binom{i}{m}\binom{k}{m}} \phi_{r0'}\varphi_{p0'}
\end{aligned}$$

$$
\begin{aligned}
&\mathcal{B}_{0,0}^{p+r-m,i+k-2m}{}_{A_1\dots A_{i+k-2m}} \\
&\times \mathcal{B}_{0,0}^{q,m}{}_{B_1\dots B_m}\mathcal{B}_{0,0}^{q,m\,B_1\dots B_m} \\
&= \sum_{r=0}^{i}\sum_{p=0}^{k}\sum_{q=0}^{m}(-1)^{k+i-r-p+m-q} \\
&\frac{\binom{i}{r}\binom{i-r}{q}\binom{r}{m-q}\binom{k}{p}\binom{k-p}{m-q}\binom{p}{q}}{\binom{i}{m}\binom{k}{m}\binom{m}{q}}\phi_{r0'}\varphi_{p0'}\mathcal{B}_{0,0}^{p+r-m,i+k-2m}{}_{A_1\dots A_{i+k-2m}}.
\end{aligned}
\tag{46}
$$

The relation (39) then gives

$$
\begin{aligned}
(\phi \overset{m,0}{\odot} \varphi)_{s0'} &= \mathcal{B}_{0,0}^{i+k-2m-s,i+k-2m} \overset{i+k-2m,0}{\odot} (\phi \overset{m,0}{\odot} \varphi) \\
&= \sum_{r=0}^{i}\sum_{p=0}^{k}\sum_{q=0}^{m}(-1)^{-r-p+m-q-s}\frac{\binom{i}{r}\binom{i-r}{q}\binom{r}{m-q}\binom{k}{p}\binom{k-p}{m-q}\binom{p}{q}}{\binom{i}{m}\binom{k}{m}\binom{m}{q}\binom{i+k-2m}{i+k-2m-s}}\phi_{r0'}\varphi_{p0'}\delta_r^{m+s-p} \\
&= \sum_{p=0}^{k}\sum_{q=0}^{m}(-1)^{q}\frac{\binom{i}{m+s-p}\binom{i+p-m-s}{q}\binom{m+s-p}{m-q}\binom{k}{p}\binom{k-p}{m-q}\binom{p}{q}}{\binom{i}{m}\binom{k}{m}\binom{m}{q}\binom{i+k-2m}{i+k-2m-s}}\phi_{(m+s-p)0'}\varphi_{p0'} \\
&= \sum_{p=0}^{k} G_{i,k}^{m,p,s}\phi_{(s+m-p)0'}\varphi_{p0'}.
\end{aligned}
\tag{47}
$$

The primed indices gives an analogous expansion and the combination yields (42). □

Lemma 12 *The GHP components of fundamental spinor operators (26) on $\phi \in \mathcal{S}_{k,l}$ take the form*

$$
\begin{aligned}
(\mathscr{D}\phi)_{ij'} = {} & (\text{þ} - (k-i)\rho - (l-j)\bar{\rho})\phi_{(i+1)(j+1)'} + (\text{þ}' - (i+1)\rho' - (j+1)\bar{\rho}')\phi_{ij'} \\
& - (\eth - (k-i)\tau - (j+1)\bar{\tau}')\phi_{(i+1)j'} - (\eth' - (i+1)\tau' - (l-j)\bar{\tau})\phi_{i(j+1)'} \\
& + (k-i-1)\kappa\phi_{(i+2)(j+1)'} - (k-i-1)\sigma\phi_{(i+2)j'} - i\sigma'\phi_{(i-1)(j+1)'} \\
& + i\kappa'\phi_{(i-1)j'} + (l-j-1)\bar{\kappa}\phi_{(i+1)(j+2)'} - (l-j-1)\bar{\sigma}\phi_{i(j+2)'} \\
& - j\bar{\sigma}'\phi_{(i+1)(j-1)'} + j\bar{\kappa}'\phi_{i(j-1)'},
\end{aligned}
\tag{48a}
$$

$$
\begin{aligned}
(\mathscr{C}\phi)_{ij'} = {} & \big(-(k-i+1)(\text{þ} + i\rho - (l-j)\bar{\rho})\phi_{i(j+1)'} + i(\text{þ}' + (k-i+1)\rho' \\
& - (j+1)\bar{\rho}')\phi_{(i-1)j'}x + (k-i+1)(\eth + i\tau - (j+1)\bar{\tau}')\phi_{ij'} \\
& - i(\eth' + (k-i+1)\tau' - (l-j)\bar{\tau})\phi_{(i-1)(j+1)'} \\
& - (k-i+1)(k-i)\kappa\phi_{(i+1)(j+1)'} + (k-i+1)(k-i)\sigma\phi_{(i+1)j'} \\
& - i(i-1)\sigma'\phi_{(i-2)(j+1)'} + i(i-1)\kappa'\phi_{(i-2)j'} - (k-i+1)(l-j-1)\bar{\kappa}\phi_{i(j+2)'} \\
& - i(l-j-1)\bar{\sigma}\phi_{(i-1)(j+2)'} + (k-i+1)j\bar{\sigma}'\phi_{i(j-1)'} \\
& + ij\bar{\kappa}'\phi_{(i-1)(j-1)'}\big)/(k+1),
\end{aligned}
\tag{48b}
$$

$$
\begin{aligned}
(\mathscr{C}^{\dagger}\phi)_{ij'} = {} & \big(-(l-j+1)(\text{þ} + j\bar{\rho} - (k-i)\rho)\phi_{(i+1)j'} + j(\text{þ}' \\
& + (l-j+1)\bar{\rho}' - (i+1)\rho')\phi_{i(j-1)'} \\
& + (l-j+1)(\eth' + j\bar{\tau} - (i+1)\tau')\phi_{ij'}
\end{aligned}
$$

$$
\begin{aligned}
&\quad - j(\eth + (l-j+1)\bar{\tau}' - (k-i)\tau)\phi_{(i+1)(j-1)'} - (l-j+1)(l-j)\bar{\kappa}\phi_{(i+1)(j+1)'} \\
&\quad + (l-j+1)(l-j)\bar{\sigma}\phi_{i(j+1)'} - j(j-1)\bar{\sigma}'\phi_{(i+1)(j-2)'} \\
&\quad + j(j-1)\bar{\kappa}'\phi_{i(j-2)'} \\
&\quad - (l-j+1)(k-i-1)\kappa\phi_{(i+2)j'} - j(k-i-1)\sigma\phi_{(i+2)(j-1)'} \\
&\quad + (l-j+1)i\sigma'\phi_{(i-1)j'} + ij\kappa'\phi_{(i-1)(j-1)'}\big)/(l+1), \qquad (48c)\\
(\mathscr{T}\phi)_{ij'} &= \big((k+1-i)(l+1-j)(\text{þ} + i\rho + j\bar{\rho})\phi_{ij'} \\
&\quad + (k+1-i)j(\eth + i\tau + (l-j+1)\bar{\tau}')\phi_{i(j-1)'} \\
&\quad + i(l+1-j)(\eth' + (k-i+1)\tau' + j\bar{\tau})\phi_{(i-1)j'} \\
&\quad + ij(\text{þ}' + (k-i+1)\rho' + (l-j+1)\bar{\rho}')\phi_{(i-1)(j-1)'} \\
&\quad + (k-i+1)(k-i)(l+1-j)\kappa\phi_{(i+1)j'} + (k-i+1)(k-i)j\sigma\phi_{(i+1)(j-1)'} \\
&\quad + i(i-1)(l+1-j)\sigma'\phi_{(i-2)j'} + i(i-1)j\kappa'\phi_{(i-2)(j-1)'} \\
&\quad + (k+1-i)(l+1-j)(l-j)\bar{\kappa}\phi_{i(j+1)'} + i(l+1-j)(l-j)\bar{\sigma}\phi_{(i-1)(j+1)'} \\
&\quad + (k+1-i)j(j-1)\bar{\sigma}'\phi_{i(j-2)'} + ij(j-1)\bar{\kappa}'\phi_{(i-1)(j-2)'}\big)/\big((k+1)(l+1)\big) \qquad (48d)
\end{aligned}
$$

Proof To prove (48a), we start by expanding the argument of $\mathscr{D}\phi$ using (40) and contract with a symmetric basis as in (41),

$$
\begin{aligned}
(\mathscr{D}\phi)_{ij'} &= \mathcal{B}^{k-1-i,k-1}_{l-1-j,l-1} \overset{k,l}{\odot} (\mathscr{D}\phi) \\
&= \sum_{n=0}^{k}\sum_{m=0}^{l}(-1)^{k-n+l-m}\binom{k}{n}\binom{l}{m}\mathcal{B}^{k-1-i,k-1}_{l-1-j,l-1} \overset{k,l}{\odot} (\mathscr{D}(\phi_{nm'}\mathcal{B}^{n,k}_{m,l})). \qquad (49)
\end{aligned}
$$

Next, we use the Leibniz rule (31d), but switch to the GHP connection $\Theta_{AA'}$ (so the fundamental spinor operators are with respect to $\Theta_{AA'}$ instead of $\nabla_{AA'}$) as the GHP components and the basis elements are GHP weighted,

$$
\mathscr{D}(\phi_{nm'} \underset{k,l}{\overset{0,0}{\odot}} \mathcal{B}^{n,k}_{m,l}) = \mathcal{B}^{n,k}_{m,l} \overset{1,1}{\odot} \mathscr{T}\phi_{nm} + \phi_{nm} \overset{0,0}{\odot} \mathscr{D}\mathcal{B}^{n,k}_{m,l}. \qquad (50)
$$

From (36) and (37) we have

$$
\mathscr{T}\phi_{nm'} = (\text{þ}\,\phi_{nm'})\mathcal{B}^{0,1}_{0,1} - (\eth\,\phi_{nm'})\mathcal{B}^{0,1}_{1,1} - (\eth'\,\phi_{nm'})\mathcal{B}^{1,1}_{0,1} + (\text{þ}'\,\phi_{nm'})\mathcal{B}^{1,1}_{1,1}, \qquad (51)
$$

and

$$
\mathscr{D}\mathcal{B}^{n,k}_{m,l} = n\Gamma \overset{1,1}{\odot} \mathcal{B}^{n-1,k}_{m,l} + (k-n)\Gamma' \overset{1,1}{\odot} \mathcal{B}^{n+1,k}_{m,l} + m\overline{\Gamma} \overset{1,1}{\odot} \mathcal{B}^{n,k}_{m-1,l} + (l-m)\overline{\Gamma'} \overset{1,1}{\odot} \mathcal{B}^{n,k}_{m+1,l}. \qquad (52)
$$

Inserting (51), (52) back into (50) and expanding Γ, Γ' into the basis we can use the contraction rules

$$\mathcal{B}^{n,k}_{m,l} \overset{1,1}{\odot} \mathcal{B}^{0,1}_{0,1} = \frac{mn}{kl}\mathcal{B}^{n-1,k-1}_{m-1,l-1}, \tag{53a}$$

$$\mathcal{B}^{n,k}_{m,l} \overset{1,1}{\odot} \mathcal{B}^{1,1}_{0,1} = -\frac{m(k-n)}{kl}\mathcal{B}^{n,k-1}_{m-1,l-1}, \tag{53b}$$

$$\mathcal{B}^{n,k}_{m,l} \overset{1,1}{\odot} \mathcal{B}^{0,1}_{1,1} = -\frac{n(l-m)}{kl}\mathcal{B}^{n-1,k-1}_{m,l-1}, \tag{53c}$$

$$\mathcal{B}^{n,k}_{m,l} \overset{1,1}{\odot} \mathcal{B}^{1,1}_{1,1} = \frac{(k-n)(l-m)}{kl}\mathcal{B}^{n,k-1}_{m,l-1}, \tag{53d}$$

which are easily verified by expanding out the symmetries. The result can now be substituted into (49). Each term has a full contraction of the form (39) which cancels the double sum due to the δ factors. After some elementary algebra, the end result is given by (48a). The other expansions can be verified along the same lines, the only minor computation that needs to be done is the analog of (52) and (53). □

Example 3 For $\phi \in \mathcal{S}_{2s,0}$, (48c) corresponds to the dyad components of the spin-s field equation, c.f. [9, Equation (4.12.44)]. For $\phi \in \mathcal{S}_{1,0}$, (48d) corresponds to the dyad components of the twistor equation, c.f. [9, Equation (4.12.46)].

3 SymSpin: a computer algebra implementation in xAct

The *xAct* [8] suite for *Mathematica* is an open source project mainly devoted to symbolic computation in differential geometry and tensor algebra. In this section we introduce our contributed package *SymSpin* [5] which contains the formalism of Sect. 2. For syntax and more examples, see *SymSpinDoc.nb* on that page.

3.1 Loading the package and defining structures

Load the package, define a four dimensional manifold M4, and Lorentzian metric with

```
In := <<xAct`SymSpin`
      $DefInfoQ=False;
      DefManifold[M4,4,{a,b,c,d}]
      DefMetric[{1,3,0},g[-a,-b],CD]
```
(54)

By default the valence numbers are displayed for each operator and complex conjugates are written with †. To keep the notation the same as in the rest of the paper, we can change the display form with

```
In := SetOptions[DefAbstractIndex,PrintAs->PrimeDagger];
      SetOptions[DefSpinor, PrintDaggerAs->AddBar];                    (55)
      SetOptions[DefFundSpinOperators,ShowValenceInfo
      ->False];
```

Define the spin structure, initialize *SymSpin* and define the fundamental spinor operators with

```
In := DefSpinStructure[g,Spin,{A,B,C,F,G,H,P,Q,R},ϵ,
      σ,CDe,{";","∇"},
      SpinorPrefix->SP,SpinorMark->"S"]                                (56)
      InitSymSpin[σ];
      DefFundSpinOperators[CDe];
```

3.2 Example: coefficients

Assume that K, L and M are symmetric spinor fields, and we want to find under which conditions of K, L and M the equation

$$0 = K_{AB}{}^{FH} L_F{}^C \varphi_{HC} + M_{(A}{}^C \varphi_{B)C}. \tag{57}$$

holds for all symmetric spinor fields φ. The following calculation leads to the conditions

$$K^G{}_{(ABC} L_{|G|F)} = 0, \qquad M_{AB} = \tfrac{1}{2} K^{CF}{}_{AB} L_{CF}. \tag{58}$$

We first define the symmetric spinor fields. For clarity we have added the valence numbers to the names of the spinors, but not the display form.

```
In := DefSymmetricSpinor[φ20,2,0,Spin,"φ"]
      DefSymmetricSpinor[K40,4,0,Spin,"K"]
      DefSymmetricSpinor[L20,2,0,Spin,"L"]                             (59)
      DefSymmetricSpinor[M20,2,0,Spin,"M"]
```

One can start with the indexed version of the spinor equation.

```
In := OriginalEq=0==K40[-A,-B,F,H]L20[-F,C]φ20[-H,-C]
      +ImposeSym[M20[-A,C]*φ20[-B,-C]]                                 (60)
```

Out = $0 == K_{AB}{}^{FH} L_F{}^C \varphi_{HC} + \underset{(13)}{Sym}[M\varphi]_A{}^C{}_{BC}$

To convert this to the new formalism, we need the irreducible decomposition of the product of the L and ϕ spinor.

```
In := IrrDecomposeSymMult[L20,φ20,{0,0}]
```

$$\text{Out} = \; L_{AB}\varphi_{CF} == -\underset{(13)(24)}{Sym}[\epsilon(L\overset{1,0}{\odot}\varphi)]_{ACBF} + (L\overset{0,0}{\odot}\varphi)_{ABCF} + \tfrac{1}{3}\underset{(13)(24)}{Sym}[\epsilon\epsilon]_{ACBF}(L\overset{2,0}{\odot}\varphi) \tag{61}$$

It is convenient to work with the expanded and canonicalized version

```
In := L20φ20IrrDecEq=ToCanonical@ExpandSym@%
```

$$\text{Out} = \; L_{AB}\varphi_{CF} == (L\overset{0,0}{\odot}\varphi)_{ABCF} - \tfrac{1}{4}\epsilon_{BF}(L\overset{1,0}{\odot}\varphi)_{AC} - \tfrac{1}{4}\epsilon_{BC}(L\overset{1,0}{\odot}\varphi)_{AF} - \tfrac{1}{4}\epsilon_{AF}(L\overset{1,0}{\odot}\varphi)_{BC} \tag{62}$$
$$-\tfrac{1}{4}\epsilon_{AC}(L\overset{1,0}{\odot}\varphi)_{BF} + \tfrac{1}{6}\epsilon_{AF}\epsilon_{BC}(L\overset{2,0}{\odot}\varphi) + \tfrac{1}{6}\epsilon_{AC}\epsilon_{BF}(L\overset{2,0}{\odot}\varphi)$$

To work efficiently we turn the original equation into an index-free version. One could also use the index-free version as a starting point.

```
In := IndexFreeEq=ToIndexFree[ToCanonical@ContractMetric[OriginalEq
      /.EqToRule@L20φ20IrrDecEq]//.SymHToSymMultRule]
      /.MultScalToSymMultRule[Spin]/.SortSymMult[Not@FreeQ[#,φ20]&]
```

$$\text{Out} = \; 0 == M\overset{1,0}{\odot}\varphi + K\overset{2,0}{\odot}L\overset{1,0}{\odot}\varphi \tag{63}$$

We can turn the spinor valued equation into a scalar equation by contracting it with a dummy spinor T to turn the free indices into contracted dummy indices. This dummy spinor is defined by

```
In := DefSymmetricSpinor[T20,2,0,Spin,"T"]
```

(64)

As the field φ and the dummy spinor T both should be arbitrary, we see that the irreducible components of their product can be treated as independent arbitrary fields. For convenience we make a list of them with

```
In := IrrDecComps=SymMult[T20,#,0,Spin][φ20]&/@Range[0,2]
```

$$\text{Out} = \; \{(T\overset{0,0}{\odot}\varphi),\;(T\overset{1,0}{\odot}\varphi),\;(T\overset{2,0}{\odot}\varphi)\} \tag{65}$$

We can now contract our index-free equation with T.

```
In := SymMult[T20,2,0]/@IndexFreeEq
```

$$\text{Out} = \; 0 == T\overset{2,0}{\odot}M\overset{1,0}{\odot}\varphi + T\overset{2,0}{\odot}K\overset{2,0}{\odot}L\overset{1,0}{\odot}\varphi \tag{66}$$

Commute T inside, so that T is directly contracted with the field φ, so we obtain the independent spinors in the list `IrrDecComps`.

In := **`%//.CommuteSymMultRuleIn[T20]`**

$$\text{Out} = \quad 0 == -M \overset{2,0}{\odot} T \overset{1,0}{\odot} \varphi + K \overset{4,0}{\odot} L \overset{1,0}{\odot} T \overset{0,0}{\odot} \varphi + \tfrac{1}{2} K \overset{4,0}{\odot} L \overset{0,0}{\odot} T \overset{1,0}{\odot} \varphi \tag{67}$$

Now, these independent spinors are moved out and to the left.

In := **`%/.SortSymMultReverse[MemberQ[IrrDecComps,#]&]`**
`//.Flatten[CommuteSymMultRuleOut/@IrrDecComps]`

$$\text{Out} = \quad 0 == -(T \overset{1,0}{\odot} \varphi) \overset{2,0}{\odot} M + (T \overset{0,0}{\odot} \varphi) \overset{4,0}{\odot} K \overset{1,0}{\odot} L + \tfrac{1}{2} (T \overset{1,0}{\odot} \varphi) \overset{2,0}{\odot} K \overset{2,0}{\odot} L \tag{68}$$

From this one can conclude that the coefficients of $(T \overset{1,0}{\odot} \varphi)$ and $(T \overset{0,0}{\odot} \varphi)$ both have to be zero.

As a convenience, we have implemented all of the steps from the index-free equation to the final list of equations in one function.

In := **`ExtractCoeffsIndexFree[IndexFreeEq,φ20]`**

$$\text{Out} = \quad \{0 == (K \overset{1,0}{\odot} L),\ \ 0 == -M + \tfrac{1}{2} K \overset{2,0}{\odot} L\} \tag{69}$$

This can be translated back to the indexed form with

In := **`ToIndexed/@%`**

$$\text{Out} = \quad \{0 == \underset{(2346)}{Sym}[KL]^{G}{}_{ABCGF},\ \ 0 == \tfrac{1}{2} K^{CF}{}_{AB} L_{CF} - M_{AB}\} \tag{70}$$

Performing this kind of calculation in the indexed form would require expansions of symmetries and several steps of irreducible decompositions of different products. This new method was heavily used in [7].

3.3 Example: derivatives

To also demonstrate how to work with derivatives we use the previously defined field φ and define a valence $(3, 2)$ field ψ via

In := **`DefSymmetricSpinor[ψ32,3,2,Spin,"ψ"]`** (71)

The covariant derivative

In := **`CDe[-A,-A†]@ψ32[-B,-C,-F,-B†,-C†]`**

$$\text{Out} = \quad \nabla_{AA'} \psi_{BCFB'C'} \tag{72}$$

can be decomposed into the fundamental spinor operators with

In := `%==ToFundSpinOp[%]`

$$\text{Out}= \nabla_{AA'}\psi_{BCFB'C'} == -\tfrac{1}{3}\bar{\epsilon}_{A'C'}(\mathscr{C}\psi)_{ABCFB'} - \tfrac{1}{3}\bar{\epsilon}_{A'B'}(\mathscr{C}\psi)_{ABCFC'} - \tfrac{1}{4}\epsilon_{AF}(\mathscr{C}^{\dagger}\psi)_{BCA'B'C'} - \tfrac{1}{4}\epsilon_{AC}(\mathscr{C}^{\dagger}\psi)_{BFA'B'C'} - \tfrac{1}{4}\epsilon_{AB}(\mathscr{C}^{\dagger}\psi)_{CFA'B'C'} + \tfrac{1}{12}\epsilon_{AF}\bar{\epsilon}_{A'C'}(\mathscr{D}\psi)_{BCB'} + \tfrac{1}{12}\epsilon_{AF}\bar{\epsilon}_{A'B'}(\mathscr{D}\psi)_{BCC'} + \tfrac{1}{12}\epsilon_{AC}\bar{\epsilon}_{A'C'}(\mathscr{D}\psi)_{BFB'} + \tfrac{1}{12}\epsilon_{AC}\bar{\epsilon}_{A'B'}(\mathscr{D}\psi)_{BFC'} + \tfrac{1}{12}\epsilon_{AB}\bar{\epsilon}_{A'C'}(\mathscr{D}\psi)_{CFB'} + \tfrac{1}{12}\epsilon_{AB}\bar{\epsilon}_{A'B'}(\mathscr{D}\psi)_{CFC'} + (\mathscr{T}\psi)_{ABCFA'B'C'} \tag{73}$$

Commutators can be handled like

In := `DivCDe@CurlDgCDe@`ψ`32`

$$\text{Out}= (\mathscr{D}\mathscr{C}^{\dagger}\psi) \tag{74}$$

In := `%==(%/.CommuteOp[DivCDe,CurlDgCDe])`

$$\text{Out}= (\mathscr{D}\mathscr{C}^{\dagger}\psi) == \tfrac{2}{3}\mathscr{C}^{\dagger}\mathscr{D}\psi + \Psi\overset{3,0}{\odot}\psi + 2\Phi\overset{2,1}{\odot}\psi \tag{75}$$

Derivatives of products can also be handled efficiently

In := `CurlDgCDe@SymMult[`φ`20,1,0]@`ψ`32`

$$\text{Out}= (\mathscr{C}^{\dagger}\varphi\overset{1,0}{\odot}\psi) \tag{76}$$

In := `%==(%/.SymMultLeibnizRules[CDe])`

$$\text{Out}= (\mathscr{C}^{\dagger}\varphi\overset{1,0}{\odot}\psi) == \tfrac{2}{3}\psi\overset{2,0}{\odot}\mathscr{T}\varphi - \tfrac{5}{9}\psi\overset{1,0}{\odot}\mathscr{C}^{\dagger}\varphi - \tfrac{1}{3}\varphi\overset{2,0}{\odot}\mathscr{T}\psi + \tfrac{5}{6}\varphi\overset{1,0}{\odot}\mathscr{C}^{\dagger}\psi \tag{77}$$

4 Conclusions and discussion

In this work, we introduced an algebra on symmetric 2-spinors and the corresponding *SymSpin* package for the *Mathematica* suite *xAct*. In various research projects of the authors this algebra turned out to be a very efficient way to perform calculations. For example in [7] it is used to derive conditions on the spacetime for the existence of second order symmetry operators for the massive Dirac equation. This greatly simplified the calculations compared to the earlier approach [4], where only parts of the formalism were used to investigate symmetry operators for the massless Dirac and the Maxwell equations. Potential future applications include higher order perturbation theory as well as classification of symmetry operators for other field equations.

The formalism is very efficient for cases where each spinor appears only once in each product. Choosing a preferred ordering of the factors in each product, one can use the relations in Theorem 3 to rewrite them in a canonical form. However, if a spinor appears multiple times in a product the relations in Theorem 3 can give non-trivial equations where a term of the same form can appear both in the left and right hand sides as well as in several equations. Solving these equations, it should be possible to develop a method to write such products in a canonical form. So far, we treat the cases needed (i.e. for specific valences) separately, and plan to continue the development of these tools for the general case in the future.

Acknowledgements The authors are grateful to Simon Jacobsson for testing of the *xAct* implementation and to Teake Nutma for LaTeX typesetting code of *Mathematica* expressions.

Funding Open access funding provided by Chalmers University of Technology.

Data availability No data was used or produced in this work.

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